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Projection, Search, and Optimality in Fractional Factorial Experiments

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Applied Statistics

by

Zongpeng Zheng

December 2014

Dissertation Committee:
Dr. Subir Ghosh, Chairperson
Dr. Jun Li
Dr. Gregory J. Palardy

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2014

The Dissertation of Zongpeng Zheng is approved:

Committee Chairperson

University of California, Riverside

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ABSTRACT OF THE DISSERTATION

Projection, Search, and Optimality in Fractional Factorial Experiments

by

Zongpeng Zheng

Doctor of Philosophy, Graduate Program in Applied Statistics
University of California, Riverside, December 2014
Dr. Subir Ghosh, Chairperson

We propose a general Up-Down method to search for efficient 2^m fractional factorial designs in fitting a class of models when the number of factors is *m*, and the number of runs is *n*. The orthogonal array designs exist for some specific values of *n*. The orthogonal array designs are optimal under the resolution assumptions. The proposed Up-Down method searches for efficient designs having the number of runs in between two values of *n* for orthogonal array designs satisfying a resolution assumption. We present the efficient resolution III designs obtained by the Up-Down method for $3 \le m \le 10$ and a range of practical values of *n*. While many of these designs are found to be the global optimal resolution III designs by exhaustive computer search, the other designs are near global optimal designs. For *m*=4 and 5, we compare our designs with the optimal resolution III+*k* (*k*=0,1,2,…) designs in Ghosh and Tian (2006). Moreover, we utilize the method to obtain unbalanced Up-Res V designs performing slightly better than the

balanced optimal fractional factorial designs (BOFFD) given in Srivastava and Chopra (1971) with respect to A- and D-optimality criteria. For a given *n*, all our designs are isomorphic having same optimality properties. For general *m* and *n*, the conditions are derived for obtaining such isomorphic designs with respect to Trace and Determinant.

Several interesting projection properties are known in the literature for orthogonal arrays and in particular for the Plackett-Burman (PB) designs. In this dissertation, the projection properties are investigated for both orthogonal and non-orthogonal array designs under different model assumptions. The structure of the variance-covariance matrix for the estimates of the model parameters is characterized. The optimality properties of these designs are also investigated. For m=5, we consider seven 12-run designs $d_i, i=1,...,7$ and a collection of classes of models. The designs $d_i, i=1,...,5$ are balanced arrays of full strength, d_6 and d_7 are orthogonal arrays of strength 2. The designs d_6 and d_7 are two non-isomorphic designs obtained from the PB design by projecting 11 factors onto 5 factors. Overall, our designs d_1 and d_3 are at the top of their performances. By projection, all possible t ($\leq m$) factors out of m factors are considered. As t increases from 2, to 3 and 4, the design d_1 becomes better and better compared to the design d_3. When t=5, the design d_3 is optimal under resolution III model. For fitting resolution III plus k (k=1,2,3) models, the design d_1 again becomes better and better compared to the design d_3 as k increases.

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Chapter 1

Introduction to Factorial Design

1.1 Factorial Design (FD)

Many scientific investigations involve the study of effects of two or more factors on a response variable simultaneously. Factorial design is essential for experiments in such investigations. It allows us to take all level combinations of the factors into consideration simultaneously rather than one at a time.

A typical complete 2^m factorial design is given by a design matrix that consists of 2^m rows and m columns. Each row is a level combination of the m factors representing a condition under which our response will be measured, simply called "run". Each column represents a potential factor of interest, and takes value 0 at its low level and 1 at its high level. Table 1.1 shows a simple case of the complete 2^3 factorial design; Table 1.2 shows another example of a complete 3^2 factorial design. A more general design is called a general factorial design or a mixed-level factorial design which is very similar to the 2^m factorial design except that it allows the factors to have different numbers of levels. A typical example of mixed-level factorial design is given in Table 1.3.

Table 1.1 Complete 2^3 factorial design

	Factors			
Run(condition)	A	B	C	Observations
1	1	1	1	$y_{11}\ldots,y_{1n}$
2	1	1	0	$y_{21}\ldots,y_{2n}$
3	1	0	1	$y_{31}\ldots,y_{3n}$
4	1	0	0	$y_{41}\ldots,y_{4n}$
5	0	1	1	$y_{51}\ldots,y_{5n}$
6	0	1	0	$y_{61}\ldots,y_{6n}$
7	0	0	1	$y_{71}\ldots,y_{7n}$
8	0	0	0	$y_{81}\ldots,y_{8n}$

Table 1.2 Complete 3^2 factorial design

	Factors		
Run(condition)	A	B	Observations
1	0	0	$y_{11}\ldots,y_{1n}$
2	0	1	$y_{21}\ldots,y_{2n}$
3	0	2	$y_{31}\ldots,y_{3n}$
4	1	0	$y_{41}\ldots,y_{4n}$
5	1	1	$y_{51}\ldots,y_{5n}$
6	1	2	$y_{61}\ldots,y_{6n}$
7	2	0	$y_{71}\ldots,y_{7n}$
8	2	1	$y_{81}\ldots,y_{8n}$
9	2	2	$y_{91}\ldots,y_{9n}$

Table 1.3 A mixed-level $2^1 3^1$ factorial design

	Factors		
Run(condition)	A	B	Observations
1	0	0	$y_{11}\ldots,y_{1n}$
2	1	1	$y_{21}\ldots,y_{2n}$
3	0	2	$y_{31}\ldots,y_{3n}$
4	1	0	$y_{41}\ldots,y_{4n}$
5	0	1	$y_{51}\ldots,y_{5n}$
6	1	2	$y_{61}\ldots,y_{6n}$

1.2 Fractional Factorial Design

The above designs are called complete factorial designs because they include all the level combinations of the factors. The complete 2^m factorial designs become prohibitively large because the number of treatment combinations grows by power of 2. For example, if *m*=10, the number of treatment combinations is $2^{10} = 1024$. Fractional factorial designs are used to overcome this problem by choosing a fraction of the complete factorial experiments. Such a design is feasible based on the following three principles.

I. Hierarchical Ordering Principle

The Hierarchical Ordering Principle states that higher order effects are often smaller in magnitude and hence less important than lower order effects, such as the main-effects and two-factor interactions. According to the Hierarchical Ordering Principle, it is reasonable to assume that certain high-order interactions are negligible so we can obtain the information on the main effects and low-order interactions by running only a fraction of the complete factorial experiment.

II. Effects Sparsity Principle

The Effects Sparsity Principle states that a system or process that contains many factors is likely to be driven primarily by only a few of these factors and their lower interactions.

III. Effects Heredity Principle

The Effects Heredity Principle states that if a higher order effect is important, then at least one of the lower order effects also must be important. For example, if a two-factor interaction is important, then at least one of the main effects also must be important. According to this principle, a model that contains higher order interactions but does not contain any of its parent factors is not appropriate.

A typical half fractional factorial design, 2^{m-1} design, can be obtained by choosing half runs from a 2^m complete factorial design. Likewise, a typical 2^{m-2} fractional factorial design can be obtained by choosing one fourth of the runs from a 2^m complete factorial design. For example, if $m = 5$, we can construct a 2^{5-1} fractional factorial design by taking all the runs that satisfy I=ABCDE. That is, the five-factor interaction ABCDE has its contrast coefficient 1 in all the chosen runs. Similarly, we can construct a 2^{5-2} fractional factorial design by taking all the runs that satisfy I=ABC=ADE=BCDE. Here, I=ABC=ADE=BCDE is called defining relation. From the above defining relation, we know that the contrast coefficients are the same for the effect

A and BC, and hence their effects cannot be separated. In other words, the estimators of A and BC are identical, or A=BC. These two effects are said to be aliased with each other.

A good fractional factorial design should have a defining relation such that the lower effects will not be aliased with each other. Resolution was introduced by Box and Hunter (1961) for this purpose. The length of the lowest order interaction in the defining relation is called resolution. The properties of designs with resolution III, IV, V are shown in the following.

Resolution III

In a resolution III design, main effects are aliased with second- and higher order interactions but not with each other. This allows us to estimate all the main effects, assuming all the second- and higher order effects are negligible.

Resolution IV

In a resolution IV design, main effects are aliased with third- and higher order interactions, but second-order interactions may be aliased with each other. This allows us to estimate all the main effects and some second-order interactions assuming the remaining second- and higher order interactions are negligible.

Resolution V

In a resolution V design, main effects are aliased with four-factor and higher order interactions, and two factor interactions are aliased with three-factor interactions and higher order interactions. This allows us to estimate all the effects up to second-order interactions assuming all the third- and higher order interactions are negligible.

So far, the fractional factorial design has been obtained by purposely choosing runs which share a common defining relation. This type of fractional factorial design is called a regular design. A fractional factorial design that has no defining relation is called a non-regular design. Table 1.4 and Table 1.5 give two examples of regular fractional factorial design. Table 1.6 gives an example of a non-regular fractional factorial design. In a non-regular fractional factorial design, the aliasing structure is complex.

Table 1.4 A one-half fractional of 2^4 factorial design which takes the runs from a complete 2^4 factorial design based on I=ABCD

Run	A	B	C	D=ABC
1	+	+	+	+
2	+	+	-	-
3	+	-	+	-
4	+	-	-	+
5	-	+	+	-
6	-	+	-	+
7	-	-	+	+
8	-	-	-	-

Table 1.5 A one-fourth fractional of 2^5 factorial design which takes the runs from a complete 2^5 factorial design based on I=ABD=ACE=BCDE

Run	A	B	C	D=AB	E=AC
1	-	-	-	+	+
2	+	-	-	-	-
3	-	+	-	-	+
4	+	+	-	+	-
5	-	-	+	+	-
6	+	-	+	-	+
7	-	+	+	-	-
8	+	+	+	+	+

Table 1.6 A non-regular design obtained by choosing half rows from a 2^4 complete factorial design based on I=ABCD combined by choosing a one-fourth rows from the same 2^4 complete factorial design but based on I=-ABCD. This design has no defining relation.

