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ON INTERACTING LOCAL QUANTUM FIELDS DESCRIBING MANY MASSES AND SPINS*

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ABSTRACT

We investigate the possible restrictions of locality on interacting quantum fields, which create or destroy many-particle states with different masses and spins. First we look at the vacuum expectation value of the commutator (or anticommutator) of two Heisenberg fields. Demanding that these fields transform as representations of the homogeneous Lorentz group and using the CPT theorem we pro· › that locality gives no relations between the different spectral functions, and the relations between the masses as functions of the spin are the same as those found by Abarbanel and Frishman for free fields. Then with the help of the Yang-Feldmann equations we look at the problem in perturbation theory. We find that demanding locality at each order in perturbation theory in a manner consistent with the general results found above, imposes restrictions on the possible forms of these interactions.

I. INTRODUCTION

Recently Abarbanel and Frishman[1] investigated the possible restrictions of locality on quantum fields which create or destroy single-particle states with different masses and spins and transforming under various representations of the homogeneous Lorentz group. Locality is imposed by the vanishing of the commutators (or anticommutators) of these fields at spacelike separations. Restrictions were sought among the masses considered as functions of the spin.

They reached the conclusion that in the case where the field is a finite dimensional irreducible Lorentz tensor, locality imposes no such restrictions, provided the usual connection between spin and statistics is satisfied. When the field transforms as a unitary irreducible representation of the homogeneous Lorentz group, however, the requirement of locality leads to the restriction that the masses are independent of the spin and all equal, assuming the masses are bounded below by some $m_o > 0$ for all spins.

However, the question as to whether their results persist when the fields undergo "interactions" was left open. This problem can be discussed in two different ways and this will be done in the present work.

First we look at the vacuum expectation value of the commutator (or anticommutator) of two Heisenberg fields, which now create or destroy many-particle states with different masses and spins. In this general case the requirement of locality could impose not only restrictions on the masses of these states considered as functions of the spin, but could also give relations between the different spectral functions associated with the various spin components of the fields. Demanding that these fields transform as representations of the homogeneous Lorentz group and using the CPT theorem we are able to prove that locality gives no relations between these spectral functions and the relations between the masses as functions of the spin are the same as for free fields.[1] This proof will be presented in Sec. II.

Another way is to look at this problem in perturbation theory. Without refering to any field equation we assume a "model interaction" and calculate the commutator (or anticommutator) of two interacting fields with the help of the Yang-Feldmann equations to some order in the coupling constant. Demanding locality at each order in perturbation theory in a manner consistent with the general results of the previous section imposes restrictions on the possible forms of these interactions. This is consistent with facts already known about Lagrangians having fields describing spins greater than $\frac{1}{2}$[2]. This approach will be discussed in Sec. III.

The conclusions will be found in the last paragraph.

Throughout the paper we adhere to the same convention and notation as in Ref. 1.

II. LOCALITY AND HEISENBERG FIELDS

In this section we present the proof, that the requirement of locality gives no relations between the different spectral functions and that the relations between the masses as functions of the spin are the same for interacting fields as for free fields.

Consider the vacuum expectation value of the product of two Heisenberg fields $\phi_{j\sigma}(x)$ and $\phi^{+}_{j'\sigma'}(y)$,

$$\langle 0|\, \phi_{j\sigma}(x)\, \phi^{+}_{j'\sigma'}(y)\, |0\rangle \,. \tag{2.1}$$

The fields $\phi_{j\sigma}(x)$ have the simple Lorentz transformation property

$$U[\Lambda]\, \phi_{j\sigma}(x)\, U[\Lambda]^{-1} = \sum_{j',\sigma'} D_{j\sigma;j'\sigma'}[\Lambda^{-1}]\, \phi_{j'\sigma'}(\Lambda x) \,. \tag{2.2}$$

