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Finite generators for countable group actions; Finite index pairs of equivalence relations; Complexity measures for recursive programs

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Finite generators for countable group actions; Finite index pairs of equivalence relations; Complexity measures for recursive programs

A dissertation submitted in partial satisfaction
of the requirements for the degree
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by

Anush Tserunyan

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Abstract of the Dissertation

Finite generators for countable group actions; Finite index pairs of equivalence relations; Complexity measures for recursive programs

by

Anush Tserunyan

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2013

Professor Alexander S. Kechris, Co-chair

Professor Itay Neeman, Co-chair

Part 1: Consider a continuous action of a countable group G on a Polish space X. A countable Borel partition $\mathcal{P}$ of X is called a *generator* if $G\mathcal{P} := \{gP : g \in G, P \in \mathcal{P}\}$ generates the Borel σ-algebra of X. It was asked by Benjamin Weiss in '87 whether the nonexistence of an invariant probability measure implies the existence of a finite generator. The main result of this part is obtaining a positive answer to this question in case X is σ-compact (in particular, when X is locally compact). We also show that finite generators always exist modulo a meager set, answering positively a question raised by Alexander Kechris in the mid-'90s.

Part 2: We investigate pairs of countable Borel equivalence relations $E \subseteq F$, where E is of finite index in F. Our main focus is the well-known problem of whether the treeability of E implies that of F: we provide various reformulations of it and reduce it to one natural universal example. In the measure-theoretic context, assuming that F is ergodic, we characterize the case when E is normal. Finally, in the ergodic case, we characterize the equivalence relations that arise from almost free actions of virtually free groups.

Part 3: We consider natural complexity measures for recursive programs from given primitives and derive inequalities between them, answering a question asked by Yiannis Moschovakis.

The dissertation of Anush Tserunyan is approved.

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2013

To my mom and dad, Irina and Vardan, and my sister, Arevik

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Acknowledgments

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Vita

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Publications

Anush Tserunyan, *Finite generators for countable group actions in the Borel and Baire category settings*, submitted (2012)

Anush Tserunyan, *Characterization of a class of graphs related to pairs of disjoint matchings*, Discrete Mathematics **309** (2009), no. 4, 693-713

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Introduction

The current thesis consists of three unrelated parts, each representing a separate research project. Parts 1 and 2 were done under the supervision of my advisor Alexander Kechris, and they fall into the general area of *descriptive set theory*, more specifically, the study of definable equivalence relations and group actions with applications to *ergodic theory* and *topological dynamics*. Part 3 was done under the supervision of Yiannis Moschovakis, and it lies in *complexity theory*; more specifically, it concerns recursive programs from given primitives and relations between different complexity measures. Below I give a brief introduction to the aforementioned general areas of research without going into the research projects and contributions of this thesis. The latter are contained in the following three parts, each of which is self-contained and starts with an extensive introduction to the research project it represents, providing background, motivation and the main results.

Descriptive set theory and definable equivalence relations

Descriptive set theory (DST) combines techniques from set theory, topology, analysis, recursion theory and other areas of mathematics to study definable subsets of $\mathbb{R}$ or, more generally, of any Polish space (see [Kec95]). Examples of such sets include Borel, analytic (projections of Borel), co-analytic (complement of analytic), etc. The framework of Polish spaces being used is justified by its robustness since, by Kuratowski's theorem, Polish spaces of the same cardinality are Borel isomorphic. A typical example (one of the first) of a theorem in DST is Souslin's theorem that states that if a set is both analytic and co-analytic, then it is Borel. At its earlier stage, a central interest in DST was investigating the regularity properties of definable sets such as the perfect set property (being countable or containing a perfect set, a version of Continuum Hypothesis that Cantor proved for closed sets), measurability and the Baire property. As it turned out, all these properties are satisfied by analytic sets, but curiously enough, whether they hold for the projections of co-analytic sets is already independent from ZFC.

For the past twenty years, a major focus of descriptive set theory has been the study of equivalence relations on Polish spaces that are definable when viewed as sets of pairs (e.g. orbit equivalence relations of continuous actions of Polish groups are analytic). This study is motivated by foundational questions such as understanding the nature of complete classification of mathematical objects (measure preserving transformations, unitary operators, Riemann surfaces, etc.) up to some notion of equivalence (isomorphism, conjugacy, etc.) and creating a mathematical framework for measuring the complexity of such classification problems. Due to its broad scope, it has natural interactions with other areas of mathematics, such as ergodic theory and topological dynamics, functional analysis and operator algebras, representation theory, topology, model theory and recursion theory.

The following definition makes precise what it means for one classification problem to be easier (not harder) than another.

Definition. *Let E and F be equivalence relations on Polish spaces X and Y, respectively. We say that E is Borel reducible to F and write $E \leq_B F$ if there is a Borel map $f : X \to Y$ such that for all $x_0, x_1 \in X$, $x_0 E x_1 \iff f(x_0) F f(x_1)$.*

We call E *smooth* (or *concretely classifiable*) if it is Borel reducible to the identity relation $\mathrm{id}(X)$ on some (equivalently, any) Polish space X. An example of such an equivalence relation is the similarity relation of matrices; indeed, if $J(A)$ denotes the Jordan canonical form of a matrix $A \in \mathbb{R}^{n^2}$, then for $A, B \in \mathbb{R}^{n^2}$, we have $A \sim B \iff J(A) = J(B)$. It is not hard to check that the computation of $J(A)$ is Borel, so $J : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is a Borel reduction of $\sim$ to $\mathrm{id}(\mathbb{R}^{n^2})$, and hence $\sim$ is smooth. Another (much more involved) example is the isomorphism of Bernoulli shifts, which, by Ornstein's famous theorem, is reduced to the equality on $\mathbb{R}$ by the map assigning to each Bernoulli shift its entropy.

However, many equivalence relations that appear in mathematics are nonsmooth. For example, the equivalence relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$ of eventual equality of binary sequences can be easily shown to be nonsmooth, using measure-theoretic or Baire category arguments. The following theorem, known as the General Glimm-Effros dichotomy [HKL90], shows that in fact containing $\mathbb{E}_0$ is the only obstruction to smoothness:

Theorem (Harrington-Kechris-Louveau '90)**.** *Let E be a Borel equivalence relation on a Polish space X. Then either E is smooth, or else $E_0 \sqsubseteq_c E$.*

Here, $\sqsubseteq_c$ means that there is an injective continuous reduction. This theorem was one of the first major victories of descriptive set theory in the study of equivalence relations. It in particular implies that $\mathbb{E}_0$ is the easiest among all nonsmooth Borel equivalence relations in the sense of Borel reducibility. Besides its foundational importance in the theory of Borel equivalence relations, it also generalized earlier important results of Glimm and Effros. By now, several other dichotomy theorems have been proved and general methods of placing a given equivalence relation among others in the Borel reducibility hierarchy have been developed. However, there are still many fascinating open problems left, and many parts of the Borel reducibility hierarchy are yet to be understood.

Among Borel equivalence relations, an essential role is played by countable Borel equivalence relations, i.e. those whose equivalence classes are countable. A Borel action of a countable group on a Polish space induces such an equivalence relation (the orbit equivalence relation), and conversely, the Feldman-Moore theorem states that all of the countable Borel equivalence relations arise in this fashion. Thus, although often originating in ergodic theory or topological dynamics, problems about countable group actions naturally fall into the context of equivalence relations.

In Part 1 we study the question of the existence of finite generators[1] for actions of countable groups in the Borel and Baire category settings. Part 2, however, concerns finite index extensions of countable equivalence relations and the question of whether the class of treeable equivalence relations is closed under this operation.

[1] Certain kinds of partitions of the space on which the group acts.

Complexity theory and recursive programs from given primitives

Complexity theory is a very active area of research that lies in the intersection of mathematics and theoretical computer science. One of its main focuses is classifying computational problems according to their inherent difficulty; for example, the minimum number of steps required.

When we want to establish lower bounds for some measure of computational complexity, the standard methodology is to fix a rigorously defined model of computation, such as Turing machines or random access machines, and to specify a representation of the input, e.g. unary or binary coding for natural numbers, adjacency matrices for graphs, etc. Depending on the problem, it is often convenient to use one or another model of computation in obtaining lower bounds, and thus, there is a need to compare lower bounds established for different models of computation. We can use the fact that one model of computation can be simulated by another and this simulation is typically polynomial-time. This resolves the issue if the lower bounds under consideration are not sensitive to polynomial-time perturbations: for example, in case of super-polynomial or exponential lower bounds.

However, the issue remains if the lower bounds are smaller, e.g. logarithmic or linear. In this case, the computational complexity heavily depends on what is considered as *one step* in the given model of computation. In other words, what are the given primitives (functions and relations) in that model. For example, the primitives of a Turing machine are the functions that increment or decrement the pointer i (the position of the head) by one and switch the i^{th} bit of the binary representation of the input from 0 to 1, or vice versa. Thus, it is convenient to consider a general model of computation that does not have a fixed set of primitives, but rather allows specifying one for each individual algorithm. Such a model is that of recursive programs, and it was extensively used in [MvdD04], [MvdD09] and [Mos]. Instead of defining recursive programs, I will give an example considered in [MvdD04] and leave the rigorous definition for Part 3.

The following is the Euclidean algorithm specified by a recursive program:

$$\gcd(a,b) = \begin{cases} b & \text{if } \mathrm{rem}(a,b) = 0 \\ \gcd(b, \mathrm{rem}(a,b)) & \text{otherwise} \end{cases} . \qquad (a \geq b \geq 1)$$

Here the primitives are the relation of equality to 0 and the function $\mathrm{rem}(a,b)$, which computes the remainder in the division of a by b. It is easy to see that this algorithm requires at most $3 \log_2 a$ steps (counting each call to primitives as one step). In [MvdD04], the authors conjecture that this algorithm is, up to a constant, the best algorithm among all algorithms that compute $\gcd(a,b)$, and they show a lower bound of $\frac{1}{10} \log_2 \log_2 a$ for all such algorithms.

In [Mos], different measures of complexity for recursive programs are considered, as often different methods may provide lower bounds for different measures of complexity. Hence, it is important to investigate the relations between these complexity measures, and this is the topic of Part 3. The main result is that (roughly speaking) the actual complexity of a recursive program on a given input comes from the *number of calls to primitives* made by the program, and not from *the logical operations* (such as "`if ... then ... else ...`"): those only add a constant factor that depends on the length of the code of the program and not the input.

Part 1

Finite generators for countable group actions

CHAPTER I

Introduction to finite generators and the main results

1 Background and motivation

Throughout this part of the thesis, let G denote a countably infinite discrete group. Let X be a Borel G-space, i.e. a standard Borel space equipped with a Borel action of G.

Consider the following game: Player I chooses a finite or countable Borel partition $\mathcal{P} = \{P_n\}_{n<k}$ of X, $k \le \infty$, then Player II chooses $x \in X$ and Player I tries to guess x by asking questions to Player II regarding which piece of the partition x lands in when moved by a certain group element. More precisely, for every $g \in G$, Player I asks to which P_n does gx belong and Player II gives $n_g < k$ as an answer. Whether or not Player I can uniquely determine x from the sequence $(n_g)_{g \in G}$ of responses depends on how cleverly he chose the partition $\mathcal{P}$. A partition is called a generator if it guarantees that Player I will determine x correctly no matter which x Player II chooses. Here is the precise definition, which also explains the terminology.

Definition 1.1 (Generator)**.** *Let $k \le \infty$ and $\mathcal{P} = \{P_n\}_{n<k}$ be a Borel partition of X (i.e. each P_n is Borel). $\mathcal{P}$ is called a generator if the set of its G-translates $G\mathcal{P} := \{gP_n : g \in G, n < k\}$ generates the Borel σ-algebra of X. We also call $\mathcal{P}$ a k-generator, and, if k is finite, a finite generator.*

For each $k \le \infty$, we give k^G the product topology and let G act by shift on k^G. For a Borel partition $\mathcal{P} = \{P_n\}_{n<k}$ of X, let $f_{\mathcal{I}} : X \to k^G$ be defined by $x \mapsto (n_g)_{g \in G}$, where n_g is such that $gx \in P_{n_g}$. This is often called the *symbolic representation map* for the process $(X, G, \mathcal{P})$. Clearly, $f_{\mathcal{I}}$ is a Borel G-map and, for every $x \in X$, $f_{\mathcal{I}}(x)$ is the sequence of responses of

Player I in the above game. Based on this we have the following.

Observation 1.2. *Let $k \leq \infty$ and $\mathcal{P} = \{A_n\}_{n<k}$ be a Borel partition of X. The following are equivalent:*

(1) $\mathcal{P}$ is a generator.

(2) $G\mathcal{P}$ separates points, i.e. for all distinct $x, y \in X$ there is $A \in G\mathcal{P}$ such that $x \in A \nLeftrightarrow y \in A$.

(3) $f_{\mathcal{I}}$ is one-to-one.

In all of the arguments below, we use these equivalent descriptions of a finite generator without comment.

Given a Borel G-map $f : X \to k^G$ for some $k \leq \infty$, define a partition $\mathcal{P}_f = \{P_n\}_{n<k}$ by $P_n = f^{-1}(V_n)$, where $V_n = \{\alpha \in k^G : \alpha(1_G) = n\}$. Note that $f_{\mathcal{P}_f} = f$. This and the above observation imply the following.

Observation 1.3. *For $k \leq \infty$, X admits a k-generator if and only if there is a Borel G-embedding of X into k^G.*

1.1 Countable generators

In [Wei87], it was shown that every aperiodic (i.e. having no finite orbits) $\mathbb{Z}$-space admits a countable generator. This was later generalized to any countable group in [JKL02].

Theorem 1.4 (Weiss, Jackson-Kechris-Louveau)**.** *Every aperiodic Borel G-space X admits a countable generator. In particular, there is a Borel G-embedding of X into $\mathbb{N}^G$.*

This is sharp in the sense that we could not hope to obtain a finite generator solely from the aperiodicity assumption because of measure-theoretic obstructions. Indeed, the Kolmogorov-Sinai theorem (see) implies that measure-preserving actions of $\mathbb{Z}$ with infinite entropy cannot have a finite generator, and there a lot of such actions (e.g. the action of $\mathbb{Z}$ on $[0,1]^{\mathbb{Z}} \setminus A$ by shift, where A is the set of periodic points and the measure is the product of the Lebesgue measure).

Thus, the question of existence of countable generators is completely resolved, and the current part of this thesis concerns the existence of finite generators.

1.2 Entropy and finite generators

Generators arose in the study of entropy in ergodic theory. Let (X,μ,T) be a dynamical system, i.e. (X,μ) is a standard probability space and T is a Borel measure preserving automorphism of X. We can interpret the above game as follows:

- X is the set of possible pictures of the world,
- $\mathcal{P}$ is an experiment that Player I conducts,
- the point $x \in X$ that Player II chooses is the true picture of the world,
- T is the unit of time.

Assume that $\mathcal{P}$ is finite (indeed, we want our experiment to have finitely many possible outcomes). Player I repeats the experiment every day and Player II tells its outcome. The goal is to find the true picture of the world x with probability 1. This happens exactly when $\mathcal{P}$ is a generator a.e.

The entropy of the experiment $\mathcal{P} = \{P_n\}_{n<k}$ is defined by

$$h_\mu(\mathcal{P}) = -\sum_{n<k} \mu(P_n) \log \mu(P_n),$$

and intuitively, it measures our probabilistic uncertainty about the outcome of the experiment. For example, if for some $n < k$, P_n had probability 1, then we would be probabilistically certain that the outcome is going to be in P_n. On the other hand, if all of P_n had probability $\frac{1}{k}$, then our uncertainty would be the highest. Equivalently, according to Shannon's interpretation, $h_\mu(\mathcal{P})$ measures how much information we gain from learning the outcome of the experiment. Thus, the higher the entropy the "smarter" the experiment.

We now define the time average of the entropy of $\mathcal{P}$ by

$$h_\mu(\mathcal{P},T) = \lim_{n\to\infty} \frac{1}{n} h_\mu(\bigvee_{i<n} T^i \mathcal{P}),$$

where $\bigvee$ denotes the joint of the partitions (the least common refinement). The sequence in the limit is decreasing and hence the limit always exists and is finite (see [Gla03] or [Rud90]).

Finally the entropy of the dynamical system (X, μ, T) is defined as the supremum over all (finite) experiments:

$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(\mathcal{P}, T),$$

and it could be finite or infinite. Now it is plausible that if $\mathcal{P}$ is a finite generator, then $h_\mu(\mathcal{P}, T)$ should be all the information there is to obtain about the dynamics of X and hence $\mathcal{P}$ achieves the supremum above. This is indeed the case as the following theorem (see Theorem 14.33 in [Gla03], for example) shows.

Theorem 1.5 (Kolmogorov-Sinai, '58-59)**.** *If $\mathcal{P}$ is a finite generator modulo μ-NULL, then $h_\mu(T) = h_\mu(\mathcal{P}, T)$. In particular, $h_\mu(T) \leq \log(|\mathcal{P}|) < \infty$.*

Here μ-NULL denotes the σ-ideal of μ-null sets and, by definition, a statement holds modulo a σ-ideal $\mathfrak{I}$ if it holds on $X \setminus Z$, for some $Z \in \mathfrak{I}$. We will also use this for MEAGER, the σ-ideal of meager sets in a Polish space.

In case of ergodic systems, i.e. dynamical systems where every (measurable) invariant set is either null or co-null, the converse of Kolmogorov-Sinai theorem is true (see [Kri70]):

Theorem 1.6 (Krieger, '70)**.** *Suppose (X, μ, T) is ergodic. If $h_\mu(T) < \log k$, for some $k \geq 2$, then there is a k-generator modulo μ-NULL.*

2 Questions and answers

2.1 Weiss's question and potential dichotomy theorems

Now let X be just a Borel $\mathbb{Z}$-space with no measure specified. By the Kolmogorov-Sinai theorem, if there exists an invariant Borel probability measure on X with infinite entropy, then X does not admit a finite generator. What happens if we remove this obstruction? More precisely:

Question 2.1. *Let X be a Borel $\mathbb{Z}$-space. If X does not admit any invariant Borel probability measure of infinite entropy, does it have a finite generator?*

The following seemingly simpler question was first asked in [Wei87]:

Question 2.2 (Weiss, '87)**.** *Let X be a Borel $\mathbb{Z}$-space. If X does not admit any invariant Borel probability measure, does it have a finite generator?*

It is shown below in Section 14 these two questions are actually equivalent, and thus, a positive answer to Weiss's question would imply the following dichotomy theorem:

Theorem 14.5. *Suppose the answer to Question 2.2 is positive and let X be an aperiodic Borel $\mathbb{Z}$-space. Then exactly one of the following holds:*

(1) there exists an invariant Borel probability measure with infinite entropy;

(2) X admits a finite generator.

We remark that the nonexistence of an invariant *ergodic* probability measure of infinite entropy does not guarantee the existence of a finite generator. For example, let X be a direct sum of uniquely ergodic actions $\mathbb{Z} \curvearrowright X_n$ such that the entropy h_n of each X_n is finite but $h_n \to \infty$. Then X does not admit an invariant ergodic probability measure with infinite entropy since otherwise it would have to be supported on one of the X_n, contradicting unique ergodicity. Neither does X admit a finite generator since that would contradict Krieger's theorem applied to X_n, for large enough n.

However, assuming again that the answer to 2.2 is positive for $G = \mathbb{Z}$, we prove the following dichotomy suggested by Kechris:

Theorem 14.3. *Suppose the answer to Question 2.2 is positive and let X be an aperiodic Borel $\mathbb{Z}$-space. Then exactly one of the following holds:*

(1) there exists an invariant ergodic Borel probability measure with infinite entropy,

(2) there exists a partition $\{Y_n\}_{n\in\mathbb{N}}$ of X into invariant Borel sets such that each Y_n has a finite generator.

The proofs of these dichotomies use the Ergodic Decomposition Theorem and a version of Krieger's theorem together with Theorem 13.10 about separating the equivalence classes of a smooth equivalence relation.

2.2 Weiss's question for an arbitrary group and an answer

Because Questions 2.1 and 2.2 are equivalent, we may focus on answering the latter. Moreover, since the statement of Question 2.2 does not use the notion of entropy, one may as well state it for an arbitrary countable group G as it is done in [JKL02]:

Question 2.3 (Weiss '87, Jackson-Kechris-Louveau '02)**.** *Let G be a countable group and let X be a Borel G-space. If X does not admit any invariant Borel probability measure, does it have a finite generator?*

In order to state our answer, we need the following:

Definition 2.4. *Let X be a Borel G-space and denote its Borel σ-algebra by $\mathfrak{B}(X)$. For a topological property P (e.g. Polish, σ-compact, etc.), we say that X admits a P topological realization, if there exists a Hausdorff second countable topology on X satisfying P such that it makes the G-action continuous and its induced Borel σ-algebra is equal to $\mathfrak{B}(X)$.*

We remark that every Borel G-space admits a Polish topological realization (this is actually true for an arbitrary Polish group, but it is a highly non-trivial result of Becker and Kechris, see 5.2 in [BK96]).

The main result of this part of the thesis is a positive answer to Question 2.3 in case X has a σ-compact realization:

Theorem 9.5. *Let X be a Borel G-space that admits a σ-compact realization. If there is no G-invariant Borel probability measure on X, then X admits a Borel 32-generator.*

For example, 2.3 has a positive answer when G acts continuously on a locally compact or even σ-compact Polish space.

Remark. The number 32 in the above theorem comes from the fact that the generator is constructed as the partition generated by 5 Borel sets.

Remark. The fact that a concrete numerical bound of 32 is obtained in the conclusion of the above theorem is still somewhat surprising. However, Robin Tucker-Drob pointed out that if Question 2.3 has a positive answer, then automatically there is a uniform finite bound on the number generators; indeed, otherwise, there is an unbounded sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that for each $n \in \mathbb{N}$, there is Borel G-spaces X_{k_n} that

(i) does not admit an invariant probability measure,

(ii) admits an n-generator,

(iii) does not admit a k-generator for $k < k_n$.

Then, letting X be the disjoint union of X_{k_n}, $n \in \mathbb{N}$, we see that X still does not admit an invariant probability measure, but neither does it admit a finite generator, contradicting the fact that the answer to Question 2.3 is positive.

Before explaining the idea of the proof of the above theorem, we present previously known results as well as other related results obtained in this part of the thesis.

2.3 The measure-theoretic setting and weakly wandering sets

The following result gives a positive answer to a version of Question 2.3 in the measure-theoretic context (see [Kre70] for $G = \mathbb{Z}$ and [Kun74] for arbitrary G).

Theorem 2.5 (Krengel, Kuntz, '74)**.** *Let X be a Borel G-space and let μ be a quasi-invariant Borel probability measure on X (i.e. G preserves the μ-null sets). If there is no invariant Borel probability measure absolutely continuous with respect to μ, then X admits a 2-generator modulo μ-NULL.*

The proof uses a version of the Hajian-Kakutani-Itô theorem (see [HK64] and [HI69]), which states that the hypothesis of the Krengel-Kuntz theorem is equivalent to the existence of a weakly wandering set (see Definition 11.1) of positive measure. We show in Section 11 that having a weakly wandering (or even just *locally* weakly wandering) set of full saturation implies the existence of finite generators in the Borel context (Theorem 11.5).

However, it was shown by Eigen-Hajian-Nadkarni in [EHN93] that the analogue of the Hajian-Kakutani-Itô theorem fails in the Borel context. In Section 15, we strengthen this result by showing that it fails even in the context of Baire category (Corollary 15.11). This result is a consequence of a criterion for non-existence of non-meager weakly wandering sets (Theorem 15.7), and it implies a negative answer to the following question asked in [EHN93] (question (ii) on page 9):

Question 2.6 (Eigen-Hajian-Nadkarni, '93). *Let X be a Borel $\mathbb{Z}$-space. If X does not admit an invariant probability measure, is there a countably generated (by Borel sets) partition of X into invariant sets, each of which admits a weakly wandering set of full saturation?*

Ben Miller pointed out that he also had obtained a negative answer to this question in his Ph.D. thesis, see Example 3.13 in [Mil08].

2.4 The Baire category setting

In the mid-'90s, Kechris asked whether an analogue of the Krengel-Kuntz theorem holds in the context of Baire category (see 6.6.(B) in [JKL02]), more precisely:

Question 2.7 (Kechris, mid-'90s). *Does every aperiodic Polish G-space admit a finite generator on an invariant comeager set?*

The nonexistence of invariant measures is not mentioned in the hypothesis of the question because it is automatic in the context of Baire category, due to the following (cf. Theorem 13.1 in [KM04]):

Theorem 2.8 (Kechris-Miller, '04). *For any aperiodic Polish G-space, there is an invariant* *comeager set that does not admit any invariant probability measure.*

Thus, a positive answer to Question 2.3 for all Borel G-spaces would imply a positive answer to this question.

We give an affirmative answer to Question 2.7:

Theorem 12.2. *Any aperiodic Polish G-space admits a 4-generator on an invariant comeager set.*

The proof of this uses the Kuratowski-Ulam method introduced in the proofs of Theorems 12.1 and 13.1 in [KM04]. This method was inspired by product forcing and its idea is as follows. Suppose we want to prove the existence of an object A that satisfies a certain condition on a comeager set (in our case a finite partition). We give a parametrized construction of such objects A_α, where the parameter α ranges over $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ (or any other Polish space), and then try to show that for comeager many values of α, A_α has the desired property Φ on a comeager set. In other words, we want to prove

$$\forall^* \alpha \forall^* x \Phi(\alpha, x),$$

where $\forall^*$ means "for comeager many". Now the key point is that the Kuratowski-Ulam theorem allows us to switch the order of the quantifiers and prove

$$\forall^* x \forall^* \alpha \Phi(\alpha, x)$$

instead. The latter is often an easier task since it allows one to work locally with a fixed $x \in X$.

Now that we have advertised the method, let us point out that a "blind" application of it would not give us the statement of Theorem 12.2. Indeed, assume for a moment that we have found a parametrized construction of finite partitions $\mathcal{P}_\alpha$, for $\alpha \in \mathbb{N}^{\mathbb{N}}$, and let

$$\Phi(\mathcal{P}_\alpha, x, y) \equiv \text{"if } x \neq y \text{, then } G\mathcal{P}_\alpha \text{ separates } x \text{ and } y\text{"}.$$

If we apply the Kuratowski-Ulam method to this Φ, we will get that for comeager many $\alpha \in \mathbb{N}^{\mathbb{N}}$, we have:

$$\forall^*(x, y) \in X^2 \ \Phi(\mathcal{P}_\alpha, x, y),$$

while we want a comeager set $D \subseteq X$ such that

$$\forall (x, y) \in D^2 \ \Phi(\mathcal{P}_\alpha, x, y).$$

The problem is that a 2-dimensional comeager set may not contain a square of a 1-dimensional comeager set. To get around this, we transform our 2-dimensional problem into two 1-dimensional problems.

2.5 Finitely traveling sets and finite generators

Lastly, we give a positive answer to a version of Question 2.3 with slightly stronger hypothesis. It is not hard to prove (see 6.7) that for a Borel G-space X, the nonexistence of invariant probability measures on X is equivalent to the existence of so-called traveling sets of full saturation (Definition 6.1). We define a slightly stronger notion of a locally finitely traveling set (Definition 10.2), and show in 10.5 that if there exists such a set of full saturation, then X admits a 32-generator. The proof uses the machinery developed for proving Theorem 9.5.

2.6 Nadkarni's theorem

We now present an equivalent condition to the hypothesis of Question 2.3, i.e. to the nonexistence of invariant measures. It was proved by Nadkarni in [Nad91] and it is the analogue of Tarski's theorem about paradoxical decompositions (see [Wag93]) for countably additive measures.

Let X be a Borel G-space and denote the set of invariant Borel probability measures on X by $\mathcal{M}_G(X)$. Also, for $S \subseteq X$, let $[S]_G$ denote the saturation of S, i.e. $[S]_G = \bigcup_{g \in G} gS$.

The following definition makes no reference to any invariant measure on X, yet provides a sufficient condition for the measure of two sets to be equal (resp. $\leq$ or $<$).

Definition 2.9. *Two Borel sets $A, B \subseteq X$ are said to be equidecomposable (denoted by $A \sim B$) if there are Borel partitions $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ of A and B, respectively, and $\{g_n\}_{n \in \mathbb{N}} \subseteq G$ such that $g_n A_n = B_n$. We write $A \preceq B$ if $A \sim B' \subseteq B$, and we write $A \prec B$ if moreover*

$[B \setminus B']_G = [B]_G$.

The following explains the above definition.

Observation 2.10. *Let $A, B \subseteq X$ be Borel sets.*

(a) If $A \sim B$, then $\mu(A) = \mu(B)$ for any $\mu \in \mathcal{M}_G(X)$.

(b) If $A \preceq B$, then $\mu(A) \leq \mu(B)$ for any $\mu \in \mathcal{M}_G(X)$.

(c) If $A \prec B$, then either $\mu(A) = \mu(B) = 0$ or $\mu(A) < \mu(B)$ for any $\mu \in \mathcal{M}_G(X)$.

Definition 2.11. *A Borel set $A \subseteq X$ is called compressible if $A \prec A$.*

It is clear from the observation above that if a Borel set $A \subseteq X$ is compressible, then $\mu(A) = 0$ for all $\mu \in \mathcal{M}_G(X)$. In particular, if X itself is compressible then $\mathcal{M}_G(X) = \varnothing$. Thus compressibility is an apparent obstruction to having an invariant probability measure. It turns out that it is the only one:

Theorem 2.12 (Nadkarni, '91)**.** *Let X be a Borel G-space. There is an invariant Borel probability measure on X if and only if X is not compressible.*

The proof of this first appeared in [Nad91] for $G = \mathbb{Z}$ and is also presented in Chapter 4 of [BK96] for an arbitrary countable group G. Although we don't explicitly use this theorem in our arguments, we use ideas from its proof.

2.7 Outline of the proof of Theorem 9.5

In our attempt to positively answer Question 2.3, we take the non-constructive approach and try to prove the contrapositive:

$$\text{No finite generator} \Rightarrow \exists \text{ an invariant probability measure.}$$

When constructing an invariant measure (e.g. Haar measure), one usually needs some notion of "largeness" so that X is "large" (e.g. having nonempty interior, being incompressible). So we aim at something like this:

$$\begin{array}{ccc} \text{No 32-generator} & & \exists \text{ an invariant probability measure} \\ \quad\searrow & & \nearrow\quad \\ & X \text{ is not "small"} = X \text{ is "large"} & \end{array}$$

In the definition of equidecomposability of sets A and B, the partitions $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ belong to the Borel σ-algebra. For $i \geq 1$, we define a finer notion of equidecomposability by restricting the Borel σ-algebra to *some* σ-algebra that is generated by the G-translates of i-many Borel sets. In this case we say that A and B are *i-equidecomposable* and denote by $A \sim_i B$. In other words, $A \sim_i B$ if i-many Borel sets are enough to generate a G-invariant σ-algebra that is sufficiently fine to carve out partitions $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ witnessing $A \sim B$.

As before, we say that a set A is *i-compressible* if $A \prec_i A$. Taking i-compressibility as our notion of "smallness", we prove the following:

$$\begin{array}{ccc} \text{No 32-generator} & & \exists \text{ an invariant probability measure} \\ (1)\searrow & & \nearrow(2) \\ & X \text{ is not 4-compressible} & \end{array}$$

We prove the contrapositive of Step (1). More precisely, assuming i-compressibility, we construct a 2^{i+1}-generator by hand, thus obtaining:

$$\text{No } 2^5\text{-generator} \Rightarrow X \text{ is not 4-compressible.}$$

Step (2) is proving an analog of Nadkarni's theorem for i-compressibility:

$$X \text{ is not 4-compressible} \Rightarrow \exists \text{ an invariant probability measure.}$$

To accomplish this step, firstly, we show that i-compressibility is indeed a notion of "smallness", i.e. that the set of i-compressible sets (roughly speaking) forms a σ-ideal. The difficulty here is to prevent i from growing when taking unions.

Secondly, we assume that X is not 4-compressible and give a construction of a measure reminiscent of the one in the proof of Nadkarni's theorem or the existence of Haar measure. But unfortunately, our proof only yields a family of *finitely additive* invariant probability measures (here, we cannot prevent i from growing when taking countable unions). However, with the additional assumption that X is σ-compact, we are able to concoct a countably additive invariant probability measure out of this family of finitely additive measures, thus obtaining Theorem 9.5.

2.8 Open questions

Here are some open questions that arose in this research. Let X denote a Borel G-space.

(A) Is X being compressible equivalent to X being i-compressible for some $i \geq 1$?

(B) Does the existence of a traveling complete section imply the existence of a locally finitely traveling complete section?

(C) Can we get a 2-generator instead of a 32-generator in Theorem 9.5?

A positive answer to any of these questions would imply a positive answer to Question 2.3 since (A) is just a rephrasing of Question 2.3 because of 7.7 and for (B), it follows from 6.5 and 10.5.

3 Organization

In Chapter II, we develop the theory of i-compressibility and establish its connection with the existence of finite generators and nonexistence of certain finitely additive invariant probability measures. More particularly, in Section 4 we give the definition of $\mathcal{I}$-equidecomposability and prove the important property of orbit-disjoint countable additivity (see 4.9), which is what makes $\mathfrak{C}_i$ (defined below) a σ-ideal. In Sections 5 and 6 we define the notions of i-compressibility and i-traveling sets and establish their connection. Finally, in Section 7, we show how to construct a finite generator using an i-traveling complete section[1]. In Section 8, we prove the main theorem, which provides means of constructing finitely additive invariant measures (Corollary 8.3) that are non-zero on a given non-i-compressible set.

In the following chapter, we give two applications of Corollary 8.3, namely 9.5 and 10.5, where the former is the main result of this part stated above and the latter is the result discussed in Subsection 2.5. Also, Section 11 provides various examples of i-compressible actions involving locally weakly wandering sets.

Chapter IV contains various somewhat unrelated results. Section 12 establishes the existence of a 4-generator on an invariant comeager set. In Section 13, we show that given a smooth equivalence relation E on X with $E \supseteq E_G$, there exists a finite partition $\mathcal{P}$ such that $G\mathcal{P}$ separates points in different classes of E; in fact, we give an explicit construction of such $\mathcal{P}$. This result is then used in the following section, where we establish the potential dichotomy theorems mentioned above (14.3 and 14.5). Finally, in Section 15 we develop a criterion for non-existence of non-meager weakly wandering sets and derive a negative answer to Question 2.6.

[1] A complete section is a set that meets every orbit (equivalently, has full saturation).

CHAPTER II

The theory of i-compressibility: connections with finite generators and finitely additive invariant measures

Throughout this chapter let X be a Borel G-space and E_G be the orbit equivalence relation on X. For a set $A \subseteq X$ and G-invariant set $P \subseteq X$, let $A^P := A \cap P$.

For an equivalence relation E on X and $A \subseteq X$, let $[A]_E$ denote the saturation of A with respect to E, i.e. $[A]_E = \{x \in X : \exists y \in A(xEy)\}$. In case $E = E_G$, we use $[A]_G$ instead of $[A]_{E_G}$.

Let $\mathfrak{B}$ denote the (proper) class of all Borel subsets of standard Borel spaces, i.e.

$$\mathfrak{B} = \{B : B \text{ is a Borel subset of some standard Borel space } X\}.$$

Also, let Γ be a class σ-algebra of subsets of standard Borel spaces containing $\mathfrak{B}$ and closed under Borel preimages, i.e. if X, Y are standard Borel spaces and $f : X \to Y$ is a Borel map, then for a subset $A \subseteq Y$, if $A \in \Gamma$ then $f^{-1}(A)$ is also in Γ. For example, $\Gamma = \mathfrak{B}$, $\sigma(\mathbf{\Sigma}^1_1)$, universally measurable sets.

For a standard Borel space X, let $\Gamma(X)$ denote the set of all subsets of X that belong to Γ. In particular, $\mathfrak{B}(X)$ denotes the set of all Borel subsets of X.

4 The notion of $\mathcal{I}$-equidecomposability

A countable partition of X is called Borel if all the sets in it are Borel. For a finite Borel partition $\mathcal{I} = \{A_i : i < k\}$ of X, let $F_{\mathcal{I}}$ denote the equivalence relation of not being separated by $G\mathcal{I} := \{gA_i : g \in G, i < k\}$, more precisely, $\forall x, y \in X$,

$$xF_{\mathcal{I}}y \Leftrightarrow f_{\mathcal{I}}(x) = f_{\mathcal{I}}(y),$$

where $f_{\mathcal{I}}$ is the symbolic representation map for $(X, G, \mathcal{I})$ defined above. Note that if $\mathcal{I}$ is a generator, then $F_{\mathcal{I}}$ is just the equality relation.

For $A \subseteq X$, put

$$\Gamma(X) \restriction_A = \{A' \subseteq A : \exists B \in \Gamma(X) \ (A' = B \cap A)\}.$$

Also, for an equivalence relation E on X and $A, B \subseteq X$, say that A is E-invariant relative to B or just $E \restriction_B$-invariant if $[A]_E \cap B = A \cap B$.

Definition 4.1 ($\mathcal{I}$-equidecomposability)**.** *Let $A, B \subseteq X$, and $\mathcal{I}$ be a finite Borel partition of X. A and B are said to be equidecomposable with Γ pieces (denote by $A \sim^{\Gamma} B$) if there are $\{g_n\}_{n \in \mathbb{N}} \subseteq G$ and partitions $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ of A and B, respectively, such that for all $n \in \mathbb{N}$*

- $g_n A_n = B_n$,
- $A_n \in \Gamma(X) \restriction_A$ *and* $B_n \in \Gamma(X) \restriction_B$.

If moreover,

- A_n *and* B_n *are $F_{\mathcal{I}}$-invariant relative to A and B, respectively,*

then we will say that A and B are $\mathcal{I}$-equidecomposable with Γ pieces and denote it by $A \sim_{\mathcal{I}}^{\Gamma} B$. If $\Gamma = \mathfrak{B}$, we will not mention Γ and will just write $\sim$ and $\sim_{\mathcal{I}}$.

Note that for any finite Borel partition $\mathcal{I}$ of X and Borel sets $A, B \subseteq X$, A and B are $\mathcal{I}$-equidecomposable if and only if $f_{\mathcal{I}}(A)$ and $f_{\mathcal{I}}(B)$ are equidecomposable (although the images of Borel sets under $f_{\mathcal{I}}$ are analytic, they are Borel relative to $f_{\mathcal{I}}(X)$ due to the Lusin Separation Theorem for analytic sets). Also note that if $\mathcal{I}$ is a generator, then $\sim_{\mathcal{I}}$ coincides with $\sim$.

Observation 4.2. *Below let $\mathcal{I}, \mathcal{I}_0, \mathcal{I}_1$ denote finite Borel partitions of X, and $A, B, C \in \Gamma(X)$.*

(a) (Quasi-transitivity) If $A \sim^{\Gamma}_{\mathcal{I}_0} B \sim^{\Gamma}_{\mathcal{I}_1} C$, then $A \sim^{\Gamma}_{\mathcal{I}} C$ with $\mathcal{I} = \mathcal{I}_0 \vee \mathcal{I}_1$ (the least common refinement of $\mathcal{I}_0$ and $\mathcal{I}_1$).

(b) ($F_{\mathcal{I}}$-disjoint countable additivity) Let $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ be partitions of A and B, respectively, into Γ sets such that $\forall n \neq m$, $[A_n]_{F_{\mathcal{I}}} \cap [A_m]_{F_{\mathcal{I}}} = [B_n]_{F_{\mathcal{I}}} \cap [B_m]_{F_{\mathcal{I}}} = \varnothing$. If $\forall n \in \mathbb{N}$, $A_n \sim^{\Gamma}_{\mathcal{I}} B_n$, then $A \sim^{\Gamma}_{I} B$.

If $A \sim B$, then there is a Borel isomorphism ϕ of A onto B with $\phi(x) E_G x$ for all $x \in A$; namely $\phi(x) = g_n x$ for all $x \in A_n$, where A_n, g_n are as in Definition 2.9. It is easy to see that the converse is also true, i.e. if such ϕ exists, then $A \sim B$. In Proposition 4.5 we prove the analogue of this for $\sim^{\Gamma}_{\mathcal{I}}$, but first we need the following lemma and definition that take care of definability and $F_{\mathcal{I}}$-invariance, respectively.

For a Polish space Y, $f : X \to Y$ is said to be Γ-measurable if the preimages of open sets under f are in Γ. For $A \in \Gamma(X)$ and $h : A \to G$, define $\hat{h} : A \to X$ by $x \mapsto h(x)x$.

Lemma 4.3. *If $h : A \to G$ is Γ-measurable, then the images and preimages of sets in Γ under $\hat{h}$ are in Γ.*

Proof. Let $B \subseteq A$, $C \subseteq X$ be in Γ. For $g \in G$, set $A_g = h^{-1}(g)$ and note that $\hat{h}(B) = \bigcup_{g \in G} g(A_g \cap B)$ and $\hat{h}^{-1}(C) = \bigcup_{g \in G} g^{-1}(gA_g \cap C)$. Thus $\hat{h}(B)$ and $\hat{h}^{-1}(C)$ are in Γ by the assumptions on Γ. □

The following technical definition is needed in the proofs of 4.5 and 4.9.

Definition 4.4. *For $A \subseteq X$ and a finite Borel partition $\mathcal{I}$ of X, we say that $\mathcal{I}$ is A-sensitive or that A respects $\mathcal{I}$ if A is $F_{\mathcal{I}}$-invariant relative to $[A]_G$, i.e. $[A]^{[A]_G}_{F_{\mathcal{I}}} = A$.*

For example, if $\mathcal{I}$ is finer than $\{A, A^c\}$, then $\mathcal{I}$ is A-sensitive. Note that if $A \sim_{\mathcal{I}} B$ and A respects $\mathcal{I}$, then so does B.

Proposition 4.5. *Let $A, B \in \Gamma(X)$ and let $\mathcal{I}$ be a Borel partition of X that is A-sensitive. Then, $A \sim^{\Gamma}_{\mathcal{I}} B$ if and only if there is an $F_{\mathcal{I}}$-invariant Γ-measurable map $\gamma : A \to G$ such that $\hat{\gamma}$ is a bijection between A and B. We refer to such γ as a witnessing map for $A \sim^{\Gamma}_{\mathcal{I}} B$. The same holds if we delete "$F_{\mathcal{I}}$-invariant" and "$\mathcal{I}$" from the statement.*

Proof. $\Rightarrow$: If $\{g_n\}_{n\in\mathbb{N}}$, $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are as in Definition 4.1, then define $\gamma : A \to G$ by setting $\gamma\restriction_{A_n} \equiv g_n$.

$\Leftarrow$: Let γ be as in the lemma. Fixing an enumeration $\{g_n\}_{n\in\mathbb{N}}$ of G with no repetitions, put $A_n = \gamma^{-1}(g_n)$ and $B_n = g_n A_n$. It is clear that $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}}$ are partitions of A and B, respectively, into Γ sets. Since γ is $F_{\mathcal{I}}$-invariant, each A_n is $F_{\mathcal{I}}$-invariant relative to A and hence relative to $P := [A]_G = [B]_G$ because A respects $\mathcal{I}$. It remains to show that each B_n is $F_{\mathcal{I}}$-invariant relative to B. To this end, let $y \in [B_n]_{F_{\mathcal{I}}} \cap B$ and thus there is $x \in A_n$ such that $y F_{\mathcal{I}} g_n x$. Hence $z := g_n^{-1} y \; F_{\mathcal{I}} \; g_n^{-1} g_n x = x$ and therefore $z \in A_n$ because A_n is $F_{\mathcal{I}}$-invariant relative to P. Thus $y = g_n z \in B_n$. □

In the rest of the subsection we work with $\Gamma = \mathfrak{B}$.

Next we prove that $\mathcal{I}$-equidecomposability can be extended to $F_{\mathcal{I}}$-invariant Borel sets. First we need the following separation lemma for analytic sets[1]:

Lemma 4.6 (Invariant analytic separation)**.** *Let E be an analytic equivalence relation on X. For any disjoint family $\{A_n\}_{n\in\mathbb{N}}$ of E-invariant analytic sets, there exists a disjoint family $\{B_n\}_{n\in\mathbb{N}}$ of E-invariant Borel sets such that $A_n \subseteq B_n$.*

Proof (Vaught). We give the proof for two disjoint E-invariant analytic sets A_0, A_1 since this easily implies the statement for countably many. Recursively define analytic sets $C_n \subseteq X$ and Borel sets $D_n \subseteq X$ such that for every $n \in \mathbb{N}$ we have

(i) $A_0 \subseteq C_n \subseteq D_n \subseteq C_{n+1} \subseteq A_1^c$,

[1] My original argument used Π^1_1 reflection principles, but it was pointed out to me by Shashi Srivastava that one could use analytic separation instead. I chose to present this latter argument here since analytic separation may be more transparent for non-logicians than Π^1_1 reflection principles.

(ii) C_n is E-invariant.

To do this, let $C_0 = A_0$, and, assuming that C_n is defined, define D_n, C_{n+1} as follows: since C_n and A_1 are disjoint analytic sets, there is a Borel set D_n separating them (by the Lusin separation theorem), i.e. $D_n \supseteq C_n$ and $D_n \cap A_1 = \varnothing$. Let $C_{n+1} = [D_n]_E$, and note that C_{n+1} is analytic and disjoint from A_1 since A_1 is E-invariant and disjoint from D_n. This finishes the construction.

Now let $B = \bigcup_{n \in \mathbb{N}} D_n$; hence B is Borel, contains A_0 and is disjoint from A_1. On the other hand, $B = \bigcup_{n \in \mathbb{N}} C_n$ and thus is E-invariant. □

Proposition 4.7 ($F_{\mathcal{I}}$-invariant extensions). *Let $\mathcal{I}$ be a Borel partition of X and let $A, B \subseteq X$ be Borel sets. If $A \sim_{\mathcal{I}} B$, then there exists Borel sets $A' \supseteq A$ and $B' \supseteq B$ such that A', B' are $F_{\mathcal{I}}$-invariant and $A' \sim_{\mathcal{I}} B'$. In fact, if $\{g_n\}_{n \in \mathbb{N}}, \{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ witness $A \sim_{\mathcal{I}} B$, then there are $F_{\mathcal{I}}$-invariant Borel partitions $\{A'_n\}_{n \in \mathbb{N}}, \{B'_n\}_{n \in \mathbb{N}}$ of A' and B' respectively, such that $g_n A'_n = B'_n$ and $A'_n \supseteq A_n$ (and hence $B'_n \supseteq B_n$).*

Proof. Let $\{g_n\}_{n \in \mathbb{N}}, \{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ be as in Definition 4.1 and put $\overline{A}_n = [A_n]_{F_{\mathcal{I}}}$. It is easy to see that for $n \neq m \in \mathbb{N}$,

(i) $\overline{A}_n \cap \overline{A}_m = \varnothing$;

(ii) $g_n \overline{A}_n \cap g_m \overline{A}_m = \varnothing$.

