This is the authors' final version. The official version can be found at
https://doi.org/10.1061/9780784481479.002

A Constitutive Model Controlling Damping for 2D and 3D Site Response

Samuel Yniesta, Ph.D.,[1] Scott J. Brandenberg, Ph.D., P.E., M.ASCE[2]

[1] Dept. of Civil, Geological and Mining Engineering, Polytechnique Montréal, Montréal QC, Canada H3T 1J4 ; e-mail: samuel.yniesta@polymtl.ca

[2] Dept. of Civil and Environmental Engineering, Univ. of California, Los Angeles, CA 90095. E-mail: sjbrandenberg@ucla.edu

ABSTRACT

Modulus reduction and damping curves are commonly used as input parameters in site response analyses. Until very recently, constitutive models for nonlinear site response analysis could not accurately capture both the desired modulus reduction and damping behavior due to flaws in the assumed functional form for the backbone curve, and/or flaws in the unload-reload relationship. New relationships that solve these issues have recently been proposed, however, they are restricted to defining the in-plane stress-strain relationship for a single plane of shear, and are therefore only appropriate for 1D site response. This paper presents a multi-axial generalization of one of these 1D models, enabling its use in 2D and 3D site response. The 1D model is first briefly described, followed by the multi-axial generalization in terms of stress and strain invariants. The modulus reduction and damping curves are defined in terms of stress ratios rather than shear strains, which allows the model to better capture changes in confining pressure during undrained loading. An example is presented to illustrate fundamental differences in multi-axial versus 1D loading.

INTRODUCTION

Constitutive models for 1D ground response analysis use a set of loading/unloading rules defined based on input modulus reduction and damping curves (MRD). Advanced bounding surface plasticity soil models for 2D and 3D dynamic simulations are often formulated in a stress-ratio space, where the plastic modulus is a function of the distance between the current stress ratio and the stress ratio of an image point on the bounding surface (e.g. Dafalias and Manzari (2004), Boulanger and Ziotopoulou (2015)). These models can be adjusted to provide desired G/G_{max} and D versus γ_c behavior, but it is a complex and difficult task. Other models use multiple nested yield surfaces defined based on a backbone curve calculated from an input modulus reduction curve (e.g., Prevost 1985, Elgamal et al. 2003), without offering the possibility of controlling the damping behavior. The inclusion of MRD curves in a multidimensional plasticity framework is further complicated by the pressure-dependence of the curves, because changes in effective stress during loading should result in a modification of the dynamic curves.

This paper focuses on the extension of a 1D model using MRD curves as input parameters to a full 3D model. The 3D model uses the MRD curves formulated in terms of stress ratios to integrate them into a multidimensional framework. The model provides an exact match to the input modulus reduction and damping curves, whereas other models do not provide a perfect match. The complete formulation of the 3D model is not presented here for the sake of brevity, but all the equations are available in Yniesta (2016).

FORMULATION OF THE 1D CONSTITUTIVE MODEL

The constitutive model presented herein is based on a 1D constitutive model that uses a coordinate transformation technique to match input MRD curves. All the constitutive equations of the 1D model are published in Yniesta et al. (2017), and only the most important are repeated here.

The model is formulated in terms of shear stress (τ) and shear strain (γ), and is composed of a constitutive law for initial loading, and a set of rules for unloading-reloading. The initial loading is controlled by the modulus reduction curve. Discrete points of the backbone curve are defined based on the modulus reduction curve, and a cubic spline fit is used to ensure that the functional form of the backbone curve goes through all the discrete points. Any modulus reduction curve can be modified to match a shear strength by using the Yee et al (2013) method.

