THE COMMUTATION RELATIONS OF QUANTUM MECHANICS

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ABSTRACT

The mathematical and physical meaning of the commutation relations of nonrelativistic quantum mechanics is discussed in terms of the representation of translations, Galilean transformations, and rotations of the coordinate system by unitary transformations acting on the unitary vector space of quantum states.

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INTRODUCTION

The discussion of this paper is confined to statements concerning part of the conceptual structure of the nonrelativistic quantum mechanics of particles, even though the arguments may be extended to the discussion of relativistic quantum field theories. This restriction makes it possible to study the essential points that are involved without the use of cumbersome formulae.

Most treatises on quantum mechanics include among the various postulates of the theory a statement of the fundamental commutation relations between the Cartesian components of the coordinate and the canonical momentum of a particle:

$$(x_i, p_j) = i \hbar \delta_{ij} . \tag{1}$$

Quite naturally, a great deal of attention is paid to the physical consequences of these relations as expressed by the Heisenberg uncertainty principle. However, with few exceptions,[1,2] there is little discussion of the mathematical and physical ideas which underlie them. These ideas are concerned with the representation of translations, Galilian transformations, and rotations of the coordinate system by unitary transformations acting on the unitary vector space of quantum states.

The author has discussed the commutation relations with many physicists during the past few years and has found that only the most sophisticated among them are familiar with the ideas involved. The present review is concerned with an attempt to present them in a simple and concise fashion to a wider audience. It should be remarked here that this situation has been clearly recognized by Schwinger,[3] who has given a concise and complete statement of the laws of quantum physics in terms of his general dynamical principle, the quantum analogue of Hamilton's principle. His discussion has not appeared in textbook form, however. Furthermore, Schwinger deals with the most general situation appropriate to relativistic, localizable field theories. Consequently, it is not easy to divide his arguments into their various parts in order to clearly recognize the concepts that are involved because the generality of the problem that he attacks requires the use of elaborate mathematical techniques, which are not necessary for the analysis of the simpler problem to be discussed here.

THE RELATION BETWEEN THE COORDINATE SYSTEM AND THE UNITARY VECTOR SPACE OF QUANTUM STATES

The basic postulates of quantum mechanics assert that a physical system is described by a vector which is an element of a linear unitary vector space and that observables are represented by Hermitian operators whose eigenvectors may be used to define a coordinate system in this space. They also assert that if $|A'\rangle$ is an eigenvector corresponding to the eigenvalue A' of an observable A, then the probability that a measurement of A will lead to A' when the system is in the state $|\psi\rangle$ is the absolute square of the scalar product $\langle A' | \psi\rangle$. This leads to the requirement that $\langle\psi | \psi\rangle$ be unity and is, in fact, the reason why the transformations of quantum theory must be unitary.[4] It also shows that states $|\psi\rangle$ which differ by a phase factor $e^{i\alpha}$ are equivalent.

To describe motion one must be able to represent the basic motions of the physical coordinate system, i.e. translations,[5] Galilean transformations, and rotations, by corresponding unitary transformations acting on the space of quantum states. Once this kinematical problem has been solved, the transition to dynamics may be made by relating the infinitesimal generators of these transformations to the Lagrangian of the system.

TRANSLATIONS

First consider the representation of displacements of the coordinate system by a fixed amount a_i. The eigenvalues of the coordinate operator x_i label the position of a particle, and therefore, under this displacement, corresponding eigenvalues must be related by

$$x'^{(2)}_i = x'''^{(1)}_i - a_i \quad , \tag{2}$$

where the labels 1 and 2 refer to the two different systems. If the system was described by a state vector $|\psi\rangle$, this changes into $|\psi'\rangle$ under the transformation, and $|\psi'\rangle$ is related to $|\psi\rangle$ by a unitary transformation

$$|\psi'\rangle = U\,|\psi\rangle \quad . \tag{3}$$

This transformation may be determined by the condition:

$$\langle\psi'\,|\,x_i\,|\psi'\rangle = \langle\psi\,|\,x_i - a_i\,|\psi\rangle \quad . \tag{4}$$