In (2.1) we introduce a complete set of states and make use of the translation invariance of the theory:

$$\begin{aligned} &\langle 0|\phi_{j\sigma}(x)\, \phi^{+}_{j'\sigma'}(y)\,|0\rangle \\ &= \sum_{j'',\sigma''} \int \frac{d^3p}{2p_0(j'')} \langle 0|\phi_{j\sigma}(0)\,|\vec{p}m(j'')j''\sigma''\rangle\langle\vec{p}m(j'')j''\sigma''|\phi^{+}_{j'\sigma'}(0)\,|0\rangle \; e^{-ip(x-y)} \\ &= \sum_{j'',\sigma''} \int \frac{d^3p}{2p_0(j'')} \langle 0|\phi_{j\sigma}(0)\,|\vec{p}m(j'')j''\sigma''\rangle\langle 0|\phi_{j'\sigma'}(0)\,|\vec{p}m(j'')j''\sigma''\rangle^{*} e^{-ip(x-y)} \,. \end{aligned} \tag{2.3}$$

The physical states $|\vec{p}m(j)j\sigma\rangle$ are characterized by their three-momentum $\vec{p}$, the spin j, its projection on the z-axis σ, and the mass $m(j)$, which we allow to be a function of j. The four-momentum of the state is such that $p^2 = p_o^{\,2}(j) - \vec{p}^{\,2} = m^2(j)$. Under a homogeneous Lorentz transformation Λ, these states transform as

$$U[\Lambda]|\vec{p}m(j)j\sigma\rangle = \sum_{\sigma'} |\vec{\Lambda p}\, m(j)j\sigma'\rangle\, D^j_{\sigma'\sigma}[R_W], \tag{2.4}$$

where R_W is the Wigner rotation,

$$R_W = L^{-1}(\vec{\Lambda u})\ \Lambda\ L(\vec{u}), \qquad \vec{u} = \vec{p}/m(j), \tag{2.5}$$

and $D^j[R_W]$ is the usual rotation matrix for spin j.

To boost the four-vector $(\vec{p}, m(j))$ into $(\vec{0}, m(j))$, one chooses for Λ the inverse pure Lorentz transformation:

$$\Lambda = L^{-1}(\vec{u}) . \tag{2.6}$$

Using in (2.3) the transformation properties (2.4) and (2.2) with Λ given by (2.6), we obtain

$$\begin{aligned} &\langle 0|\phi_{j\sigma}(x)\ \phi^{+}_{j'\sigma'}(y)|0\rangle \\ &= \sum_{\substack{j'',\sigma'' \\ j_1,\sigma_1 \\ j_2,\sigma_2 \\ \sigma_3,\sigma_4}} \int \frac{d^3p}{2p_0(j'')}\ D_{j\sigma;j_1\sigma_1}[L(\vec{u})]\ D^{j''}_{\sigma_3\sigma''}[R_W]\ D_{j'\sigma';j_2\sigma_2}[L(\vec{u})]^*\ D^{j''}_{\sigma_4\sigma''}[R_W]^* \\ &\quad \times \langle 0|\phi_{j_1\sigma_1}(0)|\vec{0}m(j'')j\sigma_3\rangle\ \langle 0|\phi_{j_2\sigma_2}(0)|\vec{0}m(j'')j''\sigma_4\rangle^*\ e^{-ip(x-y)} . \end{aligned} \tag{2.7}$$

Because of rotational invariance, the matrix elements in (2.7) have the following form:

$$\langle 0|\phi_{j\sigma}(0)|\vec{0}m(j'')j''\sigma''\rangle = \delta_{jj''}\delta_{\sigma\sigma''}\ F(m(j''); j'') . \tag{2.8}$$

Since the rotation matrices are unitary, we have

$$\sum_{\sigma''} D^{j''}_{\sigma_3\sigma''}\ [R_W]\ D^{j''}_{\sigma''\sigma_4}\ [R_W]^+ = \delta_{\sigma_3\sigma_4} . \qquad (2.9)$$

Thus we find for (2.7)

$$\langle 0|\phi_{j\sigma}(x)\ \phi^+_{j'\sigma'}(y)|0\rangle$$

$$= \sum_{j'',\sigma''}\int\frac{d^3p}{2p_0(j'')}\ |F(m(j'');\ j'')|^2\ D_{j\sigma;j''\sigma''}\ [L(\tfrac{\vec{p}}{m(j'')})]\ D^+_{j''\sigma'';j'\sigma'}$$

$$\times [L(\tfrac{\vec{p}}{m(j'')})]\ e^{-ip(x-y)} . \qquad (2.10)$$

Now we consider the vacuum expectation value

$$\langle 0|\phi^+_{j'\sigma'}(y)\ \phi_{j\sigma}(x)|0\rangle \quad . \qquad (2.11)$$

