A General Framework for the Capacity Analysis of Wireless Ad Hoc Networks

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***Abstract*—In this paper, we introduce a general framework for computing the throughput capacity of wireless ad-hoc networks under all kinds of information dissemination modalities. We consider point-to-point communication for unicast, muticast, broadcast and anycast routing under the physical model assumption. The general communication is denoted as (n, m, k)-cast where n is the number of nodes in the network, m is the number of destinations on (n, m, k)-cast group and $k(k \leq m)$ is the number of destinations that receive packets from the source in each (n, m, k)-cast group. For example, $(m = k = 1)$ and $(m = k = n)$ represent unicast and broadcast routings respectively. We demonstrate that the upper bound of throughput capacity is given by $O(\sqrt{m}(\sqrt{n}k)^{-1})$ bits/second. The lower bound of throughput capacity is computed as $\Omega(\sqrt{m}(nkd(n))^{-1})$, $\Omega((nkd^2(n))^{-1})$ and $\Omega(n^{-1})$ bits/second when $m = O(d^{-2}(n))$, $\Omega(k) = (d^{-2}(n)) = O(m)$ and $\Omega(d^{-2}(n)) = k$ respectively, where $d(n)$ is a network parameter. The upper bound capacity is achieved based on an (n, m, k)-cast tree constructed for routing and transport capacity while the lower bound capacity is achieved based on TDMA scheme and connected cell graph along (n, m, k)-cast tree.**

I. Introduction

The seminal work by Gupta and Kumar [1] motivated many researchers to investigate further the capacity of wireless ad hoc networks. Recent research activities focused on the capacity of wireless ad hoc networks for different types of information dissemination such as unicast, broadcast (e.g., [2]–[4]) and multicast (e.g., [5]–[7]). Computation of all kinds of information dissemination plays an important role in understanding the fundamental limits of wireless ad hoc networks. Recent work [8] has shown that all forms of information dissemination in wireless ad hoc networks can be unified into a single (n, m, k)-cast model. (n, m, k)-cast is a general communication model where n is the number of nodes in the network, m is the number of destinations in an (n, m, k)-cast group and $k(k \leq m)$ is the number of destinations that receive packets from the source in each (n, m, k)-cast group. In doing so, unicast routing, broadcast routing, multicast routing and various forms of anycast routing can be defined when $(m = k = 1)$, $(m = k = n)$, $(m = k \leq n)$ and $(k \leq m \leq n)$, respectively. However, such prior work [8] concentrated on the protocol model, where every node in the network has the same transmission range $r(n)$. The physical model is more realistic channel model than the protocol model. This paper presents the capacity of (n, m, k)-cast communication under the physical model assumption. We assume that a constant data rate is guaranteed under the physical model as long as the signal to interference plus noise ratio (SINR) is greater than a constant non-zero value.

Section II summarizes prior work on the capacity of wireless ad hoc networks. The network model is introduced in Section III. In Section IV-A, we present the capacity of (n, m, m)-casting, which corresponds to unicasting, broadcasting or multicasting. Under this condition, we show that[1] $O\left(1/(\sqrt{mn})\right)$ bits per second is an upper bound, while $\Omega\left(1/(nd(n)\sqrt{m})\right)$ and $\Omega\left(1/n\right)$ are the lower bounds for the capacity of (n, m, m)-casting when $m = O(d^{-2}(n))$ and $m = \Omega(d^{-2}(n))$, respectively. Section IV-B addresses the capacity of (n, m, k)-casting. We demonstrate that $O\left(\sqrt{m}/\left(k\sqrt{n}\right)\right)$ is the upper bound of (n, m, k)-cast. In case of the lower bound, there are three capacity regions according to the range of the parameter $d(n)$. When $m = O(d^{-2}(n))$, $\Omega(k) = d^{-2}(n) = O(m)$ and $\Omega(d^{-2}(n)) = k$, the corresponding lower bounds are $\Omega\left(\sqrt{m}/\left(nkd(n)\right)\right)$, $\Omega\left(1/\left(nkd^2(n)\right)\right)$ and $\Omega\left(1/n\right)$, respectively. When $m = k = \Theta(1)$, this result also matches the well known results on the throughput capacity of ad hoc networks for unicasting under the physical model by Gupta and Kumar. Section V concludes this paper and presents some implications of our results.