Put $\overline{A} = [A]_{F_{\mathcal{I}}}$ and note that $\{\overline{A}_n\}_{n \in \mathbb{N}}$ is a partition of $\overline{A}$. Although $\overline{A}_n$ and $\overline{A}$ are $F_{\mathcal{I}}$-invariant, they are analytic and in general not Borel. We obtain Borel analogues of these sets using invariant analytic separation as follows: by Lemma 4.6 applied to $\{A_n\}_{n \in \mathbb{N}}$ and $\{g_n A_n\}_{n \in \mathbb{N}}$, there are pairwise disjoint families $\{C_n\}_{n \in \mathbb{N}}$ and $\{D_n\}_{n \in \mathbb{N}}$ of $F_{\mathcal{I}}$-invariant Borel sets such that $C_n \supseteq A_n$ and $D_n \supseteq g_n A_n$. Taking $A'_n = C_n \cap g_n^{-1} D_n$, we see that $\{A'_n\}_{n \in \mathbb{N}}$ is a pairwise disjoint family of $F_{\mathcal{I}}$-invariant Borel sets such that $A'_n \supseteq A_n$. Moreover, $\{g_n A'_n\}_{n \in \mathbb{N}}$ is also a pairwise disjoint family. Thus, taking $B'_n = g_n A'_n$, we are done. □

Lemma 4.8 (Orbit-disjoint unions)**.** *Let $A_k, B_k \in \mathfrak{B}(X)$, $k = 0, 1$, be such that $[A_0]_G$ and $[A_1]_G$ are disjoint and put $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$. If $\mathcal{I}$ is an A, B-sensitive finite Borel partition of X such that $A_k \sim_{\mathcal{I}} B_k$ for $k = 0, 1$, then $A \sim_{\mathcal{I}} B$. Moreover, if $\gamma_0 : A_0 \to G$ is a Borel map witnessing $A_0 \sim_{\mathcal{I}} B_0$, then there exists a Borel map $\gamma : A \to G$ extending γ_0 that witnesses $A \sim_{\mathcal{I}} B$.*

Proof. First assume without loss of generality that $X = [A]_G$ $(= [B]_G)$ since the statement of the lemma is relative to $[A]_G$. Thus A, B are $F_{\mathcal{I}}$-invariant.

Applying 4.7 to $A_0 \sim_{\mathcal{I}} B_0$, we get $F_{\mathcal{I}}$-invariant $A_0' \supseteq A_0, B_0' \supseteq B_0$ such that $A' \sim_{\mathcal{I}} B'$. Moreover, by the second part of the same lemma, if $\gamma_0 : A_0 \to G$ is a witnessing map for $A_0 \sim_{\mathcal{I}} B_0$, then there is a witnessing map $\delta : A_0' \to G$ for $A' \sim_{\mathcal{I}} B'$ extending γ_0. Put $C = A_0' \cap A$ and note that C is $F_{\mathcal{I}}$-invariant since so are A_0' and A. Finally, put $\overline{A}_0 = \{x \in C : C^{[x]_G} = A^{[x]_G} \wedge \hat{\delta}(C^{[x]_G}) = B^{[x]_G}\}$ and note that $\overline{A}_0 \supseteq A_0$ since $\delta \supseteq \gamma_0$ and $[A_0]_G \cap [A_1]_G = \varnothing$.

Claim. *$\overline{A}_0$ is $F_{\mathcal{I}}$-invariant.*

Proof of Claim. First note that for any $F_{\mathcal{I}}$-invariant $D \subseteq X$ and $z \in X$, $[D^{[z]_G}]_{F_{\mathcal{I}}} = D^{[[z]_{F_{\mathcal{I}}}]_G}$. Furthermore, if $D \subseteq C$, then $[\hat{\delta}(D)]_{F_{\mathcal{I}}} = \hat{\delta}([D]_{F_{\mathcal{I}}})$ since $\hat{\delta}$ and its inverse map $F_{\mathcal{I}}$-invariant sets to $F_{\mathcal{I}}$-invariant sets.

Now take $x \in \overline{A}_0$ and let $Q = [[x]_{F_{\mathcal{I}}}]_G$. Since A, B, C are $F_{\mathcal{I}}$-invariant, $C^Q = [C^{[x]_G}]_{F_{\mathcal{I}}} = [A^{[x]_G}]_{F_{\mathcal{I}}} = A^Q$. Furthermore, $\hat{\delta}(C^Q) = \hat{\delta}([C^{[x]_G}]_{F_{\mathcal{I}}}) = [\hat{\delta}(C^{[x]_G})]_{F_{\mathcal{I}}} = [B^{[x]_G}]_{F_{\mathcal{I}}} = B^Q$. Thus, $\forall y \in [x]_{F_{\mathcal{I}}}$, $C^{[y]_G} = A^{[y]_G}$ and $\hat{\delta}(C^{[y]_G}) = B^{[y]_G}$; hence $[x]_{F_{\mathcal{I}}} \subseteq \overline{A}_0$. ⊣

Put $\overline{A}_1 = A \setminus \overline{A}_0$, $\alpha_0 = \delta\restriction_{\overline{A}_0}$, $\alpha_1 = \gamma_1\restriction_{\overline{A}_1}$, where γ_1 is a witnessing map for $A_1 \sim_{\mathcal{I}} B_1$. It is clear from the definition of $\overline{A}_0$ that $\overline{A}_0$ is E_G-invariant relative to A and hence $[\overline{A}_0]_G \cap [\overline{A}_1]_G = \varnothing$. Thus, for $k = 0, 1$, it follows that α_k witnesses $\overline{A}_k \sim_{\mathcal{I}} \overline{B}_k$, where $\overline{B}_k = \hat{\alpha_k}(\overline{A}_k)$. Furthermore, it is clear that $B^{[\overline{A}_k]_G} = \overline{B}_k$ and, since $[\overline{A}_0]_G \cup [\overline{A}_1]_G = X$, $\overline{B}_0 \cup \overline{B}_1 = B$. Now since $\overline{A}_k$ are $F_{\mathcal{I}}$-invariant, $\gamma = \alpha_0 \cup \alpha_1$ is $F_{\mathcal{I}}$-invariant and hence witnesses $A \sim_{\mathcal{I}} B$. Finally, $\alpha_0\restriction_{A_0} = \delta\restriction_{A_0} = \gamma_0$ and hence $\alpha_0 \supseteq \gamma_0$. □

Proposition 4.9 (Orbit-disjoint countable unions). *For $k \in \mathbb{N}$, let $A_k, B_k \in \mathfrak{B}(X)$ be such that $[A_k]_G$ are disjoint and put $A = \bigcup_{k\in\mathbb{N}} A_k$, $B = \bigcup_{k\in\mathbb{N}} B_k$. Suppose that $\mathcal{I}$ is an A, B-sensitive finite Borel partition of X such that $A_k \sim_{\mathcal{I}} B_k$ for all k. Then $A \sim_{\mathcal{I}} B$.*

Proof. We recursively apply Lemma 4.8 as follows. Put $\overline{A}_n = \bigcup_{k\leq n} A_k$ and $\overline{B}_n = \bigcup_{k\leq n} B_k$. Inductively define Borel maps $\gamma_n : \bigcup_{k\leq n} A_k \to G$ such that γ_n is a witnessing map for $\overline{A}_n \sim_{\mathcal{I}} \overline{B}_n$ and $\gamma_n \sqsubseteq \gamma_{n+1}$. Let γ_0 be a witnessing map for $A_0 \sim_{\mathcal{I}} B_0$. Assume γ_n is defined. Then γ_{n+1} is provided by Lemma 4.8 applied to $\overline{A}_n$ and A_{n+1} with γ_n as a witness for $\overline{A}_n \sim_{\mathcal{I}} \overline{B}_n$. Thus $\gamma_n \sqsubseteq \gamma_{n+1}$ and γ_{n+1} witnesses $\overline{A}_{n+1} \sim_{\mathcal{I}} \overline{B}_{n+1}$.

Now it just remains to show that $\gamma := \bigcup_{n\in\mathbb{N}} \gamma_n$ is $F_{\mathcal{I}}$-invariant since then it follows that γ witnesses $A \sim_{\mathcal{I}} B$. Let $x, y \in A$ be $F_{\mathcal{I}}$-equivalent. Then there is n such that $x, y \in \overline{A}_n$. By induction on n, γ_n is $F_{\mathcal{I}}$-invariant and, since $\gamma \downharpoonright_{\overline{A}_n} = \gamma_n$, $\gamma(x) = \gamma(y)$. □

Corollary 4.10 (Finite quasi-additivity). *For $k = 0, 1$, let $A_k, B_k \in \mathfrak{B}(X)$ be such that $A_0 \cap A_1 = B_0 \cap B_1 = \varnothing$ and put $A = A_0 \cup A_1$, $B = B_0 \cup B_1$. Let $\mathcal{I}_k$ be an A_k, B_k-sensitive finite Borel partition of X. If $A_0 \sim_{\mathcal{I}_0} B_0$ and $A_1 \sim_{\mathcal{I}_1} B_1$, then $A \sim_{\mathcal{I}_0 \vee \mathcal{I}_1} B$.*

Proof. Put $\mathcal{I} = \mathcal{I}_0 \vee \mathcal{I}_1$, $P = [A_0]_G \cap [A_1]_G$, $Q = [A_0]_G \setminus [A_1]_G$ and $R = [A_1]_G \setminus [A_0]_G$. Then A_k^P, B_k^P respect $\mathcal{I}$, and thus $[A_0]^P_{F_{\mathcal{I}}} \cap [A_1]^P_{F_{\mathcal{I}}} = \varnothing$, $[B_0]^P_{F_{\mathcal{I}}} \cap [B_1]^P_{F_{\mathcal{I}}} = \varnothing$. Hence $A^P \sim_{\mathcal{I}} B^P$ since the sets that are $F_{\mathcal{I}}$-invariant relative to A_k^P are also $F_{\mathcal{I}}$-invariant relative to A^P, and the same is true for B_k^P and B^P. Also, $A^Q \sim_{\mathcal{I}} B^Q$ and $A^R \sim_{\mathcal{I}} B^R$ because $A^Q = A_0$, $B^Q = B_0$, $A^R = A_1$, $B^R = B_1$. Now since P, Q, R are pairwise disjoint, it follows from Proposition 4.9 that $A \sim_{\mathcal{I}} B$. □

5 The notion of i-compressibility

For a finite collection $\mathcal{F}$ of subsets of X, let $\langle \mathcal{F} \rangle$ denote the partition of X generated by $\mathcal{F}$.

Definition 5.1 (i-equidecomposibility). *For $i \geq 1$, $A, B \subseteq X$, we say that A and B are i-equidecomposable with Γ pieces (write $A \sim_i^{\Gamma} B$) if there is an A-sensitive partition $\mathcal{I}$ of X*

generated by i Borel sets such that $A \sim_{\mathcal{I}}^{\Gamma} B$. For a collection $\mathcal{F}$ of Borel sets, we say that $\mathcal{F}$ witnesses $A \sim_i^{\Gamma} B$ if $|\mathcal{F}| = i$, $\mathcal{I} := \langle \mathcal{F} \rangle$ is A-sensitive and $A \sim_{\mathcal{I}}^{\Gamma} B$.

Remark. In the above definition, it might seem more natural to have i be the cardinality of the partition $\mathcal{I}$ instead of the cardinality of the collection $\mathcal{F}$ generating $\mathcal{I}$. However, our definition above of i-equidecomposability is needed in order to show that the collection $\mathfrak{C}_i$ defined below forms a σ-ideal. More precisely, the presence of $\mathcal{F}$ is needed in the definition of i^*-compressibility, which ensures that the partition $\mathcal{I}$ in the proof of 5.7 is B-sensitive.

For a family $\mathcal{F}$ of subsets of X, let $\sigma_G(\mathcal{F})$ denote the σ-algebra generated by $G\mathcal{F}$.

Remark. Slawek Solecki pointed out that for $i \geq 1$ and Borel sets $A, B \subseteq X$, $A \sim_i B$ if and only if $A \sim B$ and the partitions $\{A_n\}_{n \in \mathbb{N}}$, $\{B_n\}_{n \in \mathbb{N}}$ witnessing the equidecomposability of A and B can be taken from a σ-algebra generated by the G-translates of i-many Borel sets. More precisely, $A \sim_i B$ if and only if there are a family $\mathcal{F}$ of i-many Borel sets, a sequence $\{g_n\}_{n \in \mathbb{N}} \subseteq G$, and partitions $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ of A and B, respectively, such that $A_n, B_n \in \sigma_G(\mathcal{F})$ and $g_n A_n = B_n$. Thus, i-equidecomposability is obtained from equidecomposability by restricting the Borel σ-algebra to *some* σ-algebra generated by the G-translates of i-many Borel sets. Finally, note that every instance of $\sim_i$ uses a (potentially) different σ-algebra.

For $i \geq 1$, $A, B \subseteq X$, we write $A \preceq_i^{\Gamma} B$ if there is a Γ set $B' \subseteq B$ such that $A \sim_i^{\Gamma} B'$. If moreover $[A \setminus B]_G = [A]_G$, then we write $A \prec_i^{\Gamma} B$. If $\Gamma = \mathfrak{B}$, we simply write $\sim_i, \preceq_i, \prec_i$.

Definition 5.2 (i-compressibility)**.** *For $i \in \mathbb{N}$, $A \subseteq X$, we say that A is i-compressible with Γ pieces if $A \prec_i^{\Gamma} A$.*

Unless specified otherwise, we will be working with $\Gamma = \mathfrak{B}$, in which case we simply say i-compressible.

For a collection of sets $\mathcal{F}$ and a G-invariant set P, set $\mathcal{F}^P = \{A^P : A \in \mathcal{F}\}$. We will use the following observations without mentioning.

Observation 5.3. *Let $i, j \geq 2$, $A, A', B, B', C \in \mathfrak{B}$. Let $P \subseteq [A]_G$ denote a G-invariant Borel set and $\mathcal{F}, \mathcal{F}_0, \mathcal{F}_1$ denote finite collections of Borel sets.*

(a) If $A \sim_i B$ then $A^P \sim_i B^P$.

(b) If $\mathcal{F}$ witnesses $A \sim_i B$, then so does $\mathcal{F}^{[A]_G}$.

(c) If $A \sim_i B \sim_j C$, then $A \sim_{(i+j)} C$. In fact, $\mathcal{F}_0$ and $\mathcal{F}_1$ witness $A \sim_i B$ and $B \sim_j C$, respectively, then $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ witnesses $A \sim_{(i+j)} C$.

(d) If $A \preceq_i B \preceq_j C$, then $A \preceq_{(i+j)} C$. If one of the first two $\preceq$ is $\prec$ then $A \prec_{(i+j)} C$.

(e) If $A \sim_i B$ and $A' \sim_j B'$ with $A \cap A' = B \cap B' = \varnothing$, then $A \cup A' \sim_{(i+j)} B \cup B'$.

Proof. Part (e) follows from 4.10, and the rest follows directly from the definition of i-equidecomposability and 4.2. □

Lemma 5.4. *If a Borel set $A \subseteq X$ is i-compressible, then so is $[A]_G$. In fact, if $\mathcal{F}$ is a finite collection of Borel sets witnessing the i-compressibility of A, then it also witnesses that of $[A]_G$.*

Proof. Let $B \subseteq A$ be a Borel set such that $[A \setminus B]_G = [A]_G$ and $A \sim_i B$. Furthermore, let $\mathcal{I}$ be an A, B-sensitive partition generated by a collection $\mathcal{F}$ of i Borel sets such that $A \sim_{\mathcal{I}} B$. Let $\gamma : A \to G$ be a witnessing map for $A \sim_I B$. Put $A' = [A]_G$, $B' = B \cup (A' \setminus A)$ and note that A', B' respect $\mathcal{I}$. Define $\gamma' : A' \to G$ by setting $\gamma' \restriction_{A' \setminus A} = id \restriction_{A' \setminus A}$ and $\gamma' \restriction_A = \gamma$. Since A' respects $\mathcal{I}$ and $id \restriction_{A' \setminus A}, \gamma$ are $F_{\mathcal{I}}$-invariant, γ' is $F_{\mathcal{I}}$-invariant and thus clearly witnesses $A' \sim_{\mathcal{I}} B'$. □

The following is a technical refinement of the definition of i-compressibility that is (again) necessary for $\mathfrak{C}_i$, defined below, to be a σ-ideal.

Definition 5.5 (i^*-compressibility)**.** *For $i \geq 1$, we say that a Borel set A is i^*-compressible if there is a Borel set $B \subseteq A$ such that $[A \setminus B]_G = [A]_G =: P$, $A \sim_i B$, and the latter is witnessed by a collection $\mathcal{F}$ of Borel sets such that $B \in \mathcal{F}^P$.*

Finally, for $i \geq 1$, put

$$\mathfrak{C}_i = \{A \subseteq X : \text{there is a } G\text{-invariant Borel set } P \supseteq A \text{ such that } P \text{ is } i^*\text{-compressible}\}.$$

Lemma 5.6. *Let $i \geq 1$ and $A \subseteq X$ be Borel. If $A \prec_i A$, then $A \in \mathfrak{C}_{i+1}$.*

Proof. Setting $P = [A]_G$ and applying 5.4, we get that $P \prec_i P$, i.e. there is $B \subseteq P$ such that $[P \setminus B]_G = P$ and $P \sim_i B$. Let $\mathcal{F}$ be a collection of Borel sets witnessing the latter fact. Then $\mathcal{F}' = \mathcal{F} \cup \{B\}$ witnesses $P \sim_{(i+1)} B$ and contains B. □

Proposition 5.7. *For all $i \geq 1$, $\mathfrak{C}_i$ is a σ-ideal.*

Proof. We only need to show that $\mathfrak{C}_i$ is closed under countable unions. For this it is enough to show that if $A_n \in \mathfrak{B}(X)$ are i^*-compressible G-invariant Borel sets, then so is $A := \bigcup_{n \in \mathbb{N}} A_n$.

We may assume that A_n are pairwise disjoint since we could replace each A_n by $A_n \setminus (\bigcup_{k<n} A_k)$. Let $B_n \subseteq A_n$ be a Borel set and $\mathcal{F}_n = \{F_k^n\}_{k<i}$ be a collection of Borel sets with $(F_0^n)^{A_n} = B_n$ such that $\mathcal{F}_n$ witnesses $A_n \sim_i B_n$ and $[A_n \setminus B_n]_G = A_n$. Using part (b) of 5.3, we may assume that $\mathcal{F}_n^{A_n} = \mathcal{F}_n$; in particular, $F_0^n = B_n$.

Put $B = \bigcup_{n \in \mathbb{N}} B_n$ and $F_k = \bigcup_{n \in \mathbb{N}} F_k^n$, $\forall k < i$; note that $F_0 = B$. Set $\mathcal{F} = \{F_k\}_{k<i}$ and $\mathcal{I} = \langle \mathcal{F} \rangle$. Since $B \in \mathcal{F}$ and A is G-invariant, $\mathcal{I}$ is A, B-sensitive. Furthermore, since $\mathcal{F}^{A_n} = \mathcal{F}_n$, $A_n \sim_{\mathcal{I}} B_n$ for all $n \in \mathbb{N}$. Thus, by 4.9, $A \sim_{\mathcal{I}} B$ and hence A is i^*-compressible. □

6 Traveling sets

Definition 6.1. *Let $A \in \Gamma(X)$.*

- *We call A a traveling set with Γ pieces if there exists pairwise disjoint sets $\{A_n\}_{n \in \mathbb{N}}$ in $\Gamma(X)$ such that $A_0 = A$ and $A \sim^\Gamma A_n$, $\forall n \in \mathbb{N}$.*
- *For a finite Borel partition $\mathcal{I}$, we say that A is $\mathcal{I}$-traveling with Γ pieces if A respects $\mathcal{I}$ and the above condition holds with $\sim^\Gamma$ replaced by $\sim_{\mathcal{I}}^\Gamma$.*

- *For $i \geq 1$, we say that A is i-traveling if it is $\mathcal{I}$-traveling for some A-sensitive partition $\mathcal{I}$ generated by a collection of i Borel sets.*

Definition 6.2. *For a set $A \subseteq X$, a function $\gamma : A \to G^{\mathbb{N}}$ is called a travel guide for A if $\forall x \in A, \gamma(x)(0) = 1_G$ and $\forall (x,n) \neq (y,m) \in A \times \mathbb{N}$, $\gamma(x)(n)x \neq \gamma(y)(m)y$.*

For $A \in \Gamma(X)$, a Γ-measurable map $\gamma : A \to G^{\mathbb{N}}$ and $n \in \mathbb{N}$, set $\gamma_n := \gamma(\cdot)(n) : A \to G$ and note that γ_n is also Γ-measurable.

Observation 6.3. *Suppose $A \in \Gamma(X)$ and $\mathcal{I}$ is an A-sensitive finite Borel partition of X. Then A is $\mathcal{I}$-traveling with Γ pieces if and only if it has a Γ-measurable $F_{\mathcal{I}}$-invariant travel guide.*

Proof. Follows from definitions and Proposition 4.5. □

Now we establish the connection between compressibility and traveling sets.

Lemma 6.4. *Let $\mathcal{I}$ be a finite Borel partition of X, $P \in \Gamma(X)$ be a Borel G-invariant set and let A, B be Γ subsets of P. If $P \sim^{\Gamma}_{\mathcal{I}} B$, then $P \smallsetminus B$ is $\mathcal{I}$-traveling with Γ pieces. Conversely, if A is $\mathcal{I}$-traveling with Γ pieces, then $P \sim^{\Gamma}_{\mathcal{I}} (P \smallsetminus A)$. The same is true if we replace $\sim^{\Gamma}_{\mathcal{I}}$ and "$\mathcal{I}$-traveling" with $\sim^{\Gamma}$ and "traveling", respectively.*

Proof. For the first statement, let $\gamma : X \to G$ be a witnessing map for $X \sim^{\Gamma}_{\mathcal{I}} B$. Put $A' = X \smallsetminus B$ and note that A' respects $\mathcal{I}$ since so does P and hence B. We show that A' is $\mathcal{I}$-traveling. Put $A_n = (\hat{\gamma})^n(A')$, for each $n \geq 0$. It follows from injectivity of $\hat{\gamma}$ that A_n are pairwise disjoint. For all n, recursively define $\delta_n : A' \to G$ as follows

$$\begin{cases} \delta_0 = \gamma \restriction_{A'} \\ \delta_{n+1} = \gamma \circ \hat{\delta}_n \end{cases}.$$

It follows from $F_{\mathcal{I}}$-invariance of γ that each δ_n is $F_{\mathcal{I}}$-invariant. It is also clear that $\hat{\delta}_n = (\hat{\gamma})^n$ and hence δ_n is a witnessing map for $A' \sim^{\Gamma}_{\mathcal{I}} A_n$. Thus A' is i-traveling with Γ pieces.

For the converse, assume that A is $\mathcal{I}$-traveling and let $\{A_n\}_{n \in \mathbb{N}}$ be as in Definition 6.1. In particular, each A_n respects $\mathcal{I}$ and $A_n \sim^{\Gamma}_{\mathcal{I}} A_m$, for all $n, m \in \mathbb{N}$. Let $P' = \bigcup_{n \in \mathbb{N}} A_n$ and

$B' = \bigcup_{n\geq 1} A_n$. Since $A_n \sim^{\Gamma}_{\mathcal{I}} A_{n+1}$, part (b) of 4.2 implies that $P' \sim^{\Gamma}_{\mathcal{I}} B'$. Moreover, since $P \setminus P' \sim^{\Gamma}_{\mathcal{I}} P \setminus P'$, we get $P \sim^{\Gamma}_{\mathcal{I}} (B' \cup (P \setminus P')) = P \setminus A$. □

For a G-invariant set P and $A \subseteq P$, we say that A is a complete section for P if $[A]_G = P$. The above lemma immediately implies the following.

Proposition 6.5. *Let $P \in \Gamma(X)$ be G-invariant and $i \geq 1$. P is i-compressible with Γ pieces if and only if there exists a complete section for P that is i-traveling with Γ pieces. The same is true with "i-compressible" and "i-traveling" replaced by "compressible" and "traveling".*

We need the following lemma in the proofs of 6.7 and 6.8.

Lemma 6.6. *Suppose $A \subseteq X$ is an invariant analytic set that does not admit an invariant Borel probability measure. Then there is an invariant Borel set $A' \supseteq A$ that still does not admit an invariant Borel probability measure.*

Proof. Let $\mathcal{M}$ denote the standard Borel space of G-invariant Borel probability measures on X (see Section 17 in [Kec95]). Let $\Phi \subseteq Pow(X)$ be the following predicate:

$$\Phi(W) \Leftrightarrow \forall \mu \in \mathcal{M}(\mu(W) = 0).$$

Claim. *There is a Borel set $B \supseteq A$ with $\Phi(B)$.*

Proof of Claim. By the dual form of the First Reflection Theorem for $\mathbf{\Pi}^1_1$ (see the discussion following 35.10 in [Kec95]), it is enough to show that Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. To this end, let Y be a Polish space and $D \subseteq Y \times X$ be analytic. Then, for any $n \in \mathbb{N}$, the set

$$H_n = \{(\mu, y) \in \mathcal{M} \times Y : \mu(D_y) > \frac{1}{n}\},$$

is analytic by a theorem of Kondô-Tugué (see 29.26 of [Kec95]), and hence so are the sets $H'_n := \text{proj}_Y(H_n)$ and $H := \bigcup_{n\in\mathbb{N}} H'_n$. Finally, note that

$$\{y \in Y : \Phi(A_y)\} = \{y \in Y : \exists \mu \in \mathcal{M} \exists n \in \mathbb{N}(\mu(A_y) > \frac{1}{n})\}^c = H^c,$$

and so $\{y \in Y : \Phi(A_y)\}$ is $\mathbf{\Pi}^1_1$. ⊣

Now put $A' = (B)_G$, where $(B)_G = \{x \in B : [x]_G \subseteq B\}$. Clearly, A' is an invariant Borel set, $A' \supseteq A$, and $\Phi(A')$ since $A' \subseteq B$ and $\Phi(B)$. □

Proposition 6.7. *Let X be a Borel G-space. The following are equivalent:*

(1) X is compressible with universally measurable pieces;

(2) There is a universally measurable complete section that is a traveling set with universally measurable pieces;

(3) There is no G-invariant Borel probability measure on X;

(4) X is compressible with Borel pieces;

(5) There is a Borel complete section that is a traveling set with Borel pieces.

Proof. Equivalence of (1) and (2) as well as (4) and (5) is asserted in 6.5, (4)⇒(1) is trivial, and (3)⇒(4) follows from Nadkarni's theorem (see 2.12). It remains to show (1)⇒(3). To this end, suppose $X \sim^{\Gamma} B$, where $B^c = X \setminus B$ is a complete section and Γ is the class of universally measurable sets. If there was a G-invariant Borel probability measure μ on X, then $\mu(X) = \mu(B)$ and hence $\mu(B^c) = 0$. But since B^c is a complete section, $X = \bigcup_{g \in G} gB^c$, and thus $\mu(X) = 0$, a contradiction. □

Now we prove an analogue of this for i-compressibility.

Proposition 6.8. *Let X be a Borel G-space. For $i \geq 1$, the following are equivalent:*

(1) X is i-compressible with universally measurable pieces;

(2) There is a universally measurable complete section that is an i-traveling set with universally measurable pieces;

(3) There is a partition $\mathcal{I}$ of X generated by i Borel sets such that $Y = f_{\mathcal{I}}(X) \subseteq |\mathcal{I}|^G$ does not admit a G-invariant Borel probability measure;

(4) X is i-compressible with Borel pieces;

(5) There is a Borel complete section that is a i-traveling set with Borel pieces.

Proof. Equivalence of (1) and (2) as well as (4) and (5) is asserted in 6.5 and (4)⇒(1) is trivial. It remains to show (1)⇒(3)⇒(5).

(1)⇒(3): Suppose $X \sim_{\mathcal{I}}^{\Gamma} B$, where $B^c = X \setminus B$ is a complete section, $\mathcal{I}$ is a partition of X generated by i Borel sets, and Γ denotes the class of universally measurable sets. Let $\gamma : X \to G$ be a witnessing map for $X \sim_i^{\Gamma} B$. By the Jankov-von Neumann uniformization theorem (see 18.1 in [Kec95]), $f_{\mathcal{I}}$ has a $\sigma(\mathbf{\Sigma}_1^1)$-measurable (hence universally measurable) right inverse $h : Y \to X$. Define $\delta : Y \to G$ by $\delta(y) = \gamma(h(y))$ and note that δ is universally measurable being a composition of such functions. Letting $B' = \hat{\delta}(Y)$, it is straightforward to check that $\hat{\delta} \circ f_{\mathcal{I}} = f_{\mathcal{I}} \circ \hat{\gamma}$ and thus $B' = f_{\mathcal{I}}(\hat{\gamma}(X)) = f_{\mathcal{I}}(B)$. Now it follows that δ is a witnessing map for $Y \sim^{\Gamma} B'$ and hence Y is compressible with universally measurable pieces. Finally, (1)⇒(3) of 6.7 implies that Y does not admit an invariant Borel probability measure.

(3)⇒(5): Assume Y is as in (3). Then by Lemma 6.6, there is a Borel G-invariant $Y' \supseteq Y$ that does not admit a G-invariant Borel probability measure. Viewing Y' as a Borel G-space, we apply (3)⇒(4) of 6.7 and get that Y' is compressible with Borel pieces; thus there is a Borel $B' \subseteq Y'$ with $[Y' \setminus B']_G = Y'$ such that $Y' \sim B'$. Let $\delta : Y' \to G$ be a witnessing map for $Y' \sim B'$. Put $B = f_{\mathcal{I}}^{-1}(B')$ and $\gamma = \delta \circ f_{\mathcal{I}}$. By definition, γ is $F_{\mathcal{I}}$-invariant. In fact, it is straightforward to check that γ is a witnessing map for $X \sim_{\mathcal{I}} B$ and $[X \setminus B]_G = [f_{\mathcal{I}}^{-1}(Y \setminus B')]_G = f_{\mathcal{I}}^{-1}([Y \setminus B']_G) = f_{\mathcal{I}}^{-1}(Y) = X$. Hence X is $\mathcal{I}$-compressible. □

We now give an example of a 1-traveling set. First we need some definitions.

Definition 6.9. *Let X be a Borel G-space and $A \subseteq X$ be Borel. A is called*

- *aperiodic if it intersects every orbit in either 0 or infinitely many points;*
- *a partial transversal if it intersects every orbit in at most one point;*
- *smooth if there is a Borel partial transversal $T \subseteq A$ such that $[T]_G = [A]_G$.*

Proposition 6.10. *Let X be an aperiodic Borel G-space and $T \subseteq X$ be Borel. If T is a partial transversal, then T is $\langle T \rangle$-traveling.*

Proof. let $G = \{g_n\}_{n \in \mathbb{N}}$ with $g_0 = 1_G$. For each $n \in \mathbb{N}$, define $\bar{n} : X \to \mathbb{N}$ and $\gamma_n : T \to G$ recursively in n as follows:

$$\begin{cases} \bar{n}(x) = \text{the least } k \text{ such that } g_k x \notin \{\hat{\gamma}_i(x) : i < n\} \\ \gamma_n(x) = g_{\bar{n}(x)} \end{cases}.$$

Clearly, $\bar{n}$ and γ_n are well-defined and Borel. Define $\gamma : T \to G^{\mathbb{N}}$ by setting $\gamma(\cdot)(n) = \gamma_n$. It follows from the definitions that γ is a Borel travel guide for T and hence, T is a traveling set. It remains to show that γ is $F_{\mathcal{I}}$-invariant, where $\mathcal{I} = \langle T \rangle$. For this it is enough to show that $\bar{n}$ is $F_{\mathcal{I}}$-invariant, which we do by induction on n. Since it trivially holds for $n = 0$, we assume it is true for all $0 \leq k < n$ and show it for n. To this end, suppose $x, y \in T$ with $x F_{\mathcal{I}} y$, and assume for contradiction that $m := \bar{n}(x) < \bar{n}(y)$. Thus it follows that $g_m y = \hat{\gamma}_k(y) \in \hat{\gamma}_k(T)$, for some $k < n$. By the induction hypothesis, $\hat{\gamma}_k(T)$ is $F_{\mathcal{I}}$-invariant and hence, $g_m x \in \hat{\gamma}_k(T)$, contradicting the definition of $\bar{n}(x)$. □

Corollary 6.11. *Let X be an aperiodic Borel G-space. If a Borel set $A \subseteq X$ is smooth, then $A \in \mathfrak{C}_1$.*

Proof. Let $P = [A]_G$ and let T be a Borel partial transversal with $[T]_G = P$. By 6.10, T is $\mathcal{I}$-traveling, where $\mathcal{I} = \langle T \rangle$. Hence, $P \sim_{\mathcal{I}} P \setminus T$, by Lemma 6.4. This implies that P is 1^*-compressible since $\mathcal{I} = \langle T^c \rangle$ and $P \setminus T \in \{T^c\}^P$. □

7 Constructing finite generators using i-traveling sets

Lemma 7.1. *Let $A \in \mathfrak{B}(X)$ be a complete section and $\mathcal{I}$ be an A-sensitive finite Borel partition of X. If A is $\mathcal{I}$-traveling (with Borel pieces), then there is a Borel $2|\mathcal{I}|$-generator. If moreover $A \in \mathcal{I}$, then there is a Borel $(2|\mathcal{I}| - 1)$-generator.*

Proof. Let γ be an $F_{\mathcal{I}}$-invariant Borel travel guide for A. Fix a countable family $\{U_n\}_{n \in \mathbb{N}}$ generating the Borel structure of X and let $B = \bigcup_{n \geq 1} \hat{\gamma}_n(A \cap U_n)$. By Lemma 4.3, each $\hat{\gamma}_n$

maps Borel sets to Borel sets and hence B is Borel. Set $\mathcal{J} = \langle B \rangle$, $\mathcal{P} = \mathcal{I} \vee \mathcal{J}$ and note that $|\mathcal{P}| \le 2|\mathcal{I}|$. A and B are disjoint since $\{\hat{\gamma}_n(A)\}_{n\in\mathbb{N}}$ is a collection of pairwise disjoint sets and $\hat{\gamma}_0(A) = A$; thus if $A \in \mathcal{I}$, $|\mathcal{P}| \le 1 + 2(|\mathcal{I}| - 1) = 2|\mathcal{I}| - 1$. We show that $\mathcal{P}$ is a generator, that is $G\mathcal{P}$ separates points in X.

Let $x \ne y \in X$ and assume they are not separated by $G\mathcal{I}$, thus $xF_{\mathcal{I}}y$. We show that $G\mathcal{J}$ separates x and y. Because A is a complete section, multiplying x by an appropriate group element, we may assume that $x \in A$. Since A respects $\mathcal{I}$, A is $F_{\mathcal{I}}$-invariant and thus $y \in A$. Also, because γ is $F_{\mathcal{I}}$-invariant, $\gamma_n(x) = \gamma_n(y)$, $\forall n \in \mathbb{N}$. Let $n \ge 1$ be such that $x \in U_n$ but $y \notin U_n$. Put $g = \gamma_n(x)(= \gamma_n(y))$. Then $gx = \hat{\gamma}_n(x) \in \hat{\gamma}_n(A \cap U_n)$ while $gy = \hat{\gamma}_n(y) \notin \hat{\gamma}_n(A \cap U_n)$. Hence, $gx \in B$ and $gy \notin B$ because $\gamma_m(A) \cap \gamma_n(A) = \varnothing$ for all $m \ne n$ and $gy = \hat{\gamma}_n(y) \in \hat{\gamma}_n(A)$. Thus $G\mathcal{J}$ separates x and y. □

Now 6.8 and 7.1 together imply the following.

Proposition 7.2. *Let X be a Borel G-space and $i \ge 1$. If X is i-compressible then there is a Borel 2^{i+1}-generator.*

Proof. By 6.8, there exists a Borel i-traveling complete section A. Let $\mathcal{I}$ witness A being i-traveling and thus, by Lemma 7.1, there is a $2|\mathcal{I}| \le 2 \cdot 2^i = 2^{i+1}$-generator. □

Example 7.3. For $2 \le n \le \infty$, let $\mathbb{F}_n$ denote the free group on n generators and let X be the boundary of $\mathbb{F}_n$, i.e. the set of infinite reduced words. Clearly, the product topology makes X a Polish space and $\mathbb{F}_n$ acts continuously on X by left concatenation and cancellation. We show that X is 1-compressible and thus admits a Borel $2^2 = 4$-generator by Proposition 7.2. To this end, let a, b be two of the n generators of $\mathbb{F}_n$ and let X_a be the set of all words in X that start with a. Then $X = (X_{a^{-1}} \cup X^c_{a^{-1}}) \sim_{\mathcal{I}} Y$, where $Y = bX_{a^{-1}} \cup aX^c_{a^{-1}}$ and $\mathcal{I}\langle X_{a^{-1}}\rangle$. Hence $X \sim_1 Y$. Since $X \smallsetminus Y \supseteq X_{a^{-1}}$, $[X \smallsetminus Y]_{\mathbb{F}_n} = X$ and thus X is 1-compressible.

Now we obtain a sufficient condition for the existence of an embedding into a finite Bernoulli shift.

Corollary 7.4. *Let X be a Borel G-space and $k \in \mathbb{N}$. If there exists a Borel G-map $f : X \to k^G$ such that $Y = f(X)$ does not admit a G-invariant Borel probability measure, then there is a Borel G-embedding of X into $(2k)^G$.*

Proof. Let $\mathcal{I} = \mathcal{I}_f$ and hence $f = f_{\mathcal{I}}$. By (3)⇒(5) of 6.8 (or rather the proof of it), X admits a Borel $\mathcal{I}$-traveling complete section. Thus by Lemma 7.1, X admits a $2|\mathcal{I}| = 2k$-generator and hence, there is a Borel G-embedding of X into $(2k)^G$. □

Lemma 7.5. *Let $\mathcal{I}$ be a partition of X into n Borel sets. Then $\mathcal{I}$ is generated by $k = \lceil \log_2(n) \rceil$ Borel sets.*

Proof. Since $2^k \geq n$, we can index $\mathcal{I}$ by the set $\mathbf{2^k}$ of all k-tuples of $\{0,1\}$, i.e. $\mathcal{I} = \{A_\sigma\}_{\sigma \in \mathbf{2^k}}$. For all $i < k$, put

$$B_i = \bigcup_{\sigma \in \mathbf{2^k} \wedge \sigma(i) = 1} A_\sigma.$$

Now it is clear that for all $\sigma \in \mathbf{2^k}$, $A_\sigma = \bigcap_{i<k} B_i^{\sigma(i)}$, where $B_i^{\sigma(i)}$ is equal to B_i if $\sigma(i) = 1$, and equal to B_i^c, otherwise. Thus $\mathcal{I} = \langle B_i : i < k \rangle$. □

Proposition 7.6. *If X is compressible and there is a Borel n-generator, then X is $\lceil \log_2(n) \rceil$-compressible.*

Proof. Let $\mathcal{I}$ be an n-generator and hence, by Lemma 7.5, $\mathcal{I}$ is generated by i Borel sets. Since $G\mathcal{I}$ separates points in X, each $F_{\mathcal{I}}$-class is a singleton and hence $X \prec X$ implies $X \prec_{\mathcal{I}} X$. □

From 7.2 and 7.6 we immediately get the following corollary, which justifies the use of i-compressibility in studying Question 2.3.

Corollary 7.7. *Let X be a Borel G-space that is compressible (equivalently, does not admit an invariant Borel probability measure). X admits a finite generator if and only if X is i-compressible for some $i \geq 1$.*

8 Finitely additive invariant measures and i-compressibility

This section is mainly devoted to the following theorem, together its corollaries and proof.

Theorem 8.1. *Let X be a Borel G-space. If X is aperiodic, then there exists a function $m : \mathfrak{B}(X) \times X \to [0,1]$ satisfying the following properties for all $A, B \in \mathfrak{B}(X)$:*

(a) $m(A, \cdot)$ is Borel;

(b) $m(X, x) = 1$, $\forall x \in X$;

(c) If $A \subseteq B$, then $m(A, x) \leq m(B, x)$, $\forall x \in X$;

(d) $m(A, x) = 0$ off $[A]_G$;

(e) $m(A, x) > 0$ on $[A]_G$ modulo $\mathfrak{C}_4$;

(f) $m(A, x) = m(gA, x)$, for all $g \in G$, $x \in X$ modulo $\mathfrak{C}_3$;

(g) If $A \cap B = \varnothing$, then $m(A \cup B, x) = m(A, x) + m(B, x)$, $\forall x \in X$ modulo $\mathfrak{C}_4$.

Remark. A version of this theorem is what lies at the heart of the proof of Nadkarni's theorem. The conclusions of our theorem are modulo $\mathfrak{C}_4$, which is potentially a smaller σ-ideal than the σ-ideal of sets contained in compressible Borel sets used in Nadkarni's version. However, the price we pay for this is that part (g) asserts only finite additivity instead of countable additivity asserted by Nadkarni's version.

Before proceeding with the proof of this theorem, we draw a couple of corollaries. Theorem 8.1 will only be used via Corollary 8.3.

Definition 8.2. *Let X be a Borel G-space. $\mathcal{B} \subseteq \mathfrak{B}(X)$ is called a Boolean G-algebra, if it is a Boolean algebra, i.e. is closed under finite unions and complements, and is closed under the G-action, i.e. $G\mathcal{B} = \mathcal{B}$.*

Corollary 8.3. *Let X be a Borel G-space and let $\mathcal{B} \subseteq \mathfrak{B}(X)$ be a countable Boolean G-algebra. For any $A \in \mathcal{B}$ with $A \notin \mathfrak{C}_4$, there exists a G-invariant finitely additive probability measure μ on $\mathcal{B}$ with $\mu(A) > 0$. Moreover, μ can be taken such that there is $x \in A$ such that $\forall B \in \mathcal{B}$ with $B \cap [x]_G = \varnothing$, $\mu(B) = 0$.*

Proof. Let $A \in \mathcal{B}$ be such that $A \notin \mathfrak{C}_4$. We may assume that $X = [A]_G$ by setting the (to be constructed) measure to be 0 outside $[A]_G$.

If X is not aperiodic, then by assigning equal point masses to the points of a finite orbit, we will have a probability measure on all of $\mathfrak{B}(X)$, so assume X is aperiodic.

Since $\mathfrak{C}_4$ is a σ-ideal and $\mathcal{B}$ is countable, Theorem 8.1 implies that there is a $P \in \mathfrak{C}_4$ such that (a)-(g) of the same theorem hold on $X \smallsetminus P$ for all $A, B \in \mathcal{B}$. Since $A \notin \mathfrak{C}_4$, there exists $x_A \in A \smallsetminus P$. Hence, letting $\mu(B) = m(B, x_A)$ for all $B \in \mathcal{B}$, conditions (b),(f) and (g) imply that μ is a G-invariant finitely additive probability measure on $\mathcal{B}$. Moreover, since $x_A \in [A]_G \smallsetminus P$, $\mu(A) = m(A, x_A) > 0$. Finally, the last assertion follows from condition (d). □

Corollary 8.4. *Let X be a Borel G-space. For every Borel set $A \subseteq X$ with $A \notin \mathfrak{C}_4$, there exists a G-invariant finitely additive Borel probability measure μ (defined on all Borel sets) with $\mu(A) > 0$.*

Proof. The statement follows from 8.3 and a standard application of the Compactness Theorem of propositional logic. Here are the details.

We fix the following set of propositional variables

$$\mathcal{P} = \{P_{A,r} : A \in \mathfrak{B}(X), r \in [0,1]\},$$

with the following interpretation in mind:

$$P_{A,r} \Leftrightarrow \text{"the measure of } A \text{ is} \geq r\text{"}.$$

Define the theory T as the following set of sentences: for each $A, B \in \mathfrak{B}(X)$, $r, s \in [0,1]$ and $g \in G$,

(i) "$P_{A,0}$" $\in T$;

(ii) if $r > 0$, then "$\neg P_{\varnothing,r}$" $\in T$;

(iii) if $s \geq r$, then "$P_{A,s} \to P_{A,r}$" $\in T$;

(iv) if $A \cap B = \varnothing$, then "$(P_{A,r} \wedge P_{B,s}) \to P_{A\cup B,r+s}$", "$(\neg P_{A,r} \wedge \neg P_{B,s}) \to \neg P_{A\cup B,r+s}$" $\in T$;

(v) "$P_{X,1}$" $\in T$;

(vi) "$P_{A,r} \to P_{gA,r}$" $\in T$.

If there is an assignment of the variables in $\mathcal{P}$ satisfying T, then for each $A \in \mathfrak{B}(X)$, we can define

$$\mu(A) = \sup\{r \in [0,1] : P_{A,r}\}.$$

Note that due to (i), μ is well defined for all $A \in \mathfrak{B}(X)$. In fact, it is straightforward to check that μ is a finitely additive G-invariant probability measure. Thus, we only need to show that T is satisfiable, for which it is enough to check that T is finitely satisfiable, by the Compactness Theorem of propositional logic (or by Tychonoff's theorem).

Let $T_0 \subseteq T$ be finite and let $\mathcal{P}_0$ be the set of propositional variables that appear in the sentences in T_0. Let $\mathcal{B}$ denote the Boolean G-algebra generated by the sets that appear in the indices of the variables in $\mathcal{P}_0$. By 8.3, there is a finitely additive G-invariant probability measure μ defined on $\mathcal{B}$. Consider the following assignment of the variables in $\mathcal{P}_0$: for all $P_{A,r} \in \mathcal{P}_0$,

$$P_{A,r} :\Leftrightarrow \mu(A) \geq r.$$

It is straightforward to check that this assignment satisfies T_0, and hence, T is finitely satisfiable. $\square$

We now start working towards the proof of Theorem 8.1, following the general outline of Nadkarni's proof of Theorem 2.12. The construction of $m(A,x)$ is somewhat similar to that of Haar measure. First, for sets A, B, we define a Borel function $[A/B] : X \to \mathbb{N} \cup \{-1, \infty\}$

that basically gives the number of copies of $B^{[x]_G}$ that fit in $A^{[x]_G}$ when moved by group elements (piecewise). Then we define a decreasing sequence of complete sections (called a fundamental sequence below), which serves as a gauge to measure the size of a given set.