This leads to

$$U^{-1}\,x_i\,U = x_i - a_i \quad . \tag{5}$$

Now one may express U as the exponential of an Hermitian operator D which is clearly a function of the displacement a_i :

$$U = e^{i D(a_i)} . \tag{6}$$

One must have

$$D(0) = 0 ,$$

and consequently the Taylor expansion of D has the form

$$D(a_i) = \sum_{n=1}^{\infty} \frac{a_i^{\ n}}{n!} \frac{d^n D(a_i)}{d a_i} \Bigg|_{a_i=0} . \tag{7}$$

For infinitesimal displacements, only the first term is important, and so it is convenient to set

$$d_i = \frac{d D(a_i)}{d a_i} \Bigg|_{a_i=0} , \tag{8}$$

and to write

$$U = 1 + i d_i a_i . \tag{9}$$

Consequently, one has

$$(1 - i d_i a_i) x_i (1 + i d_i a_i) = x_i - a_i \tag{10}$$

or

$$(x_i, d_i) = i . \tag{11}$$

This relation defines the infinitesimal generator d_i which was desired and shows that when x_i is diagonal, d_i may be represented as

$$d_i = -i \frac{d}{dx_i} . \tag{12}$$

It may be shown that the general form [Eq. (7)] is not required to yield all displacements that may be achieved by a continuous change from the identity (no displacement at all), but that an arbitrary displacement may be written as

$$U(a_i) = e^{i a_i d_i} . \tag{13}$$

As mentioned previously, Eq. (13) is a purely kinematical statement. The transition to dynamics takes place when one makes the fundamental hypothesis that the momentum operator p_i is given by

$$p_i = \hbar d_i = \frac{\partial L}{\partial \dot{x}_i} , \tag{14}$$

where L is the Lagrangian function.

Clearly a similar argument might be used to discuss the representation of time displacements. This would, however, be incorrect since the time is merely a parameter and may not be regarded as a dynamical variable of the system. It is interesting to note that this situation which mars the structure of nonrelativistic quantum mechanics is not present in relativistic quantum field theory, where particles are described by field operators that are functions of position relative to the coordinate system. These positional coordinates (which include time) are therefore only parameters.

From the foregoing remarks it should be clear that the state vector in nonrelativistic quantum mechanics is to be regarded as a function of time which changes according to dynamical laws. The dynamical law must be expressed as a unitary transformation by postulating Schroedinger's equation

$$i \hbar \frac{d}{\partial t} | \psi t \rangle = H | \psi t \rangle . \tag{15}$$

Thus the time does not express any kinematical features of the system.

GALILEAN TRANSFORMATIONS

Nonrelativistic quantum mechanics satisfies a principle of relativity with respect to Galilean transformations. If one considers two inertial coordinate systems moving relative to each other with velocity v_i, which were coincident at $t = 0$, it is clear that the eigenvalues of the momentum operator p_i which give the momentum of the particle relative to the two inertial frames must be related by

$$p'^{(2)}_i = p'^{(1)}_i - m v_i , \tag{16}$$

where m is the mass of the particle, and the labels 1 and 2 refer to the two different inertial coordinate systems. It is also necessary to recognize that the eigenvalues of the coordinate operator x_i are related by

$$x'^{(2)}_i = x'^{(1)}_i - v t . \tag{17}$$

The transformation between the two inertial frames is now to be represented by a unitary transformation acting on the state vector $|\psi\rangle$ of the system:

$$|\psi'\rangle = U | \psi\rangle . \tag{18}$$

The conditions which determine U are

$$\langle\psi' | p_i | \psi'\rangle = \langle\psi | p_i - m v_i | \psi\rangle \tag{19}$$

and

$$\langle \psi' \mid x_i \mid \psi' \rangle = \langle \psi \mid x_i - v_i t \mid \psi \rangle . \tag{20}$$