Before we apply the same procedure as above, we relate the matrix elements $\langle 0|\phi^+_{j\sigma}(0)|\vec{p}m(j'')\ j''\sigma''\rangle$, which appear when we introduce a complete set of states in (2.11), to those in (2.3). This we achieve by using the CPT transformation law[2] for the fields $\phi_{j\sigma}(x)$:

$$CPT\ \phi_{j\sigma}(x)\ T^{-1}\ P^{-1}\ C^{-1} = \eta_C\ \eta_P\ \eta_T\ \phi^+_{j\sigma}(-x) \quad . \qquad (2.12)$$

Going through the same calculation, we find the same expression (2.10) with the exception that x and y are interchanged.

We introduce the function

$$P_{j\sigma;j'\sigma'}(\vec{p}) = \sum_{j'',\sigma''} (\vec{p}^2 + m^2(j''))^{-\frac{1}{2}}\ |F(m(j'');\ j'')|^2\ D_{j\sigma;j''\sigma''}$$

$$\times [L(\tfrac{\vec{p}}{m(j'')})]\ D^+_{j''\sigma'';j'\sigma'}\ [L(\tfrac{\vec{p}}{m(j'')})] , \qquad (2.13)$$

where $|F(m(j''); j'')|^2$ represents a generalized spectral function, which depends on the masses and the spins. Then we can write the vacuum expectation value of a commutator (or anticommutator) of two Heisenberg fields in the form

$$\langle 0|[\phi_{j\sigma}(x), \phi^{+}_{j'\sigma'}(y)]_{\epsilon}|0\rangle = \frac{1}{2}\int d^3p\, P_{j\sigma;j'\sigma'}(\vec{p})\, [e^{-ip(x-y)} + \epsilon\, e^{ip(x-y)}] , \tag{2.14}$$

where $\epsilon = \pm 1$ for an anticommutator (commutator).

From here on, the remainder of the proof follows exactly that of Abarbanel and Frishman[1]. For the sake of clarity we repeat here the main points and for the details we refer the reader to Ref. 1.

In the case where the field $\phi_{j\sigma}(x)$ transforms under the finite dimensional non-unitary representations of the homogeneous Lorentz group, the function $P_{j\sigma;j'\sigma'}(\vec{p})$ in (2.13) becomes

$$P_{j\sigma;j'\sigma'}(\vec{p}) = \sum_{j'',\sigma''} (\vec{p}^2 + m^2(j''))^{-\frac{1}{2}}\, |F(m(j'');j'')|^2\, \langle j\sigma|e^{-i\hat{p}\cdot\vec{K}\,\theta(j'')}|j''\sigma''\rangle$$

$$\times \langle j''\sigma''|e^{-i\hat{p}\cdot\vec{K}\,\theta(j'')}|j'\sigma'\rangle , \tag{2.15}$$

where $\sinh\theta(j'') = |\vec{p}|/m(j'')$. Note that here the operators $\vec{K}$ are anti-hermitian. We now consider the quantity $P^{(a,b)}_{j\sigma(J);j'\sigma'}(\vec{p})$ for the irreducible representations $[a, b]$:

$$P^{(a,b)}_{j\sigma(J);j'\sigma'}(\vec{p}) = \sum_{\sigma''} \langle j\sigma|e^{-i\hat{p}\cdot\vec{K}\,\theta(J)}|J\sigma''\rangle\langle J\sigma''|\, e^{-i\hat{p}\cdot\vec{K}\,\theta(J)}|j'\sigma'\rangle. \tag{2.16}$$

One can prove that this quantity $P^{(a,b)}_{j\sigma(J);j'\sigma'}(\vec{p})$ is a polynomial in the components of p and when $p \to -p$, it picks up a phase $(-)^{2(a+b)}$.