Run	A	B	C	D	
1	+	+	+	+	
2	+	+	-	-	
3	+	-	+	-	
4	+	-	-	+	I=ABCD
5	-	+	+	-	
6	-	+	-	+	
7	-	-	+	+	
8	-	-	-	-	
9	+	+	-	+	
10	+	-	+	+	I= -ABCD
11	-	+	-	-	
12	-	-	+	-	

1.3 Array for FD

1.3.1 Orthogonal Array (OA)

In C. R. Rao (1947), an orthogonal array of size N, m constraints, s levels and strength t, denoted $OA(N, s^m, t)$, is defined as a $N \times m$ matrix X of s symbols such that all the ordered t-tuples of the symbols occur equally often as row vectors of any $N \times t$ submatrix of X. Obviously, N must be of the form λs^t, where λ is usually called the index of the orthogonal array.

When applied to factorial designs, each $OA(N, s^m, t)$ defines a N-run factorial design for m s-level factors, with the symbols representing factor levels, columns representing factors and rows representing factor-level combinations. An important class of orthogonal arrays is the so-called regular fractional factorial designs which are constructed by using defining relations. It is well known that a regular fractional factorial design of resolution $t+1$ is an orthogonal array of strength t. An example of a regular 2^{4-1} fractional factorial design of resolution four given in Table 1.4 is an $OA(8, 2^4, 3)$. The other class of orthogonal arrays is the non-regular fractional factorial designs which are not constructed by defining relations. An example of an important non-regular $OA(12, 2^{11}, 2)$ is given in Table 1.7. It is the well-known 12-run Plackett-Burman (PB) design (Plackett and Burman, 1946) which will be discussed in detail in later chapters.

Table 1.7 A non-regular fractional factorial design: $OA(12, 2^{11}, 2)$

Run	1	2	3	4	5	6	7	8	9	10	11
1	+	+	-	+	+	+	-	-	-	+	-
2	-	+	+	-	+	+	+	-	-	-	+
3	+	-	+	+	-	+	+	+	-	-	-
4	-	+	-	+	+	-	+	+	+	-	-
5	-	-	+	-	+	+	-	+	+	+	-
6	-	-	-	+	-	+	+	-	+	+	+
7	+	-	-	-	+	-	+	+	-	+	+
8	+	+	-	-	-	+	-	+	+	-	+
9	+	+	+	-	-	-	+	-	+	+	-
10	-	+	+	+	-	-	-	+	-	+	+
11	+	-	+	+	+	-	-	-	+	-	+
12	-	-	-	-	-	-	-	-	-	-	-

1.3.2 Balanced Array (BA)

In Chakravarti (1956), Srivastava and Chopra (1973), a balanced array of size N, m constraints, s levels and strength t, denoted $BA(N, m, s, t)$, is defined as a $N \times m$ matrix X of s symbols ($0, 1, \ldots, s-1$) such that every $N \times t$ submatrix of X contains the ordered t-tuples of the symbol vector $(x_1, x_2, \ldots x_t)$, $\lambda_{x_1 x_2 \ldots x_t}$ times, where $x_i = 0, 1, \ldots, s-1$; $i = 1, 2, \ldots, t$; and $\lambda_{x_1 x_2 \ldots x_t}$ is invariant under any permutation of $(x_1, x_2, \ldots x_t)$. When $\lambda_{x_1 x_2 \ldots x_t} = \lambda$ for all $(x_1, x_2, \ldots x_t)$, X is called an orthogonal array with index λ. Balanced array is a generalization of balanced incomplete block (BIB) design. This is true because the incidence matrix of a BIB design is a balanced array of strength 2 with 2 symbols.

In case of $s = 2$, a matrix X of size $N \times m$ with elements -1 and 1 is said to be a balanced array of strength t, if for every $N \times t$ submatrix X_0 of X and for every (-1,1) vector $\underline{v}$ of size $(1 \times t)$ we have

$$\lambda(\underline{v}, X_0) = \lambda_i,\ i = 0,1,...t,$$

for all such matrices X_0, where $\lambda(\underline{v}, X_0)$ denotes the frequency of $\underline{v}$ appears as a row of X_0, λ_i is a nonnegative integer, and where the vector $\underline{v}$ is of weight *i*. The weight of a vector is the number of ones in the vector. The vector $(\lambda_0, \lambda_1, ..., \lambda_t)$ is called the index set of the array.

In this thesis, when *s*=2, we define $S_i, i = 0,1,...,m$ to be a set that contains all runs with *i* factors observed at the high level (which is represented by +1) and $m - i$ factors observed at the low level (which is represented by -1). The number of runs in S_i is $\begin{pmatrix} m \\ i \end{pmatrix}$ and $S_i = -S_{m-i}$ by this definition. For illustration purpose, the sets S_0, S_1, and S_2 are shown below.

$$S_0 = (-1,-1,...,-1),\ S_1 = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & -1 & \cdots & -1 \\ -1 & -1 & 1 & \cdots & -1 \\ & \vdots & & & \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix},\ S_2 = \begin{pmatrix} 1 & 1 & -1 & -1 & \cdots & -1 \\ 1 & -1 & 1 & -1 & \cdots & -1 \\ & \vdots & & & & \\ 1 & -1 & -1 & -1 & \cdots & 1 \\ & \vdots & & & & \\ -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

A balanced array can be composed of runs from complete sets of S_i's. An example of a balanced array with $m = 4$, $N = 12$, shown in Figure 1.1 contains runs from complete sets of S_0, S_1, S_2, and S_4.

Figure 1.1 A $BA(N = 12, m = 4, s = 2, t = 4)$ with $\lambda_0 = \lambda_1 = \lambda_2 = 1, \lambda_3 = 0, \lambda_4 = 1$

$$
\begin{array}{c|cccc|}
\boxed{S_0} \leftarrow & - & - & - & - \\
 & + & - & - & - \\
\boxed{S_1} \; \Bigg\{ & - & + & - & - \\
 & - & - & + & - \\
 & - & - & - & + \\
 & + & + & - & - \\
 & + & - & + & - \\
\boxed{S_2} \; \Bigg\{ & + & - & - & + \\
 & - & + & + & - \\
 & - & + & - & + \\
 & - & - & + & + \\
\boxed{S_4} \leftarrow & + & + & + & + \\
\end{array}
$$

1.3.3 Box-Tyssedal Array (BT)

In Box and Tyssedal (1996), an array of size N, m constraints, s levels is said to be of projectivity p if it is a $N \times m$ matrix X of s symbols such that all the ordered p-tuples of the symbols occur at least once as row vector of any $N \times p$ submatrix of X. In factorial designs, a design is said to be of projectivity p if the projection onto every subset of p factors contains a complete s^p factorial design, possibly with some runs replicated. Therefore, when there are no more than p important factors, no matter what these factors are, the projection of such design onto the important factors allows all the factorial effects to be estimated. A BT of projectivity 3 is given in Table 1.8.

Table 1.8 A BT of projectivity 3

Run	A	B	C	D	
1	+	+	+	+	D=ABC
2	+	+	-	-	
3	+	-	+	-	
4	+	-	-	+	
5	-	+	+	-	
6	-	+	-	+	
7	-	-	+	+	
8	-	-	-	-	
9	+	+	+	-	D= -ABC
10	-	-	-	+	

1.4 Robust FD against Incomplete Data

A complete 2^m factorial design allows estimating the general mean, main effects, 2-factor interactions, …, up to m-factor interaction. However, in practice, only a small set of these parameters might be of interest. If so, can we use the 2^m factorial with missing any t runs to estimate them? Such a question raises the consideration on the robustness of factorial designs against incomplete data.

Ghosh (1978, 1979) considered the above problem under the following ordinary linear model:

$$
\begin{aligned}
&E(\boldsymbol{y}) = \boldsymbol{X\xi} \\
&V(\boldsymbol{y}) = \boldsymbol{\sigma}^2 \boldsymbol{I}_N \qquad (1.1)\\
&Rank\,\boldsymbol{X} = v
\end{aligned}
$$

where $\boldsymbol{y}(N \times 1)$ is a vector of observations, $\boldsymbol{X}(N \times v)$ is a known matrix, $\boldsymbol{\xi}(v \times 1)$ is a vector of fixed unknown parameters and $\boldsymbol{\sigma}^2$ is a constant which may or may not be known.

Let T be the underlying design corresponding to $\boldsymbol{y}$. Then, T is said to be robust against missing of any t runs if the $(\underline{N\text{-}t} \times v)$ matrix obtained from $\boldsymbol{X}$ by omitting any t rows has rank v. Obviously, N must at least be $v+t$. Suppose $N=v+k$, where $k(\geq t)$ is a positive integer. Since $Rank\,\boldsymbol{X} = v$, there exist k linearly independent vectors $\mathbf{C}_i^{\mathbf{T}} = (c_{i1},..., c_{i\mathbf{N}})$, $i=1,...,k$, with real elements satisfying

$$\mathbf{C}_i^{\mathbf{T}} X = 0$$

Consider the $(k \times N)$ matrix

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1t} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2t} & \cdots & c_{2N} \\ c_{k1} & c_{k2} & \cdots & c_{kt} & \cdots & c_{kN} \end{bmatrix} \qquad (1.2)$$

whose i-th row is $\mathbf{C}_i^{\mathrm{T}}$ and furthermore, *Rank* $\mathbf{C} = k.$ Ghosh (1979) defined that a matrix is said to have the property P_t if any its t columns are linearly independent, based on which he characterized the robustness property in the following theorem.

Theorem 1.1 (Ghosh, 1979) Let T be a design under (1.1) with N=v+k observations, where $k(\geq t)$ a positive integer. Then, T is robust against missing of any t observations if and only if the matrix **C**, defined in (1.2), has the property P_t .

