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Essays on Time-Inconsistent and Endogenous Discounting in  
Macroeconomic Models

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by

Sarah Lynne Salvador Daway

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Dissertation Committee:

Dr. Jang-Ting Guo, Co-Chairperson  
Dr. Richard M.H. Suen, Co-Chairperson  
Dr. R. Robert Russell  
Dr. Aman Ullah

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The Dissertation of Sarah Lynne Salvador Daway is approved:

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Committee Co-Chairperson

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Committee Co-Chairperson

University of California, Riverside

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## ABSTRACT OF THE DISSERTATION

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by

Sarah Lynne Salvador Daway

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Dr. Jang-Ting Guo, Co-Chairperson

Dr. Richard M. H. Suen, Co-Chairperson

Empirical evidence shows that the exponential discount function employed in standard macroeconomic models falls short on two counts: it implies time-consistent preferences and assumes that the discount factor is a constant, exogenous parameter. As discounting behavior is crucial in the determination of individual intertemporal choice, it seems reasonable to expect that deviating from the standard exponential discounting behavior would significantly affect aggregate outcomes. Thus so, the main question of this dissertation is, “What are the macroeconomic and welfare consequences of time-inconsistent and endogenous discounting?” In this endeavor, this research focuses on three important applications: 1) on resource allocation and welfare in a social planner’s economy; 2) on raising the retirement age in an economy where time-consistent and time-inconsistent discounters co-exist; and 3) on monetary policy effectiveness in a standard neoclassical growth model. Indeed, the results show that moving beyond exponential discounting affects both macroeconomic and welfare outcomes to the extent that policy recommendations based on exponential discounting must at the very least be taken with a grain of salt.



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# Chapter 1

## Introduction

“And indeed there will be time...  
Time for you and time for me,  
And time yet for a hundred indecisions,  
And for a hundred visions and revisions,  
Before the taking of a toast and tea.”  
*Excerpt from “The Love Song of J. Alfred Prufrock”*  
*By T.S. Elliot (1920)*

That choices are embedded in the space-time continuum conjures up a portrait of the individual, weighing tradeoffs at every point in time – not the least of which is the tradeoff between present and future utilities. While prudence prods him to consider the future, impatience, uncertainty, bounded rationality or plain myopia urges him to often place more consequence on current utility. Whether he saves more or consumes more depends on the relative value that he imputes on the future. And this personal valuation of the future often has implications that go beyond the individual’s welfare.

In economic models, the individual’s valuation of these intertemporal tradeoffs is embodied by what is called the discount factor or the rate of time preference. While a higher discount factor reflects a higher estimation of future well-being, a higher rate of time preference (RTP) indicates greater impatience, and thus, a greater bias for current interests. The RTP, in particular, is the notion upon which the Neoclassical rate of interest rests: a positive interest rate is required in order to induce impatient individuals to save, and ultimately invest in growth-enhancing activities. Indeed, Fisher (1930) equates the real rate of interest to the rate of time preference: while the former is viewed as the objective cost of investing, the latter is its subjective cost. Thus, the higher the RTP, the more myopic the individual is or the higher his subjective cost of investing is, and would, as a consequence, prefer greater consumption today at the expense of lower savings and foregone opportunities for greater capital or wealth accumulation. Taken in the aggregate, these individual

intertemporal decisions based on time preference can have significant welfare and efficiency implications.

However, the standard RTP has two main shortcomings: it implies time-consistent preferences and is modeled as a constant, exogenous parameter, contrary to empirical evidence. On the one hand, experimental studies show that the individual alters the way he discounts intertemporal tradeoffs with every shift in temporal perspective. Specifically, he tends to exhibit greater patience when making a choice regarding an event that will happen in the distant future, but once the distant future comes about, he suddenly views the same choice with greater impatience. On the other hand, the RTP is neither constant nor exogenous, as a number of studies find that wealthier households save more, thereby indicating that the RTP declines with wealth. Studies that relax these standard assumptions are able to resolve certain conundrums and hitherto unexplained empirical regularities, as shall be discussed below.

Accordingly, this research seeks to go beyond exponential discounting in macroeconomic models, and to analyze the repercussions of non-exponential discounting on issues such as welfare, stability, fiscal and monetary policies. The main research question is, “What are the macroeconomic consequences of a time-inconsistent and endogenous RTP?” In this endeavor, I focus on three important applications: 1) on resource allocation and welfare in a social planner’s economy (Chapter 2); 2) on monetary policy effectiveness (Chapter 3); and 3) on raising the retirement age in an economy where time-consistent and time-inconsistent discounters co-exist (Chapter 4).

The rest of this chapter presents a review of the related literature on non-exponential discounting. In doing so, we first dwell briefly on the psychology of time discounting or time preference, and present the experimental evidence against an exponential discount function. We then focus on the following two strands that go beyond exponential discounting in the macroeconomics literature. Along the first strand, we encounter the hyperbolic discount function and its more analytically tractable analogue, the quasi-hyperbolic or quasi-geometric discount function that Laibson (1994, 1997) popularized. These discounting

functions imply time-inconsistent preferences that render standard dynamic optimization analyses invalid. Consequently, the decision maker’s problem has to be treated as a game amongst his current and future selves, and the corresponding solution concept is what is known as Markov perfect equilibrium. The second strand models the RTP as either an increasing or decreasing function of individual or aggregate income, consumption or wealth. The assumption that the RTP increases with income is called increasing marginal impatience (IMI). Otherwise, it is called decreasing marginal impatience (DMI). In this, we argue that the DMI assumption is the one supported by empirical evidence.

## 1 Review of Related Literature

### 1.1 Psychological Motives and Intertemporal Choice

While the impatience intrinsic to human nature engenders in him the tendency to only account for current utility, “the effective desire of accumulation”<sup>1</sup> influences him to impute positive weights on future utilities, albeit to a lesser degree compared to that on current utility. Frederick, Loewenstein and O’Donoghue (2002) (FLO (2002), *hereafter*) made a fine distinction between these two conflicting tendencies. They called the individual’s penchant for favoring current utility over future utility as time preference, while the tendency of the individual to ascribe lesser values to future utilities is known as time discounting. For the most part, these are taken as two sides of the same coin, and shall be used interchangeably.

The parameter known as the “rate of time preference” (RTP) has been viewed to encapsulate impatience, and thus the factors that affect intertemporal decision-making. The RTP reflects two notions: one, the relative worth imputed to either present or future consumption depends on the relative level of consumption, which then accounts for the nonlinearity of the indifference curves in the two-period consumption space (Fisher, 1930); and two, even with at par levels of present and future consumption (i.e.,  $c_t = c_{t+1}$ ), the assigned values need not be equal. Although it was upon these notions that both Friedman (1976)

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<sup>1</sup>See John Rae’s (1834) “The Sociological Theory of Capital.”



and Stigler (1987) equated what is known as the “pure rate of time preference” with the intertemporal rate of marginal substitution only along the 45-degree line, Becker and Mulligan (1997) asserted that the modern concept of “time preference” encompasses all points in the  $(c_t, c_{t+1})$  space.

A number of psychological factors have been identified to explain intertemporal behavior. John Rae (1834) asserted that the desire to accumulate is either promoted by the motive of leaving bequests and the human proclivity to employ self-control, or is limited by the uncertainty of human existence and the thrill of immediate gratification. From these factors, FLO identified two disparate perspectives that dominate the landscape of intertemporal choice: the “anticipatory-utility” and “abstinence” perspectives. The “anticipatory-utility” perspective was favored by Jevons (1888), who assumed that the individual is only concerned with his immediate utility. That the individual thus forgoes current utility must be because he anticipates a more than commensurate increase in his expected or “anticipal” future utility. In contrast, the “abstinence” perspective starts with the individual ascribing equal values to current and future utilities. The only reason that current utility would have more weight than future utility would be due to the pain or miseries of abstaining from current consumption. Whether “anticipatory-utility” or “abstinence” perspective however, the underlying driving force still appears to be ubiquitous impatience.

The classical economists Böhm-Bawerk (1891) and Fisher (1930) added to the psychological factors mentioned above. Böhm-Bawerk recognized the individual’s inherent limited ability and foresight to assess his future well-being. Thus, he always makes the unavoidable mistake of undervaluing his future utility. In modern parlance, this human limitation can be identified as “bounded rationality.” Fisher also pointed out the importance of fashion in determining time preference, as “fashion” spurs the individual to save in order to “keep up with the Joneses.” In the same token, it is often fashion that pushes the “millionaires to live in an ostentatious manner” (Fisher, 1930, p. 87), which is reminiscent of Veblen’s “conspicuous consumption.”

Both authors moreover identified “objective factors” such as wealth and risk. Fisher,

in particular, distinguishes between the “rational” and “irrational” influences of poverty on the individual’s behavior. The “rational” part urges the individual to give more weight to his current needs in order to survive another day, while the “irrational” side makes him even more impatient to consume as “the pressure of present needs blinds a person to the needs of the future” (1930, p. 72).

## 1.2 Exponential Discounting and Time Consistency

Following Samuelson’s (1937) discounted utility model, the RTP or alternatively, the discount factor, has been often modeled as a fixed, exogenous, catch-all parameter that does not really capture the different psychological motives that are understood to affect intertemporal utility maximization. The standard discounted utility (DU) model is of the following form:

$$U_t(c_t, c_{t+1}, \dots, c_{t+T}) = \sum_{i=0}^T \left( \frac{1}{1+\rho} \right)^i u(c_{t+i}), \quad (1)$$

where  $U_t(\bullet)$  is time-separable and is the individual’s lifetime utility discounted to time  $t$ ;  $u(c_{t+i})$ ,  $i = 0, 1, \dots, T$ , the instantaneous utility or the felicity function; and  $\rho$ , the constant, pure RTP. The fixed ratio,  $1/(1+\rho)$ , is the discount factor. The entire discount function  $1/(1+\rho)^i$  is of the exponential form or what is also known as the geometric discount function.

It is widely known that stationary discounting implies time-consistent preferences. Formally, time-consistent intertemporal preferences mean that for any two consumption profiles  $(c_1, \dots, c_T)$  and  $(c'_1, \dots, c'_T)$ ,  $U(c_1, \dots, c_T) \geq U(c'_1, \dots, c'_T)$  if and only if  $U_j(c_j, \dots, c_T) \geq U_j(c'_j, \dots, c'_T)$  for  $j = 2, \dots, T$ , given  $c_1 \geq c'_1$ . In words, this translates into: the future selves (of the individual) agree with the past self’s decision or preference. This also implies that past histories are irrelevant to the current decision problem of the individual.

Evidences against time consistency abound, however. Classic examples range from unrealized New Year’s resolutions and unsuccessful diet plans to dynamic inconsistencies in monetary policy regimes, where the policymaker makes a prior commitment to no inflation,

only to renege on this promise later on, in order to achieve higher output. Yet in spite of the theoretical reservations and empirical criticisms against constant or geometric discounting, this has been widely used in the literature of intertemporal choice (FLO, 2002). As we shall see more clearly later on, the reason, more than anything else, is analytical tractability.

### 1.3 Behavioral Experiments and Discounting Patterns

It was Strotz (1956) who was credited with first making the observation that individuals demonstrate more patience over long-run tradeoffs than they do over short-run ones from a particular time viewpoint. This is in contrast to the DU model, which predicts evenhandedness in the individual's temporal perspective. Several behavioral experiments confirm this by reporting a declining RTP (or discount rate) over the time horizon (FLO, 2002; Soman, et al., 2005; Laibson, 1998; Ainslie, 1992; Loewenstein and Prelec, 1992). The hyperbolic discount function is of the form

$$(1 + \alpha\tau)^{-\gamma/\alpha}, \tag{2}$$

where  $\alpha, \gamma > 0$ . The associated discount rate or RTP is given by  $\gamma/(1 + \alpha\tau)$ .<sup>2</sup>

A hyperbolic discount function implies that if we ask the individual what amount he would demand in  $T$  years to make him indifferent to receiving an  $X$  amount today, the implicit RTP should be observed to be lower, the farther the horizon  $T$ . Alternatively, this implies that the individual becomes more impatient the closer the temporal distance of the future alternative is to his present – i.e., a larger future amount would be needed for him to wait say, a month from now than the amount that he would require if he were to choose between getting the reward a year from now and postponing his reward to a year and a month from now.

Hyperbolic discounting, however, generates time-inconsistent preferences. Studies found that if the experimenter asks the individual regarding his preference over, say, \$50 today

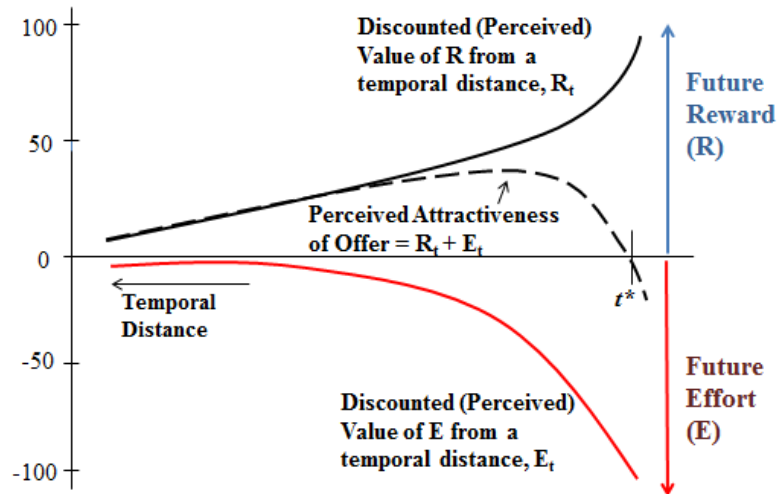
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<sup>2</sup>This is computed as  $-\beta'(\tau)/\beta(\tau)$  for any discount function  $\beta(\tau)$ .

or \$100 a year from now, the individual would often take the \$50 today. However, if he were asked to choose between \$50 in nine years' time or \$100 in 10 years, the individual would switch his preference to \$100 in ten years. It is in this connection, that hyperbolic discounting is purported to be consistent with human behavior that displays excessive short-run impatience relative to the long run from some contemporaneous perspective. Angeletos et al. (2001) mentioned that this specific discount function explains undersaving, addiction, procrastination, problems of self-control and other behavioral outcomes that are viewed to be anomalous if one were to adhere to the DU model.

The implied preference reversal under hyperbolic discounting is evident in Figure 1.1 below.<sup>3</sup>

Figure 1.1: Perceived Rewards and Effort with Hyperbolic Discounting



We observe that the further one goes away leftward from a critical time value,  $t^*$ , an offer that was not palatable to the individual at a very short temporal distance (all values to the right of  $t^*$ ) becomes gradually attractive as the agent's time perspective shifts to the long run. Accordingly, the perceived attractiveness of the offer (computed as the value

<sup>3</sup>See Soman et al. (2005).

of the reward minus the cost of the effort of procuring the reward) goes from negative to positive as the temporal distance increases.

## 1.4 Macroeconomics beyond Exponential Discounting

The preceding discussion has established that except perhaps for analytical and computational convenience, there is neither strong *a priori* nor *a posteriori* justification for assuming a fixed, time-consistent RTP or equivalently, the exponential discount function. In fact, in a model with heterogeneous agents, Becker (1980) shows that a constant time preference has the inconceivable implication of a degenerate wealth distribution: all society's wealth is amassed by the most patient individual for a given real interest rate.

This section presents two interesting strands pursued in the macroeconomic literature to model intertemporal discounting.<sup>4</sup> The more recent strand incorporates Laibson's (1994, 1997) quasi-hyperbolic discount function (also known as quasi geometric discount function) into representative-agent models (Krusell, Kuruşçu and Smith, 2001; Krusell and Smith, 2001; Maliar and Maliar, 2004). The older strand looks at the implications of endogenizing the RTP in growth models.

### 1.4.1 First Strand: Quasi-Hyperbolic Discounting

For analytical tractability, Laibson (1994, 1997) favored a discrete-time discounting function that has the qualitative properties of the hyperbolic function. He called this the quasi-hyperbolic discount function, which is specified below:

$$\gamma(\Delta t) = \begin{cases} 1, & \Delta t = 0 \\ \beta\delta^{\Delta t}, & \Delta t > 0 \end{cases}, \quad (3)$$

where  $\Delta t$  is the delay (i.e., the time between the individual's present and the time he enjoys his future utility); and the constants  $\beta \in (0, 1)$  and  $\delta \in (0, 1]$ . From Equation (3), we

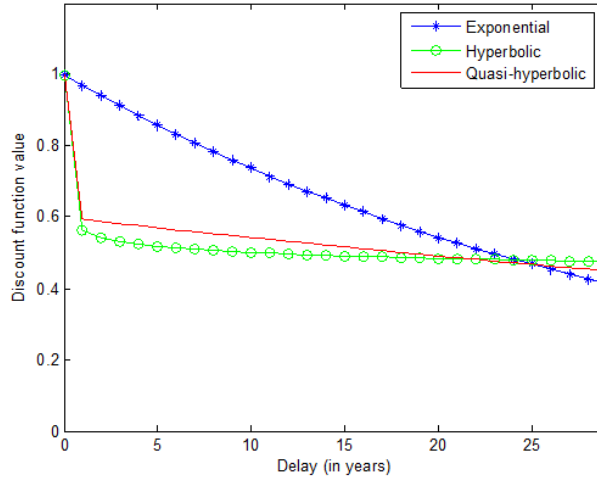
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<sup>4</sup>A third strand deals with recursive utility. But for the purpose at hand, we restrict ourselves to the two we mention presently.

can show that the discount rate between the present and the next period is  $((1 - \beta\delta)/\beta\delta)$ ; while the per-period discount rate or RTP between any two periods in the “long run” is  $((1 - \beta\delta)/\beta)$  which is less than the “short-run” discount rate,  $((1 - \beta\delta)/\beta\delta)$ . This indicates that from a contemporary standpoint, the individual views that next period with less patience than when he views the farther future ( $\Delta t > 0$ ). Thus, the quasi-hyperbolic function preserves the “present bias” that the hyperbolic function implies.

Figure 1.2 plots the hyperbolic, quasi-hyperbolic and exponential discount functions. We follow Laibson (1997) in letting  $\beta = .97$  in the exponential discount function,  $\alpha = 10^5$  and  $\gamma = 5000$  in (2); and  $\delta = .6$ ,  $\beta = .99$  in (3). It can be observed from the graph that for these specific values, quasi-hyperbolic discounting does well in tracking hyperbolic discounting.

Figure 1.2: Discount Functions



Under quasi-hyperbolic preferences, the agent’s lifetime utility can be formulated as:

$$U(c_t) = u(c_t) + \delta \sum_{j=1}^{\infty} \beta^j u(c_{t+j}), \quad (4)$$

where  $\delta\beta$  is called the short-run discount factor, while  $\beta \in (0, 1)$  is the long-run discount factor. However, maximizing (4) with respect to a budget constraint cannot be done using the usual Lagrangian method. This is because in spite of the greater tractability that

this functional form affords when compared to the hyperbolic function, it also inherits the dynamic-inconsistent preferences implied by the latter. To illustrate, from the perspective of the individual at any time  $t$ , the marginal rates of substitution between  $(t+1)$  and  $(t+2)$  is:

$$MRS_{(t+1,t+2)} = \beta \frac{u'(c_{t+2})}{u'(c_{t+1})}. \quad (5)$$

However, come period  $(t+1)$ , the individual's perspective changes and

$$MRS_{(t+1,t+2)} = \delta\beta \frac{u'(c_{t+2})}{u'(c_{t+1})}. \quad (6)$$

Thus, the usual dynamic optimization methods cannot be applied indiscriminately to obtain a meaningful solution when the agent is unable to commit. To address this, economists have treated this problem as a game that the current individual plays with his future selves. Laibson (1996) showed that for any finite-horizon game of this type, a unique subgame-perfect equilibrium that is Markov perfect exists. Moreover, the consumption rule is time-dependent and linear in wealth. Krusell, Kuruscu and Smith (2002) extend this analysis to the infinite-horizon setup, where they restrict the equilibrium set to only admit *first-order* Markov perfect equilibria in order to derive a unique solution to the standard, neoclassical growth model.

With regard to empirical performance, Laibson (1998) reported that quasi-hyperbolic discounting explains the stylized correlations between measured patience and the variables age, income and wealth. Moreover, Angeletos, et al. (2001) compared quasi-hyperbolic discounting against the standard geometric or exponential discounting and found that the former explains the following behavioral patterns consistent with the data:

- Households with hyperbolic preferences prefer to keep their wealth in the form of illiquid assets, as a mechanism for self-control.<sup>5</sup>

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<sup>5</sup>They lifted their data from the Survey of Consumer Finances (1995).

- They tend to have more credit card debt, which is an indication of excessive short-run impatience that demands immediate satisfaction.
- They are less successful in smoothing consumption, as most of their wealth is tied up in illiquid assets. Thus, their income and consumption tend to exhibit high degrees of co-movement – and even predictable changes in both income and consumption, which the standard model incorrectly predicts to be nil.<sup>6</sup>
- This co-movement between income and consumption is more pronounced when liquid, labor income ceases upon retirement.

#### 1.4.2 Second Strand: Endogenous Time Preference

Earlier attempts were made to model an endogenous RTP to incorporate the “objective” factors mentioned by Bohm-Bawerk (1891) and Fisher (1930). In particular, Uzawa (1968), Lucas and Stokey (1984) and Epstein (1987) linked the RTP to consumption; while Laidler (1969) modeled the RTP as a function of wealth.

We identify three prototype specifications for the RTP in intertemporal optimization models. The first prototype models the RTP as an increasing function of individual consumption. The second specification is directly in contrast to the first one in that the RTP decreases with consumption. The third specifies it as an increasing function of aggregate consumption. The first prototype is the most widely used and is identified as increasing marginal impatience (IMI), while the second one is known as decreasing marginal impatience (DMI). Whether IMI or DMI has crucial implications on the stability of the dynamic system in question.

##### *First Prototype: IMI in Individual Consumption*

It was Uzawa (1968) who first modeled time preference as an increasing function of current consumption. Along a given constant consumption path, this means that the RTP

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<sup>6</sup>To verify this, they used data from the Panel Study of Income Dynamics.



ultimately becomes a function of current and future consumption. The theoretical justification for this was provided by Epstein (1987) who asserted that this assumption yields both a necessary and sufficient condition for a stable steady state. The intuition is developed in the following manner: when consumption is below its steady-state level (along a constant path of adjustment), the RTP would be below the real interest rate as the RTP would decrease with consumption. This, in turn, increases the accumulation of wealth, which induces a persistent increase in consumption until it again reaches the steady state.

Put differently, the stability result should not be surprising in this light: suppose otherwise that the agent becomes more patient as consumption increases. As the economy then grows over time, the agent becomes more patient and hence saves more. Consequently, the economy grows even faster, away from the steady state.

A glaring implication of this is that since consumption increases with the individual's utility and wealth, the richer the individual, the more impatient he becomes. Although this implication clashes with Becker and Mulligan's (1997)<sup>7</sup> observations and with Fisher's (1930) intuition that impatience is more characteristic of low-income individuals (especially those with near-subsistence incomes), a majority in the literature has modeled the RTP as an increasing function of consumption, income or wealth.

Shi and Epstein (1993) took habit formation into account by constructing the RTP as a function of an index for past consumption. The implication of this is that in equilibrium, time preference is now influenced not only by current and future consumption, but also by past consumption along any constant consumption path. For sound intuition, the authors drew from Fisher's (1930) argument for habit formation: an individual who is used to a parsimonious lifestyle is more apt to be patient and to save, while an individual used to lavish living is more predisposed to impatient consumption. Fisher further discussed that the intergenerational implication is that prudent parents accustomed to frugal living save and amass wealth by habit, while the children accustomed to the luxurious life spends beyond their means, so that the time path of wealth is characterized by cycles of accumulation and

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<sup>7</sup>See further discussion in the following subsection.

decline.<sup>8</sup>

Table 1 below presents the Uzawa (1968) and the Shi and Epstein (1999) specifications:

Table 1.1: Intertemporal Utility Functions

Uzawa (1968)	Shi and Epstein (1999)
$\int_0^\infty u(c_t) e^{-\int_0^t \rho(c_v) dv} dt$	$\int_0^\infty u(c_t) e^{-\int_0^t \rho(h_v) dv} dt$
	$h(v) = \gamma \int_{-\infty}^v (c_\tau) e^{\gamma(\tau-v)} d\tau$

The variable  $h$  stands for the habit index, which is a weighted average of past consumption. The imputed weight  $\gamma e^{\gamma(T-t)}$  decreases at the rate  $\gamma$  as one goes back further in time. The IMI assumption works to stabilize the system locally by reinforcing the stabilizing effects associated with the Uzawa specification: when consumption has been below the steady-state level for some period, the RTP will be lower than the real of interest rate as habit sets in, which entails a positive growth rate of consumption, as wealth is built up through patience.

Kompas and Abdel-Razeq (2001) showed that the Uzawa transformation that makes it convenient for the RTP to be transformed into a constant at every point on a constant equilibrium path, is only valid for a special type of autonomous systems and that employing this transformation generates errors in the derivation of the first-order conditions. They thus proceeded with analyzing the more complex monetary system of four differential equations using an RTP that depends on the entire time path of consumption. Consequently,  $\rho' > 0$  is a measure of a “path effect.” This means that an increase in the entire time path of consumption (i.e., present and future consumption) will induce the individual to put more weight on current consumption. This result sounds more intuitive in that it conforms to the consumption smoothing motive.

Beginning with Obstfeld (1990), there has been an influx of open-economy models with

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<sup>8</sup>See Fisher (1930, p. 337-339).

endogenous discounting. As with the closed-economy case, a fixed RTP generates insupportable predictions regarding the distribution of wealth: in the long run, all the world's wealth accrues to the most patient country. Many of the papers written in this literature adapt the assumption of IMI in order to achieve stability. These papers include Mendoza (1991), Schmitt-Grohé and Uribe (2002), Kim and Kose (1999) and Meng (2006).

*Second Prototype: DMI in Individual Consumption*

As noted above, though the IMI assumption is employed to ensure unconditional stability, it produces a counterintuitive description of preference behavior, and has therefore been subject to numerous criticisms (Koopmans, 1986; Barro and Sala-i-Martin, 1995; Das, 2002; Lahiri, 2002). In contrast, the counter-assumption known as decreasing marginal impatience (DMI)<sup>9</sup> has the triad advantage of being 1) more intuitive, 2) capable of generating a stable steady state given certain conditions, and 3) consistent with empirical findings

Indeed, Becker and Mulligan (1997) link the RTP to the agent's level of income or wealth by building a model of time preference that is determined by the "propinquity of future pleasures." Their reasoning stems from the observation that the individual is boundedly rational and can only therefore imagine or anticipate the future with myopia, which limits his ability to assess his future well-being. Aware of this shortcoming, the individual takes measures to reduce this myopia by spending resources on endeavors that would increase the "propinquity of future pleasures," and thus, allow him a better assessment of future welfare. They showed that the wealthier the individual, the more he can afford to "invest in patience," which provides a theoretical justification for the DMI assumption.

Das (2003) shows that in a standard neoclassical, non-monetary, exogenous growth model with the RTP as a decreasing function of real consumption, the steady state can be saddle-path stable for as long as decreasing returns to capital are sufficiently large. The intuition is as follows: under DMI, a shock to the economy that increases consumption above its steady-state level reduces the individual's degree of impatience, and thus, the subjective

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<sup>9</sup>RTP is a decreasing function of consumption, real wealth or lifetime utility.

cost of investing. This incites the individual to accumulate more capital, and thus wealth, which stimulates higher consumption, which in turn, reduces the RTP further. In order to pull the system back to equilibrium, the marginal product of capital should diminish at a rate that is faster than the rate at which the subjective cost of investing is declining so that the economy eventually reaches a point where the subjective cost of further capital accumulation becomes prohibitive.

On the empirical side, Lawrence (1991) and Samwick (1998) both report that the RTP is higher for poorer households. Indeed, studies show that wealthier households save more (Huggett and Ventura, 2000 and Dynan, Skinner and Zeldes, 2004), which is indicative of greater patience in households that are better endowed.

Other papers have also delved into the effects of DMI on issues such as capital taxation, income distribution and asset pricing (Das et al., 2004; Nath, 2006; and Hirose and Ikeda, 2008) and have shown that the DMI assumption generates more reasonable implications. Moreover, as the IMI assumption is theoretically shown to imply a positive relation between the money growth rate (and thus, the inflation rate) and economic growth, it seems reasonable to conjecture that the DMI assumption would be consistent with the opposite. The intuition for the latter is as follows, when inflation reduces either real consumption, real income or real wealth, the agent becomes more impatient, which then discourages him from saving. In the aggregate, this slows down capital accumulation, which adversely affects aggregate output. A startlingly realistic implication of employing DMI is that for a given real interest rate, the rich only get richer, while the poor get poorer (Das et al., 2004).

### *Third Prototype: Aggregate Variables*

A practical difficulty with the Uzawa-Epstein specification is that endogenizing the RTP in terms of individual consumption or wealth requires another co-state variable, which complicates the dynamic analysis. In this light, Ogawa (1993) modeled the RTP as a function of average labor income, while Schmitt-Grohé and Uribe (2003) specified it as an increasing function of the average instantaneous utility. Meng (2006) found justification

for this in accordance with Rae (1834), who asserted that the individual’s time preference is a product of his culture or of social norms. Aggregate variables are then viewed in this light as socially-determined. Theoretically speaking, the beauty of this assumption is in its ability to reduce the analytical dimension of the problem by allowing us to treat the RTP as a constant with respect to the optimization procedure.

In particular, Meng (2006) modeled the RTP as a function of the aggregate values of output and consumption (i.e.,  $\rho(C, Y)$ ). He assumed that  $\rho_C > 0$  while  $\rho_Y < 0$ . The former condition means that the individual becomes more impatient as society consumes more. Meng (2006) noted that this can be linked to “jealousy effects”: for a given level of aggregate income, the individual discounts future utility less and consumes more now in order to “keep up with the Joneses.” As such, we can also perhaps call this the “conspicuous consumption” effect. The latter condition is in accordance with the DMI assumption: the higher the standard of living of society or the wealthier the society is, the more patient the individual becomes. These two necessary conditions present two opposing forces:  $\rho_C > 0$  serves as a stabilizing force, while  $\rho_Y < 0$ , a destabilizing force. The characterization of these two forces is in line with the IMI literature.

A locally unique steady-state equilibrium occurs if  $(\rho_C f_k + \rho_Y f_k - f_{kk}) > 0$ . In the special case where there are no diminishing returns to capital, i.e.,  $f_{kk} = 0$ , the conditions for local determinacy are either  $\rho_C > 0$  and  $\rho_Y = 0$  or  $\rho_Y > 0$ , which are both consistent with the IMI literature. Moreover, when the system is reduced to the fixed RTP case (i.e.,  $\rho_C = \rho_Y = 0$  and  $f_{kk} < 0$ ) the equilibrium is stable and locally determinate.

Guo and Janko (2007) modeled the discount factor as an increasing function not only of economy-wide levels of current consumption and labor hours, but also of aggregate labor hours in the last period to account for the “internal habit formation in labor supply.” Coupled with variable capital utilization the fluctuations generated by a technology shock are able to match the business cycle fluctuations of post-1981 Canada in an open-economy setting.

## 2 Roadmap of the Dissertation

The rest of the dissertation is divided into three other chapters. Chapter 2 takes off from the work of Krusell, Kuruşçu and Smith (2002), and presents the analyses for a stochastic, infinite-horizon, neoclassical growth model with labor-leisure choice under a social planner endowed with quasi-geometric preferences. We compare the welfare in this setup to that derived under a social planner with the standard, exponential discounting preference. We also assess the impacts of a stochastic (technology) shock on the key economic variables in the system. Our results show that relative welfare depends on the parameter that determines the social planner's degree of short-run impatience.

In Chapter 3, we analyze the long-run macroeconomic and welfare effects of raising the retirement age and compares these to outcomes from social security privatization in the context of a dynamic overlapping generations model where rational and quasi-hyperbolic agents co-exist. This is an offshoot of the rising concerns over the imminent retirement of the Baby Boom generation that has spurred developed countries to raise their respective retirement ages in an attempt to save the unfunded social security system. We find that mandatorily raising the retirement age works like another commitment device that induces time-inconsistent agents to increase lifetime savings, so that higher aggregate welfare gains accrue to the economy inhabited by a greater fraction of quasi-hyperbolic agents. All agents benefit from this reform under a regime that keeps the level of pension benefits constant. Moreover, in this setting, we observe that social security privatization is most desirable in a mixed population of rational and quasi-hyperbolic discounters. Indeed, retired quasi-hyperbolic agents gain after privatization in a mixed economy, whereas they lose in an economy purely composed of either rational or quasi-hyperbolic discounters. This is due to the pecuniary externalities generated in a mixed economy that allow better consumption smoothing by quasi-hyperbolic agents upon the elimination of the pre-commitment mechanism provided by social security.