Assume throughout that X is an aperiodic Borel G-space (although we only use the aperiodicity assumption in 8.15 to assert that smooth sets are in $\mathfrak{C}_1$).

8.1 Measuring the size of a set relative to another

Lemma 8.5 (Comparability). *$\forall A, B \in \mathfrak{B}(X)$, there is a partition $X = P \cup Q$ into G-invariant Borel sets such that for any A, B-sensitive finite Borel partition $\mathcal{I}$ of X, $A^P \prec_{\mathcal{I}} B^P$ and $B^Q \preceq_{\mathcal{I}} A^Q$.*

Proof. It is enough to prove the lemma assuming $X = [A]_G \cap [B]_G$ since we can always include $[B]_G \setminus [A]_G$ in P and $X \setminus [B]_G$ in Q.

Fix an enumeration $\{g_n\}_{n \in \mathbb{N}}$ for G. We recursively construct Borel sets A_n, B_n, A'_n, B'_n as follows. Set $A'_0 = A$ and $B'_0 = B$. Assuming A'_n, B'_n are defined, set $B_n = B'_n \cap g_n A'_n$, $A_n = g_n^{-1} B_n$, $A'_{n+1} = A'_n \setminus A_n$ and $B'_{n+1} = B'_n \setminus B_n$.

It is easy to see by induction on n that for any A, B-sensitive $\mathcal{I}$, A_n, B_n are $F_{\mathcal{I}}$-invariant since so are A, B. Thus, setting $A^* = \bigcup_{n \in \mathbb{N}} A_n$ and $B^* = \bigcup_{n \in \mathbb{N}} B_n$, we get that $A^* \sim_{\mathcal{I}} B^*$ since $B_n = g_n A_n$.

Let $A' = A \setminus A^*$, $B' = B \setminus B^*$ and set $P = [B']_G$, $Q = X \setminus P$.

Claim. $[A']_G \cap [B']_G = \varnothing$.

Proof of Claim. Assume for contradiction that $\exists x \in A'$ and $n \in \mathbb{N}$ such that $g_n x \in B'$. It is clear that $A' = \bigcap_{k \in \mathbb{N}} A'_k$, $B' = \bigcap_{k \in \mathbb{N}} B'_k$; in particular, $x \in A'_n$ and $g_n x \in B'_n$. But then $g_n x \in B_n$ and $x \in A_n$, contradicting $x \in A'$. $\dashv$

Let $\mathcal{I}$ be an A, B-sensitive partition. Then $A^P = (A^*)^P$ and hence $A^P \prec_{\mathcal{I}} B^P$ since $(A^*)^P \sim_{\mathcal{I}} (B^*)^P \subseteq B^P$ and $[B^P \setminus (B^*)^P]_G = [B']_G = P = [B^P]_G$. Similarly, $B^Q = (B^*)^Q$ and hence $B^Q \preceq_{\mathcal{I}} A^Q$ since $(B^*)^Q \sim_{\mathcal{I}} (A^*)^Q \subseteq A^Q$. □

Definition 8.6 (Divisibility)**.** *Let $n \le \infty$, $A, B, C \in \mathfrak{B}(X)$ and $\mathcal{I}$ be a finite Borel partition of X.*

- *Write $A \sim_{\mathcal{I}} nB \oplus C$ if there are Borel sets $A_k \subseteq A$, $k < n$, such that $\{A_k\}_{k<n} \cup \{C\}$ is a partition of A, each A_k is $F_{\mathcal{I}}$-invariant relative to A and $A_k \sim_{\mathcal{I}} B$.*
- *Write $nB \le_{\mathcal{I}} A$ if there is $C \subseteq A$ with $A \sim_{\mathcal{I}} nB \oplus C$, and write $nB \prec_{\mathcal{I}} A$ if moreover $[C]_G = [A]_G$.*
- *Write $A \le_{\mathcal{I}} nB$ if there is a Borel partition $\{A_k\}_{k<n}$ of A such that each A_k is $F_{\mathcal{I}}$-invariant relative to A and $A_k \le_{\mathcal{I}} B$. If moreover, $A_k \prec_{\mathcal{I}} B$ for at least one $k < n$, we write $A \prec_{\mathcal{I}} nB$.*

For $i \ge 1$, we use the above notation with $\mathcal{I}$ replaced by i if there is an A, B-sensitive partition $\mathcal{I}$ generated by i sets for which the above conditions hold.

Proposition 8.7 (Euclidean decomposition)**.** *Let $A, B \in \mathfrak{B}(X)$ and put $R = [A]_G \cap [B]_G$. There exists a partition $\{P_n\}_{n\le\infty}$ of R into G-invariant Borel sets such that for any A, B-sensitive finite Borel partition $\mathcal{I}$ of X and $n \le \infty$, $A^{P_n} \sim_{\mathcal{I}} nB^{P_n} \oplus C_n$ for some C_n such that $C_n \prec_{\mathcal{I}} B^{P_n}$, if $n < \infty$.*

Proof. We repeatedly apply Lemma 8.5. For $n < \infty$, recursively define R_n, P_n, A_n, C_n satisfying the following:

(i) R_n are invariant decreasing Borel sets such that $nB^{R_n} \le_{\mathcal{I}} A^{R_n}$ for any A, B-sensitive $\mathcal{I}$;

(ii) $P_n = R_n \setminus R_{n+1}$;

(iii) $A_n \subseteq R_{n+1}$ are pairwise disjoint Borel sets such that for any A, B-sensitive $\mathcal{I}$, every A_n respects $\mathcal{I}$ and $A_n \sim_{\mathcal{I}} B^{R_{n+1}}$;

(iv) $C_n \subseteq P_n$ are Borel sets such that for any A, B-sensitive $\mathcal{I}$, every C_n respects $\mathcal{I}$ and $C_n \prec_{\mathcal{I}} B^{P_n}$.

Set $R_0 = R$. Given R_n, $\{A_k\}_{k<n}$ satisfying the above properties, let $A' = A^{R_n} \setminus \bigcup_{k<n} A_k$. We apply Lemma 8.5 to A' and B^{R_n}, and get a partition $R_n = P_n \cup R_{n+1}$ such that $(A')^{P_n} \prec_{\mathcal{I}} B^{P_n}$ and $B^{R_{n+1}} \preceq_{\mathcal{I}} (A')^{R_{n+1}}$. Set $C_n = (A')^{P_n}$. Let $A_n \subseteq (A')^{R_{n+1}}$ be such that $B^{R_{n+1}} \sim_{\mathcal{I}} A_n$. It is straightforward to check (i)-(iv) are satisfied.

Now let $R_\infty = \bigcap_{n\in\mathbb{N}} R_n$ and $C_\infty = (A \setminus \bigcup_{n\in\mathbb{N}} A_n)^{R_\omega}$. Now it follows from (i)-(iv) that for all $n \leq \infty$, $\{A_k^{P_n}\}_{k<n} \cup \{C_n\}$ is a partition of A^{P_n} witnessing $A^{P_n} \sim_{\mathcal{I}} nB \oplus C_n$, and for all $n < \infty$, $C_n \prec B^{P_n}$. □

For $A, B \in \mathfrak{B}(X)$, let $\{P_n\}_{n\leq\infty}$ be as in the above proposition. Define

$$[A/B](x) = \begin{cases} n & \text{if } x \in P_n, n < \infty \\ \infty & \text{if } x \in P_\infty \text{ or } x \in [A]_G \setminus [B]_G \\ 0 & \text{if } x \in [B]_G \setminus [A]_G \\ -1 & \text{otherwise} \end{cases} .$$

Note that $[A/B] : X \to \mathbb{N} \cup \{-1, \infty\}$ is a Borel function by definition.

8.2 Properties of [A/B]

Lemma 8.8 (Infinite divisibility ⇒ compressibility). *Let $A, B \in \mathfrak{B}(X)$ with $[A]_G = [B]_G$, and let $\mathcal{I}$ be a finite Borel partition of X. If $\infty B \preceq_{\mathcal{I}} A$, then $A \prec_{\mathcal{I}} A$.*

Proof. Let $C \subseteq A$ be such that $A \sim_{\mathcal{I}} \infty B \oplus C$ and let $\{A_k\}_{k<\infty}$ be as in Definition 8.6. $A_k \sim_{\mathcal{I}} B \sim_{\mathcal{I}} A_{k+1}$ and hence $A_k \sim_{\mathcal{I}} A_{k+1}$. Also trivially $C \sim_{\mathcal{I}} C$. Thus, letting $A' = \bigcup_{k<\infty} A_{k+1} \cup C$, we apply (b) of 4.2 to A and A', and get that $A \sim_{\mathcal{I}} A'$. Because $[A \setminus A']_G = [A_0]_G = [B]_G = [A]_G$, we have $A \prec_{\mathcal{I}} A$. □

Lemma 8.9 (Ambiguity ⇒ compressibility). *Let $A, B \in \mathfrak{B}(X)$ and $\mathcal{I}$ be a finite Borel partition of X. If $nB \preceq_{\mathcal{I}} A \prec_{\mathcal{I}} nB$ for some $n \geq 1$, then $A \prec_{\mathcal{I}} A$.*

Proof. Let $C \subseteq A$ be such that $A \sim_{\mathcal{I}} nB \oplus C$ and let $\{A_k\}_{k<n}$ be a partitions of $A \setminus C$ witnessing $A \sim_{\mathcal{I}} nB \oplus C$. Also let $\{A'_k\}_{k<n}$ be witnessing $A \prec_{\mathcal{I}} nB$ with $A'_0 \prec_{\mathcal{I}} B$. Since

$A'_k \preceq_{\mathcal{I}} B \sim_{\mathcal{I}} A_k$, $A'_k \preceq_{\mathcal{I}} A_k$, for all $k < n$ and $A'_0 \prec_{\mathcal{I}} A_0$. Note that it follows from the hypothesis that $[A]_G = [B]_G$ and hence $[A_0]_G = [A]_G$ since $[A_0]_G = [B]_G$. Thus it follows from (b) of 4.2 that $A = \bigcup_{k<n} A'_k \prec_{\mathcal{I}} \bigcup_{k<n} A_k \subseteq A$. □

Proposition 8.10. *Let $n \in \mathbb{N}$ and $A, A', B, P \in \mathfrak{B}(X)$, where P is invariant.*

(a) $[A/B] \in \mathbb{N}$ on $[B]_G$ modulo $\mathfrak{C}_3$.

(b) If $A \subseteq A'$, then $[A/B] \leq [A'/B]$.

(c) If $[A/B] = n$ on P then $nB^P \preceq_{\mathcal{I}} A^P \prec_{\mathcal{I}} (n+1)B^P$, for any finite Borel partition $\mathcal{I}$ that is A, B-sensitive. In particular, $nB^P \preceq_2 A^P \prec_2 (n+1)B^P$ by taking $\mathcal{I} = \langle A, B \rangle$.

(d) For $n \geq 1$, if $A^P \prec_i nB^P$, then $[A/B] < n$ on P modulo $\mathfrak{C}_{i+1}$;

(e) If $A^P \subseteq [B]_G$ and $nB^P \preceq_i A^P$, then $[A/B] \geq n$ on P modulo $\mathfrak{C}_{i+1}$.

Proof. For (a), notice that 8.8 and 5.6 imply that $P_\infty \in \mathfrak{C}_3$. (b) and (c) follow from the definition of $[A/B]$. For (d), let $\mathcal{I}$ be an A, B-sensitive partition of X generated by i Borel sets such that $A^P \prec_{\mathcal{I}} nB^P$, and put $Q = \{x \in P : [A/B](x) \geq n\}$. By (c), $nB^Q \preceq_{\mathcal{I}} A^Q$. Thus, by Lemma 8.9, $A^Q \prec_{\mathcal{I}} A^Q$ and hence, by Lemma 5.6, $[A^Q]_G = Q \in C_{i+1}$.

For (e), let $\mathcal{I}$ be an A, B-sensitive partition of X generated by i Borel sets such that $nB^P \preceq_{\mathcal{I}} A^P$, and put $Q = \{x \in P : [A/B](x) < n\}$. By (c), $A^Q \prec_{\mathcal{I}} nB^Q$. Thus, by Lemma 8.9, $A^Q \prec_{\mathcal{I}} A^Q$ and hence, by Lemma 5.6, $[A^Q]_G = Q \in C_{i+1}$. □

Lemma 8.11 (Almost cancelation)**.** *For any $A, B, C \in X$,*

$$[A/B][B/C] \leq [A/C] < ([A/B]+1)([B/C]+1)$$

on $R := [B]_G \cap [C]_G$ modulo $\mathfrak{C}_4$.

Proof. Let $\mathcal{I} = \langle A, B, C \rangle$.

$[A/B][B/C] \leq [A/C]$: Fix integers $i, j > 0$ and let $P = \{x \in X : [A/B](x) = i \wedge [B/C](x) = j\}$. Since $i, j > 0$, $P \subseteq [A]_G \cap [B]_G \cap [C]_G$ and we work in P. By (c) of 8.10, $iB \preceq_{\mathcal{I}} A$ and $jC \preceq_{\mathcal{I}} B$. Thus it follows that $ijC \preceq_{\mathcal{I}} A$ and hence $[A/C] \geq ij$ modulo $\mathfrak{C}_4$ by (e) of 8.10.

$[A/C] < ([A/B]+1)([B/C]+1)$: By (a) of 8.10, $[A/C],[A/B],[B/C] \in \mathbb{N}$ on R modulo $\mathfrak{C}_3$. Fix $i,j \in \mathbb{N}$ and let $Q = \{x \in R : [A/B](x) = i \wedge [B/C](x) = j\}$. We work in Q. By (c) of 8.10, $A \prec_{\mathcal{I}} (i+1)B$ and $B \prec_{\mathcal{I}} (j+1)C$. Thus $A \prec_{\mathcal{I}} (i+1)(j+1)C$ and hence $[A/C] < (i+1)(j+1)$ modulo $\mathfrak{C}_4$ by (d) of 8.10. □

Lemma 8.12 (Invariance)**.** *For $A, F \in \mathfrak{B}(X)$, $\forall g \in G, [A/F] = [gA/F]$, modulo $\mathfrak{C}_3$.*

Proof. We may assume that $X = [A]_G \cap [F]_G$. Fix $g \in G$, $n \in \mathbb{N}$, and put $Q = \{x \in X : [gA/F](x) = n\}$. We work in Q. Let $\mathcal{I} = \langle A, F\rangle$ and hence A, gA, F respect $\mathcal{I}$. By (c) of 8.10, $nF \preceq_{\mathcal{I}} gA$. But clearly $gA \sim_{\mathcal{I}} A$ and hence $nF \preceq_{\mathcal{I}} A$. Thus, by (e) of 8.10, $[A/F] \geq n = [gA/F]$, modulo $\mathfrak{C}_3$. By symmetry, $[gA/F] \geq [A/F]$ (modulo $\mathfrak{C}_3$) and the lemma follows. □

Lemma 8.13 (Almost additivity)**.** *For any $A, B, F \in X$ with $A \cap B = \varnothing$, $[A/F] + [B/F] \leq [A \cup B/F] \leq [A/F] + [B/F] + 1$ modulo $\mathfrak{C}_4$.*

Proof. Let $\mathcal{I} = \langle A, B, F\rangle$.

$[A/F]+[B/F] \leq [A \cap B/F]$: Fix $i,j \in \mathbb{N}$ not both 0, say $i > 0$, and let $S = \{x \in X : [A/F](x) = i \wedge [B/F](x) = j\}$. Since $i > 0$, $S \subseteq [A]_G \cap [F]_G$ and we work in S. By (c) of 8.10, $iF^S \preceq_{\mathcal{I}} A^S$ and $jF^S \preceq_{\mathcal{I}} B^S$. Hence $(i+j)F^S \preceq_{\mathcal{I}} (A \cup B)^S$ and thus, by (e) of 8.10, $[A \cup B/F] \geq i + j$, modulo $\mathfrak{C}_4$.

$[A \cap B/F] \leq [A/F] + [B/F] + 1$: Outside $[F]_G$, the inequality clearly holds. Fix $i,j \in \mathbb{N}$ and let $M = \{x \in [F]_G : [A/F](x) = i \wedge [B/F](x) = j\}$. We work in M. By (c) of 8.10, $A \prec_{\mathcal{I}} (i+1)F$ and $B \prec_{\mathcal{I}} (j+1)F$. Thus it is clear that $A \cup B \prec_{\mathcal{I}} (i+j+2)F$ and hence $[A \cup B/F] < i+j+2$, modulo $\mathfrak{C}_4$, by (d) of 8.10. □

8.3 Fundamental sequence

Definition 8.14. *A sequence $\{F_n\}_{n\in\mathbb{N}}$ of decreasing Borel complete sections with $F_0 = X$ and $[F_n/F_{n+1}] \geq 2$ modulo $\mathfrak{C}_3$ is called fundamental.*

Proposition 8.15. *There exists a fundamental sequence.*

Proof. Take $F_0 = X$. Given any complete Borel section F, its intersection with every orbit is infinite modulo a smooth set (if the intersection of an orbit with a set is finite, then we can choose an element from each such nonempty intersection in a Borel way and get a Borel transversal). Thus, by 6.11, F is aperiodic modulo $\mathfrak{C}_1$. Now use Lemma 13.1 to write $F = A \cup B, A \cap B = \varnothing$, where A, B are also complete sections. Let now P, Q be as in Lemma 8.5 for A, B, and hence $A^P \prec_2 B^P, B^Q \preceq_2 A^Q$ because we can take $\mathcal{I} = \langle A, B \rangle$. Let $A' = A^P \cup B^Q, B' = B^P \cup A^Q$. Then $F = A' \cup B', A' \cap B' = \varnothing$, $A' \preceq B'$ and A' is also a complete Borel section. By (e) of 8.10, $[F/A'] \geq 2$ modulo $\mathfrak{C}_3$. Iterate this process to inductively define F_n. □

8.4 The definition and properties of $m(A, x)$

Fix a fundamental sequence $\{F_n\}_{n \in \mathbb{N}}$ and for any $A \in \mathfrak{B}(X), x \in X$, define

$$m(A, x) = \lim_{n \to \infty} \frac{[A/F_n](x)}{[X/F_n](x)}, \tag{†}$$

if the limit exists, and 0 otherwise. In the above fraction we define $\frac{\infty}{\infty} = 1$. We will prove in Proposition 8.17 that this limit exists modulo $\mathfrak{C}_4$. But first we need a lemma.

Lemma 8.16. *For any $A \in \mathfrak{B}(A)$,*

$$\lim_{n \to \infty} [A/F_n] = \begin{cases} \infty & \text{on } [A]_G \\ 0 & \text{on } X \setminus [A]_G \end{cases}, \text{ modulo } \mathfrak{C}_4.$$

Proof. The part about $X \setminus [A]_E$ is clear, so work in $[A]_E$, i.e. assume $X = [A]_G$. By (a) of 8.10 and 8.11, we have

$$\infty > [F_1/A] \geq [F_1/F_n][F_n/A] \geq 2^{n-1}[F_n/A], \text{ modulo } \mathfrak{C}_4,$$

which holds for all n at once since $\mathfrak{C}_4$ is a σ-ideal. Thus $[F_n/A] \to 0$ modulo $\mathfrak{C}_4$ and hence, as $[F_n/A] \in \mathbb{N}$, $[F_n/A]$ is eventually 0, modulo $\mathfrak{C}_4$. So if

$$B_k := \{x \in [A]_G : [F/A](x) = 0\},$$

then $B_k \nearrow X$, modulo $\mathfrak{C}_4$. Now it follows from Lemma 8.5 that $[A/F_k] > 0$ on B_k modulo $\mathfrak{C}_4$. But

$$[A/F_{k+n}] \geq [A/F_k][F_k/F_{k+n}] \geq 2^n[A/F_k], \text{ modulo } \mathfrak{C}_4,$$

so for every k, $[A/F_n] \to \infty$ on B_k modulo $\mathfrak{C}_4$. Since $B_k \nearrow X$ modulo $\mathfrak{C}_4$, we have $[A/F_n] \to \infty$ on X, modulo $\mathfrak{C}_4$. □

Proposition 8.17. *For any Borel set $A \subseteq X$, the limit in (†) exists and is positive on $[A]_G$, modulo $\mathfrak{C}_4$.*

Proof.

Claim. *Suppose $B, C \in \mathfrak{B}(X)$, $i \in \mathbb{N}$ and $D_i = \{x \in X : [C/F_i](x) > 0\}$. Then*

$$\overline{\lim} \frac{[B/F_n]}{[C/F_n]} \leq \frac{[B/F_i] + 1}{[C/F_i]}$$

on D_i, modulo $\mathfrak{C}_4$.

Proof of Claim. Working in D_i and using Lemma 8.11, $\forall j$ we have (modulo $\mathfrak{C}_4$)

$$[B/F_{i+j}] \leq ([B/F_i] + 1)([F_i/F_{i+j}] + 1)$$
$$[C/F_{i+j}] \geq [C/F_i][F_i/F_{i+j}] > 0,$$

so

$$\frac{[B/F_{i+j}]}{[C/F_{i+j}]} \leq \frac{[B/F_i] + 1}{[C/F_i]} \cdot \frac{[F_i/F_{i+j}] + 1}{[F_i/F_{i+j}]} \leq \frac{[B/F_i] + 1}{[C/F_i]} \cdot (1 + \frac{1}{2^j}),$$

from which the claim follows. ⊣

Applying the claim to $B = A$ and $C = X$ (hence $D_i = X$), we get that for all $i \in \mathbb{N}$

$$\overline{\lim_{n\to\infty}} \frac{[A/F_n](x)}{[X/F_n](x)} \leq \frac{[A/F_i](x) + 1}{[X/F_i](x)} (\text{modulo } \mathfrak{C}_4).$$

Thus

$$\overline{\lim_{n\to\infty}} \frac{[A/F_n]}{[X/F_n]} \leq \underline{\lim}_{i\to\infty} \frac{[A/F_i] + 1}{[X/F_i]} = \underline{\lim}_{i\to\infty} \frac{[A/F_i]}{[X/F_i]}$$

since $\lim_{i\to\infty} \frac{1}{[X/F_i]} = 0$.

To see that $m(A,x)$ is positive on $[A]_E$ modulo $\mathfrak{C}_4$ we argue as follows. We work in $[A]_G$. Applying the above claim to $B = X$ and $C = A$, we get

$$\frac{1}{m(A,x)} = \lim_{n\to\infty} \frac{[X/F_n]}{[A/F_n]} \leq \frac{[X/F_i]+1}{[A/F_i]} < \infty \text{ on } D_i \text{ (modulo } \mathfrak{C}_4).$$

Thus $m(A,x) > 0$ on $\cup_{i\in\mathbb{N}} D_i$, modulo $\mathfrak{C}_4$. But $D_i \nearrow [A]_G$ because $[A/F_i] \to \infty$ as $i \to \infty$, and hence $m(A,x) > 0$ on $[A]_G$ modulo $\mathfrak{C}_4$. □

8.5 Proof of Theorem 8.1

Fix $A, B \in \mathfrak{B}(X)$. The fact that $m(A,x) \in [0,1]$ and parts (b) and (d) follow directly from the definition of $m(A,x)$. Part (a) follows from the fact that $[A/F_n]$ is Borel for all $n \in \mathbb{N}$. (c) follows from (b) of Lemma 8.10, and (e) and (f) are asserted by 8.17 and 8.12, respectively.

To show (g), we argue as follows. By Lemma 8.13, $[A/F_n] + [B/F_n] \leq [A \cup B/F_n] \leq [A/F_n] + [B/F_n] + 1$, modulo $\mathfrak{C}_4$, and thus

$$\frac{[A/F_n]}{[X/F_n]} + \frac{[B/F_n]}{[X/F_n]} \leq \frac{[A \cup B/F_n]}{[X/F_n]} \leq \frac{[A/F_n]}{[X/F_n]} + \frac{[B/F_n]}{[X/F_n]} + \frac{1}{[X/F_n]},$$

for all n at once, modulo $\mathfrak{C}_4$ (using the fact that $\mathfrak{C}_4$ is a σ-ideal). Since $[X/F_n] \geq 2^n$, passing to the limit in the inequalities above, we get $m(A,x) + m(B,x) \leq m(A \cup B, x) \leq m(A,x) + m(B,x)$. QED (Thm 8.1)

CHAPTER III

Applications of the theory of i-compressibility

In this chapter we establish the existence of finite generators for compressible actions in various cases using the developments of the previous chapter.

9 Finite generators in the case of σ-compact spaces

In this section we prove that the answer to Question 2.3 is positive in case X has a σ-compact realization. To do this, we first prove Proposition 9.2, which shows how to construct a countably additive invariant probability measure on X using a finitely additive one. We then use 8.3 to conclude the result.

For the next two statements, let X be a second countable Hausdorff topological space equipped with a continuous action of G.

Lemma 9.1. *Let $\mathcal{U} \subseteq Pow(X)$ be a countable base for X closed under the G-action and finite unions/intersections. Let ρ be a G-invariant finitely additive probability measure on the G-algebra generated by $\mathcal{U}$. For every $A \subseteq X$, define*

$$\mu^*(A) = \inf\{\sum_{n \in \mathbb{N}} \rho(U_n) : U_n \in \mathcal{U} \ \wedge \ A \subseteq \bigcup_{n \in \mathbb{N}} U_n\}.$$

Then:

(a) μ^* is a G-invariant outer measure.

(b) If $K \subseteq X$ is compact, then K is metrizable and μ^* is a metric outer measure on K (with respect to any compatible metric).

Proof. It is a standard fact from measure theory that μ^* is an outer measure. That μ^* is G-invariant follows immediately from G-invariance of ρ and the fact that $\mathcal{U}$ is closed under the action of G.

For (b), first note that by Urysohn metrization theorem, K is metrizable, and fix a metric on K. If $E, F \subseteq K$ are a positive distance apart, then so are $\bar{E}$ and $\bar{F}$. Hence there exist disjoint open sets U, V such that $\bar{E} \subseteq U$, $\bar{F} \subseteq V$. Because $\bar{E}$ and $\bar{F}$ are compact, U, V can be taken to be finite unions of sets in $\mathcal{U}$ and therefore $U, V \in \mathcal{U}$.

Now fix $\epsilon > 0$ and let $W_n \in \mathcal{U}$, be such that $E \cup F \subseteq \bigcup_n W_n$ and

$$\sum_n \rho(W_n) \leq \mu^*(E \cup F) + \epsilon \leq \mu^*(E) + \mu^*(F) + \epsilon. \qquad (*)$$

Note that $\{W_n \cap U\}_{n\in\mathbb{N}}$ covers E, $\{W_n \cap V\}_{n\in\mathbb{N}}$ covers F and $W_n \cap U, W_n \cap V \in \mathcal{U}$. Also, by finite additivity of ρ,

$$\rho(W_n \cap U) + \rho(W_n \cap V) = \rho(W_n \cap (U \cup V)) \leq \rho(W_n).$$

Thus

$$\mu^*(E) + \mu^*(F) \leq \sum_n \rho(W_n \cap U) + \sum_n \rho(W_n \cap V) \leq \sum_n \rho(W_n),$$

which, together with $(*)$, implies that $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$ since ϵ is arbitrary. □

Proposition 9.2. *Suppose there exist a countable base $\mathcal{U} \subseteq Pow(X)$ for X and a compact set $K \subseteq X$ such that the G-algebra generated by $\mathcal{U} \cup \{K\}$ admits a finitely additive G-invariant probability measure ρ with $\rho(K) > 0$. Then there exists a countably additive G-invariant Borel probability measure on X.*

Proof. Let $K, \mathcal{U}$ and ρ be as in the hypothesis. We may assume that $\mathcal{U}$ is closed under the G-action and finite unions/intersections. Let μ^* be the outer measure provided by Lemma 9.1 applied to $\mathcal{U}$, ρ. Thus μ^* is a metric outer measure on K and hence all Borel subsets of K are μ^*-measurable (see 13.2 in [Mun53]). This implies that all Borel subsets of $Y = [K]_G = \bigcup_{g\in G} gK$ are μ^*-measurable because μ^* is G-invariant. By Carathéodory's theorem, the restriction of μ^* to the Borel subsets of Y is a countably additive Borel measure on Y, and we extend it to a Borel measure μ on X by setting $\mu(Y^c) = 0$. Note that μ is G-invariant and $\mu(Y) \le 1$.

It remains to show that μ is nontrivial, which we do by showing that $\mu(K) \ge \rho(K)$ and hence $\mu(K) > 0$. To this end, let $\{U_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$ cover K. Since K is compact, there is a finite subcover $\{U_n\}_{n<N}$. Thus $U := \bigcup_{n<N} U_n \in \mathcal{U}$ and $K \subseteq U$. By finite additivity of ρ, we have

$$\sum_{n\in\mathbb{N}} \rho(U_n) \ge \sum_{n<N} \rho(U_n) \ge \rho(U) \ge \rho(K),$$

and hence, it follows from the definition of μ^* that $\mu^*(K) \ge \rho(K)$. Thus $\mu(K) = \mu^*(K) > 0$. □

Corollary 9.3. *Let X be a second countable Hausdorff topological G-space whose Borel structure is standard. For every compact set $K \subseteq X$ not in $\mathfrak{C}_4$, there is a G-invariant countably additive Borel probability measure μ on X with $\mu(K) > 0$.*

Proof. Fix any countable base $\mathcal{U}$ for X and let $\mathcal{B}$ be the Boolean G-algebra generated by $\mathcal{U} \cup \{K\}$. By Corollary 8.3, there exists a G-invariant finitely additive probability measure ρ on $\mathcal{B}$ such that $\rho(K) > 0$. Now apply 9.2. □

As a corollary, we derive the analogue of Nadkarni's theorem for $\mathfrak{C}_4$ in case of σ-compact spaces.

Corollary 9.4. *Let X be a Borel G-space that admits a σ-compact realization. $X \notin \mathfrak{C}_4$ if and only if there exists a G-invariant countably additive Borel probability measure on X.*

Proof. $\Leftarrow$: If $X \in \mathfrak{C}_4$, then it is compressible in the usual sense and hence does not admit a G-invariant Borel probability measure.

$\Rightarrow$: Suppose that X is a σ-compact topological G-space and $X \notin \mathfrak{C}_4$. Then, since X is σ-compact and $\mathfrak{C}_4$ is a σ-ideal, there is a compact set K not in $\mathfrak{C}_4$. Now apply 9.3. □

Remark. For a Borel G-space X, let $\mathcal{K}$ denote the collection of all subsets of invariant Borel sets that admit a σ-compact realization (when viewed as Borel G-spaces). Also, let $\mathfrak{C}$ denote the collection of all subsets of invariant compressible Borel sets. It is clear that $\mathcal{K}$ and $\mathfrak{C}$ are σ-ideals, and what 9.4 implies is that $\mathfrak{C} \cap \mathcal{K} \subseteq \mathfrak{C}_4$.

Theorem 9.5. *Let X be a Borel G-space that admits a σ-compact realization. If there is no G-invariant Borel probability measure on X, then X admits a Borel 32-generator.*

Proof. By 9.4, $X \in \mathfrak{C}_4$ and hence, X is 4-compressible. Thus, by Proposition 7.2, X admits a Borel 2^5-generator. □

Example 9.6. Let $LO \subseteq 2^{\mathbb{N}^2}$ denote the Polish space of all countable linear orderings and let G be the group of finite permutations of elements of $\mathbb{N}$. G is countable and acts continuously on LO in the natural way. Put $X = LO \setminus DLO$, where DLO denotes the set of all dense linear orderings without endpoints (copies of $\mathbb{Q}$). It is straightforward to see that DLO is a G_δ subset of LO and hence, X is F_σ. Therefore, X is in fact σ-compact since LO is compact being a closed subset of $2^{\mathbb{N}^2}$. Also note that X is G-invariant.

Let μ be the unique measure on LO defined by $\mu(V_{(F,<)}) = \frac{1}{n!}$, where $(F,<)$ is a finite linearly ordered subset of $\mathbb{N}$ of cardinality n and $V_{(F,<)}$ is the set of all linear orderings of $\mathbb{N}$ extending the order $<$ on F. As shown in [GW02], μ is the unique invariant measure for the action of G on LO and $\mu(X) = 0$. Thus there is no G-invariant Borel probability measure on X and hence, by the above theorem, X admits a Borel 32-generator.

10 Finitely traveling sets

Let X be a Borel G-space.

Definition 10.1. *Let $A, B \in \mathfrak{B}(X)$ be equidecomposable, i.e. there are $N \leq \infty$, $\{g_n\}_{n<N} \subseteq G$ and Borel partitions $\{A_n\}_{n<N}$ and $\{B_n\}_{n<N}$ of A and B, respectively, such that $g_n A_n = B_n$ for all $n < N$. A, B are said to be*

- *locally finitely equidecomposable (denote by $A \sim_{lfin} B$), if $\{A_n\}_{n<N}, \{B_n\}_{n<N}, \{g_n\}_{n<N}$ can be taken so that for every $x \in A$, $A_n \cap [x]_G = \varnothing$ for all but finitely many $n < N$;*

- *finitely equidecomposable (denote by $A \sim_{fin} B$), if N can be taken to be finite.*

The notation $\prec_{\text{fin}}$, $\prec_{\text{lfin}}$ and the notions of finite and locally finite compressibility are defined analogous to Definitions 2.9 and 2.11.

Definition 10.2. *A Borel set $A \subseteq X$ is called (locally) finitely traveling if there exists pairwise disjoint Borel sets $\{A_n\}_{n \in \mathbb{N}}$ such that $A_0 = A$ and $A \sim_{fin} A_n$ ($A \sim_{lfin} A_n$), $\forall n \in \mathbb{N}$.*

Proposition 10.3. *If X is (locally) finitely compressible then X admits a (locally) finitely traveling Borel complete section.*

Proof. We prove for finitely compressible X, but note that everything below is also locally valid (i.e. restricted to every orbit) for a locally compressible X.

Run the proof of the first part of Lemma 6.4 noting that a witnessing map $\gamma : X \to G$ of finite compressibility of X has finite image and hence the image of each δ_n (in the notation of the proof) is finite, which implies that the obtained traveling set A is actually finitely traveling. □

Proposition 10.4. *If X admits a locally finitely traveling Borel complete section, then $X \in \mathfrak{C}_4$.*

Proof. Let A be a locally finitely traveling Borel complete section and let $\{A_n\}_{n \in \mathbb{N}}$ be as in Definition 10.2. Let $\mathcal{I}_n = \{C_k^n\}_{k \in \mathbb{N}}$, $\mathcal{J}_n = \{D_k^n\}_{k \in \mathbb{N}}$ be Borel partitions of A and A_n, respectively, that together with $\{g_k^n\}_{k \in \mathbb{N}} \subseteq G$ witness $A \sim_{\text{lfin}} A_n$ (as in Definition 10.1). Let $\mathcal{B}$ denote the Boolean G-algebra generated by $\{X\} \cup \bigcup_{n \in \mathbb{N}} (\mathcal{I}_n \cup \mathcal{J}_n \cup \{A_n\})$.

Now assume for contradiction that $X \notin \mathfrak{C}_4$ and hence, $A \notin \mathfrak{C}_4$. Thus, applying Corollary 8.3 to A and $\mathcal{B}$, we get a G-invariant finitely additive probability measure μ on $\mathcal{B}$ with $\mu(A) > 0$. Moreover, there is $x \in A$ such that $\forall B \in \mathcal{B}$ with $B \cap [x]_G = \varnothing$, $\mu(B) = 0$.

Claim. $\mu(A_n) = \mu(A)$*, for all* $n \in \mathbb{N}$.

Proof of Claim. For each n, let $\{C_{k_i}^n\}_{i < K_n}$ be the list of those C_k^n such that $C_k^n \cap [x]_G \neq \varnothing$ ($K_n < \infty$ by the definition of locally finitely traveling). Set $B = A \setminus (\bigcup_{i < K_n} C_{k_i}^n)$ and note that by finite additivity of μ,

$$\mu(A) = \mu(B) + \sum_{i<K_n} \mu(C^n_{k_i}).$$

Similarly, set $B' = A_n \setminus (\bigcup_{i<K_n} D^n_{k_i})$ and hence

$$\mu(A_n) = \mu(B') + \sum_{i<K_n} \mu(D^n_{k_i}).$$

But $B \cap [x]_G = \varnothing$ and $B' \cap [x]_G = \varnothing$, and thus $\mu(B) = \mu(B') = 0$. Also, since $g^n_{k_i} C^n_{k_i} = D^n_{k_i}$ and μ is G-invariant, $\mu(C^n_{k_i}) = \mu(D^n_{k_i})$. Therefore

$$\mu(A) = \sum_{i<K_n} \mu(C^n_{k_i}) = \sum_{i<K_n} \mu(D^n_{k_i}) = \mu(A_n).$$

$\dashv$

This claim contradicts μ being a probability measure since for large enough N, $\mu(\bigcup_{n<N} A_n) = N\mu(A) > 1$, contradicting $\mu(X) = 1$. □

This, together with 7.2, implies the following.

Corollary 10.5. *Let X be a Borel G-space. If X admits a locally finitely traveling Borel complete section, then there is a Borel 32-generator.*

11 Locally weakly wandering sets and other special cases

Assume throughout the section that X is a Borel G-space.

Definition 11.1. *We say that $A \subseteq X$ is*

- *weakly wandering with respect to $H \subseteq G$ if $(hA) \cap (h'A) = \varnothing$, for all distinct $h, h' \in H$;*
- *weakly wandering, if it is weakly wandering with respect to an infinite subset $H \subseteq G$ (by shifting H, we can always assume $1_G \in H$);*
- *locally weakly wandering if for every $x \in X$, $A^{[x]_G}$ is weakly wandering.*

For $A \subseteq X$ and $x \in A$, put

$$\Delta_A(x) = \{(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}} : g_0 = 1_G \wedge \forall n \neq m (g_n A^{[x]_G} \cap g_m A^{[x]_G} = \varnothing)\},$$

and let $F(G^{\mathbb{N}})$ denote the Effros space of $G^{\mathbb{N}}$, i.e. the standard Borel space of closed subsets of $G^{\mathbb{N}}$ (see 12.C in [Kec95]).

Proposition 11.2. *Let $A \in \mathfrak{B}(X)$.*

(a) $\forall x \in X$, $\Delta_A(x)$ is a closed set in $G^{\mathbb{N}}$.

(b) $\Delta_A : A \to F(G^{\mathbb{N}})$ is $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable and hence universally measurable.

(c) Δ_A is F_A-invariant, i.e. $\forall x, y \in A$, if $x F_A y$ then $\Delta_A(x) = \Delta_A(y)$.

(d) If $s : F(G^{\mathbb{N}}) \to G^{\mathbb{N}}$ is a Borel selector (i.e. $s(F) \in F$, $\forall F \in F(G^{\mathbb{N}})$), then $\gamma := s \circ \Delta_A$ is a $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable F_A- and G-invariant travel guide. In particular, A is a 1-traveling set with $\sigma(\boldsymbol{\Sigma}^1_1)$-pieces.

Proof. (a) $\Delta_A(x)^c$ is open since being in it is witnessed by two coordinates.

(b) For $s \in G^{<\mathbb{N}}$, let $B_s = \{F \in F(G^{\mathbb{N}}) : F \cap V_s \neq \varnothing\}$, where $V_s = \{\alpha \in G^{\mathbb{N}} : \alpha \sqsupseteq s\}$. Since $\{B_s\}_{s\in G^{<\mathbb{N}}}$ generates the Borel structure of $F(G^{\mathbb{N}})$, it is enough to show that $\Delta_A^{-1}(B_s)$ is analytic, for every $s \in G^{<\mathbb{N}}$. But $\Delta_A^{-1}(B_s) = \{x \in X : \exists (g_n)_{n\in\mathbb{N}} \in V_s [g_0 = 1_G \wedge \forall n \neq m g_n (A^{[x]_G} \cap g_m A^{[x]_G} = \varnothing)]\}$ is clearly analytic.

(c) Assume for contradiction that $x F_A y$, but $\Delta_A(x) \neq \Delta_A(y)$ for some $x, y \in A$. We may assume that there is $(g_n)_{n\in\mathbb{N}} \in \Delta_A(x) \setminus \Delta_A(y)$ and thus $\exists n \neq m$ such that $g_n A^{[y]_G} \cap g_m A^{[y]_G} \neq \varnothing$. Hence $A^{[y]_G} \cap g_n^{-1} g_m A^{[y]_G} \neq \varnothing$ and let $y', y'' \in A^{[y]_G}$ be such that $y'' = g_n^{-1} g_m y'$. Let $g \in G$ be such that $y' = gy$.

Since $y' = gy$, $y'' = g_n^{-1} g_m g y$ are in A, $x F_A y$, and A is F_A-invariant, $gx, g_n^{-1} g_m g x$ are in A as well. Thus $A^{[x]_G} \cap g_n^{-1} g_m A^{[x]_G} \neq \varnothing$, contradicting $g_n A^{[y]_G} \cap g_m A^{[y]_G} = \varnothing$ (this holds since $(g_n)_{n\in\mathbb{N}} \in \Delta_A(x)$).

(d) Follows from parts (b) and (c), and the definition of Δ_A.

□

Theorem 11.3. *Let X be a Borel G-space. If there is a locally weakly wandering Borel complete section for X, then X admits a Borel 4-generator.*

Proof. By part (d) of 11.2 and 6.8, X is 1-compressible. Thus, by 7.2, X admits a Borel 2^2-finite generator. □

Observation 11.4. *Let $A = \bigcup_{n\in\mathbb{N}} W_n$, where each W_n is weakly wandering and put $W_n' = W_n \setminus \bigcup_{i<n}[W_i]_G$. Then $A' := \bigcup_{n\in\mathbb{N}} W_n'$ is locally weakly wandering and $[A]_G = [A']_G$.*

Corollary 11.5. *Let X be a Borel G-space. If X is the saturation of a countable union of weakly wandering Borel sets, X admits a Borel 3-generator.*

Proof. Let $A = \bigcup_{n\in\mathbb{N}} W_n$, where each W_n is weakly wandering. By 11.4, we may assume that $[W_n]_G$ are pairwise disjoint and hence A is locally weakly wandering. Using countable choice, take a function $p : \mathbb{N} \to G^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}$, $p(n) \in \bigcap_{x\in W_n} \Delta_{W_n}(x)$ (we know that $\bigcap_{x\in W_n} \Delta_{W_n}(x) \neq \varnothing$ since W_n is weakly wandering).

Define $\gamma : A \to G^{\mathbb{N}}$ by

$$x \mapsto \text{the smallest } k \text{ such that } p(k) \in \Delta_A(x).$$

The condition $p(k) \in \Delta_A(x)$ is Borel because it is equivalent to $\forall n, m \in \mathbb{N}, y, z \in A \cap [x]_G, p(k)(n)y = p(k)(m)z \Rightarrow n = m \wedge x = y$; thus γ is a Borel function. Note that γ is a travel guide for A by definition. Moreover, it is F_A-invariant because if $\Delta_A(x) = \Delta_A(y)$ for some $x, y \in A$, then conditions $p(k) \in \Delta_A(x)$ and $p(k) \in \Delta_A(y)$ hold or fail together. Since Δ_A is F_A-invariant, so is γ. Hence, Lemma 7.1 applied to $\mathcal{I} = \langle A \rangle$ gives a Borel $(2 \cdot 2 - 1)$-generator. □

Remark. The above corollary in particular implies the existence of a 3-generator in the presence of a weakly wandering Borel complete section. (For a direct proof of this, note that if W is a complete section that is weakly wandering with respect to $\{g_n\}_{n\in\mathbb{N}}$ with $g_0 = 1_G$ and $\{U_n\}_{n\in\mathbb{N}}$ is a family generating the Borel sets, then $\mathcal{I} =< W, \bigcup_{n\geq 1} g_n(W \cap U_n) >$ is a generator and $|\mathcal{I}| = 3$.) This can be viewed as a Borel version of the Krengel-Kuntz theorem (see 2.5) in the sense that it implies a version of the latter (our result gives a 3-generator instead of a 2-generator). To see this, let X be a Borel G-space and μ be a quasi-invariant measure on X such that there is no invariant measure absolutely continuous with respect to μ. Assume first that the action is ergodic. Then by the Hajian-Kakutani-Itô theorem, there exists a weakly wandering set W with $\mu(W) > 0$. Thus $X' = [W]_G$ is conull and admits a 3-generator by the above, so X admits a 3-generator modulo μ-NULL.

For the general case, one can use Ditzen's Ergodic Decomposition Theorem for quasi-invariant measures (Theorem 5.2 in [Mil08]), apply the previous result to μ-a.e. ergodic piece, combine the generators obtained for each piece into a partition of X (modulo μ-NULL) and finally apply Theorem 13.10 to obtain a finite generator for X. Each of these steps requires a certain amount of work, but we will not go into the details.

Example 11.6. Let $X = \mathcal{N}$ (the Baire space) and $\tilde{E}_0$ be the equivalence relation of eventual agrement of sequences of natural numbers. We find a countable group G of homeomorphisms of X such that $E_G = \tilde{E}_0$. For all $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = |t|$, let $\phi_{s,t} : X \to X$ be defined as follows:

$$\phi_{s,t}(x) = \begin{cases} t^\frown y & \text{if } x = s^\frown y \\ s^\frown y & \text{if } x = t^\frown y \\ x & \text{otherwise} \end{cases},$$

and let G be the group generated by $\{\phi_{s,t} : s, t \in \mathbb{N}^{<\mathbb{N}}, |s| = |t|\}$. It is clear that each $\phi_{s,t}$ is a homeomorphism of X and $E_G = \tilde{E}_0$. Now for $n \in \mathbb{N}$, let $X_n = \{x \in X : x(0) = n\}$ and let $g_n = \phi_{0,n}$. Then X_n are pairwise disjoint and $g_n X_0 = X_n$. Hence X_0 is a weakly wandering set and thus X admits a Borel 3-generator by Corollary 11.5.