Equations (19) and (20) lead immediately to

$$U^{-1} p_i U = p_i - m v_i \tag{21}$$

and

$$U^{-1} x_i U = x_i - v_i t . \tag{22}$$

Suppose that one first studies Eq. (21) by temporarily ignoring condition (22). One may then write U in the form

$$U = e^{i g_i m v_i} , \tag{23}$$

where g_i is the infinitesimal generator of the transformation. Upon passing to the case of infinitesimal v_i , one finds from Eq. (21) that

$$(g_i, p_i) = -i . \tag{24}$$

Consequently, the infinitesimal generator may be expressed as

$$g_i = - i \frac{d}{d p_i} . \tag{25}$$

Now Eq. (25) is a purely kinematical statement so that the connection with dynamics must be made by the assertion that the generator g_i is identical with the negative of the coordinate operator

$$x_i = - \hbar g_i . \tag{26}$$

One may now return to the problem of representing the Galilean transformation. One must exhibit a unitary transformation U which is determined by Eqs. (21) and (22). Since Eq. (21) by itself would lead to a unitary transformation U_1 of the form

$$U_1 = e^{\frac{i}{\hbar} v_i t p_i} , \tag{27}$$

one is lead to a study of the composite transformation $U_1 U_2$:

$$U_1 U_2 = e^{\frac{i}{\hbar} a_i p_i} e^{\frac{i}{\hbar} b_i x_i} , \tag{28}$$

where $a_i = v t$ and $b_i = -m v_i$. If this is applied to Eq. (21), one finds

$$\begin{aligned} U_2^{-1} U_1^{-1} p_i U_1 U_2 &= e^{-\frac{i}{\hbar} b_i x_i} e^{-\frac{i}{\hbar} a_i p_i} p_i e^{\frac{i}{\hbar} a_i p_i} e^{\frac{i}{\hbar} b_i x_i} \\ &= p_i - m v_i . \end{aligned} \tag{29}$$

Now checking Eq. (22), one can write

$$\begin{aligned} U_2^{-1} U_1^{-1} g_i U_1 U_2 &= e^{-\frac{i}{\hbar} b_i x_i} e^{-\frac{i}{\hbar} a_i p_i} x_i e^{\frac{i}{\hbar} a_i p_i} e^{\frac{i}{\hbar} b_i x_i} \\ &= x_i - v_i t . \end{aligned} \tag{30}$$

Thus the unitary transformation

$$U = U_1 U_2 \tag{31}$$

does indeed represent the Galilean transformation.

It is at this point that one comes upon an interesting and somewhat surprizing situation, for if one considers the unitary transformation

$$U' = U_2 U_1 , \tag{32}$$

it may be immediately verified that U' also satisfies the conditions required by Eqs. (21) and (22). One is therefore lead to the conclusion that the state vectors $|\psi'\rangle$ and $|\psi''\rangle$ defined by

$$|\psi'\rangle = U\,|\,\psi\,\rangle \tag{33}$$

and

$$|\psi''\rangle = U'|\,\psi\,\rangle \tag{34}$$

actually represent the same physical situation. This possibility can exist only because of the probability hypothesis of quantum mechanics which asserts that only the modulus of the state vector has a physical meaning. Weyl described this situation by saying only the rays of the vector space were physically significant.[1] A ray is defined by

$$|R\rangle = e^{i\,\alpha}\,|\,\psi\,\rangle\ , \tag{35}$$

where α is an arbitrary real number. All state vectors which satisfy Eq. (35) lie on the same ray.

Upon returning to Eqs. (33) and 34), one may therefore conclude that

$$|\,\psi''\,\rangle = e^{i\,\alpha}\,|\,\psi'\,\rangle\ , \tag{36}$$

or that $|\,\psi''\,\rangle$ and $|\,\psi'\,\rangle$ lie on the same ray. Consequently the unitary transformations are commutative in the sense that

$$U_1(a)\,U_2(b) = e^{i\,\gamma(a,\,b)}\,U_2(b)\,U_1(a)\quad . \tag{37}$$

Weyl asserted that quantum kinematics is described by an Abelian group of "rotations" of the rays associated with the vector space. With this hypothesis, he then showed that one is led to the fundamental commutation relations. This is easily seen by letting a_i, b_i be infinitesimal. In this case

$$\gamma(a, b) = a_i b_i \left. \frac{\partial^2 \gamma}{\partial a_i \, \partial b_i} \right|_{\substack{a=0 \\ b=0}} , \tag{38}$$

sin $\gamma(0, 0) = 1$. Consequently, one finds

$$(1 + \frac{i}{\hbar} a_i \, p_i)(1 + \frac{i}{\hbar} b_i \, x_i) = (1 + i \, a_i b_i \, \gamma'')(1 + \frac{i}{\hbar} b_i \, g_i)(1 + \frac{i}{\hbar} a_i \, p_i),$$

or

$$\frac{a_i \, b_i}{\hbar^2} (x_i, p_i) = i \, a_i \, b_i \, \gamma'' \quad . \tag{39}$$