In Chapter 4, we deal with the long-run implications of decreasing marginal impatience

on money superneutrality and stability in two popular monetary models: the money-in-the utility (MIU) and cash in-advance (CIA) models. We find money non-superneutrality even in the case where the CIA constraint is solely imposed on consumption. However, the inflation-growth nexus is determined by how the rate of time preference is specified. The numerical results mirror the inflation-growth literature: inflation negatively affects growth in the MIU models and in the models with a CIA on investment, while the Mundell-Tobin effect emerges in models where consumption is subject to a CIA constraint. Moreover, saddle-path stability characterizes the equilibria in all the cases considered.

## Chapter 2

# On the Allocative and Welfare Effects of Quasi-Hyperbolic Discounting

### 1 Introduction

In the wake of Samuelson (1937), the geometric discount function has been used as the standard in intertemporal choice models. Under this method of discounting, preferences are time-consistent, the fundamental welfare theorems hold and the conventional dynamic optimization procedures apply. Recent behavioral experiments, however, report that the individual exhibits time-inconsistent behavior in that he alters the way he discounts intertemporal tradeoffs with every shift in time perspective.<sup>1</sup> Thus, from a given temporal standpoint, the individual's degree of patience in evaluating short-run tradeoffs differs from his degree of patience in evaluating long-run tradeoffs. To model this revealed pattern of discounting, the literature proposes two discount functions: the hyperbolic discount function<sup>2</sup> and the more analytically tractable quasi-geometric discount function proposed by Laibson (1994, 1997).<sup>3</sup> The resulting difficulty is that time-inconsistent discounting renders invalid the usual dynamic optimization methods. Furthermore, it creates a wedge between the social planner's solution and the decentralized equilibrium so that the fundamental welfare theorems break down.<sup>4</sup>

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<sup>1</sup>See, for example, the empirical findings of Ainslie (1992), Loewenstein and Prelec (1992), Angeletos, *et al.* (2001), Frederick, *et al.* (2002) and Soman *et al.* (2005). For the theoretical results, see the seminal works of Strotz (1956), Phelps and Pollack (1968) and Peleg and Yaari (1972).

<sup>2</sup>Under hyperbolic discounting, the discount factor for events  $\tau$  periods away is given by  $(1 + \gamma\tau)^{-\frac{\eta}{\gamma}}$ , where  $\gamma, \eta > 0$ . The implied discount rate monotonically decreases with the length of the delay in the event,  $\tau$ .

<sup>3</sup>Under quasi-geometric discounting, long-run patience is measured by the long-run discount factor,  $\beta \in (0, 1)$ , while short-run patience is measured by the short-run discount factor,  $\delta\beta$ , where  $\delta \in (0, \infty)$ . In contrast, under geometric discounting, there is no difference between the long-run and short-run discount factors: they both equal  $\beta$ .

<sup>4</sup>The general idea is that with time-inconsistent preferences, the future generations do not agree with the prior ones. Consequently, the individual's decision at any prior time period imposes an externality on future generations' decisions. See, for example, Caplin and Leahy (2004) and Luttmer and Mariotti (2007).



To arrive at a time-consistent solution to the standard, deterministic, infinite-horizon, neoclassical growth model under quasi-geometric discounting with fixed labor supply, Krusell, Kuruşçu and Smith (2002) (KKS, hereafter) reformulate the individual’s dynamic choice problem into a game amongst the intertemporal selves using a solution concept called Markov perfect equilibrium.<sup>5</sup> By using specific functional forms and further confining the solution set to admit only first-order Markov perfect equilibria that are limits to the finite-horizon equilibria in Laibson (1994, 1997), they were able to find closed-form solutions to both the social planner’s and the representative agent’s problems.<sup>6</sup> In each scenario, the optimal policy rule for capital accumulation determined by the current self is a first-order Markov strategy that leaves none of his future selves with the incentive to deviate from it. Given the corresponding policy rules, KKS show that in spite of the absence of any external friction in the economy, welfare under *laissez faire* strictly surpasses welfare under a benevolent, social planner in the presence of internal conflict amongst the different temporal selves.

In this paper, we focus on another fundamental issue, which to our knowledge, has not been yet addressed in the literature: we make comparisons between the allocations and the welfare levels implied by quasi-geometric and geometric discounting. Here, we take as the benchmark the outcome generated under the geometric discount function. Doing so ultimately enables us to make a quantitative assessment of the impact of time-inconsistent discounting on resource allocation and welfare. To further enrich the analysis, we modify the KKS economy in two significant respects, while preserving the availability of closed-form solutions. One, we incorporate labor-leisure choice, which allows us to analyze how labor supply elasticity influences our results. Two, we include a technology shock in the production function, which permits us to evaluate how the technology shock affects the impulse responses of some key macroeconomic variables and how the persistence of the

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<sup>5</sup>Along the same strand in the literature are Krusell and Smith (2001), Maliar and Maliar (2005) and Maliar, Maliar and Valli (2008).

<sup>6</sup>They further assert that the solutions are unique in each case.

shock affects the welfare comparisons.

We observe that our analyses hinge on the degree of short-run patience (or impatience), which is measured by the discount factor,  $\delta\beta$ , where  $\beta \in (0, 1)$  is the geometric discount factor, and  $\delta \in (0, \infty)$  controls the degree of short-run patience. When  $\delta = 1$ , we recover the standard degree of short-run patience under geometric discounting. However, when  $\delta < 1$ , we say that the planner is endowed with excessive short-run impatience, whereas when  $\delta > 1$ , he exhibits excessive short-run patience. In other words, excessive short-run patience (impatience) on the part of the quasi-geometric planner signifies that he discounts his short-run tradeoffs less (more) than the geometric planner.

Comparing across allocations, we find that the steady-state labor hours, capital and output derived under quasi-geometric discounting exceed those derived under geometric discounting when the (quasi-geometric) planner possesses excessive short-run patience. The intuition is that with excessive short-run patience, the (quasi-geometric) planner has a higher saving rate, as he now prefers future consumption over current consumption, *ceteris paribus*. Since consumption and leisure are complements, the reduction in current consumption consequently reduces leisure, which implies an increase in hours worked. Greater labor hours, in turn, enable greater capital accumulation and output. The opposite ensues with excessive short-run impatience. Moreover, the difference between the two allocations is augmented as labor supply elasticity increases. This happens because the more elastic labor supply is, the more responsive labor hours are to the degree of patience. Thus, with perfect labor supply elasticity, the most (least) patient economy works the most (least). Capital accumulation and output behave in a similar manner as both are positively related to labor hours.

Meanwhile, quasi-geometric consumption is strictly greater than geometric consumption only over a certain range of degrees of excessive short-run patience. The more elastic labor hours are, the greater is this range. The reason is that there are two opposing marginal effects of greater patience on consumption: the negative effect of a higher saving rate, on the one hand, and the positive effects of greater capital accumulation and labor hours, on

the other hand. We can think of the former as a “substitution effect” (of marginal patience on consumption), while the two latter effects can be construed together as a “wealth effect.” When the quasi-geometric planner is excessively patient, the “wealth effect” dominates the “substitution effect” up to a certain degree. However, at some higher degree of excessive patience, too much savings enables the “substitution effect” to overcome the “wealth effect.” As labor supply elasticity increases, the labor hours effect – and thus, the “wealth effect” – becomes stronger, thereby allowing quasi-geometric consumption to exceed geometric consumption over a wider range of degrees of excessive short-run patience.

Looking at the current generation’s welfare, we observe that quasi-geometric welfare is strictly greater than geometric welfare when the planner is endowed with excessive short-run impatience, while the opposite occurs when the planner has excessive short-run patience. This happens because the current planner, who is sovereign over the current resource allocation, manipulates resources between consumption and savings in a manner that would favor contemporary (future) welfare relative to future (contemporary) welfare when he is excessively impatient (patient). Thus, greater current welfare redounds to the more impatient economy.

Furthermore, we observe that a greater elasticity of labor supply widens the gap between current quasi-geometric and current geometric welfare, while a more persistent technology shock causes it to shrink. The former effect occurs as the greater responsiveness of labor hours to a given degree of excessive patience (or impatience) due to a higher elasticity of labor supply enhances the effects of labor hours and capital accumulation on current, quasi-geometric welfare. Thus, a higher elasticity of labor supply increases the difference between the welfare levels of the excessively patient (or impatient) quasi-geometric economy and the geometric economy, which has the standard degree of patience. The latter effect happens as a more persistent technology shock increases expected future welfare, which in turn, increases current welfare. As the more patient economy values the future more than the less patient one, it is the former that derives a greater advantage from a more persistent shock. Thus, a greater persistence of the technology shock enables the more

patient economy to “catch up with” the more myopic one.

Making intergenerational comparisons, when the planner is excessively impatient at the onset, we observe a tradeoff between current and future welfare as the planner becomes less excessively impatient. In contrast, when the planner is excessively patient to begin with, both current and future welfare are reduced as he becomes more excessively patient. The reason is that when the current planner is excessively impatient, current welfare is favored over future welfare, causing the economy to save, invest and work less than it would under the standard degree of patience associated with geometric discounting. Consequently, decreasing the degree of impatience would increase savings, investment and labor hours in a manner that would increase future welfare at the expense of current welfare. However, when the planner is excessively patient to begin with, the economy already saves, invests and works too much. Thus, pushing the degree of patience towards greater excess would only serve to reduce not only current welfare, but also future welfare.

The quantitative results confirm that time-inconsistent discounting can have substantial effects on allocations and welfare. That quasi-geometric discounting yields outcomes that are markedly different from those generated by the widely used geometric discounting implies that the welfare analyses of economic policies based on geometric discounting must be received with caution. If economic agents indeed evaluate tradeoffs in accordance with quasi-geometric discounting, policy evaluations based on the standard geometric discounting might even generate some spurious conclusions and consequently, misguided proposals for policy reforms.<sup>7</sup>

The rest of the paper is structured as follows: Section 2 presents the stochastic KKS model with labor-leisure choice. Section 3 analyzes the steady-state allocations and the effects of labor supply elasticity. Section 4 compares the welfare derived under quasi-geometric discounting to the welfare generated under geometric discounting for both the current and future generations, and further looks into the effects of the persistence of a shock on the wel-

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<sup>7</sup>See, for instance, Diamond and Köszegi (2003) and Schwarz and Sheshinski (2007), and Kumru and Thanopoulos (2008).

fare gap between the current, quasi-geometric generation and the contemporary geometric generation. Finally, Section 5 summarizes and proffers some concluding remarks.

## 2 The Model

Our benevolent social planner is assumed to have the same time-inconsistent preferences as the typical agent in the economy. More specifically, the planner’s discount function takes the form of Laibson’s (1996, 1997) quasi-geometric discount function.<sup>8</sup> Accordingly, the planner’s intertemporal maximization problem is defined as

$$\max u(c_0, h_0) + \delta E_0[\beta u(c_1, h_1) + \beta^2 u(c_2, h_2) + \beta^3 u(c_3, h_3) + \dots] \quad (\text{P1})$$

$$\text{subject to } c_t + k_{t+1} = f(z_t, k_t, h_t), \quad t = 0, 1, 2, 3, \dots,$$

where  $c_t$ ,  $k_t$ ,  $h_t$  and  $z_t$  denote period- $t$  consumption, capital, labor hours and the technology shock, respectively. The standard discount factor,  $\beta \in (0, 1)$ , is called the long-run discount factor, while  $\delta\beta$  is the short-run discount factor, where  $\delta$  determines the degree of short-run patience. When  $\delta = 1$ , we recover the planner’s problem under geometric discounting. In this light, we say that the planner is characterized by the standard degree of patience. The cases  $\delta < 1$  and  $\delta > 1$ , indicate excessive short-run impatience and excessive short-run patience, respectively. We allow  $\delta$  to be greater than unity to account for instances where the individual does exhibit excessive patience in viewing his short-run prospects, prodding him to introspect, “I have been waiting already for some time: I might as well wait more now than postpone the inevitable to a later date.”<sup>9</sup>

The main difficulty with quasi-geometric discounting is that it renders the social plan-

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<sup>8</sup>This is also known as the quasi-hyperbolic discount function.

<sup>9</sup>Alternatively, we might just have an individual who cares for his children (and his children’s children) so much more than he cares for his own well-being. Here we stand back and say, “*De gustibus non est disputandum.*”

ner's problem non-recursive, which in turn, disallows the use of the more convenient Lagrangian method in solving the planner's problem. To illustrate, using the Lagrangian method indiscriminately yields the following Euler equations from the planner's perspective at time  $t$ :

$$\text{for } t, t + 1 : u_c(c_t, h_t) = \delta\beta E_t[u_c(c_{t+1}, h_{t+1})f_k(k_{t+1}, h_{t+1}, z_{t+1})], \quad (1)$$

$$\text{for } t + 1, t + 2 : u_c(c_{t+1}, h_{t+1}) = \beta E_t[u_c(c_{t+2}, h_{t+2})f_k(k_{t+2}, h_{t+2}, z_{t+2})]. \quad (2)$$

When  $\delta < 1$ , the planner at time  $t$  ascribes a lower discounted value on next-period marginal utility (or short-run marginal utility) than on a later future one (or long-run marginal utility). In this case, he considers the period  $t + 1$ , as his short-run future, while  $t + 2$  onwards is his long-run future. From (1) and (2), we observe that the short-run discount factor,  $\delta\beta$ , which the planner imputes to the short-run tradeoff between ( $t$  and  $t + 1$ ) is less than the long-run discount factor,  $\beta$ , which he ascribes to the long-run tradeoff (between  $t + 1$  and  $t + 2$ ). However, at the dawn of  $t + 1$ , the planner (or his self at  $t + 1$ ) suffers a preference reversal and suddenly views the intertemporal tradeoff between  $t + 1$  and  $t + 2$  with greater impatience, as  $t + 2$  now becomes his short-run (immediate) future, that is:<sup>10</sup>

$$\text{for } t + 1, t + 2 : u_c(c_{t+1}, h_{t+1}) = \delta\beta E_t[u_c(c_{t+2}, h_{t+2})f_k(z_{t+2}, k_{t+2}, h_{t+2})]. \quad (3)$$

Thus, without any strategy-proof, commitment mechanism in place, the social planner's optimal choices of  $\{(c_{t+j}, h_{t+j})\}_{j=1}^{\infty}$ , made at any time  $t$ , would be disregarded by his future selves. More succinctly, the optimal solutions are time-inconsistent. To obtain a time-consistent solution without commitment to (P1), we follow KKS in reformulating it as a

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<sup>10</sup>When  $\delta > 1$ , the reverse happens: the planner at time  $t$  ascribes a higher discounted value on long-run tradeoffs than on short-run ones.

dynamic programming problem using a solution concept called Markov perfect equilibrium.

In the attainment of a closed-form solution, the following functional assumptions are crucial:

$$u(c, h) = \log(c) - A \frac{h^{1+\chi}}{1+\chi}, \quad A > 0 \quad (4)$$

$$f(z, k, h) = zk^\alpha h^{1-\alpha}, \quad (5)$$

$$\log(z_t) = \rho \log(z_{t-1}) + \nu_t, \quad (6)$$

where (4) is the utility function, which is additively separable in  $c$  and  $h$ . The parameter  $\chi \geq 0$  is the inverse of labor supply elasticity. The stochastic Cobb-Douglas production function is given by (5), where  $\alpha \in (0, 1)$ , and  $k$  is assumed to depreciate fully in every period. In (6), the stochastic variable  $\log(z)$  follows a stationary AR(1) process, where  $\rho \in (0, 1)$  is the persistence parameter and  $\nu_t \sim \text{iid}(0, 1)$ .

In order to take into account preference-switching for every change in temporal perspective, the contemporary planner (at time 0) is made to play a game against his future selves by solving the following dynamic programming problem:

$$V_0(k, z) = \max_{k', h} \left\{ \log(zk^\alpha h^{1-\alpha} - k') - A \frac{h^{1+\chi}}{1+\chi} + \delta \beta EV(k', z') \right\} \quad (7)$$

where

$$V(k, z) = \log(zk^\alpha h^{1-\alpha} - g(k, z)) - A \frac{h^{1+\chi}}{1+\chi} + \beta EV(g(k, z), z'), \quad (8)$$

where  $V_0(\cdot)$  is the time-0 planner's expected optimal lifetime utility discounted to the current period (at time 0), while  $V(\cdot)$  corresponds to the value function of his future selves. In other words, the current planner defines his own allocation problem in (7), while taking into account (8), which he regards as the problem of his future selves. When  $\delta = 1$ , we recover the standard social planner's problem under geometric discounting. We also note that  $g(k, z)$  is the optimal decision rule for next-period capital and that the variable  $x'$

represents the next-period value of  $x$ . The first-order Markov assumption is embodied by the time-invariant policy function for capital that depends solely on current values of capital and the technology shock.

The time-0 planner (or current planner) conjectures that his future selves will use the policy rule  $k' = g(k, z)$ , which solves (8). Denoting the current planner's solution to (7) by  $\tilde{g}(k, z)$ , the time-consistent solution entails that  $g(k, z) = \tilde{g}(k, z)$ , for all  $k$ , and  $z$ . In other words, without any mechanism for enforcing commitment in place, the social planner chooses the optimal sequence of capital such that none of his future selves will have the incentive to deviate from it. Otherwise, time inconsistency ensues: the policy function  $\tilde{g}(k, z)$  will be used only by the current planner, while his future selves will follow the policy rule  $g(k, z)$ .

**Proposition 1** *Given (4) to (6), the (first-order) Markov perfect solution to the quasi-geometric planner's problem under a no-commitment regime consists of:*

1.  $V(k, z) = a + b \log(k) + d \log(z)$ , where  $b = \frac{\alpha}{1-\alpha\beta}$  and  $d = \frac{1}{(1-\alpha\beta)(1-\rho\beta)}$ .
2.  $g_1(k_1, z_1) = \frac{\alpha\beta\delta}{1-\alpha\beta(1-\delta)} z k_1^\alpha h_1^{1-\alpha}$ , where the subscript 1 denotes values derived under quasi-geometric discounting
3.  $h_1 = \left[ \frac{1-\alpha}{A} \left( \frac{1-\alpha\beta(1-\delta)}{1-\alpha\beta} \right) \right]^{\frac{1}{1+\alpha}}$ .

**Proof.** See Appendix A. ■

Given our parametric assumptions, we are able to derive the time-consistent, closed-form solution to the stochastic planner's problem. Indeed, the optimal decision rule for next-period capital is a stochastic, Markov strategy that depends only on current values of capital and the technology shock, where the constant,  $\frac{\alpha\beta\delta}{1-\alpha\beta(1-\delta)}$ , is the economy's marginal propensity to save. However, we observe that the optimal labor supply is constant and independent of the technology shock. This result arises from the specific forms of the utility and production functions, which allow the income and substitution effects to cancel out each other.<sup>11</sup>

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<sup>11</sup>The key assumptions are additive separability of the utility function,  $\log(c)$  and the Cobb-Douglas



Analogously, we can derive the geometric planner's solution when  $\delta = 1$ . In particular, the optimal policy functions under this standard method of discounting are given by  $g_2 = \alpha\beta z k_2^\alpha h_2^{1-\alpha}$  and  $h_2 = [(1 - \alpha)/(A(1 - \alpha\beta))]^{\frac{1}{1+x}}$ , where the subscript 2 indicates values obtained under geometric discounting.

We thus observe that the key to the difference between the quasi-geometric and the geometric solutions is the planner's degree of short-run patience. Accordingly, in the next section, we compare the corresponding steady-state allocations and welfare levels for different values of  $\delta$ .

### 3 The Effects of Short-Run Patience on Allocations

#### 3.1 Steady-State Allocations

In the steady state, we obtain the following expressions for labor hours, the savings rate, capital and consumption under quasi-geometric discounting:

$$h_1^* = \left[ \frac{1 - \alpha}{A} \left( \frac{1 - \alpha\beta(1 - \delta)}{1 - \alpha\beta} \right) \right]^{\frac{1}{1+x}} \quad (9)$$

$$s_1^* = \frac{\alpha\beta\delta}{1 - \alpha\beta(1 - \delta)} \quad (10)$$

$$k_1^* = \left( \frac{\alpha\beta\delta}{1 - \alpha\beta(1 - \delta)} \right)^{\frac{1}{1-\alpha}} h_1^* \quad (11)$$

$$c_1^* = \frac{1 - \alpha\beta}{1 - \alpha\beta(1 - \delta)} k_1^{*\alpha} h_1^{*(1-\alpha)} \quad (12)$$

$$y_1^* = k_1^{*\alpha} h_1^{*(1-\alpha)} \quad (13)$$

The corresponding equilibrium values for the geometric planner follow when  $\delta = 1$ .

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production function.

**Proposition 2** *When  $\delta < 1$ , the steady-state levels of savings, capital, labor hours, and output under quasi-geometric discounting are less than the corresponding steady-state levels under geometric discounting. When  $\delta > 1$ , the opposite ensues..*

**Proof.** It can be shown that for all  $\delta$ ,  $\frac{ds_1^*}{d\delta}, \frac{dk_1^*}{d\delta}, \frac{dh_1^*}{d\delta}, \frac{dy_1^*}{d\delta} > 0$ . We can thus conclude that for  $\delta \geq 1$ ,  $s_1^* \geq s_2^*$ ,  $k_1^* \geq k_2^*$  and  $h_1^* \geq h_2^*$ . ■

It stands to reason that the economy, which exhibits excessive short-run patience ( $\delta > 1$ ) saves more, works more, accumulates more capital, and thus produces more output in the steady state than the economy characterized by the standard level of patience ( $\delta = 1$ ), while the individual with excessive short-run impatience ( $\delta < 1$ ) behaves otherwise. This is because the more patient the individual is, the more willing he is to forego current consumption in favor of future consumption. He accordingly saves more, which reduces current consumption. As current consumption and leisure are complements, the reduction in current consumption increases labor hours.<sup>12</sup> Furthermore, since capital accumulation is an increasing function of labor hours, the increase in labor hours spurs capital accumulation, and consequently, output.

For consumption, it is not as straightforward to obtain the critical value for  $\delta$ . This can be seen from decomposing the marginal effects of  $\delta$  on  $c_1^*$ :

$$\frac{dc_1^*}{d\delta} = \underbrace{(1 - \alpha)(1 - s_1^*) \frac{y_1^*}{h_1^*} \frac{dh_1^*}{d\delta}}_{\text{labor hours effect (+)}} + \underbrace{\alpha(1 - s_1^*) \frac{y_1^*}{k_1^*} \frac{dk_1^*}{d\delta}}_{\text{capital accumulation effect (+)}} - \underbrace{y_1^* \frac{ds_1^*}{d\delta}}_{\text{savings effect (-)}} \quad (14)$$

The labor hours effect (the first term) and the capital accumulation effect (the second term) are positive while the savings effect (the third term) is negative. From here, the intuition is clear: greater short-run patience increases labor hours, which increases output and thus, consumption. However, while greater patience implies a higher capital level,

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<sup>12</sup>It can be shown that the intra-temporal marginal rate of substitution between consumption and leisure yields the expression,  $h_1 = \left[ \frac{\alpha}{A(1-s_1)} \right]^{\frac{1}{1+\alpha}}$ , which shows clearly that the increase in savings increases labor hours.

which increases consumption, greater patience also spurs savings, which reduces current consumption. This implies that increasing short-run patience can increase  $c_1^*$  up to a critical value of  $\delta$ , above which  $c_1^*$  starts decreasing with greater short-run patience.

### 3.2 The Effects of Labor Supply Elasticity

We expect the degree of labor supply elasticity to play a role in determining the extent to which labor hours, capital, output and consumption derived under quasi-geometric discounting differ from those obtained under geometric discounting. We remark that as labor hours become more inelastic, the stochastic social planner's problem (P1) approaches the case without labor-leisure choice.<sup>13</sup>

To aid our analysis, we choose some standard parametric values.<sup>14</sup> We let the share of capital in income,  $\alpha$ , to be equal to .36. The long-run discount factor,  $\beta$ , is set equal to .99, which is consistent with an average quarterly real interest rate of around 1%.<sup>15</sup> In accordance with King, Plosser and Rebelo (1988), we choose  $\chi = .25$  to be the benchmark value so that labor supply elasticity equals four. Finally, we calibrate the preference parameter  $A$  to yield the steady-state labor hours under geometric discounting,  $h_2$ , to equal 1/3.<sup>16</sup>

Figure 2.1 below plots  $h_1^*$ ,  $k_1^*$  and  $y_1^*$  over different values of  $\delta$ . Since  $h_2^*$ ,  $k_2^*$  and  $y_2^*$  are invariant to  $\delta$ , these plots also indicate the magnitudes of the differences between  $h_1^*$  and  $h_2^*$ ,  $k_1^*$  and  $k_2^*$ , and  $y_1^*$  and  $y_2^*$ . In particular, the steeper the line, the greater is the gap between the quasi-geometric allocation and its geometric counterpart. We observe that the gap increases as labor supply elasticity increases from  $\frac{1}{\chi} = .25$  to  $\frac{1}{\chi} = 4$ . This is because the greater the labor supply elasticity, the more responsive labor hours become to

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<sup>13</sup>This happens as the  $\lim_{\chi \rightarrow \infty} h_1^* = 1$ .

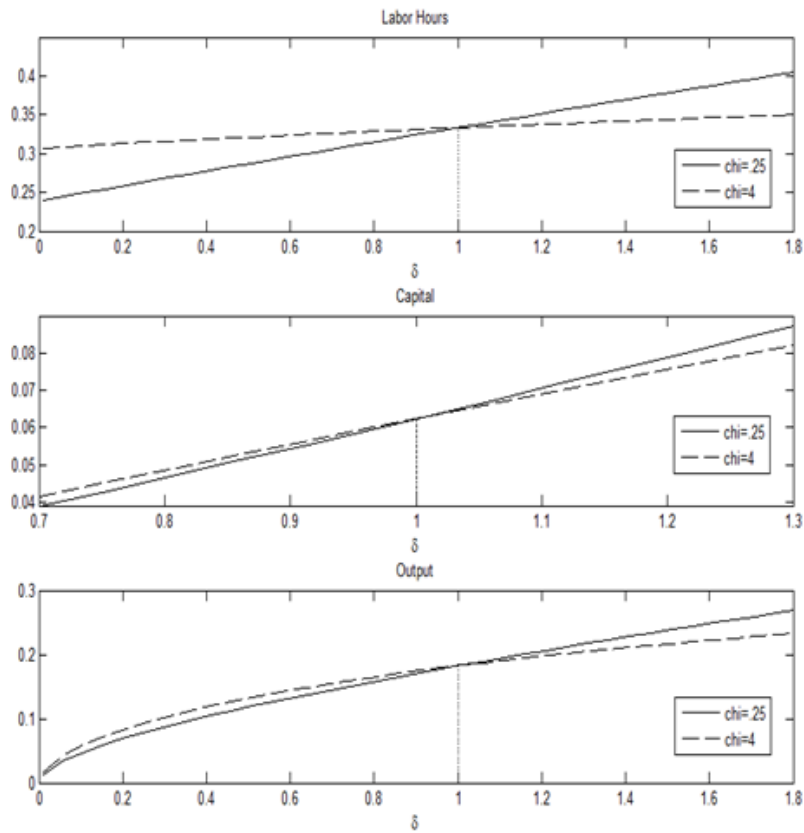
<sup>14</sup>Since we set the steady-state value of  $z^*$  to be equal to 1, the technology shock plays no role in the following analyses.

<sup>15</sup>At the deterministic steady-state, the real interest rate,  $r = \frac{1}{\beta} - 1$ . With a quarterly real interest rate of around 1%,  $\beta \approx .99$ .

<sup>16</sup>We rationalize this by considering that the average person usually works 8 hours in a 24-hour day. As the individual is endowed with 1 unit of time, which he can use either working or at leisure, he spends 1/3 of his total time endowment working.

the degree of short-run patience.<sup>17</sup> We observe the same behavior of capital and output in response to labor supply elasticity for a given degree of patience, as they both increase with labor hours. We can thus conclude that in the steady state, the economy characterized by both excessive short-run impatience and the most elastic labor hours, works the least, accumulates the least capital stock and produces the least amount of output. In contrast, the economy endowed with excessive short-run patience and the most elastic labor hours, works the most, accumulates the greatest capital stock and produces the most amount of output.

Figure 2.1: The Allocative Effects of Short-Run Patience

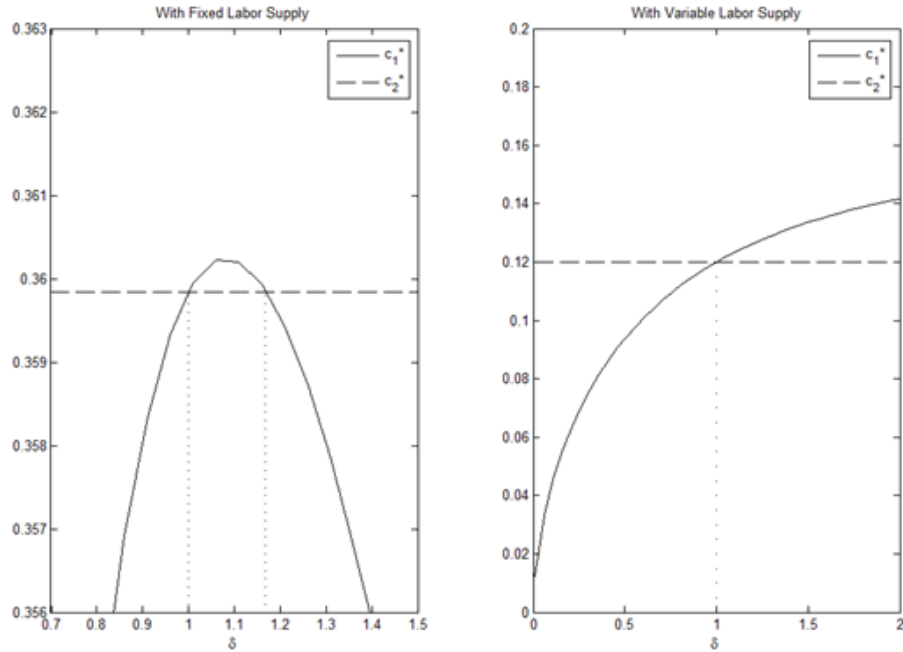



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<sup>17</sup>From the optimality condition that the marginal rate of substitution between consumption and leisure equals the marginal product of labor, we can show that  $\frac{dh_1}{d\delta} = \frac{h_1}{(1+\chi)(1-s_1)} \frac{ds_1}{d\delta}$ . From here it is clear that the greater the elasticity of labor supply (or the lower the value of  $\chi$ ), the greater is  $dh_1/d\delta$  or the more responsive  $h_1$  is to  $\delta$ .

Figure 2.2 plots consumption as a function of the degree of short-run patience. The left-hand panel shows the case when labor supply is perfectly inelastic while the right-hand panel presents the case when labor supply is elastic ( $\chi = .25$ ).

Figure 2.2: The Allocative Effects of Short-Run Patience on Consumption



It is apparent in both cases that increasing  $\delta$  can increase  $c_1^*$  up to a critical value of  $\delta$ , above which  $c_1^*$  starts decreasing as  $\delta$  increases. When labor supply is inelastic, there is only a narrow range of  $\delta \in (1, 1.168)$  over which  $c_1^*$  exceeds  $c_2^*$ , but this range widens with a greater labor supply elasticity. This happens because when labor supply is perfectly inelastic, the labor hours effect drops out from (12). For some  $\delta > 1$ , the more patient economy saves more and accumulates more capital, which then supports a higher level of steady-state consumption. However, at some value of  $\delta > 1.168$ , the overly patient economy saves too much – so much so that  $c_1^*$  is driven below  $c_2^*$ . Now, allowing labor hours to be more elastic enables us to recover the labor hours effect, which works in the same direction as the capital accumulation effect to overcome the contrary impact of greater savings on

consumption. The greater the elasticity of labor supply, the stronger the labor hours effect, which then permits a wider range of  $\delta$ -values that can support quantities of quasi-geometric consumption that are greater than geometric consumption. Indeed, the right-hand panel in Figure 2.2 shows that  $c_1^* > c_2^*$  for a greater range of  $\delta$  once we allow labor supply elasticity to increase.<sup>18</sup>

## 4 The Effects of Short-run Patience on Welfare

### 4.1 The Welfare Gap of the Current Generation

The future selves' inability to commit to prescribed actions by the current planner prevents us from using the notion of Pareto optimality in assessing welfare, as this necessitates the planner's ability to make costless reallocations across time. These reallocations would be clearly non-binding in the face of the internal conflict generated by time-inconsistent preferences under a no-commitment regime.

How do we then evaluate welfare? As the current planner has the same preferences as the typical agent in the economy and is, moreover, sovereign over resource allocation in his own time period (i.e., at time 0), we can make a case for measuring the welfare of the current generation using  $V_0(k, z)$ , which is the current planner's maximum expected lifetime discounted (to time 0) utility from "playing the game" against his future selves. From this, we can proceed to compare the current welfare derived under quasi-geometric discounting to that derived under geometric discounting.