Specifically, Ghosh (1979) provided the conditions for **C** to have property P_1 and P_2 under general factorial experiments in Table 1.9. Moreover, we give equivalent conditions under two-level factorial experiments in Table 1.10.

Table 1.9 Property P_t in general designs

Property	Conditions need to be satisfied
P_1	$\left(\mathrm{c}_{1j},\mathrm{c}_{2j},\ldots,\mathrm{c}_{kj}\right) \neq \left(0,0,\ldots,0\right)$ *for* $(j=1,\ldots,N)$
P_2	$\left(\mathrm{c}_{1j},\mathrm{c}_{2j},\ldots,\mathrm{c}_{kj}\right) \neq \left(0,0,\ldots,0\right)$ *for* $(j=1,\ldots,N)$ $\left(\mathrm{c}_{1j},\mathrm{c}_{2j},\ldots,\mathrm{c}_{kj}\right) \neq \mathrm{w}\left(\mathrm{c}_{1j'},\mathrm{c}_{2j'},\ldots,\mathrm{c}_{kj'}\right)$ *where* $j \neq j'$, $(j, j'=1,\ldots,N)$

Table 1.10 Property P_t under two-level factorial experiments

Property	Conditions need to be satisfied
P_1	No null vector as a column.
P_2	a. P_1; b. Every submatrix with 2 columns must have 2 distinct rows which are not complement of each other. (easy to prove)
P_3	P_2 (see Proof 1.1 in Appendix)

In the rest of this section, we consider a full 2^3 factorial design T (as shown below) with factors A, B, and C under model (1.1), and use Theorem 1.1 to study its robustness property in different cases, depending on which parameters are of interests.

$$\mathbf{T} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix} \end{array}$$

Case 1, Suppose we are interested in all the 2-factor interactions besides the main effects, then $\boldsymbol{X}$ in model (1.1) is a 8×7 matrix with columns corresponding to μ, A, B, C, AB, AC, BC, and $Rank\, X = 7$. By Theorem 1.1, we conclude that design T is robust against missing of any t=1 observation, and **C** $(k \times N)$=(1, -1, -1, 1, -1, 1, 1, -1) is the transpose of vector corresponding to the effect ABC and has the property P_1.

Case 2, Suppose we are only interested in the two 2-factor interactions (AB, AC), then $\boldsymbol{X}(8 \times 6)$ is a known matrix corresponding to μ, A, B, C, AB, AC and $Rank\, \boldsymbol{X} = 6$. By Theorem 1.1, we conclude that design T is robust against missing of any t=1 observation. The matrix $\mathbf{C}(k \times N) = \begin{array}{c} ABC \\ BC \end{array} \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}_{(2 \times 8)}$ is the transpose of vectors corresponding to effects ABC and BC. Matrix **C** has the property P_1 but not P_2.

Case 3, Suppose we are only interested in one 2-factor interaction (AB), then $\boldsymbol{X}(8\times 5)$ is a known matrix corresponding to μ, A, B, C, AB and $Rank\,\boldsymbol{X}=5$. By Theorem 1.1, we conclude that design T is robust against missing of any t=1 observation. The matrix $\mathbf{C}(k\times N)=\begin{array}{c} ABC \\ BC \\ AC \end{array}\begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix}_{(3\times 8)}$ is the transpose of vectors corresponding to effects ABC, BC, and AC. Matrix **C** has the property P_1 but not P_2.

Case 4, Suppose we are not interested in any 2-factor interactions, then $\boldsymbol{X}(8\times 4)$ is a known matrix corresponding to μ, A, B, C and $Rank\,\boldsymbol{X}=4$. By Theorem 1.1, we conclude that design T is robust against missing of any t=3 observations. The matrix $\mathbf{C}(k\times N)=\begin{array}{c} ABC \\ BC \\ AC \\ AB \end{array}\begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}_{(4\times 8)}$ is the transpose of vectors corresponding to effects ABC, BC, AC, and AB. Matrix **C** has the property P_1, P_2, P_3 but not P_4.

Case 5, Suppose we are only interested in two main effects (A, B), then $\boldsymbol{X}(8\times 3)$ is a known matrix corresponding to μ, A, B and $Rank\,\boldsymbol{X}=3$. By Theorem 1.1, we conclude that design T is robust against missing of any t=3 observations. The matrix

$$\mathbf{C}(k \times N) = \begin{array}{c} ABC \\ BC \\ AC \\ AB \\ C \end{array} \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}_{(5\times 8)}$$

is the transpose of vectors corresponding to effects ABC, BC, AC, AB and C. Matrix **C** has the property P_1, P_2, P_3 but not P_4.

1.5 Search Linear Model in FD

The usefulness of fractional factorial experiments is based on the assumption that high order interactions are negligible. However, J.N. Srivastava (1975) questioned the validity of this assumption and pointed out there do exist some experiments containing a few non-negligible effects which are assumed negligible and are difficult to be pinpointed in advance. For this reason, he divided the factorial effects into three categories: (1) effects of interest, i.e., those we want to estimate anyway, (2) effects that are certainly negligible, and (3) the remaining effects most of which are actually negligible, but a few of which may be non-negligible.

In view of the above, he expressed a set of N observations, given by the $N \times 1$ vector $\boldsymbol{y}$, in terms of the following linear model:

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + e \tag{1.3}$$

$$E(e) = 0, \quad \text{var}(e) = \boldsymbol{\sigma}^2\boldsymbol{I}_N \tag{1.4}$$

where e is a $N \times 1$ vector of measurement error, $\boldsymbol{X}_i(N \times v_i)\,(i = 1,2)$ are the design matrices, $\boldsymbol{\beta}_1$ is a $v_1 \times 1$ vector of unknown parameters consisting of the effects in category (1), $\boldsymbol{\beta}_2$ is a $v_2 \times 1$ vector of unknown parameters consisting of the effects in category (3) among which at most k elements are assumed to be non-negligible. Let T be the design that gives rise to the observations Y in (1.3). If $\boldsymbol{y}$ and hence $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are such that we can estimate $\boldsymbol{\beta}_1$ and the k non-negligible effects of $\boldsymbol{\beta}_2$, then T is said to be a search design of resolving power $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, k\}$ and the model (1.3) is said to be a search linear model. When $\boldsymbol{\beta}_2 = 0$, the search linear model becomes an ordinary linear model. A fundamental property of such design is characterized by Theorem 1.2.

Theorem 1.2 (Srivastava, 1975)

(i) Given model (1.3), a necessary condition for a design T to be a search design of resolving power $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, k\}$ is that for every submatrix $\boldsymbol{X}_{20}(N \times 2k)$ of $\boldsymbol{X}_2$, the augmented matrix $(\boldsymbol{X}_1 : \boldsymbol{X}_{20})$ is of full rank, that is, $Rank(\boldsymbol{X}_1 : \boldsymbol{X}_{20}) = v_1 + 2k$.

(ii) In the noiseless case, that is, for e=0 in (1.3) and (1.4), the condition is also sufficient.

To search for the k non-negligible effects, Srivastava (1975, 1977) suggested conducting model identification and model discrimination as follows.

Model identification

Consider the following class of $w = \binom{v_2}{k}$ models from (1.3):

$$\begin{aligned}
&M1: \quad E(\boldsymbol{y}) = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_{21}\boldsymbol{\beta}_{21}; \\
&M2: \quad E(\boldsymbol{y}) = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_{22}\boldsymbol{\beta}_{22}; \\
&\ \ \vdots \qquad\qquad \vdots \\
&Mw: \quad E(\boldsymbol{y}) = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_{2w}\boldsymbol{\beta}_{2w}.
\end{aligned} \tag{1.5}$$

where $\boldsymbol{\beta}_{21}, \boldsymbol{\beta}_{22}, ..., \boldsymbol{\beta}_{2w}$ are all the $w = \binom{v_2}{k}$ possible vectors when choosing k elements in $\boldsymbol{\beta}_2$, and $\boldsymbol{X}_{21}, \boldsymbol{X}_{22}, ..., \boldsymbol{X}_{2w}$ are the corresponding submatrices in $\boldsymbol{X}_2$. To fit model Mi, $rank(\boldsymbol{X}_1, \boldsymbol{X}_{2i}) = v_1 + k$ should be satisfied for $i = 1,2,...,w$. When e=0 , the model Mi fits the data best if and only if $\boldsymbol{\beta}_{2i}$ contains exactly the k non-negligible effects.

Model discrimination

To do the model discrimination, we need to consider every pair of the models in (1.5). For each pair, we consider a bigger model by combining the two selected models together. Take the pair of $M1$ and $M2$ as an example, the combined model is:

$$M1 + M2: \ E(\boldsymbol{y}) = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_{21}\boldsymbol{\beta}_{21} + \boldsymbol{X}_{22}^*\boldsymbol{\beta}_{22}^*$$

If $\boldsymbol{\beta}_{21}$ and $\boldsymbol{\beta}_{22}$ do not have any common parameters, then $\boldsymbol{\beta}_{22}^* = \boldsymbol{\beta}_{22}, \boldsymbol{X}_{22}^* = \boldsymbol{X}_{22}$, and the combined model becomes $E(\boldsymbol{y}) = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_{21}\boldsymbol{\beta}_{21} + \boldsymbol{X}_{22}\boldsymbol{\beta}_{22}$. To fit this model, $rank(\boldsymbol{X}_1 : \boldsymbol{X}_{21} : \boldsymbol{X}_{22}^*) = rank(\boldsymbol{X}_1 : \boldsymbol{X}_{21} : \boldsymbol{X}_{22}) = v_1 + 2k$ should be satisfied.