If γ'' is chosen as $\hbar^{-1}$, one may conclude

$$(x_i, p_i) = i \hbar \quad . \tag{40}$$

From a purely physical point of view the situation may be summed up by saying that one may make the translation first and then the momentum change, or vice versa. It seems evident that either way should lead to the same physical properties.

As the reader will have noticed, many salient points have been omitted from the foregoing discussion. Some of them will be discussed in a later section, since they do not at this point fall into the scheme of this paper.

ROTATIONS

In order to complete the study of nonrelativistic quantum kinematics, it is necessary to represent the only other possible type of motion that can occur--rotation. While it is clear from the foregoing discussion that one might immediately infer that the infinitesimal generator of rotations is the usual angular-momentum operator, it seems more in the spirit of this paper to treat rotations in the same way as translations and Galilean transformations.

Consider, therefore, a rotation of the coordinate system by an amount specified by the rotation matrix S_{ij}. The effect of this rotation is to alter the eigenvalues of the coordinate operator x_i according to the relation

$$x'^{(2)}_i = S_{ij}\, x'^{(1)}_j \quad . \tag{41}$$

Accordingly, this transformation induces a change of the state vector of the system $|\psi\rangle$ given by

$$|\psi'\rangle = U\, |\, \psi \rangle \quad , \tag{42}$$

where the unitary transformation U is determined by

$$\langle \psi' \,|\, x_i \,|\, \psi' \rangle = S_{ij} \langle \psi \,|\, x_j \,|\, \psi \rangle \quad . \tag{43}$$

Consequently, one finds

$$U^{-1}\, x_i\, U = S_{ij}\, x_j \quad . \tag{44}$$

If, as in the foregoing sections, one considers only infinitesimal rotations, one may write

$$S_{ij} = \delta_{ij} + \Omega_{ij} \, , \tag{45}$$

where Ω_{ij} is an antisymmetric matrix

$$\Omega_{ij} = -\Omega_{ji} . \tag{46}$$

This matrix is related to the infinitesimal angle of rotation θ_i ($\theta_i = \omega_i \delta t$) by the relation

$$\Omega_{ij} = \epsilon_{ijk} \theta_k , \tag{47}$$

where ϵ_{ijk} is the usual alternating symbol of tensor analysis.

The unitary transformation U , on the other hand, may be written in terms of its infinitesimal generator as

$$U = e^{i \tau_k \theta_k} , \tag{48}$$

where summation over i is now intended. For infinitesimal θ_i one therefore finds

$$U = 1 + i \tau_i \theta_i . \tag{49}$$

Upon using this relation in conjunction with Eqs. (44) and (45), one finds that

$$-i(\tau_i \theta_i , x_j) = \Omega_{ki} x_k , \tag{50}$$

and from Eq. (29), one obtains

$$-i (\tau_i \theta_i , x_j) = - \epsilon_{ijk} x_k \theta_i . \tag{51}$$

Since the angle of rotation θ_k is arbitrary, one may conclude that

$$(\tau_i , x_j) = i \epsilon_{ijk} x_k . \tag{52}$$

Since this is a kinematical statement only, one must make the connection with dynamics by comparing it with Eqs. (14) and (24) or by making, independently, the hypothesis that

$$\tau_k \hbar = L_k$$

and (53)

$$L_k = -i \epsilon_{kji} x_j p_i \quad .$$

In either case, one finds that the generator of infinitesimal rotations is

$$L_k = -i \hbar \epsilon_{kji} x_j \frac{\partial}{\partial x_i} \quad , \tag{54}$$

which is just the angular-momentum operator. An elementary calculation leads to the commutation rules between various components of the angular momentum

$$(L_i, L_j) = i \hbar \epsilon_{ijk} L_k \quad . \tag{55}$$

It is instructive to discuss three simple cases which illustrate the typical problems with which nonrelativistic quantum mechanics is concerned. These are concerned with scalar, spinor, and vector functions of the position operator. Under a rotation, S_{ij} , these scalar functions transform as

$$\langle \psi' \mid \phi(x) \mid \psi' \rangle = \langle \psi \mid \phi(S^{-1} x) \mid \psi \rangle \quad . \tag{56}$$