**Proposition 3** *Under our parametric assumptions, the current planners' value functions under quasi- and geometric discounting are given, respectively, by*

$$\begin{aligned}
 V_0(k, z; \delta) = & a_1 + \frac{\alpha(1 - \alpha\beta(1 - \delta))}{1 - \alpha\beta} \log(k) \\
 & + \frac{(1 - \rho\beta)[1 - \alpha\beta(1 - \delta)] + \beta\delta\rho}{(1 - \alpha\beta)(1 - \rho\beta)} \log(z)
 \end{aligned} \tag{15}$$

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<sup>18</sup>When labor supply elasticity equals four,  $c_1^* < c_2^*$  when  $\delta > 52$ .

$$V_0(k, z; 1) = a_2 + \frac{\alpha}{1 - \alpha\beta} \log(k) + \frac{1}{(1 - \alpha\beta)(1 - \rho\beta)} \log(z) \quad (16)$$

where

$$a_1 = \frac{1 - \beta(1 - \delta)}{1 - \beta} \log(1 - s_1) + \frac{\alpha\beta\delta}{(1 - \beta)(1 - \alpha\beta)} \log(s_1) - A \frac{1 - \beta(1 - \delta)}{(1 - \beta)(1 + \chi)} h_1^{1+\chi} + \frac{(1 - \alpha)((1 - \beta)[1 - \alpha\beta(1 - \delta)] + \beta\delta)}{(1 - \alpha\beta)(1 - \beta)} \log(h_1)$$

$$a_2 = \frac{1}{1 - \beta} \log(1 - s_2) + \frac{\alpha\beta}{(1 - \beta)(1 - \alpha\beta)} \log(s_2) - A \frac{1}{(1 - \beta)(1 + \chi)} h_2^{1+\chi} + \frac{1 - \alpha}{(1 - \alpha\beta)(1 - \beta)} \log(h_2)$$

**Proof.** See Appendix A. ■

The optimal, expected, lifetime utility discounted to time 0, can thus be collapsed into (15) for the quasi-geometric planner and (16) for the geometric planner. We denote the welfare of the current generation under a planner with quasi-geometric preferences by  $V_0(k, z; \delta)$ , while we denote the contemporary welfare under a planner with geometric preferences by  $V_0(k, z; 1)$ .

In making welfare comparisons, we define the current welfare gap as  $G(\delta) \equiv V_0(k, z; \delta) - V_0(k, z; 1)$ . This welfare gap can be construed as a quantitative measure of the difference in welfare generated by time-inconsistent discounting. To facilitate the analysis, we let  $k = k_2^*$  and  $z = z^* = 1$ .<sup>19</sup>

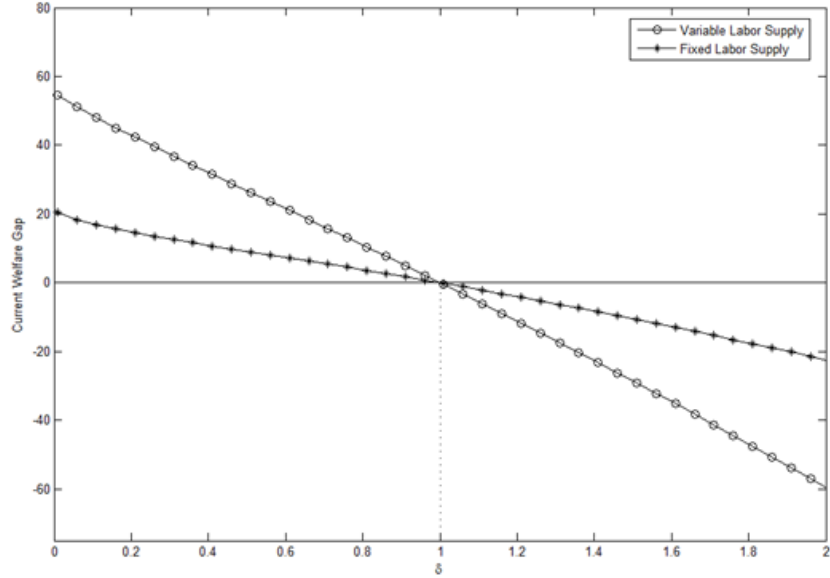
Using our given set of parameters, we graph the overall effects of a marginal increase in  $\delta$  on the welfare gap in Figure 2.3. We observe that the welfare gap monotonically decreases with an increase in short-run patience. This is because the more patient the current planner

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<sup>19</sup>We obtain the same qualitative results when we let  $k = k_1^*$  or any  $k \in (0, 1)$ .

is, the lesser the weight he imputes on current welfare relative to future welfare. Moreover, when  $\delta < 1$ , the current generation of the economy with the quasi-geometric planner gains greater welfare than the contemporary economy with the more patient, geometric planner. The opposite occurs when  $\delta > 1$ , as this signifies that the quasi-geometric planner is now more patient than the geometric planner.

Figure 2.3: The Welfare Effects of Short-Run Patience



To gain further insight, we decompose the marginal effects of short-run patience on the current welfare gap. From (15) and (16) we have:

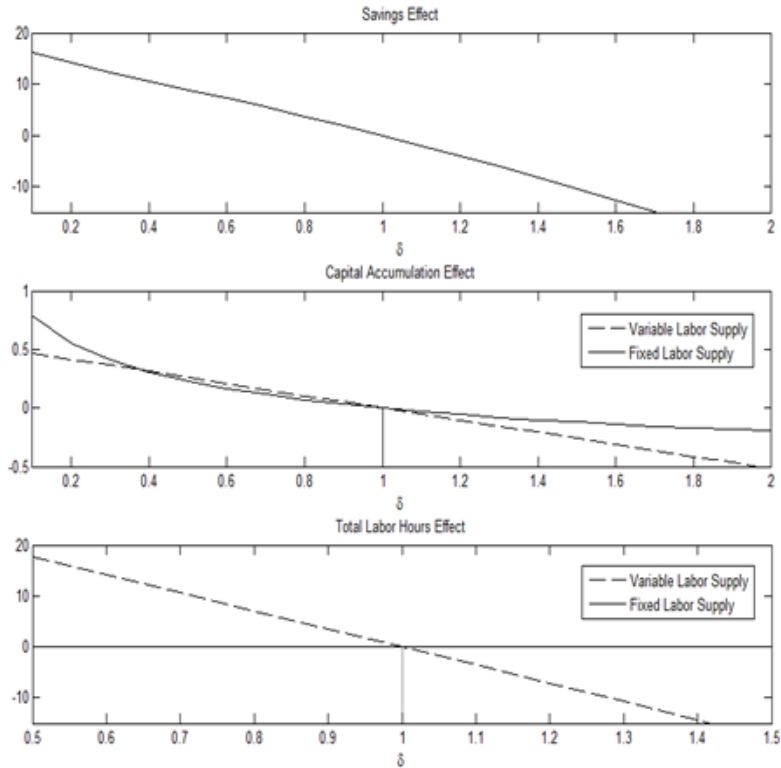
$$\begin{aligned}
 \frac{dG(\delta)}{d\delta} = & \underbrace{-\frac{\beta(1-2\alpha)}{(1-\beta)(1-\alpha\beta)} \frac{1}{s_1} \frac{ds_1}{d\delta}}_{\text{savings effect (-)}} + \underbrace{\frac{\alpha^2\beta [\log(k_2^*)]}{1-\alpha\beta}}_{\text{capital accumulation effect (-)}} \\
 & + \underbrace{\frac{(1-\alpha) \left\{ [(1-\beta)\alpha\beta + \beta] \log(h_1) + [(1-\beta)(1-\alpha\beta(1-\delta)) + \beta\delta] \frac{dh_1}{h_1 d\delta} \right\}}{(1-\alpha\beta)(1-\beta)} - \frac{Ah_1^\chi [\beta h_1 + (1-\beta(1-\delta))(1+\chi)] \frac{dh_1}{d\delta}}{(1-\beta)(1+\chi)}}_{\text{total labor hours effect (+/-)}}
 \end{aligned} \tag{17}$$

From (17), we observe that the effects of a marginal increase in short-run patience on



the current welfare gap can be separated into the savings effect, the capital accumulation effect and the total labor hours effect. We plot these distinct effects in Figure 2.4. Both the savings effect and the capital effect are unambiguously negative due to the adverse impact of greater savings and capital accumulation on contemporary consumption, which both reduce the welfare of the current planner. Thus, with fixed labor hours (which is consistent with the KKS specification), (17) is negative. The total labor hours effect can be decomposed into two terms, where the first term can be positive or negative and the second term is unambiguously negative and accounts for the marginal disutility of labor. Given our parametric constraints, however, Figure 4 shows that the total labor hours effect is negative, so that (17) is negative for a wide range of  $\delta$ -values.

Figure 2.4: The Effects of Short-Run Patience on the Welfare Gap



When  $\delta < 1$ , the quasi-geometric economy is more impatient than the geometric economy. Accordingly, the quasi-geometric economy saves less, works less and accumulates less

capital than the geometric economy. In this case, (17) implies that the adverse effects of savings, capital accumulation and total labor hours on the current welfare of the quasi-geometric economy are weaker than on that of the geometric economy. Both lower savings and lower capital accumulation allow the current self to enjoy greater consumption, which in turn increases contemporary utility. Likewise, lesser work hours reduces the marginal disutility of effort, which also increases current utility. These together allow the more impatient (quasi-geometric) planner (at time 0) to gain a higher welfare than his more patient (geometric) contemporary.

When  $\delta > 1$ , the quasi-geometric economy is now more patient than its geometric counterpart. Consequently, the quasi-geometric economy saves more, works more and accumulates more capital than the now less patient, geometric economy. This time, however, excessive short-run patience augments the negative effects of savings on consumption and of the disutility of labor on current welfare, resulting in a strictly lower level of current welfare for the more patient planner. In this light, excessive short-run patience appears to be like a “vice”<sup>20</sup> in that it causes the economy to save and work too much, which results in a lower welfare level than that under a more impatient planner.

We further note that Figure 2.3 also shows that increasing labor supply elasticity from  $\chi = 0$  to  $\chi = 4$  increases the welfare gap in absolute value terms (except when  $\delta = 1$ ). This occurs as a higher labor supply elasticity implies a greater sensitivity of labor hours to the degree of short-run patience: the negative effect of a greater disutility of effort is further reduced the more elastic labor hours are when  $\delta < 1$ , while the same negative effect is further enhanced the more elastic labor hours are when  $\delta > 1$ .

## 4.2 The Welfare Gap of the Future Generation

It is natural to inquire about what happens to the welfare of the future generations, particularly that of Generation 1. In this connection, we define the “future” welfare gap as  $\tilde{G}(\delta) \equiv V(k, z; \delta) - V(k, z; 1)$ , where  $V(k, z; \delta)$  and  $V(k, z; 1)$  are the Generation-1 value

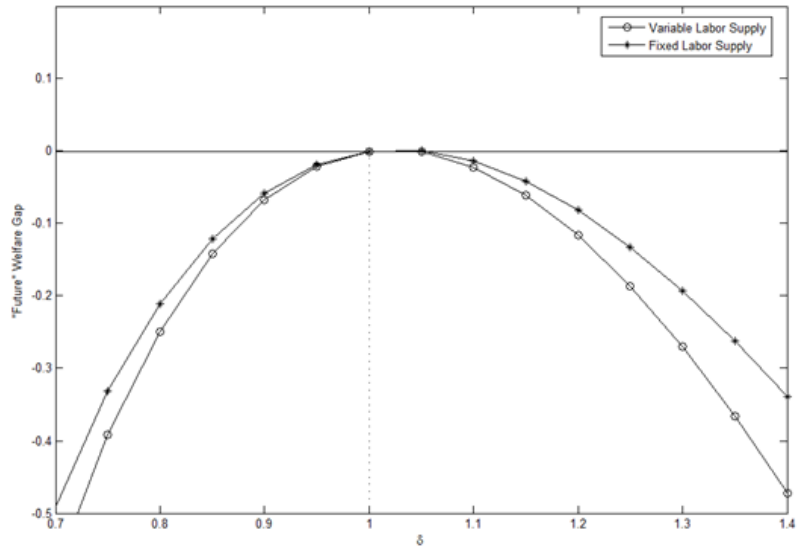
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<sup>20</sup>As opposed to being a virtue.

functions under quasi-geometric and geometric discounting, respectively.

Figure 2.5 plots the “future” welfare gap against  $\delta$ . Except when  $\delta = 1$ , geometric welfare surpasses quasi-geometric welfare for all  $\delta$ . Moreover, when  $\delta < 1$ , the “future” welfare gap increases with  $\delta$  and decreases with it when  $\delta > 1$ .

Figure 2.5: The Effects of Short-Run Patience on the Future Welfare Gap



We now make intergenerational comparisons. As  $V(k, z; 1)$  is not a function of  $\delta$ , the welfare of the future generation under quasi-geometric discounting behaves as the “future” welfare gap does with respect to  $\delta$ . We further recall that for Generation 0, quasi-geometric welfare always decreases with  $\delta$ . These observations combined imply that when  $\delta < 1$ , reducing the degree of excessive short-run impatience creates a tradeoff between current welfare and future welfare: it reduces current welfare but increases future welfare. This is because when the economy is endowed with excessive short-run impatience, the current generation’s savings, investment and work hours are too low (compared to the standard case) to begin with. Thus, reducing excessive short-run impatience would increase savings, investment and labor hours, which would benefit the future generation at the expense of

the current one.

In contrast, when  $\delta > 1$ , both current and future welfare decrease with  $\delta$ . This occurs because with excessive short-run patience, the economy already saves, invests and works too much. Hence, further pushing the degree of patience to greater excess would only serve to reduce not only current, but also future welfare.

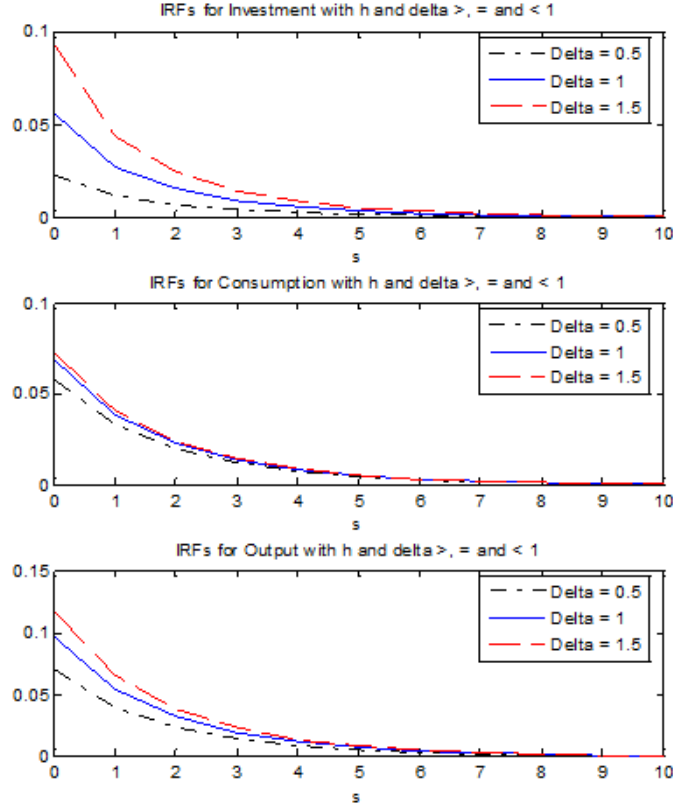
## 5 The Effects of a Technology Shock

In the following subsections, we analyze the effects of a positive technology shock on the impulse responses of consumption, investment and output and the effects of the persistence of a technology shock on the welfare gap.

### 5.1 On Output, Consumption and Investment

We perturb the system with a unit change to the technology shock  $\log(z)$ , and obtain the impulse response functions for investment ( $I$ ), consumption ( $C$ ) and output ( $Y$ ), using their respective optimal policy functions. The results are reported in Figures 2.6 and 2.7. Figure 2.6 below compares the impulse response functions for  $C$ ,  $I$  and  $Y$  for different values of  $\delta$ . As expected, the variables corresponding to the highest degree of short-run patience (i.e., the one endowed with the higher  $\delta$ ), are the most responsive to the technology shock. This is in line with our previous results: as the saving rate, and capital both increase monotonically with  $\delta$ , it is to be expected that the impulse responses of next-period capital, and thus investment (with full depreciation) would be higher when  $\delta > 1$ . The same can be said for output, as it is an increasing function of capital. We thus observe that, *ceteris paribus*, the more patient planner invests more in response to a technology shock. This, together with the result that greater patience encourages higher labor hours, translates into a higher output response to a given stochastic shock.

Figure 2.6: Impulse Responses for  $I$ ,  $C$  and  $Y$



With regard to the impulse responses for consumption, there is barely variation with respect to different values of  $\delta$ . This can be verified from

$$\frac{dc_1^*}{d\delta} = \underbrace{(1 - \alpha)(1 - s_1^*) \frac{y_1^*}{h_1^*} \frac{dh_1^*}{d\delta}}_{(+)} + \underbrace{\alpha(1 - s_1^*) \frac{y_1^*}{k_1^*} \frac{dk_1^*}{d\delta}}_{(+)} - \underbrace{y_1^* \frac{ds_1^*}{d\delta}}_{(-)}$$

where the first term is positive as , the second term is positive, while the third term is negative From these, the intuition is clear: intuitively, greater short-run patience increases labor hours, which increases output and thus, consumption. However, while greater patience implies higher current capital (or last-period investment), which increases consumption, greater patience also spurs savings, which reduces current consumption. Unlike in the case with fixed labor supply, however, the labor hours effect and the capital effect work together

to overcome the contrary effect of savings on contemporary consumption. Thus, we expect a greater response of consumption to a unit technology shock with labor-leisure choice.

Figure 2.7: Impulse Responses for  $I$ ,  $C$  and  $Y$

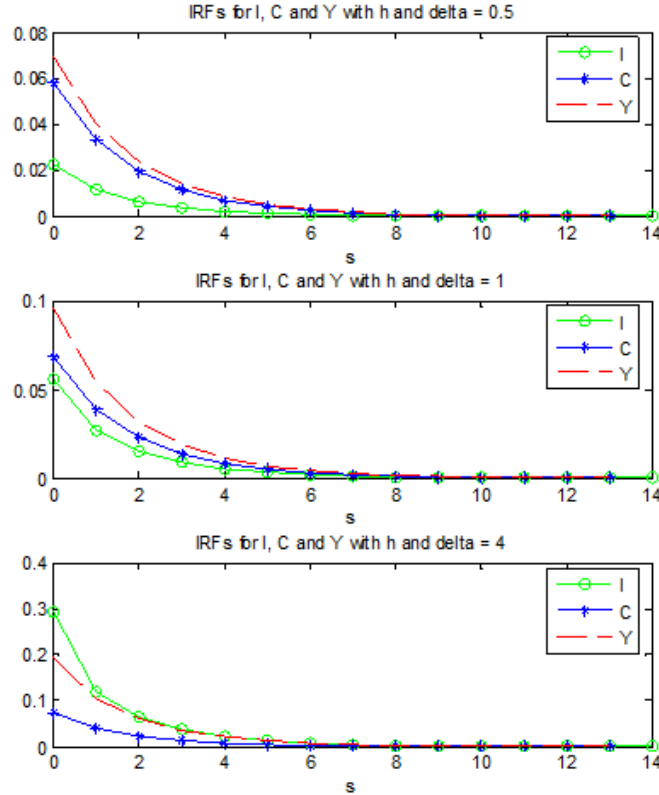


Figure 2.7 above groups together the impulse responses for consumption, investment and output for different values of  $\delta$  in order to compare their relative sensitivities to the technology shock, for a given degree of short-run patience. Compared to output and consumption, investment is not as responsive to the technology shock for lower values of  $\delta$ , unlike in the standard real business cycle (RBC) model where investment is the most reactive to a given shock. The most likely reason for this is the assumption of full depreciation. It is only when the degree of patience is quite high (say,  $\delta = 4$ ) that the relative rankings mimic that of the standard RBC model. Thus, with full depreciation of capital, the individual should exhibit extreme short-run patience for greater capital accumulation, and thus, investment to have

a greater initial response than output and consumption. This is only to be expected as the optimal labor supply is a constant. Thus, introducing labor-leisure choice into the social planner’s problem does not seem alter the relative dynamics of the key economic variables.<sup>21</sup>

## 5.2 On Persistence, Patience and the Current Welfare Gap

In the previous sections, we fixed  $z$  to equal unity in order to allow us to focus on the effects of  $\delta$  on welfare. In this subsection, however, we allow the persistence of the shock ( $\rho$ ) to vary while we fix  $\delta$  to enable us to analyze the marginal impact of  $\rho$  on the current welfare gap.<sup>22</sup>

**Proposition 4** *For our given set of parameters, a more persistent technology shock reduces the difference between the current quasi-geometric welfare and the current geometric welfare.*

**Proof.** The expression  $\frac{\partial G(\cdot)}{\partial \rho} = -\frac{\beta(1-\delta)}{(1-\alpha\beta)(1-\rho\beta)} \log(z) \geq 0$  if and only if  $\delta \geq 1$ . Since  $V_0(k, z; \delta) - V_0(k, z; 1) \leq 0$  when  $\delta \geq 1$ , the current welfare gap becomes less negative as  $\rho$  increases when  $\delta > 1$ , and becomes less positive as  $\rho$  increases when  $\delta < 1$ . ■

A more highly persistent shock increases the expected future value function, which redounds to increasing current welfare. As it is the more patient economy that places a greater value on the future, we can anticipate that the longer the duration of a beneficial technology shock, the greater will be its expected future welfare. When  $\delta > 1$ , the more patient, quasi-geometric economy imputes a greater value on expected future welfare than the more myopic geometric economy, so that even though  $V_0(k, z; \delta) < V_0(k, z; 1)$  at the onset, raising  $\rho$  allows  $V_0(k, z; \delta)$  to approach  $V_0(k, z; 1)$ . When  $\delta < 1$ , the opposite argument holds: in this case, a higher  $\rho$  enables  $V_0(k, z; 1)$  to approximate  $V_0(k, z; \delta)$ . It thus appears that excessive patience is rewarded with a more persistent technology shock, as this enables the more patient economy to “catch up with” the less patient one.

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<sup>21</sup>Indeed, we look at the impulse response functions for the case without labor-leisure choice and we find that the relative rankings are preserved in both groupings. The only difference is that the impulse responses are greater with labor-leisure choice.

<sup>22</sup>For the future generation, the persistence of the technology shock affects quasi-geometric and geometric welfare in the same manner so that the “future” welfare gap remains unaffected by  $\rho$ .

## 6 Concluding Remarks

This paper mainly contributes to the existing literature on the implications of time-inconsistent discounting by comparing the (steady-state) allocations and welfare levels generated under quasi-geometric discounting to those derived under the time-consistent, geometric discounting. By reconstructing the stochastic, infinite-horizon planner’s problem with labor-leisure choice and quasi-geometric discounting into a game between the current planner and his future selves, we obtain the time-consistent, closed-form solution within the category of first-order Markov perfect equilibria. We then observe that the planner’s degree of short-run patience is the key to the resulting comparisons between the quasi-geometric and geometric solutions.

Our analyses unveil four striking results. First, the quasi-geometric allocations (i.e., labor hours, capital and output) are below their geometric counterparts when the planner is characterized by excessive short-run patience, while the opposite ensues with excessive short-run impatience. Second, from the current generation’s standpoint, quasi-geometric welfare is strictly lower than geometric welfare when the economy is endowed with excessive short-run patience, whereas the reverse outcome arises with excessive short-run impatience. Moreover, comparing between generations, we find that reducing excessive short-run impatience creates a welfare tradeoff between the current and future generations, but further increasing excessive short-run patience becomes detrimental to the welfare of both. Third, a higher elasticity of labor supply increases the differences between the quasi-geometric and geometric allocations and welfare levels. Fourth, a more persistent technology shock enables the more patient economy to “catch up with” the less patient one so that the gap between the current quasi-geometric and geometric welfare is reduced as the persistence parameter is increased.

That quasi-geometric and geometric discounting yield very different results indicates that the choice of the discount function matters in making welfare analyses. Needless to say, policy recommendations derived from conclusions based on the standard geometric



discount function should be taken with a grain of salt, as these may not hold with time-inconsistent preferences. As a case in point, an unfunded social security system is claimed to be welfare-reducing under geometric discounting due to its distortionary effects on savings and labor supply decisions.<sup>23</sup> However, with time-inconsistent preferences, Kumru and Thanapoulos (2008) provide some theoretical and empirical verification that the existence of a social security system can be welfare-enhancing, particularly in the face of severe preference reversals.

Finally, we close by pointing out an interesting direction for future research. As the second fundamental welfare theorem does not hold with quasi-geometric discounting, we would have to solve for the quasi-geometric decentralized equilibrium in order to ascertain how the quasi-geometric outcome fares against the geometric case. Doing so would involve the use of a numerical algorithm, as solving the decentralized economy's problem entails the derivation of the aggregate policy functions for both capital and labor, aside from the corresponding individual policy rules. This additional complication prevents the attainment of a closed-form solution. A promising direction lies in developing the grid-based Euler equation method in Maliar and Maliar (2005), which yields a unique interior solution to the intertemporal-choice problem with quasi-geometric discounting for a wide range of parameter values.

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<sup>23</sup>See Diamond (1965), Auerbach and Kotlikoff (1987), Imrohoroglu *et al.* (1995), and Huggett and Ventura (1999).

## Chapter 3

# On the Macroeconomic and Welfare Effects of Raising the Retirement Age

## 1 Introduction

The imminent retirement of the Baby Boom generation and the problems this would pose on claiming post-retirement benefits have given rise to concerns over the long-run solvency of the public pension system. As an offshoot of this issue, an extensive literature that evaluates the efficiency and welfare implications of the unfunded social security system has emerged. Most of the studies in this literature conclude that the unfunded system is welfare-reducing and that privatization is the key to avoiding the public pension debt problem (Kotlikoff, 1996, 1998; Imrohoroglu et al., 1995, 1999, 2003). The gains from privatization include the elimination of both the distortions to savings and labor supply decisions and the relaxation of borrowing constraints imposed by the mandatory contributions to the pension fund (Diamond, 1965; Kotlikoff, 1998; and Nishiyama and Smetters, 2007). However, aside from the political obstacles that stand in its way, the costs of transitioning to a privatized pension system are reported to be prohibitive (Bernstein, 2010).

A seemingly more politically viable alternative involves increasing the retirement age at which agents become eligible for claiming full social security benefits. Indeed, other developed countries that are in a similar quandary due to an ageing population structure have already made legislative headways towards delaying retirement (Cendrowicz, 2010).<sup>1</sup> This proposal is argued to augment the revenues of the unfunded pension system in three ways: 1) by expanding the working base that supports the retired sector of the population; 2) by reducing the the total benefits owed pensioners through the subsequent decrease in

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<sup>1</sup>In Ireland and the U.K., the legal retirement age is legislated to increase to 68 by 2028 and 2046, respectively. In Spain and Germany, the projected increase is from 65 to 67 years old, while France plans to raise the age of eligibility for claiming full pension benefits from 60 to 62 by the year 2018 (Cendrowicz, 2010).

the fraction of retired agents in the population; and 3) by actuarially reducing the size of the pool of pensioners, as less retirees would then be expected to actually survive to claim benefits at a more advanced age (Weller, 2010). Moreover, the subsequent increase in aggregate labor supply is alleged to increase total production and enhance the standard of living in the economy (Mermin and Steuerle, 2006). However, dissenting arguments from younger cohorts include greater lifetime tax liabilities and higher unemployment rates (Rainsford, 2011; and Okello and Charlton, 2010),<sup>2</sup> while the reservations of older workers against such a reform stem largely from the unpalatable prospect of staying longer in the labor force.

Consequent questions arise: who gain and who lose from raising the retirement age? How will such a policy reform affect the relevant macroeconomic aggregates? How will the entire economy fare after the reform? We address these questions in the context of a dynamic general equilibrium model with overlapping generations. Our primary objective is to analyze the long-run macroeconomic and welfare effects of raising the retirement age,<sup>3</sup> and to compare these outcomes to those from social security privatization. As far as we know, this is the first attempt in the literature at developing a theoretical framework to address these specific issues in the U.S.<sup>4</sup> To preserve the availability of a closed-form solution, we abstract from income uncertainty and borrowing constraints and only allow for mortality risk.

Intra-cohort heterogeneity is introduced into the model via economic agents that exhibit

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<sup>2</sup>Indeed, younger cohorts in France and Spain have taken to the streets in violent protests, claiming that such a reform would put them at a disadvantage against older and more seasoned workers (Rainsford, 2011; and Okello and Charlton, 2010).

<sup>3</sup>In the rest of the text, we will be using the phrases “raising the retirement age,” “postponing retirement” and “delaying retirement interchangeably” to refer to the law-mandated increase in the age at which retired agents can claim full pension benefits. It is 65 years for agents born in or before 1937, while it increases in two-month increments until 66 for agents born from 1943 to 1954. The subsequent cohorts born from 1955 to 1960 have full retirement ages that increase by two months starting from age 66 (Duggan et al., 2001).

<sup>4</sup>Diaz-Jimenez and Diaz-Saavedra (2009) study the welfare implications of delaying retirement in Spain for three years in a multiperiod, general equilibrium, overlapping generations model with purely rational agents. They find that this policy reform is sufficient to address the long-run viability issues of the Spanish social security system. Moreover, they find this reform to be welfare-improving after 2014.

either time-consistent or time-inconsistent discounting behavior.<sup>5</sup> It is in this setting that the existence of social security can be welfare-enhancing for the more present-biased, time-inconsistent agents, who need some sort of pre-commitment device that would compel them to save in their earlier working years, when they are most tempted to spend impulsively, for their latter years in retirement, when reduced incomes would constrain their ability to consume (Akerloff, 1998; Kumru and Thanapoulos, 2007; Fehr et al., 2008). In modeling time-inconsistent discounting behavior, we follow most of the literature in employing the quasi-hyperbolic discount (QHD) function proposed by Laibson (1994). This discount function is purported to explain some observed puzzles in consumption behavior such as myopic undersaving (Laibson, 1997) and the abrupt drop in consumption upon retirement (Diamond and Koszegi, 2003).<sup>6</sup> Although previous studies like Imrohoroglu et al. (2003) and Fehr et al. (2008) have dealt with the long-run welfare consequences of social security privatization with either rational (time-consistent) or hyperbolic agents (time-inconsistent), none has yet done so in a model economy where both rational and hyperbolic agents co-exist. However, an analytical segregation of rational from hyperbolic agents fails to account for the aggregate and welfare externalities generated by the presence of hyperbolic agents. In the present setup, we unveil these externalities by allowing the fraction of hyperbolic agents to vary.

We consider a policy experiment in which we raise the legal retirement age (for full benefits eligibility) from 65 to 70 years old.<sup>7</sup> In instituting this reform, we either hold constant the social security tax rate or the per capita amount of social security benefits.

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<sup>5</sup>Empirical evidence suggests that from a given temporal perspective, agents are present-biased or that they discount short-run tradeoffs more than they do long-run tradeoffs. See, for instance, Ainslie (1992), Loewenstein and Prelec (1992), Angeletos, et al. (2001), Frederick, et al. (2002) and Soman et al. (2005). For the theoretical results, see the seminal works of Strotz (1956), Phelps and Pollack (1968) and Peleg and Yaari (1972).

<sup>6</sup>Diamond and Koszegi (2003) employ a partial equilibrium model with quasi-hyperbolic discounters, who are allowed the decision of timing their retirement. They find that the quasi-hyperbolic discounter is tempted to retire earlier than what his earlier self had planned originally, thereby leading to lower post-retirement consumption.

<sup>7</sup>Although the Social Security Amendments of 1983 started increasing the full retirement age from 65 to 67 years old, only agents born in and after 1960 are fully affected by this reform (Duggan, et al., 2007). We thus stick with the baseline case of 65 years old for the purpose at hand.

In both cases, we observe that postponing retirement has two effects on agents. First, agents would now have to work more years, which increases the aggregate labor supply. Second, their incentive to save is dampened due to their curtailed retirement period. For the aggregate economy, the first effect raises the marginal productivity of capital, and hence, the demand for aggregate capital, while the second effect leads to a decline in the supply of aggregate capital. Both effects contribute to raising the equilibrium real interest rate. Our numerical analysis shows that postponing retirement would lead to an increase in aggregate labor and a decline in aggregate capital in both regimes. Hence, the overall effect on aggregate output is ambiguous, belying the claim that a mandated increase in the retirement age unconditionally increases production and the standard of living in the economy. However, the decline in aggregate capital under a fixed-tax rate regime is more severe than that under a constant-benefits regime due to the greater liquidity constraints imposed by the higher present value of taxes in the former regime. In contrast, aggregate labor hours increase more in the latter regime, as holding the amount of per capita benefits fixed allows a lower corresponding social security tax rate, which increases the after-tax real wage rate and in turn induces a greater aggregate labor supply. Thus, the effects of a lower aggregate capital dominate under a constant-tax rate regime, while the effects of a higher aggregate labor dominate under a constant-benefits reform. Accordingly, aggregate output and aggregate consumption decrease under a constant-tax rate regime, while these increase under a constant-benefits regime.

