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DESIGN OF EIGHT-BAR LINKAGES WITH AN APPLICATION TO RECTILINEAR MOTION

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ABSTRACT

This paper explores an automated methodology to design eight-bar linkages for five positions, by constraining the user defined 4R open chain and 6R closed chain using RR dyads. The application in focus is that of rectilinear motion. The paper lays down the rules for systematic selection of link pairs for the application of RR constraints, that help automate the synthesis of eight-bar linkages. The methodology performs a random search within the user specified tolerance zones around the task specifications to increase the number of candidate linkage solutions. These linkages are then subjected to forward kinematic analysis using the Dixon determinant elimination procedure to find all possible linkage configurations over the range of motion of the input link. Linkages that have all the task configurations on one branch ensure their smooth movement through them. The result is an array of defect-free eight-bars that can perform approximate rectilinear motion. This method provides increased flexibility and control over the synthesized linkage compared to other known rectilinear motion linkages, owing to the ability to specify the backbone chain. A couple of examples are presented to illustrate the results.

INTRODUCTION

In this paper we introduce an automated methodology for the design of eight-bar linkages for five finitely separated task positions. We explore the constraining of both 4R(revolute) open chain and 6R closed chain using RR constraints to synthesize eight-bar linkages. Note that the design algorithm can work for any task positions, but since the application in focus for this paper is rectilinear motion, all the five positions are specified in a straight line. This will ensure zero deviation from rectilinear motion at least in these five positions. Synthesized linkage solutions will be ranked based on their accuracy to perform rectilinear motion, and presented to the user.

The input to the design algorithm is a set of five task positions, a 4R (revolute joint) open chain or 6R closed chain robot that can achieve these task positions along with tolerances as shown in Fig.1. A 4R open chain is a four degree of freedom (DOF) robot and requires three RR constraints, to synthesize a single DOF eight-bar linkage. Similarly a 6R closed chain is a three DOF robot and requires two RR constraints to be applied. We lay down rules that automate the systematic selection of all the correct link pairs, across which an RR constraint can be applied. It is found that in case of the 4R open chain, there are 100 ways in which three RR constraints can be applied to yield a maximum of 3951 eight-bar linkages. In case of 6R closed chain, there are 32 ways in which two RR constraints can be applied to yield a maximum of 340 eight-bar linkages.

Following the synthesis, each linkage is analyzed to find the forward kinematic solutions. For a particular input link angle, each forward kinematics solution depicts a linkage configuration, which represents, a specific way to assemble the various links of

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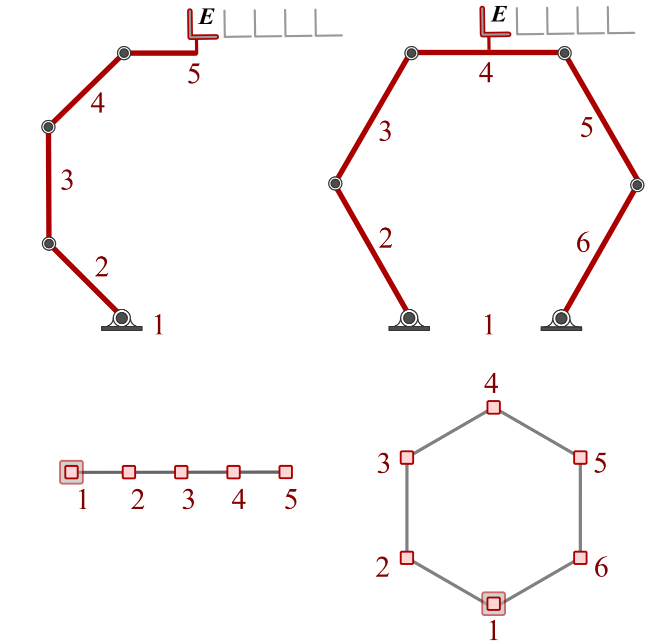


FIGURE 1. USER DEFINED 4R OPEN CHAIN AND 6R CLOSED CHAIN ARE SHOWN WITH THEIR GRAPHS. LINK 1 IS THE GROUND LINK AND LINK *E* IS THE END-EFFECTOR LINK.

the eight-bar linkage. As the input link is rotated in the range, all the possible forward kinematic solutions are tracked and then sorted into branches. A linkage qualifies as a defect-free linkage, if all the five linkage configurations at the five task positions, lie on the same branch. This strategy of finding defect-free eight-bar linkages ensures smooth movement of the end-effector through the five task positions.

Providing the user freedom to select the 4R open chain and the 6R closed chain permits greater control over the synthesized linkage. It is important to note that the probability of finding defect-free eight-bar linkages is dependent on the selection of these backbone chains. In order to generate more solutions, we run the design algorithm iteratively over randomized task positions and randomized ground pivots for the two backbone chains.

LITERATURE SURVEY

James Watt invented the first approximate straight line motion generating linkage in 1784. Later, Roberts and Tchebicheff also invented ones with greater accuracy. Note that in this paper, a straight line linkage is considered as one, which ensures that only a point on the linkage traverses along a straight line. We refer to rectilinear motion as one, where an entire link in the linkage traverses in a straight line. In 1864, M. Peaucellier invented the first exact straight line eight-bar linkage. Later Hart, Sylvester and Kempe came up with their own versions of ex-

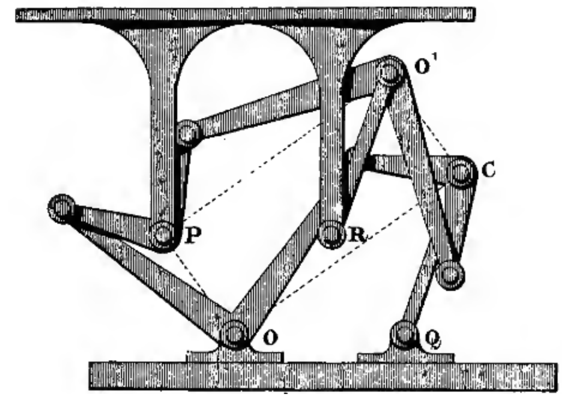


FIGURE 2. KEMPE'S EXACT RECTILINEAR MOTION EIGHT-BAR LINKAGE

act straight line linkages. Kempe [1, 2] also invented two ten-bars and one eight-bar exact rectilinear motion linkage, which is shown in Fig.2.

Shaffer and Krause (1962) [3] proposed a general method for generating controlled rectilinear motion to any desired accuracy, by expressing the desired motion in terms of a Fourier series. Zhao and Feng (2008) [4] proposed a synthesis method for a spatial rectilinear motion mechanism with application to automobile suspension. Their mechanism is similar to the Sarrus linkage which is also a spatial rectilinear motion linkage invented by Sarrus in 1853. Compliant linkages like the double parallelogram flexure mechanism have been explored to generate exact rectilinear motion for small range of motion, refer Clay(1937) [5], Jones (1962) [6] and Awtar(2013) [7].

Hain (1967) [8] developed a graphical synthesis method to synthesize an eight-bar linkage with two of its coupler links producing simultaneous rectilinear motion in different directions for four finite positions. Soni (1973) [9] proposed a technique to synthesize an eight-bar linkage for a variety of motions (function, path, motion), along with cases involving constraints on the input and output angle. Subbian and Flugrad (1994) [10] used continuation methods (secant parameter homotopy) to synthesize an eight-bar to reach six precision points by combining three triads. Angeles and Chen (2008) [11] developed a method to synthesize an eight-bar that can reach upto 11 poses exactly. Their method couples two four-bar legs, to guide a coupler through 11 task positions. Our work follows the design procedure of constraining the user defined open or closed chain using RR constrains, introduced by Soh and McCarthy (2007) [12, 13] and further developed by Sonawale and McCarthy [14, 15]. Soh and Ying (2013) [16] followed this procedure to design an eight-bar linkage with prismatic joints.