II. Related Work

Gupta and Kumar [1] computed the capacity of wireless ad hoc networks for n static nodes and the multi-pair unicast routing assumption. They derived the transport capacity in random and arbitrary networks for the protocol and physical models. As a result of this, there have been many contributions that try to improve the capacity of wireless networks. Furthermore, there are recent advances on the study of wireless ad hoc networks for various routing schemes such as multicast and broadcast.

Franceschetti et al. [9] showed that enhanced throughput capacity is possible under the physical model by utilizing highway paths based on percolation theory. They proved that, with a long-range routing scheme, the upper and lower bounds

[1] Θ, Ω and O are the standard order bounds.

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of throughput capacity in random wireless networks have the same order of $\Theta\left(1/\sqrt{n}\right)$ under the physical model assumption. Furthermore, Ozgur et al. [10] showed that hierarchical MIMO cooperation provides linear scaling laws for wireless ad hoc networks. Toumpis and Goldsmith [11] studied capacity regions of wireless ad hoc networks with an arbitrary number of nodes and topology. They showed that combining multihop routing, the ability for concurrent transmissions, and successive interference cancelation at the receiver side, significantly increase the capacity of ad hoc networks. Garcia-luna-aceves et al. [12] proved that the throughput capacity under physical model can be increased by a factor of $\Theta\left((\log n)^{\frac{\alpha-2}{2\alpha}}\right)$ compared to Gupta and Kumar's result when nodes are equipped with multiple packet reception and a successive interference cancelation decoding scheme.

In addition, there have been research results for various kinds of information dissemination schemes such as broadcast and multicast. Tavli [2] showed that $\Theta(n^{-1})$ is an upper bound per node broadcast capacity in an arbitrary network. Zheng [3] studied the behavior of information dissemination in power-constrained wireless networks in terms of the broadcast capacity and information diffusion rate in both random extended and dense networks. Keshavarz et al. [4] extended Zheng's work by considering the interference effect in general wireless networks and proposed the most general case for broadcast capacity results with multi hop routing under the protocol model. In [13], they extended the broadcast capacity for the physical model and the generalized physical model that can be derived from Shannon's formula [14]. Jacquet and Rodolakis [5] showed that, in massively dense networks, the multicast capacity can be decreased by a factor of $O(\sqrt{n})$ compared to the unicast capacity result [1]. Li et al. [7] unified the capacity of wireless ad-hoc networks utilizing unicast, multicast, and broadcast routing schemes. More recent work [8] provided a general framework for the capacity of wireless ad-hoc networks and for all forms of information dissemination including anycast and manycast under the protocol model.

III. NETWORK MODEL AND PRELIMINARIES

We consider a random wireless dense network where n nodes are distributed according to the Poisson point process over a unit square area. In this network model, the density of the network goes to infinity as the number of nodes increase. The channel is defined based on the path-loss propagation model. In addition to this, we employ the physical model introduced by Gupta and Kumar [1] to analyze the capacity for dense networks. Let X_i and $X_{R(i)}$ denote the location of a node i and its receiving node respectively. Then SINR between X_i and $X_{R(i)}$ is defined as

$$SINR = \frac{\frac{P}{|X_i - X_{R(i)}|^\alpha}}{N + \sum_{k\neq i} \frac{P}{|X_k - X_{R(i)}|^\alpha}}, \tag{1}$$

where N is the ambient noise power and $X_k(i \neq k)$ is the interfering node. The following definitions and lemmas describe the basic notion for our analysis of the (n,m,k)-cast capacity.

Definition 3.1: Physical Model
In this analysis, a successful transmission occurs if SINR$\geq \beta$. Thus if SINR$\geq \beta$ at the receiver, the data rate between the transmitter-receiver pair is W bits/second.