In terms of welfare, we find that in both cases, this policy reform benefits retirees the most and the new entrants to the workforce the least. This is because the extension of the working-age horizon increases the present values of the lifetime taxes of the youngest cohorts the most. Indeed, when the social security tax rate is held constant, the young workers (aged 20 to 34 years old) even incur welfare losses from the reform. In contrast, all agents gain under a constant-benefits regime. However, retirees gain more under a constant social security tax rate as per capita social security benefits increase by a considerable percentage due to both the reduction in the fraction of retired agents sharing in the pension

fund pool and the direct increase in the social security tax revenues from the extra years of mandatory work. In the aggregate, the economy gains more under a policy regime that holds the per capita amount of pension benefits constant.

Meanwhile, social security privatization primarily relaxes the liquidity constraints facing agents from the elimination of the social security tax rate. This induces greater savings, which increases aggregate capital. Also, phasing out the social security tax rate increases the after-tax real wage rate, which prompts agents to work more and the aggregate labor supply to rise. As a result, aggregate output and aggregate consumption increase. Furthermore, the youngest workers benefit the most while the retirees benefit the least from privatization. The younger workers gain significantly from the relaxation of their liquidity constraints upon the abolition of social security taxes, while retirees gain to a lesser extent due to the complete elimination of their pension benefits. The aforementioned results are comparable to those in Imrohoroglu et al. (2003).

Comparing the two alternative proposals, younger cohorts benefit considerably more while retirees benefit considerably less under social security privatization than under either one of the aforementioned regimes that postpones retirement. It thus appears that if the institution of reforms were made subject to a vote, the younger workers would vote for privatization while the older workers and retirees would vote for postponing retirement.

We observe that hyperbolic agents indeed impose pecuniary externalities on the rest of the economy: the real interest rate exhibits an inverted U-shaped pattern as the fraction of hyperbolic agents increases. As the economy becomes more populated by hyperbolic agents, aggregate undersaving worsens, which causes the real interest rate to rise. However, as hyperbolic agents further crowd the economy, the real interest rate rises sufficiently so that the income effect induces agents to work less, which then reduces aggregate labor in equilibrium. Due to factor complementarity in production, the resulting decrease in the equilibrium demand for aggregate labor is coupled by a decline in the demand for aggregate capital, which now tends to reduce the equilibrium real interest rate.

Under a reform that raises the retirement age, the higher aggregate welfare gain accrues

to the economy inhabited by a greater fraction of hyperbolic agents. This suggests that the mandatory postponement of retirement works like a commitment device that compels the myopic hyperbolic agents to increase lifetime savings and consequently experience greater welfare gains. In the context of social security privatization, the aggregate welfare gain follows the real interest rate pattern: when the real interest rate rises as the fraction of hyperbolic agents only starts to increase, it is the more numerous rational agents that are able to take advantage of the higher real interest rate and realize steeper consumption paths so that aggregate welfare increases. However, as the economy becomes more populated by the more present-biased hyperbolic agents and the real interest rate starts falling, the corresponding aggregate welfare gain also starts declining. Our parametric economy suggests that there is indeed no monotonic relationship between welfare gains and the fraction of hyperbolic agents: social security privatization is most desirable with a labor force composed of half-rational and half-hyperbolic agents. Moreover, we find that retired hyperbolic agents gain after privatization in a mixed economy, while they lose in an economy purely composed of time-inconsistent individuals. This is because the higher resultant real interest rate in a mixed economy enables better consumption-smoothing by hyperbolic agents, who need to save more than their rational counterparts, in response to the elimination of the pre-commitment mechanism against undersaving provided by social security. These outcomes show that in making welfare evaluations, it is not sufficient to consider an economy composed purely either of rational or hyperbolic agents.

Although it appears that social security privatization yields a greater aggregate welfare gain than raising the retirement age due to the subsequent increase in aggregate savings that allows aggregate consumption to increase in equilibrium, the reality that problematic countries are heading towards postponing retirement and not privatization is a positive confirmation that the welfare, efficiency and even political costs of implementing the former might not be as egregious as that of the latter.

The rest of the paper is structured as follows: Section 2 describes the model economy. Section 3 explains the calibration procedure. Section 4 presents the macroeconomic and

welfare results from raising the retirement and privatization. Finally, Section 5 proffers some concluding remarks.

## 2 The Model Economy

In this section, we present the model setup. The economy is composed of three sectors: a production sector, a consumer sector and government.

### 2.1 The Production Sector

The typical firm's production function is characterized by a constant returns to scale Cobb-Douglas technology:<sup>8</sup>

$$F(K, L) = BK^\alpha L^{1-\alpha},$$

where  $K$  is aggregate capital,  $L$  denotes aggregate labor,  $\alpha \in (0, 1)$  is the share of aggregate capital in output and  $B > 0$  represents the constant total factor productivity parameter. The depreciation rate of capital is a constant and is given by  $d_K \in (0, 1)$ . To maximize profits, the firm rents capital and hires labor to the point where the respective net marginal products equal  $r$ , for capital and the real wage rate,  $w$ , for labor. These are given, respectively, by the following:

$$r = \alpha BK^{\alpha-1} L^{1-\alpha} - d_K, \tag{1}$$

$$w = (1 - \alpha)BK^\alpha L^{-\alpha}. \tag{2}$$

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<sup>8</sup>As the firm's problem is static, we omit time subscripts.



## 2.2 Demographics

In each period, a generation of new agents is born. The number of agents at time  $t$  is given by  $N_t = (1 + n)^t$ , for  $t \geq 0$  and  $n > 0$ . Agents in the same generation are not identical. They differ in terms of the degree of short-run impatience, which will be discussed in greater detail in the next subsection.

Each agent can live at most  $J + 1$  periods. Conditional on being alive at age  $j$ , the probability of being alive in the next period is  $\psi_{j+1} \in (0, 1)$ . Death is certain at the terminal age: hence,  $\psi_{J+1} = 0$ . The unconditional probability of surviving up to age  $j$  is given by

$$s_j \equiv \left( \prod_{m=0}^j \psi_m \right), \quad \text{for } j \in \{0, 1, \dots, J\},$$

where  $s_0 = \psi_0 = 1$ .

Let  $\Theta_{j,t}$  denote the number of age- $j$  agents at time  $t$ . This is given by

$$\Theta_{j,t} = s_j N_{t-j} = s_j (1 + n)^{-j} N_t. \quad (3)$$

The share of age- $j$  agents at time  $t$  is then given by

$$\mu_j \equiv \frac{\Theta_{j,t}}{\sum_{l=0}^J \Theta_{l,t}} = \frac{s_j (1 + n)^{-j}}{\sum_{l=0}^J s_l (1 + n)^{-l}}, \quad (4)$$

which is constant over time and where  $\sum_{j=0}^J \mu_j = 1$ . The age structure of this economy can be computed by using

$$\mu_{j+1} \equiv \frac{\psi_{j+1} \mu_j}{1 + n}, \quad (5)$$

with  $\mu_0 \equiv [\sum_{l=0}^J s_l (1 + n)^{-l}]^{-1}$ . The economy is thus distinguished by a constant demographic structure.

Since we are interested in stationary equilibria in which all individual-level variables are constant over time, we will omit the time index  $t$  in the subsequent sections. Agents will then be solely identified according to their age indexed by  $j$ .

### 2.3 Preferences and Intra-cohort Heterogeneity

In each period, each agent derives utility from current consumption and experiences disutility from working. The utility of being deceased is normalized to zero. Let  $c_j^i$  and  $h_j^i$ , respectively denote the consumption and working hours of a type- $i$  agent at age  $j$ . The agent's lifetime utility, from age-0 perspective, is represented by

$$U_0^i \equiv u(c_0^i, h_0^i) + \delta_i \beta \left[ \sum_{j=1}^J \beta^{j-1} s_j u(c_j^i, h_j^i) \right], \quad (6)$$

where  $\beta \in (0, 1)$  and  $\delta_i \in (0, 1]$ . The parameter  $\beta$  is the long-run discount factor, while  $\delta_i \beta$  is the short-run discount factor and is assumed to differ across agents. Assume that there are  $I$  types of consumers. Each group is characterized by a different short-run discount factor  $\delta_i \beta \in (0, 1)$ . The size of group  $i$  is given by  $\lambda_i \in (0, 1)$ , so that  $\sum_{i=1}^I \lambda_i = 1$ .

When  $\delta_i = 1$ , a type- $i$  agent has the standard, exponential discounting preference. We follow the literature in calling him a “rational agent.” When  $\delta_i < 1$ , agent  $i$  has the quasi-hyperbolic discounting (QHD) preference. In this case, we refer to him as a “hyperbolic agent.”

The period utility function is

$$u(c, h) = \frac{\left( c - A \frac{h^{1+\theta}}{1+\theta} \right)^{1-\sigma}}{1-\sigma}, \quad \text{with } \sigma, \theta, A > 0,$$

where  $\sigma$  is the coefficient of relative risk aversion and  $\theta$  is the inverse of the intertemporal elasticity of substitution in labor supply. This utility function is known in the literature as the GHH preference (which is an acronym for Greenwood-Hercowitz-Huffman preference). We employ this utility form as the characteristic absence of the income effect on labor supply lends itself to the existence of a closed-form solution.

## 2.4 Budget Constraint

All agents start working at age 0. Retirement is mandatory at age  $J_R < J$ . In each period, each age- $j$  agent receives the same amount of labor endowment  $\varepsilon_j > 0$ . The sequence  $\{\varepsilon_j\}_{j=0}^{J_R-1}$  is intended to capture the observed life-cycle earnings profile. For the sake of convenience, we take  $\varepsilon_j = 0$  for  $j \geq J_R$ .

All agents are subject to four taxes: a consumption tax, a labor income tax, a capital income tax and a social security tax. These tax rates are denoted respectively by  $\tau_c, \tau_h, \tau_k, \tau_{ss} \in (0, 1)$ . The social security tax is imposed on labor income alone. Each retired agent receives the same amount of social security benefits,  $x$ . Given the utility function, the individual's labor supply is age-dependent, but is independent of  $\delta_i$ , i.e.,  $h_j^i = h_j$  for all  $i$ .<sup>9</sup> Each agent then contributes  $\tau_{ss}w\varepsilon_jh_j$  units of income to social security, which does not depend on  $\delta_i$ . Even if we then assume that  $x$  is a function of past contributions, it will still be independent of  $\delta_i$ .

There is no private annuity market in this economy.<sup>10</sup> As a result, agents who die prematurely would leave behind accidental bequests. We assume that these unintended bequests are divided equally among all surviving agents. Let  $b > 0$  be the amount of accidental bequests received by each surviving agent.

For  $j \in \{0, 1, \dots, J\}$ , define  $y_j$  according to

$$y_j = (1 - \tau_h - \tau_{ss})w\varepsilon_j\chi_jh_j + (1 - \chi_j)x + b$$

where  $w > 0$  is a constant real wage rate.<sup>11</sup> We let the after-tax wage rate be denoted by  $\tilde{w} = (1 - \tau_h - \tau_{ss})w$ .  $\chi_j$  is an indicator function which equals 1 if  $j < J_R$  and 0 otherwise.

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<sup>9</sup>This is formally established in the proposition below.

<sup>10</sup>This is a common assumption in the literature and can be rationalized by Friedman and Warshawsky's (1990) observation that the private annuity market is small due to adverse selection problems. Moreover, agents may opt out of this market due to a bequest motive or a self-insurance motive against large medical or nursing homes expenses. See Imrohorglu et al. (2003) for further discussion.

<sup>11</sup>We follow the literature in assuming constant prices because we will focus on stationary equilibria in the quantitative analysis.

The individual budget constraint is given by

$$(1 + \tau_c)c_j^i + a_{j+1}^i = [1 + (1 - \tau_k)r]a_j^i + y_j, \quad (7)$$

with  $a_0^i \geq 0$  given, and  $a_{J+1}^i = 0$ .

## 2.5 The Consumer's Problem

The consumer's problem is to choose the sequence  $\left\{ \left( c_j^i, h_j^i, a_{j+1}^i \right) \right\}_{j=0}^J$  so as to maximize his expected lifetime discounted utility in (6), subject to the budget constraint in (7). However, since the agent's future selves disagree with his current self, we cannot use the standard Lagrangian method in this instance, as time-inconsistent discounting renders the agent's problem non-recursive. To illustrate, using the Lagrangian method indiscriminately yields the following Euler equations from the planner's perspective at time  $t$ :

$$\text{for } t, t + 1 : u_c(c_t^i, h_t^i) = \delta_i \beta E_t[u_c(c_{t+1}^i, h_{t+1}^i)(1 + r_{t+1})] \quad (8)$$

$$\text{for } t + 1, t + 2 : u_c(c_{t+1}^i, h_{t+1}^i) = \beta E_t[u_c(c_{t+2}^i, h_{t+2}^i)(1 + r_{t+2})] \quad (9)$$

When  $\delta_i < 1$ , agent  $i$  at time  $t$  ascribes a lower discounted value on next-period marginal utility (or short-run marginal utility) than on a later future one (or long-run marginal utility). In this case, he considers the period  $t + 1$ , as his short-run future, while  $t + 2$  onwards is his long-run future. From (8) and (9), we observe that the short-run discount factor,  $\delta_i \beta$ , which agent  $i$  imputes to the short-run tradeoff (between  $t$  and  $t + 1$ ) is less than the long-run discount factor,  $\beta$ , which he ascribes to the long-run tradeoff (between  $t + 1$  and  $t + 2$ ). However, at the dawn of  $t + 1$ ; the agent (or his self at  $t + 1$ ) suffers a preference reversal and suddenly views the intertemporal tradeoff between  $t + 1$  and  $t + 2$  with greater impatience, as  $t + 2$  now becomes his short-run (immediate) future, that is:

$$\text{for } t + 1, t + 2 : u_c(c_{t+1}^i, h_{t+1}^i) = \delta_i \beta E_t[u_c(c_{t+2}^i, h_{t+2}^i)(1 + r_{t+2})] \quad (10)$$

Thus, without any strategy-proof commitment mechanism in place, agent  $i$ 's optimal choice,  $\{c_{j+k}^i, h_{j+k}^i\}_{k=0}^J$ , made at any time  $t$ ; would be disregarded by his future selves. More succinctly, the optimal solutions are time-inconsistent.

To arrive at a time-consistent solution, we reformulate the agent's problem into a game between the current self and his future temporal selves using the solution concept of subgame perfect equilibrium. We thus solve the agent's problem using backward induction. For analytical tractability, we restrict the solution set to that characterized by first-order Markov equilibria. See Krusell, Kuruscu and Smith (2003) for a more involved discussion on this type of equilibrium.

The dynamic programming problem of the current self of agent  $i$  at age  $j = 1, \dots, J$  can be defined recursively as follows:

$$\max_{c_j^i, h_j^i} \left\{ u(c_j^i, h_j^i) + \delta_i \beta \psi_{j+1} \tilde{V}_{j+1}(a_{j+1}^i) \right\}, \quad (11)$$

subject to (3), where

$$\tilde{V}_{j+1}(a_{j+1}^i) = u(\tilde{c}_{j+1}^i, \tilde{h}_{j+1}^i) + \beta \psi_{j+2} \tilde{V}_{j+2}(\tilde{a}_{j+2}^i),$$

where  $\tilde{c}_{j+1}^i$ ,  $\tilde{h}_{j+1}^i$  and  $\tilde{a}_{j+2}^i$  denote the current self's conjecture regarding his future self's optimal decisions.  $\tilde{V}(\cdot)$  corresponds to the value function of the future selves. This can further be interpreted as the agent's long-run utility.

## 2.6 Life-Cycle Profiles

Let  $\tilde{r} \equiv (1 - \tau_k)r$  be the after-tax interest rate and define  $q \equiv (1 + \tilde{r})^{-1}$ . Let  $\Omega_j$  be the present value of all future incomes starting from age  $j$ , i.e.,

$$\Omega_j^i \equiv \sum_{l=j}^J q^{l-j} y_l^i, \quad \text{for } i = 1, \dots, I.$$

In the proposition below, we present the closed-form solution to the consumer's problem.

In particular, we specify how to obtain agent  $i$ 's optimal allocation sequence,

$$\left\{ \left( c_j^i, h_j^i, a_{j+1}^i \right) \right\}_{j=0}^J.$$

**Proposition.** *Given the tax rates  $\{\tau_c, \tau_k, \tau_h, \tau_{ss}\}$ , the transfers  $\{x, b\}$  and the prices  $\{\tilde{w}, \tilde{r}\}$ , the life-cycle profile  $\left\{ \left( c_j^i, h_j^i, a_{j+1}^i \right) \right\}$  for a typical type- $i$ , age- $j$  agent satisfies*

$$h_j^i = \begin{cases} \left( \frac{\tilde{w} \varepsilon_j}{A} \right)^{\frac{1}{\theta}}, & \text{for } j = 0, 1, \dots, J_R - 1 \\ 0, & \text{otherwise} \end{cases}, \quad (12)$$

$$c_j^i = \frac{1}{\Lambda_j^i} \left[ \frac{(1 + \tilde{r}) a_j^i + \Omega_j^i}{1 + \tau_c} - \Psi_j^i \right], \quad (13)$$

$$c_{j+1}^i = \Phi_{j+1}^i c_j^i + \Delta_{j+1}^i, \quad (14)$$

for  $j \in \{0, 1, \dots, J\}$  and where

$$\Phi_{j+1}^i \equiv [\beta \psi_{j+1} (1 + \tilde{r})]^{\frac{1}{\sigma}} \left[ \frac{\delta_i \Gamma_{j+1}^i}{\Lambda_{j+1}^i} \right]^{\frac{1}{\sigma}}, \quad (15)$$

$$\Gamma_j^i \equiv 1 + \beta \psi_{j+1} (\Phi_{j+1}^i)^{1-\sigma} \Gamma_{j+1}^i, \quad (16)$$

$$\Lambda_j^i \equiv 1 + q \Phi_{j+1}^i \Lambda_{j+1}^i, \quad (17)$$

$$\Delta_{j+1}^i = \begin{cases} \frac{(\tilde{w})^{\frac{1+\theta}{\theta}}}{A^{\frac{1}{\theta}}(1+\theta)} \left[ (\varepsilon_{j+1})^{\frac{1+\theta}{\theta}} - \Phi_j^i (\varepsilon_j)^{\frac{1+\theta}{\theta}} \right], & \text{for } j = 0, \dots, J_R - 1 \\ -\frac{(\tilde{w})^{\frac{1+\theta}{\theta}}}{A^{\frac{1}{\theta}}(1+\theta)} \left[ \Phi_j^i (\varepsilon_j)^{\frac{1+\theta}{\theta}} \right], & \text{for } j = J_R \\ 0, & \text{for } j = J_R + 1, \dots, J \end{cases}, \quad (18)$$

$$\Psi_j^i \equiv \begin{cases} q \sum_{k=j}^{J_R-1} q^{k-j} (\Lambda_{k+1}^i \Delta_{k+1}^i), & \text{for } j = 0, \dots, J_R - 1 \\ 0, & \text{for } j = J_R, \dots, J \end{cases}, \quad (19)$$

where  $\Phi_j^i = [\delta_i \beta \psi_j (1 + \tilde{r})]^{\frac{1}{\sigma}}$ ,  $\Gamma_j^i = 1$  and  $\Lambda_j^i = 1$ .

**Proof.** See Appendix B. ■

Equation (12) gives the optimal decision rule for labor hours. Due to the specific utility function used, it only depends on the constant real wage rate and the age-dependent productivity parameter,  $\varepsilon_j$ . Equation (13) shows that at age  $j$ , agent  $i$  consumes a fraction  $(\Lambda_j^i)^{-1}$  of the resources available to him. Equation (14) presents how consumption evolves over the life cycle. For rational agents, it can be shown that  $\Phi_{j+1}^i = [\beta \psi_{j+1} (1 + \tilde{r})]^{\frac{1}{\sigma}}$ ,<sup>12</sup> which, if not for the presence of  $\psi_{j+1}$ , would generate a monotonically increasing consumption profile without labor-leisure choice.

In the absence of labor-leisure choice,  $\Delta_{j+1}^i = 0$ . In this case,  $\Phi_{j+1}^i$  represents the growth factor of individual consumption between age  $j$  and age  $j+1$ , as is the case during the agent's retirement years. During his working years (i.e., for  $j < J_R$ ), (18) can be rewritten as

$$\Delta_{j+1}^i = \frac{(\tilde{w})^{\frac{1+\theta}{\theta}}}{A^{\frac{1}{\theta}}(1+\theta)} \left[ \left( \frac{\varepsilon_{j+1}}{\varepsilon_j} \right)^{\frac{1+\theta}{\theta}} - \Phi_{j+1}^i \right] \geq 0 \text{ iff } \left( \frac{\varepsilon_{j+1}}{\varepsilon_j} \right)^{\frac{1+\theta}{\theta}} \geq \Phi_{j+1}^i, \quad (20)$$

where the term  $\left( \frac{\varepsilon_{j+1}}{\varepsilon_j} \right)^{\frac{1+\theta}{\theta}}$  is the growth factor of labor earnings and thus, measures the steepness of the life-cycle earnings profile. If the growth factor of labor earnings is greater than the growth factor of consumption without labor-leisure choice (i.e., if  $\Delta_{j+1}^i > 0$ ), then

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<sup>12</sup>This can be shown as follows: suppose indeed that when  $\delta_i = 1$ ,  $\Phi_{j+1}^i = [\beta \psi_{j+1} (1 + \tilde{r})]^{\frac{1}{\sigma}}$ , which implies that  $\Gamma_{j+1}^i = \Lambda_{j+1}^i$  and thus,  $\Gamma_j^i = \Lambda_j^i$ . Substituting for  $\Phi_{j+1}^i$  from (13) into (12), we obtain  $\Phi_{j+1}^i = [\beta \psi_{j+1} (1 + \tilde{r})]^{\frac{1}{\sigma}}$ .

$c_{j+1}^i > \Phi_{j+1}^i c_j$ . This means that as the agent becomes sufficiently productive, he is better able to earn more income, which then enables him to save more to raise the level of future consumption above that without labor-leisure choice.

Notice that upon retirement at age  $J_R$ ,  $\Delta_{j+1}^i < 0$ . This means that consumption falls abruptly below the “trend”  $\Phi_{j+1}^i c_j^i$  at age  $J_R$ . This is consistent with the empirical literature that observes that the sharp discontinuous drop in consumption upon retirement is too large to be accounted for by the standard life-cycle model.<sup>13</sup> Although Bernheim et al. (2001) conjecture that this may be due to time-inconsistent behavior, we here observe that both rational and hyperbolic agents experience a steep decline in consumption upon retirement due to the way labor-leisure choice enters the utility function.

## 2.7 Welfare

In order to evaluate welfare in the post-reform era, we compute for  $\phi_j^i$ , which is the amount of resources in the initial equilibrium required for the age- $j$ , type- $i$  agent to attain his long-run utility after the reform. Alternatively, we can interpret  $\phi_j^i$  as the age- $j$ , type- $i$  agent’s required proportional increase in consumption net of the term  $Ah(\tilde{w}; \varepsilon_j)^{1+\chi} / (1+\chi)$  (which we interpret as the consumption-equivalent amount that is necessary to work) to make him as well off in the initial equilibrium as after the reform. Let the subscripts  $s$  and  $r$  denote the status quo and the post-reform period, respectively. Accordingly, age- $j$ , type- $i$  agent’s pre-reform and post-reform long-run utilities can be denoted by where  $V(a_{j+1,s}^i)$  and  $V(a_{j+1,r}^i)$  respectively. We thus compute the individual welfare measure as

$$\phi_j^i = \left[ \frac{V(a_{j+1,r}^i)}{V(a_{j+1,s}^i)} - 1 \right] \times 100, \quad (21)$$

A  $\phi_j^i = 1$  means that agent  $i$  of age  $j$  requires one percent more resources before the reform in order to attain his long-run utility after the reform. Alternately, a positive  $\phi_j^i$  implies that the post-reform scenario yields greater welfare, while a negative  $\phi_j^i$  indicates otherwise.

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<sup>13</sup>See Laibson (1998).



The aggregate welfare measure is:

$$\bar{\phi} = \sum_{i=1}^I \lambda_i \sum_{j=1}^J \mu_j \phi_j^i, \quad (22)$$

which is a weighted average of the welfare changes across all agents of different ages and  $\delta$ -types.

## 2.8 Government

The government is assumed to be infinitely-lived. To finance its operational expenditures, denoted by  $G_t$ , it collects revenues by taxing consumption at the rate,  $\tau_c$ , and both capital and labor incomes at the rates,  $\tau_k$  and  $\tau_h$ , respectively. The government's budget equation satisfies:

$$G_t = \sum_{i=1}^I \lambda_i \sum_{j=1}^J \mu_j [\tau_c c_j^i + \tau_h w h_j + \tau_k r a_{j+1}^i] N_t. \quad (23)$$

The unfunded social security system is mandated to be self-financing. Its budget constraint is given by

$$x \sum_{j=J_R}^J s_j (1+n)^{-j} = \tau_{ss} \sum_{j=0}^{J_R-1} s_j (1+n)^{-j} w \varepsilon_j h_j, \quad (24)$$

where the left-hand side represents the total social security benefits meted out to the current pensioners, while the right-hand side describes the total mandatory contributions made by the current workforce.

## 2.9 Competitive Equilibrium

Given the set of government policy parameters  $\{\tau_c, \tau_h, \tau_k, \tau_{ss}\}$ , a competitive equilibrium is an allocation sequence,  $\left\{ \left( c_j^i, h_j^i, a_{j+1}^i \right) \right\}_{j=0, \dots, J}^{i=1, \dots, I}$ , a constant amount of unintended bequest,  $b$ , and a price system  $\{r, w\}$  and  $\{K_t, L_t\}_{t=0}^{\infty}$ , such that  $\forall t$  :

1. The allocation sequence solves the consumer's problem defined in (11).

2. Firms maximize profits by satisfying the conditions (1) and (2).
3. All markets clear, i.e.,

$$N_t \sum_{j=0}^{J_R-1} s_j (1+n)^{-j} \varepsilon_j h_j = L_t,$$

$$N_t \sum_{i=1}^I \lambda_i \sum_{j=0}^J s_j (1+n)^{-j} a_{j+1}^i = K_{t+1}.$$

4. The budgets of the government and the unfunded social security system are balanced, as given by (23) and (24) respectively.
5. The amount of per capita unintended bequests satisfies

$$b = \frac{1}{1+n} \sum_{i=1}^I \lambda_i \sum_{j=0}^{J-1} (1+\tilde{r})(1-\psi_{j+1}) \mu_j a_{j+1}^i. \quad (25)$$

Equation (25) above is derived as follows: at time  $t$ , the number of type- $i$  agents at age  $j$  is given by  $s_j(1+n)^{-j}N_t$ . Each of these agents choose to have  $a_{j+1}^i$  units of assets in the next period. However, in the next period, a fraction  $(1-\psi_{j+1})$  of these agents die. The total unclaimed assets that they leave behind (including interest payments) are given by  $(1+\tilde{r})(1-\psi_{j+1})s_j(1+n)^{-j}N_t a_{j+1}^i$ . Thus, the total amount of accidental bequests can be obtained by summing across age groups and across types:

$$\sum_{i=1}^I \lambda_i \sum_{j=0}^{J-1} (1+\tilde{r})(1-\psi_{j+1}) s_j (1+n)^{-j} N_t a_{j+1}^i.$$

These unclaimed assets are divided evenly among the surviving agents in the next period. As the size of the population in the next period is  $\sum_{j=0}^J s_j(1+n)^{-j}N_{t+1}$ . Hence, the amount of bequests received by each agent is determined by

$$\begin{aligned}
b &= \frac{\sum_{i=1}^I \lambda_i \sum_{j=0}^{J-1} (1 + \tilde{r})(1 - \psi_{j+1}) s_j (1 + n)^{-j} a_{j+1} N_t}{\sum_{j=0}^J s_j (1 + n)^{-j} N_{t+1}} \\
&= \frac{1}{1 + n} \sum_{i=1}^I \lambda_i \sum_{j=0}^{J-1} (1 + \tilde{r})(1 - \psi_{j+1}) \mu_j a_{j+1}^i.
\end{aligned}$$

### 3 Calibration

In order to execute quantitative analyses, we calibrate the baseline economy. A model period is assumed to be one year.

#### 3.1 Demographic Parameters

Agents are assumed to be “born” and start making economic decisions at age 20 ( $j = 0$ ). They are assumed to live until the real-time age of 85 ( $J = 65$ ), which is chosen to match the percentage of retired agents in the population, which is about 18% in 2006.<sup>14</sup> The baseline age at which a retired agent can claim full pension benefits is 65 ( $J_R = 45$ ). The sequence of conditional survival probabilities  $\{\psi_j\}_{j=1}^J$  is calculated from the United States Life Tables in 2006. The series of cohort shares  $\{\mu_j\}_{j=1}^J$  is constructed using (1). The population growth rate is set at  $n = .012$ , which is the long-run average population growth rate in the United States over the last 50 years (Imrohoroglu, 2003). The age-specific productivity index,  $\varepsilon_j$ , is computed from the average hourly earnings reported by the Bureau of Labor Statistics in 2006. We normalize to 1 the average hourly earnings of the agent at age 20 and subsequently determine the age- $j$  agent’s productivity by taking the ratio of his average hourly earnings to that of the agent at age 20.

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<sup>14</sup>This is computed from the Bureau of Labor Statistics data.

### 3.2 Preference Parameters

In the benchmark model, we restrict the number of  $\delta$ -types of agents in the economy,  $I = 2$ . We let  $\delta_i \in \{.7, 1\}$ . The long-run discount factor,  $\beta$ , is chosen to match an annual real interest rate of around 6%. The coefficient of relative risk aversion,  $\sigma$ , is set at 1.5, which is within the range of values identified in Mehra and Prescott (1985). The inverse of the intertemporal elasticity of substitution (IES) in labor supply,  $\theta$ , is chosen to be 3.5, which yields an IES in labor supply that falls within the range found in Macurdy (1985). The preference parameter  $A$  is chosen to generate labor hours that approximate 1/3.

### 3.3 Technology Parameters

We let the share of capital in U.S. output,  $\alpha$ , equal .36. The depreciation rate of capital,  $d_K$ , is chosen to attain a capital-output ratio ( $K/Y$ ) of 2.52, which is estimated in Imrohoroglu et al. (2003).

### 3.4 Government

We follow McDaniel (2007) in approximating the respective tax rates in the U.S. economy: the consumption tax rate is set at 0.0825, the capital income tax rate at 0.4 and the labor income tax rate at 0.2. The social security tax rate,  $\tau_{ss}$ , is calibrated at 0.062, which is the employee's share stipulated in the Federal Insurance Contributions Act Tax. We follow Imrohoroglu et al. (2003) in fixing government spending as a fraction of GDP ( $G/Y$ ) at 0.18. In the following experiments, we keep all tax rates and the level of government spending as a fraction of output as constants.

Table 1 below summarizes the parameter values employed in the baseline model.

**Table 3.1**

Parameterization of the Baseline Model

<b>Demographics</b>			
population growth rate	$n$		0.012
maximum age	$J$		65
retirement age	$J_R$		45
conditional survival probabilities	$\psi_j$		U.S. Life Tables, 2006
efficiency profile	$\varepsilon_j$		Bureau of Labor Statistics
<b>Preferences</b>			
long-run discount factor	$\beta$		0.998
short-run discount factor parameter	$\delta_i$		{.7, 1}
coefficient of relative risk aversion	$\sigma$		1.5
inverse of IES in labor supply	$\theta$		3.5
<b>Technology</b>			
capital share in output	$\alpha$		0.36
depreciation rate of captial	$d_K$		0.0785
<b>Government</b>			
consumption tax rate	$\tau_c$		.0825
labor income tax rate	$\tau_h$		0.4
capital income tax rate	$\tau_k$		0.2
social security tax rate	$\tau_{ss}$		0.062
<b>Targeted Variables</b>			
before-tax real interest rate	$r$		0.064
capital-output ratio	$K/Y$		2.52
government purchases as a fraction of output	$G/Y$		0.18
retired agents as a fraction of the population	—		0.18

## 4 Quantitative Results

In this section, we present the quantitative implications of the model. We start by looking at the consumption profiles generated by the baseline parameterization and show that these profiles are consistent with the empirical findings on consumption at retirement. We then proceed to answer the following questions:

1. What are the macroeconomic effects of raising the retirement age?
2. Who gain and who lose from such a reform?
3. Based on the aggregate welfare measure, will the entire economy gain or lose from such a reform?
4. How does social security privatization compare to such a reform?
5. What sort of externalities do hyperbolic agents generate in the economy?