If $\boldsymbol{\beta}_{21}$ and $\boldsymbol{\beta}_{22}$ have common parameters, then $\boldsymbol{\beta}_{22}^{*}$ contains all the parameters in $\boldsymbol{\beta}_{22}$ but not in $\boldsymbol{\beta}_{21}$ and $\boldsymbol{X}_{22}^{*}$ is the matrix corresponding to $\boldsymbol{\beta}_{22}^{*}$. To fit this model, $rank(\boldsymbol{X}_1 : \boldsymbol{X}_{21} : \boldsymbol{X}_{22}^{*}) = v_1 + k + k^*$, should be satisfied, where k^* equals to the number of parameters in $\boldsymbol{\beta}_{22}^{*}$.

It deserves to point out that the rank conditions shown in Theorem 1.2 guarantees that all the rank conditions for model identification and discrimination are satisfied. Therefore, the search designs introduced above are able to identify the models as well as discriminate the models and hence they are able to identify the "best" model.

1.6 Thesis Outline

In Chapter 2, projection in factorial experiments will be introduced and existing literature about the projection property of some famous factorials will be reviewed. In Chapter 3, projection will be studied from the perspective of statistical modeling and the projection property will be characterized under linear models. In Chapter 4, we will study the projection of PB design as well as BA designs. The projection property between this two types of design will be compared. In Chapter 5, we exhaustively search for the optimal resolution V designs of 2^m series for m=4 and a range of practical values of n. In Chapter 6, we propose a Up method to obtain unbalanced Up-Res V designs for m=5 performing slightly better than the balanced optimal fractional factorial designs (BOFFD) given in Srivastava and Chopra (1971) with respect to A- and D-optimality criteria. For a given n, all our designs are isomorphic having same optimality properties. For general m and n, the conditions are derived for obtaining such isomorphic designs with respect to Trace and Determinant in Chapter 7. In Chapter 8, we propose a general Up-Down method to search for efficient 2^m fractional factorial designs in fitting a class of models when the number of factors is m, and the number of runs is n. We present the efficient resolution III designs obtained by the Up-Down method for $3 \le m \le 10$ and a range of practical values of n.

Chapter 2

Projection

2.0 Main Results

In this chapter, we briefly review the major developments in projections. We discuss orthogonal projection and non-orthogonal projection separately. In the meantime, we present some existing results regarding projection, and our own results on projection from balanced arrays as well.

2.1 Introduction

At the initial stage of an investigation, an experimenter may come up with a large number of potential factors with the goal for identifying a few important factors by screening out the unimportant factors. An experimental design to serve this purpose is known as a "screening design". The projection of a screening design identifies the important factors and their possible interactions that can be estimated. Specifically, suppose there are m factors of interest, out of which t factors are important. A helpful screening design $T(n \times m)$ is such that the $\binom{m}{t}$ projections of T onto any of its t factors have good property to identify the t important factors as well as their interactions.

The literature of the projection on screening design can be divided into two categories: (1) orthogonal projection (Rao, 1947) and (2) non-orthogonal projection (Cheng, 1995; Wang and Wu, 1995).

2.2 Orthogonal Projection

An orthogonal projection is made when the projected design is an orthogonal array. This is the case when we project from $OA(N, s^m, t)$ onto any of its *t* factors. This projection has an important property, i.e., it allows estimating all the main effects and interactions of any *t* factors when the other *m-t* factors are ignored.

Moreover, an $OA(N, s^m, t)$ with t=2l allows estimating all the *m* main effects and interactions involving at most *l* factors when all the interactions involving more than *l* factors are negligible. And an $OA(N, s^m, t)$ with t=2l-1 allows estimating all the *m* main effects and interactions involving at most *l*-1 factors when all the interactions involving more than *l* factors are negligible. For example, an $OA(16, 2^5, 4)$ allows estimating all the 5 main effects as well as all the 2-factor interactions, and an $OA(8, 2^4, 3)$ allows estimating all the 4 main effects.

2.3 Non-Orthogonal Projection

A non-orthogonal projection is made when the projected design is not an orthogonal array. This is the case when we project from $OA(N, s^m, t)$ onto any of its more than t factors. Moreover, this is often the case when we project from balanced arrays or Box-Tyssedal arrays.

Cheng (1995) studied the projection of an $OA(N, 2^m, t)$ onto any of its $t+1$ factors, and found out such a projection must be one of the following three types.

Type I: one or more copies of the complete 2^{t+1} factorial.

Type II: one or more copies of a half-replicate of 2^{t+1}.

Type III: combination of type I and type II.

Moreover, which type does the projection belong to depends on the value of N. If $N \geq 2^{t+1}$, type I, II and III are all possible; if $N < 2^{t+1}$, only type II is possible. Take the $OA(12, 2^{11}, 2)$ in Table 1.7 as an example, the projection onto its t+1=3 factor columns 3, 4 and 5, labeled by factors A, B and C is as follows after rearranging the runs.

Run	3 A	4 B	5 C
11	+	+	+
3	+	+	-
5	+	-	+
9	+	-	-
1	-	+	+
6	-	+	-
7	-	-	+
8	-	-	-
10	+	+	-
2	+	-	+
4	-	+	+
12	-	-	-

A complete 2^3 factorial

A half replicate of 2^3 with I= -ABC

From the above table, we see that this projection belongs to type III. In fact, this projection property holds for every 3 factor columns.

However, the projection of $OA(N, s^m, t)$ onto more than t+1 factors does not have the above property. Take the $OA(12, 2^{11}, 2)$ in Table 1.7 as an example, the projection onto its factor columns 3, 4, 5 and 6, after rearranging runs is

Run	3 A	4 B	5 C	6 D	
1	-	+	+	+	
2	+	-	+	+	
3	+	+	-	+	One "-", three "+" , runs 2 and 5 are identical
11	+	+	+	-	
5	+	-	+	+	
4	-	+	+	-	
6	-	+	-	+	Two "-", two "+"
10	+	+	-	-	
7	-	-	+	-	
8	-	-	-	+	One "+", three "-"
9	+	-	-	-	
12	-	-	-	-	Four "-"

From the above table, we see that this projection does not belong to any of the three types. It does not either yield a complete 2^4 factorial or have any defining relation among these four factors.

Wang and Wu (1995) went further to study the projection of 12-run PB design. Since the 12-run PB design is an $OA(12, 2^{11}, 2)$ with strength t=2, the projection onto any t=2 factors is three copies of complete 2^2 factorial. This is an orthogonal projection as defined in the previous section.

The projections onto any $t+1=3$ factors, labeled by A, B and C, have the following properties:

(1) Always consist of a complete 2^3 factorial plus a half-replicate of 2^3 with defining relation either I=ABC or I=-ABC. As aforementioned, these projections are type III projections in Cheng (1995).

(2) Among all the $\binom{11}{3} = 165$ projections, 55 have a complete 2^3 factorial plus a half-replicate of 2^3 with defining relation I=ABC, whereas the other 110 have a complete 2^3 factorial plus a half-replicate of 2^3 with defining relation I=-ABC.

(3) The complete 2^3 factorial allows estimating all the 8 factorial effects. The half-replicate of 2^3 provides the capability of estimating all the 3 main effects, and so reinforcing the main effects estimates.

The projections onto any $t + 2 = 4$ factors have the following properties:

(1) All the $\binom{11}{4} = 330$ projected designs are a resolution V plan that allows estimating the general mean, all the 4 main effects and $\binom{4}{2} = 6$ two-factor interactions if assuming that the third and higher-order interactions are negligible. This property is referred to as hidden projection property. For example, the projection onto columns 3, 4, 5, 6 from PB design has a design matrix $\boldsymbol{X}$ with determinant $det(\boldsymbol{X}'\boldsymbol{X}) = 77309411328 > 0$. Therefore, the projected design is a resolution V plan. In fact, we checked that $det(\boldsymbol{X}'\boldsymbol{X}) = 77309411328 > 0$ is true for all the 330 projections.

μ	3 A	4 B	5 C	6 D	34 AB	35 AC	…	36 CD
1	-1	1	1	1	-1	-1	…	1
1	1	-1	1	1	-1	1	…	1
1	1	1	-1	1	1	-1	…	-1
1	1	1	1	-1	1	1	…	-1
1	1	-1	1	1	-1	1	…	1
1	-1	1	1	-1	-1	-1	…	-1
1	-1	1	-1	1	-1	1	…	-1
1	1	1	-1	-1	1	-1	…	1
1	-1	-1	1	-1	1	-1	…	-1
1	-1	-1	-1	1	1	1	…	-1
1	1	-1	-1	-1	-1	-1	…	1
1	-1	-1	-1	-1	1	1	…	1

(2) All these 330 projections can be obtained from one to another by permuting runs and/or changing the signs of all the entries in the same factor. For example, the projection onto factor columns 3, 4, 5, 6, labeled by A, B, C and D can be obtained by permuting runs of the projection onto factor columns 1, 2, 3, 4, labeled by E, F, G, and H as follows.

Run	1 E	2 F	3 G	4 H
1	+	+	-	+
2	-	+	+	-
3	+	-	+	+
4	-	+	-	+
5	-	-	+	-
6	-	-	-	+
7	+	-	-	-
8	+	+	-	-
9	+	+	+	-
10	-	+	+	+
11	+	-	+	+
12	-	-	-	-

permuting runs

Run	1 E	2 F	3 G	4 H
10	-	+	+	+
11	+	-	+	+
1	+	+	-	+
9	+	+	+	-
3	+	-	+	+
2	-	+	+	-
4	-	+	-	+
8	+	+	-	-
5	-	-	+	-
6	-	-	-	+
7	+	-	-	-
12	-	-	-	-

Run	3 A	4 B	5 C	6 D
1	-	+	+	+
2	+	-	+	+
3	+	+	-	+
11	+	+	+	-
5	+	-	+	+
4	-	+	+	-
6	-	+	-	+
10	+	+	-	-
7	-	-	+	-
8	-	-	-	+
9	+	-	-	-
12	-	-	-	-

(3) Every projection yields a design that consists of 11 distinct runs and 1 replicated run. For example, the projection onto factor columns 3, 4, 5, 6 has two identical runs 2 and 5: (+ - + +).