Therefore we can take $P^{(a,b)}_{j\sigma(J);j'\sigma'}(\vec{p})$ out of the locality integral and the vacuum expectation value of the commutator (or anticommutator) becomes a finite number of derivatives on the usual causal function $\Delta(x-y)$ and therefore, vanishes for $x - y$ spacelike. Thus the vacuum expectation value of the commutator of a field transforming as $[a,b]$ and its adjoint, taking the usual connection between spin and statistics, $\epsilon = -(-)^{2(a+b)}$, can be written as

$$\langle 0|[\phi_{j\sigma}(x),\ \phi^{+}_{j'\sigma'}(y)]_{\epsilon}\ |0\rangle$$

$$= \sum_{J} |F(m(J);\ J)|^{2}\ P^{(a,b)}_{j\sigma(J);j'\sigma'}(\vec{i\partial})\ [i(2\pi)^{3}\ \Delta\ (x-y,\ m^{2}(J))]. \quad (2.17)$$

This quantity vanishes for $x-y$ spacelike and thus establishes locality for the finite dimensional case, without any conditions on the spectral functions $|F(m(J);\ J)|^{2}$ and on the mass spectrum as a function of spin.

If the fields $\phi_{j\sigma}(x)$ transform under the unitary, irreducible representations of the homogeneous Lorentz group, the function $P_{j\sigma;j'\sigma'}(\vec{p})$ in (2.13) becomes

$$P_{j\sigma;j'\sigma'}(\vec{p}) = \sum_{j'',\sigma''} (\vec{p}^{2} + m^{2}(j''))^{-\frac{1}{2}}\ |F(m(j'');j'')|^{2}$$

$$\times \langle j\sigma|e^{-i\hat{p}\cdot\vec{K}\,\theta(j'')}|j''\sigma''\rangle\ \langle j''\sigma''|e^{i\hat{p}\cdot\vec{K}\,\theta(j'')}|j'\sigma'\rangle, \qquad (2.18)$$

where the operators $\vec{K}$ are now hermitian. If we look at the vacuum expectation value of the equal time commutator in (2.14), then in the case of Bose statistics (Fermi statistics) we have to consider the antisymmetric (symmetric) function $P_{j\sigma;j'\sigma'}(\vec{p}) - P_{j\sigma;j'\sigma'}(-\vec{p})$ $(P_{j\sigma;j'\sigma'}(\vec{p}) + P_{j\sigma;j'\sigma'}(-\vec{p}))$

and require that it be a polynomial of degree $2N + 1$ $(2N)$ for locality. If we make the physical assumption that the mass spectrum is bounded below, $m(j) \geqslant m_o > 0$, these conditions lead to the conclusion that the only mass spectrum allowed is that of equal masses: $m(j) = m$. Again there are no relations between the spectral functions. They are only constrained to be polynomials in j.

III. INTERACTING FIELDS IN PERTURBATION THEORY

It is well-known[2] that for conventional Lagrangians involving fields of spin higher than 0 and $\frac{1}{2}$ that one has to add so called contact terms to insure the locality of the Hamiltonian densities $H(x)$:

$$[H(x), H(y)] = 0 \quad \text{for} \quad (x - y)^2 < 0. \tag{3.1}$$

We shall show in this section that certainly not every manifestly Lorentz invariant Lagrangian is consistent with locality to each order in perturbation theory, when these Lagrangians involve fields describing many masses and spins. We do this by studying the commutators (or anticommutators) of interacting fields in perturbation theory and demanding that they vanish for spacelike separations to every order in a manner consistent with the general results of the previous section concerning mass spectra.

We construct the interacting field $\phi_{j\sigma}(x)$ with the help of the Yang-Feldmann equation

$$\phi_{j\sigma}(x) = \phi_{j\sigma}^{in}(x) + \int d^4x' \, \Delta_R(x - x'; m^2(j)) \, \Gamma_{j\sigma}(x'), \tag{3.2}$$

where $\Delta_R(x)$ is the retarded singular function, $\Delta_R(x) = -\theta(x)\,\Delta(x)$. Equation (3.1) can be viewed as the integrated form of the equation of motion

$$(\Box_x + m^2(j)) \, \phi_{j\sigma}(x) = \Gamma_{j\sigma}(x), \tag{3.3}$$

where $\Gamma_{j\sigma}(x)$ is some combination of field operators expressing the interaction and transforming like $\phi_{j\sigma}(x)$ under the Lorentz group. Now the commutator (or anticommutator) takes the following form:

$$[\phi_{j\sigma}(x),\ \phi^{+}_{j'\sigma'}(y)]_{\epsilon} = [\phi^{in}_{j\sigma}(x),\ \phi^{in^+}_{j'\sigma'}(y)]_{\epsilon}$$