### 4.1 Consumption Profiles

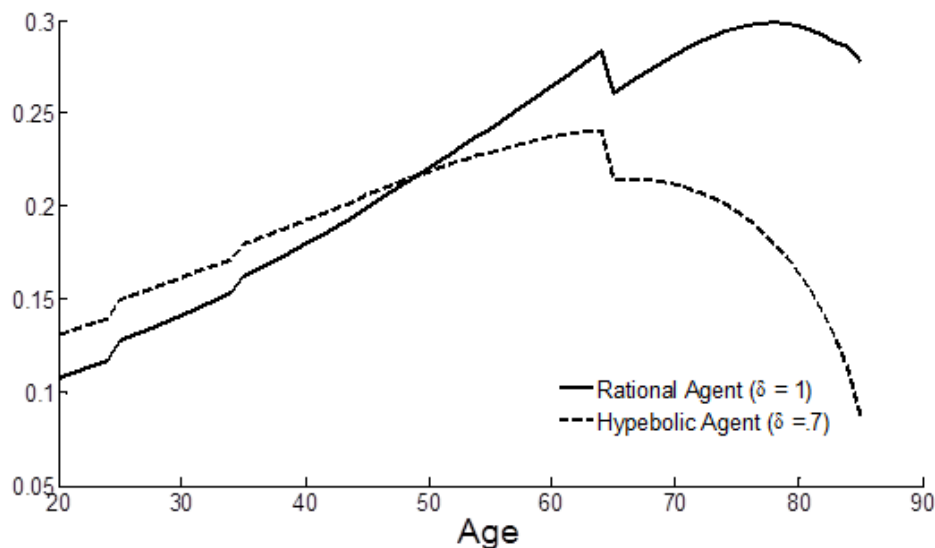
Figure 1 shows the consumption profiles of both rational ( $\delta = 1$ ) and hyperbolic ( $\delta = 0.7$ ) agents. As expected, the consumption path of the rational agent is steeper than that of the hyperbolic agent's. The reason is that faced with the same real interest rate, the more myopic hyperbolic agent opts to consume more and save less at every age than his rational counterpart. The hyperbolic agent's consistent "undersaving" in relation to the rational agent's results not only in a flatter consumption profile, but also in a consumption trajectory that starts turning downward immediately after retirement. In contrast, the rational agent's post-retirement consumption path only starts declining near death.

The empirical estimates for the reduction in consumption upon retirement range from 9% to 20%.<sup>15</sup> For the retired rational agent, consumption drops abruptly by 8.51%, but for his hyperbolic counterpart, consumption falls by 10.88% right after retirement.

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<sup>15</sup>See, among others, Ameriks, et al. (2007), Hurd and Rohwedder (2005), Bernheim, et al. (2001) and Hamermesh (1984).

Figure 3.1: Consumption over the Life Cycle



## 4.2 Raising the Retirement Age

In this section, we increase  $J_R$ , the age at which retired agents can start claiming full social security benefits, from 65 to 70 years old. In doing this, we implement two regimes: we either keep the social security tax rate,  $\tau_{ss}$ , at the pre-reform level or hold the per capita amount of social security benefits,  $x$ , constant. In both cases, the new retirement policy reduces the share of social security claimants from around 18% to 12.42%.

Table 2 below presents the effects of these regimes on some macroeconomic aggregates. In both experiments, we observe a reduction in the post-reform real wage rate due to the increase in the supply of aggregate labor hours. Due to factor complementarity in production, the increase in the equilibrium demand for aggregate labor also increases the demand for aggregate capital, which increases the real interest rate. When  $\tau_{ss}$  is held constant, raising the retirement age increases the present values of expected lifetime incomes across all agent types in two ways: the extension of the working-age horizon increases the

present value of lifetime labor income, and the reduction in the fraction of retired agents increases the amount of per capita benefits upon retirement by 42.25%. This substantial increase in expected lifetime incomes induces agents to reduce savings. Thus, aggregate savings decline, which further increases the real interest rate and reduces the amount of accidental bequests by 15.11%. In equilibrium, aggregate capital declines. The resultant decline in aggregate capital is sufficient to reduce aggregate output by 2.70%, which in turn, leads to a reduction of aggregate consumption by 2.02%.

When the amount of per capita social security benefits,  $x$ , is held constant, aggregate capital still declines but by a smaller percentage than that under constant  $\tau_{ss}$ . The reason is that although expected lifetime labor incomes increase, there is no corresponding increase in  $x$ , which dissuades agents from reducing savings as much as in the case with constant  $\tau_{ss}$  so that aggregate capital also does not fall as much. This consequently prevents the equilibrium real interest rate from rising as much as in the constant- $\tau_{ss}$  regime. Furthermore, as raising the retirement age reduces the number of retirees that share in the pool of pension benefits, keeping the amount of per capita social security benefits constant allows for a lower social security tax rate,  $\tau_{ss}$ , which can be essentially interpreted as partial privatization of social security. Thus, raising the retirement age while keeping the level of social security benefits constant essentially constitutes a mixture of two policy reforms. The reduction in  $\tau_{ss}$  increases the after-tax real wage rate, which encourages agents to work more so that the equilibrium aggregate labor hours increase by more than that in the previous experiment. In this case, the per capita amount of unintended bequests increases with the increase in the real interest rate, in spite of the reduction in aggregate capital. Aggregate output increases by 3.39%, which in turn, raises aggregate consumption by 3.51%.



**Table 3.2**

Macroeconomic Effects of Raising the Retirement Age (in %)

Regime	$r$	$w$	$\tau_{ss}$	$x$	$b$	$K$	$L$	$Y$	$C$
Fixed $\tau_{ss}$	15.61	-7.84	0	42.25	-15.11	-15.84	5.58	-2.70	-2.02
Fixed $x$	7.47	-3.97	-33.30	0	2.21	-3.80	7.67	3.39	3.51

Table 3 shows the welfare effects of raising the retirement age from 65 to 70 years old for the two experiments. With the increase in the real interest rate, consumption profiles across agent types become steeper after the reform as the after-tax real interest rate increases, which raises  $\Phi_{j+1}^i$ . Although,  $\Delta_{j+1}^i$  (i.e., the relationship between the steepness of the earnings profile and the growth factor of consumption in the absence of labor-leisure choice) declines, which tends to reduce future consumption, its negative impact is but miniscule.

**Table 3.3**

Welfare Effects of Raising the Retirement Age (in %)

Constant $\tau_{ss}$			Constant $x$		
	$\delta = .7$	$\delta = 1$		$\delta = .7$	$\delta = .1$
age 20-24	-3.11	-3.13	age 20-24	2.71	2.70
age 25-34	-1.44	-1.24	age 25-34	3.55	3.65
age 35-44	0.81	1.30	age 35-44	4.67	4.91
age 45-54	3.07	3.86	age 45-54	5.78	6.17
age 55-64	5.35	6.45	age 55-64	6.89	7.42
age 65-85	8.70	10.22	age 65-85	8.50	9.22
$\bar{\phi}$	2.63		$\bar{\phi}$	5.54	

In both experiments, it is the groups of retired agents that gain the most advantage in raising the mandatory retirement age even when retirement benefits are held constant, as the extra years of work allows agents to accumulate more pre-retirement assets, which enables them to consume more during retirement. Moreover, raising the retirement age reduces the number of retirees who would split the pension benefits so that when  $\tau_{ss}$  is held constant, the amount of social security benefits per retiree,  $x$ , increases. We thus observe that greater gains accrue to the retired agents under a constant  $\tau_{ss}$ . However, the young and middle-aged workers (aged 20 to 54 years old) experience higher welfare gains under a constant- $x$  regime, as they are the ones that benefit more in terms of lower present values of lifetime taxes owed from the reduction of the social security tax rate. Indeed, the younger cohorts (aged 20 to 34 years old) even lose from the policy reform under a constant  $\tau_{ss}$  mainly because they face the greatest increases in the present values of their lifetime taxes (due to the extension of their working-age horizon) and the highest declines in the present values of lifetime unintended bequests. Accordingly, the overall economy gains more under a policy reform that holds the amount of pension benefits,  $x$ , constant.

Except for the youngest workers (aged 20-24 years old), the less myopic agents benefit more or lose less from the reform, as they are the ones whose savings respond more to a higher equilibrium real interest rate and thus have steeper consumption profiles. The youngest hyperbolic workers garner greater welfare gains (or incur less welfare losses) as myopia induces them into borrowing, so that the lower post-reform equilibrium real interest rate reduces their interest payments on debt.

#### 4.2.1 Alternative Discounting Specification

In order to check the robustness of our results, we also allow the long-run discount factors of both types of agents to differ. We let  $\beta_r = .99$  for the rational agent, which is the usual value used in the macroeconomics literature for time-consistent agents. For the hyperbolic agent, we calibrate his long-run discount factor,  $\beta_h$  in order to achieve equilibrium in the capital market. The value we obtain is 1.007, which implies that the hyperbolic agent is

more patient in the long-run (from any current time perspective).

The macroeconomic effects of raising the retirement age from 65 to 70 years old are reported in Table 4 below. We observe that the values we get in this exercise are quite similar to the previous results: aggregate output and consumption decline when the social security tax rate is held constant due to the effective increases in the present values of the agents' lifetime taxes; while aggregate output and consumption go up when the per capita amount of social security benefits is held constant.

**Table 3.4**

Macroeconomic Effects of Raising the Retirement Age (in %)

Regime	$r$	$w$	$\tau_{ss}$	$x$	$b$	$K$	$L$	$Y$	$C$
Fixed $\tau_{ss}$	15.74	-7.89	0	42.14	-15.74	-16.00	5.56	-2.77	-2.10
Fixed $x$	7.54	-4.01	-33.27	0	1.97	-3.90	7.66	3.35	3.46

The welfare effects of raising the retirement age are tabulated in Table 5 below. As in the previous calibration experiment, every agent gains when the per capita social security benefits are held constant, while the youngest cohorts (ages 20-34) lose from the reform when the social security tax rate is kept the same. The order of gains is still the same: the youngest cohorts gain the least, as they experience the highest increases in the present values of their lifetime taxes, while the oldest cohorts gain the most. However, in both experiments, we find that the young and middle-aged hyperbolic agents (ages 20 to 54 years old) gain more than their rational counterparts; while the old and retired hyperbolic agents gain less than the rational agents of the same age groups. This is in contrast to the previous result, where rational agents across all age groups uniformly gained more than their hyperbolic counterparts. The reason is that in the previous case, the rational agent was consistently more patient both in the short run and in long run from a given temporal perspective, and this allowed the rational agent to invariably save more and thus, enjoy

more long-run consumption and welfare, than the hyperbolic agent. Yet in this instance, the hyperbolic agent is characterized by greater long-run patience than the rational agent from any given time perspective. When the hyperbolic agent is young, his long-run horizon stretches out before him, so that that he is more patient than the rational agent in the long run gives him a greater advantage. However, as he becomes older and his long-run horizon starts to shorten, “long-run advantage” diminishes, as we observe that the difference between the welfare effects for the two types of agents decreases with age until age 64. Near and upon retirement, his remaining lifespan is short enough: because he is more impatient in the short run than the rational agent, he now tends to consume more and save less, which yields him a lower level of long-run welfare than that of his rational counterpart’s.

**Table 3.5**

Welfare Effects of Raising the Retirement Age (in %)

Constant $\tau_{ss}$			Constant $x$		
	$\delta = .7$	$\delta = 1$		$\delta = .7$	$\delta = .1$
age 20-24	-2.17	-4.13	age 20-24	3.18	2.18
age 25-34	-0.61	-2.12	age 25-34	3.96	3.20
age 35-44	1.51	0.59	age 35-44	5.01	4.55
age 45-54	3.67	3.29	age 45-54	6.07	5.88
age 55-64	5.88	6.00	age 55-64	7.14	7.20
age 65-85	9.18	9.90	age 65-85	8.73	9.06
$\bar{\phi}$	2.64		$\bar{\phi}$	5.54	

### 4.3 Privatizing Social Security

Table 6 summarizes the macroeconomic effects of social security privatization. We observe a considerable increase in aggregate savings of 54.05%, as privatization induces agents of all types to augment private savings in lieu of the reduction in the “forced savings” mandated

by the social security system. This increase in aggregate savings, in turn, increases the amount of unintended bequests in the economy by 58.70%. In equilibrium, the increase in the supply of capital due to the increase in aggregate savings reduces the real interest rate by 21.13%. The consequent increase in the equilibrium demand for capital is matched by an increase in the demand of labor, since capital and labor are complements in production. Although the reduction in the after-tax real wage rate induces an increase in aggregate labor supply, the outward shift in the demand for labor dominates so that the equilibrium real wage rate increases by 14.29%. Together, the increases in the aggregate levels of capital and labor increase output (by 21.50%) and thus, aggregate consumption (by 19.08%).

**Table 3.6**

Macroeconomic Effects of Social Security Privatization (in %)

$r$	$w$	$b$	$K$	$L$	$Y$	$C$
-21.13	14.29	58.70	54.05	6.31	21.50	19.08

Although the reduction in the post-reform real interest rate causes flatter consumption profiles, all agents still gain from privatization. This happens due to the upward shifting of consumption profiles across agent types (indexed by  $\delta$ ), as post-reform incomes rise due to the increase in labor incomes and the per capita amount of unintended bequests. As flatter consumption profiles mean higher consumption levels when young and lower consumption levels when old, privatization in this case implies greater welfare gains for younger workers than that for older workers and retired agents. Indeed, Table 7 shows that the youngest workers favor the pension system the least, while the retired agents favor it the most. As in Fehr et al. (2008), the youngest workers gain the most from having the highest reduction of the present value of their lifetime taxes from the elimination of the social security tax rate and from experiencing the greatest increase in the present value of lifetime unintended bequests, which together allow them to increase their lifetime consumption

the most. The retirees are disadvantaged by privatization the most, as this eliminates retirement benefits. This causes an abrupt reduction in post-retirement income, which severely constrains consumption. The surviving retirees lose further from privatization, as the social security system provides, what Imrohoroglu et al. (2003) refer to as an “actuarial reward for survival.” This ranking of welfare gains is consistent with Imrohoroglu et al. (2003) although in their setup even the retired rational agents gain from privatization mainly due to the presence of liquidity constraints that are relaxed upon the elimination of social security.<sup>16</sup> On an aggregate level, welfare gains accrue to the entire economy.

Except for the youngest workers, the more present-biased workers gain more from the privatization of social security. The youngest, more myopic workers gain less than their less myopic counterparts because greater myopia induces them to incur greater current debt, which leaves them less resources for next-period consumption. For the rest of the older, more myopic workers, the increase in disposable incomes due to the elimination of the social security tax, encourages them to increase consumption more than their less myopic counterparts so that they are able to experience higher welfare gains. Moreover, under a no social security regime, the rest of the more myopic agents know that the abrupt decline in their retirement consumption due to myopia would be aggravated by the absence of the commitment mechanism that the “forced savings” under social security afforded them. Since it is the hyperbolic agents that value post-retirement consumption more (or discount post-retirement consumption less than their rational counterparts), they are the ones that increase their asset holdings more all throughout their middle-age and latter working days. Consequently, consumption levels for the more myopic retirees do not fall as much.

We thus observe that younger cohorts benefit considerably more while retirees benefit considerably less under social security privatization than under either one of the aforementioned regimes that postpones retirement. The younger workers gain significantly from the relaxation of their liquidity constraints upon the abolition of social security taxes, while re-

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<sup>16</sup>Indeed for the same reason, Fehr et al. (2008) find that relaxing borrowing constraints reduces the long-run welfare gains from privatization.

irees either lose or gain to a lesser extent due to the complete elimination of their pension benefits. It thus appears that if the institution of these reforms were put to a vote, the younger workers would vote for privatization while the older workers and retirees would vote for postponing retirement. The voting outcome would depend on the welfare weights used in aggregating welfare. In this instance, using a simple weighted average places privatization at a greater advantage.

**Table 3.7**

Welfare Effects of Social Security Privatization (in %)

	$\delta = .7$	$\delta = 1$
age 20 – 24	18.28	18.32
age 25 – 34	15.66	15.36
age 35 – 44	12.26	11.54
age 45 – 54	8.99	7.90
age 55 – 64	5.85	4.41
age 65 – 85	1.50	-0.35
$\bar{\phi}$	9.89	

#### 4.4 Changing the Fraction of QHD Agents

In order to ascertain the externalities imposed by the existence of time-inconsistent discounters on the rest of the economy, we systematically change the fraction of hyperbolic agents while still allowing the policy reforms to take place. We denote the fraction of hyperbolic agents in the economy by  $\lambda_1 \in (0, 1)$ .

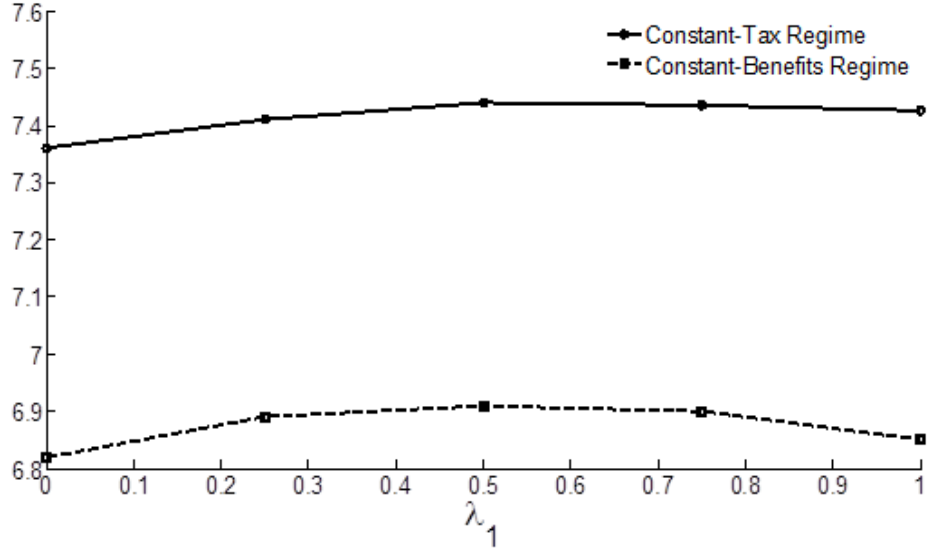
#### 4.4.1 Raising the Retirement Age

Figure 2 below shows that in both experiments, increasing the fraction of hyperbolic agents,  $\lambda_1$ , increases the equilibrium real interest rate up to a critical value and decreases it above this value. This happens because as the economy becomes more populated with myopic hyperbolic agents, undersaving is aggravated, which causes the equilibrium real interest rate,  $r$ , to rise. However, further increasing  $\lambda_1$ , which raises  $r$  even further, allows agents to work less via the income effect. This leads to a reduction in the equilibrium level of aggregate labor hours. The reduction in the demand for labor reduces the demand for aggregate capital, which consequently, tends to reduce the equilibrium real interest rate for higher values of  $\lambda_1$ . This is key to explaining the welfare patterns that are presented below.

In Table 8, we observe the welfare effects of raising the retirement age as we increase  $\lambda_1$  under a constant- $\tau_{ss}$  regime. When  $\lambda_1$  is low enough, the younger hyperbolic workers lose less than their rational cohorts after the reform. This is because increasing  $\lambda_1$  (when it is low enough) which raises  $r$ , increases the interest debt of the younger hyperbolic workers so that they could not afford to reduce their savings as much as their rational counterparts in response to the mandatory increase in the retirement age. This allows the former group to support a greater increase in equilibrium consumption, and thus, higher welfare gains. However, as  $\lambda_1$  increases further, the real interest rate starts decreasing, which lowers the interest debt payments of the young hyperbolic workers so that greater myopia now induces them to reduce savings more than their rational counterparts. Accordingly, the former group's post-reform welfare losses become higher than that of the latter. For the older sector of the workforce (who are now all savers), the less present-biased agents consistently gain more than their more myopic counterparts, as it is the former group that experience higher post-reform consumption levels from having steeper consumption profiles.



Figure 3.2: Real Interest Rate vs. Population Share of QHD Agents



For the younger rational workers (aged 20 to 34 years old) and most of the hyperbolic workers (aged 20 to 54 years old), the welfare pattern that we observe is as follows: the welfare gains (losses) decrease (increase) up to an age-specific critical value,  $\hat{\lambda}_{1,j}$  (for  $j = 1, \dots, J$ ) and increase (decrease) above this critical value. The reason is that when  $r$  increases as  $\lambda_1$  rises, agents are induced to save less as the income effect dominates after the retirement age is extended. This then reduces equilibrium consumption, and thus welfare. However, when  $r$  starts falling the further  $\lambda_1$  is increased, agents are now induced to save more (still via the income effect), which allows a greater amount of consumption to be supported in equilibrium. Thus, welfare increases. For the older workers and retirees, welfare levels monotonically increase with  $\lambda_1$  even when  $\lambda_1 < \hat{\lambda}_j$ . This is because when  $r$  increases as  $\lambda_1 (< \hat{\lambda}_{1,j})$  increases, these sectors of the workforce are the ones that are affected the most from the consequent reduction in the present value of their lifetime benefits. Thus, the higher  $r$  prompts them to save more, which allows them to consume more and experience higher welfare gains as  $\lambda_1$  rises. In the same line of reasoning, when  $r$  starts falling as  $\lambda_1$

( $> \widehat{\lambda}_{1,j}$ ) rises, the present value of their lifetime benefits rise considerably so that they are able to gain increasingly more after the reform. The overall welfare gains follow a U-shaped pattern as the economy becomes more populated by hyperbolic agents. This merely reflects the pattern that we observe for a majority of the age groups.

**Table 3.8**

Welfare Effects of Raising the Retirement Age (in %, constant  $\tau_{ss}$ )

	$\lambda_1 = 0$	$\lambda_1 = 0.1$		$\lambda_1 = 0.5$		$\lambda_1 = 0.9$		$\lambda_1 = 1$
	$\delta = 1$	$\delta = .7$	$\delta = 1$	$\delta = .7$	$\delta = 1$	$\delta = .7$	$\delta = 1$	$\delta = .7$
age 20 – 24	-3.20	-3.03	-3.32	-3.11	-3.12	-2.21	-1.94	-1.78
age 25 – 34	-1.40	-1.41	-1.48	-1.44	-1.24	-0.68	-0.20	-0.32
age 35 – 44	1.01	0.78	0.99	0.81	1.30	1.40	2.16	1.68
age 45 – 54	3.42	2.97	3.46	3.07	3.86	3.52	4.57	3.73
age 55 – 64	5.83	5.17	5.94	5.35	6.45	5.69	7.04	5.84
age 65 – 85	9.32	8.37	9.53	8.70	10.22	8.94	10.70	9.01
$\bar{\phi}$	2.55	2.54		2.63		2.92		3.07

Table 9 presents the results under a constant  $x$ . We generally observe the same U-shaped pattern of welfare gains (as  $\lambda_1$  increases) for rational agents aged 20 to 44 years old and for hyperbolic aged 20 to 54 years old. Also, just like in the previous experiment, the older rational workers experience monotonic increases in welfare as  $\lambda_1$  is increased. However, for the older and retired hyperbolic workers, welfare starts decreasing as  $\lambda_1$  is further increased above a certain threshold. This happens because unlike in the previous experiment where the corresponding reduction in  $r$  is met by a substantial increase in the present value of lifetime benefits, which enables retirees to increase equilibrium consumption and thus welfare, the hyperbolic retiree's welfare gains, in this case, start decreasing as  $r$  goes down due to two reasons: 1) the increase in the present value of lifetime pension

claims is not as high under a constant- $x$  regime than under a constant- $\tau_{ss}$  regime; and 2) myopia induces a reduction in savings in the face of even a slight increase in the present value of lifetime benefits. Accordingly, the older and retired hyperbolic agents experience falling consumption levels and thus, welfare gains, as  $\lambda_1$  approaches unity. In the aggregate, the welfare level from raising the retirement age increases as the economy becomes more populated with hyperbolic agents. Unlike in the previous regime where the aggregate welfare gain decreases as  $\lambda_1$  is increased below a certain critical level, in this case, the combination of a lower social security tax rate and an increase in  $r$  (due to a higher  $\lambda_1$ ) sufficiently reduces the present value of lifetime taxes so that the economy's welfare gains monotonically increase with  $\lambda_1$ .

**Table 3.9**

Welfare Effects of Raising the Retirement Age (in %, constant  $x$ )

Age	$\lambda_1 = 0$	$\lambda_1 = 0.10$		$\lambda_1 = 0.50$		$\lambda_1 = 0.90$		$\lambda_1 = 1$
	$\delta = 1$	$\delta = .7$	$\delta = 1$	$\delta = .7$	$\delta = 1$	$\delta = .7$	$\delta = 1$	$\delta = .7$
20 – 24	2.93	3.18	2.73	2.71	2.70	3.24	3.69	3.52
25 – 34	3.72	3.94	3.58	3.55	3.65	3.95	4.49	4.16
35 – 44	4.77	4.95	4.72	4.67	4.91	4.90	5.57	5.03
45 – 54	5.81	5.95	5.84	5.78	6.17	5.86	6.67	5.91
55 – 64	6.83	6.94	6.95	6.89	7.42	6.83	7.66	6.81
65 – 85	8.29	8.37	8.54	8.50	9.22	8.27	9.38	8.14
$\bar{\phi}$	5.42	5.44		5.54		5.60		5.62

In both cases, we observe that the highest aggregate welfare gains accrue to the economy with purely hyperbolic agents. This is because it is the more myopic, hyperbolic agents that benefit more in terms of greater increases in lifetime labor incomes from having the retirement age raised mandatorily. For time-inconsistent agents, the external imposition

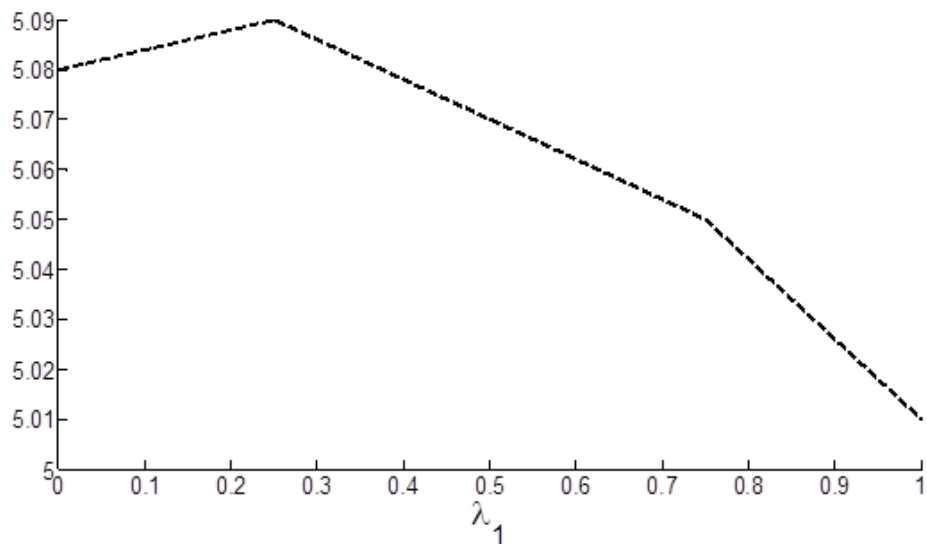
of an extended working-period horizon functions like another pre-commitment device that constrains them to increase lifetime savings.

#### 4.4.2 Privatizing Social Security

In Figure 3 below, we observe that as the fraction of hyperbolic agents is increased up to some critical value,  $\tilde{\lambda}_1$ , the real interest rate increases. However, further increasing  $\lambda_1$  above  $\tilde{\lambda}_1$ , reduces the real interest rate. As in the preceding simulated reform, this can be explained as follows: as the economy becomes inhabited by more myopic hyperbolic agents, aggregate undersaving becomes more severe, which pushes up the real interest rate,  $r$ . When  $r$  increases, both substitution and income effects come to play. While the substitution effect occasions an increase in savings due to the reduction in the relative price of future consumption, the income effect motivates a reduction in savings due to the increase in interest income. When  $r$  further rises as  $\lambda_1$  continues to increase, the substitution effect starts dominating, as there are now more hyperbolic agents, who have to save more in response to privatization. This now tends to reduce  $r$ . Moreover, increasing  $r$  further induces agents to reduce their labor hours so that the equilibrium level of aggregate labor hours decreases. The consequent decline in the equilibrium aggregate demand for labor reduces the aggregate demand for capital due to labor-capital complementarity in the production function. This also tends to reduce the equilibrium real interest rate as more and more hyperbolic agents start dominating the population.

Table 3.10 below shows that as the hyperbolic agents start crowding the economy (i.e., when  $\lambda_1 > .5$ ), rational agents start gaining more from privatization than the hyperbolic agents. This happens because the real interest rate falls sufficiently enough to dissuade the more myopic hyperbolic agent from increasing savings as much as that of his rational neighbor. This then allows the latter to experience a higher consumption level in equilibrium.

Figure 3.3: Real Interest Rate vs. Population Share of QHD Agents



For rational agents, increasing  $\lambda_1$  increases their welfare gains up to a critical level,  $\tilde{\phi}_{ij}$ , but reduces these gains above it. The reason is that when  $r$  increases as  $\lambda_1$  initially rises, these agents save more and are able to support a greater level of consumption in equilibrium. However, further increasing  $\lambda_1$ , which starts reducing the real interest rate, induces these agents to reduce savings, which reduces equilibrium consumption and thus, welfare. In this light, we remark that a sufficiently small number of hyperbolic agents generates positive pecuniary externalities on their rational counterparts, while one too many of these agents in the population imposes negative pecuniary externalities on rational agents. For hyperbolic agents, welfare decreases successively as myopia becomes more endemic in the economy.

The overall welfare gain,  $\bar{\phi}$ , decreases with  $\lambda_1$  below a critical value, while it increases with  $\lambda_1$  above this critical level. This pattern reflects the rise and fall of the equilibrium real interest rate as  $\lambda_1$  is increased (after its initial decline due to privatization). For our given set of parameters, a mixed population composed of half rational and half myopic agents generates the greatest overall welfare gains from privatization. It thus appears that

social security privatization becomes even more attractive in an economy composed of both rational and hyperbolic agents.

**Table 3.10**

Welfare Effects of Privatizing Social Security while Increasing  $\lambda_1$  (in %)

	$\lambda_1 = 0$	$\lambda_1 = 0.10$		$\lambda_1 = 0.5$		$\lambda_1 = 0.90$		$\lambda_1 = 1$
	$\delta = 1$	$\delta = .7$	$\delta = 1$	$\delta = .7$	$\delta = 1$	$\delta = .7$	$\delta = 1$	$\delta = .7$
age 20 – 24	18.25	19.32	18.31	18.28	18.78	16.47	17.51	15.74
age 25 – 34	15.19	16.59	15.26	15.66	15.36	14.03	14.73	13.39
age 35 – 44	11.25	13.07	11.35	12.26	11.84	10.84	11.11	10.29
age 45 – 54	7.53	9.72	7.64	8.99	8.14	7.73	7.60	7.25
age 55 – 64	3.99	6.51	4.11	5.85	4.60	4.68	4.18	4.24
age 65 – 85	-0.80	2.11	-0.66	1.50	-0.18	0.36	-0.58	-0.04
$\bar{\phi}$	9.14	9.43		9.89		9.50		8.40

We further note that in this particular parameterization, the retired hyperbolic agents in a mixed-population economy gain from privatization while retired hyperbolic agents loses in an economy purely populated by hyperbolic agents (i.e., when  $\lambda_1 = 1$ ). This is in line with our earlier observation that the higher  $r$  due to the pecuniary externalities generated by hyperbolic agents in a mixed economy allows these agents to save more in response to privatization. The higher post-reform savings by hyperbolic agents then allows them to support higher post-reform levels of retirement consumption.

## 5 Concluding Comments

This paper mainly contributes to the existing literature on the long-run viability of the social security system by evaluating the macroeconomic and welfare consequences of raising the

retirement age in the U.S and comparing these effects to those from privatizing the public pension system. We do this using a 65-period, general equilibrium, overlapping generations economy inhabited by both rational and quasi-hyperbolic agents, who face mortality risks. The co-existence of both rational and quasi-hyperbolic discounters allows us to ascertain the aggregate and welfare externalities generated by the latter agent type. By abstracting from borrowing constraints and income uncertainty, we derive an analytical solution to the system using the solution concept of first-order Markov perfect equilibria.

In raising the retirement age from 65 to 70 years old, we consider two alternative regimes: we either hold constant the social security tax rate or keep the amount of per capita social security benefits constant. In both cases, aggregate capital decreases, as the increases in the expected lifetime labor incomes induce agents to reduce savings, while aggregate labor increases from the compulsory extension of the working-age horizon. Accordingly, the reform's impacts on aggregate output and aggregate consumption are ambiguous, in contrast to the expectation that raising the retirement age would raise the standard of living. Our numerical results show that under a constant-benefits regime, aggregate output and consumption increase, while these decrease under a constant-tax rate regime.

On an aggregate level, this policy reform is welfare-increasing. However, under a constant-social security tax rate regime, the youngest workers incur welfare losses from the reform primarily due to the increase in their expected discounted lifetime tax liabilities and a considerable reduction in the amount of unintended bequests. Under a constant-benefits regime, agents of all ages and discounting types gain, as the subsequent reduction in the social security tax rate significantly relaxes their liquidity constraints. The quantitative results thus imply that postponing retirement under the latter regime is preferable to the former regime, unless some compensation scheme is instituted for the younger cohorts.

The presence of hyperbolic agents in the economy indeed imposes a pecuniary externality on the rest of the economy. In particular, we observe that the real interest rate exhibits an inverted U-shaped pattern as the fraction of hyperbolic agents is increased. As the percentage of hyperbolic agents increases, aggregate undersaving worsens, which causes the

real interest rate to rise. A rising real interest rate, however, depresses the real wage rate, which induces agents to work less so that equilibrium labor hours go down. Labor-capital complementarity in production reduces the demand for aggregate capital subsequent to the decrease in the demand for aggregate labor. This tends to reduce the equilibrium real interest rate as hyperbolic agents begin to crowd the economy.