(4) No defining relation exists among all the four factors as well as any three of them.

Since PB design is of practical importance, we will study its projection property in details later in Chapter 4.

2.4 Projection from Balanced Arrays

As aforementioned in Chapter 1, BA is a generalization to OA. However, unlike the OA, there is little literature available to study the projection on BA. Therefore, in this section, we are going to focus on BA and study its projection property. Notice that S_i is the set of all distinct row vectors that contain *i* (+1)'s and (*m*-*i*) (-1)'s, in the following we will consider various BAs by combining S_0, S_1, S_2 and S_m in different ways.

(I) Considering the BAs of combining S_1 with S_0 and/or S_m, we have the results in Table 2.1.

Table 2.1 BAs of combining S_1, with S_0, S_m

BA	Number of runs	Results
$S_1 \cup S_0$	1+*m*	can estimate (can): 1, μ, *m* MEs can't estimate (can't): 1, μ, (*m*-*k*) MEs, any 1 interaction of them (k =1,…, m-2)
$S_1 \cup S_m$	1+*m*	can: 1, μ, *m* MEs; 2, μ, (*m*-1) MEs, any 1 interaction of them; can't: 1, μ, (*m*-*k*) MEs, any 2 interactions of them (k =1,…, m-3)
$S_1 \cup S_m \cup S_0$	2+*m*	can: 1, μ, *m* MEs, any 1 interaction of them can't: 1, μ, (*m*-*k*) MEs, any 2 interactions of them (k =1,…, m-3)

From Table 2.1, we can see that:

1. Design $S_1 \cup S_m$ is preferred to $S_1 \cup S_0$.

2. Design $S_1 \cup S_m \cup S_0$ has its advantage by adding one treatment to design $S_1 \cup S_m$ or $S_1 \cup S_0$ to make any one interaction estimable in addition to μ, m MEs.

3. When k is big, the above three designs have bad projection properties since their projected designs have too many replications so that they cannot estimate more than one interaction.

(II) Considering the BA S_2, we have the results in Table 2.2.

Table 2.2 The BA S_2

Design	Number of runs	Projected onto	Results
S_2	$\binom{m}{2}$	m	can't: μ, m MEs
		m-1	1, can: μ, (m-1) MEs, any 1 t-fi (t=2,…,m-2), but can't: μ, (m-1) MEs, and the (m-1)-fi; 2, can: μ, (m-1) MEs, any w 2-fi's (w=1,…, $\binom{m-1}{2}-1$), but can't: μ, (m-1) MEs, all $\binom{m-1}{2}$ 2-fi's
		m-2	can: μ, (m-2) MEs, all $\binom{m-2}{2}$ 2-fi's

Table 2.3 shows that by adding S_m to the design S_2, the resulting design $S_2 \cup S_m$ can solve the three "can't" in Table 2.1.

Table 2.3 BA of $S_2 \cup S_m$

Design	Number of runs	Results
$S_2 \cup S_m$	$1+\binom{m}{2}$	can: 1, μ, m MEs 2, μ, (m-1) MEs, the (m-1)-fi ($m \geq 4$) 3, μ, (m-1) MEs, all $\binom{m-1}{2}$ 2-fi's

(III) Notice that when projecting S_2 onto (m-1) factors, the resulting design is $S_1 \cup S_2$, we consider the BA of $S_1 \cup S_2$ and establish the results in Table 2.4.

Table 2.4 BA of $S_1 \cup S_2$

Design	Number of runs	Results
$S_1 \cup S_2$	$\binom{m}{1}+\binom{m}{2}$	can: 1, μ, m MEs, any $\binom{m}{2}$-1 2-fi's 2, μ, (m-1) MEs, all $\binom{m-1}{2}$ 2-fi's 3, μ, m MEs, any q 3-fi's. When m=3,4,5,6,7, then q=0,3,9,7,6, respectively.

(IV) Notice that when projecting design S_2 onto fewer than (m-1) factors or when projecting design $S_1 \cup S_2$ onto (m-1) factors, the resulting design is $S_0 \cup S_1 \cup S_2$ (with replications), we consider the BA of $S_0 \cup S_1 \cup S_2$ and establish the results in Table 2.5.

Table 2.5 BA of $S_0 \cup S_1 \cup S_2$

Design	Number of runs	Results
$S_0 \cup S_1 \cup S_2$	$1+\binom{m}{1}+\binom{m}{2}$	can: 1, μ, *m* MEs, all 2-fi's 2, μ, *m* MEs, any *q* 3-fi's. When *m*=3,4,5,6,7, then *q*=1,4,10,7,7, respectively.

Chapter 3

Characterization of Projection Using Linear Models

3.0 Main Results

In this chapter, we are going to study the projection of factorial design from the perspective of statistical modeling and characterize the projection property by using linear models.

3.1 Motivation

Let's consider the linear model (3.1) under a 2^3 factorial experiment. The three factors are denoted by x_1, x_2 and x_3, and each of them has two levels. When projecting onto factor x_1, model (3.1) can be written as (3.2).

$$\begin{aligned} E(y) &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123} x_1 x_2 x_3 \qquad (3.1) \\ &= (\beta_0 + \beta_2 x_2 + \beta_3 x_3 + \beta_{23} x_2 x_3) + (\beta_1 + \beta_{12} x_2 + \beta_{13} x_3 + \beta_{123} x_2 x_3) x_1 \\ &= \beta_0^{(1)} + \beta_1^{(1)} x_1 \qquad (3.2) \end{aligned}$$

where

$$\begin{aligned} \beta_0^{(1)} &= \beta_0 + \beta_2 x_2 + \beta_3 x_3 + \beta_{23} x_2 x_3 \\ \beta_1^{(1)} &= \beta_1 + \beta_{12} x_2 + \beta_{13} x_3 + \beta_{123} x_2 x_3 \end{aligned}$$

Then, for a fixed set of (x_2, x_3), there are only two possible level combinations (conditions) of the three factors as shown in the following under which the response is measured.

(x_1, x_2, x_3)	$E(y(x_1, x_2, x_3))$
$(1, x_2, x_3)$	$\beta_0^{(1)} + \beta_1^{(1)}$
$(-1, x_2, x_3)$	$\beta_0^{(1)} - \beta_1^{(1)}$

Therefore,

$$\beta_0^{(1)} = E(\frac{y(1, x_2, x_3) + y(-1, x_2, x_3)}{2})$$

and
$$\beta_1^{(1)} = E(\frac{y(1, x_2, x_3) - y(-1, x_2, x_3)}{2}).$$

Notice that, in $\beta_0^{(1)}(x_2, x_3) = \beta_0 + \beta_2 x_2 + \beta_3 x_3 + \beta_{23} x_2 x_3$, since (x_2, x_3) can take four possible values, $\beta_0^{(1)}(x_2, x_3)$ can also take four possible values as shown in the following.

(x_2, x_3)	$\beta_0^{(1)}(x_2, x_3)$
(1, 1)	$\beta_0 + \beta_2 + \beta_3 + \beta_{23}$
(1, -1)	$\beta_0 + \beta_2 - \beta_3 - \beta_{23}$
(-1, 1)	$\beta_0 - \beta_2 + \beta_3 - \beta_{23}$
(-1, -1)	$\beta_0 - \beta_2 - \beta_3 + \beta_{23}$

Therefore, $\beta_0 = \dfrac{\beta_0^{(1)}(1,1) + \beta_0^{(1)}(1,-1) + \beta_0^{(1)}(-1,1) + \beta_0^{(1)}(-1,-1)}{4}$. If assuming $\beta_2 = \beta_3 = \beta_{23} = 0$, then $\beta_0^{(1)}(x_2, x_3) = \beta_0$. Similar results can be obtained for $\beta_1^{(1)} = \beta_1 + \beta_{12}x_2 + \beta_{13}x_3 + \beta_{123}x_2x_3$.

3.2 Characterization of Projection Using Linear Models

Suppose the observation vector $\boldsymbol{y}(n \times 1)$ can be expressed as the following linear model

$$E(\boldsymbol{y}) = \boldsymbol{X\beta} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2$$

where $\boldsymbol{X}(n \times (p_1 + p_2))$ is the design matrix and $\boldsymbol{\beta}((p_1 + p_2) \times 1)$ is the corresponding vector of parameters, and $p_1 \le n \le p_1 + p_2$. Depending on what dimension (say p_1) and what elements desired to be projected onto, $\boldsymbol{X}$ can be partitioned into $\boldsymbol{X}_1(n \times p_1)$ and $\boldsymbol{X}_2(n \times p_2)$, and $\boldsymbol{\beta}$ can be partitioned into $\boldsymbol{\beta}_1(p_1 \times 1)$ and $\boldsymbol{\beta}_2(p_2 \times 1)$ accordingly. Suppose the projected design matrix $\boldsymbol{X}_1(n \times p_1)$ is of full column rank, i.e., $Rank(\boldsymbol{X}_1) = p_1$, we can establish the following proposition.

Proposition 3.1

Case 1, when $\boldsymbol{\beta}_2 = 0$, then $E(\boldsymbol{y}) = \boldsymbol{X}_1\boldsymbol{\beta}_1$.

Case 2, when $\boldsymbol{\beta}_2 \neq 0$, n=p_1 : $E(\boldsymbol{y}) = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 = \boldsymbol{X}_1\boldsymbol{\beta}_1^*$, where $\boldsymbol{\beta}_1^* = \boldsymbol{\beta}_1 + \boldsymbol{X}_1^{-1}\boldsymbol{X}_2\boldsymbol{\beta}_2$.