Upon unloading the model uses a rotation and a translation of the τ-γ coordinate system, to control the shape of the stress-strain curve (Figure 1a). The new coordinate system is defined by the coordinate of the target reversal point (τ_R-γ_R) and previous reversal points (τ_L-γ_L). The later are picked among the previous reversal points based on loading history and are used to define the rotation (θ) and the translation (γ_0,τ_0) of the coordinate system:

$$\theta = \tan^{-1}\frac{\tau_R - \tau_L}{\gamma_R - \gamma_L}$$

$$\gamma_0 = \frac{\gamma_R + \gamma_L}{2}$$

$$\tau_0 = \frac{\tau_R + \tau_L}{2}$$

In the τ'-γ' transformed coordinate system, the new coordinate can be calculated based on the following equations:

$$\begin{pmatrix} \gamma' \\ \tau' \end{pmatrix} = \begin{pmatrix} (\gamma - \gamma_0)\cos\theta + (\tau - \tau_0)\sin\theta \\ -(\gamma - \gamma_0)\sin\theta + (\tau - \tau_0)\cos\theta \end{pmatrix} \qquad \text{Eq. 1}$$

In the τ'-γ' transformed coordinate system the stress-strain curve follows a biquadratic equation (figure 1b) which is defined to satisfy the three following conditions:

- loops close and repeat, and the model exhibits no cyclic degradation. This implies that cycles of same strain amplitude have the same starting and ending stress-strain points, and the shape of each loop is identical.
- the area of the loop satisfies the damping curve. A cyclic strain amplitude (γ_c) for the current loop is defined based on the target and previous reversal points ($\gamma_c = |\gamma_R - \gamma_L|/2$), and the equation of the stress strain curve satisfies the damping ratio at said cyclic strain amplitude, by controlling the area *A* under the curve (figure 1b).
- the curve is concave, so that the shape of the loops is realistic. This condition is related to the functional form of the stress strain curve which could create an unrealistic convex curve.

The new stress is found by solving the following equation with the Ridder's method, which is derived by combining the equation of the stress-strain curve in the τ'-γ' system, and equation 1.

$$\tau = [(\gamma - \gamma_0)\cos\theta + (\tau - \tau_0)\sin\theta]\sin\theta + [a((\gamma - \gamma_0)\cos\theta + (\tau - \tau_0)\sin\theta)^4 + b((\gamma - \gamma_0)\cos\theta + (\tau - \tau_0)\sin\theta)^2 + c]\cos\theta + \tau_0$$

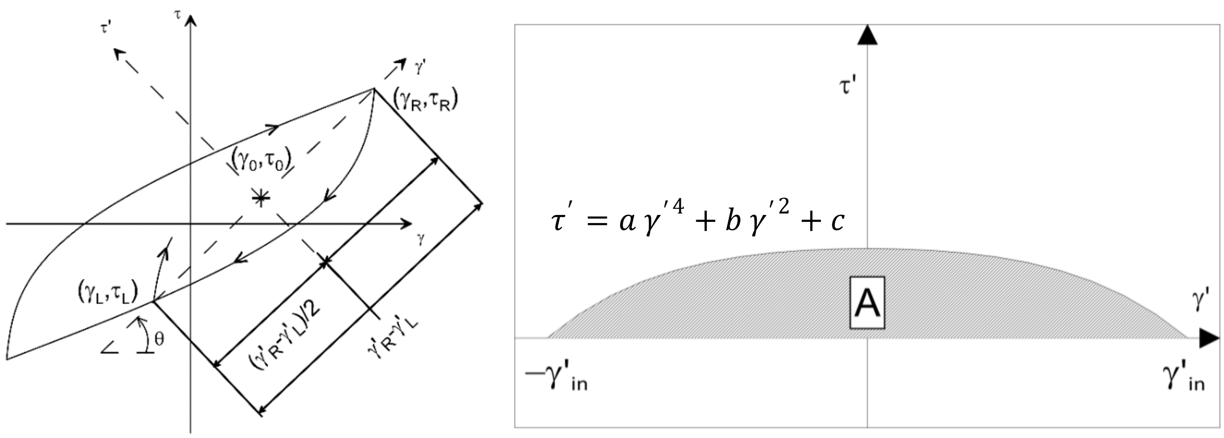

Figure 1 Rotation and translation of the coordinate system (a), and loop in the coordinate system (b) from Yniesta et al. (2017)

DYNAMIC CURVES AS A FUNCTION OF STRESS RATIOS

Dynamic curves (i.e. MRD curves) are formulated in terms of shear strains because they are derived from cyclic laboratory testing equipment where shear strains are directly measured. Research has shown that MRD curves depend on soil type, effective stress (e.g. Darendeli 2001), number of cycles (Matasovic and Vucetic 1995) and strain rate (Matesic and Vucetic 2003). By contrast, advanced constitutive models are commonly formulated in stress-ratio space (e.g., Dafalias and Manzari 2004). Modeling the modulus reduction behavior in stress-ratio space is therefore attractive for multi-axial models because it would be consistent with convention.