Definition 3.2: Feasible Throughput capacity
In a dense random wireless ad hoc network with n nodes in which each source node transmits its packets to k out of m destinations, the per node (n,m,k)-cast throughput capacity is defined as

$$C_{m,k}(n) = \frac{1}{n}\sum_{i=1}^{n} \lambda^i_{m,k}(n) \tag{2}$$

where $\lambda^i_{m,k}(n)$ is the throughput capacity of source i transmitting packets to k out of its m chosen destinations in a network of n nodes, and with all such k nodes receiving the information within a finite time interval.

Definition 3.3: Order of throughput capacity
$C_{m,k}(n)$ is said to be of order $\Theta(f(n))$ bits/second if there exist deterministic positive constants c and c' such that

$$\begin{cases} \lim_{n\to\infty} \text{Prob } (C_{m,k}(n) = cf(n) \text{ is feasible}) = 1 \\ \liminf_{n\to\infty} \text{Prob } (C_{m,k}(n) = c'f(n) \text{ is feasible}) < 1. \end{cases} \tag{3}$$

Definition 3.4: Transport capacity
The transport capacity [1] in a random wireless network is defined as the maximum bit-meters per second which can be achieved in aggregate by optimally operating the network. Therefore,

$$C_T = \sup \sum_{i\neq j} C_{ij}|X_i - X_j| \tag{4}$$

where C_{ij} is the data rate defined from each node i to each node j.

Definition 3.5: Euclidean Minimum Spanning Tree (EMST): Consider a connected undirected graph $G = (V,E)$ where V and E are sets of vertices and edges in the graph G, respectively. The EMST of G is a spanning tree of G with the total minimum Euclidean distance between connected vertices of this tree.

Definition 3.6: (n,m,k)-cast tree: An (n,m,k)-cast tree is a minimum set of nodes that connect a source node of an (n,m,k)-cast with all its intended m destinations, in order for the source to send information to k of those destinations. The selection of k out of m is optimum.
We can also define (n,m,m)-cast tree (i.e., when $m = k$) as a m-cast tree in a similar manner.

Definition 3.7: Minimum Euclidean (n,m,k)-cast Tree (MEMKT): The MEMKT of an (n,m,k)-cast is an (n,m,k)-cast tree in which the k destinations that receive information from the source among the m receivers of the (n,m,k)-cast have the minimum total Euclidean distance. When $k = m$, we denote by minimum Euclidean m-cast tree (MEMT) an (n,m,m)-cast tree with a total minimum Euclidean distance.

Lemma 3.8: Let $f(x)$ denote the node probability distribution function in the network area. Then, for large values of n and $d > 1$, the $\overline{\|\text{EMST}\|}$ is tight bounded as

$$\overline{\|\text{EMST}\|} \;=\; \Theta\left(c(d)n^{\frac{d-1}{d}}\int_{R^d} f(x)^{\frac{d-1}{d}}dx\right), \qquad (5)$$

where d is the dimension of the network. Note that both $c(d)$ and the integral are constants and not functions of n. When $d = 2$, then $\overline{\|\text{EMST}\|} = \Theta\left(\sqrt{n}\right)$.

IV. The Capacity In Physical Model

A. The Capacity of (n, m, m)-Cast

In order to compute the capacity of (n, m, k)-casting under the physical model, we first derive the capacity of (n, m, m)-cast. The (n, m, m)-cast model corresponds to unicasting, multicasting, and broadcasting when $m = 1$, $m < n$, and $m = n$, respectively.

1) Upper Bound: Gupta and Kumar derived the transport capacity as the following lemma [1].

Lemma 4.1: Assuming that each node can transmit at W bits/second over a wireless channel shared by all nodes, the transport capacity for an arbitrary network where n nodes are arbitrarily located over an area of A is $\Theta(W\sqrt{An})$ bits-meter/sec.

According to the physical model in Definition 3.1, the transmission range between any two nodes in (n, m, m)-cast depends on the SINR at the receiver side. Hence, given that the successful communication condition is SINR$\geq \beta$, successful communication can only occur between transmitter-receiver pairs that satisfy this condition. It was shown in [15] that such successful communication condition for the physical model in random networks can be translated into the successful communication criterion for the protocol model in an arbitrary network when $\beta = (1 + \Delta)^{\alpha}$.