In order to ensure the design of an eight-bar linkage is usable, we analyze its movement through the five task positions.

Our approach uses the Dixon determinant elimination procedure described in Wampler (2001) [17] and Neilson and Roth (1999) [18], to find all the solutions to the forward kinematics problem. They refer to this procedure as solving an input/output problem for planar linkages. Dhingra et al.(2000) [19] showed that the displacement analysis problem for the eight-bar mechanism can be reduced into a univariate polynomial devoid of any extraneous roots. Apart from the elimination methods, there are numerical methods ranging from Newton's method, which finds a solution near an initial guess, to more sophisticated methods like polynomial continuation, that can find all possible solutions.

Analysis of an eight-bar linkage using the Dixon determinant approach requires the selection of three loops and their corresponding loop equations. For this paper, we use the procedure developed by Parrish et al.(2013) [20, 21]. Plecnik, Sonawale and McCarthy [22, 23] demonstrated the effectiveness of random variation of the task positions within tolerance zones, to increase the number of candidate linkages in the design of spatial and spherical linkages. Our formulation is intended to find defect-free eight-bars by verifying the synthesized linkages for branch defects. This ensures smooth movement through the five task positions.

SYNTHESIS OF EIGHT-BAR LINKAGES

This section explains the automated synthesis procedure used for the design of eight-bar linkages by constraining the two backbone chains, 4R open chain and 6R closed chain, respectively. Our synthesis procedure begins with the specification of a set of five task positions and the backbone chains. Next, we perform inverse kinematics on the two chains at the five tasks to extract five positions for each link. Following that the design algorithm identifies all the possible link pairs across which an RR constraint can be applied for the given chain. The synthesis of an RR constraint across a pair of links, whose five relative positions are known can be found in McCarthy [13] and Sonawale [14]. Each RR constraint can have a maximum of four solutions. Using all possible combinations of these, the algorithm synthesizes numerous eight-bar linkage solutions.

Inverse Kinematics of the 4R Open Chain

Here we discuss the process of calculating the joint angles for the 4R open chain robot, namely, angles $\theta_{2j}, \theta_{3j}, \theta_{4j}, \theta_{5j}$, $j = 1,\ldots,5$, for the links a_2, a_3, a_4, a_5, when the end effector is in each of the five task positions, refer Fig.3. Note that all the link angles are all measured from the horizontal axis of the fixed frame $\boldsymbol{F}$. Angles θ_{5j}, $j = 1,\ldots,5$, made by the end effector link a_5, are obtained directly from the lines connecting the origins of the five task positions and the five points C_4.

As part of the 4R chain specification, the user specifies the ground pivot C_1, end-effector pivot C_4 and the common link

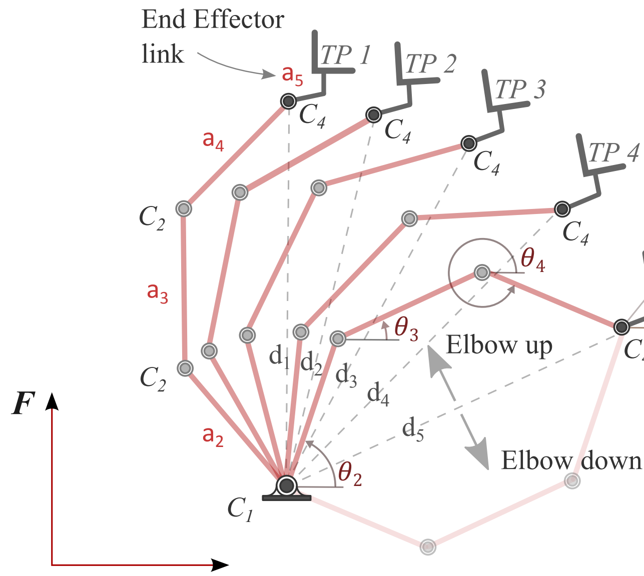


FIGURE 3. THE 4R CHAIN $C_1,\ldots,C_4$ IS SHOWN IN ALL THE FIVE TASK POSITIONS LABELED $TP1,\ldots,TP5$. LINKS a_2, a_3 AND a_4 HAVE THE SAME LENGTH a AND ARE SYMMETRICALLY POSITIONED WITH RESPECT TO THE SEGMENTS d_i, $i = 1,\ldots,5$.

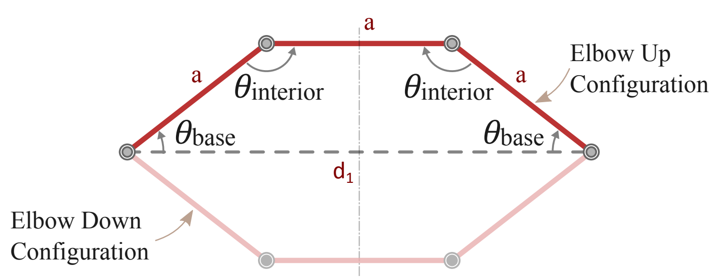


FIGURE 4. TWO SYMMETRIC CONFIGURATIONS, ELBOW UP AND ELBOW DOWN, ARE SHOWN FOR THE 4R CHAIN IN THE FIRST POSITION.

length a for the links (a_2, a_3, a_4). In addition, the user also specifies whether the elbow is up or down. Using this information, we first find the location of the point C_4 in the fixed frame $\boldsymbol{F}$ in all the five task positions. This gives us the distances $d_i, i = 1,\ldots,5$ between points C_1 from C_4 in each of the five task positions as shown in Fig.3. Next, for each of the distances d_i, the links are symmetrically arranged with respect to it. Consider the case of distance d_1 as shown in Fig.4. Since we have odd number of links of length a, to be arranged, the rule used is that the center link (here second link) is parallel and symmetric with respect to d_1, such that all the interior angles are equal and the two base angles are also equal. The two equations that are used to solve

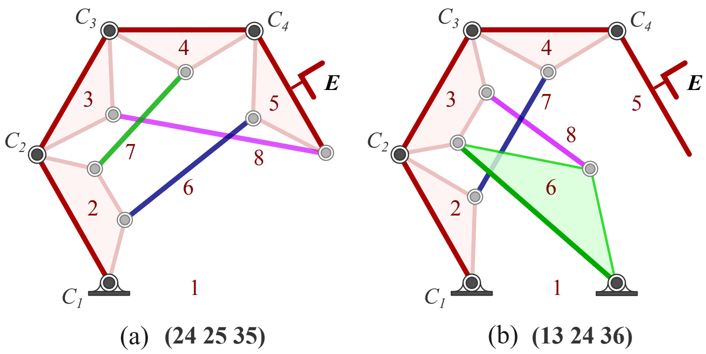


FIGURE 5. (a) THREE RR CONSTRAINTS APPLIED BETWEEN LINK PAIRS $\{(2,4),(2,5),(3,5)\}$ RESULT IN A STRUCTURE FORMED BY THE LINKS $\{2,3,4,5,6,7,8\}$ AND THIS STRUCTURE CAN FREELY ROTATE WITH RESPECT TO LINK 1 ABOUT PIVOT C_1. (b) THREE RR CONSTRAINTS APPLIED BETWEEN LINK PAIRS $\{(1,3),(2,4),(3,6)\}$ RESULTS IN A STRUCTURE FORMED BY LINKS $\{1,2,3,4,6,7,8\}$, AND LINK 5 CAN ROTATE FREELY WITH RESPECT TO THE STRUCTURE ABOUT PIVOT C_4.

for interior angle $\theta_{interior}$ and base angle θ_{base} are,

$$2*a*Cos(\theta_{base})+a = d_1, \quad (1)$$

$$2*\theta_{base}+2*\theta_{interior} = 2\pi. \quad (2)$$

The first equation Eq.(1) is summation of the x coordinates for the pivots of the 4R chain. The second equation Eqn.(2) gives a relation between the two angles, θ_{base} and $\theta_{interior}$, using the fact that the sum of the interior angles of an n sided polygon is given by $(n-2)*\pi$.