Example 11.7. Let $X = 2^{\mathbb{N}}$ (the Cantor space) and E_t be the tail equivalence relation on X, that is $xE_ty \Leftrightarrow (\exists n, m \in \mathbb{N})(\forall k \in \mathbb{N})x(n+k) = y(m+k)$. Let G be the group generated by $\{\phi_{s,t} : s, t \in 2^{<\mathbb{N}}, s \perp t\}$, where $\phi_{s,t}$ are defined as above. To see that $E_G = E_t$ fix $x, y \in X$ with $x E_t y$. Thus there are nonempty $s, t \in 2^{<\mathbb{N}}$ and $z \in X$ such that $x = s \frown z$ and $y = t \frown z$. If $s \perp t$, then $y = \phi_{s,t}(x)$. Otherwise, assume say $s \sqsubseteq t$ and let $s' \in 2^{<\mathbb{N}}$ be such that $s \perp s'$ (exists since $s \neq \varnothing$). Then $s' \perp t$ and $y = \phi_{s',t} \circ \phi_{s,s'}(x)$.

Now for $n \in \mathbb{N}$, let $s_n = \underbrace{11...1}_{n} 0$ and $X_n = \{x \in X : x = s_n \frown y, \text{ for some } y \in X\}$. Note that s_n are pairwise incompatible and hence X_n are pairwise disjoint. Letting $g_n = \phi_{s_0, s_n}$, we see that $g_n X_0 = X_n$. Thus X_0 is a weakly wandering set and hence X admits a Borel 3-generator.

Using the function Δ defined above, we give another proof of Proposition 6.10.

Proposition 6.10. *Let X be an aperiodic Borel G-space and $T \subseteq X$ be Borel. If T is a partial transversal then T is $\langle T \rangle$-traveling.*

Proof. By definition, T is locally weakly wandering.

Claim. *Δ_T is Borel.*

Proof of Claim. Using the notation of the proof of part (b) of 11.2, it is enough to show that $\Delta_T^{-1}(B_s)$ is Borel for every $s \in G^{<\mathbb{N}}$. But since $\forall x \in T$, $T \cap [x]_G$ is a singleton, $\Delta_T(x) \in B_s$ is equivalent to $s(0) = 1_G \wedge (\forall n < m < |s|)\ s(m)x \neq s(n)x$. The latter condition is Borel, hence so is $\Delta_T^{-1}(B_s)$. ⊣

By part (d) of 11.2, $\gamma = s \circ \Delta_T$ is a Borel F_T-invariant travel guide for T. □

Corollary 11.8. *Let X be a Borel G-space. If X is smooth and aperiodic, then it admits a Borel 3-generator.*

Proof. Since the G-action is smooth, there exists a Borel transversal $T \subseteq X$. By 6.10, T is $\langle T \rangle$-traveling. Thus, by 7.1, there is a Borel $(2 \cdot 2 - 1)$-generator. □

Lastly, in case of smooth free actions, a direct construction gives the optimal result as the following proposition shows.

Proposition 11.9. *Let X be a Borel G-space. If the G-action is free and smooth, then X admits a Borel 2-generator.*

Proof. Let $T \subseteq X$ be a Borel transversal. Also let $G \setminus \{1_G\} = \{g_n\}_{n\in\mathbb{N}}$ be such that $g_n \neq g_m$ for $n \neq m$. Because the action is free, $g_nT \cap g_mT = \varnothing$ for $n \neq m$.

Define $\pi : \mathbb{N} \to \mathbb{N}$ recursively as follows:

$$\pi(n) = \begin{cases} \min\{m : g_m \notin \{g_{\pi(i)} : i < n\}\} & \text{if } n = 3k \\ \min\{m : g_m, g_m g_k \notin \{g_{\pi(i)} : i < n\}\} & \text{if } n = 3k+1 \\ \text{the unique } l \text{ s.t. } g_l = g_{\pi(3k+1)} g_k & \text{if } n = 3k+2 \end{cases}.$$

Note that π is a bijection. Fix a countable family $\{U_n\}_{n\in\mathbb{N}}$ generating the Borel sets and put $A = \bigcup_{k\in\mathbb{N}} g_{\pi(3k)}(T \cap U_k) \cup \bigcup_{k\in\mathbb{N}} g_{\pi(3k+1)}T$. Clearly, A is Borel, and we show that $\mathcal{I} = \langle A \rangle$ is a generator. Fix distinct $x, y \in X$. Note that since T is a complete section, we can assume that $x \in T$.

First assume $y \in T$. Take k with $x \in U_k$ and $y \notin U_k$. Then $g_{\pi(3k)}x \in g_{\pi(3k)}(T \cap U_k) \subseteq A$ and $g_{\pi(3k)}y \in g_{\pi(3k)}(T \setminus U_k)$. However $g_{\pi(3k)}(T \setminus U_k) \cap A = \varnothing$ and hence $g_{\pi(3k)}y \notin A$.

Now suppose $y \notin T$. Then there exists $y' \in T^{[y]_G}$ and k such that $g_k y' = y$. Now $g_{\pi(3k+1)}x \in g_{\pi(3k+1)}T \subseteq A$ and $g_{\pi(3k+1)}y = g_{\pi(3k+1)}g_k y' = g_{\pi(3k+2)}y' \in g_{\pi(3k+2)}T$. But $g_{\pi(3k+2)}T \cap A = \varnothing$, hence $g_{\pi(3k+1)}y \notin A$. □

Corollary 11.10. *Let H be a Polish group and G be a countable subgroup of H. If G admits an infinite discrete subgroup, then the translation action of G on H admits a 2-generator.*

Proof. Let G' be an infinite discrete subgroup of G. Clearly, it is enough to show that the translation action of G' on H admits a 2-generator. Since G' is discrete, it is closed. Indeed, if d is a left-invariant compatible metric on H, then $B_d(1_H, \epsilon) \cap G' = \{1_H\}$, for some $\epsilon > 0$. Thus every d-Cauchy sequence in G' is eventually constant and hence G' is closed. This implies that the translation action of G' on H is smooth and free (see 12.17 in [Kec95]), and hence 11.9 applies. □

CHAPTER IV

Other results

This chapter is self-contained. It establishes various results concerning finite generators, as well as other kinds of partitions and weakly wandering sets.

12 Finite generators on comeager sets

Throughout this section let X be an aperiodic Polish G-space. We use the notation $\forall^*$ to mean "for comeager many x".

The following lemma proves the conclusion of Lemma 13.8 for any group on a comeager set. Below, we use this lemma only to conclude that there is an aperiodically separable comeager set, while we already know from 13.7 that X itself is aperiodically separable. However, the proof of the latter is more involved, so we present this lemma to keep this section essentially self-contained.

Lemma 12.1. *There exists $A \in \mathfrak{B}(X)$ such that $G\langle A \rangle$ separates points in each orbit of a comeager G-invariant set D, i.e. $f_A \downharpoonright_{[x]_G}$ is one-to-one, for all $x \in D$.*

Proof. Fix a countable basis $\{U_n\}_{n \in \mathbb{N}}$ for X with $U_0 = \varnothing$ and let $\{A_n\}_{n \in \mathbb{N}}$ be a partition of X provided by Lemma 13.1. For each $\alpha \in \mathcal{N}$ (the Baire space), define

$$B_\alpha = \bigcup_{n \in \mathbb{N}} (A_n \cap U_{\alpha(n)}).$$

Claim. $\forall^* \alpha \in \mathcal{N} \forall^* z \in X \forall x, y \in [z]_G (x \neq y \Rightarrow \exists g \in G (gx \in B_\alpha \not\Leftrightarrow gy \in B_\alpha))$.

Proof of Claim. By Kuratowski-Ulam, it is enough to show the statement with places of quantifiers $\forall^* \alpha \in \mathcal{N}$ and $\forall^* z \in X$ switched. Also, since orbits are countable and countable intersection of comeager sets is comeager, we can also switch the places of quantifiers $\forall^* \alpha \in \mathcal{N}$ and $\forall x, y \in [z]_G$. Thus we fix $z \in X$ and $x, y \in [z]_G$ with $x \neq y$ and show that $C = \{\alpha \in \mathcal{N} : \exists g \in G \ (gx \in B_\alpha \not\Leftrightarrow gy \in B_\alpha)\}$ is dense open.

To see that C is open, take $\alpha \in C$ and let $g \in G$ be such that $gx \in B_\alpha \not\Leftrightarrow gy \in B_\alpha$. Let $n, m \in \mathbb{N}$ be such that $gx \in A_n$ and $gy \in A_m$. Then for all $\beta \in \mathcal{N}$ with $\beta(n) = \alpha(n)$ and $\beta(m) = \alpha(m)$, we have $gx \in B_\beta \not\Leftrightarrow gy \in B_\beta$. But the set of such β is open in $\mathcal{N}$ and contained in C.

For the density of C, let $s \in \mathbb{N}^{<\mathbb{N}}$ and set $n = |s|$. Since A_n is a complete section, $\exists g \in G$ with $gx \in A_n$. Let $m \in \mathbb{N}$ be such that $gy \in A_m$. Take any $t \in \mathbb{N}^{\max\{n,m\}+1}$ with $t \sqsupseteq s$ satisfying the following condition:

Case 1: $n > m$. If $gy \in U_{s(m)}$ then set $t(n) = 0$. If $gy \notin U_{s(m)}$, then let k be such that $gx \in U_k$ and set $t(n) = k$.

Case 2: $n \leq m$. Let k be such that $gx \in U_k$ but $gy \notin U_k$ and set $t(n) = t(m) = k$.

Now it is easy to check that in any case $gx \in B_\alpha \not\Leftrightarrow gy \in B_\alpha$, for any $\alpha \in \mathcal{N}$ with $\alpha \sqsupseteq t$, and so $\alpha \in C$ and $\alpha \sqsupseteq s$. Hence C is dense. $\dashv$

By the claim, $\exists \alpha \in \mathcal{N}$ such that $D = \{z \in X : \forall x, y \in [z]_G \text{ with } x \neq y, G\langle B_\alpha \rangle \text{separates } x \text{ and } y\}$ is comeager and clearly invariant, which completes the proof. $\square$

Theorem 12.2. *Any aperiodic Polish G-space admits a 4-generator on an invariant comeager set.*

Proof. Let A and D be provided by Lemma 12.1. Throwing away an invariant meager set from D, we may assume that D is dense G_δ and hence Polish in the relative topology. Therefore, we may assume without loss of generality that $X = D$.

Thus A aperiodically separates X and hence, by 13.3, there is a partition $\{A_n\}_{n \in \mathbb{N}}$ of X into F_A-invariant Borel complete sections (the latter could be inferred directly from Corollary 13.9 without using Lemma 12.1). Fix an enumeration $G = \{g_n\}_{n \in \mathbb{N}}$ and a countable basis

$\{U_n\}_{n\in\mathbb{N}}$ for X. Denote $\mathcal{N}_2 = (\mathbb{N}^2)^{\mathbb{N}}$ and for each $\alpha \in \mathcal{N}_2$, define

$$B_\alpha = \bigcup_{n\geq 1} (A_n \cap g_{(\alpha(n))_0} U_{(\alpha(n))_1}).$$

Claim. $\forall^* \alpha \in \mathcal{N}_2 \forall^* x \in X \forall l \in \mathbb{N} \exists n, k \in \mathbb{N}(\alpha(n) = (k, l) \wedge g_k x \in A_n)$.

Proof of Claim. By Kuratowski-Ulam, it is enough to show that $\forall x \in X$ and $\forall l \in \mathbb{N}$, $C = \{\alpha \in \mathcal{N}_2 : \exists k, n \in \mathbb{N}(\alpha(n) = (k, l) \wedge g_k x \in A_n)\}$ is dense open.

To see that C is open, note that for fixed $n, k, l \in N$, $\alpha(n) = (k, l)$ is an open condition in $\mathcal{N}_2$.

For the density of C, let $s \in (\mathbb{N}^2)^{<\mathbb{N}}$ and set $n = |s|$. Since A_n is a complete section, $\exists k \in \mathbb{N}$ with $g_k x \in A_n$. Any $\alpha \in \mathcal{N}_2$ with $\alpha \sqsupseteq s$ and $\alpha(n) = (k, l)$ belongs to C. Hence C is dense. ⊣

By the claim, there exists $\alpha \in \mathcal{N}_2$ such that $Y = \{x \in X : \forall l \in \mathbb{N}\ \exists k, n \in \mathbb{N}\ (\alpha(n) = (k, l) \wedge g_k x \in A_n)\}$ is comeager. Throwing away an invariant meager set from Y, we can assume that Y is G-invariant dense G_δ.

Let $\mathcal{I} = \langle A, B_\alpha \rangle$, and so $|\mathcal{I}| \leq 4$. We show that $\mathcal{I}$ is a generator on Y. Fix distinct $x, y \in Y$. If x and y are separated by $G\langle A \rangle$ then we are done, so assume otherwise, that is $x F_A y$. Let $l \in \mathbb{N}$ be such that $x \in U_l$ but $y \notin U_l$. Then there exists $k, n \in \mathbb{N}$ such that $\alpha(n) = (k, l)$ and $g_k x \in A_n$. Since $g_k x F_A g_k y$ and A_n is F_A-invariant, $g_k y \in A_n$. Furthermore, since $g_k x \in A_n \cap g_k U_l$ and $g_k y \notin A_n \cap g_k U_l$, $g_k x \in B_\alpha$ while $g_k y \notin B_\alpha$. Hence $G\langle B_\alpha \rangle$ separates x and y, and thus so does $G\mathcal{I}$. Therefore $\mathcal{I}$ is a generator. □

Corollary 12.3. *Let X be a Polish G-space. If X is aperiodic, then it is 2-compressible modulo MEAGER.*

Proof. By Theorem 13.1 in [KM04], X is compressible modulo MEAGER. Also, by the above theorem, X admits a 4-generator modulo MEAGER. Thus 7.6 implies that X is 2-compressible modulo MEAGER. □

13 Separating smooth-many invariant sets

Assume throughout that X is a Borel G-space.

Lemma 13.1. *If X is aperiodic then it admits a countably infinite partition into Borel complete sections.*

Proof. The following argument is also given in the proof of Theorem 13.1 in [KM04]. By the marker lemma (see 6.7 in [KM04]), there exists a vanishing sequence $\{B_n\}_{n\in\mathbb{N}}$ of decreasing Borel complete sections, i.e. $\bigcap_{n\in\mathbb{N}} B_n = \varnothing$. For each $n \in \mathbb{N}$, define $k_n : X \to \mathbb{N}$ recursively as follows:

$$\begin{cases} k_0(x) &= 0 \\ k_{n+1}(x) &= min\{k \in \mathbb{N} : B_{k_n(x)} \cap [x]_G \nsubseteq B_k\} \end{cases},$$

and define $A_n \subseteq X$ by

$$x \in A_n \Leftrightarrow x \in A_{k_n(x)} \setminus A_{k_{n+1}(x)}.$$

It is straightforward to check that A_n are pairwise disjoint Borel complete sections. □

For $A \in \mathfrak{B}(X)$, if $\mathcal{I} = \langle A \rangle$ then we use the notation F_A and f_A instead of $F_{\mathcal{I}}$ and $f_{\mathcal{I}}$, respectively.

We now work towards strengthening the above lemma to yield a countably infinite partition into F_A-invariant Borel complete sections.

Definition 13.2 (Aperiodic separation)**.** *For Borel sets $A, Y \subseteq X$, we say that A aperiodically separates Y if $f_A([Y]_G)$ is aperiodic (as an invariant subset of the shift 2^G). If such A exists, we say that Y is aperiodically separable.*

Proposition 13.3. *For $A \in \mathfrak{B}(X)$, if A aperiodically separates X, then X admits a countably infinite partition into Borel F_A-invariant complete sections.*

Proof. Let $Y = \{y \in 2^G : |[y]_G| = \infty\}$ and hence $f_A(X)$ is a G-invariant subset of Y. By Lemma 13.1 applied to Y, there is a partition $\{B_n\}_{n\in\mathbb{N}}$ of Y into Borel complete sections.

Thus $A_n = f_{\mathcal{I}}^{-1}(B_n)$ is a Borel F_A-invariant complete section for X and $\{A_n\}_{n\in\mathbb{N}}$ is a partition of X. □

Let $\mathfrak{A}$ denote the collection of all subsets of aperiodically separable Borel sets.

Lemma 13.4. *$\mathfrak{A}$ is a σ-ideal.*

Proof. We only have to show that if Y_n are aperiodically separable Borel sets, then $Y = \bigcup_{n\in\mathbb{N}} Y_n \in \mathfrak{A}$. Let A_n be a Borel set aperiodically separating Y_n. Since A_n also aperiodically separates $[Y_n]_G$ (by definition), we can assume that Y_n is G-invariant. Furthermore, by taking $Y_n' = Y_n \setminus \bigcup_{k<n} Y_k$, we can assume that Y_n are pairwise disjoint. Now letting $A = \bigcup_{n\in\mathbb{N}}(A_n \cap Y_n)$, it is easy to check that A aperiodically separates Y. □

Let $\mathfrak{S}$ denote the collection of all subsets of smooth sets. By a similar argument as the one above, $\mathfrak{S}$ is a σ-ideal.

Lemma 13.5. *If X is aperiodic, then $\mathfrak{S} \subseteq \mathfrak{A}$.*

Proof. Let $S \in \mathfrak{S}$ and hence there is a Borel transversal T for $[S]_G$. Fix $x \in S$ and let $y \neq z \in [x]_G$. Since T is a transversal, there is $g \in G$ such that $gy \in T$, and hence $gz \notin T$. Thus $f_T(y) \neq f_T(z)$, and so $f_T([x]_G)$ is infinite. Therefore T aperiodically separates $[S]_G$. □

For the rest of the section, fix an enumeration $G = \{g_n\}_{n\in\mathbb{N}}$ and let F_A^n be following equivalence relation:

$$yF_A^n z \Leftrightarrow \forall k < n(g_k y \in A \leftrightarrow g_k z \in A).$$

Note that F_A^n has no more than 2^n equivalence classes and that $yF_A z$ if and only if $\forall n(yF_A^n z)$.

Lemma 13.6. *For $A, Y \in \mathfrak{B}(X)$, A aperiodically separates Y if and only if* $(\forall x \in Y)(\forall n)(\exists y, z \in Y^{[x]_G})[yF_A^n z \wedge \neg(yF_A z)]$.

Proof. $\Rightarrow$: Assume that for all $x \in Y$, $f_A([x]_G)$ is infinite and thus $F_A\restriction_{[x]_G}$ has infinitely many equivalence classes. Fix $n \in \mathbb{N}$ and recall that F_A^n has only finitely many equivalence classes. Thus, by the Pigeon Hole Principle, there are $y, z \in Y^{[x]_G}$ such that $yF_A^n z$ yet $\neg(yF_A z)$.

$\Leftarrow$: Assume for contradiction that $f_A(Y^{[x]_G})$ is finite for some $x \in Y$. Then it follows that $F_A = F_A^n$, for some n, and hence for any $y, z \in Y^{[x]_G}$, $yF_A^n z$ implies $yF_A z$, contradicting the hypothesis. □

Theorem 13.7. *If X is an aperiodic Borel G-space, then $X \in \mathfrak{A}$.*

Proof. By Lemma 13.1, there is a partition $\{A_n\}_{n\in\mathbb{N}}$ of X into Borel complete sections. We will inductively construct Borel sets $B_n \subseteq C_n$, where C_n should be thought of as the set of points *colored* (black or white) at the n^{th} step, and B_n as the set of points colored *black* (thus $C_n \setminus B_n$ is colored *white*).

Define a function $\# : X \to \mathbb{N}$ by $x \mapsto m$, where m is such that $x \in A_m$. Fix a countable family $\{U_n\}_{n\in\mathbb{N}}$ of sets generating the Borel σ-algebra of X.

Assuming that for all $k < n$, C_k, B_k are defined, let $\bar{C}_n = \bigcup_{k<n} C_k$ and $\bar{B}_n = \bigcup_{k<n} B_k$. Put $P_n = \{x \in A_0 : \forall k < n(g_k x \in \bar{C}_n) \wedge g_n x \notin \bar{C}_n\}$ and set $F_n = F^n_{\bar{B}_n} \restriction_{P_n}$, that is for all $x, y \in P_n$,

$$yF_n z \Leftrightarrow \forall k < n(g_k y \in \bar{B}_n \leftrightarrow g_k z \in \bar{B}_n).$$

Now put $C_n' = \{x \in P_n : \#(g_n x) = \min \#((g_n P_n)^{[x]_G})\}$, $C_n'' = \{x \in C_n' : \exists y, z \in (C_n')^{[x]_G}(y \neq z \wedge yF_n z)\}$ and $C_n = g_n C_n''$. Note that it follows from the definition of P_n that C_n is disjoint from $\bar{C}_n$.

Now in order to define B_n, first define a function $\bar{n} : X \to \mathbb{N}$ by

$$x \mapsto \text{ the smallest } m \text{ such that there are } y, z \in C_n'' \cap [x]_G \text{ with } yF_n z, y \in U_m \text{ and } z \notin U_m.$$

Note that $\bar{n}$ is Borel and G-invariant. Lastly, let $B_n' = \{x \in C_n'' : x \in U_{\bar{n}(x)}\}$ and $B_n = g_n B_n'$. Clearly, $B_n \subseteq C_n$. Now let $B = \bigcup_{n\in\mathbb{N}} B_n$ and $D = [\bigcup_{n\in\mathbb{N}}(C_n' \setminus C_n'')]_G$. We show that B aperiodically separates $Y := X \setminus D$ and $D \in \mathfrak{S}$. Since $\mathfrak{S} \subseteq \mathfrak{A}$ and $\mathfrak{A}$ is an ideal, this will imply that $X \in \mathfrak{A}$.

Claim 1. *$D \in \mathfrak{S}$.*

Proof of Claim. Since $\mathfrak{S}$ is a σ-ideal, it is enough to show that for each n, $[C'_n \setminus C''_n]_G \in \mathfrak{S}$, so fix $n \in \mathbb{N}$. Clearly, $(C'_n \setminus C''_n)^{[x]_G}$ is finite, for all $x \in X$, since there can be at most 2^n pairwise F_n-nonequivalent points. Thus, fixing some Borel linear ordering of X and taking the smallest element from $(C'_n \setminus C''_n)^{[x]_G}$ for each $x \in C'_n \setminus C''_n$, we can define a Borel transversal for $[C'_n \setminus C''_n]_G$. ⊣

By Lemma 13.6, to show that B aperiodically separates Y, it is enough to show that $(\forall x \in Y)(\forall n)(\exists y, z \in [x]_G)[y F_B^n z \wedge \neg(y F_B z)]$. Fix $x \in Y$.

Claim 2. $(\exists^\infty n)(C''_n)^{[x]_G} \neq \varnothing$.

Proof of Claim. Assume for contradiction that $(\forall^\infty n)(C''_n)^{[x]_G} = \varnothing$. Since $x \notin D$, it follows that $(\forall^\infty n) P_n^{[x]_G} = \varnothing$. Since A_0 is a complete section and $\bar{C}_0 = \varnothing$, $P_0^{[x]_G} \neq \varnothing$. Let N be the largest number such that $P_N^{[x]_G} \neq \varnothing$. Thus for all $n > N$, $C_n^{[x]_G} = \varnothing$ and hence for all $n > N$, $\bar{C}_n^{[x]_G} = \bar{C}_{N+1}^{[x]_G}$. Because $C_N^{[x]_G} \neq \varnothing$, there is $y \in A_0^{[x]_G}$ such that $\forall k \leq N(g_k y \in \bar{C}_{N+1})$; but because $P_{N+1}^{[x]_G} = \varnothing$, $g_{N+1} y$ must also fall into $\bar{C}_{N+1}$. By induction on $n > N$, we get that for all $n > N$, $g_n y \in \bar{C}_n$ and thus $g_n y \in \bar{C}_{N+1}$.

On the other hand, it follows from the definition of C'_n that for each n, $(C'_n)^{[x]_G}$ intersects exactly one of A_k. Thus $\bar{C}_{N+1}^{[x]_G}$ intersects at most $N+1$ of A_k and hence there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $\bar{C}_{N+1}^{[x]_G} \cap A_k = \varnothing$. Since $\exists^\infty n(g_n y \in \bigcup_{k \geq K} A_k)$, $\exists^\infty n(g_n y \notin \bar{C}_{N+1})$, a contradiction. ⊣

Now it remains to show that for all $n \in \mathbb{N}$, $(C''_n)^{[x]_G} \neq \varnothing$ implies that $\exists y, z \in [x]_G$ such that $y F_B^n z$ but $\neg(y F_B z)$. To this end, fix $n \in \mathbb{N}$ and assume $(C''_n)^{[x]_G} \neq \varnothing$. Thus there are $y, z \in (C''_n)^{[x]_G}$ such that $y F_n z$, $y \in U_{\bar{n}(x)}$ and $z \notin U_{\bar{n}(x)}$; hence, $g_n y \in B_n$ and $g_n z \notin B_n$, by the definition of B_n. Since C_k are pairwise disjoint, $B_n \subseteq C_n$ and $g_n y, g_n z \in C_n$, it follows that $g_n y \in B$ and $g_n z \notin B$, and therefore $\neg(y F_B z)$. Finally, note that $F_n = F_B^n \downharpoonright_{P_n}$ and hence $y F_B^n z$. □

Corollary 13.8. *Suppose all of the nontrivial subgroups of G have finite index (e.g. $G = \mathbb{Z}$), and let X be an aperiodic Borel G-space. Then there exists $A \in \mathfrak{B}(X)$ such that $G\langle A \rangle$ separates points in each orbit, i.e. $f_A \downharpoonright_{[x]_G}$ is one-to-one, for all $x \in X$.*

Proof. Let A be a Borel set aperiodically separating X (exists by Theorem 13.7) and put $Y = f_A(X)$. Then $Y \subseteq 2^G$ is aperiodic and hence the action of G on Y is free since the stabilizer subgroup of every element must have infinite index and thus is trivial. But this implies that for all $y \in Y$, $f_A^{-1}(y)$ intersects every orbit in X at no more than one point, and hence f_A is one-to-one on every orbit. □

From 13.3 and 13.7 we immediately get the following strengthening of Lemma 13.1.

Corollary 13.9. *If X is aperiodic, then for some $A \in \mathfrak{B}(X)$, X admits a countably infinite partition into Borel F_A-invariant complete sections.*

Theorem 13.10. *Let X be an aperiodic G-space and let E be a smooth equivalence relation on X with $E_G \subseteq E$. There exists a partition $\mathcal{P}$ of X into 4 Borel sets such that $G\mathcal{P}$ separates any two E-nonequivalent points in X, i.e. $\forall x, y \in X(\neg(xEy) \to f_{\mathcal{P}}(x) \neq f_{\mathcal{P}}(y))$.*

Proof. By Corollary 13.9, there is $A \in \mathfrak{B}(X)$ and a Borel partition $\{A_n\}_{n\in\mathbb{N}}$ of X into F_A-invariant complete sections. For each $n \in \mathbb{N}$, define a function $\bar{n} : X \to \mathbb{N}$ by

$$x \mapsto \text{the smallest } m \text{ such that } \exists x' \in A_0^{[x]_G} \text{ with } g_m x' \in A_n.$$

Clearly, $\bar{n}$ is Borel, and because all of A_k are F_A-invariant, $\bar{n}$ is also F_A-invariant, i.e. for all $x, y \in X$, $xF_Ay \to \bar{n}(x) = \bar{n}(y)$. Also, $\bar{n}$ is G-invariant by definition.

Put $A_n' = \{x \in A_0 : g_{\bar{n}(x)}x \in A_n\}$ and note that A_n' is F_A-invariant Borel since so are $\bar{n}$, A_0 and A_n. Moreover, A_n' is clearly a complete section. Define $\gamma_n : A_n' \to A_n$ by $x \mapsto g_{\bar{n}(x)}x$. Clearly, γ_n is Borel and one-to-one.

Since E is smooth, there is a Borel $h : X \to \mathbb{R}$ such that for all $x, y \in X$, $xEy \leftrightarrow h(x) = h(y)$. Let $\{V_n\}_{n\in\mathbb{N}}$ be a countable family of subsets of $\mathbb{R}$ generating the Borel σ-algebra of $\mathbb{R}$ and put $U_n = h^{-1}(V_n)$. Because each equivalence class of E is G-invariant, so is h and hence so is U_n.

Now let $B_n = \gamma_n(A_n' \cap U_n)$ and note that B_n is Borel being a one-to-one Borel image of a Borel set. It follows from the definition of γ_n that $B_n \subseteq A_n$. Put $B = \bigcup_{n\in\mathbb{N}} B_n$ and

$\mathcal{P} = \langle A, B \rangle$; in particular, $|\mathcal{P}| \le 4$. We show that $\mathcal{P}$ is what we want. To this end, fix $x, y \in X$ with $\neg(xEy)$. If $\neg(xF_A y)$, then $G\langle A \rangle$ (and hence $G\mathcal{P}$) separates x and y.

Thus assume that $xF_A y$. Since $h(x) \ne h(y)$, there is n such that $h(x) \in V_n$ and $h(y) \notin V_n$. Hence, by invariance of U_n, $gx \in U_n \wedge gy \notin U_n$, for all $g \in G$. Because A'_n is a complete section, there is $g \in G$ such that $gx \in A'_n$ and hence $gy \in A'_n$ since A'_n is F_A-invariant. Let $m = \bar{n}(gx)$ $(= \bar{n}(gy))$. Then $g_m gx \in B_n$ while $g_m gy \notin B_n$ although $g_m gy \in \gamma_n(A'_n) \subseteq A_n$. Thus $g_m gx \in B$ but $g_m gy \notin B$ and therefore $G\mathcal{P}$ separates x and y. □

14 Potential dichotomy theorems

In this section we prove dichotomy theorems assuming Weiss's question has a positive answer for $G = \mathbb{Z}$. In the proofs we use the Ergodic Decomposition Theorem (see [Far62], [Var63]) and a Borel/uniform version of Krieger's finite generator theorem, so we first state both of the theorems and sketch the proof of the latter.

For a Borel G-space X, let $\mathcal{M}_G(X)$ denote the set of G-invariant Borel probability measures on X and let $\mathcal{E}_G(X)$ denote the set of ergodic ones among those. It is not hard to show that $\mathcal{M}_G(X)$ and $\mathcal{E}_G(X)$ are Borel subsets of $P(X)$ (the standard Borel space of Borel probability measures on X) and thus are themselves standard Borel spaces.

Ergodic Decomposition Theorem 14.1 (Farrell, Varadarajan)**.** *Let X be a Borel G-space. If $\mathcal{M}_G(X) \ne \varnothing$ (and hence $\mathcal{E}_G(X) \ne \varnothing$), then there is a Borel surjection $x \mapsto e_x$ from X onto $\mathcal{E}_G(X)$ such that:*

(i) $xE_G y \Rightarrow e_x = e_y$;

(ii) For each $e \in \mathcal{E}_G(X)$, *if* $X_e = \{x \in X : e_x = e\}$ *(hence X_e is invariant Borel), then* $e(X_e) = 1$ *and* $e\restriction_{X_e}$ *is the unique ergodic invariant Borel probability measure on* X_e;

(iii) For each $\mu \in \mathcal{M}_G(X)$ *and* $A \in \mathfrak{B}(X)$, *we have* $\mu(A) = \int e_x(A) d\mu(x)$.

For the rest of the section, let X be a Borel $\mathbb{Z}$-space.

For $e \in \mathcal{E}_{\mathbb{Z}}(X)$, if we let h_e denote the entropy of $(X, \mathbb{Z}, e)$, then the map $e \mapsto h_e$ is Borel. Indeed, if $\{\mathcal{P}_k\}_{k\in\mathbb{N}}$ is a refining sequence of partitions of X that generates the Borel σ-algebra of X, then by 4.1.2 of [Dow11], $h_e = \lim_{k\to\infty} h_e(\mathcal{P}_k, \mathbb{Z})$, where $h_e(\mathcal{P}_k, \mathbb{Z})$ denotes the entropy of $\mathcal{P}_k$. By 17.21 of [Kec95], the function $e \mapsto h_e(\mathcal{P}_k)$ is Borel and thus so is the map $e \mapsto h_e$.

For all $e \in \mathcal{E}_{\mathbb{Z}}(X)$ with $h_e < \infty$, let N_e be the smallest integer such that $\log N_e > h_e$. The map $e \mapsto N_e$ is Borel because so is $e \mapsto h_e$.

Krieger's Finite Generator Theorem 14.2 (Uniform version)**.** *Let X be a Borel $\mathbb{Z}$-space. Suppose $\mathcal{M}_{\mathbb{Z}}(X) \neq \varnothing$ and let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem. Assume also that all measures in $\mathcal{E}_{\mathbb{Z}}(X)$ have finite entropy and let $e \mapsto N_e$ be the map defined above. Then there is a partition $\{A_n\}_{n\leq\infty}$ of X into Borel sets such that*

(i) A_∞ is invariant and does not admit an invariant Borel probability measure;

(ii) For each $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $\{A_n \cap X_e\}_{n<N_e}$ is a generator for $X_e \setminus A_\infty$, where $X_e = \rho^{-1}(e)$.

Sketch of Proof. Note that it is enough to find a Borel invariant set $X' \subseteq X$ and a Borel $\mathbb{Z}$-map $\phi : X' \to \mathbb{N}^{\mathbb{Z}}$, such that for each $e \in \mathcal{E}_{\mathbb{Z}}(X)$, we have

(I) $e(X \setminus X') = 0$;

(II) $\phi\restriction_{X_e\cap X'}$ is one-to-one and $\phi(X_e \cap X') \subseteq (N_e)^{\mathbb{Z}}$, where $(N_e)^{\mathbb{Z}}$ is naturally viewed as a subset of $\mathbb{N}^{\mathbb{Z}}$.

Indeed, assume we had such X' and ϕ, and let $A_\infty = X \setminus X'$ and $A_n = \phi^{-1}(V_n)$ for all $n \in \mathbb{N}$, where $V_n = \{y \in \mathbb{N}^{\mathbb{Z}} : y(0) = n\}$. Then it is clear that $\{A_n\}_{n\in\mathbb{N}}$ satisfies (ii). Also, (I) and part (ii) of the Ergodic Decomposition Theorem imply that (i) holds for A_∞.

To construct such a ϕ, we use the proof of Krieger's theorem presented in [Dow11], Theorem 4.2.3, and we refer to it as Downarowicz's proof. For each $e \in \mathcal{E}_{\mathbb{Z}}(X)$, the proof constructs a Borel $\mathbb{Z}$-embedding $\phi_e : X' \to N_e^{\mathbb{Z}}$ on an e-measure 1 set X'. We claim that this construction is uniform in e in a Borel way and hence would yield X' and ϕ as above.

Our claim can be verified by inspection of Downarowicz's proof. The proof uses the existence of sets with certain properties and one has to check that such sets exist with the properties satisfied for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$ at once. For example, the set C used in the proof of Lemma 4.2.5 in [Dow11] can be chosen so that for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $C \cap X_e$ has the required properties for e (using the Shannon-McMillan-Brieman theorem). Another example is the set B used in the proof of the same lemma, which is provided by Rohlin's lemma. By inspection of the proof of Rohlin's lemma (see 2.1 in [Gla03]), one can verify that we can get a Borel B such that for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $B \cap X_e$ has the required properties for e. The sets in these two examples are the only kind of sets whose existence is used in the whole proof; the rest of the proof constructs the required ϕ "by hand". □

Theorem 14.3 (Dichotomy I)**.** *Suppose the answer to Question 2.3 is positive and let X be an aperiodic Borel $\mathbb{Z}$-space. Then exactly one of the following holds:*

(1) there exists an invariant ergodic Borel probability measure with infinite entropy;

(2) there exists a partition $\{Y_n\}_{n \in \mathbb{N}}$ of X into invariant Borel sets such that each Y_n has a finite generator.

Proof. We first show that the conditions above are mutually exclusive. Indeed, assume there exist an invariant ergodic Borel probability measure e with infinite entropy and a partition $\{Y_n\}_{n \in \mathbb{N}}$ of X into invariant Borel sets such that each Y_n has a finite generator. By ergodicity, e would have to be supported on one of the Y_n. But Y_n has a finite generator and hence the dynamical system $(Y_n, \mathbb{Z}, e)$ has finite entropy by the Kolmogorov-Sinai theorem (see 1.5). Thus so does $(X, \mathbb{Z}, e)$ since these two systems are isomorphic (modulo e-NULL), contradicting the assumption on e.

Now we prove that at least one of the conditions holds. Assume that there is no invariant ergodic measure with infinite entropy. Now, if there was no invariant Borel probability measure at all, then, since the answer to Question 2.3 is assumed to be positive, X would admit a finite generator, and we would be done. So assume that $\mathcal{M}_{\mathbb{Z}}(X) \neq \varnothing$ and let

$\{A_n\}_{n\leq\infty}$ be as in Theorem 14.2. Furthermore, let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem. Set $X' = X \smallsetminus A_\infty$, $Y_\infty = A_\infty$, and for all $n \in \mathbb{N}$,

$$Y_n = \{x \in X' : N_{e_x} = n\},$$

where the map $e \mapsto N_e$ is as above. Note that the sets Y_n are invariant since ρ is invariant, so $\{Y_n\}_{n\leq\infty}$ is a countable partition of X into invariant Borel sets. Since Y_∞ does not admit an invariant Borel probability measure, by our assumption, it has a finite generator.

Let E be the equivalence relation on X' defined by ρ, i.e. $\forall x, y \in X'$,

$$xEy \Leftrightarrow \rho(x) = \rho(y).$$

By definition, E is a smooth Borel equivalence relation with $E \supseteq E_\mathbb{Z}$ since ρ respects the $\mathbb{Z}$-action. Thus, by Theorem 13.10, there exists a partition $\mathcal{P}$ of X' into 4 Borel sets such that $\mathbb{Z}\mathcal{P}$ separates any two points in different E-classes.

Now fix $n \in \mathbb{N}$ and we will show that $\mathcal{I} = \mathcal{P} \vee \{A_i\}_{i<n}$ is a generator for Y_n. Indeed, take distinct $x, y \in Y_n$. If x and y are in different E-classes, then $\mathbb{Z}\mathcal{P}$ separates them and hence so does $\mathbb{Z}\mathcal{I}$. Thus we can assume that xEy. Then $e := \rho(x) = \rho(y)$, i.e. $x, y \in X_e = \rho^{-1}(e)$. By the choice of $\{A_i\}_{i\in\mathbb{N}}$, $\{A_n \cap X_e\}_{n<N_e}$ is a generator for X_e and hence $\mathbb{Z}\{A_i\}_{i<N_e}$ separates x and y. But $n = N_e$ by the definition of Y_n, so $\mathbb{Z}\mathcal{I}$ separates x and y. □

Proposition 14.4. *Let X be a Borel $\mathbb{Z}$-space. If X admits invariant ergodic probability measures of arbitrarily large entropy, then it admits an invariant probability measure of infinite entropy.*

Proof. For each $n \geq 1$, let μ_n be an invariant ergodic probability measure of entropy $h_{\mu_n} > n2^n$ such that $\mu_n \neq \mu_m$ for $n \neq m$, and put

$$\mu = \sum_{n\geq1} \frac{1}{2^n}\mu_n.$$

It is clear that μ is an invariant probability measure, and we show that its entropy h_μ is infinite. Fix $n \geq 1$. Let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem and put $X_n = \rho^{-1}(\mu_n)$. It is clear that $\mu_m(X_n) = 1$ if $m = n$ and 0 otherwise.

For any finite Borel partition $\mathcal{P} = \{A_i\}_{i=1}^k$ of X_n, put $A_0 = X \smallsetminus X_n$ and $\bar{\mathcal{P}} = \mathcal{P} \cup \{A_0\}$. Let T be the Borel automorphism of X corresponding to the action of $1_{\mathbb{Z}}$, and let $h_\nu(\mathcal{I})$ and $h_\nu(\mathcal{I}, T)$ denote, respectively, the static and dynamic entropies of a finite Borel partition $\mathcal{I}$ of X with respect to an invariant probability measure ν. Then, with the convention that $\log(0) \cdot 0 = 0$, we have

$$h_\mu(\bar{\mathcal{P}}) = -\sum_{i=0}^k \log(\mu(A_i))\mu(A_i) \geq -\sum_{i=1}^k \log(\mu(A_i))\mu(A_i) = -\sum_{i=1}^k \log(\frac{1}{2^n}\mu_n(A_i))\frac{1}{2^n}\mu_n(A_i)$$
$$\geq -\frac{1}{2^n}\sum_{i=1}^k \log(\mu_n(A_i))\mu_n(A_i) = \frac{1}{2^n} h_{\mu_n}(\bar{\mathcal{P}}).$$

Since $\mathcal{P}$ is arbitrary and X_n is invariant, it follows that

$$h_\mu(\bar{\mathcal{P}}, T) = \lim_{m\to\infty} \frac{1}{m} h_\mu(\bigvee_{j<m} T^j \bar{\mathcal{P}}) \geq \frac{1}{2^n} \lim_{m\to\infty} \frac{1}{m} h_{\mu_n}(\bigvee_{j<m} T^j \bar{\mathcal{P}}) = \frac{1}{2^n} h_{\mu_n}(\bar{\mathcal{P}}, T).$$

Now for any finite Borel partition $\mathcal{I}$ of X, it is clear that $h_{\mu_n}(\mathcal{I}) = h_{\mu_n}(\bar{\mathcal{P}})$ (and hence $h_{\mu_n}(\mathcal{I}, T) = h_{\mu_n}(\bar{\mathcal{P}}, T)$), for some $\mathcal{P}$ as above. This implies that

$$h_\mu \geq \sup_{\mathcal{P}} h_\mu(\bar{\mathcal{P}}, T) \geq \frac{1}{2^n} \sup_{\mathcal{P}} h_{\mu_n}(\bar{\mathcal{P}}, T) = \frac{1}{2^n} \sup_{\mathcal{I}} h_{\mu_n}(\mathcal{I}, T) = \frac{1}{2^n} h_{\mu_n} > n,$$

where $\mathcal{P}$ and $\mathcal{I}$ range over finite Borel partitions of X_n and X, respectively. Thus $h_\mu = \infty$. $\square$

Theorem 14.5 (Dichotomy II). *Suppose the answer to Question 2.2 is positive and let X be an aperiodic Borel $\mathbb{Z}$-space. Then exactly one of the following holds:*

(1) there exists an invariant Borel probability measure with infinite entropy;

(2) X admits a finite generator.

Proof. The Kolmogorov-Sinai theorem implies that the conditions are mutually exclusive, and we prove that at least one of them holds. Assume that there is no invariant measure with infinite entropy. If there was no invariant Borel probability measure at all, then, by our assumption, X would admit a finite generator. So assume that $\mathcal{M}_{\mathbb{Z}}(X) \neq \varnothing$ and let $\{A_n\}_{n\leq\infty}$ be as in Theorem 14.2. Furthermore, let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem. Set $X' = X \smallsetminus A_\infty$ and $X_e = \rho^{-1}(e)$, for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$.

By our assumption, A_∞ admits a finite generator $\mathcal{P}$. Also, by 14.4, there is $N \geq 1$ such that for all $e \in \mathcal{E}_\mathbb{Z}(X)$, $N_e \leq N$ and hence $\mathcal{Q} := \{A_n\}_{n<N}$ is a finite generator for X_e; in particular, $\mathcal{Q}$ is a partition of X'. Let E be the following equivalence relation on X:

$$xEy \Leftrightarrow (x, y \in A_\infty) \vee (x, y \in X' \wedge \rho(x) = \rho(y)).$$

By definition, E is a smooth equivalence relation with $E \supseteq E_\mathbb{Z}$ since ρ respects the $\mathbb{Z}$-action and A_∞ is $\mathbb{Z}$-invariant. Thus, by Theorem 13.10, there exists a partition $\mathcal{J}$ of X into 4 Borel sets such that $\mathbb{Z}\mathcal{J}$ separates any two points in different E-classes.

We now show that $\mathcal{I} := \langle \mathcal{J} \cup \mathcal{P} \cup \mathcal{Q} \rangle$ is a generator. Indeed, fix distinct $x, y \in X$. If x and y are in different E-classes, then $\mathbb{Z}\mathcal{J}$ separates them. So we can assume that xEy. If $x, y \in A_\infty$, then $\mathbb{Z}\mathcal{P}$ separates x and y. Finally, if $x, y \in X'$, then $x, y \in X_e$, where $e = \rho(x)$ $(= \rho(y))$, and hence $\mathbb{Z}\mathcal{Q}$ separates x and y. □

Remark. It is likely that the above dichotomies are also true for any amenable group using a uniform version of Krieger's theorem for amenable groups, cf. [DP02], but I have not checked the details.

15 A condition for non-existence of non-meager weakly wandering sets

Throughout this section let X be a Polish $\mathbb{Z}$-space and T be the homeomorphism corresponding to the action of $1 \in \mathbb{Z}$.

Observation 15.1. *Let $A \subseteq X$ be weakly wandering with respect to $H \subseteq \mathbb{Z}$. Then A is weakly wandering with respect to*

(a) any subset of H;

(b) $r + H$, $\forall r \in \mathbb{Z}$;

(c) $-H$.

Definition 15.2. *Let $d \geq 1$ and $F = \{n_i\}_{i<k} \subseteq \mathbb{Z}$, where $n_0 < n_1 < ... < n_{k-1}$ are increasing. F is called d-syndetic if $n_{i+1} - n_i \leq d$ for all $i < k-1$. In this case we say that the length of F is $n_{k-1} - n_0$ and denote it by $\|F\|$.*

Lemma 15.3. *Let $d \geq 1$ and $F \subseteq \mathbb{Z}$ be a d-syndetic set. For any $H \subseteq \mathbb{Z}$, if $|H| = d+1$ and $\max(H) - \min(H) < \|F\| + d$, then F is not weakly wandering with respect to H (viewing $\mathbb{Z}$ as a $\mathbb{Z}$-space).*

Proof. Using (b) and (c) of 15.1, we may assume that H is a set of non-negative numbers containing 0. Let $F = \{n_i\}_{i<k}$ with n_i increasing.