Figure 2 (Yniesta and Brandenberg 2016) presents MRD curves calculated from three different empirical relationships for sand, clay, and peat, at different confining pressure (i.e. effective stress). When plotted against shear strain the curves are strongly pressure-dependent (Figure 2a, b and c), and the influence is more pronounced for sand. At every strain level, a corresponding shear stress (τ) can be calculated from the modulus reduction curve and the maximum shear modulus. A stress ratio (η) can then be calculated by dividing τ by the mean confining pressure (p'). When the G/G_{max} and $D\text{-}D_{min}$ curves are plotted against the stress ratios, they become essentially independent of the confining pressure. Note that D_{min} is subtracted from the strain dependent damping because $D\text{-}D_{min}$ vs. η is independent of the confining pressure, but D vs. η is not. This concept has proven to be true for the three studied relationships, with a wide range of input parameters, consistent with each model's database.

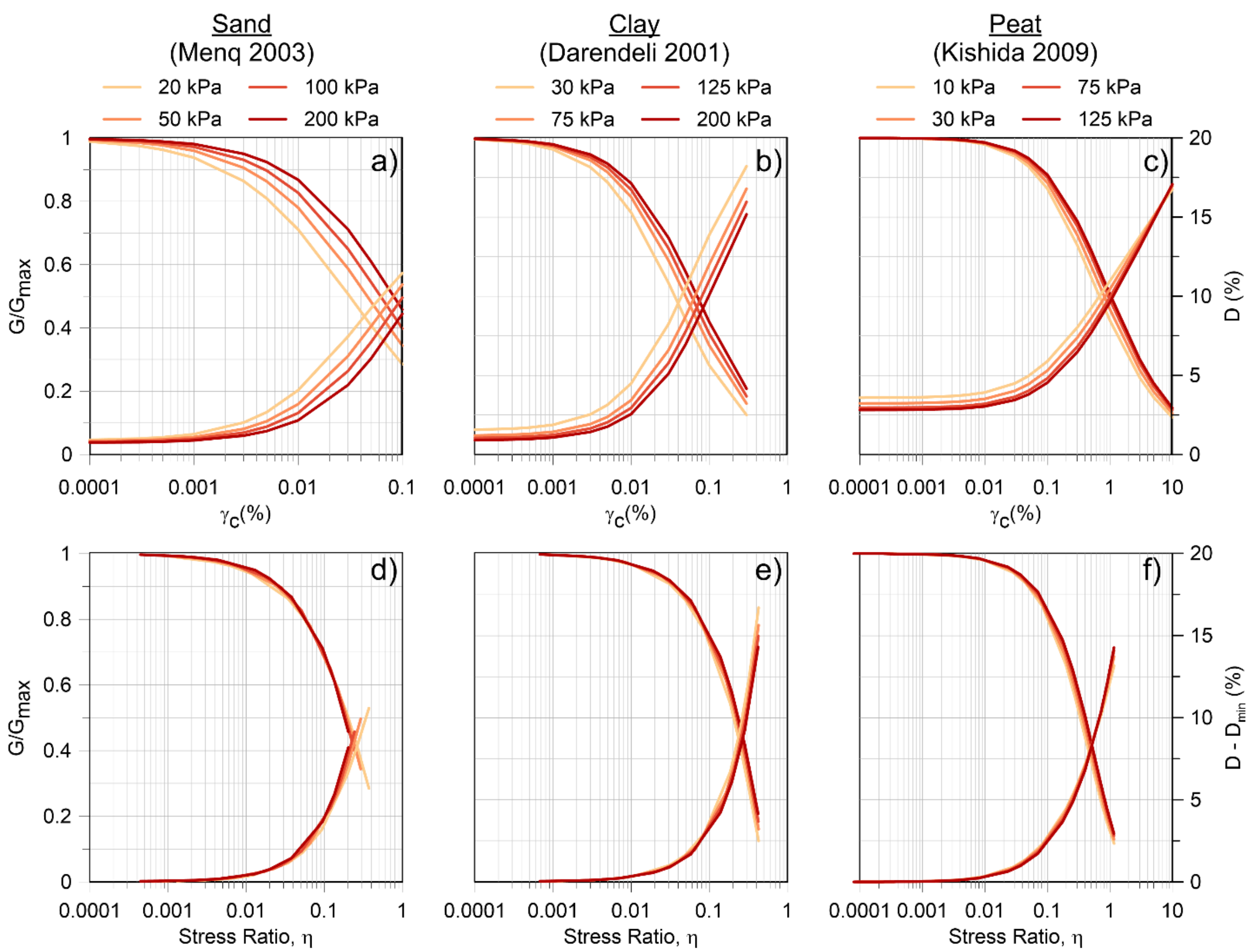

Figure 2 MRD curves for different soils at different confining stresses, in terms of shear strain (a, b and c) and stress ratio (d, e, and f) from Yniesta and Brandenberg (2016)

FORMULATION OF THE 3D MODEL AND INTEGRATION OF MRD CURVES

The 3D model presented here defines modulus reduction and damping curves in terms of stress ratios and uses similar unloading/reloading rules as the 1D model to control damping behavior. The multi-axial generalization is achieved by formulating the model in terms of deviatoric stress and strain invariants (q and ε_q, respectively) and the stress ratios ($\eta=q/p'$), calculated from the full stress (σ) and strain (ε) tensors, rather than the shear stress and strain, according to the following equations:

$$\boldsymbol{s} = \boldsymbol{\sigma} - p'\boldsymbol{I} \; ; J_2 = \frac{1}{2}(\boldsymbol{s}:\boldsymbol{s}) \; ; q = [3J_2]^{1/2}$$

$$\boldsymbol{\varepsilon_d} = \boldsymbol{\varepsilon} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} \; ; \varepsilon_q = \left[\frac{2}{3}(\boldsymbol{\varepsilon_d}:\boldsymbol{\varepsilon_d})\right]^{1/2}$$

Where I is the first invariant of the Cauchy stress tensor, J_2 is the second invariant of the deviatoric stress tensor, and a bold font indicates a tensor. Based on the previous definitions, an equivalent shear stress (τ) and shear strain (γ) can be calculated from the deviatoric stress (q) and strain (ε_q) representing the stress-strain conditions in a simple shear test. By definition, q and ε_q are positive but τ and γ are not (figure 3), so a sign needs to be assigned to q ($sign\tau$) and ε_q ($sign\gamma$) so that the unloading/reloading rules can be used. Both variables are initially equal to 1, and they can only be equal to 1 or -1. $sign\tau$ changes when only the sign of $\dot{q}$ changes (point B on the figure 3), and $sign\gamma$ changes when only the sign of $\dot{\varepsilon_q}$ changes (point C on the figure). An overall change of loading direction is detected when both $\dot{q}$ and $\dot{\varepsilon_q}$ change signs (point A on the figure). The following equations are used to relate the one-dimensional stress and strain quantities to the stress and strain invariants: $\tau = q \cdot sign\tau/\sqrt{3}$ and $\gamma = \sqrt{3} \cdot \varepsilon_q \cdot sign\gamma$. Note that since $sign\tau$ and $sign\gamma$ evolve based on $\dot{q}$ and $\dot{\varepsilon_q}$ respectively, prior knowledge of τ and γ is not necessary. These variables are subsequently used to assign a sign to q and ε_q without having to compute τ and γ.