The corresponding transformation for spinors is

$$\langle \psi' \mid \psi_\alpha(x) \mid \psi' \rangle = \langle \psi \mid {\Lambda_{\alpha\beta}}^{-1} \psi_\beta(S^{-1} x) \mid \psi \rangle$$

(57)

$$\Lambda^{-1} \sigma_i \Lambda = S_{ij} \sigma_j$$

where σ_i is the Pauli spin operator. For vectors the transformation is

$$\langle \psi' \mid A_i(x) \mid \psi' \rangle = S_{ij} \langle \psi \mid A_j(S^{-1} x) \mid \psi \rangle \quad . \tag{58}$$

For the scalar case, the unitary transformation is

$$U^{-1} \phi(x) U = \phi(S^{-1} x) \quad . \tag{59}$$

Thus when the rotation is infinitesimal, one finds, upon writing

$$U = (1 + \frac{i}{\hbar} J \cdot \theta) \quad , \tag{60}$$

that

$$-\frac{i}{\hbar} [J_i \cdot \theta_i, \phi] = \frac{\partial \phi}{\partial x_k} \epsilon_{kji} x_j \theta_i \tag{61}$$

or

$$[J_i, \phi] = -i \epsilon_{ijk} x_j p_k \quad . \tag{62}$$

In this case one finds $J_i = L_i$. Thus, the angular momentum carried by a scalar field is purely orbital.

The spinor case is more interesting. One writes

$$\left.\begin{aligned} U &= 1 + \frac{i}{\hbar} J_i \theta_i \\ \Lambda &= 1 + \frac{i}{2} \sigma_i \theta_i \end{aligned}\right\} \quad . \tag{63}$$

Consequently, one has

$$-\frac{i}{\hbar} [J_i \cdot \theta_i, \psi] = -\frac{i}{2} \sigma_i \theta_i \psi + \epsilon_{kji} x_j \theta_i \frac{\partial \psi}{\partial x_k} \tag{64}$$

or

$$[J_i, \psi] = [-i \hbar \epsilon_{ijk} x_j \frac{\partial}{\partial x_k} + i \frac{\hbar}{2} \sigma_i] \psi . \tag{65}$$

In other words, one has

$$J_i = L_i + \frac{\hbar}{2} \sigma_i . \tag{66}$$

A number of details regarding the explicit construction of the spin matrices have been omitted from this argument for the sake of brevity. They may be obtained however by an application of the methods of this paper.

For the case of the vector field, one finds that

$$[J_i, A_m] = [L_i \delta_{m\ell} - i \hbar \epsilon_{im\ell}] A_\ell . \tag{67}$$

Consequently, one can write

$$J_i = L_i + S_i , \tag{68}$$

where

$$(S_i)_{jk} = - i \hbar \epsilon_{ijk} . \tag{69}$$

One easily verifies that

$$(\sum_i S_i^2)_{jk} = 2 \hbar^2 \delta_{jk} , \tag{70}$$

which is the standard result that the vector field describes an intrinsic spin of $\hbar$.

The foregoing discussion may be extended to the case of higher-rank tensor fields provided that sufficient attention is paid to the question of irreducibility of the acquired representations. It does not seem profitable to discuss more complicated situations in this paper.

WEYL'S THEORY OF QUANTUM KINEMATICS[1]

Weyl's theory of quantum kinematics shows the close connection between the probability hypothesis of quantum mechanics and the commutation rules. The probability hypothesis leads to the conclusion that one is not concerned with the vectors of the representation space, but only with the rays of this space. Weyl observed that the commutation rules of quantum mechanics imply that the operators x_i and p_i are the infinitesimal generators of an Abelian group of "rotations" of the rays of the state vector space. He then investigated the general conditions that are required in order to set up an irreducible unitary representation of an Abelian group of ray rotations.