In (n, m, m)-cast communication, when a node transmits a packet, we can assume two different approaches to compute the capacity [16]. We can either assume that, for each transmission, only a single node receives the packet or multiple nodes within an area of transmission range receive the packet. The former concept is called unicast communication while the latter approach corresponds to broadcast [16]. Keshavarz et al. used these two concepts to compute the multicast capacity in wireless ad hoc networks for both cases. In this paper, we compute the upper bound (n, m, m)-cast throughput capacity when each transmitter is only allowed to transmit packets to a single relay or destination based on the unicast concept.

Lemma 4.2: The per-node throughput capacity of (n, m, m)-cast in dense wireless ad-hoc networks is upper bounded by $O\left(\frac{1}{n} \times \frac{\sup \Sigma_{i\neq j} d_{ij}C_{ij}}{\overline{\|\text{MEMT}\|}}\right)$ under the physical model.

Proof: Given that the throughput capacity for the node i is defined as $\lambda^i_{m,m}(n)$, the throughput capacity in aggregate is equal to $nC_{m,m}(n) = \sum_{i=1}^{n} \lambda^i_{m,m}(n)$. To find out the per-node throughput capacity, we define $d^i_{m,m}(n)$ as the total distance that the generated bits from the node i travel to its m destinations. Now it is obvious that the total bit-distance product in (n, m, m)-cast should be upper bounded by the transport capacity in the network. Therefore,

$$\sum_{i=1}^{n} \lambda^i_{m,m}(n)d^i_{m,m}(n) \leq \sup \sum_{i\neq j} d_{ij}C_{ij}, \qquad (6)$$

Since $d^i_{m,m}(n) \geq \overline{\|\text{MEMT}\|}$, the following inequality can be derived.

$$\overline{\|\text{MEMT}\|}\sum_{i=1}^{n} \lambda^i_{m,m}(n) \leq \sum_{i=1}^{n} \lambda^i_{m,m}(n)d^i_{m,m}(n) \qquad (7)$$

Combining the above two inequalities and the definition of (n, m, m)-cast capacity, we arrive at

$$\overline{\|\text{MEMT}\|}nC_{m,m}(n) \leq \sup \sum_{i\neq j} d_{ij}C_{ij}. \qquad (8)$$

■

Next, we derive the upper bound of the transport capacity for the random wireless network under physical model.

Lemma 4.3: The transport capacity for random networks under the physical model is $\Theta\left(W\sqrt{An}\right)$ bit-meters per second.

Proof: From [15], we know that the successful communication condition under the physical model in a random network is related to the protocol model in an arbitrary network. Accordingly, the upper-bound transport capacity for a random network under the physical model is $\Theta\left(W\sqrt{An}\right)$, which was first proved by Gupta and Kumar in [1]. ■

Based on these observations, the following theorem states the upper bound for the throughput capacity of (n, m, m)-cast.

Theorem 4.4: In a dense wireless ad hoc network with (n, m, m)-cast, the upper bound per node throughput capacity under the physical model is given by

$$C_{m,m}(n) = O\left(1/\sqrt{nm}\right). \qquad (9)$$

Proof: Assuming that there are $m+1$ nodes in (n, m, m)-cast tree, it is obvious that $\overline{\|\text{MEMT}\|}$ is equal to $\Theta\left(\sqrt{m}\right)$ from (5). Now the proof is immediate by replacing $\overline{\|\text{MEMT}\|}$ with $\Theta\left(\sqrt{m}\right)$ and combining Lemmas 4.2 and 4.3. ■

Adopting the broadcast concept for the network, a transmitter can simultaneously deliver packets to multiple destinations or relays spread over an area where the successful communication in the physical model is satisfied. Thus, to find out the upper bound of the throughput capacity based on the broadcast concept, we have to consider the consumed area used to route packets from source to destinations as a channel usage instead of the $\overline{\|\text{MEMT}\|}$. Recently in [16], Keshavarz showed that the upper bound of the multicast per node throughput capacity is $C_{m,m}(n) = O\left(1/n\right)$ when we utilize broadcast concept in the network. Due to page limitations, the analysis will be the subject of future work.