The two equations are then solved numerically using Newton's method with the initial value of 45° for θ_{base} and 135° for $\theta_{interior}$. Using the solutions for angles θ_{base} and $\theta_{interior}$ for each of the d_i, $i,\ldots,5$, corresponding to each of the five task positions, the five 4R chains can be constructed as shown in Fig.3.

While the symmetrical arrangement of the links $\{a_2,a_3,a_4\}$ provides a convenient method to perform the inverse kinematics, it results in synthesis of redundant RR constraints. This problem is solved by inducing some asymmetry in each of the five 4R chains. If we consider each 4R chain as a four-bar linkage with ground link as d_i, $j=1,\ldots,5$, we can then offset the crank by a small percentage of the crank angle to offset the symmetry.

Eight-bar Synthesis by Constraining the 4R chain

We discuss several rules in this section to automate the process of finding sets of three link pairs, across which three RR constraints can be applied to constrain the 4R chain. We start with an array of links representing the 4R chain as $\mathbf{p_1} = \{1,2,3,4,5\}$, refer Fig.1. In addition, we also specify a list of link pairs, between which joints exist in the 4R chain as $\mathbf{j} = \{(1,2),(2,3),(3,4),(4,5)\}$.

In order to apply an RR constraint, we need to pick two links from $\mathbf{p_1}$, to form a link pair. This is same as selecting two out of five which could be done in ${}^5C_2 = 10$ ways. The solution list is given as $\mathbf{l_1} = \{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}$. Note that list $\mathbf{l_1}$ also includes the link pairs from $\mathbf{j}$. If an RR constraint is applied between any these link pairs from $\mathbf{j}$, a structure will be formed. So, removing these four link pairs from $\mathbf{l_1}$, we get 6 valid links pairs $\mathbf{f_1} = \{(1,3),(1,4),(1,5),(2,4),(2,5),(3,5)\}$, that could be used to apply the first RR constraint. We refer to these RR constraints as independent RR constraints and the link pairs themselves as independent pairs.

The application of first RR constraint adds link 6 and the second RR constraint adds link 7 respectively, thus making list $\mathbf{p_1}$ to become $\mathbf{p_2} = \{1,2,3,4,5,6,7\}$. We observe that for the application of the third and final RR constraint, the link list available is $\mathbf{p_2}$. Next, we select 2 out of 7 links, which could be done in ${}^7C_2 = 21$ ways. Again, this list contains the link pairs from $\mathbf{j}$. After removing these, we get a list of 17 pairs represented as $\mathbf{f_2} = \{(1,3),(1,4),(1,5),(1,6),(1,7),(2,4),(2,5),(2,6),(2,7),(3,5),(3,6),(3,7),(4,6),(4,7),(5,6),(5,7),(6,7)\}$. We refer to the RR constraints applied between the link pairs involving links 6 and 7 as dependent RR constraints and the link pairs themselves as dependent pairs.

Next we select a set of three link pairs out of 17 from the list $\mathbf{f_2}$, for applying the 3 RR constraints. This could be done in ${}^{17}C_3 = 680$ ways represented by $\mathbf{g_1}$. A lot of these sets are either invalid or not desirable. The rules for eliminating such sets will be discussed next.

1. First rule for an open chain is to remove the sets from $\mathbf{g_1}$, that do not include the ground link 1 or the end-effector link 5. In Fig.5(a), the set of link pairs $\{(2,4),(2,5),(3,5)\}$ does not include link 1. This forms a structure with links $\{2,3,4,5,6,7,8\}$, which can freely rotate with respect to link 1 about pivot C_1. In Fig.5(b), the the set of link pairs $\{(1,3),(2,4),(3,6)\}$ does not include link 5. Here links $\{1,2,3,4,6,7,8\}$ form a structure and the end-effector link 5 is free to rotate with respect to this structure about pivot C_4. Removing such cases, reduces the number of sets to 296, represented by $\mathbf{g_2}$.
2. The next rule is to remove the sets from $\mathbf{g_2}$ that do not contain even one link pair from the list of independent pairs $\mathbf{f_1}$. This arises from the fact that links 6,7 are a result of the first and second RR constraints. So for applying the first constraint, the links 6 and 7 don't even exist. Hence the first RR constraint has to be applied between independent link pairs from $\mathbf{f_1}$. As an example, consider the case of set $\{(1,6),(1,7),(5,6)\}$. Here the first RR constraint is applied between links 1 and 6, Now there is no link 6 to start with, it

will emerge only when the first RR constraint link is applied. Hence this set is invalid. Elimination of such cases reduces the number of sets of link pairs to 264, represented by $\mathbf{g_3}$.

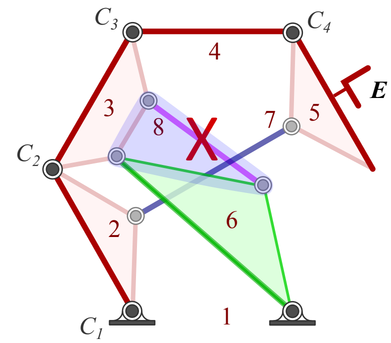
The next condition we impose is to remove sets from $\mathbf{g_3}$ that have link 1 appearing more than once. Leaving such solutions will in fact increase the number of synthesized solutions. But adding one more ground pivot, by allowing link 1 to appear more than once in the set, will result in three ground pivots overall for the synthesized eight-bar linkage. This is not desirable to us as two out of three ground pivot locations are beyond the control of the user. Removing such cases form $\mathbf{g_3}$ reduces the number of sets to 188, represented by $\mathbf{g_4}$. Next we order all the 188 sets of the three link pairs, so that the ones with more number of independent link pairs from $\mathbf{f_1}$ are ranked higher. Sets that have three independent link pairs take the top positions, followed by the ones that have two independent pairs, and to follow them are ones with only one independent pair.

Once the list $\mathbf{g_4}$ is sorted, each set of the three link pairs is sorted internally according to the second link number. For example, the set $\{(1,3),(1,7),(5,6)\}$ is sorted to $\{(1,3),(5,6),(1,7)\}$, because the application of the second RR constraint between link pair $(1,7)$ is invalid as link 7 doesn't exist yet. But after sorting the set, for applying the third RR constraint between link pair $(1,7)$, link 7 has now been generated from the second RR constraint between link pair $(5,6)$.