Claim. *$\forall h \in H$, $(h+F) \cap [n_{k-1}, n_{k-1}+d) \neq \varnothing$.*

Proof of Claim. Fix $h \in H$. Since $0 \leq h < \|F\| + d$,

$$n_0 + h < n_0 + (\|F\| + d) = n_{k-1} + d.$$

We prove that there is $0 \leq i \leq k-1$ such that $n_i + h \in [n_{k-1}, n_{k-1}+d)$. Otherwise, because $n_{i+1} - n_i \leq d$, one can show by induction on i that $n_i + h < n_{k-1}, \forall i < k$, contradicting $n_{k-1} + h \geq n_{k-1}$. ⊣

Now $|H| = d+1 > d = |\mathbb{Z} \cap [n_{k-1}, n_{k-1}+d)|$, so by the Pigeon Hole Principle there exists $h \neq h' \in H$ such that $(h+F) \cap (h'+F) \neq \varnothing$ and hence F is not weakly wandering with respect to H. □

Definition 15.4. *Let $d, l \geq 1$ and $A \subseteq X$. We say that A contains a d-syndetic set of length l if there exists $x \in X$ such that $\{n \in \mathbb{Z} : T^n(x) \in A\}$ contains a d-syndetic set of length $\geq l$. This is equivalent to $\bigcap_{n \in F} T^n(A) \neq \varnothing$, for some d-syndetic set $F \subseteq \mathbb{Z}$ of length $\geq l$.*

For $A \subseteq X$, define $s_A : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by

$$d \mapsto \sup\{l \in \mathbb{N} : A \text{ contains a } d\text{-syndetic set of length } l\}.$$

Also, for infinite $H \subseteq \mathbb{Z}$, define a width function $w_H : \mathbb{N} \to \mathbb{N}$ by

$$d \mapsto \min\{\max(H') - \min(H') : H' \subseteq H \wedge |H'| = d+1\}.$$

Proposition 15.5. *If $A \subseteq X$ is weakly wandering with respect to an infinite $H \subseteq \mathbb{Z}$ then $\forall d \in \mathbb{N}, s_A(d) + d \leq w_H(d)$.*

Proof. Let H be an infinite subset of $\mathbb{Z}$ and $A \subseteq X$, and assume that $s_A(d) + d > w_H(d)$ for some $d \in \mathbb{N}$. Thus $\exists x \in X$ such that $\{n \in \mathbb{Z} : T^n(x) \in A\}$ contains a d-syndetic set F of length l with $l + d > w_H(d)$ and $\exists H' \subseteq H$ such that $|H'| = d + 1$ and $\max(H') - \min(H') = w_H(d)$. By Lemma 15.3 applied to F and H', F is not weakly wandering with respect to H' and hence neither is A. Thus A is not weakly wandering with respect to H. □

Corollary 15.6. *If $A \subseteq X$ contains arbitrarily long d-syndetic sets for some $d \geq 1$, then it is not weakly wandering.*

Proof. If A and d are as in the hypothesis, then $s_A(d) = \infty$ and hence, by Proposition 15.5, A is not weakly wandering with respect to any infinite $H \subseteq \mathbb{Z}$. □

Theorem 15.7. *Let X be a Polish G-space. Suppose for every nonempty open $V \subseteq X$ there exists $d \geq 1$ such that V contains arbitrarily long d-syndetic sets, i.e. $\bigcap_{n \in F} T^n(V) \neq \varnothing$ for arbitrarily long d-syndetic sets $F \subseteq \mathbb{Z}$. Then X does not admit a non-meager Baire measurable weakly wandering subset.*

Proof. Let A be a non-meager Baire measurable subset of X. By the Baire property, there exists a nonempty open $V \subseteq X$ such that A is comeager in V. By the hypothesis, there exists arbitrarily long d-syndetic sets $F \subseteq \mathbb{Z}$ such that $\bigcap_{n \in F} T^n(V) \neq \varnothing$. Since A is comeager in V and T is a homeomorphism, $\bigcap_{n \in F} T^n(A)$ is comeager in $\bigcap_{n \in F} T^n(V)$, and hence $\bigcap_{n \in F} T^n(A) \neq \varnothing$ for any F for which $\bigcap_{n \in F} T^n(V) \neq \varnothing$. Thus A also contains arbitrarily long d-syndetic sets and hence, by Corollary 15.6, A is not weakly wandering. □

Corollary 15.8. *Let X be a Polish G-space. Suppose for every nonempty open $V \subseteq X$ there exists $d \geq 1$ such that $\{T^{nd}(V)\}_{n \in \mathbb{N}}$ has the finite intersection property. Then X does not admit a non-meager Baire measurable weakly wandering subset.*

Proof. Fix nonempty open $V \subseteq X$ and let $d \geq 1$ such that $\{T^{nd}(V)\}_{n \in \mathbb{N}}$ has the finite intersection property. Then for every N, $F = \{kd : k \leq N\}$ is a d-syndetic set of length Nd and $\bigcap_{n \in F} T^n(V) \neq \varnothing$. Thus Theorem 15.7 applies. □

Lemma 15.9. *Let X be a generically ergodic Polish G-space. If there is a non-meager Baire measurable locally weakly wandering subset then there is a non-meager Baire measurable weakly wandering subset.*

Proof. Let A be a non-meager Baire measurable locally weakly wandering subset. By generic ergodicity, we may assume that $X = [A]_G$. Throwing away a meager set from A we can assume that A is G_δ. Then, by (d) of 11.2, there exists a $\sigma(\mathbf{\Sigma}_1^1)$-measurable (and hence Baire measurable) G-invariant travel guide $\gamma : A \to G^{\mathbb{N}}$. By generic ergodicity, γ must be constant on a comeager set, i.e. there is $(g_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}}$ such that $Y := \gamma^{-1}((g_n)_{n \in \mathbb{N}})$ is comeager. But then $W := A \cap Y$ is non-meager and is weakly wandering with respect to $\{g_n\}_{n \in \mathbb{N}}$. □

Let $X = \{\alpha \in 2^{\mathbb{N}} : \alpha$ has infinitely many 0-s and 1-s$\}$ and T be the odometer transformation on X. We will refer to this $\mathbb{Z}$-space as the odometer space.

Corollary 15.10. *The odometer space does not admit a non-meager Baire measurable locally weakly wandering subset.*

Proof. Let $\{U_s\}_{s \in 2^{<\mathbb{N}}}$ be the standard basis. Then for any $s \in 2^{<\mathbb{N}}$, $T^d(U_s) = U_s$ for $d = |s|$. Thus $\{T^{nd}(U_s)\}_{n \in \mathbb{N}}$ has the finite intersection property, in fact $\bigcap_{n \in \mathbb{N}} T^{nd}(U_s) = U_s$. Hence, we are done by 15.8 and 15.9. □

The following corollary shows the failure of the analogue of the Hajian-Kakutani-Itô theorem in the context of Baire category as well as gives a negative answer to Question 2.6.

Corollary 15.11. *There exists a generically ergodic Polish $\mathbb{Z}$-space Y (namely an invariant dense G_δ subset of the odometer space) with the following properties:*

(i) there does not exist an invariant Borel probability measure on Y;

(ii) there does not exist a non-meager Baire measurable locally weakly wandering set;

(iii) there does not exist a Baire measurable countably generated partition of Y into invariant sets, each of which admits a Baire measurable weakly wandering complete section.

Proof. By the Kechris-Miller theorem (see 2.8), there exists an invariant dense G_δ subset Y of the odometer space that does not admit an invariant Borel probability measure. Now (ii) is asserted by Corollary 15.10. By generic ergodicity of Y, for any Baire measurable countably generated partition of Y into invariant sets, one of the pieces of the partition has to be comeager. But then that piece does not admit a Baire measurable weakly wandering complete section since otherwise it would be non-meager, contradicting (ii). □

Part 2

Finite index pairs of countable Borel equivalence relations

CHAPTER I

Introduction to countable equivalence relations and the main results

1 Countable equivalence relations and subrelations

In this chapter, we give a brief survey of countable Borel equivalence relations and discuss subequivalence relations, focusing on those of finite index. We also define classes of hyperfinite and treeable equivalence relations, and discuss the question of whether they are closed under finite index extensions. Along the way, we state the main results of this part as they become relevant.

Before we begin, let us recall the basic definitions and notation of the theory of definable equivalence relations.

Definition 1.1. *A a standard Borel space X is an uncountable set X equipped with a σ-algebra that is the Borel σ-algebra of some Polish topology on X.*

Because any two uncountable Polish spaces are Borel isomorphic, it does not matter which Polish topology is used in the above definition. Thus, we work with Borel spaces when only the Borel structure is relevant.

Definition 1.2. *Let E, F be equivalence relations on standard Borel spaces X, Y, respectively. A Borel map $f : X \to Y$ is called a Borel reduction of E to F if for all $x_0, x_1 \in X$, we have*

$$x_0 E x_1 \iff f(x_0) F f(x_1).$$

We say that E is Borel reducible to F, and write $E \leq_B F$, if there is a Borel reduction f of E to F. If there is an injective Borel reduction f, then we write $E \sqsubseteq_B F$, and if moreover, $f(X)$ is an F-invariant subset of Y, then we write $E \sqsubseteq_B^* F$.

Definition 1.3. *A Borel equivalence relation on a standard Borel X is called countable (finite) if each equivalence class is countable (finite).*

For an equivalence relation E on a set X and $x \in X$, let $[x]_E$ denote the equivalence class of x. Also, for $A \subseteq X$, put

$$[A]_E = \{x \in X : [x]_E \cap A \neq \varnothing\}$$
$$(A)_E = \{x \in X : [x]_E \subseteq A\}.$$

We call $[A]_E$ the *E-saturation* of A and $(A)_E$ the *E-hull* of A. Note that if E is a countable Borel equivalence relation on a standard Borel space X and $A \subseteq X$ is a Borel set, then, by the Luzin-Novikov theorem, $[A]_E$ and $(A)_E$ are also Borel. A subset $A \subseteq$ is called a *complete section* for E (or an E-complete section) if $[A]_E = X$.

1.1 Borel actions of countable groups

Let X be a standard Borel space. Every Borel action $\Gamma \curvearrowright X$ of a countable group Γ on X induces such an equivalence relation, namely the orbit equivalence relation E_Γ^X, defined by

$$xE_G^X y \iff \exists \gamma \in \Gamma, \gamma x = y,$$

for $x, y \in X$. We often write E_Γ for E_Γ^X if X is clear from the context. The following theorem shows that these are the only examples of countable Borel equivalence relations (see Theorem 1.3 of [KM04] or [FM77]).

Theorem 1.4 (Feldman-Moore)**.** *For every countable Borel equivalence relation E on a standard Borel space X, there is a countable group Γ and a Borel action $\Gamma \curvearrowright X$ such that $E = E_\Gamma^X$. Moreover, Γ can be taken such that*

$$xEy \iff \exists \gamma \in G, \gamma^2 = 1 \text{ and } \gamma x = y,$$

for all $x, y \in X$.

Using basic descriptive set theory, one can show that any Borel action $\Gamma \curvearrowright X$ of a countable group Γ on X has a Polish realization, that is: there is a Polish topology on X, whose Borel structure coincides with the Borel structure of X, and the action is continuous with respect to this topology. Thus, given a Borel action $\Gamma \curvearrowright X$, we may assume whenever we need that X is a Polish space and the action is continuous.

For a countable group Γ, an important action is the shift action $s : \Gamma \curvearrowright X^\Gamma$, where X is a standard Borel space. It is defined as follows:

$$\gamma \cdot^s u(\alpha) = u(\gamma^{-1}\alpha),$$

for $\gamma, \alpha \in \Gamma$ and $u \in X^\Gamma$.

Proposition 1.5. *For an uncountable standard Borel space X and a countable group Γ, the shift action $s : \Gamma \curvearrowright X^\Gamma$ is universal among all Borel actions of Γ, that is: for any other Borel action $a : \Gamma \curvearrowright Y$ on a standard Borel space Y, there is an equivariant Borel embedding $f : Y \hookrightarrow X^\Gamma$.*

Proof. Without loss of generality we may assume that $Y \subseteq X$. Define $f : Y \to X^\Gamma$ by $y \to (\gamma \cdot^a y)_{\gamma \in \Gamma}$. It is straightforward to check that this f satisfies the conclusion of the proposition. □

1.2 Universal countable equivalence relations

Let $\mathbb{F}_n$ denote the free group on n generators, for $n \leq \infty$. Using the Feldman-Moore theorem and Proposition 1.5, one easily sees that the orbit equivalence relation F of the shift action of $\mathbb{F}_\infty$ on $(2^{\mathbb{N}})^{\mathbb{F}_\infty}$ is a universal countable Borel equivalence relation. Indeed, let E is a countable Borel equivalence relation on some standard Borel X. By the Feldman-Moore theorem, there is a countable group Γ and a Borel action $\Gamma \curvearrowright X$ such that $E^X_\Gamma = E$. Because $\mathbb{F}_\infty$ surjects onto Γ, we may lift this action of Γ to that of $\mathbb{F}_\infty$ on X. Thus E is induced by a Borel action

$a : \mathbb{F}_\infty \curvearrowright X$. Now Proposition 1.5 gives an injective equivariant map $f : X \to (2^{\mathbb{N}})^{\mathbb{F}_\infty}$ and thus $E \sqsubseteq_B^* F$.

Using a bit more coding, it was shown in [DJK94] that the orbit equivalence relation $\mathbb{E}_\infty$ of the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ is also universal. More precisely:

Theorem 1.6 (Dougherty-Jackson-Kechris). *For any countable Borel equivalence relation E, $E \sqsubseteq_B \mathbb{E}_\infty$.*

1.3 Notation and tools

For a standard Borel space X, let $\mathrm{Aut}(X)$ denote the group of Borel automorphisms of X. For a countable Borel equivalence relation E on X, we denote by $[E]$ the subgroup of $\mathrm{Aut}(X)$ of automorphisms that act within the E-classes, i.e. $f(x)Ex$, for all $x \in X$. $[E]$ is referred to as the full group of E. For $T \in [E]$, let E_T (or E_T^X) denote the orbit equivalence relation induced by the action of T, namely:

$$xE_Ty \iff \exists n \in \mathbb{Z}(T^n(x) = y),$$

for $x, y \in X$.

Countable equivalence relations are often considered in the measure theoretic context, namely, in the presence of a probability measure μ on X. We say that E is μ-measure preserving or μ is E-invariant if every $T \in [E]$ preserves μ. Equivalently, using Feldman-Moore, if $E = E_\Gamma^X$ for some Borel action $\Gamma \curvearrowright X$ of a countable group Γ, then this action is μ-measure preserving. We say that E is μ-ergodic (or equivalently, μ is E-ergodic) if any E-invariant Borel set $A \subseteq X$ is either μ-null or μ-conull.

Presence of a probability measure (especially if it is invariant) serves as a powerful tool for studying countable equivalence relations as it enables quantitative arguments, pigeon hole principle, $0-1$ law (in case of ergodicity), etc.

When X is a Polish space, a similar tool is Baire category, although, as we explain below, it is not as useful for countable Borel equivalence relations.

1.4 Subequivalence relations

For equivalence relations E, F on X, E is said to be a subequivalence relation of F if $E \subseteq F$ (as subsets of X^2). Equivalently, we say that F is an extension of E. In this case, each F-class is a union of E-classes.

If F is a countable equivalence relation, then there are at most countably many E-classes within each F-class. When there are only finitely many E-classes in every F-class, we say that E is a finite index subequivalence relation of F and denote this by $[F : E] < \infty$. When the number of E-classes on every F-class is exactly $i \leq \infty$, we say that the index of E in F is i and write $[F : E] = i$. We also write $[F : E] \leq i$ ($< i$) when the number of E-classes on every F-class is $\leq i$ ($< i$). Thus $[F : E] \leq i$ and $[F : E] = j$ for some $j \leq i$ mean two different things.

Definition 1.7. *A pair (E, F) of countable Borel equivalence relations is called nested if $E \subseteq F$. We say that (E, F) is a finite index pair if $E \subseteq F$ and $[F : E] < \infty$.*

Finite index pairs are the main objects of study of the current part of the thesis.

Example 1.8. Let $X = 2^{\mathbb{N}}$ and recall from the introduction the equivalence relation $\mathbb{E}_0$ on X defined by

$$x \mathbb{E}_0 y \iff \exists n \in \mathbb{N} \forall m \geq n (x(m) = y(m)),$$

for $x, y \in X$. Define $T : X \to X$ by $T(x)(n) = 1 - x(n)$, for $x \in X$, $n \in \mathbb{N}$. Clearly, T is an involution and each E_T-class has exactly two elements. Also note that T takes $\mathbb{E}_0$-classes to $\mathbb{E}_0$-classes. Thus, the smallest equivalence relation F_0 containing E_T and $\mathbb{E}_0$ is an index-2 extension of $\mathbb{E}_0$.

Note that for every finite index pair (E, F), there is a countable partition $\{X_n\}_{n \in \mathbb{N}}$ of X into F-invariant Borel sets such that $[F \restriction_{X_n} : E \restriction_{X_n}] = n$: simply take

$$X_n = \{x \in X : \text{there are exactly } n \text{ distinct } E\text{-classes in } [x]_F\}.$$

Thus, questions about finite index pairs can be reduced to questions about index-i pairs for $i < \infty$.

1.4.1 A universal index-i pair

Definition 1.9. *Let (E_0, F_0) and (E_1, F_1) be nested pairs with underlying standard Borel spaces X and Y, respectively. We write $(E_0, F_0) \leq_B (E_1, F_1)$ if $E_0 \leq_B E_1$ and $F_0 \leq_B F_1$ simultaneously; more precisely, there is a Borel map $f : X \to Y$ that is a reduction of E_0 to E_1, as well as a reduction of F_0 to F_1. We write $(E_0, F_0) \sqsubseteq_B (E_1, F_1)$ if there is an injective such f, and we write $(E_0, F_0) \sqsubseteq_B^* (E_1, F_1)$ if moreover $f(X)$ is F_1-invariant.*

In Section 4, we give a construction of a universal index-i pair[1], that is, a pair (E_u, F_u) on some underlying space X_u such that for any other index-i pair (E, F),

$$(E, F) \sqsubseteq_B (E_u, F_u).$$

Thus, when studying index-i pairs, it is enough to study (E_u, F_u).

1.4.2 Atomic decomposition for finite index pairs

When dealing with finite index pairs (E, F) in the presence of a probability measure μ, the proofs are simpler when μ is E-ergodic. If this is not the case, one often needs some sort of ergodic decomposition theorem. In Section 6, we prove such a theorem, and call it the F/E-atomic decomposition theorem. To state it, we need the following definition:

Definition 1.10. *Let (E, F) be a finite index pair on a standard probability space (X, μ). A μ-measurable set $A \subseteq X$ is called F/E-atomic if any $E\restriction_A$-invariant Borel subset $B \subseteq A$ is also $F\restriction_A$-invariant modulo a μ-null set; more precisely, $[B]_{F\restriction_A} \setminus B$ is μ-null.*

This definition basically says that, modulo μ-null, there is no uniform way of splitting an $F\restriction_A$-orbit into two nonempty $E\restriction_A$-invariant pieces. Note that if F is μ-ergodic, then X being F/E-atomic is equivalent to E being μ-ergodic.

Now we can state the atomic decomposition theorem:

[1] After I presented my results in a seminar, Andrew Marks pointed out that Ben Miller has a more general result, which implies the existence of such a universal pair.

Theorem 6.2. *Let (X,μ) be a standard probability space and let (E,F) be a nested pair on X with $[F:E] \leq i$, for some $i \in \mathbb{N}$. Then, there is a partition $X = \biguplus_{j<k} X_j$, $k \leq i$, into E-invariant F/E-atomic sets. Such a partition is unique up to a null set.*

1.4.3 Normality

For a countable Borel equivalence relation E, put

$$N[E] = \{T \in \mathrm{Aut}(X) : \forall x,y \in X (xEy \Leftrightarrow T(x)ET(y))\}.$$

In other words, $N[E]$ is the group of Borel automorphisms of E.

Definition 1.11. *Let (E,F) be a nested pair. E is said to be normal in F if there is a countable subgroup $\Gamma < N[E]$ whose natural action on X induces F, i.e. $F = E_\Gamma$. We denote this by $E \triangleleft F$. In this case, we also say that (E,F) is a normal pair.*

In Example 1.8 above, $T \in N[E]$ and thus F_0 is normal over $\mathbb{E}_0$. In the end of Section 3, we show that any index-2 pair is normal if we disregard a Borel set on which E and F are equal.

Many interesting nested pairs (E,F) are not normal. For example, it was shown in [Tho09] that recursive isomorphism is not a normal subequivalence relation of Turing equivalence.

In Section 7, we characterize the normal index-i subequivalence relations of a fixed ergodic countable equivalence relation F. More precisely, we prove the following:

Theorem 7.2. *Let (E,F) be an index-i pair on a standard probability space (X,μ) and suppose that F is measure preserving and ergodic. Then, the following are equivalent:*

(1) $E \triangleleft F$;

(2) F is generated by a countable group $\Delta < [F]$ of the form $H \times \Gamma$ such that $|H| = i$ and the action of Γ induces E a.e.

(3) F is generated by a Borel action of a countable group Δ that admits a normal subgroup Γ of finite index such that the action of Γ induces E a.e.

Using the convenient tool of links developed in Section 3, the proof of this theorem is fairly easy in the case when μ is E-ergodic. The general case, however, is handled using the F/E-atomic decomposition theorem mentioned above (Theorem 6.2).

1.4.4 Smooth pairs

Definition 1.12. *Let (E, F) be a nested pair on a standard Borel space X. F is said to be smooth over E if there is a Borel reduction $f : X \to X$ from F to E such that $f(x)Fx$, for all x. In other words, f takes every F-class C into one E-class D within C, thus singling out D among all E-class in C. We also call (E, F) a smooth pair.*

If (E, F) is a smooth pair and f is a witnessing function as in the definition, then taking $Y = f(X)$, we see that Y is a complete section[2] for F and $F\restriction_Y = E\restriction_Y$. Therefore, in most situations (including those considered in the current part of the thesis), F inherits the properties of E. This is the reason why many of the interesting nested pairs are not smooth.

For instance, the pair $(\mathbb{E}_0, F_0)$ considered in Example 1.8 is nonsmooth: indeed, suppose it is smooth and let f be the witnessing reduction. $\mathbb{E}_0$ is generically ergodic, so $Y = [f(X)]_{\mathbb{E}_0}$ is either meager or comeager. Note that the involution T (as in the example) is a homeomorphism. Thus, Y is meager (comeager) if and only if $T(Y)$ is meager (comeager). But $2^{\mathbb{N}}$ is a disjoint union of Y and $T(Y)$ and this contradicts the Baire category theorem.

A more interesting example is given in [TW13], where the authors prove that the relation of bi-embeddability for finitely generated groups is not smooth over the isomorphism relation.

[2]Complete section for F is a set that meets every F-class.

2 Important subclasses of countable equivalence relations

2.1 Hyperfinite equivalence relations

Definition 2.1. *A countable Borel equivalence relation is called hyperfinite if there is an increasing sequence* $(E_n)_{n\in\mathbb{N}}$ *of finite Borel equivalence relations such that* $E = \bigcup_{n\in\mathbb{N}} E_n$.

The requirement of $(E_n)_{n\in\mathbb{N}}$ being increasing is crucial in the definition since *every* countable Borel equivalence relation is a union of finite Borel equivalence relations.

The following theorem (see [DJK94]) shows that the notion of hyperfiniteness is very robust.

Theorem 2.2 (Weiss, Slaman-Steel, Dougherty-Jackson-Kechris)**.** *For a countable Borel equivalence relation* E *on a standard Borel space* X*, the following are equivalent:*

(1) E *is hyperfinite;*

(2) $E \sqsubseteq_B \mathbb{E}_0$*;*

(3) There is a Borel action of $\mathbb{Z}$ *on* X *such that* $E = E_{\mathbb{Z}}^X$.

Here we give various closure properties of the class of hyperfinite equivalence relations listed in Proposition 1.3 of [JKL02].

Proposition 2.3. *Let* E, F *be countable Borel equivalence relations on standard Borel spaces* X, Y*, respectively.*

(a) If $E \subseteq F$ *and* F *is hyperfinite, then so is* E.

(b) If $E \leq_B F$ *and* F *is hyperfinite, then so is* E.

(c) If E *is hyperfinite and* $A \subseteq X$*, then* $E\restriction_A$ *is hyperfinite.*

(d) If $A \subseteq X$ *is a complete section for* E *and* $E\restriction_A$ *is hyperfinite, then so is* E.

(e) If E, F *are hyperfinite, then so is* $E \times F$.

(f) If E is hyperfinite and F is a finite index extension of E, then F is hyperfinite.

A lot is known about hyperfinite equivalence relations. However, there are still simple questions about them, to which we do not yet know the answers. Here are two of the most notorious ones:

1. (Dougherty-Jackson-Kechris [DJK94]) Is increasing union of hyperfinite equivalence relations hyperfinite?

2. (Weiss [Wei84]) Is the orbit equivalence relation induced by a Borel action of a countable amenable group hyperfinite?

Concerning Question 2, it was proved by Jackson-Kechris-Louveau in [JKL02] that the answer is positive for finitely generated groups of polynomial growth (which are nilpotent-by-finite groups, by Gromov's theorem). It was also proved by Gao-Jackson in [GJ12] that the answer is yes for all abelian groups. Very recently, Seward and Schneider announced that they can now show that the answer is positive for all nilpotent groups, but there is no preprint available yet.

An interesting example of a hyperfinite equivalence relation is given in [JKL02]: let $GL_2(\mathbb{Z})$ act on $\mathbb{T}$ by linear transformations, identifying $\mathbb{T}$ with rays through the origin. Then the orbit equivalence relation induced by this action is hyperfinite.

In the measure theoretic context however, both of the above questions have positive answers:

1. (Dye [Dye63], Krieger [Kri69]) Let (X, μ) be a probability space and suppose E is an increasing union of hyperfinite equivalence relations on X. Then E is hyperfinite a.e.

2. (Ornstein-Weiss [OW80]) Orbit equivalence relations of measure preserving actions of amenable groups are hyperfinite a.e.

As far as Baire category is concerned, the following theorem (see Theorem 12.1 of [KM04]) shows that *every* countable Borel equivalence relation is hyperfinite modulo a meager set:

Theorem 2.4 (Hjorth-Kechris, Sullivan-Weiss-Wright, Woodin)**.** *Let E be a countable Borel equivalence relation on a Polish space X. Then there is a comeager invariant set $C \subseteq X$ such that $E\restriction_C$ is hyperfinite.*

Thus, the technique of Baire category often does not apply when studying countable Borel equivalence relations, although, of course, it depends on the property one is chasing (for example, the Generic Compressibility Theorem 13.1 of [KM04] or Theorem 12.2 of Part 1 of the current thesis).

Finally, it is worth mentioning that there are lots of non-hyperfinite countable equivalence relations. In particular, $\mathbb{E}_\infty$ is non-hyperfinite (mainly because $\mathbb{F}_2$ is not amenable), even if we disregard a null set (with respect to the natural product measure on $2^{\mathbb{F}_2}$).

2.2 Treeable equivalence relations

Let X be a standard Borel space. A Borel graph $\mathcal{G}$ on X is a Borel relation $\mathcal{G} \subseteq X^2$ that is irreflexive and symmetric. Such a graph induces an analytic equivalence relation $E_\mathcal{G}$ on X defined by setting $xE_\mathcal{G}y$ if x and y are in the same connected component of $\mathcal{G}$, i.e. there is a path in $\mathcal{G}$ from x to y, for $x, y \in X$. In this case, $\mathcal{G}$ is called a graphing of $E_\mathcal{G}$.

Definition 2.5. *A countable Borel equivalence relation E on X is called treeable if it admits an acyclic Borel graphing $\mathcal{G}$. In this case, $\mathcal{G}$ is necessarily locally countable (every vertex has only countably many neighbors) and each connected component is a tree. Thus, $\mathcal{G}$ is called a treeing of E.*

Example 2.6.

(A) All hyperfinite equivalence relations are treeable.

(B) More generally, all equivalence relations induced by free actions of countable free groups are treeable (use their Cayley graphs).

(C) For $n \geq 2$, consider the action of $GL_n(\mathbb{Z})$ on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ by linear transformations and let F_n denote the induced orbit equivalence relation. It was shown in [JKL02] that F_2 is treeable, while for $n \geq 3$, F_n is not treeable as $GL_n(\mathbb{Z})$ is a Kazhdan group [AS90].

The following theorem, proved in [JKL02], provides many more examples of treeable equivalence relations:

Theorem 2.7 (Jackson-Kechris-Louveau)**.** *Let Γ be a countable group that admits an action on a (graph-theoretic) tree by automorphisms, so that the stabilizer of every vertex is finite. Then any orbit equivalence relation induced by a free Borel action of Γ is treeable.*

Let $\text{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2}) \subseteq 2^{\mathbb{F}_2}$ denote the free part of the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$. It was also shown in [JKL02] that $\mathbb{E}_{\infty T} = \mathbb{E}_\infty \restriction_{\text{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})}$ is a universal treeable equivalence relation. More precisely:

Theorem 2.8 (Jackson-Kechris-Louveau)**.** *For any countable treeable equivalence relation E, $E \sqsubseteq_B \mathbb{E}_{\infty T}$.*

Because $\text{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$ is conull in $2^{\mathbb{F}_2}$ and $\mathbb{E}_\infty$ is not hyperfinite even when restricted to conull set, $\mathbb{E}_{\infty T}$ is not hyperfinite either. Thus, there are non-hyperfinite treeable equivalence relations (actually, there are lots of them).

2.2.1 Closure properties and finite index extensions

Here we give various closure properties of the class of treeable equivalence relations listed in Proposition 3.3 of [JKL02].

Proposition 2.9. *Let E, F be countable Borel equivalence relations on standard Borel spaces X, Y, respectively.*

(a) If $E \subseteq F$ and F is treeable, then so is E.

(b) If $E \leq_B F$ and F is treeable, then so is E.

(c) If E is treeable and $A \subseteq X$, then $E\restriction_A$ is treeable.

(d) If $A \subseteq X$ is a complete section for E and $E\restriction_A$ is treeable, then so is E.

This list of closure properties of the class of treeable equivalence relations is missing two properties that the class of hyperfinite equivalence relations enjoys, namely, the closure under products and finite index extensions. It was shown in [JKL02] and in [Ada90] for locally finite case that the former property does not hold for treeable equivalence relations; in fact, $\mathbb{E}_0 \times \mathbb{E}_{\infty T}$ and $\mathbb{E}_{\infty T} \times \mathbb{E}_{\infty T}$ are not treeable, and hence $\mathbb{E}_\infty$ is not treeable as well.

As for the closure under finite index extensions, it is still an open problem, even in the measure-theoretic context and even for index-2. This question was first raised in [JKL02], and it is what got the present author interested in studying finite index pairs in general.

Towards this question, the following corollary is drawn in [JKL02] from Theorem 2.7 (above) and a difficult result in geometric group theory (see Theorem 55 in [Coh89]) stating that virtually free groups act on trees with finite (vertex) stabilizers:

Corollary 2.10 (Jackson-Kechris-Louveau). *If a Borel equivalence relation F is induced by a free Borel action of a virtually free countable group, then F is treeable.*

In Chapter III, we study finite index extensions of treeable equivalence relations. In particular, we give a converse (in terms of cost) to the above corollary in the presence of an ergodic invariant measure (see [KM04] for the definition and theory of cost). More precisely, we prove the following characterization theorem:

Theorem 11.2. *Let F be an ergodic measure preserving countable Borel equivalence relation on a probability space (X, μ). The following are equivalent:*

(1) F is induced by a Borel almost free action of a virtually free countable group;

(2) F is treeable and admits a normal Borel subequivalence relation of finite index with integer or ∞ cost;

(3) F is induced by a Borel almost free action of a countable group Δ of the form $H \times \mathbb{F}_n$, where $|H| = i$ and $n \in \mathbb{N} \cup \{\infty\}$.

The proof of this theorem uses the characterization of normal finite index pairs mentioned above (Theorem 7.2), a criterion for treeability of finite index extensions of treeable equivalence relations given in Section 8, as well as the following theorem of Hjorth (see Corollary 1.2 of [Hjo02] or Theorems 28.2 and 28.5 of [KM04]):

Theorem 2.11 (Hjorth)**.** *Let E be an aperiodic[3] treeable measure preserving ergodic equivalence relation on a standard probability space (X, μ). If E has cost $n \in \mathbb{N} \cup \{\infty\}$, then E is induced by an almost free action of $\mathbb{F}_n$.*

2.2.2 Universal treeable-by-i pairs

We call a countable Borel equivalence relation F treeable-by-finite if it is a finite index extension of a treeable equivalence relation E, and we refer to (E, F) a treeable-by-finite pair. If $[F : E] = i$, then we call (E, F) a treeable-by-i pair.

As mentioned above, in Section 4 we construct a universal index-i pair (E_u, F_u). This E_u is defined as the orbit equivalence relation induced by a certain action of $\mathbb{F}_2$. In Section 9 we show that if we restrict E_u and F_u to the free part of this action of $\mathbb{F}_2$, then the resulting pair (E_{uT}, F_{uT}) is a universal treeable-by-i pair. In particular, we have the following:

Proposition 9.3. *Every treeable-by-i equivalence relation is treeable if and only if F_{uT} is treeable.*

The pair (E_{uT}, F_{uT}) is somewhat hard to work with as the properties we want this pair to have are basically postulated in its definition. However, in Section 14 we construct a more concrete universal treeable-by-i pair. The construction of this pair is based on the natural left action of $\mathbb{F}_2$ on the right coset space $S_\infty(\mathbb{F}_2) \setminus \mathbb{F}_2$, where $S_\infty(\mathbb{F}_2)$ is the group of all bijections from $\mathbb{F}_2$ to $\mathbb{F}_2$ and $\mathbb{F}_2$ is naturally viewed as a subgroup of $S_\infty(\mathbb{F}_2)$. Moreover,

[3] *Aperiodic means that there are no finite orbits.*

using this action, in Section 12 we construct a simple example of a treeable-by-2 system that I believe is a very good candidate for a counter-example to treeable-by-2 equivalence relations being treeable.

CHAPTER II

General and normal index-i pairs

3 Links

Throughout this section, fix $i \in \mathbb{N}$ and let (E, F) denote an index i-pair on a standard Borel space X.

Definition 3.1. *A relation $L \subseteq X^2$ is called a partial equivalence relation if there is $Y \subseteq X$ such that L is an equivalence relation on Y. We refer to this Y as the domain of L and denote by* $\mathrm{dom}(L)$. *Similarly, we call Y^c the codomain of L.*

Note that if L as above is Borel, then $\mathrm{dom}L = \{x \in X : (x, x) \in L\}$ is also Borel.

Definition 3.2. *A Borel finite partial subequivalence equivalence relation L of F is called a partial (E, F)-link if each L-equivalence class consists of i-many pairwise E-inequivalent elements. Such L is called a full (E, F)-link or just an (E, F)-link if* $\mathrm{dom}(L) = X$.

Definition 3.3. *We say that relations $R, S \subseteq X^2$ commute if $R \circ S = S \circ R$, where*

$$x(R \circ S)y \iff \exists z \in X(xRz \text{ and } zSy),$$

for $x, y \in X$. If R and S are equivalence relations, we write $R \vee S$ for their join, that is, the smallest equivalence relation containing $R \cup S$. Clearly, R and S commute if and only if $R \circ S = R \vee S$.

Note that for any (E, F)-link L, E and L commute and $F = E \vee L$.

Definition 3.4. *A Borel set $D \subseteq X$ is called (E, F)-negligible if $[F\restriction_D : E\restriction_D] < i$ and D^c is an E-complete section.*

Proposition 3.5. *There exists a partial (E,F)-link L with (E,F)-negligible codomain.*

Proof. Let $[F]^i$ denote the standard Borel space of all subsets of X of cardinality i consisting of pairwise F-equivalent elements, and put

$$\Phi = \{S \in [F]^i : \text{for any distinct } x, y \in S, x \not E y\}.$$

By Lemma 7.3 in [KM04], there is a Φ-maximal finite partial Borel subequivalence relation L of F. It is clear from the definition of Φ that L is a partial (E,F)-link. Moreover, it follows from the Φ-maximality of L that $\mathrm{dom}(L)$ is an F-complete section and thus an E-complete section, by the virtue of being a partial (E,F)-link.

Now put $Z = \mathrm{dom}(L)^c$ and let $C \subseteq Z$ be an $F\downharpoonright_Z$-class. We show that C contains less than i-many $E\downharpoonright_Z$-classes: otherwise, picking a point from each $E\downharpoonright_Z$-class within C, we can form a set $S \in \Phi$ disjoint from $\mathrm{dom}(L)$, contradicting the Φ-maximality of L. Thus $[F\downharpoonright_Z : E\downharpoonright_Z] < i$. □

For a Borel set $Y \subseteq X$, we call $T \in \mathrm{Aut}(Y)$ a partial Borel automorphism of X with domain Y and write $\mathrm{dom}(T) = Y$. We refer to $\mathrm{dom}(T)^c$ as the codomain of T. We denote by L_T the partial equivalence relation induced by the action of T.

Definition 3.6. *For a countable Borel equivalence relation E on X and $T \in \mathrm{Aut}(X)$, we say that T has index i over E if*

(i) $T^i = \mathrm{id}$,

(ii) $T^0(x), T^1(x), \ldots, T^{i-1}(x)$ *are pairwise E-inequivalent, for all $x \in X$,*

(iii) for each E-class C, $\bigcup_{j<i} T^j(C)$ is a union of exactly i-many E-classes. In other words, for every $x, y \in X$ and $j < i$, there is $k < i$ such that $T^j(x) = T^k(y)$.

Given a Borel linear order $<$ on X, we would say in addition that T respects that order if

(iv) $T^j(x) < T^{j+1}(x)$, *for all $j < i-1$ and $x \in X$.*

Finally, we say that a partial Borel automorphism T of X has index i over E if T has index i over $E\restriction_{\mathrm{dom}(T)}$.

For a Borel automorphism T of index i over E, we let $F_{[E,T]}$ denote the join of L_T and E. Note that $(E, F_{[E,T]})$ is an index-i pair and L_T is an $(E, F_{[E,T]})$-link. The following proposition shows that modulo a negligible set, all index-i pairs arise in this fashion.

Proposition 3.7. *For any index-i pair (E, F), there is a partial Borel automorphism T of X having index i over E and an (E, F)-negligible codomain. In fact, we can make sure that T respects a given Borel linear order on X.*

Proof. Let L be as in Proposition 3.5. Fix a Borel linear order $<$ on X and define T on $\mathrm{dom}(L)$ as follows:

$$T(x) = \begin{cases} \min[x]_L & \text{if } x = \max[x]_L \\ \min\{y \in [x]_L : y > x\} & \text{otherwise} \end{cases},$$

for $x \in \mathrm{dom}(L)$. □

An immediate corollary of this is the following:

Corollary 3.8. *If $[F : E] = 2$, then F is normal over E modulo a set on which F is equal to E.*

Proof. Let T be as in Proposition 3.7 and put $Y = \mathrm{dom}(T)$. Then Y^c is (E, F)-negligible and hence $F\restriction_{Y^c} = E\restriction_{Y^c}$. Also, T is an involution on Y and $T \in N[E\restriction_Y]$. Thus, $F\restriction_Y$ is normal over $E\restriction_Y$ since $F\restriction_Y = L_T \vee E\restriction_Y$. □

Definition 3.9. *If a Borel automorphism T of X has index-i over E and preserves some fixed Borel linear order on X, we call $[E, T]$ an index-i system.*

Lemma 3.10. *Let $A \subseteq X$ be a complete section for E. Then there is a Borel reduction $f : X \to A$ of E to $E\restriction_A$ such that $xEf(x)$ for all $x \in X$.*

Proof. By the Feldman-Moore theorem, there is a Borel action of a countable group Γ on X such that $E = E_\Gamma$. Fix an enumeration $\Gamma = (\gamma_n)_{n\in\mathbb{N}}$ and define $f : X \to A$ by $x \mapsto \gamma_n x$, where $n \in \mathbb{N}$ is the least such that $\gamma_n x \in A$. It is clear that this f satisfies the conclusion of the lemma. □

From this lemma and Proposition 3.7, we immediately get:

Corollary 3.11. *For any index-i pair (E, F) on X, there is an index-i system $[E', T]$ such that $(E, F) \leq_B (E', F_{[E',T]})$. In fact, $E' = E\restriction_A$, for some Borel subset $A \subseteq X$.*

4 A universal index-i pair

Recall that $\mathbb{E}_\infty$ is the orbit equivalence relation of the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$, and it is a universal countable Borel equivalence relation. The following proposition shows that in any index-i pair (E, F), we may assume without loss of generality that $E = \mathbb{E}_\infty\restriction_Y$, for some $\mathbb{E}_\infty$-invariant Borel subset Y of $2^{\mathbb{F}_2}$.

Proposition 4.1. *For any index-i pair (E, F), there is an $\mathbb{E}_\infty$-invariant Borel subset Y of $2^{\mathbb{F}_2}$ and an index-i extension F' of $\mathbb{E}_\infty\restriction_Y$ on Y such that $(E, F) \sqsubseteq_B (\mathbb{E}_\infty\restriction_Y, F')$.*

Proof. Take a Borel embedding $f : E \hookrightarrow \mathbb{E}_\infty$ and let $D \subseteq 2^{\mathbb{F}_2}$ denote the image of this embedding. Putting $Y = [D]_{\mathbb{E}_\infty}$, let $\pi : \mathbb{E}_\infty\restriction_Y \to \mathbb{E}_\infty\restriction_D$ be a Borel reduction (using Lemma 3.10). Now define an equivalence relation F' on Y as follows: for all $y_1, y_2 \in Y$,

$$y_1 F' y_2 \iff f^{-1}(\pi(y_1)) F f^{-1}(\pi(y_2)).$$

Clearly $[F' : \mathbb{E}_\infty\restriction_Y] = [F : E] = i$ and f is a Borel embedding of (E, F) into $(\mathbb{E}_\infty\restriction_Y, F')$. □

Proposition 3.7 gives a simple way of constructing a universal index-i pair (E_u, F_u). Indeed, let $\Delta = \mathbb{Z}_i * \mathbb{F}_2$ and let g denote the generator $1 \in \mathbb{Z}_i$. Let $E_{\mathbb{F}_2}$ and E_Δ denote the orbit equivalence relations of the shift actions of $\mathbb{F}_2$ and Δ on $X' = (2^{\mathbb{F}_2})^{\Delta}$, respectively. Put

$$X'' = \{y \in X' : \text{the set } \bigcup_{n<i} g[y]_{E_\Delta} \text{ is equal to a union}$$
$$\text{of exactly } i\text{-many } E_{\mathbb{F}_2}\text{-classes}\}.$$

Finally, put $X_u = (X'')_{E_{\mathbb{F}_2}}$, $F_u = E_\Delta \restriction_Y$ and $E_u = E_{\mathbb{F}_2} \restriction_Y$. It is clear from the definitions that (E_u, F_u) is an index-i pair.

Proposition 4.2. *For any index-i pair (E, F), $(E, F) \sqsubseteq_B (E_u, F_u)$.*

Proof. By Proposition 4.1, we may assume that $E = \mathbb{E}_\infty \restriction_Y$, for some $\mathbb{E}_\infty$-invariant Borel subset Y of $2^{\mathbb{F}_2}$. Thus, E is the orbit equivalence relation of the shift action of $\mathbb{F}_2$ on Y. Now let T be a partial Borel automorphism of Y given by Proposition 3.7 and extend it to a (full) Borel automorphism S of Y by setting S be the identity on $Y \setminus \text{dom}(T)$. Thus we can extend the action of $\mathbb{F}_2$ on Y to that of Δ by letting g act as S. By the choice of T, this action generates F.

Now define a map $\pi : Y \to (2^{\mathbb{F}_2})^{\Delta}$ by $y \mapsto (\gamma y)_{\gamma \in \Delta}$. Clearly, this is a Δ-equivariant Borel embedding. Using the defining properties of T (in particular, the fact that $\text{dom}(T)^c$ is (E, F)-negligible), it is straightforward to check that the image of π is contained in X_u. Thus, f witnesses $(E, F) \sqsubseteq_B (E_u, F_u)$. □

5 An important example of an index-2 system

In this section we define a pivotal example of an index-2 system that will be used below in constructing various index-i systems with treeable equivalence relations.

Fix a countable group Γ and denote by $S_\infty(\Gamma)$ the group of all bijections from Γ to Γ. The group operation is just the composition and we denote it by $\circ$. $S_\infty(\Gamma)$ is a Polish group under the pointwise convergence topology being a G_δ subset of the Polish space of all functions from Γ to Γ. We naturally view Γ as a subgroup of $S_\infty(\Gamma)$ by letting $\gamma(\alpha) = \gamma\alpha$, for $\gamma, \alpha \in \Gamma$.

Let $\Gamma \setminus S_\infty(\Gamma)$ be the space of right Γ-cosets and consider the natural action $\rho : \Gamma \curvearrowright \Gamma \setminus S_\infty(\Gamma)$ of Γ on $\Gamma \setminus S_\infty(\Gamma)$ defined by:

$$\gamma \cdot^\rho \Gamma g = \Gamma(g \circ \gamma^{-1}),$$

for $\gamma \in \Gamma$, $g \in S_\infty(\Gamma)$.

Let $R(\Gamma) < S_\infty(\Gamma)$ be the subgroup of all bijections $\Gamma \to \Gamma$ that fix the identity e of Γ (R stands for *rotations*). We will just write R if Γ is understood.

Lemma 5.1. $S_\infty(\Gamma) = \Gamma R = R\Gamma$.

Proof. We will prove only $S_\infty(\Gamma) = \Gamma R$ since the other statement is proved similarly. Fix $g \in S_\infty(\Gamma)$ and put $r = g(e)^{-1} \circ g$. Then $g = g(e) \circ r$. ☐

Thus R is a transversal for $\Gamma \setminus S_\infty(\Gamma)$ and we identify $\Gamma \setminus S_\infty(\Gamma)$ with R via the map $\pi : \Gamma \setminus S_\infty(\Gamma) \to R$ defined by

$$\pi(\Gamma g) = g(e)^{-1} \circ g,$$

for $g \in S_\infty(\Gamma)$. Let $\theta : \Gamma \curvearrowright R$ denote the π-pushforward of the action $\rho : \Gamma \curvearrowright \Gamma \setminus S_\infty(\Gamma)$. It is defined as follows:

$$\gamma \cdot^\theta r = \pi(\gamma \cdot^\rho \Gamma r) = \pi(\Gamma(r \circ \gamma^{-1})) = r(\gamma^{-1})^{-1} \circ r \circ \gamma^{-1},$$

for $\gamma \in \Gamma$ and $r \in R$. Thus,

$$\gamma \cdot^\theta r(\alpha) = r(\gamma)^{-1} \circ r(\gamma^{-1}\alpha),$$

for $\alpha \in \Gamma$.