Therefore, we conclude that the upper bound of the (n, m, m)-cast throughput capacity is $C_{m,m}(n) = O\left(\frac{1}{\sqrt{nm}}\right)$ when the unicast concept of communication is considered.

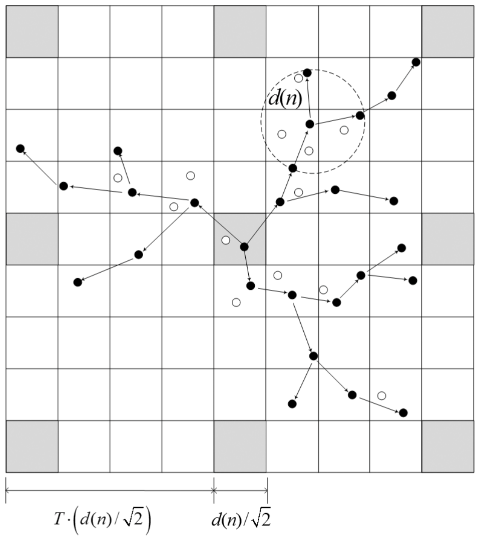

Fig. 1. Cell graph construction used to derive a lower bound on capacity. The solid dots are used to connect (n, m, m) cast tree while the blank dots do not act.

2) Lower Bound: The lower bound for (n, m, m)-cast is derived using a TDMA scheme as shown in Fig. 1. To construct the TDMA scheme, cells with the same side length of $d(n)/\sqrt{2}$ are grouped into T^2 non-interfering groups. Note that, in the physical model, there is no common communication range and in order for this scheme to work for the physical model we need to derive the condition under which the SINR condition is satisfied. By choosing a common value for $d(n)$, we derive a loose lower bound that can potentially be improved utilizing percolation theory [9]. The communication is divided into T^2 time slots. In each time slot, every node in the same group transmits packets with a common transmission power P. Furthermore, we will define P as a function of $d(n)$.

Lemma 4.5: Under the physical model, by properly choosing TDMA parameter T, a particular node in a cell can successfully transmit to any other nodes placed within a distance of $d(n)$.

Proof: First note that, in order to use a common $d(n)$, we need to assure that the physical model condition is satisfied. We showed that the physical model in a random network is equivalent to the protocol model in an arbitrary network [1], [15]. We can thus achieve the lower bound for the capacity by computing the upper bound for interference at the receiver. Figure 1. demonstrates the nodes that can simultaneously transmit in shaded cells while the physical model criterion is satisfied. Clearly, the interference is maximized when the interfering nodes have the closest distance to the receiver node, i.e., $\frac{d(n)}{\sqrt{2}}(iT-2)$ for $i \in I = \{1, 2, \ldots\}$. Therefore, the total interference experienced by each node is given by

$$\sum_{i=1}^{\infty} 8i \frac{P}{\{\frac{d(n)}{\sqrt{2}}(i \cdot T - 2)\}^{\alpha}}. \tag{10}$$

The SINR can be computed as

$$\begin{aligned} SINR &\geq \frac{\frac{P}{(d(n))^{\alpha}}}{N + \sum_{i=1}^{\infty} 8i \frac{P}{\{\frac{d(n)}{\sqrt{2}}(i \cdot T-2)\}^{\alpha}}} \\ &= \frac{P}{N\,(d(n))^{\alpha} + \frac{8P\left(\sqrt{2}\right)^{\alpha}}{T^{\alpha}} \sum_{i=1}^{\infty} \frac{i}{(i-\frac{2}{T})^{\alpha}}} \geq \beta \end{aligned} \tag{11}$$

$\sum_{i=1}^{\infty} \frac{i}{(i-\frac{2}{T})^{\alpha}}$ has a bounded value of c_1 when $\alpha \geq 2$. By solving this equation with respect to $d(n)$, we arrive at

$$d(n) \leq \left(\frac{P}{N}\left(\frac{1}{\beta} - \frac{8\left(\sqrt{2}\right)^{\alpha} c_1}{T^{\alpha}}\right)\right)^{\frac{1}{\alpha}}. \tag{12}$$