In cases where the first two link pairs or the last two link pairs have the same second link, they are sorted according to the first link. For example the first two link pairs for the set $\{(2,4),(1,4),(3,5)\}$ will be sorted to $\{(1,4),(2,4),(3,5)\}$ and the last two link pairs for $\{(1,3),(5,6),(3,6)\}$ will be sorted to $\{(1,3),(3,6)(5,6)\}$. If all the three link pairs have the same second link, example $\{(2,5),(1,5),(3,5)\}$, then the set is sorted according to the first link to give $\{(1,5),(2,5),(3,5)\}$. Note that sorting according to the first link number in these cases is only for convenience, it does not affect the linkage solutions.

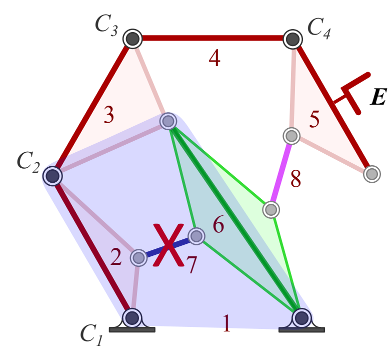
In the next step the sorted list $\mathbf{g_4}$ is divided in to two lists, namely independent sets $\mathbf{i_1}$ and dependent sets $\mathbf{d_1}$. Note that, all the three link pairs in each independent set are independent pairs from $\mathbf{f_1}$. There are 9 sets in $\mathbf{i_1}$ and 179 sets in $\mathbf{d_1}$. The dependent sets will always have at least one dependent link pair involving links 6, 7 or both.

Independent Sets: Total 12 In addition to the 9 sets in the independent sets list $\mathbf{i_1}$, there are three more sets that have independent link pairs and are feasible. They are $\{\{(1,3),(2,5),(2,5)\},\{(1,4),(2,5),(2,5)\},\{(1,5),(2,5),(2,5)\}\}$. The reason our algorithm is unable to find these sets is because they involve repetitions of the link pairs. This is a subject of future research. Thus, the total number of feasible independent sets of the three link pairs are 12, represented by $\mathbf{i_2}$.



(13 25 36)

FIGURE 6. APPLICATION OF THE FIRST RR CONSTRAINT BETWEEN LINKS 1 AND 3, RESULTS IN THE NEWLY GENERATED LINK 6, TO BE ADJACENT TO LINKS 1 AND 3. THEREFORE APPLICATION OF THE THIRD RR CONSTRAINT BETWEEN THE LINKS 3 AND 6, RESULTS IN A STRUCTURE AND HENCE $\{(1,3),(2,5),(3,6)\}$ IS INVALID.

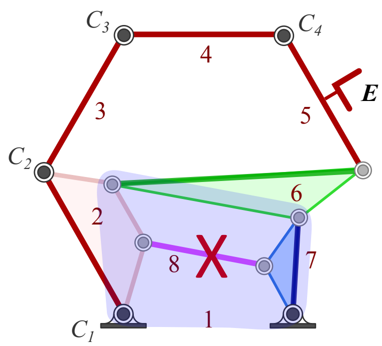


(13 26 56)

FIGURE 7. APPLICATION OF THE FIRST RR CONSTRAINT BETWEEN LINKS 1 AND 3, RESULTS IN THE FORMATION OF A FOUR-BAR WITH LINKS $\{1,2,3,6\}$. SINCE THE SECOND LINK PAIR IS $(2,6)$ IS PART OF THE FOUR-BAR, ADDING AN RR CONSTRAINT BETWEEN THE LINKS OF THE FOUR-BAR WILL RESULT IN A STRUCTURE. HENCE THE SET $\{(1,3),(2,6),(5,6)\}$ IS INVALID.

Dependent Sets: Total 86 Out of the 179 dependent sets of the three link pairs $\mathbf{d_1}$, some are invalid. The following rules are used to eliminate the invalid sets.

3. The third elimination rule for open chains is that the second last link, here link 7, cannot appear twice. The reason for this is that, link 7 is generated from the second RR constraint. So link 7 is available only for the application of the third RR constraint, which is the last one. Elimination of these cases reduces the number of dependent sets to 150,



(25 16 27)

FIGURE 8. APPLICATION OF THE FIRST TWO RR CONSTRAINTS BETWEEN THE LINK PAIRS (2,5) AND (1,6), RESULTS IN THE FORMATION OF A FOUR-BAR CONSISTING OF LINKS {1,2,6,7}. NOW THE THIRD LINK PAIR (2,7) HAS BOTH THE LINKS PART OF THIS FOUR-BAR. THEREFORE ADDING AN RR CONSTRAINT BETWEEN THE LINKS OF THE FOUR-BAR WILL RESULT IN A STRUCTURE. HENCE THE SET {(2,5),(1,6),(2,7)} IS INVALID.

represented by $\mathbf{d_2}$.

4. The fourth rule is to remove the sets in which the dependent link pairs turn out to be adjacent to each other. Consider for example {(1,3),(2,5),(3,6)} shown in Fig.6. Application of the first RR constraint between links 1 and 3, results in the newly generated link 6 to be adjacent to both links 1 and 3. Therefore, application of the third RR constraint between links 3 and 6 results in a structure between the links {3,6,8}. Hence this set is invalid. Removing such cases reduces the number of sets to 97, represented by $\mathbf{d_3}$.
5. The fifth rule is to remove the sets, which have a dependent link pair that is part of a four-bar sub-loop, generated by the RR constraints, applied between previous link pairs. Consider for example set {(1,3),(2,6),(5,6)} shown in Fig.7. Application of the first RR constraint between links 1 and 3 generates a new link 6. This new link results in the formation of a four-bar sub-loop consisting of links {1,2,3,6}. Now the second link pair (2,6) has both its links part of this four-bar, and hence adding an RR constraint between these links will result in a structure. Hence the set {(1,3),(2,6),(5,6)} is invalid. Removing such cases reduces the number of sets to 87, represented by $\mathbf{d_4}$.
6. The sixth and last rule is to remove the sets which have a dependent link pair that is part of a four-bar sub-loop generated by a combination of RR constraints, applied between previous link pairs. Consider for example the set {(2,5),(1,6),(2,7)}, shown in Fig.8. Application of the first two RR constraints between link pairs (2,5) and (1,6), results in the formation of a four-bar sub-loop consisting of links {1,2,6,7}. Here we observe that link 6 and link 7 are contributed by the first and second RR constraints respectively. Hence we call it formation of four-bar by combination. Now the third link pair (2,7) has both the links part of this four-bar. Therefore adding an RR constraint between these links of the four-bar will result in a structure. Hence, the set {(2,5),(1,6),(2,7)} is invalid. Removing such cases reduces the number of sets to 84, represented by $\mathbf{d_5}$.

Again, there are two more feasible dependent sets of three link pairs, that involve repetition of link pairs and hence were not found by our automated algorithm. These sets are {{(1,3),(5,6),(5,6)}, and {(1,5),(3,6),(3,6)}}. Thus the total number of feasible dependent sets are 88, represented by $\mathbf{d_5}$.

Finally, the total number of feasible sets of three link pairs, for the application of 3RR constraints, are equal to $12+88=100$. Once all the possible sets of three link pairs are identified, the next step is to synthesize RR constraints between the link pairs. For each set, using all the possible combinations of the RR constraint solutions, we synthesize a maximum of 64 eight-bar linkages. Overall, the 100 sets of three link pairs can synthesize a maximum of 3951 eight-bar solutions. For more details about the 100 sets of three link pairs, refer Sonawale [15]. Using Randomization of the task positions within acceptable variations during each iteration, we can generate more candidate linkage solutions. The design algorithm relays this list of candidate linkages to the analysis routine to check for branch defects, which is explained in the Analysis section.