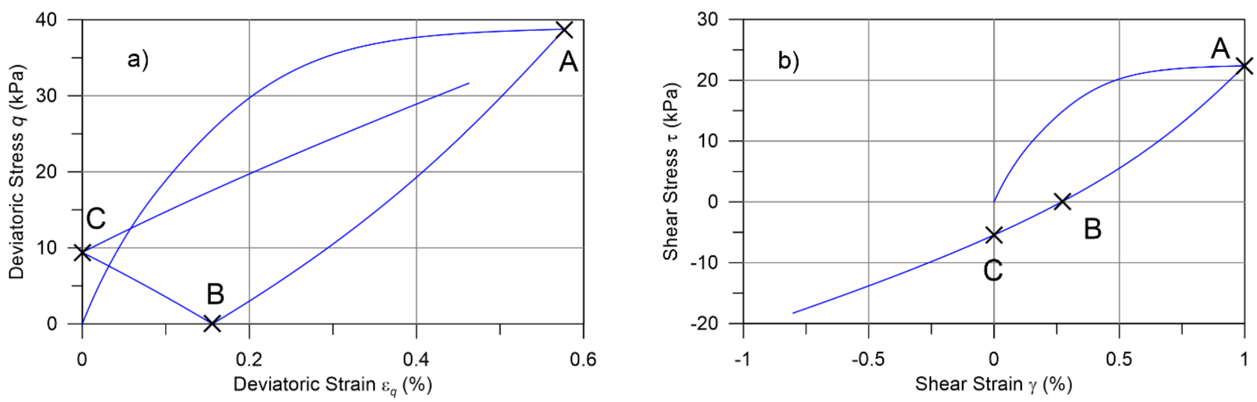

Figure 3 Evolution of the signs of q and ε_q during cyclic loading

Upon unloading the stress-strain curve is controlled in a modified coordinate system. When a change of loading direction is detected, the original coordinate system q-ε_q is rotated and translated based on the previous and target reversal points (figure 4). The later are selected based on rules similar to the 1D case, except that reversal points are tracked in terms of stress ratios η and deviatoric strains ε_q rather than shear stress and strain τ-γ. Note that in this 3D formulation, τ

and γ are never computed because the entire model is entirely formulated in terms of q-ε_q that have been assigned a sign. The translation (ε_{q_0}, q_0) and rotation (θ) of the system are calculated as follow:

$$\varepsilon_{q_0} = \frac{\varepsilon_{q.R} + \varepsilon_{q.L}}{2}$$

$$q_0 = \frac{\eta_R \cdot p' + \eta_L \cdot p'}{2}$$

$$\theta = \tan^{-1}\frac{\eta_R \cdot p' - \eta_L \cdot p'}{(\varepsilon_{q.R} - \varepsilon_{q.L})} \quad if\ load = 1$$

The coordinate in both systems are linked by the following relationships:

$$\begin{pmatrix} \varepsilon_q{}' \\ q' \end{pmatrix} = \begin{pmatrix} (\varepsilon_q - \varepsilon_{q_0})\cos\theta + (q - q_0)\sin\theta \\ -(\varepsilon_q - \varepsilon_{q_0})\sin\theta + (q - q_0)\cos\theta \end{pmatrix} \qquad \text{Eq. 2}$$

In the modified q'-ε'_q coordinate system the stress-strain curve follows a biquadratic equation ($q' = a\,\varepsilon_q{}'^4 + b\,\varepsilon_q{}'^2 + c$), and the parameters a, b and c are calculated based on three conditions as in the 1D model, except that the damping value to match is picked from the D-D_{min} curve and depends on a cyclic stress ratio amplitude ($\eta_{eq} = |\eta_R - \eta_L|/2$). The small strain damping D_{min} can be added using a viscous damping formulation (e.g. Rayleigh 1945), or by using an empirical relationship defining D_{min} as a function of p'.