Equation (11) can be also solved with respect to T.

$$T \geq \left(\frac{8c_1\left(\sqrt{2}\right)^{\alpha}}{\frac{1}{\beta} - \frac{N(d(n))^{\alpha}}{P}}\right)^{\frac{1}{\alpha}} \tag{13}$$

Therefore, any node in a cell can successfully transmit to any other node placed within a distance of $d(n)$ when T is satisfies in Eq. (13). It also implies that T is a function of α, β and $d(n)$. In this paper, the TDMA parameter that satisfies (13) is denoted as $T(\alpha, \beta, d(n))$. As mentioned earlier, we choose the transmit power as a function of transmission range, i.e., $P = k(d(n))^{\alpha}$ where k is a constant value. Under this assumption, the TDMA parameter is not a function of n. ■

Next we show that there exists a minimum $d_{\min}(n)$ which guarantees a connected cell graph for any arbitrary $\|\text{MEMT}\|$ under the physical model.

Lemma 4.6: If $d(n) = \Omega\left(\sqrt{\log(n)/n}\right)$ and the condition in (13) is satisfied, the cell graph is connected under the physical model based on our TDMA scheme.

Proof: It was proved in [17] that the longest edge M_n of the nearest neighbor graph (NNG) has the following property.

$$\lim P[\pi \cdot n \cdot M_n^2 - \log(n) \leq a] = \exp(-e^{-a}), a \in R \tag{14}$$

If a is an increasing function of n, the probability that $M_n \leq \sqrt{\frac{\log(n)+a}{n\pi}}$ goes to 1 as n tends to infinity. In this paper we will set a as $\log n \cdot (\pi - 1)$.

It is also proved in [17] that the longest edge of NNG is asymptotically the same as the Euclidean minimal spanning tree. Thus, by defining the side length of the cell as $d(n) \geq M_n$ and setting up the condition $T(\alpha, \beta, d(n))$ to guarantee successful transmissions, a particular node can successfully relay packets to its adjacent nodes existing within $d(n)$. This also implies that the minimum guaranteed $d_{\min}(n)$ is equal to $\sqrt{\log(n)/n}$. Therefore, any two neighboring nodes on $\|\text{MEMT}\|$ can be connected based on our TDMA scheme if the side length of cells is greater than $d_{\min}(n)/\sqrt{2}$.

Now it is obvious that two neighboring cells are connected under the physical model if the distance from a particular node to its receiver nodes in the neighboring cells is within $d(n)$. Otherwise, a particular node should exploit relay nodes in the same cell to connect two neighboring cells. Since we already know that with high probability a particular node can find relay nodes when $d(n) \geq d_{\min}(n)$, we can construct a connected cell graph by connecting relay nodes on $\overline{\|\text{MEMT}\|}$ in a multi-hop fashion. ■

Next we prove that, based on our TDMA scheme, any two neighboring cells can be connected in finite hops through the nodes on $\overline{\|\text{MEMT}\|}$.

Lemma 4.7: Assume that nodes u and v are located in two adjacent cells in an MEMT. Then, the number of hops between these two nodes are a constant value.

Proof: Since the graph is a connected graph, then there is always a path between u and v. Further, the number of hops in any cell is at most two. Therefore, there is a finite number of relays between these two nodes either directly between the two adjacent cells or through some of the eight cells surrounding any single cell. ■

Now it is obvious that, based on our TDMA scheme, the condition for $d(n)$ is $\left(\sqrt{\frac{\log(n)}{n}}\right) \leq d(n) \leq \left(\frac{P}{N}\left(\frac{1}{\beta} - \frac{8\left(\sqrt{2}\right)^{\alpha} c_1}{T^{\alpha}}\right)\right)^{\frac{1}{\alpha}}$ in order to assure connectivity and physical model criterion in the network. It is easy to show that the upper bound is always greater than the lower bound when $\left(\frac{k}{N}\left(\frac{1}{\beta} - \frac{8\left(\sqrt{2}\right)^{\alpha} c_1}{T^{\alpha}}\right)\right)^{\frac{1}{\alpha}} \geq 1$. With this condition for $d(n)$, we have shown that $T(\alpha, \beta, d(n))$ does not increase with n as n goes to infinity. The result implies that our TDMA scheme does not change the order of throughput capacity.