Eight-bar Synthesis by Constraining the 6R chain

The inverse kinematics of the 6R closed chain for five task positions can be found in McCarthy [13] and Sonawale [14]. For constraining the 6R closed chain to a single DOF eight-bar linkage, we need to apply two RR constraints. We next develop rules for the closed chain, to automate the process of finding sets of two link pairs, across which two RR constraints can be applied.

Here we will use the same letters to represent several lists as in case of constraining the 4R chain. We start with an array of links representing the 6R chain as $\mathbf{p_1}=\{1,2,3,4,5,6\}$, refer Fig.1. In addition, we also specify a list of link pairs, between which joints exist in the 6R chain as $\mathbf{j}=\{(1,2),(2,3),(3,4),(4,5),(5,6),(1,6)\}$. To apply an RR constraint, we need to pick two links out of six in $\mathbf{p_1}$ which could be done in ${}^6C_2=15$ ways, represented by $\mathbf{l_1}$.

Now we remove the six link pairs $\mathbf{j}$ from $\mathbf{l_1}$, to get 9 valid links pairs $\mathbf{f_1}=\{(1,3),(1,4),(1,5),(2,4),(2,5),(2,6),(3,5),(3,6),(4,6)\}$, that could be used to apply the first RR constraint. We refer to these RR constraints as independent RR constraints and the link pairs themselves as independent pairs.

The application of first RR constraint adds link 7 thus making list $\mathbf{p_1}$ to become $\mathbf{p_2}=\{1,2,3,4,5,6,7\}$. We observe that for the application of the second and final RR constraint, the link list available is $\mathbf{p_2}$. Next, we select 2 out of 7 links, which could be done in ${}^7C_2 = 21$ ways. Again, this list contains the link pairs from $\mathbf{j}$. After removing these, we get a list of 15 pairs represented as $\mathbf{f_2} = \{(1,3),(1,4),(1,5),(1,7),(2,4),(2,5),(2,6),(2,7),(3,5),(3,6),(3,7),(4,6),(4,7),(5,7),(6,7)\}$. We refer to the RR constraints applied between the link pairs involving link 7 as dependent RR constraints and the link pairs themselves as dependent pairs.

Next we select a set of two link pairs out of 15 from the list $\mathbf{f_2}$, for applying the two RR constraints. This could be done in ${}^{15}C_2 = 105$ ways represented by $\mathbf{g_1}$. A lot of these sets are either invalid or not desirable. The rules for eliminating such sets will be discussed next.

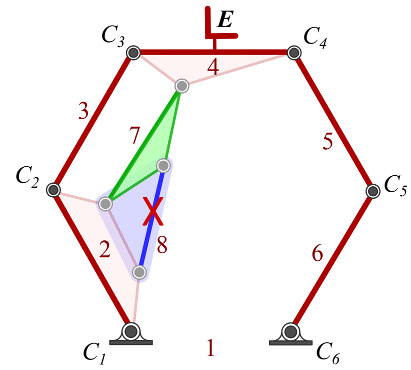
1. The first rule in closed chain, is to remove the sets from $\mathbf{g_1}$ that do not contain even one link pair from the list of independent pairs $\mathbf{f_1}$. This arises from the fact that link 7 is a result of the first RR constraint. So for applying the first constraint, the link 7 don't even exist. Hence the first RR constraint has to be applied between independent link pairs from $\mathbf{f_1}$. Elimination of such cases reduces the number of sets of link pairs to 90, represented by $\mathbf{g_2}$.

The next condition we impose is to remove sets from $\mathbf{g_2}$ that contains link 1, even once. This avoids the addition of more than the existing two ground pivots, and not have control over their location. Removing such cases form $\mathbf{g_2}$ reduces the number of sets to 45, represented by $\mathbf{g_3}$. Next we sort all the 45 sets of the two link pairs just like before. In the next step the sorted list $\mathbf{g_3}$ is divided in to two lists, namely independent sets $\mathbf{i_1}$ and dependent sets $\mathbf{d_1}$. There are 15 sets in $\mathbf{i_1}$ and 30 sets in $\mathbf{d_1}$. The dependent sets will always have one dependent link pair involving link 7.

Independent Sets: Total 17 In addition to the 15 sets in the independent sets list $\mathbf{i_1}$, there are two more sets that have independent link pairs and are feasible. They are $\{(2,5),(2,5)\}$ and $\{(3,6),(3,6)\}$. The reason our algorithm is unable to find these sets is because they involve repetitions of the link pairs. Thus, the total number of feasible independent sets of the two link pairs are 17, represented by $\mathbf{i_2}$.

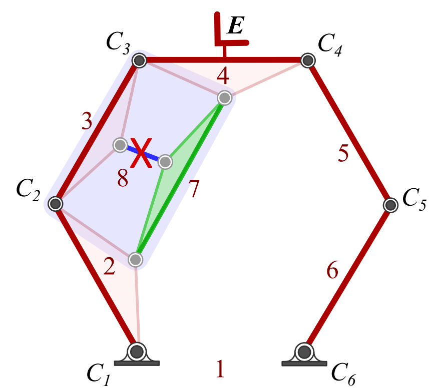
Dependent Sets: Total 15 Out of the 30 dependent sets of the two link pairs $\mathbf{d_1}$, some are invalid. The following rules are used to eliminate the invalid sets.

2. The second elimination rule for closed chains is that the second last link, here link 7, cannot appear more than once. The reason for this is that, link 7 is generated from the first RR constraint. So link 7 is available only for the application of the second RR constraint, which is the last one. No elimination takes place in case of the 6R closed chain as the first rule that alteast one link pair should be from the list of independent pairs $\mathbf{f_1}$ takes care of it.
3. The third rule is to remove the sets in which the dependent link pair turn out to be adjacent to each other. Consider for example $\{(2,4),(2,7)\}$ shown in Fig.9. Application of the first RR constraint between links 2 and 4, results in the newly generated link 7 to be adjacent to both links 2 and 4.



(24 27)

FIGURE 9. APPLICATION OF THE FIRST RR CONSTRAINT BETWEEN LINKS 2 AND 4, RESULTS IN THE NEWLY GENERATED LINK 7, TO BE ADJACENT TO LINKS 2 AND 4. THEREFORE APPLICATION OF THE SECOND RR CONSTRAINT BETWEEN THE LINKS 2 AND 7, RESULTS IN A STRUCTURE AND HENCE $\{(2,4),(2,7)\}$ IS INVALID.



(24 37)

FIGURE 10. APPLICATION OF THE FIRST RR CONSTRAINT BETWEEN LINKS 2 AND 4, RESULTS IN THE FORMATION OF A FOUR-BAR WITH LINKS $\{2,3,4,7\}$. SINCE THE SECOND LINK PAIR IS $(3,7)$ IS PART OF THE FOUR-BAR, ADDING AN RR CONSTRAINT BETWEEN THE LINKS OF THE FOUR-BAR WILL RESULT IN A STRUCTURE. HENCE THE SET $\{(2,4),(3,7)\}$ IS INVALID.

Therefore, application of the second RR constraint between links 2 and 7 results in a structure between the links $\{2,7,8\}$. Hence this set is invalid. Removing such cases reduces the number of sets to 18, represented by $\mathbf{d_2}$.