Note that R is a Polish subgroup of $S_\infty(\Gamma)$, the action θ is continuous and (R, θ) is isomorphic to $(\Gamma \setminus S_\infty(\Gamma), \rho)$ as Polish Γ-spaces. Let E_θ denote the orbit equivalence relation on R induced by the action θ.

We now define a homeomorphism $T_{\text{inv}} : R \to R$ by $r \mapsto r^{-1}$. Clearly T_{inv} is an involution, and here is how it interacts with the action of Γ:

Lemma 5.2. *For all $\gamma \in \Gamma$ and $r \in R$, $T_{\text{inv}}(\gamma \cdot^\theta r) = \delta \cdot^\theta T_{\text{inv}}(r)$, where $\delta = r(\gamma^{-1})^{-1}$.*

Proof. We simply compute:

$$\begin{aligned}T_{\text{inv}}(\gamma \cdot^{\theta} r) &= T_{\text{inv}}(r(\gamma^{-1})^{-1} \circ r \circ \gamma^{-1}) \\ &= \gamma \circ r^{-1} \circ r(\gamma^{-1}) \\ &= r^{-1}(\delta^{-1})^{-1} \circ r^{-1} \circ \delta^{-1} \\ &= \delta \cdot^{\theta} T_{\text{inv}}(r).\end{aligned}$$

□

This immediately gives:

Proposition 5.3. $T_{\text{inv}} \in N[E_\theta]$.

Proof. Fix $r_1, r_2 \in R$ such that $r_2 = \gamma \cdot^{\theta} r_1$ for some $\gamma \in \Gamma$. By Lemma 5.2, we have

$$T_{\text{inv}}(r_2) = T_{\text{inv}}(\gamma \cdot^{\theta} r_1) = \delta \cdot^{\theta} T_{\text{inv}}(r_1),$$

for $\delta = r(\gamma^{-1})^{-1}$, and thus $T_{\text{inv}}(r_1) E_\theta T_{\text{inv}}(r_2)$. □

Taking $F_\theta = E_\theta \vee L_{T_{\text{inv}}}$, we get a pair (E_θ, F_θ) with $[F_\theta : E_\theta] \leq 2$. Strictly speaking, $[E_\theta, T_{\text{inv}}]$ is not an index-2 system as T_{inv} may be fixing some E_θ-classes. However, we can always restrict to an E_θ-invariant Borel subset R' of R so that $[F_\theta \restriction_{R'} : E_\theta \restriction_{R'}] = 2$.

6 F/E-atomic decomposition

In this section we establish the F/E-atomic decomposition theorem mentioned in Chapter I.

Let (E, F) be a finite index pair defined on a standard probability space (X, μ). Recall the definition of an F/E-atomic set and note that all null sets are automatically F/E-atomic. Thus, the complete section obtained in the conclusion of the following lemma may simply be a null set. However, we will use this lemma only for F-invariant μ, in which case F-complete sections cannot be null.

Lemma 6.1. *There is an E-invariant μ-measurable F-complete section $Y \subseteq X$ that is F/E-atomic.*

Proof. Recursively define a decreasing sequence of E-invariant Borel F-complete sections Y_n and an increasing sequence d_n of reals as follows: put $Y_0 = X$ and assuming that Y_n is defined, put

$$\mathcal{A}_n = \{A \subseteq Y_n : A \text{ is a } \mu\text{-measurable } E\text{-invariant } F\text{-complete section}\}$$

and

$$d_n = \inf_{A \in \mathcal{A}_n} \mu(A).$$

Let $Y_{n+1} \in \mathcal{A}_n$ be such that $\mu(Y_{n+1}) < d_n + n^{-1}$. By definition, Y_n decreases, while d_n increases (not necessarily strictly). Put $Y = \bigcap_{n \in \mathbb{N}} Y_n$. Clearly Y is E-invariant.

Claim. *Y is an F-complete section.*

Proof of Claim. Let C be an F-class and $\mathcal{C}_n$ be the set of E-classes in C that are contained in Y_n. Thus each $\mathcal{C}_n$ is finite and nonempty, which implies that $(\mathcal{C}_n)_{n \in \mathbb{N}}$ eventually stabilizes since it is decreasing. Hence, Y contains the union of the E-classes in $\mathcal{C}_n$, for some n; and therefore, $Y \cap C \neq \varnothing$. $\dashv$

It remains to show that Y is F/E-atomic. Assume it isn't, i.e. there is a non-null μ-measurable $A \subseteq Y$ that is E-invariant but $B = [A]_{F\restriction_Y} \setminus A$ is not null. We assume without loss of generality that A is an F-complete section since otherwise, we could replace A with $(Y \setminus [A]_{F\restriction_Y}) \cup A$. Let $n \in \mathbb{N}$ be large enough to ensure $\mu(B) > n^{-1}$. Since $Y = A \uplus B$, we have

$$\mu(A) = \mu(Y) - \mu(B) < \mu(Y_n) - n^{-1} < d_n,$$

contradicting the definition of d_n as it is easy to see that $A \in \mathcal{A}_n$. $\square$

Theorem 6.2. *Let (X, μ) be a standard probability space and let (E, F) be a nested pair on X with $[F : E] \leq i$, for some $i \in \mathbb{N}$. Then, there is a partition $X = \biguplus_{j<k} X_j$, $k \leq i$, into E-invariant F/E-atomic sets. Such a partition is unique up to a null set.*

Proof. The uniqueness is clear since for any other such partition $X = \biguplus_{l<m} X'_l$, $X_j \cap X'_l$ is E-invariant and hence is either null or conull in both X_j and X'_l. Thus, modulo a null set, X_j and X'_l are either disjoint or equal, and therefore, $m = k$ and the partitions are the same up to a permutation of indices.

We prove the existence by induction on i. Let Y be as in Lemma 6.1. If Y is conull, then we are done, so suppose it is not. Because Y is an E-invariant F-complete section, we have $[F\restriction_{Y^c} : E\restriction_{Y^c}] < i$. Hence, by induction, there is a partition of Y^c into less than i-many E-invariant measurable F/E-atomic sets. Combining this with Y gives the desired partition of X. □

We call this partition the F/E-atomic decomposition of X. Here are some properties of it in case F is μ-ergodic:

Lemma 6.3. *Suppose $[F : E] = i$, F is μ-ergodic and let $X = \biguplus_{j<k} X_j$, $k \leq i$, be the F/E-atomic decomposition. Then each X_j is an F-complete section modulo μ-null, and $E\restriction_{X_j}$ is $\mu\restriction_{X_j}$-ergodic. If moreover $E \lhd F$, then $[F\restriction_{X_j} : E\restriction_{X_j}] = i/k$, for all $j < k$.*

Proof. The first statement follows immediately from the definitions. For the second, let $j, l < k$ and we show that $[F\restriction_{X_j} : E\restriction_{X_j}] = [F\restriction_{X_l} : E\restriction_{X_l}]$. By normality of E in F, there exists $h \in N[E] \cap [F]$ such that $h(X_j) \cap X_l$ is non-null. But, $\{h(X_j)\}_{j<k}$ is an F/E-atomic decomposition as well, so by the uniqueness of the latter, we have $h(X_j) = X_l$. Since h takes E-classes to E-classes, it is clear that $[F\restriction_{X_j} : E\restriction_{X_j}] = [F\restriction_{X_l} : E\restriction_{X_l}]$. □

7 Normal subequivalence relations

Throughout this section let (E, F) be an index-i pair. In case when (E, F) is normal, we have the following strengthening of Proposition 3.5:

Proposition 7.1. *If (E, F) is normal, then there is a partial (E, F)-link with (E, F)-negligible codomain such that $[\mathrm{dom}(L)^c]_E$ is F-compressible (hence so is $[\mathrm{dom}(L)^c]_F$).*

Proof. Let L be a partial (E, F)-link given by Proposition 3.5. Put

$$A = (\mathrm{dom}(L))_E \cap [\mathrm{dom}(L)^c]_F.$$

Thus A is the union of those E-classes that are fully covered by $\mathrm{dom}(L)$ but lie inside an F-class that is not fully covered by $\mathrm{dom}(L)$. Because $\mathrm{dom}(L)^c$ is (E, F)-negligible, $[A]_F = [\mathrm{dom}(L)^c]_F$.

Fix $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}} \subseteq N[E]$ that generates F and define $\pi : A \to \mathbb{N}$ by $x \mapsto$ the least n such that $\gamma_n(A \cap [x]_F) \cap A^c \neq \varnothing$. Note that such n exists for every $x \in A$ because by the definition of A, $A^c \cap [x]_F \neq \varnothing$ and the action of Γ on $[x]_F$ is transitive. Also, π is F-invariant, i.e. constant on every F-class.

Now define $g : A \to X$ by $x \mapsto \gamma_{\pi(x)} x$. Since within each F-class g is equal to γ_n for some $n \in \mathbb{N}$ and $\gamma_n \in N[E]$, g is a bijection and $g(A)$ is E-invariant. Thus, also $B := g(A) \cap A^c$ is E-invariant since so is A. Moreover, B intersects every F-orbit of $[A]_F = [\mathrm{dom}(L)]_F$ and $B \subseteq [\mathrm{dom}(L)^c]_E$. Thus it is enough to show that B is F-compressible, and in fact, we show that it is E-compressible.

Define the compressing function $\rho : B \to B$ by $x \mapsto$ the unique $y \in [x]_F$ such that $g^{-1}(x) L y$ and $x E y$. It is easy to see that ρ is one-to-one and Borel. It is also immediate from the definition that $\rho(B) \subseteq \mathrm{dom}(L)$, while for every $x \in B$, $[x]_E \setminus \mathrm{dom}(L) \neq \varnothing$ because $B \subseteq [\mathrm{dom}(L)^c]_E$. Thus, indeed, ρ is compressing, and we are done. $\square$

For the rest of this section, assume that the pair (E, F) is defined on a standard probability space (X, μ). We will work towards proving the following characterization of when E is normal in F, in case F is ergodic:

Theorem 7.2. *Let (E, F) be an index-i pair on a standard probability space (X, μ) and suppose that F is measure preserving and ergodic. Then, the following are equivalent:*

(1) $E \triangleleft F$;

(2) F is generated by countable group $\Delta < [F]$ of the form $H \times \Gamma$ such that $|H| = i$ and the action of Γ induces E a.e.

(3) F is generated by a Borel action of a countable group Δ that admits a normal subgroup Γ of finite index such that the action of Γ induces E a.e.

Because groups admitting finite index subgroups also admit further subgroups that are normal and still of finite index, we immediately get the following:

Corollary 7.3. *Suppose F is ergodic. Then, modulo a null set, F is generated by a Borel action of a countable group that admits a proper finite index subgroup if and only if F admits a proper normal subequivalence relation of finite index.*

Lemma 7.4. *Suppose $E \lhd F$ and E is μ-ergodic. Then, for any full (E,F)-link L, there exists a group $G < N[E] \cap [L]$ of order i that generates L.*

Proof. For each $h \in N[E]$, define $\hat{h} : X \to X$ by $x \mapsto$ the unique element in $[x]_L \cap [h(x)]_E$.

Claim. *$\hat{h} \in [L] \cap N[E]$, for all $h \in N[E]$.*

Proof of Claim. $\hat{h}$ is invertible as it is easy to check that the map

$$y \mapsto \text{the unique } x \in [y]_L \text{ with } h(x)Ey$$

is its inverse. Hence $\hat{h} \in [L]$. To check that $\hat{h} \in N[E]$, fix $x, y \in X$ with xEy. Then $h(x)Eh(y)$, i.e. $[h(x)]_E = [h(y)]_E$, and hence $\hat{h}(x)E\hat{h}(y)$, by the definition of $\hat{h}$. $\dashv$

By the normality of E in F, there is $H < N[E]$ that generates F. By the above claim, $G = \langle \hat{h} : h \in H \rangle < [L] \cap N[E]$. The action of G on X is almost free since for any $1_G \neq g \in G$, the set $\{x \in X : g(x) = x\}$ is E-invariant and hence null, by the ergodicity of E. To see that G generates L, fix $x, y \in X$ with xLy. Thus xFy and hence $y = h(x)$, for some $h \in H$. But then $\hat{h}(x) = y$ and $\hat{h} \in G$. It remains to show that $|G| = i$. Since the action of G is almost free and generates L, there is $x \in X$ such that the action of G restricted to $[x]_L$ is free and transitive. This implies that $|G| = i$ since $|[x]_L| = i$. □

When E is not ergodic, we use the F/E-atomic decomposition to decompose X into finitely many E-invariant pieces, on each of which E is ergodic.

Lemma 7.5. *Suppose $E \lhd F$ and F is ergodic, and let $X = \biguplus_{j<k} X_j$ be the F/E-atomic decomposition. For any full (E,F)-link L, there is $T \in N[E] \cap [L]$ of order k such that $T(X_j) = X_{j+1 \pmod k}$, for all $j < k$.*

Proof. For $h \in N[E] \cap [F]$, let $\hat{h} \in N[E] \cap [L]$ be defined as in the proof of Lemma 7.4. For every $j < k$, as in the proof of Lemma 6.3, let $h_j \in N[E] \cap [F]$ be such that $h_j(X_j) = X_{j+1 \pmod k}$ and define $T : X \to X$ as follows: for $x \in X_j$, put $T(x) = \hat{h}_j(x)$. It is clear that T satisfies the condition of the lemma. □

Proposition 7.6. *Suppose $E \lhd F$ and F is ergodic. Then, for any full (E,F)-link L, there exists a group $H < N[E] \cap [L]$ of order i that generates L.*

Proof. Let $X = \biguplus_{j<k} X_j$ be the F/E-atomic decomposition and let $T \in N[E] \cap [L]$ be as in Lemma 7.5. Since $E \restriction_{X_0}$ is ergodic, we can apply Lemma 7.4 and get a group $G < N[E \restriction_{X_0}] \cap [L \restriction_{X_0}]$ of order i/k. Now for each $g \in G$, define $\phi(g) : X \to X$ as follows: for $x \in X_j$, $\phi(g)(x) = T^j \circ g \circ T^{-j}(x)$. Note that for $g_1, g_2 \in G$, $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ and hence ϕ is a group isomorphism between G and $\phi(G)$. Also it is clear from the definitions that $\phi(G) \in N[E] \cap [L]$ and that T commutes with $\phi(G)$. Thus $H = \langle T, \phi(G) \rangle < N[E] \cap [L]$ is of order $k \cdot i/k = i$ and generates L. □

Definition 7.7. *For a full (E,F)-link L and $\gamma \in [E]$, we say that γ commutes with L if so does the graph of γ. For $\Gamma < [E]$, we say that Γ commutes with L if so does every element of Γ.*

Lemma 7.8. *Let L be a full (E,F)-link, $s : X \to X$ be a Borel selector for L, and $\alpha : \Gamma \curvearrowright s(X)$ be a Borel action of a countable group Γ that induces $F \restriction_{s(X)}$. Then there is a Borel action $\beta : \Gamma \curvearrowright X$ that induces E, commutes with L, and makes s an equivariant map. In particular, if α is free, then so is β.*

Proof. We define a Borel action $\beta : \Gamma \times X \to X$ of Γ on X as follows: for $x \in X$ and $\gamma \in \Gamma$,

$$\beta(\gamma, x) = \text{the unique } y \text{ such that } yEx \text{ and } s(y) = \alpha(\gamma, s(x)).$$

To show that this action generates E fix $x, y \in X$ with xEy. Then $s(x) F \restriction_{s(X)} s(y)$ and hence there is $\gamma \in \Gamma$ such that $s(y) = \alpha(\gamma, s(x))$. Thus $\beta(\gamma, x) = y$. □

Proof of Theorem 7.2. (1) ⇒ (2): Let L be a partial (E, F)-link as in Proposition 7.1. Since compressible sets are null, we may assume without loss of generality that L is a full link.

Let $s : X \to X$ be a Borel selector for L. Take a faithful Borel action $\alpha : \Gamma \curvearrowright s(X)$ of some countable group Γ that induces $F \restriction_{s(X)}$. By Lemma 7.8, there is a Borel action $\beta : \Gamma \curvearrowright X$ of Γ generating E, commuting with L and making s equivariant. Since the action β is faithful, we can identify Γ with a subgroup of $[E]$.

Now let $H < N[E] \cap [L]$ be as in Lemma 7.6. Because Γ commutes with L, it also commutes with H and thus $\Delta = \langle H, \Gamma \rangle = H \times \Gamma$. It is clear that Δ generates F.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): It is enough to show that each element of Δ acts as an automorphism in $N[E]$. Fix $\delta \in \Delta$ and $x, y \in X$ such that $y = \gamma x$ for some $\gamma \in \Gamma$. We need to show that $\delta y = \gamma'(\delta x)$ for some $\gamma' \in \Gamma$. But because $\Gamma \lhd \Delta$, $\delta\gamma = \gamma'\delta$ for some $\gamma' \in \Gamma$. Hence, $\delta y = \delta\gamma x = \gamma'\delta x$ and we are done. □

We close this section with the following questions.

Question 7.9 . *Let (E, F) be an index-i pair on a standard Borel space X. Is there a countable group Δ with a finite index subgroup Γ and a Borel action $\Delta \curvearrowright X$ such that $E_\Delta = F$ and $E_\Gamma = E$?*

Because groups admitting finite index subgroups also admit further subgroups that are normal and still of finite index, a positive answer to the above question implies a positive answer to the following:

Question 7.10. *With the hypothesis of the previous question, is there a Borel subequivalence relation $E' \subseteq E$ such that $E' \lhd F$ and $[E : E'] = n$ for some $n < \infty$?*

CHAPTER III

Treeable-by-finite equivalence relations

In this chapter, we investigate the question of whether a treeable-by-finite equivalence relation is treeable (see [JKL02]). Unless stated otherwise, let (E, F) denote an index-i pair on some standard Borel space.

8 A useful criterion

For a partial (E, F)-link L, put $X_L := \mathrm{dom}(L)/L$ and let F_L denote the push-forward of $F\restriction_{\mathrm{dom}(L)}$ under the natural projection map $\pi_L : \mathrm{dom}(L) \to X_L$; thus $F\restriction_{\mathrm{dom}(L)} \leq_B F_L$. Since each L-class is finite, there is a Borel right-inverse of π_L and hence $F_L \leq_B F\restriction_{\mathrm{dom}(L)}$. Therefore, $F_L \sim_B F\restriction_{\mathrm{dom}(L)}$.

Lemma 8.1. *The following are equivalent:*

(1) F is treeable;

(2) For any partial (E, F)-link L, F_L is treeable;

(3) There exists a partial (E, F)-link L with (E, F)-negligible codomain such that F_L is treeable.

Proof. (1)⇒(2): follows from the fact that $F_L \leq_B F$.

(2)⇒(3): by Proposition 3.5.

(3)⇒(1): Because $\mathrm{dom}(L)$ is an F-complete section, $F \leq_B F\restriction_{\mathrm{dom}(L)} \leq_B F_L$ and hence F itself is treeable since treeability is closed downward under Borel reductions. □

Definition 8.2. *For a countable equivalence relation E on a standard Borel space X, a subgraphing $\mathcal{G}$ of E is a Borel graph on X such that $E_{\mathcal{G}} \subseteq E$.*

For a partial (E,F)-link L and a subgraphing $\mathcal{G}$ of E, we spell out what it means for L and $\mathcal{G}$ to commute: for all $x_0, x_1, y_0, y_1 \in X$, whenever $x_0 L x_1$, $y_0 L y_1$ and $x_k E y_k$ for $k = 0, 1$, we have

$$(x_0, y_0) \in \mathcal{G} \iff (x_1, y_1) \in \mathcal{G}.$$

Thus, if $\mathcal{G}$ is a graphing of E that commutes with L, then within every F-class the $\mathcal{G}$-connected components are isomorphic and the isomorphism is realized via the link L.

Proposition 8.3. *For any index-i pair (E,F), the following are equivalent:*

(1) F is treeable;

(2) For any partial (E,F)-link L, there is a treeing of $E\restriction_{\mathrm{dom}(L)}$ that commutes with L;

(3) There exists a partial (E,F)-link L with (E,F)-negligible codomain and a treeing $\mathcal{G}$ of $E\restriction_{\mathrm{dom}(L)}$ that commutes with L.

Proof. (1)⇒(2): If F is treeable, so is F_L. Let $\mathcal{G}_L$ be a treeing of F_L. Then, taking the pull-back of $\mathcal{G}_L$ under π_L and deleting the edges between E-inequivalent points, we get a desired treeing $\mathcal{G}$ for $E\restriction_{\mathrm{dom}(L)}$.

(2)⇒(3): by Proposition 3.5.

(3)⇒(1): If $\mathcal{G}$ is a treeing of $E\restriction_{\mathrm{dom}(L)}$ commuting with L, we define a Borel graph $\mathcal{G}_L$ on X_L as follows: for $C_0, C_1 \in X_L$, put $(C_0, C_1) \in \mathcal{G}_L$ if there are $x \in C_0$ and $y \in C_1$ with $(x,y) \in \mathcal{G}$. Because $\mathcal{G}$ commutes with L, $\mathcal{G}_L$ is a treeing of F_L. Thus, by Lemma 8.1, F is treeable. □

9 A universal treeable-by-i pair

Recall that $\mathbb{E}_{\infty T}$ is the orbit equivalence relation on $\mathrm{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$ of the shift action of $\mathbb{F}_2$. Just like Proposition 4.1, the following proposition shows that in any treeable-by-i pair

(E, F), we may assume without loss of generality that $E = \mathbb{E}_{\infty T} \restriction_Y$ for some $\mathbb{E}_{\infty T}$-invariant Borel subset Y of $\mathrm{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$.

Proposition 9.1. *For any treeable-by-i pair (E, F), there is an $\mathbb{E}_{\infty T}$-invariant Borel subset Y of $\mathrm{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$ and an index-i extension F' of $\mathbb{E}_{\infty T} \restriction_Y$ on Y such that $(E, F) \sqsubseteq_B (\mathbb{E}_{\infty T} \restriction_Y, F')$.*

Proof. Same as the proof of Proposition 4.1, using $\mathbb{E}_{\infty T}$ instead of $\mathbb{E}_\infty$. □

Using this and Proposition 3.7, we construct a universal treeable-by-i pair as follows: recall that (E_u, F_u) is defined on a certain Γ-invariant subset X_u of $(2^{\mathbb{F}_2})^\Gamma$, where $\Gamma = \mathbb{Z}_i * \mathbb{F}_2$. Put $X_{uT} = \mathrm{Free}(\mathbb{F}_2, (2^{\mathbb{F}_2})^\Gamma)$, i.e. the free part of the action of $\mathbb{F}_2$ on $(2^{\mathbb{F}_2})^\Gamma$, and let $E_{uT} = E_u \restriction_{X_{uT}}$, $F_{uT} = F_u \restriction_{X_{uT}}$.

Proposition 9.2. *For any treeable-by-i pair (E, F), $(E, F) \sqsubseteq_B (E_{uT}, F_{uT})$.*

Proof. Same as the proof of Proposition 4.2, using Proposition 9.1 instead of 4.1. □

Thus, we reduce the problem of whether treeable-by-i is treeable, to just verifying it for F_{uT}; more precisely:

Proposition 9.3. *Every treeable-by-i equivalence relation is treeable if and only if F_{uT} is treeable.*

Lastly, we consider the case $i = 2$. It is somewhat special because in this case F coincides with E on F/E-negligible sets. Thus, we have:

Lemma 9.4. *Let (E, F) be a treeable-by-2 pair on a standard Borel space X and let $A \subseteq X$ be a Borel F/E-negligible set. Then $F \restriction_{[A]_F}$ is treeable.*

Proof. $F \restriction_A = E \restriction_A$ and hence is treeable. But A is an F-complete section for $[A]_F$, so by (d) of Proposition 2.9, $F \restriction_{[A]_F}$ is treeable as well. □

Note that for any involution $T \in N[E]$, each L_T-equivalence class has either one or two elements, so $[E \vee L_T : E] \leq 2$.

Proposition 9.5. *Every treeable-by-2 equivalence relation is treeable if and only if for every $T \in N[\mathbb{E}_{\infty T}]$, $\mathbb{E}_{\infty T} \vee L_T$ is treeable.*

Proof. We only prove the right-to-left direction as the other is trivial. Let (E, F) be an index-2 pair. By Proposition 9.1, we may assume that $E = \mathbb{E}_{\infty T} \restriction_Y$ for some $\mathbb{E}_{\infty T}$-invariant Borel subset Y of $X = \mathrm{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$. Let T be a partial Borel automorphism of Y having index-2 over E as in Proposition 3.7. Hence T is an involution and $Y \setminus \mathrm{dom}(T)$ is F/E-negligible. Thus, by Lemma 9.4, $F \restriction_{Y_0}$ is treeable, where $Y_0 = [Y \setminus \mathrm{dom}(T)]_F$. Therefore, it is enough to show that $F \restriction_Z$ is treeable, where $Z = Y \setminus Y_0$.

Note that $T \restriction_Z$ is a (full) involution defined on Z and $T \restriction_Z \in N[E \restriction_Z]$. Define $T' : X \to X$ by setting $T' \restriction_Z = T \restriction_Z$ and $T' \restriction_{X \setminus Z} = \mathrm{id}_{X \setminus Z}$. It is clear that $T' \in N[\mathbb{E}_{\infty T}]$ and thus $\mathbb{E}_{\infty T} \vee L_{T'}$ is treeable. But $F \restriction_Z = (\mathbb{E}_{\infty T} \vee L_{T'}) \restriction_Z$ and hence $F \restriction_Z$ is treeable as well. □

10 Sufficient conditions

Definition 10.1. *Consider a Borel action of a countable group Γ on X and let T be a Borel automorphism of X. We say that the action of T normalizes that of Γ if viewing each element of Γ as a Borel automorphism of X, we have that for all $\gamma \in \Gamma$, there is $\gamma' \in \Gamma$ such that $T \circ \gamma \circ T^{-1} = \gamma'$. Similarly, we say that the action of T commutes with that of Γ, if $T \circ \gamma = \gamma \circ T$, for all $\gamma \in \Gamma$.*

Proposition 8.3 gives the following sufficient condition for treeability:

Proposition 10.2. *Consider a Borel free action of a countable free group Γ on a standard Borel space X and let $T \in \mathbb{N}(E_\Gamma)$ be such that $T^i = \mathrm{id}$ for some $i \in \mathbb{N}$. If T commutes with the action of Γ, then $E_\Gamma^X \vee L_T$ is treeable.*

Proof. Let $\mathcal{G}$ be the standard treeing for E_Γ^X, i.e. $\mathcal{G}$ is equal to the union of the graphs of the free generators of Γ and their inverses, viewed as elements of $\mathrm{Aut}(X)$. The fact that T commutes with Γ implies that T commutes with $\mathcal{G}$. Thus, it follows from Proposition 8.3 that $E_\Gamma^X \vee L_T$ is treeable. □

Example 10.3. Let T be the switching 0-1 involution defined on $\mathrm{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$, i.e.

$$T(x)(\gamma) = 1 - x(\gamma),$$

for $x \in \mathrm{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$ and $\gamma \in \mathbb{F}_2$. It is clear that T commutes with the action of $\mathbb{F}_2$ and hence, by the previous proposition, $\mathbb{E}_{\infty T} \vee L_T$ is treeable.

Recall the following result of Jackson-Kechris-Louveau mentioned in Chapter I:

Corollary 2.10. *If a Borel equivalence relation F is induced by a free Borel action of a virtually free countable group, then F is treeable.*

This generalizes the previous proposition and in fact implies a stronger version of it:

Corollary 10.4. *Consider a Borel free action of a countable free group Γ on a standard Borel space X and let $T \in \mathbb{N}(E_\Gamma)$ be such that $T^i = \mathrm{id}$ for some $i \in \mathbb{N}$. If T normalizes the action of Γ, then $E_\Gamma^X \vee L_T$ is treeable.*

Proof. Since the action of Γ is faithful, we may consider it as a subgroup of $\mathrm{Aut}(X)$ and let $\Delta = \langle \Gamma, T \rangle < \mathrm{Aut}(X)$. Since T is in the normalizer of Γ inside Δ, we have that $\Gamma \triangleleft \Delta$ and $[\Delta : \Gamma] = i$. Now apply Corollary 2.10. □

In Theorem 11.2 below, we prove a converse to Corollary 2.10 in the measure-theoretic context, in the case when F is ergodic.

Lastly, consider the index-2 case. Recall that $\mathbb{E}_{\infty T}$ is the orbit equivalence relation induced by the shift action of $\mathbb{F}_2$ on $\mathrm{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$. In light of Proposition 9.5 and Corollary 10.4, we ask the following:

Question 10.5 . *Does there exists an involution $T \in N[\mathbb{E}_{\infty T}]$ that does not normalize the shift action of $\mathbb{F}_2$ on* $\mathrm{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$*?*

Proposition 9.5 and Corollary 10.4 imply that a negative answer to this question will imply that all treeable-by-2 equivalence relations are treeable. However, the analogous question asked for the free part of the shift on $X^{\mathbb{F}_2}$ instead of $2^{\mathbb{F}_2}$ has a positive answer:

Proposition 10.6. *Let X be an uncountable standard Borel space. Consider the shift action $s : \mathbb{F}_2 \curvearrowright X^{\mathbb{F}_2}$ and denote the induced orbit equivalence relation on* $\mathrm{Free}(\mathbb{F}_2, X^{\mathbb{F}_2})$ *by E. There is an involution $T \in N[E]$ that neither normalizes the shift action of $\mathbb{F}_2$, nor commutes with the standard treeing of E (induced by the Cayley graph of $\mathbb{F}_2$).*

Proof. Let $a : \mathbb{F}_2 \curvearrowright Y$ be a free Borel action of $\mathbb{F}_2$ on a standard Borel space Y and denote by E_a the induced orbit equivalence relation on Y. By Proposition 1.5, there is an equivariant embedding $f : Y \to \mathrm{Free}(\mathbb{F}_2, X^{\mathbb{F}_2})$.

Now if $T_Y \in N[E_a]$ is an involution, then we can define an involution $T \in N[E]$ on $\mathrm{Free}(\mathbb{F}_2, X^{\mathbb{F}_2})$ as follows: let T be the identity on $\mathrm{Free}(\mathbb{F}_2, X^{\mathbb{F}_2}) \setminus f(Y)$, and define T on $f(Y)$ by $T(f(y)) = f(T_Y(y))$. It is clear that T_Y normalizes the action a of $\mathbb{F}_2$ on Y if and only if T normalizes the shift action of $\mathbb{F}_2$ on $\mathrm{Free}(\mathbb{F}_2, X^{\mathbb{F}_2})$. Similarly, T_Y commutes with the standard treeing of E_a if and only if T commutes with the standard treeing of E.

Therefore, it is enough to construct an example of a free Borel action of $a : \mathbb{F}_2 \curvearrowright Y$ with an involution $T_Y \in N[E_a]$ that does not normalize the action a and does not commute with the standard treeing of E_a. Such an example is given below in Corollary 12.7. □

11 A characterization of ergodic free actions of virtually free groups

Lemma 11.1. *Let (E, F) be an index-i pair on a standard probability space (X, μ) such that F is measure preserving and ergodic. If F is treeable a.e. and $C_\mu(E) = n \in \mathbb{N} \cup \{\infty\}$, then for any (E, F)-link L, there exists an almost free Borel action of $\mathbb{F}_n$ on X that induces E and commutes with L.*

Proof. Let $s : X \to X$ be a Borel selector for L and put $Y = s(X)$. Note that Y is a Borel transversal for L and put $\nu = \frac{\mu\restriction_Y}{\mu(Y)}$. Clearly, ν is $F\restriction_Y$-invariant.

Claim. *$\mu(Y) = 1/i$ and $C_\nu(F\restriction_Y) = n$.*

Proof of Claim. For the first statement, let $T \in [F]$ be defined as in the proof of Proposition 3.7. Then $\{T^k(Y)\}_{k<i}$ is a partition of X and $\mu(T^k(Y)) = \mu(Y)$ since μ is T-invariant. Thus $\mu(Y) = 1/i$.

For the second statement, apply the cost formula for a complete section (see Theorem 21.1 of [KM04]) to Y and get

$$C_{\mu\restriction_Y}(F\restriction_Y) = C_\mu(F) - \mu(X \setminus Y) = C_\mu(F) - (1 - \frac{1}{i}).$$

By Proposition 25.6 of [KM04], since F is treeable and E is an index-i subequivalence relation of F, we can express the cost of F in terms of the cost of E as follows:

$$C_\mu(F) = \frac{1}{i}(C_\mu(E) - \mu(X)) + \mu(X) = \frac{n-1}{i} + 1 = \frac{n}{i} + (1 - \frac{1}{i}).$$

Plugging this into the previous equality, we get $C_{\mu\restriction_Y}(F\restriction_Y) = n/i$. Finally,

$$C_\nu = \frac{C_{\mu\restriction_Y}(F\restriction_Y)}{\mu(Y)} = iC_{\mu\restriction_Y}(F\restriction_Y) = n.$$

Note that the calculations are still valid for the case $n = \infty$. ⊣

Because F is μ-ergodic, $F\restriction_Y$ is ν-ergodic. Thus, by Theorem 2.11 (of Hjorth), there is an almost free Borel action $\alpha : \mathbb{F}_n \curvearrowright Y$ that generates $F\restriction_Y$. Thus, by Lemma 7.8, there is an almost free action of $\mathbb{F}_n$ on X that generates E and commutes with L. □

Theorem 11.2. *Let F be an ergodic measure preserving countable Borel equivalence relation on a probability space (X, μ). The following are equivalent:*

(1) F is induced by a Borel almost free action of a virtually free countable group;

(2) F is treeable and admits a normal Borel subequivalence relation of finite index with integer or ∞ cost;

(3) F is induced by a Borel almost free action of a countable group Δ of the form $H \times \mathbb{F}_n$, where $|H| = i$ and $n \in \mathbb{N} \cup \{\infty\}$.

Proof. (1)⇒(2): Suppose that F is induced by a Borel almost free action of a countable group Δ that admits a finite index free subgroup Γ. Then, the treeability of F is the conclusion of Corollary 2.10.

Let $\Gamma' < \Gamma$ be such that $[\Delta : \Gamma'] = i < \infty$ and $\Gamma' \triangleleft \Delta$. Since subgroups of a free group are free, Γ' is free. Let $E = E_{\Gamma'}$. By (3)⇒(1) of Theorem 7.2, $E \triangleleft F$. Also, because the action of Δ is free, we have $[F : E] = i$. Finally, if $n \in \mathbb{N} \cup \{\infty\}$ is the number of the free generators of Γ', then $C_\mu(E) = n$.

(2)⇒(3): Let $E \triangleleft F$ be of finite index. By the ergodicity of F, we may assume that $[F : E] = i < \infty$. Let L be a partial (E, F)-link as in Proposition 7.1. Since $[\mathrm{dom}(L)^c]_F$ is F-compressible, it is μ-null, and hence, we can assume without loss of generality that L is a full link. By Proposition 7.6, there is $H < N[E] \cap [L]$ of order i that generates L.

Put $n = C_\mu(E) \in \mathbb{N} \cup \{\infty\}$. By Lemma 11.1, there is an almost free action of $\mathbb{F}_n$ on X that generates E and commutes with L. Thus the actions of $\mathbb{F}_n$ and H commute and induce an almost free action of $H \times \mathbb{F}_n$ on X that generates F.

(3)⇒(1): Trivial. □

12 The action $\theta : \mathbb{F}_n \curvearrowright R(\mathbb{F}_n)$

Throughout this section, let Γ be a countable free group, i.e. $\Gamma = \mathbb{F}_n$ for some $n \leq \infty$. Also let S denote the set of free generators of Γ together with their inverses.

Recall from Section 5 that $R = R(\Gamma) = \{g \in S_\infty(\Gamma) : g(e) = e\}$. Let E_θ denote the orbit equivalence relation induced by the action θ of Γ on R. Also recall that $T_{\mathrm{inv}} : R \to R$ is the involution $r \mapsto r^{-1}$ and $F_\theta = E_\theta \vee L_{T_{\mathrm{inv}}}$. Let $R' = \mathrm{Free}(\Gamma, R)$ denote the free part of the action $\theta : \Gamma \curvearrowright R$.

Thus, $E'_\theta := E_\theta \restriction_{R'}$ is treeable, and we let $\mathcal{G}'$ be its standard treeing; that is:

$$(r_1, r_2) \in \mathcal{G}' \iff \exists \gamma \in S(\gamma \cdot^\theta r_1 = r_2).$$

for $r_1, r_2 \in R'$.

It is an open question whether $F'_\theta := F_\theta \restriction_{R'}$ is treeable, and this is the simplest and most natural example of a treeable-by-2 equivalence relation[1] that I am aware of, for which the answer is unknown.

For a relation $D \subseteq R^2$, put $D^{-1} = \{(r_1^{-1}, r_2^{-1}) : (r_1, r_2) \in D\}$. In this notation, Proposition 8.3 translates to the following:

Proposition 12.1. *F'_θ is treeable if and only if there is a treeing $\mathcal{G}$ of E'_θ such that $\mathcal{G}^{-1} = \mathcal{G}$.*

We now consider different subgroups of R and investigate the interaction of T_{inv} and the action θ on these subgroups.

12.1 Group-automorphisms of Γ

Let $H = H(\Gamma)$ denote the subgroup of R of all group-automorphisms of Γ, i.e. those bijections that are group homomorphisms. Note that if Γ is finitely generated, then H is countable.

Lemma 12.2. *H is equal to the set of fixed points of the action $\theta : \Gamma \curvearrowright R$.*

Proof. For $r \in R$,

$$\begin{aligned} r \text{ is a fixed point of the action } \theta &\iff \forall \gamma \in \Gamma, \gamma \cdot^\theta r = r \\ &\iff \forall \gamma, \alpha \in \Gamma, r(\gamma^{-1})^{-1} r(\gamma^{-1}\alpha) = r(\alpha) \\ &\iff \forall \gamma, \alpha \in \Gamma, r(\gamma^{-1}\alpha) = r(\gamma^{-1}) r(\alpha) \\ &\iff \forall \beta, \alpha \in \Gamma, r(\beta\alpha) = r(\beta) r(\alpha) \\ &\iff r \in H. \end{aligned}$$

□

[1] Technically, it is not treeable-by-2 since some F_θ-classes may contain only one E_θ-class, but F_θ restricted to the union of these classes is treeable, so we may discard it.

Proposition 12.3. *There is no F_θ-class $B \subseteq R'$ such that $T_{\text{inv}} \restriction_B$ normalizes the action $\theta \restriction_B : \Gamma \curvearrowright B$.*

Proof. Assume for contradiction that $T_{\text{inv}} \restriction_B$ normalizes the action $\theta \restriction_B$. Since this action is free (and hence faithful), we may view Γ as a group of homeomorphism of B. Thus T is in the normalizer of Γ inside the group $\Delta =< T, \Gamma >$, and hence $T \circ \gamma = \delta \circ T$ for some unique $\delta \in \Gamma$. Therefore, for $r \in B$, we have

$$T(\gamma \cdot^\theta r) = \delta \cdot^\theta T(r).$$

By Lemma 5.2 and freeness of the action $\theta \restriction_B$, $\delta^{-1} = r(\gamma^{-1})$. Since $r \in B$ is arbitrary, the same is true for $\alpha \cdot^\theta r$, for all $\alpha \in \Gamma$, i.e.

$$\delta^{-1} = (\alpha \cdot^\theta r)(\gamma^{-1}) = r(\alpha^{-1})^{-1} r(\alpha^{-1}\gamma^{-1}).$$

Hence, we have

$$r(\alpha^{-1}\gamma^{-1}) = r(\alpha^{-1}) r(\gamma^{-1}).$$

Because this is true for all $\gamma, \alpha \in \Gamma$, r is a group-homomorphism and hence $r \in H$. Thus, r is a fixed point of the action θ, contradicting r being in the free part of this action. □

12.2 Graph-automorphisms of Cay(Γ, S)

Let $\mathrm{Cay}(\Gamma, S)$ denote the Cayley graph of (Γ, S) and denote by $\mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ the group of automorphisms of $\mathrm{Cay}(\Gamma, S))$ (as a graph). Put $R_C(\Gamma) = R \cap \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ and we just write R_C below when Γ is understood. Clearly, R_C is a subgroup and hence, $T_{\text{inv}}(R_C) = R_C$. Moreover, since $\Gamma < \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ and R_C is a transversal for the right coset space $\Gamma \setminus \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$, R_C is invariant under the action θ of Γ. Put

- $R'_C = R' \cap R_C$,
- $E'_C = E'_\theta \restriction_{R'_C}$,
- $F'_C = F'_\theta \restriction_{R'_C}$,

- $T'_C = T_{\mathrm{inv}} \restriction_{R'_C}$,
- $\mathcal{G}'_C \coloneqq \mathcal{G}' \restriction_{R'_C}$.

Note that $\mathcal{G}'_C$ is a treeing of E'_C, and in fact, we have the following:

Proposition 12.4. *T'_C commutes with $\mathcal{G}'_C$.*

Proof. Fix $(r_1, r_2) \in \mathcal{G}'_C$ and hence $r_2 = \gamma \cdot^{\theta} r_1$ for some $\gamma \in S$. By Lemma 5.2, $T'_C(r_2) = \delta \cdot^{\theta} T'_C(r_1)$, where $\delta = r_1(\gamma^{-1})^{-1}$. Because r_1 is in R_C, $r_1(S) = S$ and thus $\delta \in S$. Therefore, $(T'_C(r_1), T_C(r_2)) \in \mathcal{G}'_C$. □

This and (2)⇒(1) of Proposition 8.3 immediately imply:

Corollary 12.5. *F'_C is treeable.*

Note that because of Proposition 12.3, the treeability of F'_C does not follow from Corollary 10.4.

It is worth mentioning that we also have a converse to Proposition 12.4:

Proposition 12.6. *If $B \subseteq R'$ is an F_θ-equivalence class such that $T_{\mathrm{inv}} \restriction_B$ commutes with $\mathcal{G}' \restriction_B$, then $B \subseteq R'_C$.*

Proof. Fix $r \in B$, $\gamma \in \Gamma$ and $u \in S$. We need to show that for some $v \in S$, $r(\gamma u) = r(\gamma)v$. Let $r_1 = \gamma^{-1} \cdot^{\theta} r$. Because $T_{\mathrm{inv}} \restriction_B$ commutes with $\mathcal{G}' \restriction_B$, and $(r_1, u^{-1} \cdot^{\theta} r_1) \in \mathcal{G}'$, there is $v \in S$ such that $v^{-1} \cdot^{\theta} T_{\mathrm{inv}}(r_1) = T_{\mathrm{inv}}(u^{-1} \cdot^{\theta} r_1)$. By Lemma 5.2 and the freeness of the action $\theta \restriction_{R'}$, $v = r_1(u)$. Since $r_1(u) = r(\gamma)^{-1} r(\gamma u)$, we get $r(\gamma)v = r(\gamma u)$. □

In Section 13, we consider the case when $\Gamma = \mathbb{F}_2$ and define an F_θ-invariant probability measure on $R_C(\mathbb{F}_2)$.

12.3 Summary

Put $R'' = R' \setminus R_C$, $E''_\theta = E_\theta \downharpoonright_{R''}$ and $F''_\theta = F_\theta \downharpoonright_{R''}$. By Proposition 12.5, F'_θ is treeable if and only if F''_θ is treeable.

From Propositions 12.3 and 12.6, we immediately get

Corollary 12.7. *The action of $T_{\text{inv}} \downharpoonright_{R''}$ does not normalize the action $\theta \downharpoonright_{R''}$ and does not commute with the standard treeing $\mathcal{G}' \downharpoonright_{R''}$ of E''_θ.*

Thus, we cannot apply either of Corollary 10.4 or Proposition 8.3, to deduce the treeability of F''_θ.

Speculation. I believe that F''_θ is not treeable and I suggest the following strategy for proving it: fix a treeing $\mathcal{G}$ of E''_θ. By playing games on the Cayley graph of Γ and using Borel determinacy[2], construct $r \in R' \setminus R_C$, for which there is $\gamma \in \Gamma$ such that $(r, \gamma \cdot^\theta r) \in \mathcal{G}$ but $(r^{-1}, (\gamma \cdot^\theta r)^{-1}) \notin \mathcal{G}$. Thus, $\mathcal{G}^{-1} \neq \mathcal{G}$, and since $\mathcal{G}$ was an arbitrary treeing of E''_θ, Proposition 12.1 implies that F''_θ is not treeable.

13 A measure-theoretic example

In this section, we consider the case $\Gamma = \mathbb{F}_2$ and define an F_θ-invariant probability measure on $R_C = R_C(\mathbb{F}_2)$, obtaining a measure-theoretic example of a treeable-by-2 system $[E'_C, T'_C]$. This example is due to Alex Furman.

By Corollary 12.5, $F'_C = E'_C \vee L_{T'_C}$ is treeable. We will show below that the action of T'_C is not finite index over the action $\theta \downharpoonright_{R'_C}$ of Γ on R'_C, and thus the treeability of F'_C does not follow from Corollary 2.10.

Let a, b be the free generators of $\mathbb{F}_2$ and put $S = \{a, a^{-1}, b, b^{-1}\}$. Because $\Gamma = \mathbb{F}_2$ is finitely generated, $\mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ is locally compact and hence admits a unique left-invariant Haar

[2]These kinds of games have been recently used by Andrew Marks in [Mar13], providing very elegant solutions to well-known open problems in Borel combinatorics.

measure ν. R_C is a compact subgroup of $\mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ and thus $\nu(R_C) < \infty$. Also, since $\Gamma R_C = \mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$, $\nu(R_C) > 0$. Therefore, taking $\mu = \frac{\nu\restriction_{R_C}}{\nu(R_C)}$, we get a Γ-invariant probability measure on R_C. In other words, Γ is a lattice in $\mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ and this implies that $\mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ is unimodular, i.e. the left and right Haar measures are equal (see, for example, Proposition 9.20 in [EW11]).