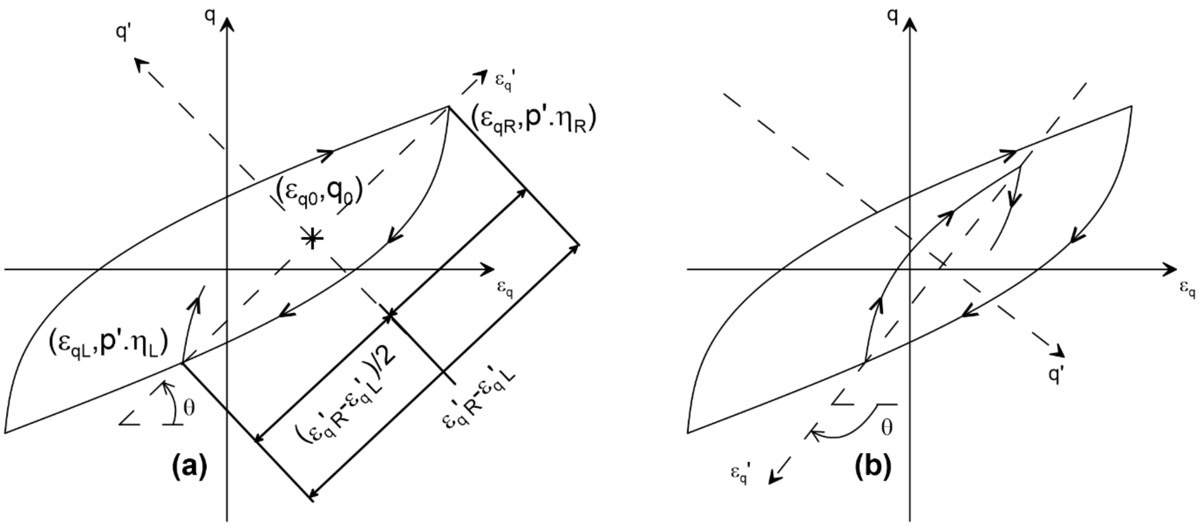

Figure 4 Transformed coordinate system upon unloading/unloading

The new deviatoric stress is calculated based on the following equation obtained by combining the biquadratic equation in the q'-ε'_q and equation 2:

$$q = \left[\left(\varepsilon_q - \varepsilon_{q_0}\right)\cos\theta + (q - q_0)\sin\theta\right]\sin\theta + \left[a\left((\varepsilon_q - \varepsilon_{q_0})\cos\theta + (q - q_0)\sin\theta\right)^4 + b\left((\varepsilon_q - \varepsilon_{q_0})\cos\theta + (q - q_0)\sin\theta\right)^2 + c\right]\cos\theta + q_0$$

The unloading-reloading rules are used whenever the current stress ratio is lower than the maximum stress ratio ever attained, otherwise the model is considered in an initial loading phase. During initial loading, the model uses a hybrid formulation where it follows the modulus reduction curve formulated in terms of stress ratios up to a limit stress ratio, and then follow a bounding surface algorithm similarly to the formulation by Dafalias and Manzari (2004). The volumetric response is coupled with the deviatoric response based on the formulation similar to Dafalias and Manzari (2004), although other flow rules could be used. The theoretical framework of bounding surface plasticity is formulated based on a decomposition of the strain increment between elastic and plastic strain increments. The work presented herein is based on total strain, but it is a simple matter to compute the elastic deviatoric strain increment based on the deviatoric stress increment and the shear modulus to obtain the plastic strain increment.

In a numerical modeling software constitutive laws are implemented to establish a relationship between stress tensor and strain increment tensor. The equations presented herein are used to calculate the new deviatoric stress from the deviatoric strain increment and derive an equivalent plastic modulus that is then used to relate deviatoric stress tensor and deviatoric strain tensor. The full stress tensor can then be reconstructed by adding back the mean pressure *p'* to the deviatoric stress tensor.

EXAMPLE

This section presents a numerical simulation involving a bidirectional drained direct simple shear stress path under a vertical effective stress of 100 kPa. A harmonic strain history was imposed in one direction, while a linear strain history was imposed in an orthogonal direction, as indicated in Fig. 6a. In addition to running the model in three dimensions, the 1D model was run in each orthogonal direction. The in-plane shear stress versus shear strain for the 3D and 1D simulations are shown in Fig. 6b and d. The stress-strain curves in these planes are more complicated for 3D loading because loading in the out-of-plane direction influences the response of the material model in the plane being plotted. In the 1D simulations, shearing occurs only in a single plane, and the material response is more traditional. The resulting stress and strain values from the pair of 1D simulations were combined, and invariants q and ε_q were computed from the combined stress paths. These were plotted with the invariants from the 3D simulation in Fig. 6c. The material response is a bit stiffer for the 1D models than for the 3D model because the out-of-plane strain softens the material response in the plane of interest in the 3D model.