4. The fourth rule is to remove the sets, which have a dependent link pair that is part of a four-bar sub-loop, generated by the RR constraints, applied between previous link pairs. Consider for example set $\{(2,4),(3,7)\}$ shown in Fig.10. Application of the first RR constraint between links 2 and 4 generates a new link 7. This new link results in the formation of a four-bar sub-loop consisting of links $\{2,3,4,7\}$. Now the second link pair $(3,7)$ has both its links part of this four-bar, and hence adding an RR constraint between these links will result in a structure. Hence the set $\{(2,4),(3,7)\}$ is invalid. Removing such cases reduces the number of sets to 15, represented by $\mathbf{d_3}$.
5. The fifth and last rule is to remove the sets which have a dependent link pair that is part of a four-bar sub-loop generated by a combination of RR constraints, applied between previous link pairs. No elimination takes place using this rule in case of the 6R closed chain, as that can happen only if the number of RR constraints require to constrain the chain are greater than two. Thus the total number of feasible dependent sets are 15, represented by $\mathbf{d_3}$.

Finally, the total number of feasible sets of two link pairs, for the application of two RR constraints, are equal to $17+15=32$. Once all the possible sets of two link pairs are identified, the next step is to synthesize RR constraints between the link pairs. For each set, using all the possible combinations of the RR constraint solutions, we can synthesize a maximum of 16 eight-bar linkages. Overall, the 32 sets of two link pairs can synthesize a maximum of 340 eight-bar solutions. For more details about the 32 sets of two link pairs, refer Sonawale [14, 15].

The design algorithm is first ran on the original task positions and user defined backbone chain, and candidate eight-bar linkage solutions if any are saved. For the following runs the task positions and the backbone chain are modified by introducing small random variations within the designer specified tolerance zones. This results in more candidate linkage solutions. The design algorithm relays this list of candidate linkages to the analysis routine to check for branch defects, which is explained in the next section.

ANALYSIS USING DIXON DETERMINANT METHOD

Analysis of an linkage is to find all possible linkage configurations for a given input angle. A particular linkage configuration can be conveniently captured by a list of joint angles also called as configuration angles. We use the analysis algorithm, developed by Parrish [20, 21], to find the forward kinematic solutions to the eight-bar linkage. Using graph theory, this algorithm first finds all the independent loops for the linkage and generates their loop closure equations with respect to a given ground connected input link. These equations are then solved using the Dixon Determinant elimination procedure to find all possible solutions for the unknown joint angles (configuration angles), for any given input angle.

The goal of the sorting algorithm is to track the development of different configurations as the input link rotates and then sort them into branches. The procedure for sorting the branches is derived from a six-bar sorting algorithm developed by Plecnik and McCarthy [24]. Each of these branches ensure that when the input link is rotated, the linkage moves smoothly in that specific configuration. Now for each branch, it is checked, if the end-effector reaches all the five task positions when the linkage moves in that configuration. If yes, then the linkage is deemed defect-free and the configuration branch is saved.

TABLE 1. FIVE TASK POSITIONS FOR CONSTRAINED 4R OPEN CHAIN

Task	Orientation (θ) (degrees)	Location(x,y)
1	0°	(0.0, 0.0)
2	0°	(22.0, 0.0)
3	0°	(50.0, 0.0)
4	0°	(78.0, 0.0)
5	0°	(100.0, 0.0)

TABLE 2. 4R CHAIN DATA

Ground Pivot (C_1)	$(50.0,-90.0,0)$
End-effector Pivot (C_4)	$(-3.54,-3.54)$
Common length (a)	45
Elbow Position	up (-1)

EXAMPLE LINKAGES

In this section we discuss a couple of approximate rectilinear motion eight-bar linkages, obtained by constraining a 4R open chain and a 6R closed chain respectively. The two linkages are then compared to verify their performance in terms of deviation of their motion from rectilinear motion.

TABLE 3. FIVE TOLERANCES ON TASK POSITIONS FOR CONSTRAINED 4R OPEN CHAIN

Task	Tolerance Data$(\Delta\theta, \Delta x, \Delta y)$
1	$(0.0°, 1.0, 0.001)$
2	$(0.1°, 5.0, 0.001)$
3	$(0.2°, 5.0, 0.001)$
4	$(0.1°, 5.0, 0.001)$
5	$(0.0°, 1.0, 0.001)$

TABLE 4. MULTI-ITERATION RUN OF THE ALGORITHM FOR CONSTRAINED 4R OPEN CHAIN

No. of Iterations	No. of linkages Synthesized	No. of defect-free Linkages	No. of useful Linkages	Time Taken
1	1577	154	14	17.8 min
10	12179	1102	108	141.94 min
100	110454	7885	853	43 hr 52 min

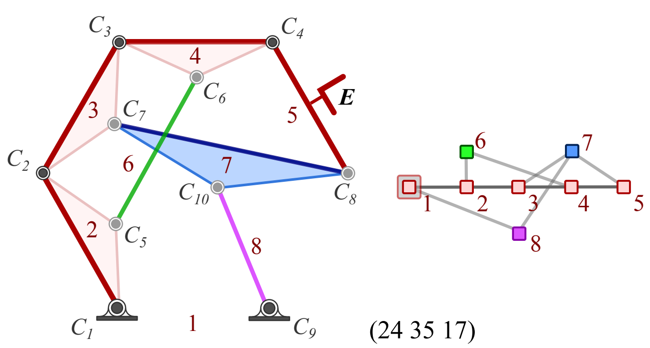


FIGURE 11. TOPOLOGY OF THE SELECTED EIGHT-BAR LINKAGE FOR THE RECTILINEAR MOTION BY CONSTRAINING THE 4R CHAIN ALONG WITH GRAPH. THE THREE RR CONSTRAINTS ARE APPLIED ACROSS LINK PAIRS $\{(2,4),(3,5),(1,7)\}$

Eight-bar Rectilinear Motion - Constrained 4R Open Chain

The five task positions selected for the rectilinear motion linkage are given in Tab.1 and the user specified 4R chain information is given in Tab.2. The tolerances for each task position is a list $(\Delta\theta, \Delta x, \Delta y)$, where $\Delta\theta$ is the tolerance on the orientation of the task position and Δx and Δy are the tolerances on the x and y coordinates of the origin for the task position. These tolerances are mentioned in Table3. Notice that since the tolerances are specified on the task positions, the accuracy of the rectilinear motion of the linkage solutions is limited to the tolerances specified. The multi-iteration run results are shown in Tab.4. The linkage solution with the least deviation from the rectilinear motion is selected for comparison and is mentioned in Tab. 5, as coordinates of the various joints $(C_1, \ldots, C_{10})$. The three RR constraints are applied between the link pairs $(2,4),(3,5)$ and $(1,7)$ respectively as shown in Fig.11. Figure 12 show the linkage movement through the five task positions.

TABLE 5. SELECTED SOLUTION FOR CONSTRAINED 4R OPEN CHAIN

Pivot	Location Data (x, y)
C_1	$(50.0, -90.0, 0)$
C_2	$(62.17, -46.68)$
C_3	$(41.35, -6.78)$
C_4	$(-3.54, -3.54)$
C_5	$(46.69, -39.86)$
C_6	$(56.43, -20.30)$
C_7	$(-4.65, 15.47)$
C_8	$(4.75, 11.67)$
C_9	$(104.60, -78.80)$
C_{10}	$(5.75, -15.33)$

Eight-bar Rectilinear Motion - Constrained 6R Closed Chain

The five task positions selected for this eight-bar rectilinear motion linkage are given in Tab.6. The user specified 6R closed chain data is given in Tab.7 and the tolerance on the ground pivots C_1 and C_6 is give in Tab.8. The eight-bar design algorithm was run with several iterations and the results are show in Tab.9. The linkage solution with the least deviation from the rectilinear motion is selected for comparison and is given in Tab.10 as coordinates of the various joints $(C_1, \ldots, C_{10})$. The topology for this linkage is shown in Fig.13 and its motion through the five task positions is show in Fig.14.