We revisit the issue of social security privatization in our model economy. Interestingly, we find that the aggregate welfare gain follows the trajectory of the real interest rate as the fraction of hyperbolic agents is increased. We thus observe that social security privatization is most desirable with a mixed population of rational and hyperbolic agents. Moreover, we find that retired hyperbolic agents gain after privatization in a mixed economy, while they lose in a economy purely composed of time-inconsistent individuals. These observations imply that ignoring the externalities generated by hyperbolic agents may lead to spurious welfare conclusions.

In the context of raising the retirement age, the highest aggregate welfare gains accrue to the economy with a greater population of hyperbolic agents. This suggests that the mandatory postponement of retirement serves as another pre-commitment device for time-inconsistent agents to increase their lifetime savings.

Comparing the welfare effects of social security privatization to those of raising the retirement age, we find that the younger workers would prefer the former, while older workers and retirees would opt for the latter reform. That the aggregate welfare gain under privatization is greater than that under delayed retirement might be primarily because we have not considered the costs of transitioning to a fully privatized pension system in order to reference the long-run analyses of Imrohorglu et al. (2003). Moreover, we have abstracted from incorporating into the model the reported physiological and psychological benefits (see Butrica et al., 2006) that older agents gain from continuing to work during the “golden age years.” Taking account of these considerations may serve to strengthen the arguments in favor of raising the legal retirement age and can thus be a subject for future research.



## Chapter 4

# On Decreasing Marginal Impatience, Money Superneutrality and Stability

## 1 Introduction

The rate of time preference (RTP) is a crucial concept in economics. It is a reflection of the impatience that characterizes human nature – and thus, the individual’s tendency to impute lesser weight on future utility. It is upon this notion that the neoclassical rate of interest rests: a positive interest rate is required in order to induce impatient individuals to save, and ultimately invest in growth-enhancing activities.<sup>1</sup> Thus so, the RTP has been often viewed to encapsulate the individual’s tastes or preferences. However, it is often assumed to be a constant, exogenous parameter. Far from being an innocuous assumption, a constant RTP is a key to preserving the Classical dichotomy between the real and nominal sectors in both monetary models with money in the utility (MIU) and with a cash-in-advance (CIA) constraint on consumption (Sidrauski, 1967; Brock, 1974; Stockman, 1981; Suen and Yip, 2005; and Chen and Guo, 2007). A fixed RTP implies the absence of a channel through which the growth rate of money, and thus inflation, can affect the real interest rate, which should, in turn, affect investment and output. This phenomenon is what is referred to as money superneutrality.

Although previous attempts have been made to endogenize the RTP in monetary models, these papers have used an assumption known as “increasing marginal impatience” (IMI), which is purported to be both counterintuitive and contra-empirical (Koopmans, 1986; Barro and Sala-i-Martin, 1995; and Becker and Mulligan, 1997) yet is used extensively in the literature for it implies an unconditionally stable equilibrium. Accordingly, the main

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<sup>1</sup>In particular, Fisher (1930) equates the real rate of interest to the RTP, given that the former is equal to the relative price of present and future consumption, while the latter is determined by the marginal rate of substitution between present and future consumption properly adjusted for by the ratio of present and future consumptions.

objective of this paper is to model the RTP using the more intuitive and pro-empirical assumption called “decreasing marginal impatience” (DMI) and to verify its implications on money superneutrality and stability in both the MIU and CIA models. To our knowledge, this is the first attempt in the endogenous time preference literature that seeks to untangle the implications of DMI on money superneutrality and stability in these classes of models.

In the next subsections, we first establish both the theoretical and empirical justifications for endogenizing the RTP and for favoring the DMI assumption; and then proceed to give a preview of the results.

### **1.1 A Brief Literature Review**

We recall that in the Sidrauski (1967a) model (with an MIU function), the net marginal product of per capita capital equals the rate of time preference (RTP). Under perfect competition, the real interest rate equals the marginal product of capital net of depreciation. Thus, a constant RTP implies a constant real interest rate that only depends on unchanging tastes, technology (embodied by the production function) and the rate of depreciation. Consequently, given the Fisher equation, we observe that any increase in the inflation rate due to expansionary monetary policy is fully absorbed by the nominal interest rate, leaving the real interest and therefore, investment untouched – thereby preserving the dichotomy between the nominal and real sectors. Moreover, it can be shown that the growth rate of money does not leave any real effects, which renders money superneutral. Stockman (1981) shows that the same result holds in an economy with a CIA constraint imposed on consumption.

In contrast, Keynes’ theory of liquidity preference proposes a short-run positive effect of money on output through the interest rate. Given the individual’s desire for a certain level of liquidity, an increase in the money supply would compel him to reduce his money holdings in favor of bonds, which consequently bids up the price of bonds and depresses the rate of interest, spurring investment and output growth.

In line with Keynesian economics, Mundell (1963)<sup>2</sup> and Tobin (1965) derived a long-run role for money in a Solow-growth economy. Given the assumptions that individuals are sensitive to the composition of their portfolios and that money bears no interest, expected inflation negatively affects the real rate of interest. This creates a channel through which investment and thus output can increase. Mundell’s mechanism can be summarized as follows: inflation generated by an increase in the growth rate of money depreciates the individual’s real money balances and reduces real wealth. This in turn, increases savings in the form of greater bond holdings (the Scitovszky-Pigou-Wicksell effect), pressuring a drop in the real interest rate. In Tobin’s model, total wealth is defined as the sum of real money and real capital holdings. Accordingly, an increase in the money growth rate causes the individuals to shift from holdings of money to capital, compelling the real rate of interest to go down and growth-enhancing investment to go up. In the literature, this negative relationship between inflation and the real interest rate is dubbed the “Mundell-Tobin effect.”

As the long-run constancy of the real interest rate depends essentially on tastes and technology, it would thus seem that an RTP that responds to a monetary shock is crucial to unraveling the Fisher equation and thus, money neutrality. Indeed, efforts to model the RTP have allowed monetary growth to affect capital accumulation and economic growth. We look at two ways in which the literature models the RTP.

A strand in the literature endogenizes the RTP by specifying it as an increasing function of consumption, real income or real wealth. This is known as increasing marginal impatience (IMI). Kompas and Abdel-Razeq (2001) and Kam (2005) show a Mundell-Tobin effect in models with MIU where the RTP is an increasing function of consumption and real wealth, respectively; while Kam (2007) reports a Mundell-Tobin effect in an economy with a CIA constraint on both consumption and investment. The IMI assumption is primarily favored in the literature because it unconditionally implies a stable equilibrium (Uzawa,

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<sup>2</sup>To verify the Fisher effect, Mundell (1967) modified Metzler’s (1951) model to account for expected inflation instead of a one-shot increase in the price level and integrated (1) Fisher’s theory of the interest rate; (2) Pigou’s theory of wealth and savings; and (3) Keynes’ liquidity preference theory.

1968; Shi and Epstein, 1983; Epstein, 1987). However, this assumption has been criticized as counterintuitive because it implies that wealthier individuals are more impatient, and thus, save less (Koopmans, 1986; Barro and Sala-i-Martin, 1995; and Becker and Mulligan, 1997).<sup>3</sup> The empirical evidence disputes this: wealthier individuals do save more (Huggett and Ventura, 2000; Dynan, Skinner and Zeldes, 2004) and that this saving behavior is related to better saving habits due to lower rates of time preference.(Chakrabarty, et al., 2008).<sup>4</sup>

In contrast, the counter-assumption known as decreasing marginal impatience (DMI)<sup>5</sup> has the triad advantage of being 1) more intuitive, 2) consistent with empirical findings and 3) capable of generating a stable steady state given certain conditions. Indeed, Lawrence (1991) and Samwick (1998) both report that the RTP is higher for poorer households. Furthermore, Das (2003) shows that in a standard neoclassical, non-monetary, exogenous growth model with the RTP as a decreasing function of real consumption, the steady state is saddle-path stable if decreasing returns to capital are sufficiently large. The intuition is as follows: under DMI, a shock to the economy that increases consumption above its steady-state level reduces the individual's degree of impatience, and thus, the subjective cost of investing. This incites the individual to accumulate more capital, and thus wealth, which stimulates higher consumption, which in turn, reduces the RTP further. In order to pull the system back to equilibrium, the marginal product of capital should diminish at a rate that is faster than the rate at which the subjective cost of investing is declining so that the economy eventually reaches a point where the subjective cost of further capital accumulation becomes prohibitive. Other papers have also delved into the effects of DMI on

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<sup>3</sup>Fisher (1930) observes that the agent with near-subsistence income gives more weight to current needs relative to future ones as the "rational" part urges him to give more weight to his current needs in order to survive another day, while the "irrational" side makes him even more impatient to consume as "the pressure of present needs blinds a person to the needs of the future." (p. 72).

<sup>4</sup>Chakrabarty, et al. (2008) make a case for allowing agents in their model to choose their patience levels by choosing the amount of health goods that they consume: the more patient the agent is, the more he would decide to spend on future-oriented health-improving goods. They test the empirical implications of their model using Australian data.

<sup>5</sup>RTP is a decreasing function of consumption, real wealth or lifetime utility.

issues such as capital taxation, income distribution and asset pricing (Das et al., 2004; Nath, 2006; and Hirose and Ikeda, 2008) and have shown that the DMI assumption generates more reasonable implications. Moreover, as the IMI assumption is theoretically shown to imply a positive relation between the money growth rate (and thus, the inflation rate) and economic growth, it seems reasonable to conjecture that the DMI assumption would be consistent with the opposite. The intuition for the latter is as follows, when inflation reduces either real consumption, real income or real wealth, the agent becomes more impatient, which then discourages him from saving. In the aggregate, this slows down capital accumulation, which adversely affects aggregate output.

For decades, the relationship between inflation and output growth has been a subject for empirical debate with little promise of a consensus being reached. In the 1960s, when inflation rates were low and output growth rates were high, inflation was purported to have a positive impact on growth (the Mundell-Tobin effect). This was supported by the anecdotal evidence in the developing countries at that time such as Israel and several growing economies in Asia and in Latin America (Bruno and Easterly, 1996). However, more formal cross-country studies did not reach a convincing consensus (See, for example, Wai (1959), Dorrance (1963) and (1966), Bhatia (1960) and Wallich (1969)). Following the stagflationary episode of the 1970s, cross-country pooled time-series regressions in the 1980s and 1990s have generally found a significantly negative relationship between inflation and growth (Kormendi and Meguire, 1985; Cardoso and Fishlow, 1989; De Gregorio, 1992; Roubini and Sala-i-Martin, 1992; Fischer, 1993; Corbo and Rojas, 1993; and Barro, 1995). However, cross-section inflation and growth regressions fail to conclusively establish an inflation-growth nexus (Levine and Renelt, 1992; Levine and Zervos, 1993 and Easterly and Bruno, 1996).

## **1.2 A Preview of the Results**

We thus evaluate the implications of DMI on the inflation-growth nexus (i.e., money superneutrality) and stability in models with money in the utility (MIU) and cash in-advance

(CIA) constraints imposed on consumption and investment. In this endeavor, we allow the RTP to be decreasing functions of some alternative variables that have been used previously in the literature: 1) real consumption (Das, 2003); 2) real capital (Becker and Mulligan, 1997); and 3) real wealth (Kam, 2005).<sup>6</sup> While the first two specifications have only been used in non-monetary models, the third one is used in an MIU model under the IMI assumption.

We find that money is indeed not superneutral even in the model with a CIA constraint imposed solely on consumption, as an endogenous RTP creates a channel through which inflation can affect real variables via the changes that it works on marginal impatience. In general, we find that the growth rate of real money – and thus, inflation – can have both first-order and second-order effects on the RTP, which ultimately affect capital accumulation and output. The first-order effect operates as follows: an increase in the money growth rate, which increases the steady-state inflation rate, reduces real money balances and thus, real wealth. Under DMI, this decreases the instantaneous RTP, which induces the agent to save less, and therefore, accumulate less capital, which in turn, reduces steady-state output. The second-order effect sets in when higher inflation, which reduces real wealth, increases the rate at which marginal impatience declines, so that the agent is induced to accumulate more capital, which raises steady-state output. Thus, the two effects work in opposite directions: whichever effect dominates determines whether or not inflation has a negative or positive effect on output. We call this total effect of inflation on marginal impatience as the *subjective cost of investing*. Thus, when the subjective cost of investing increases due to higher inflation, capital accumulation and thus, output decrease; while the opposite ensues when the subjective cost of investing decreases due to higher inflation. To determine the direction of the impact of inflation on output, we calibrate the model.

The numerical outcomes mirror the state of the inflation-growth literature: we find that even under DMI, how the RTP is specified influences how inflation affects growth. Higher

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<sup>6</sup>Becker and Mulligan (1997) espouse the concept of an RTP that is a negative function of the amount of resources that the agent spends on any object or activity that would make the future seem less remote. In this paper, we interpret this, as spending on future-oriented capital.

inflation has a negative effect on output in both the model with MIU and the one with a CIA constraint on both consumption and investment. This holds for a wide range of inflation rate values. The explanation is that inflation, which erodes the agent's real wealth, raises his marginal impatience to the point where the subjective cost of investing also increases. This discourages capital accumulation and subsequently reduces steady-state output in the economy. This negative effect of inflation on output that is channeled through the RTP, reinforces the adverse effect of inflation that works through the CIA constraint.<sup>7</sup>

For the model with a liquidity constraint imposed solely on consumption, this negative inflation-growth nexus is still observable for lower rates of inflation in cases where DMI is in terms of real capital or real wealth. However, for higher inflation rates and for the case where the RTP is a sole diminishing function of real consumption, inflation has a positive effect on output. The intuition works as follows: at lower rates of inflation, the *shadow cost of marginal patience* increases enough so that the agent is induced to intertemporally substitute future capital for more future consumption, which increases the RTP. This subsequently increases the agent's subjective cost of investing, which further reduces the steady-state levels of both capital and output. During higher inflationary episodes, the portfolio substitution effect dominates: the agent starts economizing on his real money holdings, and instead, increases his real capital assets. This reduces the subjective cost of investing, which then spurs capital accumulation and output growth. This latter effect is consistent with the Mundell-Tobin assertion that during higher inflationary episodes, agents prefer to hold their wealth in the form of capital rather than monetary assets.

The steady states generated in all the models are saddle-path stable for a wide range of inflation rate values. An increase in the instantaneous RTP causes the agent to save less, and thus, accumulate less capital in the steady state. However, the rise in the instantaneous RTP also affects the way the individual discounts future utilities *in the future*. In particular, the increase in the instantaneous RTP also raises the entire stream of future rates of time

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<sup>7</sup>The theoretical underpinnings are provided by De Gregorio (1993) and Jones and Manuelli (1993) who assert that inflation works like a tax on capital in models with cash in-advance constraints on investment.

preference. This, in turn, reduces the value of the stream of discounted future utilities to the point where the agent shifts resources away from future consumption to future-oriented capital. Consequently, steady-state capital and output rise, so that system tends towards a locally stable equilibrium.

The rest of the paper is organized as follows: Section 2 incorporates the DMI assumption into the MIU model and examines its implications on money superneutrality and stability. Section 3 presents the models with a cash-in-advance constraint on both consumption and investment and on consumption only. Section 4 concludes.

## 2 Money-in-the-Utility Model

### 2.1 The Rate of Time Preference

To facilitate the discussion, we model the rate of time preference as a decreasing function of  $s$ , which we define to be the weighted sum of real consumption,  $c$ , real capital,  $k$ , and real money holdings,  $m$ :

$$s_t = \gamma_c c_t + \gamma_k k_t + \gamma_m m_t, \quad \gamma_c, \gamma_k, \gamma_m \in [0, 1]. \quad (1)$$

When  $\gamma_c = \gamma_k = \gamma_m = 0$ , we recover the Sidrauski (1967a) model. The case  $\gamma_c = 1$ , while  $\gamma_k = \gamma_m = 0$ , gives us Das' (2003) specification for the RTP; while the case  $\gamma_k = \gamma_m = 1$ , and  $\gamma_c = 0$  is consistent with Kam (2005). When  $\gamma_k = 1$  and  $\gamma_c = \gamma_m = 0$ , we have an interpretation of the Becker and Mulligan (1997) model, where the RTP is modeled as a function of the size of the investment that the agent makes in order to bring the future closer to the present.<sup>8</sup>

The discounting function is thus given by

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<sup>8</sup> "... People may also purchase disciplinary devices, such as a piggy bank or membership in a Christmas Club which help a person sacrifice current consumption. Financial instruments such as piggy banks involve a cost-forgone interest-but can be beneficial if they are successful at diverting one's attention toward the future..." (Becker and Mulligan, 1997, p. 735)



$$\beta(t) = \int_0^t \rho(s_\tau) d\tau \quad (2)$$

where the instantaneous rate of time preference,  $\rho(\cdot)$ , is real-valued, twice continuously differentiable on  $\mathbb{R}_{++}$ . Under DMI, the following assumptions apply:  $\rho(\cdot) > 0$ ,  $\rho'(\cdot) < 0$  and  $\rho''(\cdot) > 0$  for all  $s_t > 0$ . Furthermore, we impose an upper bound,  $\rho(0) = \bar{\rho}$ . Thus, the discount factor evolves according to the law of motion:

$$\dot{\beta}_t = \rho(s_t) \quad (3)$$

## 2.2 The Agent's Problem

The economy is inhabited by a continuum of identical, infinitely-lived individuals, with the number of individuals being normalized to unity. With perfect foresight, the representative agent does a constrained maximization of his lifetime utility. His lifetime discounted utility is given by:

$$\int_0^\infty u(c_t, m_t) e^{-\beta(t)} dt, \quad (4)$$

where  $c_t$  and  $m_t$  represent his consumption and real money holdings at time  $t$ . The function  $u(\cdot)$  is the agent's instantaneous utility, which is characterized by strictly increasing marginal utility with respect to consumption and real money balances, i.e.,  $u_c > 0$  and  $u_m > 0$ , respectively; and diminishing marginal utility in consumption and real money balances:  $u_{cc} \leq 0$  and  $u_{mm} \leq 0$ , respectively. Instantaneous utility,  $u(\cdot)$ ,  $\forall c$  and  $m$ , has to be positive in order to keep the shadow cost of impatience positive, as we shall see more clearly later on.

The agent's budget constraint is:

$$c_t + \dot{m}_t = y_t + x_t - \pi_t m_t, \quad (5)$$

where  $i_t$  is investment,  $y_t$  represents exogenous income earned,  $x_t$  is lump-sum transfers and  $\pi_t$  is the inflation rate.

On the production side, output ( $y_t$ ) is produced according to the production function:

$$y_t = f(k_t), \quad (6)$$

where  $f(\cdot)$  is the neoclassical production function with  $f' \geq 0$  and  $f'' \leq 0$ . The variable  $k_t$  stands for capital, which evolves according to the law of motion:

$$\dot{k}_t = i_t - \delta k_t, \quad (7)$$

where  $\delta \in [0, 1]$  is the constant depreciation rate of capital.

To close the model, we assume that nominal money at time  $t$ ,  $M_t$ , changes according to the money supply rule:

$$M_t = M_0 e^{\mu t}, \quad M_0 > 0, \quad (8)$$

where  $\mu \neq 0$ , is the constant money growth rate rule. The corresponding seignorage from inflation is given back to the agents so that  $x_t = \mu m_t$ .

The agent thus maximizes his lifetime discounted utility given by (4) subject to constraints (1), (3), (5), (6), (7) and given the initial conditions for capital,  $k_0$ , and nominal money holdings,  $M_0$ . We employ the co-state variables  $\lambda_{\beta t}$ ,  $\lambda_{m t}$  and  $\lambda_{k t}$  for (3), (5) and (7), respectively.

The first-order conditions for an interior solution are given by

$$u_c = \lambda_{k t} - \gamma_c \lambda_{\beta t} \rho'_t, \quad (9)$$

$$\lambda_{k t} = \lambda_{m t}, \quad (10)$$

$$\dot{\lambda}_{kt} = -\lambda_{kt}(f_k - \delta - \rho_t) + \gamma_k \lambda_{\beta t} \rho_t', \quad (11)$$

$$\dot{\lambda}_{mt} = -u_m + \lambda_{mt}(\rho_t + \pi_t) + \gamma_m \lambda_{\beta t} \rho_t', \quad (12)$$

$$\dot{\lambda}_{\beta t} = \lambda_{\beta t} \rho_t - u(c_t, m_t), \quad (13)$$

with the transversality conditions

$$\lim_{t \rightarrow \infty} e^{-\beta(t)} \lambda_{kt} k_t = \lim_{t \rightarrow \infty} e^{-\beta(t)} \lambda_{mt} m_t = 0. \quad (14)$$

Condition (9) equates the marginal benefits of consumption to its net marginal cost: the marginal utility of consumption equals the utility value of real money balances. Condition (10) means that the shadow prices of capital and real balances are equal. Conditions (11), (12) and (13) define the equations for the evolution of the shadow prices of capital, real money balances and impatience, respectively.

From (10) and (11), we derive the condition:

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = f_{kt} - \delta + \pi_t + (\gamma_m - \gamma_k) \frac{\lambda_{\beta t}}{u_c} \rho_t'. \quad (15)$$

When  $\gamma_m = \gamma_k = 1$ , we have the standard condition that the marginal rate of substitution between real money balances and consumption equals the nominal interest rate.

In equilibrium, we have

$$\dot{m}_t = (\mu - \pi_t) m_t, \quad (16)$$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t, \quad (17)$$

where (16) is the real money market equilibrium condition, while (17) is the goods market

equilibrium condition.

The dynamic equations for consumption, real money holdings, capital, and the shadow price of impatience are given, respectively, by

$$\dot{k}_t = f(k_t) - \delta k_t - c_t, \quad (18)$$

$$\dot{c}_t = -\frac{1}{u_{cct}} \left\{ u_{ct}(f_{kt} - \delta - \rho_t) - \gamma_k \lambda_{\beta t} \rho'_t + u_{cm} m_t \left[ \begin{array}{l} \mu - \frac{u_{mt}}{u_{ct}} + f_{kt} - \delta \\ + (\gamma_m - \gamma_k) \frac{\lambda_{\beta t}}{u_c} \rho' \end{array} \right] \right\}, \quad (19)$$

$$\dot{m}_t = \left( \mu - \frac{u_m}{u_c} + f_k - \delta + (\gamma_m - \gamma_k) \frac{\lambda_{\beta t}}{u_c} \rho' \right) m_t, \quad (20)$$

$$\dot{\lambda}_{\beta t} = \lambda_{\beta t} \rho_t - u(c_t, m_t). \quad (21)$$

Equations (18) to (21) represent a four-by-four system of differential equations involving  $c_t$ ,  $k_t$ ,  $m_t$  and  $\lambda_{\beta t}$ . Along a convergent path,  $\lambda_{\beta t}$  represents the aggregate future utility, that is,

$$\lambda_{\beta t} = \int_t^\infty u(c_\tau, m_\tau) e^{-\int_t^\tau \rho(s_v) dv} d\tau. \quad (22)$$

Equation (22) equates the shadow cost of marginal patience at time  $t$  to the marginal benefit of foregoing current consumption, which equals the agent's foregone stream of discounted future utilities.

### 2.3 Superneutrality

In the steady-state, we obtain

$$c = f(k) - \delta k, \quad (23)$$

$$\frac{u_m}{u_c} = f_k - \delta + \pi + (\gamma_m - \gamma_k) \frac{\lambda_\beta}{u_c} \rho' \quad (24)$$

$$\pi = \mu, \quad (25)$$

$$\lambda_\beta = \frac{u(c, m)}{\rho(s)}, \quad (26)$$

$$f_k - \delta = \rho(s) + \gamma_k \frac{\lambda_\beta}{\lambda_k} \rho'(s). \quad (27)$$

Equations (23) to (27) characterize, respectively, the steady-state levels of consumption, real money balances, the inflation rate, the shadow price of capital and the real interest rate. In particular, (25) shows that the steady-state inflation rate equals the constant money growth rate. Meanwhile, (26) reveals that the steady-state shadow price of impatience equals the steady-state present value of instantaneous utility.

Equation (27) is the key to money non-superneutrality. We remark that when  $\rho$  is a constant exogenous parameter, (27) is reduced to the Sidrauski (1969) result (i.e.,  $f_k - \delta = \rho$ ), which preserves the classical dichotomy. However, with endogenous time preference, inflation can have both “first-order” and “second-order” effects on impatience, which then affects equilibrium capital accumulation and output. The “first-order” effect of inflation is channeled through the first term of the right-hand side of (27): an increase in the inflation rate erodes the value of  $s$ . The decline in  $s$ , in turn, induces the individual to accumulate less capital, which results in lower steady-state output. The second-order effect (the second term of the right-hand side of (28)) works in the opposite direction: the decline in  $s$  due to a higher inflation rate increases the rate at which marginal impatience declines (i.e.,  $\rho'$  becomes more negative) – valued at the utility value of impatience measured in terms of the utility value of capital (i.e.,  $\lambda_\beta/\lambda_k$ ) – which effectively reduces the subjective cost of investing so that the individual is encouraged to save more – thereby increasing the capital

stock, and thus, output. If the first-order effect dominates, then inflation has a negative effect on output, whereas, inflation will have a positive effect on output if the second-order effect is stronger. Incidentally, these two opposing effects are also key to the attainment of saddle-path stability, as shall be discussed in greater detail later on. We can thus consider the sum of the first- and second-order effects of marginal impatience as the subjective cost of investing.

Linearizing (23) to (27) around the steady state, we obtain the following system of equations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} dk \\ dm \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\mu, \quad (28)$$

where

$$\begin{aligned} a_{11} &= \frac{1}{u_c} \left\{ \begin{aligned} &\left\{ \omega_c - (\gamma_m - \gamma_k) \left[ \gamma_c \lambda_{\beta} \rho'' - \frac{u_{cc}}{u_c} \lambda_{\beta} \rho' \right] \right\} (f_k - \delta) \\ &- \{ u_c f_{kk} + \gamma_k (\gamma_m - \gamma_k) \lambda_{\beta} \rho' \} \\ &- \rho' (\gamma_m - \gamma_k) \{ (u_c - \gamma_c \lambda_{\beta} \rho') (f_k - \delta) - \gamma_k \lambda_{\beta} \rho' \} \end{aligned} \right\}, \\ a_{12} &= \frac{1}{u_c} \left\{ \omega_m - (\gamma_m - \gamma_k) \left[ \gamma_m \lambda_{\beta} \rho'' - \frac{u_{cm}}{u_c} \lambda_{\beta} \rho' \right] \right\} - \left( \frac{\gamma_m - \gamma_k}{u_c} \right) (u_m - \gamma_m \lambda_{\beta} \rho') \rho', \\ a_{21} &= \left\{ \begin{aligned} &(f_k - \delta) \left[ \left( \frac{\gamma_k \gamma_c}{u_c} \right) \lambda_{\beta} \rho'' - \gamma_k \left( \frac{u_{cc}}{u_c^2} \right) \lambda_{\beta} \rho' + \gamma_c \rho' \right] + \left( \frac{\gamma_k^2}{u_c} \right) \lambda_{\beta} \rho'' + \gamma_k \rho' \\ &- f_{kk} + \rho' \left( \frac{\gamma_k}{u_c} \right) \{ (u_c - \gamma_c \lambda_{\beta} \rho') (f_k - \delta) - \gamma_k \lambda_{\beta} \rho' \} \end{aligned} \right\} \\ a_{22} &= \left( \frac{\gamma_k}{u_c} \right) \left\{ \gamma_m \lambda_{\beta} \rho'' - \left( \frac{u_{cm}}{u_c} \right) \lambda_{\beta} \rho' + \rho' [u_m - \gamma_m \lambda_{\beta} \rho'] \right\} + \gamma_m \rho', \\ \omega_c &= (u_m/u_c) u_{cc} - u_{cm}, \\ \omega_m &= (u_m/u_c) u_{cm} - u_{mm}. \end{aligned}$$

The impact of the growth rate of real money on the rest of the other variables can be derived as follows:

$$\frac{dc}{d\mu} = (f_k - \delta) \frac{dk}{d\mu}, \quad (29)$$

$$\frac{dy}{d\mu} = f_k \frac{dk}{d\mu}, \quad (30)$$

and

$$\frac{d\lambda_\beta}{d\mu} = (u_c - \gamma_c \lambda_\beta \rho') \frac{dc}{d\mu} - \gamma_k \lambda_\beta \rho' \frac{dk}{d\mu} + (u_m - \gamma_m \lambda_\beta \rho') \frac{dm}{d\mu}, \quad (31)$$

where (29) to (31) describe the effect of a marginal change in the growth rate of real money on real consumption, real output and the shadow cost of marginal impatience, respectively.

## 2.4 Numerical Examples

As the signs of  $\frac{dk}{d\mu}$ ,  $\frac{dm}{d\mu}$ ,  $\frac{dc}{d\mu}$  and  $\frac{d\lambda_\beta}{d\mu}$  cannot be determined analytically, we impose the following specific functional forms on the discount factor, the utility function and the production function:

$$\rho = \bar{\rho} e^{-\theta(s)}, \quad \bar{\rho}, \theta > 0, \quad (32)$$

$$u(c, m) = \varepsilon \log(c) + (1 - \varepsilon) \log(m), \quad \varepsilon \in (0, 1) \quad (33)$$

$$f(k) = k^\alpha, \quad \alpha \in (0, 1) \quad (34)$$

We then calibrate the model. We let the share of per capita capital in output,  $\alpha$ , to be equal to .36. The capital stock is assumed to depreciate at a rate of 10% (i.e.,  $\delta = .1$ ). We fix the preference parameter,  $\varepsilon$ , to equal .5 so that the weights of consumption and real money balances in the utility are the same. To keep both the analytical and numerical exercise more tractable, we assume log utility in order to keep the utility function separable in terms of real consumption and real money holdings.<sup>9</sup> The exogenous RTP parameter,  $\bar{\rho}$ , equals .05, which is consistent with an average annual real interest rate of 5%, while  $\theta$  is set at .001.<sup>10</sup> We let the inflation rate,  $\pi$ , range from 0 to 1.

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<sup>9</sup>With a utility function of the form  $u = \frac{(c^\varepsilon m^{1-\varepsilon})^{1-\sigma} - 1}{1-\sigma}$ , we cannot allow  $\sigma > 1$ , as this would imply negative steady-state values of the shadow price of impatience,  $\lambda_\beta$ . As the case  $\sigma < 1$ , is not consistent with the empirical literature, we settle for the case  $\sigma = 1$ . Hence, the log specification. For the parameter values that were used  $u(c, m)$  is positive.

<sup>10</sup>In conducting the numerical exercise, we allowed  $\theta$  to take on the values .01, .05, .5, 1 and 5. However, these values returned negative values of the real interest rate for allowable values of  $\sigma$ .

We then allow the RTP to embody three specifications that have been previously used: as a function of consumption (Kompas and Abdel-Razeq, 2001; KA (2001), *hereafter*), of capital (Becker and Mulligan, 1997) and of real wealth (Kam, 2005 and 2007). The numerical results are reported in Table 1 below. The second column reports the results under DMI, while the third one shows that under IMI. Under DMI, we look at the effects of the money growth rate on real variables when  $\gamma_c = 1$  and  $\gamma_k = \gamma_m = 0$ , i.e., when the RTP is assumed to be a decreasing function of real consumption,  $c$  (the first sub-column). The second sub-column shows the results for when  $\gamma_k = 1$  and  $\gamma_c = \gamma_m = 0$ , i.e., when the RTP is a decreasing function of real capital,  $k$ ; while the third sub-column under DMI gives the results for the case where  $\gamma_k = \gamma_m = 1$  and  $\gamma_c = 0$ , i.e., when the RTP is a decreasing function of real wealth, defined as the sum of real capital and real money holdings,  $m$ . Under IMI, we report the corresponding results in KA (2001) and Kam (2005), which also look at these issues in the context of a standard neoclassical growth model with perfect foresight and money in the utility function.

Table 4.1 Results for the MIU Model

	DMI			IMI
	$\rho(c)$	$\rho(k)$	$\rho(k+m)$	$\rho(k+m)$ (Kam, 2005)
$dk/d\mu$	< 0	< 0	< 0	> 0
$dc/d\mu$	< 0	< 0	< 0	> 0
$dm/d\mu$	< 0	< 0	< 0	< 0
$dy/d\mu$	< 0	< 0	< 0	> 0
No. of Real Roots < 0	1	1	1	1
Valid $\pi$ -Range	[0,1]	[0,1]	[0,1]	[0,1]

CASE 1:  $\gamma_c = 1$  and  $\gamma_k = \gamma_m = 0$ . When the RTP is a decreasing function solely of



real consumption, (27) becomes

$$f_k - \delta = \rho(c). \quad (35)$$

At the onset of higher inflation due to an increase in the growth rate of money, the agent economizes on his holdings of real money balances. The reduction in the demand for real money balances reduces the agent's overall real wealth. This occasions a subsequent decline in consumption goods, which then increases impatience or the RTP. The increase in the RTP consequently reduces steady-state capital and thus, output and consumption.