Let E_C denote the orbit equivalence relation induced by the action $\theta\restriction_{R_C}$ of Γ on R_C. Recall the involution $T_{\mathrm{inv}} : R \to R$ defined by $r \mapsto r^{-1}$, and take $T_C = T_{\mathrm{inv}}\restriction_{R_C}$. Because G is unimodular, ν is preserved under the action of T_C, and hence so is μ. By Proposition 5.3, $T_C \in N[E_C]$. Thus, denoting by F_C the join of E_C and L_{T_C}, we have that $[F_C : E_C] \leq 2$ and μ is F_C-invariant.

The Haar measure μ on R_C is easy to calculate (implicitly using the uniqueness of the latter). The basic open sets in R_C are finite intersections of the sets of the form

$$U_{\gamma,\delta} = \{r \in R_C : r(\gamma) = \delta\},$$

where $\gamma, \delta \in \Gamma$. We denote by $|\gamma|$ the length of the reduced word γ. By the definition of R_C, if $|\gamma| \neq |\delta|$, $U_{\gamma,\delta} = \varnothing$.

Lemma 13.1. *For $\gamma, \delta \in \Gamma$ with $|\gamma| = |\delta| = n$, $\mu(U_{\gamma,\delta}) = 4^{-1} \cdot 3^{1-n}$.*

Proof. Simply note that for $r \in R_C$, the value of $r(\gamma)$ can be any word in $\mathbb{F}_2$ of length n, and the number of such words is $4 \cdot 3^{n-1}$. □

Proposition 13.2. *The action of Γ on R_C is almost free. In particular, E_C is treeable a.e.*

Proof. Fix a nontrivial element $\gamma \in \Gamma$, and suppose $\gamma \cdot^{\theta} r = r$, for some $r \in R_C$. Thus, for all $\alpha \in \Gamma$, we have:

$$r(\gamma^{-1})^{-1} r(\gamma^{-1}\alpha) = r(\alpha),$$

and hence

$$r(\gamma^{-1}\alpha) = r(\gamma^{-1}) r(\alpha).$$

Inductively applying this to powers of γ^{-1} in lieu of α, we get:

$$r(\gamma^{-n}) = r(\gamma^{-1})^n. \tag{13.1}$$

By above, the set of fixed points of γ is contained in

$$\bigcup_{\delta \in \Gamma, |\delta| = |\gamma|} B_{\gamma,\delta},$$

where

$$B_{\gamma,\delta} = \bigcap_{n \in \mathbb{N}} U_{\gamma^{-n},\delta^n}.$$

But $B_{\gamma,\delta}$ is clearly null due to Lemma 13.1 since $|\gamma^{-n}| \geq n$. □

We will show that T_C is index-2 over E_C a.e., but first we prove the following technical lemma. For $\gamma, \delta, \alpha \in \Gamma$, put

$$A_{\gamma,\delta,\alpha} = \{r \in U_{\gamma,\delta} : r(\gamma\alpha) = \delta r^{-1}(\alpha)\},$$

and we write $\delta \perp \alpha$ if there is $j < \min\{|\delta|, |\alpha|\}$ $\delta(j) \neq \alpha(j)$.

Lemma 13.3. *If $\gamma, \delta, \alpha \in \Gamma$ are such that $|\gamma\alpha| = |\gamma| + |\alpha|$ and $\delta \perp \alpha$, then $\mu(A_{\gamma,\delta,\alpha}) = \mu(U_{\gamma,\delta}) \cdot 3^{-|\alpha|}$.*

Proof. We may assume that $|\gamma| = |\delta|$ since otherwise $U_{\gamma,\delta} = \varnothing$ and the statement is trivial. The condition $|\gamma\alpha| = |\gamma| + |\alpha|$ means that the last symbol in γ doesn't cancel with the first symbol of α. Thus, for any $r \in U_{\gamma,\delta}$, the value of $r(\gamma\alpha)$ is equal to $\delta\alpha'$ for some $\alpha' \in \Gamma$ with $|\alpha'| = |\alpha|$. The condition $\alpha \perp \delta$ ensures that $r^{-1}(\alpha) \perp \gamma$ and hence the value of α' is independent from that of $r^{-1}(\alpha)$, i.e. whatever $r^{-1}(\alpha)$ is, there are exactly $3^{|\alpha|}$ possibilities for the value of α'. Hence the lemma follows. □

Proposition 13.4. *For a.e. $r \in R_C$, $T_C(r)$ and r are not E_C-equivalent. In particular, $[F_C : E_C] = 2$ a.e.*

Proof. Let $A = \{r \in R_C : T_C(r) E r\}$. For fixed $r \in A$, there is $\gamma \in \Gamma$ such that $T_C(r) = \gamma \cdot^\theta r$. Thus, for all $\alpha \in \Gamma$, we have $r(\gamma^{-1})^{-1} r(\gamma^{-1}\alpha) = r^{-1}(\alpha)$, and hence, $r(\gamma^{-1}\alpha) = r(\gamma^{-1}) r^{-1}(\alpha)$. Therefore, $r \in \bigcap_{\alpha \in \Gamma} A_{\gamma^{-1}, \delta, \alpha}$, where $\delta = r(\gamma^{-}1)$. Thus,

$$A \subseteq \bigcup_{\gamma, \delta \in \Gamma} \bigcap_{\alpha \in \Gamma} A_{\gamma, \delta, \alpha},$$

so it is enough to show that for fixed $\gamma, \delta \in \Gamma$, $\bigcap_{\alpha \in \Gamma} A_{\gamma, \delta, \alpha}$ is null. To this end, take $c \in \{a, a^{-1}, b, b^{-1}\} \setminus \{\delta(0), \gamma(|\gamma| - 1)\}$, and put $\alpha_n = c^n$. This ensures that $\alpha = \alpha_n$ satisfies the hypothesis of Lemma 13.3 and hence $\mu(A_{\gamma, \delta, \alpha_n}) \leq 3^{-n}$, which implies that $\bigcap_{\alpha \in \Gamma} A_{\gamma, \delta, \alpha}$ is null. □

Viewing Γ as a group of measure preserving Borel automorphisms of R_C, we let Δ be the group generated by $\Gamma \cup \{T_C\}$. We will see shortly that Γ has infinite index inside Δ and hence the action of Δ does not satisfy the hypothesis of Corollary 2.10.

Lemma 13.5. *For any nontrivial $\delta \in \Gamma$, $T_C \delta T_C \notin \Gamma < \mathrm{Aut}(R_C, \mu)$. In other words, for all $\gamma \in \Gamma$, the set $B_{\gamma, \delta} = \{r \in R_C : \delta \cdot^\theta T_C(r) = T_C(\gamma \cdot^\theta r)\}$ is not conull.*

Proof. Fix $\gamma, \delta \in \Gamma$. By Lemma 5.2 and the fact that the action $\theta \downharpoonright_{R_C}$ is almost free,

$$r \in B_{\gamma, \delta} \iff \delta = r(\gamma^{-1})^{-1} \iff r(\gamma^{-1}) = \delta^{-1} \iff r \in U_{\gamma^{-1}, \delta^{-1}}.$$

Thus, $B_{\gamma, \delta} = U_{\gamma^{-1}, \delta^{-1}}$ and hence is not conull. □

Proposition 13.6. *For nontrivial $\alpha \in \Gamma$, the cosets $(T_C \alpha T_C)\Gamma$ in Δ / Γ are pairwise distinct. In particular, $[\Delta : \Gamma] = \infty$.*

Proof. Just note that for $\alpha \neq \beta$, $(T_C \alpha T_C)^{-1}(T_C \beta T_C) = T_C \alpha^{-1} \beta T_C \notin \Gamma$, by the previous proposition. □

Nevertheless, by Corollary 12.5, F is treeable a.e.

Lastly, we note that E is ergodic as it follows from Theorem 1 in [LM92]. Thus, since $C_\mu(E) = 2$ and $L = E_{T_C}$ is a full (E, F)-link, we get the following (somewhat surprising) corollary from Lemma 11.1:

Corollary 13.7. *There is an almost free action $\alpha : \mathbb{F}_2 \curvearrowright R_C$ that generates E_C and commutes with T_C.*

14 A more natural universal treeable-by-i pair

The universal treeable-by-i pair constructed in Section 9 is somewhat hard to work with as it is just a restriction of a shift action to a set satisfying some conditions. In this section, we give a more concrete and illuminating example of such a pair.

For every $i \geq 2$ and every countable group Γ, we will define a Polish space $X_{\Gamma,i}$ with a continuous free action $\lambda : \Gamma \curvearrowright X_{\Gamma,i}$ and a homeomorphism $T_{\Gamma,i} : X_{\Gamma,i} \to X_{\Gamma,i}$ that is index-i over the orbit equivalence relation $E_{\Gamma,i}$ of the action λ. The system $[E_{\Gamma,i}, T_{\Gamma,i}]$ will be universal among its peers in the following sense:

Theorem 14.1. *Let Z be a standard Borel space equipped with a free Borel action of Γ and let T be a Borel automorphism of Z that is index-i over E_Γ^X, where E_Γ^X is the orbit equivalence relation of the action of Γ on Z. Then there is a Γ-equivariant embedding $\epsilon : Z \hookrightarrow X_{\Gamma,i}$ such that $T_{\Gamma,i} \circ \epsilon = \epsilon \circ T$.*

Let $F_{\Gamma,i}$ be the join of $E_{\Gamma,i}$ and $L_{T_{\Gamma,i}}$. Thus $(E_{\Gamma,i}, F_{\Gamma,i})$ is an index-i pair. Let

$$\biguplus_{j\leq i}(E_{\Gamma,j}, F_{\Gamma,j}) = (\biguplus_{j\leq i} E_{\Gamma,j}, \biguplus_{j\leq i} F_{\Gamma,j})$$

denote the disjoint union of the pairs $(E_{\Gamma,j}, F_{\Gamma,j})$, $j < i$. Note that if Γ is a free group, then $E_{\Gamma,i}$ is treeable since the action λ is free. Thus, $\biguplus_{j\leq i} E_{\Gamma,j}$ is treeable as well. Moreover, Theorem 14.1 implies the following

Theorem 14.2. *For any treeable-by-i pair (E, F), $(E, F) \sqsubseteq_B \biguplus_{j\leq i}(E_{\mathbb{F}_2,j}, F_{\mathbb{F}_2,j})$.*

We will prove both of these theorems below after we construct $X_{\Gamma,i}$, $\lambda : \Gamma \curvearrowright X_{\Gamma,i}$ and $\mathbb{T}_{\Gamma,i}$. First we give an outline of the construction. Take $X = 2^{\mathbb{F}_2}$, although it is not essential for the construction which uncountable standard Borel space we take since any two are Borel isomorphic. Let $<$ be a lexicographic order on X; it is a linear order and is open as a subset of X^2. Put

$$(X)^i = \{(x_j)_{j<i} \in X^i : x_j \neq x_k \text{ if } j \neq k\}.$$

Recall the group $R = R(\Gamma)$ and the action $\theta : \Gamma \curvearrowright R$. We will abuse the notation and denote the product action of Γ on R^{i-1} by θ as well; that is,

$$\gamma \cdot^\theta \vec{r} = (\gamma \cdot^\theta r_1, \gamma \cdot^\theta r_2, ..., \gamma \cdot^\theta r_{i-1}),$$

for $\gamma \in \Gamma$ and $\vec{r} = (r_1, r_2, ..., r_{i-1}) \in R^{i-1}$.

Here are the steps of the construction:

1. Define a continuous action $\tau : \Gamma \curvearrowright ((X)^i)^\Gamma$ that combines the shift action with a "twist".
2. Using the actions τ and θ, define a continuous action $\lambda : \Gamma \curvearrowright (((X)^i)^\Gamma \times R^{i-1})$ and let E_λ denote the induced orbit equivalence relation.
3. Define a homeomorphism S of $((X)^i)^\Gamma \times R^{i-1}$.
4. Take a certain G_δ subset Y of $((X)^i)^\Gamma$ invariant under the action τ, and set $X_{\Gamma,i} = Y \times R^{i-1}$.
5. Finally, put $E_{\Gamma,i} = E_\lambda \restriction_{X_{\Gamma,i}}$ and $T_{\Gamma,i} = S \restriction_{X_{\Gamma,i}}$.

Before proceeding with the construction, we define an embedding $\epsilon : Z \hookrightarrow ((X)^i)^\Gamma \times R^{i-1}$ for a given free Borel action of Γ on a standard Borel space Z together with a Borel automorphism T of Z that is index-i over $E = E^Z_\Gamma$. This will show how to define the action $\lambda : \Gamma \curvearrowright (((X)^i)^\Gamma \times R^{i-1})$ and the homeomorphism S of $((X)^i)^\Gamma \times R^{i-1}$, so that ϵ is Γ-equivariant and $S \circ \epsilon = \epsilon \circ T$, i.e. satisfies the conclusion of Theorem 14.1.

First, assume without loss of generality that $Z = X$. As usual let L_T denote the equivalence relation on X induced by the action of T, and let F denote the join of E and L_T. The following are auxiliary functions needed to define the desired embedding ϵ:

(i) $t : F \to X$ defined by $(x, y) \mapsto$ the unique element in $[x]_{L_T} \cap [y]_{E^X_\Gamma}$.

(ii) $l : X \to (X)^i$ defined by $x \mapsto (x, x_1, x_2, ..., x_{i-1})$, where $\{x, x_1, x_2, ..., x_{i-1}\} = [x]_{L_T}$ and $x_1, x_2, ..., x_{i-1}$ are in the increasing <-order. Thus, $l(x)(0) = x$ and $l(x)(j) = x_j$, for $j = 1, 2, ..., i-1$.

(iii) $q : E \to (X)^i$ by putting $q(x, y)(j) = t(y, l(x)(j))$, for $j < i$. Note that $q(x, x) = l(x)$.

(iv) $g : X \to ((X)^i)^\Gamma$ by $x \mapsto (q(x, \gamma \cdot x))_{\gamma \in \Gamma}$.

(v) $h : X \to R^{i-1}$ by putting $x \to (r_1, r_2, ..., r_{i-1})$, where $r_j(\gamma) =$ the unique $\delta \in \Gamma$ such that

$$\delta \cdot l(x)(j) = t(\gamma \cdot x, l(x)(j)),$$

for $\gamma \in \Gamma$. The uniqueness here is because the action of Γ on X is free.

Finally define $\epsilon : X \to (((X)^i)^\Gamma \times R^{i-1})$ by $x \mapsto (g(x), h(x))$. Clearly ϵ is injective since $g(x)(e)(0) = x$, where $e \in \Gamma$ is the identity of Γ.

Now we proceed with the construction of $\lambda\Gamma \curvearrowright (((X)^i)^\Gamma \times R^{i-1})$ and $S \in \mathrm{Aut}(((X)^i)^\Gamma \times R^{i-1})$.

14.1 Defining $\tau : \Gamma \curvearrowright ((X)^i)^\Gamma$

Let $s : S_\infty(\Gamma) \curvearrowright ((X)^i)^\Gamma$ denote the generalized shift action of $S_\infty(\Gamma)$ on $((X)^i)^\Gamma$ defined by

$$g \cdot^s f(\alpha) = f(g^{-1}(\alpha)),$$

for $g \in S_\infty(\Gamma)$, $f \in ((X)^i)^\Gamma$ and $\alpha \in \Gamma$. As previous sections, we consider Γ as a subgroup of $S_\infty(\Gamma)$ by letting $\gamma(\alpha) = \gamma\alpha$, for $\gamma, \alpha \in \Gamma$. Thus, if $g = \gamma \in \Gamma$ in the above definition, we get the usual shift action

$$\gamma \cdot^s f(\alpha) = f(\gamma^{-1}\alpha).$$

First, for each $\sigma \in \Sigma\{0, 1, 2, ..., i-1\}$ and $\vec{x} = (x_j)_{j<i} \in (X)^i$, put

$$\sigma'(x_0, x_1, ..., x_{i-1}) = (x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(i-1)}).$$

We also apply the above definition of σ' to $\sigma \in \Sigma\{1,2,...,i-1\}$ treating it as an element of $\Sigma\{0,1,2,...,i-1\}$ that fixes 0. We further define $\sigma'' : ((X)^i)^\Gamma \to ((X)^i)^\Gamma$ by

$$\sigma''(f)(\alpha) = \sigma' f(\alpha),$$

for $f \in ((X)^i)^\Gamma$ and $\alpha \in \Gamma$.

Next, for each $\vec{x} = (x_j)_{j<i} \in (X)^i$, let $\pi(\vec{x})$ denote the unique permutation $\sigma \in \Sigma\{1,2,...,i-1\}$ for which $(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(i-1)})$ is increasing (with respect to the order $<$). Note that the value of $\pi(\vec{x})$ does not depend on x_0. Also note that $\pi : (X)^i \to \Sigma\{1,2,...,i-1\}$ is continuous because $<$ is an open subset of X^2.

Finally, define the action $\tau : \Gamma \curvearrowright ((X)^i)^\Gamma$ by setting

$$\gamma \cdot^\tau f = \pi(\gamma \cdot^s f(e))''(\gamma \cdot^s f) = \pi(f(\gamma^{-1}))''(\gamma \cdot^s f),$$

for each $\gamma \in \Gamma$ and $f \in ((X)^i)^\Gamma$. To see that this is indeed an action, first note that for any $\sigma \in \Sigma\{1,2,...,i-1\}$, $f \in ((X)^i)^\Gamma$ and $\vec{x} \in (X)^i$, we have:

(a) σ'' and the shift action of γ on f commute, i.e. $\sigma''(\gamma \cdot^s f) = \gamma \cdot^s \sigma''(f)$;

(b) $\pi(\sigma'\vec{x}) = \pi(\vec{x})\sigma^{-1}$.

Now fix $\gamma, \delta \in \Gamma$, $f \in ((X)^i)^\Gamma$, denote $\sigma = \pi(f(\gamma^{-1}))$, and compute

$$\begin{aligned}
\delta \cdot^\tau (\gamma \cdot^\tau f) &= \pi(\gamma \cdot^\tau f(\delta^{-1}))''(\delta \cdot^s (\gamma \cdot^\tau f)) \\
&= \pi(\sigma' f(\gamma^{-1}\delta^{-1}))''(\delta \cdot^s (\sigma''(\gamma \cdot^s f))) \\
&= (\pi(f((\delta\gamma)^{-1}))\sigma^{-1})''(\sigma''(\delta \cdot^s \gamma \cdot^s f)) \\
&= (\pi(f((\delta\gamma)^{-1}))\sigma^{-1}\sigma)''((\delta\gamma) \cdot^s f) \\
&= \pi(f((\delta\gamma)^{-1}))''((\delta\gamma) \cdot^s f) \\
&= (\delta\gamma) \cdot^\tau f(\alpha).
\end{aligned}$$

Clearly, this action is continuous since so are all of the functions involved. When $i \leq 2$, the action τ coincides with the shift action of Γ on $((X)^2)^\Gamma$.

14.2 Defining $\lambda : \Gamma \curvearrowright ((X)^i)^\Gamma \times R^{i-1}$

For $\sigma \in \Sigma\{1, 2, ..., i-1\}$ and $\vec{r} = (r_1, r_2, ..., r_{i-1}) \in R^{i-1}$, we abuse the notation σ' and apply it to $\vec{r}$ as well:

$$\sigma'(r_1, r_2, ..., r_{i-1}) = (r_{\sigma(1)}, r_{\sigma(2)}, ..., r_{\sigma(i-1)}).$$

Define the action $\lambda : \Gamma \curvearrowright (((X)^i)^\Gamma \times R^{i-1})$ by setting

$$\gamma \cdot^\lambda (f, \vec{r}) = (\gamma \cdot^\tau f, \pi(\gamma \cdot^s f(e))'\vec{r}),$$

for $f \in ((X)^i)^\Gamma$ and $\vec{r} \in R^{i-1}$ (assuming that the indices of $\vec{r}$ start with 1). Unraveling the definitions, we get

$$\gamma \cdot^\lambda (f, \vec{r}) = (\sigma''(\gamma \cdot^s f), \sigma'\vec{r}),$$

where $\sigma = \pi(f(\gamma^{-1}))$.

To verify that this is an action, fix $\gamma \in \Gamma$, $(f, \vec{r}) \in ((X)^i)^\Gamma \times R^{i-1}$, denote $\sigma = \pi(f(\gamma^{-1}))$, and compute

$$\begin{aligned}
\delta \cdot^\lambda (\gamma \cdot^\lambda (f, \vec{r})) &= \delta \cdot^\lambda (\gamma \cdot^\tau f, \sigma'\vec{r}) \\
&= (\delta \cdot^\tau (\gamma \cdot^\tau f), \pi((\gamma \cdot^\tau f)(\delta^{-1}))'\sigma'\vec{r})) \\
&= ((\delta\gamma) \cdot^\tau f, (\pi(\sigma' f(\gamma^{-1}\delta^{-1}))\sigma)'\vec{r}) \\
&= ((\delta\gamma) \cdot^\tau f, (\pi(f(\gamma^{-1}\delta^{-1}))\sigma^{-1}\sigma)'\vec{r}) \\
&= ((\delta\gamma) \cdot^\tau f, \pi(f((\delta\gamma)^{-1}))'\vec{r}) \\
&= (\delta\gamma) \cdot^\lambda (f, \vec{r}).
\end{aligned}$$

We denote by E_λ the orbit equivalence relation induced by the action λ.

14.3 Defining $S : ((X)^i)^\Gamma \times R^{i-1} \to ((X)^i)^\Gamma \times R^{i-1}$

For $\vec{x} = (x_j)_{j<i} \in (X)^i$, let $\mathbf{j}(\vec{x})$ be equal to unique $j < i$ such that

$$x_j = \begin{cases} \min\{x_k : k < i\} & \text{if } x_0 = \max\{x_k : k < i\} \\ \min\{x_k : x_k > x_0, 0 < k < i\} & \text{otherwise} \end{cases}.$$

Furthermore, for every $j < i$, let $\sigma_j \in \Sigma\{0, 1, ..., i-1\}$ be the transposition that swaps 0 and j, and put

$$\rho(\vec{x}) = \pi(\sigma'_{\mathbf{j}(\vec{x})}\vec{x}).$$

Thus, when the permutation $\rho(\vec{x})$ is applied to some $\vec{y} \in (X)^i$, it permutes the coordinates $1, 2, ..., i-1$ of $\vec{y}$ by the unique permutation that is necessary to order the vector $(x'_1, x'_2, ..., x'_{i-1}) = (x_1, x_2, ..., x_{\mathbf{j}(\vec{x})-1}, x_0, x_{\mathbf{j}(\vec{x})+1}, ..., x_{i-1})$ in the increasing order with respect to $<$.

We need one more gadget before we can define S. For fixed $\vec{r} = (r_1, r_2, ..., r_{i-1}) \in R^{i-1}$, put

(i) $r_0 = \mathrm{id}_\Gamma$;

(ii) $r_{jk} = r_k \circ r_j^{-1}$, for all $j, k < i$.

Furthermore, define

$$\mathrm{swap}_j(\vec{r}) = (r_{j1}, r_{j2}, ..., r_{j(j-1)}, r_{j0}, r_{j(j+1)}, ..., r_{j(i-1)}),$$

for $j = 1, 2, ..., i-1$.

Finally, we define $S : ((X)^i)^\Gamma \times R^{i-1} \to ((X)^i)^\Gamma \times R^{i-1}$ by setting

$$S(f, \vec{r}) = ((\rho(f(e))\sigma_{\mathbf{j}(f(e))})''(r_j \cdot^s f), \rho(f(e))'\mathrm{swap}_{\mathbf{j}(f(e))}(\vec{r})),$$

for $(f, \vec{r}) \in ((X)^i)^\Gamma \times R^{i-1}$.

It is tedious to check that $S^i = \mathrm{id}$ and thus S is a homeomorphism of $((X)^i)^\Gamma \times R^{i-1}$.

14.4 Finalizing the construction

It remains to shrink the space $((X)^i)^\Gamma$ to make the action τ free and S index-i over E_λ (and not less). Put

$$Y = \{f \in ((X)^i)^\Gamma : \forall \text{ distinct } (\gamma, j), (\delta, k) \in \Gamma \times i \ (f(\gamma)(j) \neq f(\delta)(k))\}.$$

and

$$X_{\Gamma,i} = Y \times R^{i-1},$$
$$E_{\Gamma,i} = E_\lambda \restriction_{X_{\Gamma,i}},$$
$$T_{\Gamma,i} = S \restriction_{X_{\Gamma,i}} .$$

Note that the action $\tau \restriction_Y : \Gamma \curvearrowright Y$ is free and hence so is the action $\lambda \restriction_{X_{\Gamma,i}} : \Gamma \curvearrowright X_{\Gamma,i}$. Also, it is obvious that $T_{\Gamma,i}$ is index-i over $E_{\Gamma,i}$. Thus, we are finally ready to prove Theorems 14.1 and 14.2.

Proof of Theorem 14.1. Given a free Borel action of Γ on X together with a Borel automorphism T of X that is index-i over E^X_Γ, we define the embedding $\epsilon : X \hookrightarrow ((X)^i)^\Gamma \times R^{i-1}$ as above. Note that $\epsilon(X) \subseteq X_{\Gamma,i}$. The fact that ϵ is equivariant and that $\epsilon \circ T = T_{\Gamma,i} \circ \epsilon$ is a tedious verification that we will leave out as the action λ and the automorphism S were defined exactly so that this would happen. □

Proof of Theorem 14.2. We prove by induction on i. The case $i = 1$ is trivial, so assume that $i > 1$ and the statement is true for all $j < i$. Let (E, F) be a treeable-by-i pair. By Proposition 9.1, we may assume that $E = \mathbb{E}_{\infty T} \restriction_Z$ for some $\mathbb{E}_{\infty T}$-invariant Borel subset Z of $\mathrm{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$. Let T be as in Proposition 3.7 for (E, F) and put

- $Z_1 = [Z \smallsetminus \mathrm{dom}(T)]_E$,
- $Z_2 = [Z_1]_F \smallsetminus Z_1$,
- $Z_0 = Z \smallsetminus [Z_1]_F$,
- $E_k = E \restriction_{Z_k}$, for $k = 0, 1, 2$,
- $F_k = F \restriction_{Z_k}$, for $k = 0, 1, 2$.
- $T_0 = T \restriction_{Z_0}$.

Each Z_k is E-invariant, and Z_0 is F-invariant. Thus T_0 is a (full) Borel automorphism of Z_0 that has index-i over E_0, and E_0 is induced by a Borel free action of $\mathbb{F}_2$ (simply the shift action). Therefore, Theorem 14.1 provides an embedding $\epsilon : (E_0, F_0) \sqsubseteq_B^* (E_{\Gamma,i}, F_{\Gamma,i})$.

As for the pairs (E_1, F_1) and (E_2, F_2), note that $[F_k : E_k] < i$, for $k = 1, 2$, and thus, by induction, we have

$$(E_k, F_k) \sqsubseteq_B \biguplus_{j \leq i-1} (E_{\mathbb{F}_2,j}, F_{\mathbb{F}_2,j}).$$

It also follows from the construction that

$$(E_k, F_k) \sqsubseteq_B^* \biguplus_{j \leq i-1} (E_{\mathbb{F}_2,j}, F_{\mathbb{F}_2,j})$$

and the images of (E_1, F_1) and (E_2, F_2) in $\biguplus_{j \leq i-1} (E_{\mathbb{F}_2,j}, F_{\mathbb{F}_2,j})$ are disjoint. Combining all three embeddings, we get

$$(E, F) \sqsubseteq_B^* \biguplus_{j \leq i} (E_{\mathbb{F}_2,j}, F_{\mathbb{F}_2,j}),$$

as desired. Note that we get $\sqsubseteq_B^*$ only after assuming that E is defined on some invariant subset of $\mathrm{Free}(\mathbb{F}_2, 2^{\mathbb{F}_2})$, and that's why the general statement of the theorem is merely with $\sqsubseteq_B$. □

15 The case $i = 2$

It is worth considering the special case of $i = 2$ separately, as the construction of $(E_{\Gamma,2}, T_{\Gamma,2})$ is considerably simpler in this case and hence easier to investigate. Recall that $X_{\Gamma,2} = Y \times R$, where Y is a certain G_δ subset of $((X)^2)^\Gamma$, invariant under the action $\tau : \Gamma \curvearrowright ((X)^2)^\Gamma$. Note that the action τ coincides with the shift action $s : \Gamma \curvearrowright ((X)^2)^\Gamma$ and thus

$$\gamma \cdot^\lambda (f, r) = (\gamma \cdot^s f, \gamma \cdot^\theta r),$$

for $\gamma \in \Gamma$, $f \in ((X)^2)^\Gamma$ and $r \in R(\Gamma)$.

Also, for all $f = (x_\gamma, y_\gamma)_{\gamma \in \Gamma} \in Y$ and $r \in R(\Gamma)$, we have

$$T_{\Gamma,2}(f, r) = (r \cdot^s \tilde{f}, r^{-1}), \tag{$*$}$$

where $\tilde{f} = (y_\gamma, x_\gamma)_{\gamma \in \Gamma}$. In particular, $T_{\Gamma,2}$ is an involution.

From now on, we fix $\Gamma = \mathbb{F}_2$. As an immediate corollary of Theorem 14.2, we get that the following:

Corollary 15.1. *All treeable-by-2 equivalence relations are treeable if and only if $F_{\mathbb{F}_2,2}$ is treeable.*

Thus, when working on the question of weather treeable-by-finite equivalence relations are treeable, one has to understand what happens to $F_{\mathbb{F}_2,2}$.

To simplify the notation below, we use Γ for $\mathbb{F}_2$, E for $E_{\Gamma,2}$, F for $F_{\Gamma,2}$ and T for $T_{\Gamma,2}$. Recall that $X_{\mathbb{F}_2,2} = Y \times R$, for a certain subset Y of $((X)^2)^\Gamma$.

Similar to Lemma 5.2, we have the following interaction between the action λ of Γ and the action of T on $X_{\mathbb{F}_2,2}$:

Lemma 15.2. *For all $\gamma \in \Gamma$ and $(f,r) \in Y \times R$, $T(\gamma \cdot^\lambda (f,r)) = \delta \cdot^\lambda T(f,r)$, where $\delta = r(\gamma^{-1})^{-1}$.*

Proof. Using Lemma 5.2, we compute:

$$\begin{aligned}
T(\gamma \cdot^\lambda (f,r)) &= T(\gamma \cdot^s f, \gamma \cdot^\theta r) \\
&= ((\gamma \cdot^\theta r) \cdot^s (\gamma \cdot^s \tilde{f}), (\gamma \cdot^\theta r)^{-1}) \\
&= ((r(\gamma^{-1})^{-1} \circ r \circ \gamma^{-1} \circ \gamma) \cdot^s \tilde{f}, \delta \cdot^\theta r^{-1}) \\
&= (\delta \cdot^s (r \cdot^s \tilde{f}), \delta \cdot^\theta r^{-1}) \\
&= \delta \cdot^\lambda (r \cdot^s \tilde{f}, r^{-1}) \\
&= \delta \cdot^\lambda T(f,r).
\end{aligned}$$

□

Below we consider the familiar subgroups $H(\mathbb{F}_2)$ and $R_C(\mathbb{F}_2)$ of $R(\mathbb{F}_2)$ and investigate the question of whether the restrictions of $F_{\mathbb{F}_2,2}$ to $Y \times H(\mathbb{F}_2)$ and $Y \times R_C(\mathbb{F}_2)$ are treeable. Below, we omit writing $\mathbb{F}_2$ and simply write H, R_C and R.

15.1 The treeability of $F_{\mathbb{F}_2,2}$ on $Y \times H$

First note that H is countable since $\mathbb{F}_2$ is finitely generated. For fixed $h \in H$, note that $Y_h = Y \times \{h\}$ is Γ-invariant since h is a fixed point of the action $\theta : \Gamma \curvearrowright R$ by Lemma 12.2. Thus, $Z_h = Y_h \cup Y_{h^{-1}}$ is Γ-invariant and also T-invariant.

Lemma 15.3. *For $h \in H$ such that $h^{-1} \neq h$, $F \restriction_{Z_h}$ is smooth over $E \restriction_{Z_h}$, and hence treeable.*

Proof. For $h^{-1} \neq h$, $F \restriction_{Y_h} = E \restriction_{Y_h}$ because $T(Y_h) = Y_{h^{-1}}$ and $Y_h \cap Y_{h^{-1}} = \varnothing$. Thus, the set Y_h selects exactly one E-class from each F-class in Z_h, and hence, $F \restriction_{Z_h}$ is smooth over $E \restriction_{Z_h}$. In particular, $F \restriction_{Z_h} \leq_B E \restriction_{Z_h}$, and therefore, $F \restriction_{Z_h}$ is treeable since so is $E \restriction_{Z_h}$. □

If $h \in H$ is an involution, i.e. $h^{-1} = h$, then $Y_h = Z_h$. Let $T_h = T \restriction_{Y_h}$.

Lemma 15.4. *If $h \in H$ is an involution, then the action of T_h normalizes that of Γ on Y_h.*

Proof. Viewing Γ as a group of homeomorphisms of Y_h, we show that $T_h \circ \gamma = h(\gamma) \circ T_h$, for every $\gamma \in \Gamma$. Indeed, for $f \in Y$, Lemma 15.2 implies that

$$T_h(\gamma \cdot^{\lambda} (f, h)) = \delta \cdot^{\lambda} T_h(f, h),$$

where $\delta = h(\gamma^{-1})^{-1} = h(\gamma)$. □

Proposition 15.5. *$F \restriction_{Y \times H}$ is treeable.*

Proof. Since H is countable, it is enough to show that $F \restriction_{Z_h}$ for each $h \in H$. If $h \neq h^{-1}$, then $F \restriction_{Z_h}$ is treeable by Lemma 15.3. If $h = h^{-1}$, then by Lemma 15.4, T_h normalizes the action $\lambda \restriction_{Z_h} : \Gamma \curvearrowright Z_h$. Thus, since $F \restriction_{Z_h} = E \restriction_{Z_h} \vee L_{T_h}$, Corollary 10.4 implies that $F \restriction_{Z_h}$ is treeable. □

15.2 The treeability of $F_{\mathbb{F}_2,2}$ on $Y \times R_C$

Put $E_C = E \restriction_{Y \times R_C}$, $F_C = F \restriction_{Y \times R_C}$ and $T_C = T \restriction_{Y \times R_C}$.

Let a, b be the free generators of $\Gamma = \mathbb{F}_2$ and put $S = \{a^{\pm 1}, b^{\pm 1}\}$. Let $\mathcal{G}$ be the standard treeing of E induced by the free action $\lambda : \Gamma \curvearrowright Y \times R$; that is: for $u, v \in Y \times R$,

$$(u, v) \in \mathcal{G} \iff \exists \gamma \in S (\gamma \cdot^{\lambda} u = v).$$

Put $\mathcal{G}_C = \mathcal{G} \restriction_{Y \times R_C}$ and hence $\mathcal{G}_C$ is a treeing of E_C.

Lemma 15.6. *T_C commutes with $\mathcal{G}_C$.*

Proof. Fix $(u, v) \in \mathcal{G}_C$ and hence $v = \gamma \cdot^{\lambda} u$ for some $\gamma \in S$. We need to show that $(T_C(u), T_C(v)) \in \mathcal{G}_C$. Let $u = (f, r)$ and note that by Lemma 15.2, $T_C(v) = \delta \cdot^{\lambda} T_C(u)$, where $\delta = r(\gamma^{-1})^{-1}$. But $r \in R_C$, so $r(S) = S$, and thus, $\delta \in S$. Hence, $(T_C(u), T_C(v)) \in \mathcal{G}_C$. □

Proposition 15.7. *F_C is treeable.*

Proof. Let L_{T_C} be the (E_C, F_C)-link induced by T_C. Lemma 15.6 implies that L_{T_C} commutes with $\mathcal{G}_C$. Thus, by our criterion for treeability, namely: (2)⇒(1) of Proposition 8.3, F_C is treeable. □

15.3 Summary

Let $R_0 = R \setminus (H \cup R_C)$. Putting together Corollary 15.1 and Propositions 15.5 and 15.7, we get

Corollary 15.8. *All treeable-by-2 equivalence relations are treeable if and only if $F_{\mathbb{F}_2,2} \restriction_{Y \times R_0}$ is treeable.*

Speculation. The question as to whether $F_{\mathbb{F}_2,2} \restriction_{Y \times R_0}$ is treeable is open even measure-theoretically. In fact, I do not know whether there is a λ-invariant probability measure on $Y \times R_0$. Clearly Y has one, namely the product measure, and thus, the existence of a λ-invariant probability measure on $Y \times R_0$ is equivalent to the existence of a θ-invariant probability measure on R_0. Denoting by E_θ^0 the orbit equivalence relation induced by the action $\theta \restriction_{R_0} : \Gamma \curvearrowright R_0$, the latter statement is equivalent to the incompressibility of E_θ^0, which seems plausible.

Part 3

Complexity measures for recursive programs

CHAPTER I

Main definitions and results

1 Introduction to recursive programs and the main result

Recursive programs are a model of computation introduced by McCarthy in [McC63]. The syntax and semantics of these programs are essentially the same as that of the programming language `C`. These programs come with a specified list Φ of primitive functions (may be constant) such as $\Phi_a = \{0, 1, +, \cdot\}$ or $\Phi_P = \{0, S, +\}$, so we refer to the programs with primitives from Φ as Φ-programs. It was shown in [McC63] that the partial functions that can be realized by Φ_a-programs are exactly the Turing computable partial functions. Each Φ-program E consists of the main function $\mathtt{f}_0(x_1, ..., x_{k_0})$ (the head of the program) and various other functions $\mathtt{f}_j(x_1, ..., x_{k_j})$, $j \leq m$. In the body of each function, one can recursively perform the following operations:

(i) `if ... then ... else ...`,

(ii) call $\mathtt{f}_j(x_1, ..., x_{k_j})$, for $j \leq m$,

(iii) call $\phi(x_1, ..., x_n)$, for $\phi \in \Phi$.

In this part of the thesis, we consider different measures of complexity for Φ-programs, introduced in [Mos], and explore the relations between them. One such measure is the *sequential logical complexity* $l^s_E(\vec{a})$ of a Φ-program E on input $\vec{a}$ on which E halts (write $E(\vec{a})\downarrow$). Roughly speaking, $l^s_E(\vec{a})$ is the number of operations performed by the program E on input $\vec{a}$. Another such measure is the *sequential call complexity* $c^s_E(\vec{a})$ of a Φ-program E on input $\vec{a}$ with $E(\vec{a})\downarrow$; it is equal to the number of calls to primitives (i.e. functions from Φ)

during the run of $E(\vec{a})$ (disregarding the operations (i)-(ii)). By definition, $c^s_E(\vec{a}) \le l^s_E(\vec{a})$, and it was asked by Moschovakis in [Mos] whether this inequality can be reversed for a fixed program, up to some constants that depend only on the code of the program. The main result of this part of the thesis is a positive answer to this question, and here is (a somewhat informal version of) the statement of the main theorem:

Theorem. *Let E be a Φ-program. There exists a constant K (depending only on E) such that for every input $\vec{a}$, we have*

$$l^s_E(\vec{a}) \le K c^s_E(\vec{a}) + K,$$

provided that $E(\vec{a})\downarrow$.

This theorem basically says that one cannot do much without having to call a primitive function, and thus, the actual complexity of an algorithm comes from the number of calls to primitives and not the logical operations. So it is no surprise that many of the methods for obtaining complexity lower bounds actually provide bounds for the number of calls to primitives. Such examples are given in [MvdD04] and [MvdD09].

The key to the proof of the above theorem is introducing a new (auxiliary) measure of complexity (see Section 7). Using the same technique, we also obtain (in Section 9) the analog of the above theorem for the *parallel logical* and *parallel call* complexity measures.

In this chapter, we rigorously define the syntax and semantics of recursive programs, as well as the relevant complexity measures, and we leave the statements of the results and their proofs for Chapter II.

This work stemmed out of a seminar run by Yiannis Moschovakis and it is also presented in [Mos]. It will appear as a section in Moschovakis's projected book on Recursion and Complexity. The exposition here closely follows [Mos].

2 The definition of recursive programs

Below we will define the notions of vocabulary, term, structure and interpretation as we do in second order logic, using partial functions instead of total functions.

Recall that a partial function from a set A to a set B is a usual function from a subset of A to B. We denote this by $f : A \rightharpoonup B$, and we denote the domain of f by $\mathrm{dom}(f)$. We also write $f(x)\downarrow$ to mean that $x \in \mathrm{dom}(f)$. Denote by $\mathrm{Partial}(A, B)$ the set of all partial functions from A to B (including the empty function).

2.1 Syntax

We first fix a (second order) vocabulary:

- Let 0 denote a special constant symbol (to be interpreted by structures);
- Let $\mathtt{v}_0, \mathtt{v}_1, ...$ be the symbols for (ordinary) variables;
- For every $k \in \mathbb{N}$, let $\mathtt{f}_0^k, \mathtt{f}_1^k, ...$ be the symbols for variable partial functions of arity k.

Definition 2.1. *A Φ-term M is defined recursively as follows:*

$$M \coloneqq 0 \mid \mathtt{v}_i \mid \mathtt{f}_i^k(M_1, ..., M_k) \mid \phi(M_1, ..., M_k) \mid (\mathtt{if}\ M_0 = 0\ \mathtt{then}\ M_1\ \mathtt{else}\ M_2),$$

where $\mathtt{v}_i$ is an ordinary variable, $\mathtt{f}_i^k$ is a function variable, $\phi \in \Phi$ is of arity k, $M_0, M_1, ..., M_k$ are terms, and $i, k \geq 1$.

For a term M, we write

$$M(\mathtt{v}_{i_0}, \mathtt{v}_{i_1}, ..., \mathtt{v}_{i_l}, \mathtt{f}_{j_0}^{k_0}, \mathtt{f}_{j_1}^{k_1}, ..., \mathtt{f}_{j_m}^{k_m})$$

to mean that the ordinary and function variables that appear in M are among $\mathtt{v}_{i_0}, \mathtt{v}_{i_1}, ..., \mathtt{v}_{i_l}$, $\mathtt{f}_{j_0}^{k_0}, \mathtt{f}_{j_1}^{k_1}, ..., \mathtt{f}_{j_m}^{k_m}$.

A recursive Φ-program is a system of recursive equations

$$E \quad : \quad \begin{cases} \mathtt{f}_E(\vec{\mathtt{v}}_0) = \mathtt{f}_0^{k_0}(\vec{\mathtt{v}}_0) & = \quad E_0(\vec{\mathtt{v}}_0, \mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, \ldots, \mathtt{f}_m^{k_m}) \\ \mathtt{f}_1^{k_1}(\vec{\mathtt{v}}_1) & = \quad E_1(\vec{\mathtt{v}}_1, \mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, \ldots, \mathtt{f}_m^{k_m}) \\ & \vdots \\ \mathtt{f}_m^{k_m}(\vec{\mathtt{v}}_m) & = \quad E_m(\vec{\mathtt{v}}_m, \mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, \ldots, \mathtt{f}_m^{k_m}) \end{cases},$$

where

- $\mathtt{f}_E$ is a special function symbol called the head of the program and we consider it equal to $\mathtt{f}_0$,
- $\mathtt{f}_0, \mathtt{f}_1, \ldots, \mathtt{f}_m$ are distinct function variables called the *recursive variables* of E, and we will denote the vector $(\mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, \ldots, \mathtt{f}_m^{k_m})$ by $\vec{\mathtt{f}}$,
- each $E_j(\vec{\mathtt{v}}_j, \vec{\mathtt{f}})$ is a Φ-term.

Example 2.2. The following recursive program encodes Bezout's algorithm for computing $\alpha(x,y)$ and $\beta(x,y)$ for $x, y \in \mathbb{Z}$ such that $\alpha(x,y)x + \beta(x,y)y = \gcd(x,y)$:

$$E : \begin{cases} \mathtt{f}_E(x,y) = \alpha(x,y) & = \quad (\mathtt{if}\ \operatorname{rem}(x,y) = 0\ \mathtt{then}\ 0\ \mathtt{else}\ \beta(y, \operatorname{rem}(x,y))) \\ \beta(x,y) & = \quad (\mathtt{if}\ \operatorname{rem}(x,y) = 0\ \mathtt{then}\ 1 \\ & \qquad \mathtt{else}\ \alpha(y, \operatorname{rem}(x,y)) - \operatorname{iq}(x,y)\beta(y, \operatorname{rem}(x,y))) \end{cases},$$

where Φ consists of the binary function symbols 1, rem, iq and −.

For a Φ-term M, let Subterms(M) denote the set of all subterms of M (i.e. substrings that are Φ-terms themselves). Now fix E as above and let $\vec{\mathtt{f}} = (\mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, \ldots, \mathtt{f}_m^{k_m})$ denote the vector of recursive variables of E. Put

$$\operatorname{Subterms}(E) = \bigcup_{j=0}^{m} \operatorname{Subterms}(E_j(\vec{\mathtt{v}}_j, \vec{\mathtt{f}})),$$

and let $t(E) = |\operatorname{Subterms}(E)|$. For Φ-term $M(\vec{\mathtt{v}}, \vec{\mathtt{f}}) \in \operatorname{Subterms}(E)$, $\vec{\mathtt{v}}$ can be assumed to be a subvector of $\vec{\mathtt{v}}_j$ for some $j \le m$, and we will assume so below. Thus, the length n of $\vec{\mathtt{v}}$ is at most $\max\limits_{0 \le j \le m} k_j$. This will be important later.

2.2 Semantics

Definition 2.3. *Given a signature Φ consisting of function symbols, a Φ-algebra is a structure $\mathfrak{A} = (A, 0^{\mathfrak{A}}, \{\phi^{\mathfrak{A}}\}_{\phi\in\Phi})$ in the signature $(0, \Phi)$ that interprets 0 as an element $0^{\mathfrak{A}} \in A$ and every $\phi \in \Phi$ as a partial function $\phi^{\mathfrak{A}} : A \rightharpoonup A$.*

We refer to the functions in $\Phi^{\mathfrak{A}}$ as the primitives of the Φ-algebra $\mathfrak{A}$.