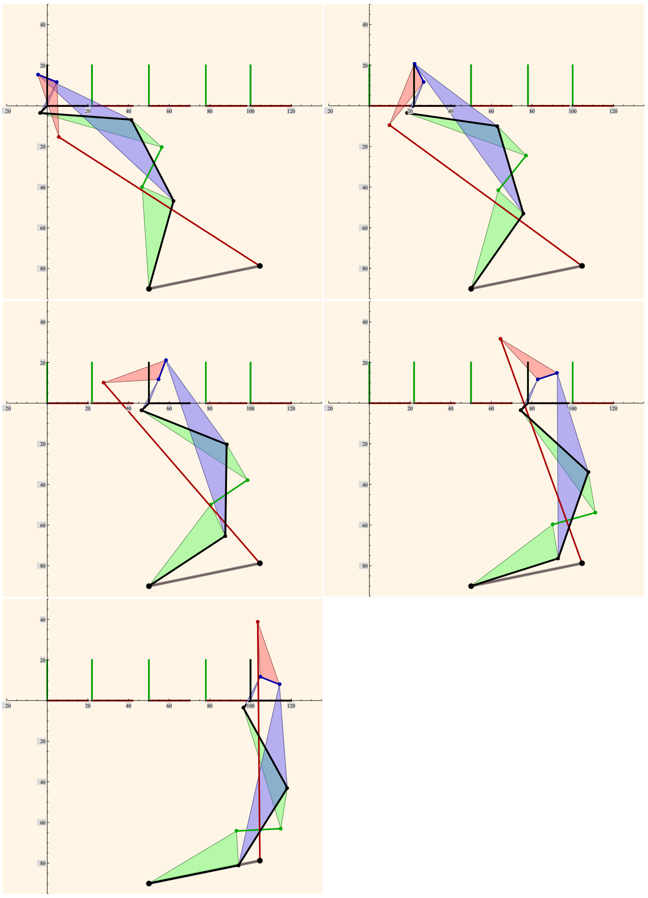


FIGURE 12. SELECTED EIGHT-BAR LINKAGE FOR RECTILINEAR MOTION, OBTAINED BY CONSTRAINING THE 4R OPEN CHAIN, IS SHOWN MOVING THROUGH THE FIVE TASK POSITIONS

Performance Comparison

Here we compare the performance of the two selected eight-bar linkages mentioned above, in terms of their accuracy to achieve rectilinear motion. The deviation from the straight line motion for the two linkages is shown in Fig.15 and the orientation deviation of the end-effector for the two linkages from the zero degree is shown in Fig.16.

It was observed that the eight-bar linkage obtained by constraining the 6R closed chain had superior performance compared to the one obtained by constraining the 4R open chain. The former had a maximum deviation (translational deviation perpendicular to the straight line) of 0.0049 mm which is 0.0049% of the travel of 100 mm compared to later's 0.0335 mm which is 0.0335% of the travel. Also the orientation deviation from the straight line for the former is 0.0083° compared to the later's 0.129°. It is to be noted that the performance of the linkages are dependent on the selection of the task positions along the rectilinear motion and also on the dimensions of the open or closed chain selected. Also, each linkage can further be optimized to provide more accurate rectilinear motion using standard optimization methods.

TABLE 6. FIVE TASK POSITIONS FOR CONSTRAINED 6R CLOSED CHAIN

Task	Orientation (θ) (degrees)	Location(x, y)
1	0°	(0.0, 0.0)
2	0°	(15.0, 0.0)
3	0°	(50.0, 0.0)
4	0°	(85.0, 0.0)
5	0°	(100.0, 0.0)

TABLE 7. 6R CLOSED CHAIN DATA

Pivot	Location Data (x, y)
C_1	$(30.0, -150.0, 0)$
C_2	$(118.078, -54.384)$
C_3	$(0.0, 0.0)$
C_4	$(30.0, 0.0)$
C_5	$(118.078, -95.616)$
C_6	$(0.0, -150)$

TABLE 8. TOLERANCES ON THE GROUND PIVOTS C_1 AND C_6 FOR CONSTRAINED 6R CLOSED CHAIN

Ground Pivot	Tolerance Data$(\Delta x, \Delta y)$
C_1	$(2.0, 2.0)$
C_6	$(2.0, 2.0)$

CONCLUSION

This paper introduces an automated methodology to design defect-free eight-bar linkages using two approaches - constraining the user defined 4R open chain and the 6R closed chain. The application in focus for this paper is that of rectilinear motion. The five task positions are specified in a straight line, which ensure zero deviation from rectilinear motion at least in these five positions. The input to the design procedure is a set of five task

TABLE 9. MULTI-ITERATION RUN OF THE ALGORITHM FOR CONSTRAINED 6R CLOSED CHAIN

No. of Iterations	No. of linkages Synthesized	No. of defect-free Linkages	No. of useful Linkages	Time Taken
1	84	52	24	0.737 min
10	881	522	239	7.084 min
100	8857	5287	2404	68.208 min

TABLE 10. SELECTED SOLUTION FOR CONSTRAINED 6R CLOSED CHAIN

Pivot	Location Data (x, y)
C_1	$(30.0, -150.0, 0)$
C_2	$(118.078, -54.384)$
C_3	$(0.0, 0.0)$
C_4	$(30.0, 0.0)$
C_5	$(118.078, -95.616)$
C_6	$(0.0, -150.0)$
C_7	$(167.641, -170.891)$
C_8	$(-52.0919, 79.7726)$
C_9	$(-13.5632, 58.2072)$
C_{10}	$(79.791, -119.927)$

positions, tolerances and the backbone chain robot. The backbone chain could be either a 4R open chain robot or a 6R closed chain robot which is capable of reaching all the five task positions.

The synthesis procedure starts with inverse kinematics of the user defined backbone chain robot when the end-effector is in each of the five task positions. This gives us five relative positions for each link. Next, link pairs are identified to apply RR constraints. In case of a 4R open chain robot, three RR constraints are required to constrain the robot to single DOF eight-bar linkage and in case of 6R closed chain robot, two RR constraints are required.