Note that the (n, m, m)-cast tree based on the TDMA scheme is a function of transmission range $d(n)$. Therefore, the optimum m-cast tree will depend on the transmission range. We define $\overline{\#\text{MEMTC}(d(n))}$ as the average number of cells that contains an (n, m, m)-cast tree. The following lemma presents the achievable lower bound capacity for the (n, m, m)-cast.

Lemma 4.8: The achievable lower bound of the (n, m, m)-cast capacity is

$$C_{m,m}(n) = \Omega\left(\frac{1}{\overline{\overline{\#\text{MEMTC}(d(n))}}} \times \frac{1}{T\left(\alpha, \beta, d(n)\right)^2 n d^2(n)}\right). \tag{15}$$

Proof: It is obvious that the maximum simultaneous transmitting cells based on TDMA scheme are at most $\frac{1}{T(\alpha,\beta,d(n))^2 \cdot d^2(n)/2}$. Lemma 4.7 proves that there is a finite number of hops to traverse from one cell to its adjacent cell. Since the total number of cells in (n, m, m)-cast is $\overline{\#\text{MEMTC}(d(n))}$, then it is easy to see that the per-node lower bound capacity is given by $\Omega\left(\frac{1}{\overline{\overline{\#\text{MEMTC}(d(n))}}} \times \frac{1}{T(\alpha,\beta)^2 n d^2(n)/2}\right)$, which proves the lemma. ■

Given the above lemma, to express the lower bound of $C_{m,m}(n)$ as a function of network parameters, we need to compute the tight bound of $\overline{\#\text{MEMTC}(d(n))}$, which we do next.

Lemma 4.9: The average number of the cells that belongs to a (n, m, m)-cast tree satisfy the following upper bound.

$$\overline{\#\text{MEMTC}(d(n))} = \min\left(\Theta\left(\sqrt{m}/d(n)\right), \Theta\left(d^{-2}(n)\right)\right) \tag{16}$$

Proof: Because the maximum number of cells in this network is equal to $\Theta\left(d^{-2}(n)\right)$, it is clear that one tight bound for $\overline{\#\text{MEMTC}(d(n))}$ is this value. That is, $\overline{\#\text{MEMTC}(d(n))}$ cannot exceed the total number of cells in the network and will cover all cells when the number of multicast destinations is large enough. On the other hand, the total Euclidean distance of the (n, m, m)-cast tree was shown earlier to be $\Theta(\sqrt{m})$. Because $d(n)$ is a network parameter that limits the the transmission range in this TDMA scheme, the number of cells for this (n, m, m)-cast tree must be $\Theta\left(\sqrt{m}/d(n)\right)$, i.e., $\overline{\#\text{MEMTC}(d(n))} = \Theta\left(\sqrt{m}/d(n)\right)$. The actual tight bound clearly is the minimum of these two extreme values in the network, which is a function of the topology of the network and this proves the lemma. ■

By combining Lemmas 4.9 and 4.8, the following theorem can be presented.

Theorem 4.10: The achievable lower bound of the (n, m, m)-cast capacity when $d(n) = \Theta\left(\sqrt{\log n/n}\right)$ is

$$C_{m,m}(n) = \begin{cases} \Omega\left(1/\sqrt{nm\log n}\right), m = O\left(n/\log n\right) \\ \Omega\left(n^{-1}\right), m = \Omega\left(n/\log n\right) \end{cases} \tag{17}$$

This is the maximum lower-bound capacity that can be attained in the network.

B. Capacity Bounds of (n, m, k)-cast

1) Upper Bound: In this section, we demonstrate the throughput capacity of (n, m, k)-cast in random wireless ad hoc networks. The proofs are very similar to those shown in the previous section. Thus, lemmas and theorems are only stated without proof for completeness of the paper.