Suppose that $|\psi\rangle$ belongs to the ray $|R\rangle$ and that one considers two "rotations" $\mathcal{S}_1$ and $\mathcal{S}_2$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are commutative, the ray obtained after the "rotations" is

$$|R'\rangle = \mathcal{S}_1 \mathcal{S}_2 \mid R\rangle = \mathcal{S}_2 \mathcal{S}_1 \mid R\rangle \quad . \tag{71}$$

If one now represents these "rotations" in terms of unitary transformations acting on the vector space,

$$\left.\begin{array}{l} U_1 \rightarrow \mathcal{S}_1 \\ \\ U_2 \rightarrow \mathcal{S}_2 \end{array}\right\} , \tag{72}$$

one realizes that U_1 and U_2 must satisfy

$$\left.\begin{array}{l} |\psi'\rangle = U_1 U_2 \mid \psi\rangle \\ \\ |\psi''\rangle = U_2 U_1 \mid \psi\rangle \end{array}\right\} \tag{73}$$

and also

$$|\psi'\rangle = e^{i\alpha} \, |\psi''\rangle \quad . \tag{74}$$

Consequently, the relation $(\mathcal{S}_1, \mathcal{S}_2) = 0$ is represented by

$$U_1 U_2 = e^{i\alpha} U_2 U_1 \quad . \tag{75}$$

It is clear from this equation that a unitary representation of an Abelian group of ray rotations can never be set up in a finite dimensional vector space, for this would require that

$$\det(U_1 U_2) = \det(e^{i\alpha} U_2 U_1) \quad , \tag{76}$$

and, consequently,

$$e^{i n \alpha} = 1 \quad . \tag{77}$$

That is, $e^{i\alpha}$ would have to be an <u>n</u>th root of unity, where n is the dimensionality of the space. Moreover, one would have the additional requirement that

$$\mathrm{tr}\; U_1 U_2 = \mathrm{tr}\; e^{i\alpha} U_2 U_1 \tag{78}$$

or that

$$e^{i\alpha} = 1 \; . \tag{79}$$

It is therefore necessary to consider a space of infinite dimensionality where Eqs. (77) and (79) need not hold true.

To investigate this problem further, one supposes that there exist infinitesimal generations, σ_i, which are appropriate to the problem so that U_1

and U_2 may be expressed as

$$\left.\begin{aligned} U_1(\tau) &= e^{i\,\tau_i\,\sigma_i} \\ U_2(\lambda) &= e^{i\,\lambda_j\,\sigma_j} \end{aligned}\right\} \quad . \tag{80}$$

The τ_i and λ_j are parameters which define U_1 and U_2, and summation in the exponent is implied over the assumed finite set of m infinitesimal generators $\sigma_1, \ldots \sigma_m$. Upon substituting Eqs. (80) into Eq. (75), one finds

$$e^{i\,\tau_i\,\sigma_i}\, e^{i\,\lambda_j\,\sigma_j} = e^{i\,\alpha(\tau,\,\lambda)}\, e^{i\,\lambda_j\,\sigma_j}\, e^{i\,\tau_i\,\sigma_i} \quad , \tag{81}$$

where the explicit dependence of α on the τ_i and λ_j has been noted.

Upon passing to the case of infinitesimal τ_i and λ_j, one may write

$$\left.\begin{aligned} e^{i\,\tau_i\,\sigma_i} &= 1 + i\,\tau_i\,\sigma_i \\ e^{i\,\lambda_j\,\sigma_j} &= 1 + i\,\lambda_j\,\sigma_j \\ e^{i\,\alpha(\tau,\,\lambda)} &= 1 + i\,\alpha(\tau,\,\lambda) \end{aligned}\right\} \tag{82}$$

and also, to sufficient accuracy,

$$\alpha(\tau,\,\lambda) = \alpha(0,\,0) + \tau_i \frac{\partial\,\alpha}{\partial\,\tau_i} + \lambda_j \frac{\partial\,\alpha}{\partial\,\lambda_j} + \frac{\tau_i\,\lambda_j}{2}\,\frac{\partial^2\,\alpha}{\partial\,\tau_i\,\partial\,\lambda_j} + \ldots \quad . \tag{83}$$

Clearly only the last term can be present, for if either τ_i or λ_j is set equal to zero, Eq. (81) reduces to an identity.