Case 3, when $\boldsymbol{\beta}_2 \neq 0$, n>p_1: $E(\boldsymbol{y}) = E\begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_{11} \\ \boldsymbol{X}_{12} \end{pmatrix}\boldsymbol{\beta}_1 + \begin{pmatrix} \boldsymbol{X}_{21} \\ \boldsymbol{X}_{22} \end{pmatrix}\boldsymbol{\beta}_2$, where $\boldsymbol{X}_{11}$ is a $p_1 \times p_1$ submatrix of $\boldsymbol{X}_1$, and $\boldsymbol{X}_{21}$ is the corresponding submatrix of $\boldsymbol{X}_2$. Since $Rank(\boldsymbol{X}_1) = p_1$, we can permute the runs appropriately to make $\boldsymbol{X}_{11}$ full rank. That is $Rank(\boldsymbol{X}_{11}) = p_1$.

Therefore,

$$E(\boldsymbol{y}_1) = \boldsymbol{X}_{11}\boldsymbol{\beta}_1 + \boldsymbol{X}_{21}\boldsymbol{\beta}_2 = \boldsymbol{X}_{11}[\boldsymbol{\beta}_1 + \boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{21}\boldsymbol{\beta}_2] = \boldsymbol{X}_{11}\boldsymbol{\beta}_1^*,$$

$$E(\boldsymbol{y}_2) = \boldsymbol{X}_{12}\boldsymbol{\beta}_1 + \boldsymbol{X}_{22}\boldsymbol{\beta}_2 = \boldsymbol{X}_{12}\boldsymbol{\beta}_1^* + [\boldsymbol{X}_{22} - \boldsymbol{X}_{12}\boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{21}]\boldsymbol{\beta}_2,$$

where

$$\boldsymbol{\beta}_1^* = \boldsymbol{\beta}_1 + \boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{21}\boldsymbol{\beta}_2.$$

Equivalently,

$$E(\boldsymbol{y}) = \begin{pmatrix} E(\boldsymbol{y}_1) \\ E(\boldsymbol{y}_2) \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_{11} \\ \boldsymbol{X}_{12} \end{pmatrix}\boldsymbol{\beta}_1^* + \begin{pmatrix} 0 \\ \boldsymbol{X}_{22} - \boldsymbol{X}_{12}\boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{21} \end{pmatrix}\boldsymbol{\beta}_2 = \boldsymbol{X}_1\boldsymbol{\beta}_1^* + \begin{pmatrix} 0 \\ \boldsymbol{X}_{22} - \boldsymbol{X}_{12}\boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{21} \end{pmatrix}\boldsymbol{\beta}_2.$$

As a result, we can establish the following theorem.

Theorem 3.1 Under **Case 3** when $Rank(\boldsymbol{X}) = p_1 = Rank(\boldsymbol{X}_{11})$, then

$$E(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{X}_{11} \\ \boldsymbol{X}_{12} \end{pmatrix} \boldsymbol{\beta}_1^* = \boldsymbol{X}_1\boldsymbol{\beta}_1^*, \text{ where } \boldsymbol{\beta}_1^* = \boldsymbol{\beta}_1 + \boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{21}\boldsymbol{\beta}_2.$$

Proof: First, because $Rank(\boldsymbol{X}) = p_1 = Rank(\boldsymbol{X}_{11})$, each of $(n - p_1)$ rows in $(\boldsymbol{X}_{12}, \boldsymbol{X}_{22})$ is linear function of the p_1 rows in $(\boldsymbol{X}_{11}, \boldsymbol{X}_{21})$, so there exist a matrix $\boldsymbol{D}$ such that $(\boldsymbol{X}_{12}, \boldsymbol{X}_{22}) = \boldsymbol{D}(\boldsymbol{X}_{11}, \boldsymbol{X}_{21})$.

Second, since $\boldsymbol{D} = \boldsymbol{X}_{12}\boldsymbol{X}_{11}^{-1}$ is unique solution of $\boldsymbol{X}_{12} = \boldsymbol{D}\boldsymbol{X}_{11}$, so $\boldsymbol{D} = \boldsymbol{X}_{12}\boldsymbol{X}_{11}^{-1}$ is also unique solution of $(\boldsymbol{X}_{12}, \boldsymbol{X}_{22}) = \boldsymbol{D}(\boldsymbol{X}_{11}, \boldsymbol{X}_{21})$.

Third, from above, we can have $\boldsymbol{X}_{22} = \boldsymbol{D}\boldsymbol{X}_{21} = \boldsymbol{X}_{12}\boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{21}$, so $\boldsymbol{X}_{22} - \boldsymbol{X}_{12}\boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{21} = 0$.

Therefore, $E\begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_{11} & 0 \\ \boldsymbol{X}_{12} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\beta}_1^* \\ \boldsymbol{\beta}_2 \end{pmatrix}$, $E(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{X}_{11} \\ \boldsymbol{X}_{12} \end{pmatrix}\boldsymbol{\beta}_1^* = \boldsymbol{X}_1\boldsymbol{\beta}_1^*$, which completes the proof.

3.3 Examples

So far, we have studied the projection of factorial designs under linear models and derived Proposition 3.1 and Theorem 3.1 to characterize the projection property. In this section, we give two examples to illustrate their usuage.

Example 3.1

Suppose in a two-level factorial experiment, we want to study the effects of six factors, say A, B, C, D, E and F, up to their two-factor interactions. And suppose we use the first six factor columns of 12-run PB design as the experimental design. Then, in the linear model $E(\boldsymbol{y}) = \boldsymbol{X\beta}$, $\boldsymbol{y}$ is a 12×1 vector, $\boldsymbol{X}$ is a 12×22 matrix whose columns are corresponding to the general mean, main effects, and two-factor interactions of these 6 factors. It can be checked that $Rank(\boldsymbol{X}) = 12$, and so up to 12 effects are estimable. By Case 2 of Proposition 3.1, we can partition $\boldsymbol{X}$ into $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ appropriately so that $E(\boldsymbol{y}) = \boldsymbol{X\beta} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 = \boldsymbol{X}_1\boldsymbol{\beta}_1^*$, where $\boldsymbol{\beta}_1^* = \boldsymbol{\beta}_1 + \boldsymbol{X}_1^{-1}\boldsymbol{X}_2\boldsymbol{\beta}_2$, and $\boldsymbol{X}_1$ is a submatrix of $\boldsymbol{X}$ with $Rank(\boldsymbol{X}_1) = 12$. Suppose we want to include the general mean, the 6 main effects and other 5 two-factor interactions in $\boldsymbol{X}_1$. Since there are $\begin{pmatrix} 6 \\ 2 \end{pmatrix} = 15$ two-factorial interactions in total, we have $\begin{pmatrix} 15 \\ 5 \end{pmatrix} = 3003$ possible choices to obtain the 5 interactions. Therefore, there are 3003 possible choices for choosing $\boldsymbol{X}_1$. By computational checking, we find that 935 choices of $\boldsymbol{X}_1$ are such that $Rank(\boldsymbol{X}_1) = 12$.

Example 3.2

With the same 6 factors and same design in Example 3.1, we consider a new linear model $E(\boldsymbol{y}) = \boldsymbol{X\beta} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2$, where all the effects associated with the 6 factors are considered in $\boldsymbol{\beta}$. Therefore, $\boldsymbol{y}$ is a 12×1 vector, and $\boldsymbol{X}$ is a 12×64 matrix. Furthermore, we set $\boldsymbol{X}_1$ as follows: Among the 6 factors, we choose 3 factors to form a full factorial and other 2 factors to form another full factorial. $\boldsymbol{X}_1$ is chosen to include all the effects associated with these two full factorials. Therefore, $\boldsymbol{X}_1$ a is 12×11 matrix, and there are $\begin{pmatrix} 6 \\ 3 \end{pmatrix}\begin{pmatrix} 6-3 \\ 2 \end{pmatrix}=60$ possible choices for it in total. For example, if we choose (A, B, C) to form the first factorial and (D, E) to form the second, $\boldsymbol{X}_1$ is the design matrix corresponding to effects $\boldsymbol{\mu}$, A, B, C, AB, AC, BC, ABC, D, E, DE. For notational convenience, we can denote this case by (A, B, C)+(D, E). Out of the 60 choices of $\boldsymbol{X}_1$, we checked that 50 satisfy $Rank(\boldsymbol{X}_1)=11$, which apply to Case 3 of Proposition 3.1 as $n>p_1$ and $Rank(\boldsymbol{X}_1)=p_1$. The remaining 10 cases that do not satisfy are listed below.

Cases Not Satisfy	Which 5 factors
1, (A,C,D)+(E,F)	(A,C,D,E,F)
2, (A,C,E)+(D,F)	(A,C,D,E,F)
3, (A,C,F)+(D,E)	(A,C,D,E,F)
4, (A,D,E)+(C,F)	(A,C,D,E,F)
5, (A,D,F)+(C,E)	(A,C,D,E,F)
6, (A,E,F)+(C,D)	(A,C,D,E,F)
7, (C,D,E)+(A,F)	(A,C,D,E,F)
8, (C,D,F)+(A,E)	(A,C,D,E,F)
9, (C,E,F)+(A,D)	(A,C,D,E,F)
10,(D,E,F)+(A,C)	(A,C,D,E,F)

Interestingly, factor B turns out to appear in the 50 satisfied cases (i.e., (B,*,*)+(*,*) or (*,*,*)+(B,*)), but not appear in the 10 unsatisfied cases (i.e.,

(*,*,*)+(*,*)). The specialty of B (factor column 2) will be studied in the following chapter.

Chapter 4

Projection Properties of PB Design and Related Designs

4.0 Main Results

In this chapter, we investigate the projection properties of PB design when projecting onto its 4 factor columns and 5 factor columns. We examine the estimability of the projected designs when fitting various models, and give some helpful results that are not available in current literature. Moreover, we compare the projected PB design of 5 columns with BAs and obtain some interesting results.