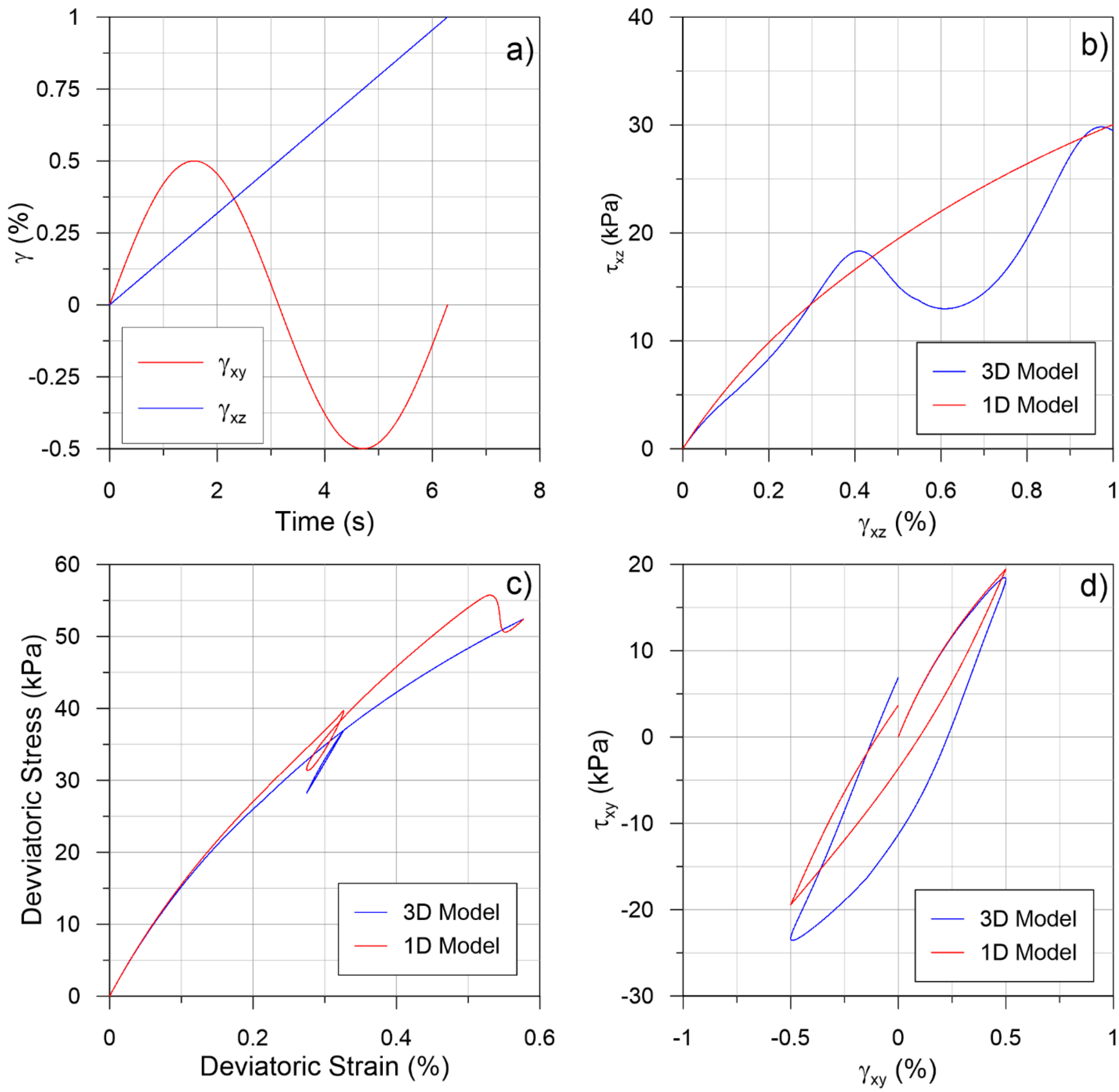

Figure 5 Bidirectional strain loading (a), stress-strain curve in the xz direction (b), in the xy direction (d), and stress strain curve of the combined response (c).