CASE 2:  $\gamma_k = 1$  and  $\gamma_c = \gamma_m = 0$ . When the RTP is a decreasing function solely of real capital, (27) becomes

$$f_k - \delta = \rho(k) + \frac{\lambda_\beta}{u_c} \rho'(k). \quad (36)$$

From Table 1, we see that when inflation kicks in, the opportunity cost of holding money balances increases. This reduces the agent's real wealth, which subsequently influences him to decrease his savings or capital holdings. This then increases the subjective cost of investing, which further reduces savings and thus, steady-state capital and output.

CASE 3:  $\gamma_k = \gamma_m = 1$  and  $\gamma_c = 0$ . When the RTP is a decreasing function solely of real capital, (27) becomes

$$f_k - \delta = \rho(k + m) + \frac{\lambda_\beta}{u_c} \rho'(k + m). \quad (37)$$

The results from Table 1 show when the RTP is a decreasing function of real wealth, a higher inflation rate has a negative impact on steady-state real money balances, consumption and capital, indicating that the first-order effect dominates the second-order effect. The intuition is as follows: at the onset of inflation, real wealth, defined as the sum of real capital and real money balances, goes down. The reduction in the level of real wealth increases marginal impatience, and thus, the subjective cost of investing, which discourages

saving. Consequently, the steady-state levels of both capital and output decline.

Under IMI, however, the same increase in the growth rate of money, which reduces real money holdings, in turn, reduces real wealth (Kam, 2005). This decreases the RTP, which then increases capital accumulation and thus, output and consumption.

We thus see that all three cases under DMI show that an increase in the rate of money growth adversely affects capital accumulation and thus, output growth. These results are consistent with the empirical evidence that shows a long-run negative relationship between inflation and growth. This result is supported by empirical studies that estimate a long-run negative relationship between inflation and output growth across countries all over the world (Kormendi and Meguire, 1985; Grier and Tullock, 1989; Levine and Renelt, 1992; Roubini and Sala-i-Martin, 1992; De Gregorio, 1993; and Barro, 1995).

## 2.5 Stability

Linearizing (11), (18), (20) and (21) around the steady state and evaluating all derivatives at the steady state, we obtain

$$\begin{bmatrix} \dot{\lambda}_{kt} \\ \dot{k}_t \\ \dot{m}_t \\ \dot{\lambda}_{\beta t} \end{bmatrix} = \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}}_B \begin{bmatrix} c_t - c \\ k_t - k \\ m_t - m \\ \lambda_{\beta t} - \lambda_{\beta} \end{bmatrix}, \quad (38)$$

where the elements of the Hessian matrix  $B$  are listed in Appendix C.

Since we have one predetermined variable,  $k_t$ , while  $m_t$ ,  $c_t$  and  $\lambda_{\beta t}$  are jump variables, local stability requires that the number of eigenvalues of  $B$  with negative real parts is not less than the number of predetermined variables in the system – in this case, one. Under IMI, Kam (2005) shows that the steady-state equilibrium is saddle-path stable for all values of  $\pi$ . The explanation is that an increase in the inflation rate, which reduces real money holdings and thus, real wealth, also reduces the RTP. The decline in the RTP, in turn,

increases capital accumulation and output, which ultimately counters the initial decline in real wealth.

Under DMI, Table 1 also shows that the steady states are stable for  $\pi \in [0, 1]$  for all specifications of the RTP. Moreover, for all cases, the steady states are locally saddle-path stable, as the number of negative real roots in each case exactly equals one. When the RTP is a function of either real capital or real wealth, local stability occurs because when real wealth decreases (due to higher inflation), the RTP increases, which discourages savings, and thus, capital accumulation and output. However, changes in the instantaneous RTP also affect that way the agent discounts the future *in the future*. In this case, the same increase in the instantaneous RTP also increases the future stream of time preference. This, consequently, reduces the discounted value of future utilities ( $\lambda_\beta$ ), which renders future consumption *in the future* less valuable to the point where the individual shifts future resources away from consumption to future-oriented capital so that steady-state capital increases. Thus, even in a monetary economy with the RTP as a decreasing function of real capital holdings or real wealth, the steady state can be saddle-path stable.

In the case where the RTP is a sole function of consumption, saddle-path stability occurs even when the second-order effects are absent. As we have seen earlier on, when consumption declines due to higher inflation, the RTP increases, which induces reductions in the steady-state levels of real capital and output and thus, further reductions in steady-state consumption. However, as steady-state consumption further decreases, the discounted value of future utilities also declines, which renders future consumption less valuable *in the future* so that the agent shifts future resources towards future-oriented capital at the expense of future consumption. This then tends to increase steady-state capital, and thus, output and consumption.

### 3 Cash-in-Advance Model

In this model, the agent does not derive direct utility from real money balances, but holds them, nonetheless, for the sole purpose of conducting transactions. In particular, he uses his real money holdings to finance either his consumption or investment. Thus, his CIA constraint takes the form generalized by Wang and Yip (1992) and Palivos et al. (1993):

$$c_t + \phi i_t \leq m_t, \quad (39)$$

where  $i_t$  is per-capita investment and  $\phi \in \{0, 1\}$ . When  $\phi = 0$ , (39) becomes the Clower constraint, while the case  $\phi = 1$  is consistent with Stockman (1981). It is assumed that the CIA constraint binds so that (39) becomes an equality.

With the rest of the variables defined as in the previous section, the representative agent solves the following constrained maximization problem:

$$\max \int_0^{\infty} u(c_t) e^{-\beta(t)} dt \quad (40)$$

$$\text{such that } \dot{m}_t = f(k_t) - \pi_t m_t + x_t - c_t - i_t, \quad (41)$$

$$\dot{k}_t = i_t - \delta k_t, \quad (42)$$

$$\dot{\beta}_t = \rho(s_t) \quad (43)$$

$$c_t + \phi i_t = m_t, \quad (44)$$

where time preference is specified as in (2) and the rest of the variables are as previously defined. The initial conditions for capital and nominal money balances are given, respectively by  $k_0$  and  $M_0$ . Constraints (41) to (43) describe the evolution of real money balances,

capital, and the discount factor, respectively.

The hamiltonian is given by

$$H_t = u(c_t) + \lambda_{mt} [f(k_t) + x_t - i_t - c_t - \pi_t m_t] + \lambda_{kt} [i_t - \delta k_t] + \psi_t [m_t - c_t - \phi i_t] - \lambda_{\beta t} \rho(s_t), \quad (45)$$

where  $\lambda_{mt}$ ,  $\lambda_{kt}$ ,  $\lambda_{\beta t}$  are the shadow prices of money, capital and impatience, respectively and  $\psi_t$  is the utility cost of relaxing the CIA constraint.

The first-order conditions for an interior solution are given by:

$$u_c = \lambda_{mt} + \psi_t + \gamma_c \lambda_{\beta t} \rho' \quad (46)$$

$$\lambda_{kt} = \lambda_{mt} + \phi \psi_t \quad (47)$$

$$\dot{\lambda}_{kt} = -\lambda_{mt} f_k + \lambda_{kt} (\rho_t + \delta) + \gamma_k \lambda_{\beta t} \rho_t' \quad (48)$$

$$\dot{\lambda}_{mt} = \lambda_{mt} (\rho_t + \pi_t) - \psi_t + \gamma_m \lambda_{\beta t} \rho_t' \quad (49)$$

$$\dot{\lambda}_{\beta t} = \lambda_{\beta t} \rho(s_t) - u(c_t) \quad (50)$$

$$TVC_1 : \lim_{t \rightarrow \infty} e^{-\beta(t)} \lambda_{kt} k_t = 0 \quad (51)$$

$$TVC_2 : \lim_{t \rightarrow \infty} e^{-\beta(t)} \lambda_{mt} m_t = 0 \quad (52)$$

Condition (46) equates the marginal utility of consumption to the sum of the cost of holding money in utils, the utility cost of relaxing the CIA constraint and a subjective

cost in terms of foregone future discounted utilities. Condition (47) shows that the price of capital in utility terms equals the utility value of real money balances plus the utility value of relaxing the CIA constraint. Conditions (48) to (50) describe the optimal time paths for the shadow prices of  $k$ ,  $m$  and  $\beta$ , respectively. Conditions (51) and (52) are the transversality conditions.

To close the model, we have

$$\dot{k}_t = f(k_t) - \delta(k_t) - c_t, \quad (53)$$

$$\dot{m}_t = (\mu - \pi_t) m_t, \quad (54)$$

where  $\mu \neq 0$  is the constant money growth rate rule. Equation (53) is the capital accumulation equation for the aggregate economy and (54) describes the evolution of real money supply.

In the following sections, we consider two cases:  $\phi = 1$  and  $\phi = 0$ . We then look at the implications of DMI on the effects of inflation on output in both cases.

### 3.1 CIA Constraint on Consumption and Investment

When  $\phi = 1$ , the CIA constraint becomes  $c_t + i_t = m_t$ , which means that real money balances have to be employed in order to procure both consumption and investment goods. In the aggregate, this implies that the real output equals the amount of real money balances floating about in the economy, i.e.,

$$y_t = m_t. \quad (55)$$

From (41) to (43) and (46) to (50), we obtain the following dynamic equations for  $c$ ,  $k$ ,  $\hat{\lambda}_m$  and  $\hat{\lambda}_\beta$ :

$$\dot{c}_t = \frac{\widehat{\lambda}_{kt}(\rho(s_t) + \delta) - \widehat{\lambda}_{mt}f_{kt} + \widehat{\lambda}_{\beta t} \left[ \gamma_k \rho'_t + \gamma_c \lambda_{\beta t} \rho \right] \left( \gamma_k \dot{k}_t + \gamma_m \dot{m}_t \right)}{u_{cc} - \gamma_c^2 \widehat{\lambda}_{\beta t} \rho}, \quad (56)$$

$$\dot{\widehat{\lambda}}_{kt} = \widehat{\lambda}_{mt}f_k + \widehat{\lambda}_{kt} [\rho(s_t) + \delta] + \gamma_k \widehat{\lambda}_{\beta t} \rho'_t, \quad (57)$$

$$\dot{\widehat{\lambda}}_{mt} = \widehat{\lambda}_{mt} [1 + \rho_t + \pi_t] - u_{ct} + \gamma_m \widehat{\lambda}_{\beta t} \rho'_t, \quad (58)$$

$$\dot{\widehat{\lambda}}_{\beta t} = \widehat{\lambda}_{\beta t} \rho(s_t) - u(c_t). \quad (59)$$

In the steady state,  $\dot{c}_t = \dot{m}_t = \dot{k}_t = \dot{\widehat{\lambda}}_{kt} = \dot{\widehat{\lambda}}_{mt} = \dot{\widehat{\lambda}}_{\beta t} = 0$ . From (47) and (57), we obtain

$$f_k - \delta = \rho(s) + \gamma_k \frac{\widehat{\lambda}_{\beta}}{\widehat{\lambda}_k} \rho' + \frac{\psi}{\widehat{\lambda}_k} f_k. \quad (60)$$

Equation (60) is akin to Equation (27) with first- and second-order effects of inflation on time preference. The third term, however, accounts for the imposition of a CIA constraint on investment. As is consistent with Stockman (1981), money can still be non superneutral even with a constant RTP (i.e.,  $\rho' = 0$ ), as changes in the inflation rate would affect the ratio of the utility value of relaxing the CIA constraint and the utility value of capital (i.e.,  $\psi/\widehat{\lambda}_k$ ), which would affect saving decisions, and thus, capital accumulation. Intuitively, even without endogenous time preference, an increase in the inflation rate, which reduces liquidity, would still reduce capital, as real money balances are required to make investments. We call this the “liquidity constraint” effect.

### 3.1.1 Superneutrality

Linearizing  $\dot{c}_t = 0$ ,  $\dot{k}_t = 0$ ,  $\dot{\widehat{\lambda}}_{mt} = 0$  and  $\dot{\widehat{\lambda}}_{\beta t} = 0$  around the steady state, we derive

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} dc \\ dk \end{bmatrix} = \begin{bmatrix} -\frac{\hat{\lambda}_m}{\hat{\lambda}_k^2(1+\rho(s)+\pi)} \\ 0 \end{bmatrix} d\mu \quad (61)$$

where

$$\begin{aligned} p_{11} &= \left\{ \begin{aligned} &\gamma_c \rho' + \gamma_c \gamma_k \left( \frac{\hat{\lambda}_\beta}{\hat{\lambda}_k} \right) \rho'' + \left[ \frac{\gamma_c (\gamma_m \lambda_\beta \rho'' + \hat{\lambda}_m \rho') - u_{cc}}{\hat{\lambda}_k^2 (1 + \rho(s) + \pi)} \right] \\ &+ \left[ \frac{\gamma_m}{\hat{\lambda}_k (1 + \rho(s) + \pi)} + \gamma_k \right] \left( \frac{u_c - \gamma_c \hat{\lambda}_\beta \rho'}{\hat{\lambda}_k \rho(s)} \right) \rho' \end{aligned} \right\} \\ p_{12} &= \left\{ \begin{aligned} &(\gamma_k + \gamma_m f_k) \left[ \rho' + \gamma_k \left( \frac{\hat{\lambda}_\beta}{\hat{\lambda}_k} \right) \rho'' + \frac{\gamma_m \hat{\lambda}_\beta \rho'' + \hat{\lambda}_\beta \rho'}{\hat{\lambda}_k^2 (1 + \rho(s) + \pi)} \right] - f_{kk} \\ &- \left[ \frac{\gamma_m}{\hat{\lambda}_k (1 + \rho(s) + \pi)} + \gamma_k \right] (\hat{\lambda}_k + \gamma_m f_k) \left( \frac{\hat{\lambda}_\beta (\rho')^2}{\hat{\lambda}_k \rho(s)} \right) \end{aligned} \right\} \\ p_{21} &= -1 \\ p_{22} &= f_k - \delta \end{aligned}$$

We determine  $\frac{dy}{d\mu}$ ,  $\frac{dm}{d\mu}$  and  $\frac{d\lambda_\beta}{d\mu}$  from

$$\frac{dy}{d\mu} = f_k \frac{dk}{d\mu}, \quad (62)$$

$$\frac{dm}{d\mu} = \frac{dy}{d\mu}, \quad (63)$$

and

$$\frac{d\hat{\lambda}_\beta}{d\mu} = \frac{1}{\rho(s)} \left[ (u_c - \gamma_c \hat{\lambda}_\beta \rho') \frac{dc}{d\mu} - \gamma_k \hat{\lambda}_\beta \rho' \frac{dk}{d\mu} + -\gamma_m \hat{\lambda}_\beta \rho' \frac{dm}{d\mu} \right]. \quad (64)$$

Equations (62) to (64) present how the growth rate of real money balances affect real output, real money balances and the shadow cost of marginal impatience, respectively.

### 3.1.2 Numerical Examples

As in the DMI model, determining the signs of  $\frac{dm}{d\mu}$ ,  $\frac{dc}{d\mu}$  and  $\frac{dk}{d\mu}$  is analytically intractable. We thus employ specific functional forms. We retain the use of (32) and (34) for the time preference and production functions, respectively. In this case, however, the instantaneous utility function becomes



$$u(c) = \log(c + \underline{c}), \tag{65}$$

where  $\underline{c} > 1$  is interpreted as subsistence consumption and is included in order to ensure that  $u(\cdot)$  is positive.

We then calibrate the model with the same benchmark parameters we employed previously. Table 2 below summarizes the results from our numerical exercise. We verify that inflation has a negative effect on capital, consumption and real money balances for the entire valid range of  $\pi$  values.<sup>11</sup> This implies that for our given set of parameters, the first-order effect and the “liquidity constraint effect” together dominate the second-order effect. Moreover, the steady-states are saddle-path stable just like in the MIU model.

Table 4.2 Results for the CIA Model for  $\phi = 1$

	$\rho(c)$	$\rho(k)$	$\rho(k + m)$
$dk/d\pi$	$< 0$	$< 0$	$< 0$
$dc/d\pi$	$< 0$	$< 0$	$< 0$
$dm/d\pi$	$< 0$	$< 0$	$< 0$
$dy/d\pi$	$< 0$	$< 0$	$< 0$
No. of Real Roots $< 0$	1	1	1
Valid $\pi$ -Range	[0,.89]	[0.88]	[0,1]

CASE 1:  $\gamma_c = 1$  and  $\gamma_k = \gamma_m = 0$ . When the RTP is a decreasing function solely of real consumption, (60) becomes:

$$f_k - \delta = \rho(c) + \frac{\psi}{\lambda_k} f_k. \tag{66}$$

When inflation increases, the demand for real money balances decreases. As money is

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<sup>11</sup>The valid range of  $\pi$  values, are those inflation rate values that are consistent with a positive net real rate of interest.

necessary for purchasing both consumption and investment goods, real consumption and capital holdings also decline. The decline in capital holdings increases the second term on the right-hand side of (66), which means that the subjective cost of investing increases to discourage savings and thus, drive down the steady-state levels of capital and output. This, we call the “liquidity constraint effect.” Moreover, as consumption goes down, the RTP goes up, which further reduces savings. Thus, the steady-state levels of capital and output decline.

CASE 2:  $\gamma_k = 1$  and  $\gamma_c = \gamma_m = 0$ . When the RTP is a decreasing function solely of real capital, (60) becomes:

$$f_k - \delta = \rho(k) + \frac{\widehat{\lambda}_\beta}{\widehat{\lambda}_k} \rho' + \frac{\psi}{\widehat{\lambda}_k} f_k. \quad (67)$$

The subsequent decline in real money balances due to inflation reduces both consumption and investment. The decline in investment reduces capital holdings, which increases the subjective cost of investing (i.e., the sum of the first and second terms in the right-hand side of (67)). The increase in the agent’s subjective cost of investing then reduces savings, and thus steady-state capital and output. This effect reinforces the liquidity constraint effect (i.e., the third term in the right-hand side of (67)).

CASE 3:  $\gamma_k = \gamma_m = 1$  and  $\gamma_c = 0$ . When the RTP is a decreasing function solely of real wealth, (60) becomes:

$$f_k - \delta = \rho(k + m) + \frac{\widehat{\lambda}_\beta}{\widehat{\lambda}_k} \rho' + \frac{\psi}{\widehat{\lambda}_k} f_k. \quad (68)$$

Higher inflation increases the RTP in two ways: 1) the decline in real money balances leads to a higher time preference and 2) the decline in the demand for real money holdings, reduces both the demand for consumption and investment, which further increases time preference. Thus, the subsequent rise in the RTP and the liquidity constraint effect together work to reduce the steady-state levels of both capital and output.

We thus observe that in all three specifications of the RTP, the negative effect of inflation on marginal impatience reinforces the CIA constraint effect. This further demonstrates that for both the MIU model and the model with a cash in-advance constraint on consumption and investment, the long-run relationship between inflation and growth we uncover seems to provide a theoretical support for the empirical observation that inflation imposes an adverse effect on economic growth.

### 3.1.3 Stability

Linearizing (53), (56), (57), (58) and (59) around the steady state and evaluating the derivatives at the steady state, we obtain the system of equations below

$$\begin{bmatrix} \dot{c}_t \\ \dot{k}_t \\ \dot{\hat{\lambda}}_{kt} \\ \dot{\hat{\lambda}}_{mt} \\ \dot{\hat{\lambda}}_{\beta t} \end{bmatrix} = \underbrace{\begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} & q_{15} \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} \\ q_{31} & q_{32} & q_{33} & q_{34} & q_{35} \\ q_{41} & q_{42} & q_{43} & q_{44} & q_{45} \\ q_{51} & q_{52} & q_{53} & q_{54} & q_{55} \end{bmatrix}}_Q \begin{bmatrix} c_t - c \\ k_t - k \\ \hat{\lambda}_{kt} - \hat{\lambda}_k \\ \hat{\lambda}_{mt} - \hat{\lambda}_m \\ \hat{\lambda}_{\beta t} - \hat{\lambda}_\beta \end{bmatrix} \quad (69)$$

where the elements of the Hessian matrix  $Q$  are listed in Appendix C.

Just like in the model with money in the utility function, we have one predetermined variable,  $k_t$ , while  $c_t$ ,  $\hat{\lambda}_{kt}$ ,  $\hat{\lambda}_{mt}$  and  $\hat{\lambda}_{\beta t}$  are jump variables. Local stability of the equilibrium then requires that the number of eigenvalues of  $Q$  with negative real parts is not less than the number of predetermined variables in the system – in this case, one. As Table 2 shows, the steady states are locally stable for  $\pi \in [0, .88]$  for all specifications of the RTP, as the number of negative real roots in each case exactly equals one. This occurs because when real wealth decreases (due to higher inflation), the RTP increases, which discourages savings, and thus, capital accumulation and output. However, changes in the instantaneous RTP also affect that way the agent discounts the future *in the future*. In this case, the same increase in the instantaneous RTP also increases the future stream of time preference. This, consequently,

reduces the discounted value of future utilities ( $\widehat{\lambda}_\beta$ ), which renders future consumption *in the future* less valuable to the point where the individual shifts future resources away from consumption to future-oriented capital so that steady-state capital increases. At the same time, as capital accumulation initially declines, the marginal product of capital increases so that the agent is induced to hold more money balances to purchase more capital goods. This tends to raise the steady-state levels of capital, output and consumption.

### 3.2 CIA Constraint on Consumption

When  $\phi = 0$ , the CIA constraint becomes  $c_t = m_t$ , which means that real money balances are only used to purchase consumption goods. Under this assumption, the following dynamic equations for  $c_t$ ,  $k_t$ ,  $\widetilde{\lambda}_{kt}$ , and  $\widetilde{\lambda}_{\beta t}$  are derived from (46) to (50), (53) and (54):

$$\dot{c}_t = \left( 1 + f_k - \delta + \mu + (\gamma_c + \gamma_m - \gamma_k) \frac{\widetilde{\lambda}_{\beta t}}{\widetilde{\lambda}_{kt}} \rho' - \frac{u_{ct}}{\widetilde{\lambda}_{kt}} \right) c_t, \quad (70)$$

$$\dot{k}_t = f(k_t) - \delta(k_t) - c_t, \quad (71)$$

$$\dot{\widetilde{\lambda}}_{kt} = -\widetilde{\lambda}_{kt}(f_k - \delta - \rho(s_t)) + \gamma_k \widetilde{\lambda}_{\beta t} \rho'_t, \quad (72)$$

$$\dot{\widetilde{\lambda}}_{\beta t} = \widetilde{\lambda}_{\beta t} \rho_t - u_t. \quad (73)$$

In the steady state,  $\dot{c}_t = \dot{k}_t = \dot{\widetilde{\lambda}}_{kt} = \dot{\widetilde{\lambda}}_{\beta t} = 0$ . From  $\dot{\widetilde{\lambda}}_{kt} = 0$ , we obtain the expression for the real interest rate

$$f_k - \delta = \rho(s) + \gamma_k \frac{\widetilde{\lambda}_\beta}{\widetilde{\lambda}_k} \rho'. \quad (74)$$

Equation (74) shows that with an endogenous RTP in a CIA model when  $\phi = 0$ , inflation can have first- and second-order effects on time preference, and can thus, affect the real

sector via its impact on saving behavior. When the RTP is a constant, exogenous parameter, (74) reduces to the Stockman (1981) result when a liquidity constraint is imposed solely on consumption.

### 3.2.1 Superneutrality

Linearizing  $\dot{c}_t = 0$ ,  $\dot{k}_t = 0$ ,  $\dot{\tilde{\lambda}}_{mt} = 0$  and  $\dot{\tilde{\lambda}}_{\beta t} = 0$  around the steady state, we get

$$\underbrace{\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix}}_D \begin{bmatrix} dc \\ dk \\ d\tilde{\lambda}_k \\ d\tilde{\lambda}_\beta \end{bmatrix} = \begin{bmatrix} \tilde{\lambda}_k \\ 0 \\ 0 \\ 0 \end{bmatrix} d\mu, \quad (75)$$

where the elements of the Hessian matrix  $D$  are listed in Appendix C.

### 3.2.2 Numerical Examples

We calibrate the model using the same functional forms and set of parameters that we used in the previous case. The results are presented in Table 3 below. We observe that in contrast to the Stockman (1981) result which shows money superneutrality under this specification, endogenous time preference creates a channel through which the growth rate of money can affect output.

CASE 1:  $\gamma_c = 1$  and  $\gamma_k = \gamma_m = 0$ . When the RTP is a decreasing function solely of real consumption, (74) becomes:

$$f_k - \delta = \rho(c). \quad (76)$$

The increase in the growth of real money raises both the opportunity cost of holding real money balances and of purchasing consumption goods. These generate two opposing effects on capital accumulation. On the one hand, as capital assets can be procured costlessly (via barter), the agent is induced to substitute capital holdings for real money holdings. On the

other hand, the increase in capital holdings raises his real wealth, which then induces him to increase consumption, and thus, his demand for real money balances in order to finance this. This wealth effect, however, tends to reduce savings, and thus, capital accumulation in the economy. Under a constant RTP, these two effects cancel out each other. However, with variable time preference (and logarithmic utility), the increase in real consumption (and thus, the demand for real money holdings) reduces marginal impatience (or increases the RTP) so that savings increase, and thus, the steady-state levels of capital and output.

Table 4.3 Results for the CIA Model when  $\phi = 0$

	$\rho(c)$	$\rho(k)$	$\rho(k+m)$
$dk/d\mu$	$> 0$	$\left\{ \begin{array}{l} < 0, \text{ for } \pi \in [0, .05] \\ > 0, \text{ for } \pi \in (.05, 1] \end{array} \right.$	$\left\{ \begin{array}{l} < 0, \text{ for } \pi \in [0, .11] \\ > 0, \text{ for } \pi \in (.11, 1] \end{array} \right.$
$dc/d\mu$	$> 0$	$\left\{ \begin{array}{l} < 0, \text{ for } \pi \in [0, .05] \\ > 0, \text{ for } \pi \in (.05, 1] \end{array} \right.$	$\left\{ \begin{array}{l} < 0, \text{ for } \pi \in [0, .11] \\ > 0, \text{ for } \pi \in (.11, 1] \end{array} \right.$
$dm/d\mu$	$> 0$	$\left\{ \begin{array}{l} < 0, \text{ for } \pi \in [0, .05] \\ > 0, \text{ for } \pi \in (.05, 1] \end{array} \right.$	$\left\{ \begin{array}{l} < 0, \text{ for } \pi \in [0, .11] \\ > 0, \text{ for } \pi \in (.11, 1] \end{array} \right.$
$dy/d\mu$	$> 0$	$\left\{ \begin{array}{l} < 0, \text{ for } \pi \in [0, .05] \\ > 0, \text{ for } \pi \in (.05, 1] \end{array} \right.$	$\left\{ \begin{array}{l} < 0, \text{ for } \pi \in [0, .11] \\ > 0, \text{ for } \pi \in (.11, 1] \end{array} \right.$
$d\tilde{\lambda}_\beta/d\mu$	$< 0$	$\left\{ \begin{array}{l} > 0, \text{ for } \pi \in [0, .05] \\ < 0, \text{ for } \pi \in (.05, 1] \end{array} \right.$	$\left\{ \begin{array}{l} > 0, \text{ for } \pi \in [0, .11] \\ < 0, \text{ for } \pi \in (.11, 1] \end{array} \right.$
No. of Real Roots $< 0$	1	1	1
Valid $\pi$ -Range	[0,1]	[0,1]	[0,1]

CASE 2:  $\gamma_k = 1$  and  $\gamma_c = \gamma_m = 0$ . When the RTP is a decreasing function solely of real consumption, (74) becomes:

$$f_k - \delta = \rho(k) + \frac{\tilde{\lambda}_\beta}{\lambda_k} \rho' . \quad (77)$$

During lower inflationary periods, Table 3 shows that the first-order effect of inflation on marginal impatience dominates: an increase in the growth rate of money increases the shadow price of real money balances, which equiproportionately increases the shadow price of real capital. This discourages the agent from holding capital assets, which increases his subjective cost of investing (i.e., the right-hand side of (77)). This, in turn decreases steady-state capital and output. In contrast, during higher inflationary periods (i.e., when  $\pi > .05$ ), the second-order effect dominates. Now, higher inflation induces the agent to alter his portfolio composition in favor of real capital assets, as these can be obtained in a relatively costless manner (through barter). This then reduces marginal impatience, and thus the subjective cost of investing, which further spurs capital accumulation. Thus, steady-state capital and output, and consequently, consumption and the demand for real money balances increase.

CASE 3:  $\gamma_k = \gamma_m = 1$  and  $\gamma_c = 0$ . When the RTP is a decreasing function solely of real consumption, (74) becomes:

$$f_k - \delta = \rho(k + m) + \frac{\tilde{\lambda}_\beta}{\lambda_k} \rho' . \quad (78)$$

Just like in the previous case, when the steady-state inflation rate is low enough (i.e., when  $\pi \leq .11$ ), increasing the growth rate of money increases the agent's flow of future discounted utilities ( $\tilde{\lambda}_\beta$ ). Consequently, the agent favors future consumption at the expense of future-oriented capital. This reduces steady-state capital, which in turn, increases the subjective cost of investing (the right-hand side of (78)). This further discourages capital accumulation, which also reduces steady-state output, consumption and real money holdings. Since real money also enters the RTP, the initial decline in real money holdings subsequent to higher inflation, also increases the RTP, so that steady-state capital and output decrease with the money growth rate over a wider range of inflation rate values than that in the

previous case. However, when inflation is high enough (i.e., when  $\pi > .11$ ), the increase in the opportunity cost of holding money reaches a point where holding more real capital assets become more attractive. This reduces the subjective cost of investing, which in turn, spurs greater investment. Steady-state capital, output, consumption and real money holdings thus increase.

We thus find that when the RTP is specified as a decreasing function of consumption, the steady-state money growth rate, and thus the inflation can have a positive effect on output. This is in line with the Mundell-Tobin effect, which asserts that at high inflation rates, agents are induced to reallocate their wealth from money to capital assets.

### 3.2.3 Stability

Linearizing (70) to (73) around the steady state, and evaluating the derivatives at the steady state, gives

$$\begin{bmatrix} \dot{c}_t \\ \dot{k}_t \\ \dot{\tilde{\lambda}}_{kt} \\ \dot{\tilde{\lambda}}_{\beta t} \end{bmatrix} = \underbrace{\begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}}_E \begin{bmatrix} c_t - c \\ k_t - k \\ \tilde{\lambda}_{kt} - \tilde{\lambda}_k \\ \tilde{\lambda}_{\beta t} - \tilde{\lambda}_\beta \end{bmatrix} \quad (79)$$

where the elements of the Hessian matrix  $E$  are listed in Appendix C.

As Table 3 reports, all cases exhibit local stability for  $\pi \in [0, 1]$ . The explanation for the cases where the RTP is a function of either real capital or real wealth is similar to the previous models: when real wealth decreases, the RTP increases, which discourages savings, and thus, capital accumulation and output. However, changes in the instantaneous RTP also affect that way the agent discounts the future *in the future*. In this case, the same increase in the instantaneous RTP also increases the future stream of time preference. This, consequently, reduces the discounted value of future utilities ( $\tilde{\lambda}_\beta$ ), which renders future



consumption *in the future* less valuable to the point where the individual shifts future resources away from consumption to future-oriented capital so that steady-state capital increases.

In the special case where the RTP is a sole decreasing function of consumption, a reduction in real consumption increases marginal impatience, which then reduces savings, and thus, the steady-state levels of capital and output. However, as consumption declines, the demand for real money balances also declines, which then increases the demand for real capital holdings via the portfolio substitution effect. This then tends to counteract the initial declines in the steady-state levels of capital and output, so that the steady-state is still locally saddle-path stable.

## 4 Conclusion

This paper examines the implications of decreasing marginal impatience on money superneutrality in two popular monetary models – the model with money in the utility and the model with a cash-in-advance constraint imposed solely on consumption or on both consumption and investment. The theoretical derivations show that the growth rate of money – and thus, the inflation rate – affects output in all models. This is explained by the presence of both first-order and second-order effects of inflation on the rate of time preference, which affects the agent’s saving behavior, and thus, capital accumulation and output. In particular, the first-order effect occurs in this manner: when the inflation rate increases, which erodes the value of real wealth and consequently, raises the instantaneous RTP, which then increases steady-state capital and output. Meanwhile, the second-order effect happens when the decline in real wealth due to higher inflation, increases the effective rate at which marginal impatience decreases. This effectively reduces the subjective cost of investing so that the agent is induced to save more. This increases steady-state capital and output. As the two effects work in opposite directions, whichever effect dominates determines whether money has a negative or positive impact on output.

The quantitative results show that an increase in the money growth rate has negative effects on output in the MIU models, the models with a CIA constraint on both consumption and investment and for lower inflation rates in the models with a CIA constraint solely on consumption. This is in line with the empirical evidence that suggests a negative relationship between inflation and output. However, an increase in the money growth rate ultimately increases output in the model with a CIA constraint imposed solely on consumption. This occurs as higher inflation induces the agent to change the composition of his wealth portfolio in favor of more capital holdings at the expense of monetary assets. As a consequence, the subjective cost of investing decreases, which further generates more capital accumulation, and thus, output growth.