4.1 Projection Properties of PB for *m*=4, 5

***m*=4**

Wang and Wu (1995) claimed that when projecting the 12-run PB design onto its four factors, the resulting projected 12×4 designs are all isomorphic to each other, and they are all Resolution V plans that can estimate the model of all the main effects as well as all the two-factor interactions. Now, we went further to study the estimability of this design by fitting another two factorial effects models.

We first randomly select four factors from the 12-run PB design, and label them by A, B, C and D. Therefore, there are $2^4 = 16$ effects associated with the four factors in total. Then, we let the first factorial effects model include the effects of $\boldsymbol{\mu}$, A, B, C, AB, AC, BC, ABC, D as well as two more effects. Therefore, there are $\binom{16-9}{2} = 21$ choices as shown in Table 4.1. We then present the estimability results by checking whether the design matrix $\boldsymbol{X}_1$ has rank 11 or not. We denote these models by (A, B, C)+(D).

Table 4.1. The projection (onto 4 columns) properties of 12-run PB design; Estimability results: all possible sets of 11 effects among which 8 are full factorial effects

Case	First 8 effects	3 other effects	$\boldsymbol{Rank}(\boldsymbol{X}_1) = \mathbf{11}$?
(A,B,C)+(D)	$\boldsymbol{\mu}$,A,B,C,AB, AC,BC,ABC	D,DA,DB	YES
		D,DA,DC	YES
		D,DB,DC	YES
		D,DA,DAB	YES
		D,DB,DAB	YES
		D,DA,DAC	YES
		D,DC,DAC	YES
		D,DB,DBC	YES
		D,DC,DBC	YES
		D,DAB,DAC	YES
		D,DAB,DBC	YES
		D,DAC,DBC	YES
		D,DA,DBC	NO
		D,DB,DAC	NO
		D,DC,DAB	NO
		D,DA,DABC	NO
		D,DB,DABC	NO
		D,DC,DABC	NO
		D,DAB,DABC	NO
		D,DAC,DABC	NO
		D,DBC,DABC	NO

Let the second factorial effects model include the effects of $\boldsymbol{\mu}$, A, B, C, D as well as k more effects, where k=4, 5 or 6. Therefore, there are $\binom{16-5}{k}$ choices for such model in total. We then present the estimability results in Table 4.2 by checking whether the design matrix $\boldsymbol{X}_1$ has rank k+5 or not.

Table 4.2 The projection (onto 4 columns) properties of 12-run PB design; Estimability results: general mean, main effects with additional interactions

First 5 effects	**Any 4 other effects**	$\boldsymbol{Rank}(\boldsymbol{X}_1)=\mathbf{9}$?
$\boldsymbol{\mu}$,A,B,C,D	$\binom{11}{4}=330$ ways	ALL YES
First 5 effects	**Any 5 other effects**	$\boldsymbol{Rank}(\boldsymbol{X}_1)=\mathbf{10}$?
$\boldsymbol{\mu}$,A,B,C,D	AB,AC,AD,BCD,ABCD	NO
	BA,BC,BD,ACD,ABCD	NO
	CA,CB,CD,ABD,ABCD	NO
	DA,DB,DC,ABC,ABCD	NO
	AB,CD,ABC,ABD,ABCD	NO
	AB,CD,CDA,CDB,ABCD	NO
	AC,BD,ACB,ACD,ABCD	NO
	AC,BD,BDA,BDC,ABCD	NO
	AD,BC,BCA,BCD,ABCD	NO
	AD,BC,ADB,ADC,ABCD	NO
	AB,AC,BC,ABC,ABCD	NO
	AB,AD,BD,ABD,ABCD	NO
	AC,AD,CD,ACD,ABCD	NO
	BC,BD,CD,BCD,ABCD	NO
	ABC,ABD,ACD,BCD,ABCD	NO
	remaining $\binom{11}{5}-15=462-15=447$ ways	YES
First 5 effects	**Any 6 other effects**	$\boldsymbol{Rank}(\boldsymbol{X}_1)=\mathbf{11}$?

μ,A,B,C,D	15 ways (from above "NO") $\times\binom{6}{1}=90$ ways; adding any interaction from remaining 11-5=6 interactions to each of the above 15 models will NOT work and these 90 models are all distinct.	NO
	AB,AC,AD,BC,ABC,BCD	NO
	AB,AC,AD,BD,ABD,BCD	NO
	AB,AC,AD,CD,ACD,BCD	NO
	BA,BC,BD,AC,ABC,ACD	NO
	BA,BC,BD,AD,ABD,ACD	NO
	BA,BC,BD,CD,BCD,ACD	NO
	CA,CB,CD,AB,ABC,ABD	NO
	CA,CB,CD,AD,ACD,ABD	NO
	CA,CB,CD,BD,BCD,ABD	NO
	DA,DB,DC,AB,ABD,ABC	NO
	DA,DB,DC,AC,ACD,ABC	NO
	DA,DB,DC,BC,BCD,ABC	NO
	AB,AC,BC,ABD,ACD,BCD	NO
	AB,AD,BD,ABC,ACD,BCD	NO
	AC,AD,CD,ABC,ABD,BCD	NO
	BC,BD,CD,ABC,ABD,ACD	NO
	AB,AC,BD,CD,ABC,BCD	NO
	AB,AC,BD,CD,ABD,ACD	NO
	AC,AD,BC,BD,ACD,BCD	NO
	AC,AD,BC,BD,ABC,ABD	NO
	AB,AD,CB,CD,ABD,BCD	NO
	AB,AD,CB,CD,ABC,ACD	NO
	AB,CD,ABC,ABD,ACD,BCD	NO
	AC,BD,ABC,ABD,ACD,BCD	NO
	AD,BC,ABC,ABD,ACD,BCD	NO
	remaining $\binom{11}{6}-(90+25)=462-115=347$ ways	YES

***m*=5**

Wang and Wu (1995) claimed that when projecting the 12-run PB design onto its five factors, the resulting projected 12×4 designs can be divided into two isomorphic groups. The projected designs in the first group contain two repeated runs such as the one includes the 1st, 2nd, 3rd, 4th and 10th factor columns, whereas the projected designs in the second group contain two mirror image runs such as the one includes the 1st, 2nd, 3rd, 4th and 5th factor columns. Denote the design in the first group to be Design 5.1 and that in the second group to be Design 5.2. Now, we study the estimability of these two designs by fitting the following two factorial effects models. We randomly label the 5 columns of the studied design by A, B, C, D and E. Therefore, there are $2^5 = 32$ factorial effects in total. Then, we let the first model to include the effects of $\boldsymbol{\mu}$, A, B, C, AB, AC, BC, ABC, D, E as well as one more interactions. Therefore, there are $\binom{32-10}{1} = 22$ choices as shown in Table 4.3 and Table 4.4. We then present the estimability of Design 5.1 in Table 4.3 and Design 5.2 in Table 4.4 respectively by checking whether the projected design matrix $\boldsymbol{X}_1$ has rank 11 or not. We denote these models by (A, B, C)+(D, E).

Table 4.3 The projection (onto 5 columns) properties of 12-run PB design. Estimability results: all possible sets of 11 effects among which 8 are full factorial effects

Design 5.1 (with two repeated runs)

Case	First 8 effects	3 other effects	$Rank(X_1) = 11$?
(A,B,C)+(D,E)	μ,A,B,C,AB, AC,BC,ABC	D,E,DE	NO
		D,E,DA	NO
		D,E,EA	NO
		D,E,DB	NO
		D,E,EB	NO
		D,E,DC	NO
		D,E,EC	NO
		D,E,DAB	NO
		D,E,EAB	NO
		D,E,DAC	NO
		D,E,EAC	NO
		D,E,DBC	NO
		D,E,EBC	NO
		D,E,DEA	NO
		D,E,DEB	NO
		D,E,DEC	NO
		D,E,DEAB	NO
		D,E,DEAC	NO
		D,E,DEBC	NO
		D,E,DABC	NO
		D,E,EABC	NO
		D,E,DEABC	NO

Table 4.4 The projection (onto 5 columns) properties of 12-run PB design. Estimability results: all possible sets of 11 effects among which 8 are full factorial effects

Under Design 5.2 (with two mirror image runs)

Case	First 8 effects	3 other effects	$Rank(X_1)=11$?
(A,B,C)+(D,E)	μ,A,B,C,AB, AC,BC,ABC	D,E,DE	YES
		D,E,DA	YES
		D,E,EA	YES
		D,E,DB	YES
		D,E,EB	YES
		D,E,DC	YES
		D,E,EC	YES
		D,E,DAB	YES
		D,E,EAB	YES
		D,E,DAC	YES
		D,E,EAC	YES
		D,E,DBC	YES
		D,E,EBC	YES
		D,E,DEA	YES
		D,E,DEB	YES
		D,E,DEC	YES
		D,E,DEAB	YES
		D,E,DEAC	YES
		D,E,DEBC	YES
		D,E,DABC	NO
		D,E,EABC	NO
		D,E,DEABC	YES

Table 4.3 and Table 4.4 can be used to explain the specialty of factor B (factor column 2) in Example 3.2. In fact, the 5 factor columns corresponding to the 50 satisfied cases (i.e., (B,*,*)+(*,*) or (*,*,*)+(B,*)) are Design 5.2 and the 5 factor columns corresponding to the 10 unsatisfied cases (i.e., (*,*,*)+(*,*)) are Design 5.1. It's noticed that the estimability results of first set of 11 parameters in Table 4.3 is NO and in Table 4.4 is YES.

Let the second model to include the effects of $\boldsymbol{\mu}$, A, B, C, D, E as well as k more interactions, where k=2, 3 or 4. Therefore, there are $\binom{32-6}{k}$ choices for the model in total. We then present the estimability of Design 5.1 in Table 4.5 and Design 5.2 in Table 4.6 by checking whether the projected design matrix $\boldsymbol{X}_1$ has rank k+6 or not.