Rules are developed that help automate the selection of all possible link pairs, across which an RR constraint can be applied. For constraining the 4R open chain, there are in all 100 such pairs possible, giving rise to maximum of 3951 eight-bar linkage

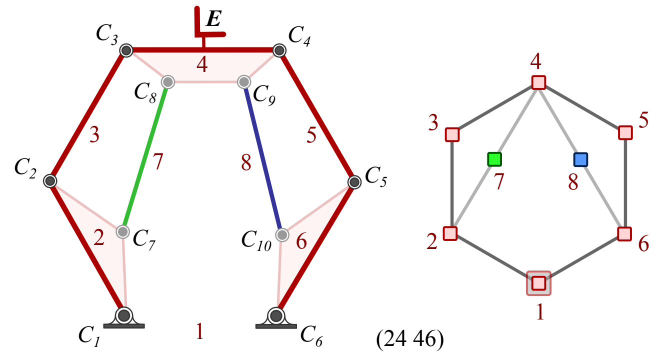


FIGURE 13. TOPOLOGY OF THE SELECTED EIGHT-BAR LINKAGE FOR THE RECTILINEAR MOTION BY CONSTRAINING THE 6R CLOSED CHAIN ALONG WITH GRAPH. THE TWO RR CONSTRAINTS ARE APPLIED ACROSS LINK PAIRS $\{(2,4),(4,6)\}$

solutions. In case of 6R closed chain, there are 32 pairs possible, giving rise to a maximum of 340 eight-bar linkage solutions. It is to be noted that in case of 4R open chain, only one RR constraint to the ground link is allowed, and in case of 6R closed chain no RR constraint to the ground is allowed by the design algorithm. These constraints are imposed by us to ensure greater control over the synthesized linkage. This method performs both type and dimensional synthesis and is topologically independent. It is observed that out of 16 topologies possible for eight-bars, 15 could be synthesized using the proposed methodology.

The synthesized eight-bar linkages can suffer from branch defects during its motion. We use the Dixon determinant elimination procedure to sort the forward kinematic solutions in to branches for the different linkage configurations. A linkage is deemed defect-free, if all the five task configurations lie on a single branch. This ensures smooth movement of the end-effector through the five task positions. An iterative process is used to run the design algorithm on toleranced task positions or ground pivots to obtain more eight-bar solutions.

Two examples for approximate rectilinear motion eight-bar linkages are presented, one obtained by constraining the 4R open chain and other by 6R closed chain. The results show the eight-bar obtained from 6R closed chain had superior performance compared to the other. The best eight-bar linkage performed approximate rectilinear motion with maximum translational deviation (perpendicular to straight line) of 0.0049 mm for a travel of 100 mm, and max orientation deviation of 0.0083° from the straight line. This methodology to synthesize approximate rectilinear motion linkages provides the designer vast number of linkage alternatives to choose from. The main advantage provided by the eight-bars designed using the proposed method is increased flexibility in terms of user specification of the backbone chain, compared to other known methods and linkages.

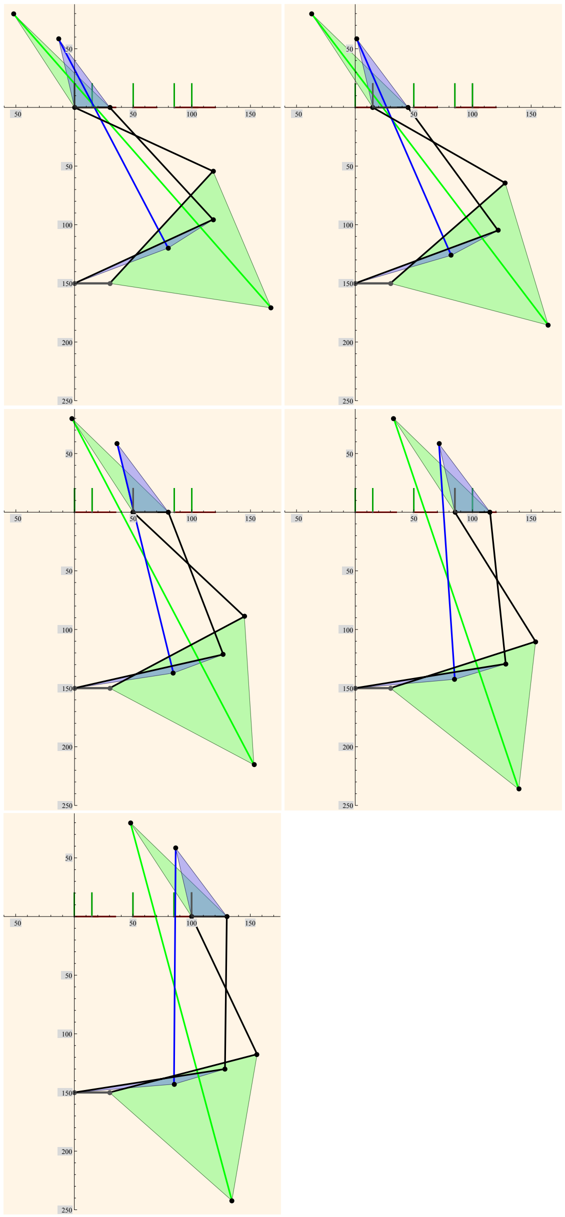


FIGURE 14. SELECTED EIGHT-BAR LINKAGE FOR RECTILINEAR MOTION, OBTAINED BY CONSTRAINING THE 6R CLOSED CHAIN, IS SHOWN MOVING THROUGH THE FIVE TASK POSITIONS

ACKNOWLEDGMENT

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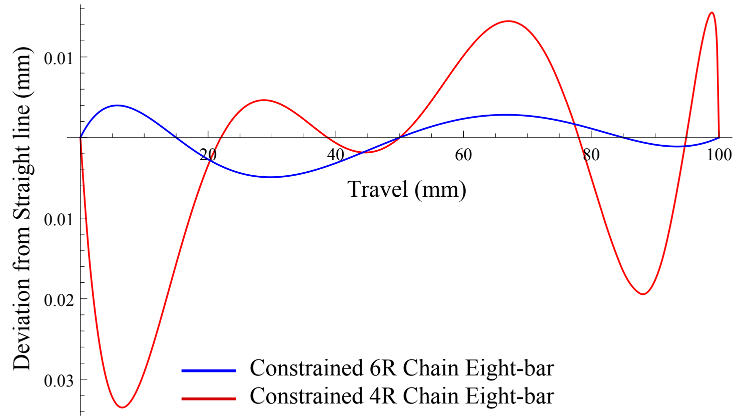

FIGURE 15. COMPARISON OF THE TRANSLATIONAL DEVIATION IN MM FROM RECTILINEAR MOTION FOR THE TWO EIGHT-BAR LINKAGES OBTAINED BY CONSTRAINING THE 6R CLOSED CHAIN AND 4R OPEN CHAIN RESPECTIVELY

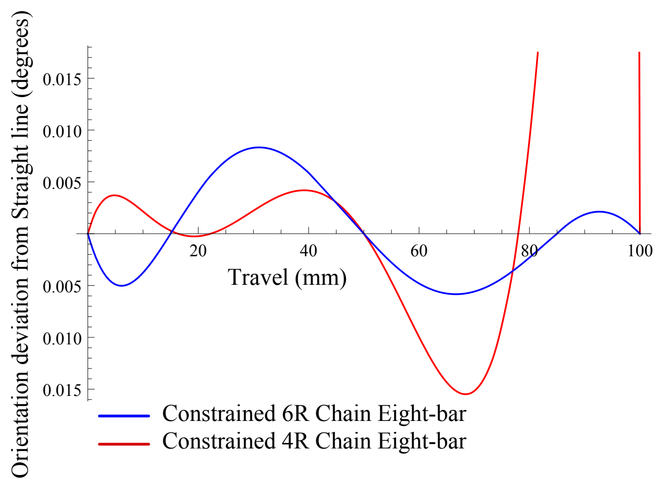

FIGURE 16. COMPARISON OF THE ORIENTATION DEVIATION IN DEGREES FROM RECTILINEAR MOTION FOR THE TWO EIGHT-BAR LINKAGES OBTAINED BY CONSTRAINING THE 6R CLOSED CHAIN AND 4R OPEN CHAIN RESPECTIVELY

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