CONCLUSION

This paper presented a multi-axial generalization of a 1D constitutive model that permits perfect matching of a desired modulus reduction and damping curve. Such models only recently were formulated in one-dimension, and we believe this is the first multi-axial model that accepts MRD curves as user-specified input parameters. Other multi-axial models have been adjusted to reasonably match specific MRD relationships, but do not permit users to input their own desired MRD curves. A simulation involving a two-dimensional strain history was compared with the uniaxial response in two orthogonal directions. The multi-axial model was significantly different from the uniaxial model, illustrating the importance of considering multi-axial loading for such problems.

The simulation presented herein involved drained loading conditions with constant mean effective stress. The model can be easily extended to undrained loading behavior by adopting an appropriate flow rule into the plasticity formulation. Yniesta (2016) provided such a formulation, but the flow rule was excluded here for simplicity.

ACKNOWLEDGMENTS

This research was funded by the National Science Foundation under Grant No. CMMI 1208170. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

REFERENCES

Boulanger, R.W., and Ziotopoulou, K. (2015). *PM4SAND (Version 3): A sand plasticity model for earthquake engineering applications.* Rep. No. UCD/CGM-15-01, Center for Geotechnical Modeling, Dept. of Civil and Environmental Engineering, Univ. of California, Davis, CA, 108.

Dafalias, Y. F., and Manzari, M. T. (2004). “Simple plasticity sand model accounting for fabric change effects.” *J. Eng. Mech.*, 10.1061/(ASCE) 0733-9399(2004)130:6(622), 622–634.

Darendeli, M. (2001). “Development of a new family of normalized modulus reduction and material damping curves.” *Ph.D. Thesis*, Dept. of Civil Eng., Univ. of Texas, Austin.

Elgamal A, Yang Z, Parra E, Ragheb (2003) “A Modeling of cyclic mobility in saturated cohesionless soils.” *Int J Plasticity*, 19:883-905.

Matasovic, N., and Vucetic, M. (1995). “Generalized cyclic degradation pore pressure generation model for clays.” *J. Geotech. Eng.*, 10.1061/(ASCE)0733-9410(1995)121:1(33), 33–42.

Matesic, L., and Vucetic, M. (2003). “Strain-rate effects on soil secant shear modulus at small cyclic strains.” *J. Geotech. Geoenviron. Eng.*, 10.1061/(ASCE)1090-0241(2003)129:6(536), 536–549.

Prevost, J.H. (1985). “A simple plasticity theory for frictional cohesionless soils,” *Soil Dyn. Earthquake Eng.*, 4(1), 9-17

Rayleigh, J.W.S and R.B. Lindsay (1945). *The theory of sound, Dover Publications,* New York

Yee E., Stewart J.P., Tokimatsu K. (2013). “Elastic and large-strain nonlinear seismic site response from analysis of vertical array recordings”, *J. Geotech. Geoenvion. Eng.,* 139 (10), 1789-1801.

Yniesta, S. (2016). *Constitutive Modeling of Peat in Dynamic Simulations.* Ph.D. Dissertation, Dept. of Civil and Environmental Eng., UCLA

Yniesta, S., S. J. Brandenberg, and A. Shafiee. (2017). "ARCS: A one dimensional nonlinear soil model for ground response analysis." *Soil Dynamics and Earthquake Engineering* 102: 75-85.

Yniesta, Samuel, and Scott J. Brandenberg (2016). "Stress-Ratio-Based Interpretation of Modulus Reduction and Damping Curves." Journal of Geotechnical and Geoenvironmental Engineering 143.1: 06016021.