Lemma 4.11: The per node throughput capacity of (n, m, k)-cast in dense wireless ad-hoc networks is upper bounded by $O\left(\frac{1}{n} \times \frac{\sup \Sigma_{ij} d_{ij} C_{ij}}{\overline{\|\text{MEMKT}\|}}\right)$

Proof: This is similar to Lemma 4.2 except that we replace MEMT with MEMKT. ■

Lemma 4.12: The average length of MEMKT has the lower bound of $\frac{\sqrt{m}k}{m}$.

Proof: The proof can be found in [8]. ■

Theorem 4.13: The per node upper bound throughput capacity of the (n, m, k)-cast in dense wireless ad hoc network under the physical model is given by

$$C_{m,k}(n) = O\left(\frac{\sqrt{m}}{\sqrt{n}k}\right). \tag{18}$$

Proof: The proof is similar to the proof of Theorem 4.4. ■

2) Lower Bound: In this section, we demonstrate the lower bound for (n, m, k)-cast based on the same approach used in Section IV-A2.

Lemma 4.14: The achievable lower bound of the (n, m, k)-cast capacity is given by

$$C_{m,m}(n) = \Omega\left(\frac{1}{\overline{\#\text{MEMKTC}(d(n))}} \times \frac{1}{T\left(\alpha, \beta, d(n)\right)^2 nd^2(n)}\right). \tag{19}$$

where $\overline{\#\text{MEMKTC}(d(n))}$ is the mean number of cells in MEMKT$(d(n))$.

Proof: The proof is similar to Lemma 4.8 except that $\overline{\#\text{MEMTC}(d(n))}$ is replaced with $\overline{\#\text{MEMKTC}(d(n))}$. ■

Lemma 4.15: The average number of cells in MEMKT$(d(n))$ tree is upper bounded as

$$\overline{\#\text{MEMKTC}(d(n))} = \begin{cases} \Theta\left(k(\sqrt{m}d(n))^{-1}\right), m = O\left(d^{-2}(n)\right) \\ \Theta\left(k\right), \Omega(k) = \left(d^{-2}(n)\right) = O(m) \\ \Theta\left(d^{-2}(n)\right), k = \Omega\left(d^{-2}(n)\right) \end{cases} \tag{20}$$

Proof: The proof is similar to the proof of lemma 4.9. ■

The maximum attainable lower bound capacity is achieved when $d_{min}(n) = \Omega\left(\sqrt{\log n/n}\right)$ is applied for $d(n)$.

Theorem 4.16: The maximum achievable lower bound for the (n, m, k)-cast capacity is

$$C_{m,k}(n) = \begin{cases} \Omega\left(\sqrt{m}/k\sqrt{n\log n}\right), m = O\left(n/\log n\right) \\ \Omega\left(1/k\log n\right), \Omega(k) = n/\log n = O(m) \\ \Omega\left(1/n\right), k = \Omega\left(n/\log n\right) \end{cases} \tag{21}$$

Proof: Combining lemmas 4.14 and 4.15 with the minimum distance parameter for $d(n)$ provide us with the result. ■

V. CONCLUSION

We have presented a general theory for the capacity of wireless ad hoc networks under the physical model. First, the (n, m, k)-casting model that was developed in [8] was adopted here to extend the results to the physical model. By doing this, a new upper bound of $O\left(\frac{\sqrt{m}}{\sqrt{nk}}\right)$ and the same lower bound similar to the results in [8] were derived. The lower bound capacity consists of three different regions with values of $\Omega\left(\frac{\sqrt{m}}{k\sqrt{n\log n}}\right)$, $\Omega\left(\frac{1}{k\log n}\right)$ and $\Omega\left(\frac{1}{n}\right)$ when $m = O(n/\log n)$, $\Omega(k) = (n/\log n) = O(m)$ and $\Omega(n/\log n) = k$, respectively. It is worth investigating as future work, if the gap in the physical model for (n, m, k)-cast can be closed using percolation theory.

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