We now define interpretation of Φ-terms. For a Φ-term $M(\vec{\mathtt{v}}, \mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, ..., \mathtt{f}_m^{k_m})$, where $\vec{\mathtt{v}} = (\mathtt{v}_1, ..., \mathtt{v}_n)$, and a structure $\mathfrak{A} = (A, 0^{\mathfrak{A}}, \{\phi^{\mathfrak{A}}\}_{\phi\in\Phi})$. Then the interpretation of M by $\mathfrak{A}$ is a partial function (map)

$$M^{\mathfrak{A}} : A^n \times \mathrm{Partial}(A^{k_0}, A) \times \mathrm{Partial}(A^{k_1}, A) \times ... \times \mathrm{Partial}(A^{k_m}, A) \to A$$

defined in the usual way by recursion on the complexity of M: we will spare the reader the rigorous definition as it is enough say that $M^{\mathfrak{A}}(\vec{a}, f_0, f_1, ..., f_m)$ is the value one obtains when substituting $\vec{\mathtt{v}}$ with $\vec{a}$ and $\mathtt{f}_j^{k_j}$ with f_j in $M(\vec{\mathtt{v}}, \mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, ..., \mathtt{f}_m^{k_m})$, for $\vec{a} \in A^n$, $f_j \in \mathrm{Partial}(A^{k_j}, A)$ and $j \le m$.

We also define a functional

$$F_M^{\mathfrak{A}} : \mathrm{Partial}(A^{k_0}, A) \times \mathrm{Partial}(A^{k_1}, A) \times ... \times \mathrm{Partial}(A^{k_m}, A) \rightharpoonup \mathrm{Partial}(A^n, A)$$

by setting

$$F_M^{\mathfrak{A}}(f_0, f_1, ..., f_m) = M^{\mathfrak{A}}(\,\cdot\,, f_0, f_1, ..., f_m),$$

for $f_j \in \mathrm{Partial}(A^{k_j}, A)$, $j \le m$.

Intuitively, it is clear that a Φ-program E defines a partial function, whose domain is the set of inputs on which E halts. Here we give a rigorous definition of this. Given a program E as above and a Φ-algebra $\mathfrak{A}$, define a system of recursive equations:

$$\begin{cases} f_0 &=& F_{E_0}^{\mathfrak{A}}(f_0, f_1, ..., f_m) \\ f_1 &=& F_{E_1}^{\mathfrak{A}}(f_0, f_1, ..., f_m) \\ & \vdots & \\ f_m &=& F_{E_m}^{\mathfrak{A}}(f_0, f_1, ..., f_m) \end{cases},$$

a solution to which is a tuple

$$(f_0, f_1, ..., f_m) \in \text{Partial}(A^{k_0}, A) \times \text{Partial}(A^{k_1}, A) \times ... \times \text{Partial}(A^{k_m}, A).$$

By Lemma 1B.1 (the Fixed Point Lemma) of [Mos], this system has a least (with respect to inclusion of partial functions) solution $\vec{f}^* = (f_0^*, f_1^*, ..., f_m^*)$ since the functionals $F_{E_j}^{\mathfrak{A}}$ are monotone and continuous. We declare f_0^* the partial function that E computes with respect to the Φ-algebra $\mathfrak{A}$, and we denote it by $f_E^{\mathfrak{A}}$. For $\vec{a} \in A^{k_0}$, we write $E^{\mathfrak{A}}(\vec{a}) \downarrow$ to mean that $f_E^{\mathfrak{A}}(\vec{a}) \downarrow$.

Fix E and $\mathfrak{A}$ as above and let $\vec{\mathtt{f}} = (\mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, ..., \mathtt{f}_m^{k_m})$ denote the vector of recursive variables of E. Given a $M(\vec{\mathtt{v}}, \vec{\mathtt{f}}) \in \text{Subterms}(E)$ and $\vec{a} \in A^n$, we call the term $M(\vec{a}, \vec{\mathtt{f}})$, obtained by substituting $\vec{\mathtt{v}}$ with $\vec{a}$, an $(\mathfrak{A}, E)$-term. We say that an $(\mathfrak{A}, E)$-term $M = M(\vec{a}, \vec{\mathtt{f}})$ converges (or is convergent) if $M^{\mathfrak{A}}(\vec{a}, \vec{f}^*) \downarrow$, where $\vec{f}^*$ is as above. We denote the value $M^{\mathfrak{A}}(\vec{a}, \vec{f}^*) \in A$ by $M_E^{\mathfrak{A}}$ (or just $\overline{M}$, if $\mathfrak{A}$ and E are understood). Finally, we denote the set of all convergent $(\mathfrak{A}, E)$-terms by $\text{Conv}(\mathfrak{A}, E)$.

For an $(\mathfrak{A}, E)$-term $M = M(\vec{a}, \vec{\mathtt{f}})$, $\vec{a} = (a_1, ..., a_n) \in A^n$, we call $a_1, ..., a_n$ the parameters of M and denote $\text{Param}(M) = \{a_1, ..., a_n\}$. We also put $\text{Param}_0(M) = \text{Param}(M) \cup \{0^{\mathfrak{A}}\}$. For a subset $B \subseteq A$, we say that M is a (B, E)-term if $\text{Param}_0(M) \subseteq B$.

3 Computation tree

For a set X and $u, v \in X^{<\infty} \setminus \{\varnothing\}$, we say that u is below v, and write $u \leq v$, if u is an initial segment of v; that is, $v = (v_1, v_2, ..., v_l)$ and $u = (v_1, v_2, ..., v_m)$ for some $m \leq l$ (note that $l, m \geq 1$). A tree T on a set X is a subset of $X^{<\infty} \setminus \{\varnothing\}$ that is closed downward, i.e. for all $u, v \in X^{<\infty} \setminus \{\varnothing\}$, $v \in T \wedge u \leq v$ implies $u \in T$. For a tree T on X and $x \in X$, we put

$$x\hat{\ }T = \{(x)\} \cup \{(x, x_1, ..., x_l) \in X^{<\infty} : (x_1, ..., x_l) \in T\}.$$

Also, for finite T, we define its depth depth(T) by setting

$$\text{depth}(T) = \max\{|u| : u \in T\} - 1,$$

where $|u|$ denotes the length of u, i.e. for $u \in X^l$, $|u| = l$.

Now fix a Φ-algebra $\mathfrak{A}$ and a Φ-program E. With each $M \in \mathrm{Conv}(\mathfrak{A}, E)$, we associate a tree $\mathcal{T}(M) = \mathcal{T}_E^{\mathfrak{A}}(M)$ on $\mathrm{Conv}(\mathfrak{A}, E)$ called the computation tree of M, which helps visualizing the process of computation (see Figure 1).

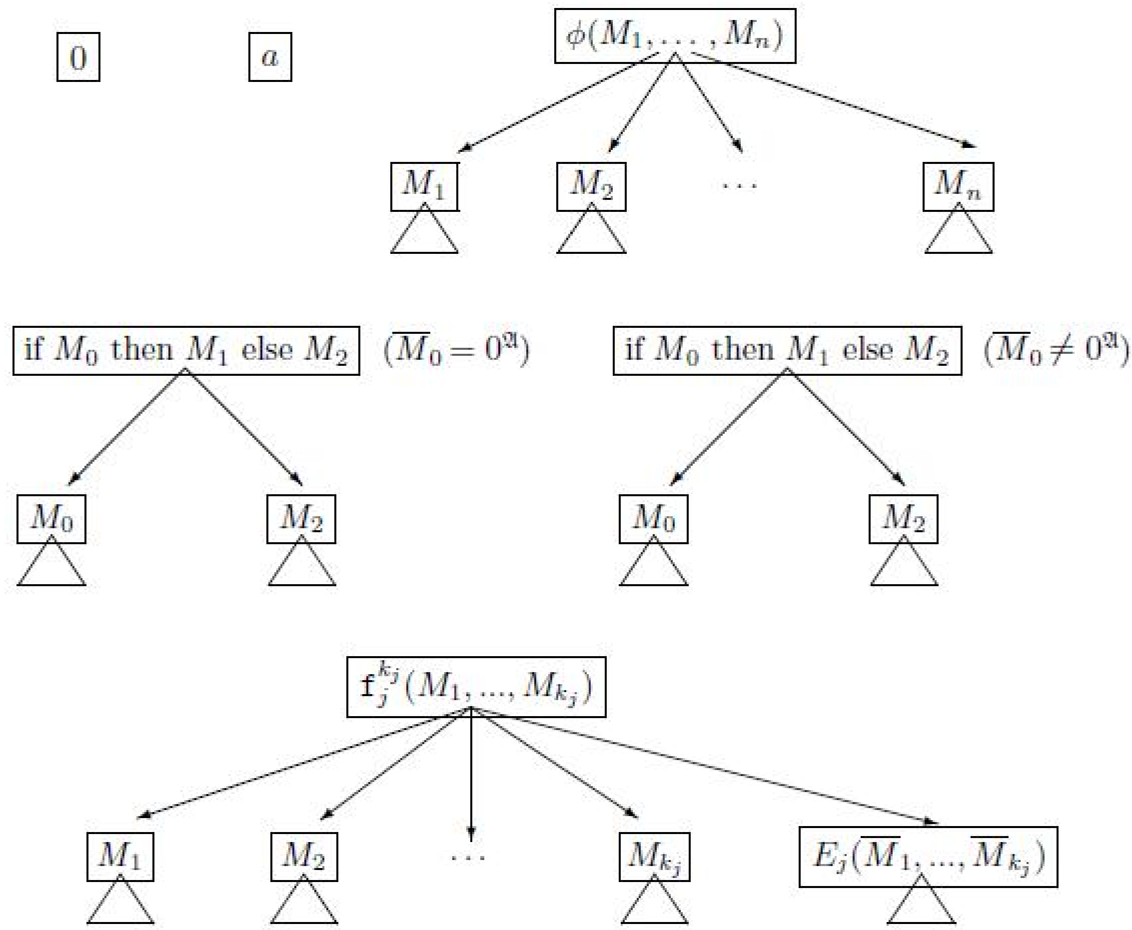


Figure 1: Computation tree

Definition 3.1. *For $M \in \mathrm{Conv}(\mathfrak{A}, E)$, we define $\mathcal{T}(M)$ by induction on the construction of M as follows:*

($\mathcal{T}1$) *if $M = 0$ or $M = a$ for some $a \in A$, then $\mathcal{T}(M) = \{(M)\}$;*

($\mathcal{T}2$) *if $M = (\texttt{if } M_0 = 0 \texttt{ then } M_1 \texttt{ else } M_2)$, for some $(\mathfrak{A}, E)$-terms M_0, M_1, M_2, then either $\overline{M}_0 = 0^{\mathfrak{A}}$ and*

$$\mathcal{T}(M) = M^\frown\mathcal{T}(M_0) \cup M^\frown\mathcal{T}(M_1),$$

or else $\overline{M}_0 \neq 0^{\mathfrak{A}}$ *and*

$$\mathcal{T}(M) = M^\frown\mathcal{T}(M_0) \cup M^\frown\mathcal{T}(M_2);$$

($\mathcal{T}3$) *if* $M = \phi(M_1, ..., M_n)$, *then* $\mathcal{T}(M) = \bigcup_{i=1}^{n} M^\frown\mathcal{T}(M_i)$;

($\mathcal{T}4$) *if* $M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j})$, *then*

$$\mathcal{T}(M) = \bigcup_{i=1}^{k_j} M^\frown\mathcal{T}(M_i) \cup M^\frown\mathcal{T}(E_j(\overline{M}_1, ..., \overline{M}_{k_j})).$$

We say that an $(\mathfrak{A}, E)$*-term* N *is in* $\mathcal{T}(M)$ *if* $(N_1, ..., N_l, N) \in \mathcal{T}(M)$ *for some* $(\mathfrak{A}, E)$*-terms* $N_1, ..., N_l$, *where* l *can be* 0.

4 Complexity measures

In this section we define various complexity measures of convergent $(\mathfrak{A}, E)$-terms, for fixed Φ-algebra $\mathfrak{A}$ and a Φ-program E.

4.1 Tree-depth complexity

For a convergent $(\mathfrak{A}, E)$-term M, $\mathcal{T}(M)$ is finite, and we define the *tree-depth complexity* $D(M) = D_{(\mathfrak{A},E)}(M)$ of M as the depth of $\mathcal{T}(M)$:

$$D(M) = D_{(\mathfrak{A},E)}(M) = \text{depth}(\mathcal{T}(M)).$$

We also define the tree-depth complexity $d_{(\mathfrak{A},E)}(\vec{a})$ of the program E on input $\vec{a} \in F_E^{\mathfrak{A}}$ by

$$d_{(\mathfrak{A},E)}(\vec{a}) = D(\mathtt{f}_E(\vec{a})).$$

This complexity measure is not of practical importance as in ($\mathcal{T}4$) of Definition 3.1 it corresponds to the complexity of an unrealistic parallel computation, where the input of a function is computed in parallel to the computation of the function on that input. However, it is a useful tool for analyzing various properties of recursive programs, and the following is a simple example of its use.

Definition 4.1. *For a program* E, *we define its total arity* $a = a(E)$ *as the maximum of the arities of its recursive variables and function symbols in* Φ. *That is:*

$$a = a(E) = \max\{\max_{1 \leq j \leq m} k_j, \max_{\phi \in \Phi} \text{arity}(\phi)\}.$$

Note that $a(E) \geq 1$ and each node in $\mathcal{T}(M)$ has at most $a(E)+1$ successors. Thus, denoting by $|\mathcal{T}(M)|$ the cardinality (number of nodes) of $\mathcal{T}(M)$, we immediately get:

Proposition 4.2. *For all* $M \in \text{Conv}(\mathfrak{A}, E)$, $|\mathcal{T}(M)| \leq (a(E)+1)^{D(M)}$.

4.2 Sequential logical complexity

For $M \in \text{Conv}(\mathfrak{A}, E)$, we define its *sequential logical complexity* $L^s(M) = L^s_{(\mathfrak{A},E)}(M)$ by induction on the construction of M as follows:

(L^s1) if $M = 0$ or $M = a$ for some $a \in A$, then $L^s(M) = 0$;

(L^s2) if $M = (\texttt{if } M_0 = 0 \texttt{ then } M_1 \texttt{ else } M_2)$, for some $(\mathfrak{A}, E)$-terms M_0, M_1, M_2, then either $\overline{M}_0 = 0^{\mathfrak{A}}$ and

$$L^s(M) = 1 + L^s(M_0) + L^s(M_1),$$

or else $\overline{M}_0 \neq 0^{\mathfrak{A}}$ and

$$L^s(M) = 1 + L^s(M_0) + L^s(M_2);$$

(L^s3) if $M = \phi(M_1, ..., M_n)$, then $L^s(M) = 1 + \sum_{i=1}^{n} L^s(M_i)$;

(L^s4) if $M = \texttt{f}_j^{k_j}(M_1, ..., M_{k_j})$, then

$$L^s(M) = 1 + \sum_{i=1}^{k_j} L^s(M_i) + L^s(E_j(\overline{M}_1, ..., \overline{M}_{k_j})).$$

We also define the sequential logical complexity $l^s_{(\mathfrak{A},E)}(\vec{a})$ of the program E on input $\vec{a} \in \text{dom}(f_E^{\mathfrak{A}})$ by

$$l^s_{(\mathfrak{A},E)}(\vec{a}) = L^s(\texttt{f}_E(\vec{a})).$$

This notion of complexity counts every step made by the program during the computation without any parallelism involved. We also have:

Proposition 4.3. *For all* $M \in \text{Conv}(\mathfrak{A}, E)$, $L^s(M) \leq |\mathcal{T}(M)|$.

Proof. Straightforward induction on the construction of M. □

4.3 Parallel logical complexity

For $M \in \mathrm{Conv}(\mathfrak{A}, E)$, we define its *parallel logical complexity* $L^p(M) = L^p_{(\mathfrak{A},E)}(M)$ by induction on the construction of M as follows:

(L^p1) if $M = 0$ or $M = a$ for some $a \in A$, then $L^p(M) = 0$;

(L^p2) if $M = (\texttt{if } M_0 = 0 \texttt{ then } M_1 \texttt{ else } M_2)$, for some $(\mathfrak{A}, E)$-terms M_0, M_1, M_2, then either $\overline{M}_0 = 0^{\mathfrak{A}}$ and

$$L^p(M) = 1 + \max\{L^p(M_0), L^p(M_1)\},$$

or else $\overline{M}_0 \neq 0^{\mathfrak{A}}$ and

$$L^p(M) = 1 + \max\{L^p(M_0), L^p(M_2)\};$$

(L^p3) if $M = \phi(M_1, ..., M_n)$, then $L^p(M) = 1 + \max_{1\leq i\leq n} L^p(M_i)$;

(L^p4) if $M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j})$, then

$$L^p(M) = 1 + \max_{1\leq i\leq k_j} L^p(M_i) + L^p(E_j(\overline{M}_1, ..., \overline{M}_{k_j})).$$

We also define the sequential logical complexity $l^p_{(\mathfrak{A},E)}(\vec{a})$ of the program E on input $\vec{a} \in \mathrm{dom}(f_E^{\mathfrak{A}})$ by

$$l^p_{(\mathfrak{A},E)}(\vec{a}) = L^p(\mathtt{f}_E(\vec{a})).$$

This notion of complexity corresponds to the time-complexity of a realistic parallel computation, and this is clear for all clauses of the above definition, except maybe for (L^p3) and (L^p4). Here is the justification for these clauses:

For (L^p3), the question may be that how can we choose to compute M_1 or M_2 before knowing whether $\overline{M}_0 = 0^{\mathfrak{A}}$ or not. It is even possible that $\overline{M}_0 = 0^{\mathfrak{A}}$ and M_2 does not converge; thus, it seems that to have a convergent computation, we have to first check whether or not $\overline{M}_0 = 0^{\mathfrak{A}}$, so that we know which of M_1 and M_2 to compute. However, we can do the following trick: start computing M_0, M_1 and M_2 in parallel, and go until M_0 halts. Now if $\overline{M}_0 = 0^{\mathfrak{A}}$, stop the computation of M_2 and resume the computation of M_1 if it hadn't halted yet. Otherwise, stop the computation of M_1 and resume the computation of M_2 if it hadn't halted yet. In this manner, we make only $\max\{LP(M_0), LP(M_1)\}$ steps in the first case, and $\max\{LP(M_0), LP(M_2)\}$ in the second, in addition to the step corresponding to processing the $(\texttt{if} \ \ldots = 0 \ \texttt{then} \ \ldots \ \texttt{else} \ \ldots)$ operation.

The justification for (L^p4) is more straightforward: we need to first compute the input $M_1, M_2, ..., M_{k_j}$ of the function, before computing the function. Hence, we add $\max_{1\leq i\leq k_j} L^p(M_i)$ to $L^p(E_j(\overline{M}_1, ..., \overline{M}_{k_j}))$ instead of taking the maximum of the two. The additional 1 is there to count the call of the function $\mathtt{f}_j^{k_j}$.

Lastly we note the following:

Proposition 4.4. *For all* $M \in \text{Conv}(\mathfrak{A}, E)$, $D(M) = \text{depth}(\mathcal{T}(M)) \leq L^p(M)$.

Proof. Straightforward induction on the construction of M. □

4.4 Sequential call complexity

For $M \in \text{Conv}(\mathfrak{A}, E)$, we define its *sequential call complexity* $C^s(M) = C^s_{(\mathfrak{A},E)}(M)$ by induction on the construction of M as follows:

(C^s1) if $M = 0$ or $M = a$ for some $a \in A$, then $C^s(M) = 0$;

(C^s2) if $M = (\texttt{if} \ M_0 = 0 \ \texttt{then} \ M_1 \ \texttt{else} \ M_2)$, for some $(\mathfrak{A}, E)$-terms M_0, M_1, M_2, then either $\overline{M}_0 = 0^{\mathfrak{A}}$ and

$$C^s(M) = C^s(M_0) + C^s(M_1),$$

or else $\overline{M}_0 \neq 0^{\mathfrak{A}}$ and

$$C^s(M) = C^s(M_0) + C^s(M_2);$$

(C^s3) if $M = \phi(M_1, ..., M_n)$, then $C^s(M) = 1 + \sum_{i=1}^n C^s(M_i)$;

(C^s4) if $M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j})$, then

$$C^s(M) = \sum_{i=1}^{k_j} C^s(M_i) + C^s(E_j(\overline{M}_1, ..., \overline{M}_{k_j})).$$

We also define the sequential call complexity $c^s_{(\mathfrak{A},E)}(\vec{a})$ of the program E on input $\vec{a} \in \text{dom}(f^{\mathfrak{A}}_E)$ by

$$c^s_{(\mathfrak{A},E)}(\vec{a}) = C^s(\mathtt{f}_E(\vec{a})).$$

This complexity measure registers the number of times the program calls a primitive (function in $\Phi^{\mathfrak{A}}$) during a sequential computation.

4.5 Parallel call complexity

For $M \in \mathrm{Conv}(\mathfrak{A}, E)$, we define its *parallel call complexity* $C^p(M) = C^p_{(\mathfrak{A},E)}(M)$ by induction on the construction of M as follows:

(C^p1) if $M = 0$ or $M = a$ for some $a \in A$, then $C^p(M) = 0$;

(C^p2) if $M = (\texttt{if } M_0 = 0 \texttt{ then } M_1 \texttt{ else } M_2)$, for some $(\mathfrak{A}, E)$-terms M_0, M_1, M_2, then either $\overline{M}_0 = 0^{\mathfrak{A}}$ and

$$C^p(M) = \max\{C^p(M_0), C^p(M_1)\},$$

or else $\overline{M}_0 \neq 0^{\mathfrak{A}}$ and

$$C^p(M) = \max\{C^p(M_0), C^p(M_2)\};$$

(C^p3) if $M = \phi(M_1, ..., M_n)$, then $C^p(M) = 1 + \max_{1 \leq i \leq n} C^p(M_i)$;

(C^p4) if $M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j})$, then

$$C^p(M) = \max_{1 \leq i \leq k_j} C^p(M_i) + C^p(E_j(\overline{M}_1, ..., \overline{M}_{k_j})).$$

We also define the sequential logical complexity $c^p_{(\mathfrak{A},E)}(\vec{a})$ of the program E on input $\vec{a} \in \mathrm{dom}(f_E^{\mathfrak{A}})$ by

$$c^p_{(\mathfrak{A},E)}(\vec{a}) = C^p(\mathtt{f}_E(\vec{a})).$$

This complexity measure counts the maximum number of times the program calls a primitive function during a parallel computation in the sense of the discussion in the end of Subsection 4.3.

CHAPTER II

Inequalities and proofs

In this chapter we explore the relations between the complexity measures defined in the previous chapter. Throughout, we fix a Φ-algebra $\mathfrak{A}$ and a program

$$E \quad : \quad \begin{cases} \mathtt{f}_E(\vec{\mathtt{v}}_0) = \mathtt{f}_0^{k_0}(\vec{\mathtt{v}}_0) & = \quad E_0(\vec{\mathtt{v}}_0, \mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, \ldots, \mathtt{f}_m^{k_m}) \\ \qquad\qquad\quad \mathtt{f}_1^{k_1}(\vec{\mathtt{v}}_1) & = \quad E_1(\vec{\mathtt{v}}_1, \mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, \ldots, \mathtt{f}_m^{k_m}) \\ & \vdots \\ \qquad\qquad\quad \mathtt{f}_m^{k_m}(\vec{\mathtt{v}}_m) & = \quad E_m(\vec{\mathtt{v}}_m, \mathtt{f}_0^{k_0}, \mathtt{f}_1^{k_1}, \ldots, \mathtt{f}_m^{k_m}) \end{cases},$$

5 The complete picture of inequalities

For a convergent $(\mathfrak{A}, E)$-term M, the following inequalities are obvious from the definitions:

$$\begin{array}{ccccc} & & C^s(M) & & \\ & \rotatebox{-135}{$\le$} & & \rotatebox{-45}{$\le$} & \\ C^p(M) & & & & L^s(M), \\ & \rotatebox{135}{$\le$} & & \rotatebox{45}{$\le$} & \\ & & L^p(M) & & \end{array}$$

This part of the current thesis concerns reversing this inequalities.

Let μ, ν be complexity measures (e.g. $\mu = L^s$ and $\nu = L^p$). We write:

- $\mu \le_{\text{lin}} \nu$ if there exist constants B_0, B_1 that depend only on the program E such that for all Φ-algebras $\mathfrak{A}$ and $M \in \text{Conv}(\mathfrak{A}, E)$, we have

$$\mu(M) \le B_1 \nu(M) + B_0.$$

- $\mu \le_{\text{exp}} \nu$ if there exists a constant B that depends only on the program E such that for all Φ-algebras $\mathfrak{A}$ and $M \in \text{Conv}(\mathfrak{A}, E)$, we have

$$\mu(M) \le B^{\nu(M)}.$$

The following proposition was proven in [Mos]:

Proposition 5.1 (Moschovakis)**.** $L^s \leq_{\exp} L^p$. *In fact, we have* $L^s(M) \leq (1 + a(E))^{L^p(M)}$ *for all* $M \in \mathrm{Conv}(\mathfrak{A}, E)$.

Proof. Follows directly from Propositions 4.3, 4.2 and 4.4 put together. □

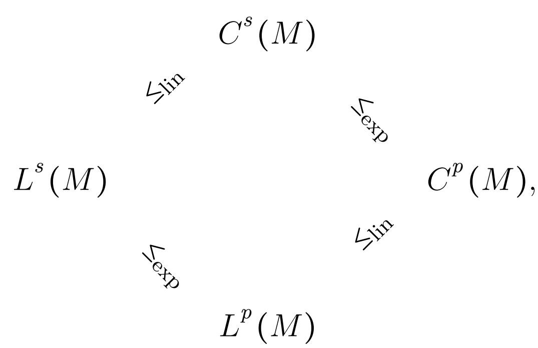
It was then asked in [Mos], whether the following hold:

(A) $L^s \leq_{\mathrm{lin}} C^s$;

(B) $L^p \leq_{\mathrm{lin}} C^p$;

(C) $C^s \leq_{\exp} C^p$.

The main result of this part of the thesis is establishing positive answers to all these questions, and thus, we have the following picture:



Note that (C) follows from (A), (B) and Proposition 5.1, so we only prove (A) and (B) below, in Sections 8 and 9, respectively. However, using similar methods, one could prove (C) directly and obtain a better constant.

6 Main idea

The first thing to notice is that if M is a convergent $(\mathfrak{A}, E)$-term, then its computation tree $\mathcal{T}(M)$ doesn't have a branch containing two equal terms, i.e. there is no $(M_1, M_2, ..., M_l) \in \mathcal{T}(M)$ with $M_i = M_j$ for some $i \neq j$. Thus, if we obtain a bound on the number of possible $(\mathfrak{A}, E)$-terms that can appear in the computation tree of $\mathtt{f}_E(\vec{a})$, for $\vec{a} \in \mathrm{dom}(f_E^{\mathfrak{A}})$, then we would get a bound on the length of the branches of the tree and thus on its size (using Proposition 4.2).

How do we find a bound on the number of possible $(\mathfrak{A}, E)$-terms that can appear in $\mathcal{T}(\mathtt{f}_E(\vec{a}))$? Here we give an outline of how.

Lemma 6.1. *For a finite subset $B \subseteq A$, there are at most $t(E) \cdot |B|^{a(E)}$-many (B, E)-terms.*

Proof. Each $M \in \mathrm{Subterms}(E)$ uses at most $a(E)$ variables. Thus, substituting these variables with elements from B, we obtain $|B|^{a(E)}$-many (B, E)-terms. □

In light of this lemma, we put

$$H = H(E) = t(E)(a(E) + 1)^{a(E)}. \tag{6.2}$$

The letter H stands for height (calling this constant height is justified by Lemma 7.2 below).

For $\vec{a} \in \mathrm{dom}(f_E^{\mathfrak{A}})$, the computation of $f_E^{\mathfrak{A}}$ on $\vec{a}$ starts with processing the $(\mathfrak{A}, E)$-term $M = E_0(\vec{a}, \vec{\mathtt{f}})$. Setting $B = \mathrm{Param}_0(\mathtt{f}_E(\vec{a})) = \{a_i : 1 \le i \le k_0\} \cup \{0^{\mathfrak{A}}\}$, we see that the computation tree of M starts with (B, E)-terms and continues this way until some branch of it hits a recursive call. By Lemma 6.1, the length of the branches containing only (B, E)-terms is at most

$$t(E) \cdot (k_0 + 1)^{a(E)} \le H.$$

How do new elements of A enter the computation?

Lemma 6.3. *If $M \in \mathrm{Conv}(\mathfrak{A}, E)$ and $C^p(M) = 0$ (equivalently $C^s(M) = 0$), then $\overline{M} \in \mathrm{Param}_0(M)$.*

Proof. Straightforward induction on the construction of M. □

This lemma says that during the computation, we only obtain new elements of A by calling a primitive $\phi \in \Phi$, and these new elements enter the computation tree via a recursive call. We isolate certain $(\mathfrak{A}, E)$-terms that correspond to this situation (called *splitting* below), and bounding the number of these terms is how we prove the desired inequalities.

7 Splitting

Definition 7.1. *An $(\mathfrak{A}, E)$-term M is called splitting if $M = \mathtt{f}_j^{k_j}(M_1, \ldots, M_{k_j})$, for some $j \le m$ and $(\mathfrak{A}, E)$-terms $M_1, \ldots, M_{k_j}$ such that $\max_{1 \le k_j} C^s(M_i) > 0$ and $C^s(E_j(\overline{M}_1, \ldots, \overline{M}_{k_j})) > 0$.*

In this definition, we can replace C^s with C^p as one being nonzero is equivalent to the other being nonzero.

For $M \in \mathrm{Conv}(\mathfrak{A}, E)$, we isolate the part of its computation tree that does not involve a splitting term; that is:

$$\mathcal{T}'(M) = \{(M_1, ..., M_l) \in \mathcal{T}(M) : \forall i, 1 \leq i \leq l (M_i \text{ is not splitting})\}.$$

The following lemma shows that the computation can only make a constant (depending only on E) number of steps before it has to encounter a splitting term:

Lemma 7.2. *For $M = M_1 \in \mathrm{Conv}(\mathfrak{A}, E)$, if $(M_1, ..., M_l) \in \mathcal{T}'(M)$, then $l \leq 2H$. In particular, the depth of $\mathcal{T}'(M)$ is at most $2H - 1$.*

Proof. Since M converges, $M_1, ..., M_l$ are distinct. Put $B = \mathrm{Param}_0(M)$. If all of M_i are (B, E)-terms, then we are done since $|B| \leq 1 + a(E)$ and thus, by Lemma 6.1, the number of distinct (B, E)-terms is bounded above by H.

Otherwise, let $n < l$ be the least number such that M_{n+1} is not a (B, E)-term; by Lemma 6.1 again,

$$n \leq H. \tag{$*$}$$

Since M_{n+1} is not a (B, E)-term, it must be that $M_n = \mathtt{f}_j^{k_j}(N_1, ..., N_{k_j})$ and $M_{n+1} = E_j(\overline{N}_1, ..., \overline{N}_{k_j})$, for some $j \leq m$ and $(\mathfrak{A}, E)$-terms $N_1, ..., N_{k_j}$. By the choice of n, $\{\overline{N}_1, ..., \overline{N}_{k_j}\} \nsubseteq B$, and thus, Lemma 6.3 implies:

$$\sum_{i=1}^{k_j} C^s(N_i) > 0.$$

But M_n is not splitting, so it must be that

$$C^s(M_{n+1}) = C^s(E_j(\overline{N}_1, ..., \overline{N}_{k_j})) = 0.$$

Hence, by Lemma 6.3, all of $M_{n+1}, ..., M_l$ are (B', E)-terms, where $B' = \mathrm{Param}_0(M_{n+1})$. Thus, by Lemma 6.1,

$$l - n \leq t(E) \cdot |B'|^{a(E)} \leq t(E) \cdot (a(E) + 1)^{a(E)} = H.$$

This and $(*)$ together imply $l \leq 2H$. ☐

Put

$$V = V(E) = (a(E) + 1)^{2H}, \tag{7.3}$$

and, for a convergent $(\mathfrak{A}, E)$-term M, put

$$v(M) = |\mathcal{T}'(M)|.$$

The above lemma implies:

Corollary 7.4. $v(M) \leq V$.

8 Sequential logical vs. sequential call complexities

We now define an auxiliary complexity measure that makes the exposition of the proof of Theorem 8.5 conceptually clear.

Definition 8.1. *For $M \in \mathrm{Conv}(\mathfrak{A}, E)$, we define its sequential splitting complexity $S^s(M) = S^s_{(\mathfrak{A},E)}(M)$ as the number of splitting terms in $\mathcal{T}(M)$.*

Lemma 8.2. *For all $M \in \mathrm{Conv}(\mathfrak{A}, E)$, $S^s(M) \leq C^s(M) \dot{-} 1$.*

Proof. The proof is by induction on the construction of M as usual, and we only write it for the case when M is splitting as $S^s(M)$ does not increase in other cases. So suppose that $M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j})$ for some $j \leq m$ and M is splitting. Then

$$\sum_{i=1}^{k_j} C^s(M_i) > 0 \text{ and } C^s(E_j(\overline{M}_1, ..., \overline{M}_{k_j})) > 0.$$

Hence, by induction, we have

$$\begin{aligned}
S^s(M) &= 1 + \sum_{i=1}^{k_j} S^s(M_i) + S^s(E_j(\overline{M}_1, ..., \overline{M}_{k_j})) \\
&\leq 1 + (\sum_{i=1}^{k_j} C^s(M_i) - 1) + (C^s(E_j(\overline{M}_1, ..., \overline{M}_{k_j})) - 1) \\
&= \sum_{i=1}^{k_j} C^s(M_i) + C^s(E_j(\overline{M}_1, ..., \overline{M}_{k_j})) - 1 \\
&= C^s(M) - 1.
\end{aligned}$$

□

Lemma 8.3. *If $M \in \mathrm{Conv}(\mathfrak{A}, E)$, then there is a (possibly empty) sequence of splitting terms $N_0, ..., N_{l-1}$ in $\mathcal{T}(M)$ such that*

$$S^s(M) = \sum_{i<l} S^s(N_i) \text{ and } L^s(M) \leq \sum_{i<l} L^s(N_i) + v(M).$$

Proof. We prove by induction on the construction of M as usual. If M itself is splitting, just take $N_0 = M$. If there are no splitting nodes in $\mathcal{T}(M)$, then $\mathcal{T}'(M) = \mathcal{T}(M)$, and the lemma trivially holds by taking $l = 0$ and using Lemma 4.3. The rest of the cases ($\mathcal{T}2$), ($\mathcal{T}3$), and ($\mathcal{T}4$) of Definition 3.1 are handled in the same manner, and we only write the proof for case ($\mathcal{T}4$) for a non-splitting M. So suppose M is non-splitting and

$$M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j}).$$

Put $l = k_j + 1$ and $M_l = E_j(\overline{M}_1, ..., \overline{M}_{k_j})$. By the induction hypothesis, there are splitting terms $N_1^i, ..., N_{n_i}^i$ in $\mathcal{T}(M)$ such that

$$S^s(M) = \sum_{1 \leq i \leq l} S^s(M_i) = \sum_{\substack{1 \leq i \leq l \\ 1 \leq p \leq n_i}} S^s(N_p^i),$$

and also

$$\begin{aligned} L^s(M) &= 1 + \sum_{1 \leq i \leq l} L^s(M_i) \\ &\leq 1 + \sum_{1 \leq i \leq l} \Big(\sum_{1 \leq p \leq n_i} L^s(N_p^i) + v(M_i) \Big) \\ &= \sum_{\substack{1 \leq i \leq l \\ 1 \leq p \leq n_i}} L^s(N_p^i) + 1 + \sum_{1 \leq i \leq l} v(M_i) \\ &= \sum_{\substack{1 \leq i \leq l \\ 1 \leq p \leq n_i}} L^s(N_p^i) + v(M). \end{aligned}$$

□

We are finally ready to prove the main lemma.

Lemma 8.4. *For every $M \in \mathrm{Conv}(\mathfrak{A}, E)$, we have:*

(a) If M is splitting, then $L^s(M) \leq ((a+1)V + 1) \cdot S^s(M)$;
(b) If M is not splitting, then $L^s(M) \leq ((a+1)V + 1) \cdot S^s(M) + V$;

where $a = a(E)$ *and* $V = V(E)$*.*

Proof. We prove (a) and (b) together by induction on the construction of M, noting that (b) is a weaker inequality than (a) and so we can use it when we invoke the induction hypothesis regardless of whether M is splitting or not.

Case 1: M is splitting. Then $M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j})$. Set $l = k_j + 1$ and $M_l = E_j(\overline{M}_1, ..., \overline{M}_{k_j})$. By the definition of $a = a(E)$, $l \leq a + 1$. Thus, using the induction hypothesis, we compute:

$$\begin{aligned} L^s(M) &= L^s(M_1) + ... + L^s(M_l) + 1 \\ &\leq ((a+1)V + 1) \cdot [S^s(M_1) + ... + S^s(M_l)] + (lV + 1) \\ &\leq ((a+1)V + 1) \cdot [S^s(M_1) + ... + S^s(M_l)] + ((a+1)V + 1) \\ &= ((a+1)V + 1) \cdot [S^s(M_1) + ... + S^s(M_l) + 1] \\ &= ((a+1)V + 1) \cdot S^s(M). \end{aligned}$$

Case 2: M is not splitting. Then, by Lemma 8.3, there are splitting terms $N_0, ..., N_{l-1}$ in $\mathcal{T}(M)$ such that

$$S^s(M) = \sum_{i<l} S^s(N_i) \text{ and } L^s(M) \leq \sum_{i<l} L^s(N_i) + v(M).$$

Thus, using Corollary 7.4 and the induction hypothesis, we compute:

$$\begin{aligned} L^s(M) &\leq L^s(N_1) + ... + L^s(N_l) + v(M) \\ &\leq ((a+1)V + 1) \cdot [S^s(N_1) + ... + S^s(N_l)] + v(M) \\ &\leq ((a+1)V + 1) \cdot S^s(M) + V. \end{aligned}$$

□

Theorem 8.5. *For every* Φ*-program* E*,* $L^s \leq_{\text{lin}} C^s$*. In particular, there exists constants* B_0, B_1 *depending on* E *such that for every* Φ*-algebra* $\mathfrak{A}$ *and every* $\vec{a} \in \text{dom}(f_E^{\mathfrak{A}})$*, we have:*

$$l^s_{(\mathfrak{A},E)}(\vec{a}) \leq B_1 \cdot c^s_{(\mathfrak{A},E)}(\vec{a}) + B_0.$$

Proof. Follows from Lemmas 8.4 and 8.2 by taking $B_0 = V(E)$ and $B_1 = (a(E) + 1)V(E) + 1$.

□

9 Parallel logical vs. parallel call complexities

In this section, we obtain the analogue of Theorem 8.5 for L^p and C^p. The proof follows the same outline as in the previous section.

Definition 9.1. *For $M \in \mathrm{Conv}(\mathfrak{A}, E)$, we define its parallel splitting complexity $S^p(M) = S^p_{(\mathfrak{A},E)}(M)$ by induction on the construction of M as follows:*

(S^p1) if $M = 0$ or $M = a$ for some $a \in A$, then $S^p(M) = 0$;

(S^p2) if $M = (\texttt{if } M_0 = 0 \texttt{ then } M_1 \texttt{ else } M_2)$, for some $(\mathfrak{A}, E)$-terms M_0, M_1, M_2, then either $\overline{M}_0 = 0^{\mathfrak{A}}$ and

$$S^p(M) = \max\{S^p(M_0), S^p(M_1)\},$$

or else $\overline{M}_0 \neq 0^{\mathfrak{A}}$ and

$$S^p(M) = \max\{S^p(M_0), S^p(M_2)\};$$

(S^p3) if $M = \phi(M_1, ..., M_n)$, then $S^p(M) = \max_{1 \le i \le n} S^p(M_i)$;

(S^p4) if $M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j})$, then either M is not splitting and

$$S^p(M) = \max_{1 \le i \le k_j} S^p(M_i) + S^p(E_j(\overline{M}_1, ..., \overline{M}_{k_j})),$$

or else M is splitting and

$$S^p(M) = 1 + \max_{1 \le i \le k_j} S^p(M_i) + S^p(E_j(\overline{M}_1, ..., \overline{M}_{k_j})).$$

Lemma 9.2. *For all $M \in \mathrm{Conv}(\mathfrak{A}, E)$, $S^p(M) \le C^p(M) \dot{-} 1$.*

Proof. The proof is by induction on the construction of M as usual, and we only write it for the case when M is splitting as $S^p(M)$ does not increase in other cases. So suppose that $M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j})$ for some $j \le m$ and M is splitting. Then

$$\max_{1 \le i \le k_j} C^p(M_i) > 0 \text{ and } C^p(E_j(\overline{M}_1, ..., \overline{M}_{k_j})) > 0.$$

Hence, by induction, we have

$$\begin{aligned} S^p(M) &= 1 + \max_{i=1}^{k_j} S^p(M_i) + S^p(E_j(\overline{M}_1, ..., \overline{M}_{k_j})) \\ &\leq 1 + (\max_{i=1}^{k_j} C^p(M_i) - 1) + (C^p(E_j(\overline{M}_1, ..., \overline{M}_{k_j})) - 1) \\ &= \max_{i=1}^{k_j} C^p(M_i) + C^p(E_j(\overline{M}_1, ..., \overline{M}_{k_j})) - 1 \\ &= C^p(M) - 1. \end{aligned}$$

□

Definition 9.3. *Call an* $(\mathfrak{A}, E)$*-term* M *a leaf if* $M = 0$ *or* $M = a$ *for some* $a \in A$.

Lemma 9.4. *If* $M \in \mathrm{Conv}(\mathfrak{A}, E)$*, then there is an* $(\mathfrak{A}, E)$*-term* N *in* $\mathcal{T}(M)$ *that is either a leaf or splitting and is such that*

$$L^p(M) \leq L^p(N) + v(M).$$

Proof. We prove by induction on the construction of M as usual. If M itself is a leaf or splitting, then take $N = M$. Otherwise, we consider cases:

Case 1: $M = \phi(M_1, ..., M_n)$. Then $L^p(M) = L^p(Mi) + 1$ for some i, $1 \leq i \leq n$, and the induction hypothesis gives us a leaf or splitting term N in $\mathcal{T}(Mi)$ such that

$$L^p(Mi) \leq L^p(N) + v(Mi).$$

Since M is not splitting,

$$\mathcal{T}'(M) = \bigcup_{i=1}^{n} M^\frown \mathcal{T}'(M)$$

and so $v(M) \geq v(M_i) + 1$. Thus, it follows that

$$L^p(M) = L^p(Mi) + 1 \leq L^p(N) + v(Mi) + 1 \leq L^p(N) + v(M).$$

Case 2: $M = (\texttt{if}\ M_0 = 0\ \texttt{then}\ M_1\ \texttt{else}\ M_2)$. The argument for this case is similar to that for Case 1, so we will skip it.

Case 3: M is not splitting and $M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j})$. Set $M' = E_j(\overline{M}_1, ..., \overline{M}_{k_j})$ and choose i such that $L^p(M_i) = \max\{L^p(M_1), ..., L^p(M_{k_j})\}$.

If $C^p(M_i) = 0$, then $L^p(M_i) \leq L^s(M_i) \leq |\mathcal{T}(M_i)| = v(M_i)$ because $\mathcal{T}(M) = \mathcal{T}'(M)$. By the induction hypothesis applied to M', there is an $(\mathfrak{A}, E)$-term N in $\mathcal{T}(M')$ that is a leaf or splitting and is such that

$$L^p(M') \leq L^p(N) + v(M').$$

Hence:

$$L^p(M) = L^p(M_i) + L^p(M') + 1 \leq v(M_i) + L^p(N) + v(M') + 1.$$

Since M is not splitting, we have $v(M) \geq v(M_i) + v(M') + 1$, so

$$L^p(M) \leq L^p(N) + v(M).$$

If $C^p(M_i) > 0$, then $C^p(M') = 0$ since M is non-splitting, and we can repeat the same argument with the roles of M_i and M' swapped. □

Lemma 9.5. *For every* $M \in \mathrm{Conv}(\mathfrak{A}, E)$*, we have:*

(a) If M *is splitting, then* $L^p(M) \leq (2V+1) \cdot S^p(M)$*;*
(b) If M *is not splitting, then* $L^p(M) \leq (2V+1) \cdot S^p(M) + V$*;*

where $V = V(E)$*.*

Proof. We prove (a) and (b) together by induction on the construction of M, noting that (b) is a weaker inequality than (a) and so we can use it when we invoke the induction hypothesis regardless of whether M is splitting or not.

Case 1: M is splitting. Then $M = \mathtt{f}_j^{k_j}(M_1, ..., M_{k_j})$. Set $M' = E_j(\overline{M}_1, ..., \overline{M}_{k_j})$ and compute:

$$\begin{aligned} L^p(M) &= \max_{1 \leq i \leq k_j} L^p(M_i) + L^p(M') + 1 \\ &\leq (2V+1) \cdot \max_{1 \leq i \leq k_j} S^p(M_i) + V + (2V+1) \cdot S^p(M') + V + 1 \\ &= (2V+1) \cdot \left[\max_{1 \leq i \leq k_j} S^p(M_i) + S^p(M') + 1 \right] \\ &= (2V+1) \cdot S^p(M). \end{aligned}$$

Case 2: M is not splitting. Then, by Lemma 9.4, there is an $(\mathfrak{A}, E)$-term N in $\mathcal{T}(M)$ that is a leaf or splitting and is such that

$$L^p(M) \leq L^p(N) + v(M).$$

Using the obvious fact that $S^p(N) \leq S^p(M)$, together with Corollary 7.4 and the induction hypothesis, we compute:

$$\begin{aligned} L^p(M) &\leq L^p(N) + v(M) \\ &\leq (2V + 1) \cdot S^p(N) + v(M) \\ &\leq (2V + 1) \cdot S^p(M) + V. \end{aligned}$$

□

Theorem 9.6. *For every Φ-program E, $L^p \leq_{\text{lin}} C^p$. In particular, there exists constants B_0, B_1 depending on E such that for every Φ-algebra $\mathfrak{A}$ and every $\vec{a} \in \text{dom}(f_E^{\mathfrak{A}})$, we have:*

$$l^p_{(\mathfrak{A},E)}(\vec{a}) \leq B_1 \cdot c^p_{(\mathfrak{A},E)}(\vec{a}) + B_0.$$

Proof. Follows from Lemmas 9.5 and 9.2 by taking $B_0 = V(E)$ and $B_1 = 2V(E) + 1$. □

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