It is convenient to set

$$\frac{1}{2} \frac{\partial^2 \alpha}{\partial \tau_i \, \partial \lambda_j} = - C_{ij} \quad . \tag{84}$$

Upon substituting Eqs. (82) into Eq. (81), one finds

$$\tau_i \, \lambda_j (\sigma_i, \sigma_j) = i \, C_{ij} \, \tau_i \, \lambda_j \tag{85}$$

or, since τ_i, λ_j are arbitrary ,

$$(\sigma_i, \sigma_j) = i \, C_{ij} \; . \tag{86}$$

There is a strong restriction on the matrix C_{ij} which is imposed by the requirement that our representation be irreducible. From Schur's lemma the only matrix which may commute with all the matrices of such a representation is the unit matrix. Consequently, one must assume that the equation

$$C_{ij} \, \tau_i \, \lambda_j = 0 \tag{87}$$

never has a solution λ_j for a given set τ_i except $\lambda_i = 0$. Thus one can write

$$\det \; C_{ij} \neq 0 \quad . \tag{88}$$

Furthermore, from Eq. (86) one sees that C is antisymmetric:

$$C_{ij} = -C_{ji} \quad . \tag{89}$$

Now such a matrix can exist only in a space of an even number of dimensions.[6] This implies that the number of infinitesimal generators σ_i

must be even:

$$m = 2f \quad , \tag{90}$$

where f is an integer.

One sees that this point how the representation is adapted to the occurrence of pairs of infinitesimal generators that occur in a canonical formalism. Furthermore, it may be shown that any matrix C_{ij} with the properties described by the last three equations may be brought into the form of blocks along the main diagonal made up from units of

$$\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \tag{91}$$

by a linear change of basis, $\sigma_i \rightarrow \sigma'_1$. If one imagines this has been done, one arrives at the commutation relations by identifying the new generators σ'_i so obtained as follows:

$$\begin{aligned} \sigma'_1 &= x_1/\hbar \; , & \sigma'_3 &= x_2/\hbar \\ \sigma'_2 &= p_1/\hbar \; , & \sigma'_4 &= p_2/\hbar \end{aligned} \quad , \text{ etc.} \tag{92}$$

It seems to the author that the main point of Weyl's investigation has been dealt with. The foregoing argument shows clearly how closely the commutation relations are connected with the probability hypothesis of quantum mechanics. The author realizes that many important mathematical questions have been heuristically treated in this paper. It is hoped that this manner of treatment will be satisfactory to the average physicist.

SUMMARY

A discussion of the fundamental commutation relations in nonrelativistic quantum mechanics has been presented which shows how closely they are connected with simple physical and mathematical requirements imposed on the theory. The method of presentation is intended to amplify and clarify arguments that lead to them by more formal means. The restriction to nonrelativistic quantum mechanics which allows a simplified discussion in terms of translations, Galilean transformations, and rotations may be removed by the following scheme:

(a) translations → translations

(b) Galilean transformations and rotations → Lorentz transformations

(c) point-particle mechanics → field theory.

This program, which is treated in the paper by Schwinger,[3] leads to the fundamental commutation relations between field operators when augmented by the demand of time-reversal invariance. The very simplicity of the requirements leading to these commutation relations suggests that an attempt to modify the commutation relations between field operators must be based on a modification of the field equations of the theory.

ACKNOWLEDGMENTS

The author is indebted to the late Professor W. Pauli, who expressed in a conversation the conviction that the fundamental commutation relations were perfect, that there was nothing to be changed.

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2. P. A. M. Dirac, The Principles of Quantum Mechanics, Third Edition, (Oxford University Press, Oxford, England, 1947), p. 89 and p. 99.

3. Julian S. Schwinger, Phys. Rev. 82, 914 (1957).

4. This remark does not apply to time reversal.

5. Translations are, of course, contained in the Galilean transformations. They are discussed separately in this paper since it seems desirable to break up the discussion in such a way that it parallels the corresponding relativistic one.

6. See reference 1, appendix 3, p. 397.