$$+ \int d^4y'\ \Delta_R(y - y'\ ;\ m^2(j'))\ [\phi^{in}_{j\sigma}(x),\ \Gamma^{+}_{j'\sigma'}(y')]_{\epsilon}$$

$$+ \int d^4y'\ \Delta_R(x - y';\ m^2(j))\ [\Gamma_{j\sigma}(y'),\ \phi^{in^+}_{j'\sigma'}(y)]_{\epsilon} \qquad (3.4)$$

$$+ \iint d^4y'd^4y''\Delta_R(x - y';\ m^2(j))\ \Delta_R(y - y'';\ m^2(j'))\ [\Gamma_{j\sigma}(y'),\ \Gamma^{+}_{j'\sigma'}(y'')]_{\epsilon}\ .$$

For commutators $(\epsilon = -1)$ a trivial solution to our problem is, of course, when $\Gamma_{j\sigma}(x)$ is a c-number function. But this is generally not the case.

Before we go into the discussion of (3.4), we should like to remark, that when the fields above are just scalar fields, then in a $\lambda\phi^3$ or $\lambda\phi^4$ theory locality can easily be demonstrated at least up to second order in λ. This is made possible by the presence of only one mass m in the problem, as we shall see later.

We wish to point out here only that the vanishing of the integrals in (3.4) for $x - y$ spacelike is not trivial and therefore puts very stringent conditions on possible choices for $\Gamma_{j\sigma}(x)$. Consider for example the case, where

$$\Gamma_{j\sigma}(x) = \lambda \sum_{j',\sigma'} \prod_{j\sigma;j'\sigma'} \phi_{j'\sigma'}(x)\ . \qquad (3.5)$$

Here $\prod_{j\sigma;j'\sigma'}(x)$ can belong to one of three classes:

1. c-number functions;
2. operator functions, made up of fields commuting with $\phi_{j\sigma}(x)$;
3. operator functions, made up of fields not commuting with $\phi_{j\sigma}(x)$.

For simplicity, and since we only intend this as a demonstration, we shall consider only classes 1 and 2. We also restrict ourselves to fields transforming under finite dimensional representations of the homogeneous Lorentz group.

Introducing (3.5) into (3.4) and looking only at the integrals to first order in λ, we get the condition, that the following integral has to vanish, where $x - y$ is spacelike:

$$\int d^4y' \Big\{ \Delta_R(y - y'; m^2(j'))\, P^{(a,b)}_{j\sigma(J);j'\sigma'}(i\vec{\partial}_{y'}) \prod_{j''\sigma'';j'\sigma'}(y')\Delta_R(x-y';m^2(J)) - \Delta_R(x-y';m^2(j)) \prod_{j\sigma;j''\sigma''}(y')\, P^{(a,b)}_{j''\sigma''(J);j'\sigma'}(i\vec{\partial}_{y'})\Delta_R(y-y';m^2(J)). \tag{3.6}$$

Here we have used the following form of the commutator for free fields, according to Ref. 1:

$$[\phi^{in}_{j\sigma}(x), \phi^{in^+}_{j'\sigma'}(y)]_\epsilon = \sum_J \frac{F^2(J)}{m^2(J)} P^{(a,b)}_{j\sigma(J);j'\sigma'}(i\vec{\partial}) \, [i(2\pi)^3 \Delta(x-y;m^2(J))]. \tag{3.7}$$

Equation (3.6) then evidently is a very stringent condition on the possible forms for $\prod_{j\sigma;j'\sigma'}(x)$, for it is not trivially satisfied.

In the case of scalar fields, the quantities P and $\prod$ have no spin dependence and only one mass appears and (3.6) is evidently zero.

IV. CONCLUSIONS

We have shown that the requirement of locality for interacting quantum fields describing many masses and spins gives no relations between the different spectral functions associated with the various spin components of the fields and the relations between the masses as functions of the spin are the same as for free fields.

Interacting quantum fields are then discussed in perturbation theory without referring to any specific field equations besides the Klein-Gordon equation. Demanding that the commutators (or anticommutators) of these fields vanish for spacelike separations to every order in a manner consistent with the general results above, puts very stringent conditions on the possible forms for the "interactions".

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FOOTNOTES AND REFERENCES

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