We also find that the steady states generated are saddle-path stable. This is because an increase in real wealth, which reduces the instantaneous RTP and ultimately increases steady-state capital, also reduces the future stream of time preference. This accordingly, increases aggregate future utility, which raises the value of future consumption to the extent where the agent favors future consumption to future-oriented capital, so that steady-state capital decreases. This shows that saddle-path stability can be achieved in a monetary economy with diminishing marginal impatience – whether money is introduced in the utility or as a liquidity constraint.

# Conclusion

At the onset, this dissertation sought to answer the question, “What are the macroeconomic and welfare consequences of time-inconsistent and endogenous discounting behaviors?” Three main applications of interest have been considered. The second chapter deals with the implications of quasi-hyperbolic discounting in a social planner’s economy. Allocative and welfare comparisons were made between the outcomes under a social planner with the time-consistent, exponential discounting function and those under a social planner with time-inconsistent, quasi-hyperbolic discounting behavior. The third chapter looks at the macroeconomic and welfare effects of raising the retirement age in a model where both exponential and quasi-hyperbolic discounters co-exist. The fourth chapter looks at the implications of endogenous discounting on money superneutrality and stability in the context of two popular monetary models, namely, the money-in-the-utility model and the cash-in-advance model.

The second chapter unveils four striking results:

- First, the quasi-geometric allocations (i.e., labor hours, capital and output) are below their geometric counterparts when the planner is characterized by excessive short-run patience, while the opposite ensues with excessive short-run impatience.
- Second, from the current generation’s standpoint, quasi-geometric welfare is strictly lower than geometric welfare when the economy is endowed with excessive short-run patience, whereas the reverse outcome arises with excessive short-run impatience. Moreover, comparing between generations, we find that reducing excessive short-run impatience creates a welfare tradeoff between the current and future generations, but further increasing excessive short-run patience becomes detrimental to the welfare of both.
- Third, a higher elasticity of labor supply increases the differences between the quasi-geometric and geometric allocations and welfare levels.

- Fourth, a more persistent technology shock enables the more patient economy to “catch up with” the less patient one so that the gap between the current quasi-geometric and geometric welfare is reduced as the persistence parameter is increased.

The third chapter shows the following non-standard outcomes in the presence of time-inconsistent discounting:

- The existence of quasi-hyperbolic agents imposes a pecuniary externality (which works through the real interest rate) on the rest of the economy.
- In raising the retirement age from 65 to 70 years old, the highest aggregate welfare gains accrue to the economy with a greater population of hyperbolic agents. This suggests that the mandatory postponement of retirement serves as another pre-commitment device for time-inconsistent agents to increase their lifetime savings.
- Social security privatization is most desirable with a mixed population of rational and hyperbolic agents.
- Also, in the context of social security privatization, we find that retired hyperbolic agents gain after privatization in a mixed economy, while they lose in a economy purely composed of time-inconsistent individuals.
- Thus, ignoring the externalities generated by hyperbolic agents may lead to spurious welfare conclusions.

The fourth chapter finds that money superneutrality depends on how the endogeneity of the discounting function is modeled:

- The money growth rate has negative effects on output and investment in the MIU models, the models with a CIA constraint on both consumption and investment and for lower inflation rates in the models with a CIA constraint solely on consumption. This is in line with the empirical evidence that suggests a negative relationship between inflation and output.

- In contrast, an increase in the money growth rate ultimately increases output in the model with a CIA constraint imposed solely on consumption. This occurs as higher inflation induces the agent to change the composition of his wealth portfolio in favor of more capital holdings at the expense of monetary assets.

These observations imply that adhering to the standard, time-consistent, exogenous exponential discounting function may lead to spurious welfare conclusions if we are to take into serious consideration the empirical evidence that the agents' discounting behavior is characterized by time-inconsistency and influenced endogenously by certain factors. An interesting direction for future research involves looking at the macroeconomic and welfare implications of discounting behavior that is both time-inconsistent and endogenous.

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## Appendix A

*Proof of Proposition 1.* Using the guess form  $V(k, z) = a + b \log(k) + d \log(z)$ , the first-order conditions of the current planner's problem yield the optimal decision rules  $k' = \frac{\beta \delta b}{1 + \beta \delta b} z k^\alpha h^{1-\alpha}$  and  $h = \left[ \frac{1-\alpha}{A} \left( 1 + \frac{\alpha \beta \delta}{1-\alpha \beta} \right) \right]^{\frac{1}{1+\chi}}$ . For these to be the time-consistent solution, these should satisfy (5):

$$\begin{aligned} V(k, z) &= a + b \log(k) + d \log(z) \\ &= \log \left[ \left( 1 - \frac{\beta \delta b}{1 + \beta \delta b} \right) z k^\alpha h^{1-\alpha} \right] - A \frac{h^{1+\chi}}{1 + \chi} \\ &\quad + E \beta \left[ a + b \log \left( \frac{\beta \delta b}{1 + \beta \delta b} z k^\alpha h^{1-\alpha} \right) + d \log(z^\rho \nu) \right] \end{aligned}$$

Further simplifying and equating coefficients, we get:

$$b = \frac{\alpha}{1 - \alpha \beta}$$

$$d = \frac{1}{(1 - \alpha \beta)(1 - \rho \beta)}$$

Inserting  $b$  into the policy function for capital gives  $k' = [\alpha \beta \delta / (1 - \alpha \beta (1 - \delta))] z k^\alpha h^{1-\alpha}$ . With full depreciation of capital, investment  $I$  equals next-period capital  $k'$ . In equilibrium,  $I = s z k^\alpha h^{1-\alpha} = k'$ . Thus, we obtain the constant saving rate  $s = \alpha \beta \delta / (1 - \alpha \beta (1 - \delta))$ . Consequently, consumption is also a constant fraction of output and can be written as  $c = (1 - s) z k^\alpha h^{1-\alpha}$ .

*Proof of Proposition 3.* The corresponding continuation period value function becomes

$$\begin{aligned} V(k, z) &= \log((1 - s) z k^\alpha h^{1-\alpha}) + \beta E \left[ \log((1 - s) z k'^\alpha h^{1-\alpha}) \right] \\ &\quad + \beta^2 E \left[ \log((1 - s) z k''^\alpha h^{1-\alpha}) \right] + \dots \end{aligned}$$

Plugging in the policy function for capital into  $V(k, z)$  we get (after much messy derivation):

$$\begin{aligned}
V(k, z) &= \frac{1 - \alpha\beta}{(1 - \beta)(1 - \alpha\beta)} \log(1 - s) + \frac{\alpha\beta}{(1 - \beta)(1 - \alpha\beta)} \log(s) \\
&+ \frac{1 - \alpha}{(1 - \alpha\beta)(1 - \beta)} \log(h) - \frac{1}{1 - \beta} A \frac{h^{1+\chi}}{1 + \chi} \\
&+ \frac{\alpha}{1 - \alpha\beta} \log(k) + \frac{1}{(1 - \alpha\beta)(1 - \rho\beta)} \log(z)
\end{aligned}$$

Thus, at time-0 the planner's value function is

$$\begin{aligned}
V_0(k, z) &= \log[(1 - s)zk^\alpha h^{1-\alpha}] - A \frac{h^{1+\chi}}{1 + \chi} + \delta\beta E[V(szk^\alpha h^{1-\alpha}, z')] \\
&= \frac{1 - \beta(1 - \delta)}{1 - \beta} \log(1 - s) + \frac{\alpha\beta\delta}{(1 - \beta)(1 - \alpha\beta)} \log(s) \\
&+ \frac{(1 - \alpha)[(1 - \beta)[1 - \alpha\beta(1 - \delta)] + \beta\delta]}{(1 - \alpha\beta)(1 - \beta)} \log(h) - Ah^{1+\chi} \left[ \frac{1 - \beta(1 - \delta)}{(1 - \beta)(1 + \chi)} \right] \\
&- Ah^{1+\chi} \left[ \frac{1 - \beta(1 - \delta)}{(1 - \beta)(1 + \chi)} \right] + \frac{\alpha[1 - \alpha\beta(1 - \delta)]}{1 - \alpha\beta} \log(k) \\
&+ \frac{(1 - \rho\beta)[1 - \alpha\beta(1 - \delta)] + \beta\delta\rho}{(1 - \alpha\beta)(1 - \rho\beta)} \log(z)
\end{aligned}$$

## Appendix B

*Proof of Proposition 1.* This proposition is proven by backward induction. Consider the retirement period  $j = J_R, \dots, J - 2, J - 1$  and  $J$ . Since  $h_j^i = 0$  for all  $i$ , it follows that  $\Delta_j^i = 0$  throughout the retirement period. Also,  $u(c_j^i, h_j^i) = u(c_j^i, 0)$  for all  $i$  and  $j$ .

At the terminal age  $J$ , optimal consumption is given by  $c_J^i = (1 + \tilde{r})a_{J-1}^i + (qy_J + y_{J-1})$  and labor hours is zero. Hence  $\Lambda_J^i = 1$ . Consider age  $J - 1$ .

The agent's problem is now given by

$$\max_{c_J^i, c_{J-1}^i, h_J^i, h_{J-1}^i} [u(c_{J-1}^i, 0) + \delta_i \beta \psi_J u(c_J^i, 0)]$$

subject to the consolidated budget constraint

$$c_{J-1}^i + qc_J^i = \frac{(1 + \tilde{r})a_{J-1}^i + (qy_J + y_{J-1})}{1 + \tau_c}. \quad (1)$$

From the Euler equation for consumption, we get

$$c_J^i = [\delta_i \beta \psi_J (1 + \tilde{r})]^{\frac{1}{\sigma}} c_{J-1}^i$$

Hence,

$$\Phi_J^i \equiv [\beta \psi_J (1 + \tilde{r})]^{\frac{1}{\sigma}} \left( \frac{\delta_i \Gamma_J^i}{\Lambda_J^i} \right)^{\frac{1}{\sigma}},$$

where  $\Lambda_J^i = 1$  and  $\Gamma_J^i = 1$ . Substituting  $c_J^i = \Phi_J^i c_{J-1}^i$  into (6) gives

$$c_{J-1}^i = \frac{(1 + \tilde{r})a_{J-1}^i + (qy_J + y_{J-1})}{(1 + \tau_c) [1 + (\Lambda_J^i) q \Phi_J^i]}.$$

Hence,  $\Lambda_{J-1}^i = 1 + (\Lambda_J^i) q \Phi_J^i$  and  $\Psi_{J-1}^i = 0$ .

Consider age  $J - 2$ . The agent's problem is now given by



$$\max_{\{c_j^i, h_j^i\}_{j=J-2}^J} \{u(c_{J-2}^i, 0) + \delta_i \beta \psi_{J-1} [u(c_{J-1}^i, 0) + \beta \psi_J u(c_J^i, 0)]\}$$

subject to

$$(1 + \tau_c)c_{J-2}^i + a_{J-1}^i = (1 + \tilde{r})a_{J-2}^i + y_{J-2}, \quad (2)$$

$$c_{J-1}^i = \frac{1}{(1 + \tau_c)\Lambda_{J-1}^i} [(1 + \tilde{r})a_{J-1}^i + \Omega_{J-1}], \quad (3)$$

and  $c_J^i = \Phi_J^i c_{J-1}^i$ . Using  $c_J^i = \Phi_J^i c_{J-1}^i$ , we can rewrite the objective function as

$$u(c_{J-2}^i, 0) + \delta_i \beta \psi_{J-1} [1 + \beta \psi_J (\Phi_J^i)^{1-\sigma}] u(c_{J-1}^i, 0).$$

The first-order condition with respect to  $a_{J-1}^i$  is

$$\begin{aligned} u'_c(c_{J-2}^i, 0) &= \delta_i \beta \psi_{J-1} [1 + \beta \psi_J (\Phi_J^i)^{1-\sigma}] u'_c(c_{J-1}^i, 0) \frac{dc_{J-1}^i}{da_{J-1}^i} \\ \implies (c_{J-2}^i)^{-\sigma} &= \frac{\delta_i \beta \psi_{J-1} (1 + \tilde{r})}{\Lambda_{J-1}^i} [1 + \beta \psi_J (\Phi_J^i)^{1-\sigma}] (c_{J-1}^i)^{-\sigma} \\ \implies c_{J-1}^i &= \underbrace{\left\{ \frac{\delta_i \beta \psi_{J-1} (1 + \tilde{r})}{\Lambda_{J-1}^i} [1 + \beta \psi_J (\Phi_J^i)^{1-\sigma}] \right\}^{\frac{1}{\sigma}}}_{\Phi_{J-1}^i} c_{J-2}^i. \end{aligned}$$

Hence,

$$\Phi_{J-1}^i = [\beta \psi_{J-1} (1 + \tilde{r})]^{\frac{1}{\sigma}} \left[ \frac{\delta_i \Gamma_{J-1}^i}{\Lambda_{J-1}^i} \right]^{\frac{1}{\sigma}},$$

where  $\Gamma_{J-1}^i \equiv 1 + \beta \psi_J (\Phi_J^i)^{1-\sigma} \Gamma_J^i$ . Combining (7) and (8) gives

$$c_{J-2}^i + q (\Lambda_{J-1}^i) \Phi_{J-1}^i c_{J-2}^i = \frac{(1 + \tilde{r})a_{J-2}^i + y_{J-2} + q(qy_J + y_{J-1})}{1 + \tau_c}$$

$$\implies c_{J-2}^i = \frac{1}{(1 + \tau_c) [1 + (\Lambda_{J-1}^i) q \Phi_{J-1}^i]} [(1 + \tilde{r}) a_{J-2}^i + \Omega_{J-2}].$$

Hence,  $\Lambda_{J-2}^i = 1 + (\Lambda_{J-1}^i) q \Phi_{J-1}^i$  and  $\Psi_{J-2}^i = 0$ .

At  $j = J_R - 1$ , the consumer's objective function is

$$\max_{\{c_k^i, h_k^i\}_{k=J_R-1}^J} \left\{ u(c_{J_R-1}^i, h_{J_R-1}^i) + \delta_i \beta \left[ \sum_{m=j+1}^J \beta^{m-j-1} \left( \prod_{d=j+1}^m \psi_d \right) u(c_m^i, h_m^i) \right] \right\}.$$

subject to

$$(1 + \tau_c) c_{J_R-1}^i + a_{J_R}^i = (1 + \tilde{r}) a_{J_R-1}^i + y_{J_R-1},$$

$$c_{J_R}^i = \frac{1}{\Lambda_{J_R}^i} \left[ \frac{(1 + \tilde{r}) a_{J_R}^i + \Omega_{J_R}^i}{1 + \tau_c} \right],$$

and

$$c_{j+1}^i = \Phi_{j+1}^i c_j^i \quad \text{for } j = J_R, \dots, J.$$

From the first-order condition for labor hours, we have:

$$u_c(c_{J_R-1}^i, h_{J_R-1}^i) \tilde{w} \varepsilon_{J_R-1} = u_h(c_{J_R-1}^i, h_{J_R-1}^i).$$

Given the specific form of the utility function,  $h_{J_R-1}^i = (\tilde{w} \varepsilon_{J_R-1} / A)^{1/\theta}$  for all  $i$ .

From the first-order condition for  $a_{J_R}^i$  :

$$(1 + \tau_c) \{ c_{J_R-1}^i + q (\Lambda_{J_R}^i) [\Phi_{J_R}^i c_{J_R-1}^i + \Delta_{J_R}^i] \} = (1 + \tilde{r}) a_{J_R-1}^i + y_{J_R-1} + q \Omega_{J_R}$$

$$\begin{aligned} \Rightarrow c_{J_{R-1}}^i &= \frac{1}{\left(1 + \left(\Lambda_{J_R}^i\right) q \Phi_{J_R}^i\right)} \left[ \frac{(1 + \tilde{r}) a_{J_{R-1}}^i + \Omega_{J_{R-1}}}{1 + \tau_c} - q \left(\Lambda_{J_R}^i\right) \Delta_{J_R}^i \right], \\ \Rightarrow c_{J_{R-1}}^i &= \frac{1}{\Lambda_{J_{R-1}}^i} \left[ \frac{(1 + \tilde{r}) a_{J_{R-1}}^i + \Omega_{J_{R-1}}}{1 + \tau_c} - \Psi_{J_{R-1}}^i \right] \end{aligned}$$

Hence,  $\Lambda_{J_{R-1}}^i = 1 + \left(\Lambda_{J_R}^i\right) q \Phi_{J_R}^i$ ,  $\Psi_{J_{R-1}}^i \equiv q \sum_{k=J_{R-1}}^{J_R-1} q^{k-j} \left(\Lambda_{k+1}^i \Delta_{k+1}^i\right)$ .

Suppose the desired results hold for ages  $j + 1, j + 2, \dots, J - 1$  and  $J$ . This means

$$c_{j+m+1} = \Phi_{j+m+1}^i c_{j+m}^i + \Delta_{j+m+1}^i, \quad (4)$$

for  $m = 1, \dots, J - j - 1$ , and

$$c_{j+1}^i = \frac{1}{\Lambda_{j+1}^i} \left[ (1 + \tilde{r}) a_{j+1}^i + \Omega_{j+1} - \Psi_{j+1}^i \right]. \quad (5)$$

We want to show that (16) holds for  $m = 0$  and  $c_j^i = \left[ (1 + \tilde{r}) a_j^i + \Omega_j - \Psi_j^i \right] / \Lambda_j^i$ . The consumer's problem at age  $j$  is given by

$$\max_{\{c_k^i, h_k^i\}_{k=j}^J} \left\{ u(c_j^i, h_j^i) + \delta_i \beta \left[ \sum_{m=j+1}^J \beta^{m-j-1} \left( \prod_{d=j+1}^m \psi_d \right) u(c_m^i, h_m^i) \right] \right\}$$

subject to (16), (17) and

$$c_j^i + a_{j+1}^i = (1 + \tilde{r}) a_j^i + y_j. \quad (6)$$

Using (16), we can rewrite the objective function as follows:

$$\begin{aligned}
& u(c_j^i, h_j^i) + \delta_i \beta \left[ \sum_{m=j+1}^J \beta^{m-j-1} \left( \prod_{d=j+1}^m \psi_d \right) u(c_m^i, h_m^i) \right] \\
= & u(c_j^i, h_j^i) + \delta_i \beta \left[ \psi_{j+1} u(c_{j+1}^i, h_{j+1}^i) + \beta \psi_{j+1} \psi_{j+2} u(c_{j+2}^i, h_{j+2}^i) + \dots \right. \\
& \left. + \beta^{J-j-1} \left( \prod_{m=j+1}^J \psi_m \right) u(c_J^i, h_J^i) \right] \\
= & u(c_j^i, h_j^i) + \delta_i \beta \psi_{j+1} \left[ \begin{array}{l} 1 + \beta \psi_{j+2} \frac{u(c_{j+2}^i, h_{j+2}^i)}{u(c_{j+1}^i, h_{j+1}^i)} + \dots \\ + \beta^{J-j-1} \left( \prod_{m=j+2}^J \psi_m \right) \frac{u(c_J^i, h_J^i)}{u(c_{j+1}^i, h_{j+1}^i)} \end{array} \right] u(c_{j+1}^i, h_{j+1}^i) \\
= & u(c_j^i, h_j^i) + \delta_i \beta \psi_{j+1} \left[ \begin{array}{l} 1 + \beta \psi_{j+2} (\Phi_{j+2})^{1-\sigma} + \dots \\ + \left( \prod_{m=j+2}^J [\beta \psi_m (\Phi_m)^{1-\sigma}] \right) \end{array} \right] u(c_{j+1}^i, h_{j+1}^i).
\end{aligned}$$

The agent's problem can now be rewritten as

$$\max_{a_{j+1}} \left\{ u(c_j^i, h_j^i) + \delta_i \beta \psi_{j+1} \left[ \begin{array}{l} 1 + \beta \psi_{j+2} (\Phi_{j+2})^{1-\sigma} + \dots \\ + \left( \prod_{m=j+2}^J [\beta \psi_m (\Phi_m)^{1-\sigma}] \right) \end{array} \right] u(c_{j+1}^i, h_{j+1}^i) \right\}$$

subject to (17) and (18)

The first-order condition with respect to  $a_{j+1}^i$  is

$$\begin{aligned}
\left( c_j^i - A \frac{(h_j^i)^{1+\theta}}{1+\theta} \right)^{-\sigma} &= \frac{\delta_i \beta \psi_{j+1} (1 + \tilde{r})}{\Lambda_{j+1}^i} \left[ \begin{array}{l} 1 + \beta \psi_{j+2} (\Phi_{j+2})^{1-\sigma} + \dots \\ + \left( \prod_{m=j+2}^J [\beta \psi_m (\Phi_m)^{1-\sigma}] \right) \end{array} \right] \\
&\times \left( c_{j+1}^i - A \frac{(h_{j+1}^i)^{1+\theta}}{1+\theta} \right)^{-\sigma}.
\end{aligned}$$

Hence,

$$c_{j+1}^i = \underbrace{\frac{\delta_i \beta \psi_{j+1} (1 + \tilde{r})}{\Lambda_{j+1}^i} \left\{ 1 + \sum_{l=j+2}^J \prod_{m=j+2}^l [\beta \psi_m (\Phi_m)^{1-\sigma}] \right\}^{\frac{1}{\sigma}}}_{\Phi_{j+1}^i} c_j^i + \left[ A \frac{(h_{j+1})^{1+\theta} - \Phi_{j+1}^i (h_j)^{1+\theta}}{1 + \theta} \right].$$

This establishes equation (8).

Combining (17) and (18) gives

$$(1 + \tau_c) c_j^i + (1 + \tau_c) \underbrace{\left\{ q \Lambda_{j+1}^i \underbrace{[\Phi_{j+1} c_j^i + \Delta_j^i]}_{c_{j+1}^i} - q \Omega_{j+1} + q \Psi_{j+1} \right\}}_{a_{j+1}^i} = (1 + \tilde{r}) a_j^i + y_j$$

$$\implies c_j^i = \frac{1}{1 + q (\Lambda_{j+1}^i) \Phi_{j+1}} \left[ \frac{(1 + \tilde{r}) a_j^i + \Omega_j}{1 + \tau_c} - \Psi_j^i \right].$$

Hence,  $\Lambda_j^i = 1 + q (\Lambda_{j+1}^i) \Phi_j^i$  and  $\Psi_j^i = q \sum_{k=j}^{J_R-1} q^{k-j} (\Lambda_{k+1}^i) \Delta_{k+1}^i$ . *Q.E.D.*

## Appendix C

### I. Stability in the MIU Model

Linearizing (11), (18), (20) and (21) yields the following system of equations:

$$\begin{bmatrix} \dot{\lambda}_{kt} \\ \dot{k}_t \\ \dot{m}_t \\ \dot{\lambda}_{\beta t} \end{bmatrix} = \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}}_B \begin{bmatrix} c_t - c \\ k_t - k \\ m_t - m \\ \lambda_{\beta t} - \lambda_{\beta} \end{bmatrix},$$

where

$$\begin{aligned} b_{11} &= -(f_k - \delta - \rho(s))u_{cc} + \gamma_c [u_c \rho' + \gamma_k \lambda_{\beta} \rho''] \\ b_{12} &= -u_c [f_{kk} - \gamma_k \rho'] + \gamma_k^2 \lambda_{\beta} \rho'' \\ b_{13} &= -(f_k - \delta - \rho(s))u_{cm} + \gamma_m [u_c \rho' + \gamma_k \lambda_{\beta} \rho''] \\ b_{14} &= \gamma_k \rho' \\ b_{21} &= -1 \\ b_{22} &= f_k - \delta \\ b_{23} &= 0 \\ b_{24} &= 0 \\ b_{31} &= -\frac{m}{u_c} \omega_c + \left( \frac{\gamma_m - \gamma_k}{u_c} \right) \left[ \gamma_c \lambda_{\beta} \rho'' - \left( \frac{u_{cc}}{u_c} \right) \lambda_{\beta} \rho' \right] \\ b_{32} &= f_{kk} m + \gamma_k \left( \frac{\gamma_m - \gamma_k}{u_c} \right) \lambda_{\beta} \rho'' \\ b_{33} &= -\frac{m}{u_c} \omega_m + \left( \frac{\gamma_m - \gamma_k}{u_c} \right) \left\{ \gamma_m \lambda_{\beta} \rho'' - \left( \frac{u_{cm}}{u_c} \right) \lambda_{\beta} \rho' \right\} \\ b_{34} &= \left( \frac{\gamma_m - \gamma_k}{u_c} \right) \rho' \\ b_{41} &= \gamma_c \lambda_{\beta} \rho' - u_c \\ b_{42} &= \gamma_k \lambda_{\beta} \rho' \\ b_{43} &= \gamma_m \lambda_{\beta} \rho' - u_m \\ b_{44} &= \rho(s). \end{aligned}$$

### II. Stability in the CIA Model with $\phi = 1$

Linearizing (55), (56), (57), (58) and (61) yields the following system of equations:

$$\begin{bmatrix} \dot{c}_t \\ \dot{k}_t \\ \dot{\widehat{\lambda}}_{kt} \\ \dot{\widehat{\lambda}}_{mt} \\ \dot{\widehat{\lambda}}_{\beta t} \end{bmatrix} = \underbrace{\begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} & q_{15} \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} \\ q_{31} & q_{32} & q_{33} & q_{34} & q_{35} \\ q_{41} & q_{42} & q_{43} & q_{44} & q_{45} \\ q_{51} & q_{52} & q_{53} & q_{54} & q_{55} \end{bmatrix}}_Q \begin{bmatrix} c_t - c \\ k_t - k \\ \widehat{\lambda}_{kt} - \widehat{\lambda}_k \\ \widehat{\lambda}_{mt} - \widehat{\lambda}_m \\ \widehat{\lambda}_{\beta t} - \widehat{\lambda}_\beta \end{bmatrix},$$

where

$$\begin{aligned} q_{11} &= \frac{1}{u_{cc} - \gamma_c^2 \widehat{\lambda}_\beta \rho''} \left\{ \gamma_c \left[ \widehat{\lambda}_k \rho' + \gamma_k \widehat{\lambda}_\beta \rho'' + (\gamma_c \widehat{\lambda}_\beta \rho' - u_c) \rho' - (\gamma_k + \gamma_m \alpha \frac{m}{k}) \widehat{\lambda}_\beta \rho'' \right] \right\} \\ q_{12} &= \frac{1}{u_{cc} - \gamma_c^2 \widehat{\lambda}_\beta \rho''} \left\{ (\gamma_k + \gamma_m f_k) \left[ \widehat{\lambda}_k \rho' + \gamma_k \widehat{\lambda}_\beta \rho'' \right] - f_{kk} \widehat{\lambda}_m + \gamma_c (\gamma_k + \gamma_m \alpha \frac{m}{k}) (f_k - \delta) \widehat{\lambda}_\beta \rho'' \right. \\ &\quad \left. + \gamma_c (\gamma_k + \gamma_m f_k) \widehat{\lambda}_\beta (\rho')^2 \right\} \\ q_{13} &= \rho(s) + \delta \\ q_{14} &= -f_k \\ q_{15} &= \gamma_k \rho' + \gamma_c \rho' \rho(s) \\ q_{21} &= -1 \\ q_{22} &= f_k - \delta \\ q_{23} &= 0 \\ q_{24} &= 0 \\ q_{25} &= 0 \\ q_{31} &= \gamma_c \left( \gamma_k \widehat{\lambda}_\beta \rho'' + \widehat{\lambda}_k \rho' \right) \\ q_{32} &= (\gamma_k + \gamma_m f_k) \widehat{\lambda}_k \rho' + \gamma_k \widehat{\lambda}_\beta \rho'' - f_{kk} \\ q_{33} &= \rho(s) + \delta \\ q_{34} &= -f_k \\ q_{35} &= \gamma_m \rho' \\ q_{41} &= \gamma_c \widehat{\lambda}_m \rho' + \gamma_c \gamma_m \widehat{\lambda}_\beta \rho'' - u_{cc} \\ q_{42} &= (\gamma_k + f_k \gamma_m) \left( \widehat{\lambda}_m \rho' + \gamma_m \widehat{\lambda}_\beta \rho'' \right) \\ q_{43} &= 0 \end{aligned}$$

$$\begin{aligned}
q_{44} &= 1 + \rho(s) + \pi \\
q_{45} &= \gamma_m \rho' \\
q_{51} &= \gamma_c \widehat{\lambda}_\beta \rho' - u_c \\
q_{52} &= (\gamma_k + \gamma_m f_k) \widehat{\lambda}_\beta \rho' \\
q_{53} &= 0 \\
q_{54} &= 0 \\
q_{55} &= \rho(s).
\end{aligned}$$

### III. Superneutrality and Stability in the CIA Model with $\phi = 0$

*Superneutrality.* Linearizing  $\dot{c}_t = 0$ ,  $\dot{k}_t = 0$ ,  $\dot{\widetilde{\lambda}}_{mt} = 0$  and  $\dot{\widetilde{\lambda}}_{\beta t} = 0$  around the steady state, we get

$$\underbrace{\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix}}_D \begin{bmatrix} dc \\ dk \\ d\widetilde{\lambda}_k \\ d\widetilde{\lambda}_\beta \end{bmatrix} = \begin{bmatrix} \widetilde{\lambda}_k \\ 0 \\ 0 \\ 0 \end{bmatrix} d\mu,$$

where

$$\begin{aligned}
d_{11} &= u_{cc} - (\gamma_c + \gamma_m)(\gamma_c + \gamma_m - \gamma_k) \widetilde{\lambda}_\beta \rho'' \\
d_{12} &= -\widetilde{\lambda}_k f_{kk} - \gamma_k (\gamma_c + \gamma_m - \gamma_k) \widetilde{\lambda}_\beta \rho'' \\
d_{13} &= -(1 + f_k - \delta + \pi) \\
d_{14} &= -(\gamma_c + \gamma_m - \gamma_k) \rho' \\
d_{21} &= -1 \\
d_{22} &= f_k - \delta \\
d_{23} &= 0 \\
d_{24} &= 0 \\
d_{31} &= (\gamma_c + \gamma_m) (\gamma_k \widetilde{\lambda}_\beta \rho'' + \widetilde{\lambda}_k \rho') \\
d_{32} &= \gamma_k (\gamma_k \widetilde{\lambda}_\beta \rho'' + \widetilde{\lambda}_k \rho')
\end{aligned}$$



$$\begin{aligned}
d_{33} &= -(f_k - \delta - \rho(s)) \\
d_{34} &= \gamma_k \rho' \\
d_{41} &= u_c - (\gamma_c + \gamma_m) \tilde{\lambda}_\beta \rho' \\
d_{42} &= -\gamma_k \tilde{\lambda}_\beta \rho' \\
d_{43} &= 0 \\
d_{44} &= -\rho(s).
\end{aligned}$$

*Stability.* Linearizing (70) to (73) around the steady state yields the following system of equations

$$\begin{bmatrix} \dot{c}_t \\ \dot{k}_t \\ \dot{\tilde{\lambda}}_{kt} \\ \dot{\tilde{\lambda}}_{\beta t} \end{bmatrix} = \underbrace{\begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}}_E \begin{bmatrix} c_t - c \\ k_t - k \\ \tilde{\lambda}_{kt} - \tilde{\lambda}_k \\ \tilde{\lambda}_{\beta t} - \tilde{\lambda}_\beta \end{bmatrix},$$

where

$$\begin{aligned}
e_{11} &= \frac{c}{\lambda_k} \left[ (\gamma_c + \gamma_m)(\gamma_c + \gamma_m - \gamma_k) \tilde{\lambda}_\beta \rho'' - u_{cc} \right] \\
e_{12} &= c f_{kk} + \frac{c}{\lambda_k} \left[ \gamma_k (\gamma_c + \gamma_m - \gamma_k) \tilde{\lambda}_\beta \rho'' \right] \\
e_{13} &= \frac{c}{\lambda_k^2} \left[ (\gamma_c + \gamma_m - \gamma_k) \tilde{\lambda}_\beta \rho' - u_c \right] \\
e_{14} &= \frac{c}{\lambda_k} [(\gamma_c + \gamma_m - \gamma_k) \rho'] \\
e_{21} &= -1 \\
e_{22} &= f_k - \delta \\
e_{23} &= 0 \\
e_{24} &= 0 \\
e_{31} &= (\gamma_c + \gamma_m) \left( \gamma_k \tilde{\lambda}_\beta \rho'' + \tilde{\lambda}_k \rho' \right) \\
e_{32} &= \gamma_k \left( \gamma_k \tilde{\lambda}_\beta \rho'' + \tilde{\lambda}_k \rho' \right) - \tilde{\lambda}_k f_{kk} \\
e_{33} &= -(f_k - \delta - \rho(s)) \\
e_{34} &= \gamma_k \rho' \\
e_{41} &= (\gamma_c + \gamma_m) \tilde{\lambda}_\beta \rho' - u_c
\end{aligned}$$

$$e_{42} = \gamma_k \tilde{\lambda}_\beta \rho^l$$

$$e_{43} = 0$$

$$e_{44} = \rho(s).$$