Table 4.5 The projection (onto 5 columns) properties of 12-run PB design; Estimability results: general mean, main effects with additional interactions
Under Design 5.1 (with two repeated runs)

First 6 effects	**2 other effects**	$\boldsymbol{Rank(X_1)=8}$?
$\boldsymbol{\mu}$,A,B,C,D,E	145 ways	NO
	$\binom{26}{2}-145=180$ ways	YES
First 6 effects	**3 other effects**	$\boldsymbol{Rank(X_1)=9}$?
$\boldsymbol{\mu}$,A,B,C,D,E	1640 ways	NO
	$\binom{26}{3}-1640=960$ ways	YES
First 6 effects	**4 other effects**	$\boldsymbol{Rank(X_1)=10}$?
$\boldsymbol{\mu}$,A,B,C,D,E	11830 ways	NO
	$\binom{26}{4}-11830=3120$ ways	YES

Table 4.6 The projection (onto 5 columns) properties of 12-run PB design; Estimability results: general mean, main effects with additional interactions
Under Design 5.2 (with two mirror image runs)

First 6 effects	2 other effects	$Rank(X_1)=8$?
μ,A,B,C,D,E	$\binom{26}{2}=325$ ways	All YES
First 6 effects	**3 other effects**	$Rank(X_1)=9$?
μ,A,B,C,D,E	30 ways	NO
	$\binom{26}{3}-30=2570$ ways	YES
First 6 effects	**4 other effects**	$Rank(X_1)=10$?
μ,A,B,C,D,E	850 ways	NO
	$\binom{26}{4}-850=14100$ ways	YES

4.2 Comparison between BAs and PB for *m*=5

In a 2^m fractional factorial experiment, the resolution plans Res III, Res III plus k, and Res V are commonly used in practice. Under Res III plus k models (Srivastava (1975)), the general mean, main effects and all possible $\binom{\binom{m}{2}}{k}$ k two-factor interactions are estimated assuming the remaining two-factor and higher order interactions are negligible.

As mentioned in the previous section, there are two non-isomorphic designs when projecting the 12-run PB design onto its 5 factors. In this section, we compare their projection and optimality properties with other five balanced array designs constructed by ourselves under different resolution models. First, we introduce the seven designs

$d_j, j = 1,...,7$. Second, we compare the seven designs for fitting Res III model and Res III plus *k* models with *k*=1, 2, 3, 4, 5 or 6 separately. Third, we project the seven designs onto their *t* factor columns, where *t*=2, 3 or 4, and compare the projected designs for fitting different resolution models.

Designs

Recall that S_i is the set of all runs with *i* factors observed at the high level (+1) and the rest *m-i* factors observed at the low level (-1). We construct 5 balanced arrays of full strength: $d_j, j = 1,...,5$ by combining S_i in 5 different ways as shown in Table 4.7. We further denote the two projected designs from 12-run PB design to be d_6 (Design 5.2) and d_7 (Design 5.1). Designs d_6, d_7 are OAs of strength 2. Interestingly, d_7 and d_3 are isomorphic to each other $(d_3 \equiv d_7)$.

In Table 4.7, we also provide the results (Yes/No) regarding the estimability of these seven designs for fitting different resolution models, where more results regarding the optimality properties will be provided in later Subsections.

Table 4.7 Designs $d_j, j = 1,...,7$ and their estimabilities of models

Design	**Runs**	Res III	Res III+*k* (*k*=1,2,3)	Res III+*k* (*k*=4,5)	*t*=2, full factorial	*t*=3, full factorial	*t*=4, Res V
d_1	$S_0 \cup S_2 \cup S_5$	Yes	Yes	Yes	Yes	Yes	Yes
d_2	$2S_0 \cup S_2$	Yes	Yes	No	Yes	No	Yes
d_3	$S_2 \cup 2S_5$	Yes	Yes	No	Yes	Yes	Yes
d_4	$S_0 \cup S_1 \cup S_4 \cup S_5$	Yes	Yes	No	Yes	Yes	No
d_5	$2S_0 \cup S_1 \cup S_4$	Yes	Yes	No	Yes	Yes	No
d_6	from PB design	Yes	Yes	No	Yes	Yes	Yes
d_7	from PB design	Yes	Yes	No	Yes	Yes	Yes

Comparison of $d_j, j = 1,...,7$

We take designs $d_j, j = 1,...,7$ to fit resolution III and resolution III plus k models. Under each model, we compare $d_j, j = 1,...,7$ with respect to Trace value (A-optimality), Determinant value (D-optimality), and Maximum Eigenvalue (E-optimality). Let the abbreviation of Trace to be "Tr", of Determinant to be "Det", of Maximum Eigenvalue to be "MEV".

Res III model

Table 4.8 compares $d_j, j = 1,...,7$ under Res III model with respect to A-, D-, and E-optimality criteria.

Table 4.8 Comparison of $d_j, j = 1,...,7$ for Res III model

Design	$(X'X)^{-1}_{6\times 6}$		
	Tr	Det$\times 10^7$	MEV
d_1	0.53	3.89	0.13
d_2	0.71	7.54	0.33
$d_3 \equiv d_7, d_6$	**0.5**	**3.35**	**0.08**
d_4	0.62	7.27	0.13
d_5	0.63	7.73	0.13

Designs $d_3 \equiv d_7$, and d_6 perform the best having the minimum "Tr", "Det", and "MEV" values.

Res III plus *k* models

The seven designs $d_j, j = 1,...,7$ not only can fit Res III model, but also Res III plus *k* (*k*=1,2,3) models. Next, we investigate how designs $d_j, j = 1,...,7$ perform when they fit the Res III plus *k* (=1,2,3) models. For a fixed *k*, there are $\binom{10}{k}$ possible Res III plus *k* models. Each model is corresponding to a particular set of *k* two-factor interaction(s).

***k* =1**

For each design d_j, we compute the "Tr", "Det" and "MEV" values for those $\binom{10}{k} = 10$ possible models and observe that the values are same from model to model. Table 4.9 compares $d_j, j = 1,...,7$ under Res III plus 1 models with respect to A-, D-, and E-optimality criteria.

Table 4.9 Comparison of $d_j, j = 1,...,7$ for Res III plus 1 models

Design	$(\boldsymbol{X}'\boldsymbol{X})^{-1}_{7\times 7}$		
	Tr	$\text{Det}\times 10^8$	MEV
d_1	**0.64**	**3.76**	0.14
d_2	0.94	9.42	0.45
$d_3 \equiv d_7, d_6$	0.67	4.19	0.20
d_4	0.72	6.81	**0.13**
d_5	0.73	7.48	**0.13**

Table 4.9 demonstrates that design d_1 performs the best for all 10 models with respect to A- and D-optimality criteria while designs d_4 and d_5 perform the best with respect to E-optimality criterion. However, the "MEV" of d_1 is very close to d_4 and d_5. It seems d_1 is a very good design.

***k* =2**

For each design d_j, we compute the arithmetic mean of "Tr" (AT optimality), "Det" (AD optimality), and "MEV" (AMEV optimality) values for those 45 models because the values may be not the same from model to model (Srivastava (1977)). Table 4.10 compares $d_j, j = 1,...,7$ under Res III plus 2 models.

Table 4.10 Overall comparison of $d_j, j = 1,...,7$ for Res III plus 2 models

Design	$(X'X)^{-1}_{8\times 8}$		
	AT	AD $\times 10^9$	AMEV
d_1	**0.774**	**4.02**	0.183
d_2	1.25	13.2	0.634
$d_3 \equiv d_7$	0.885	5.89	0.302
d_4	0.861	7.38	**0.167**
d_5	0.879	8.54	0.174
d_6	0.888	5.99	0.279

Table 4.10 demonstrates that design d_1 performs the best with respect to AT and AD optimality criteria while design d_4 performs the best with respect to AMEV optimality criterion.

***k* =3**

Similarly to k=2, the criteria values for k=3 can be computed as shown in Table 4.11, from which we can conclude that design d_1 performs the best with respect to AT, AD, and AMEV optimality criteria while d_4 performs the second best.

Table 4.11 Overall comparison of $d_j, j = 1,...,7$ for Res III plus 3 models

Design	$(\boldsymbol{X}'\boldsymbol{X})^{-1}_{9\times9}$		
	AT	AD $\times10^{10}$	AMEV
d_1	**0.959**	**4.91**	**0.272**
d_2	1.746	22.35	0.963
$d_3 \equiv d_7$	1.219	9.93	0.485
d_4	**1.062**	**9.76**	**0.277**
d_5	1.102	12.13	0.305
d_6	1.198	10.06	0.449

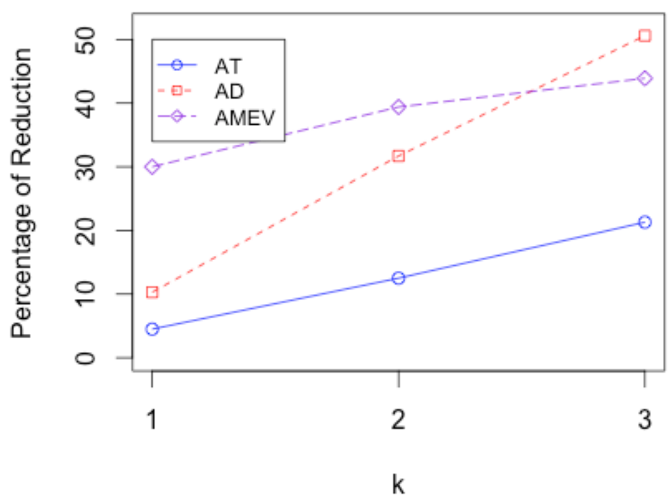
As shown in Table 4.8, design $d_3 \equiv d_7$ outperforms d_1 for fitting Res III model. However, from Table 4.9 to Table 4.11, we can see that d_1 outperforms $d_3 \equiv d_7$ for fitting Res III plus k models. In the following table, we compute the percentage of reduction in AT, AD and AMEV from $d_3 \equiv d_7$ to d_1 for k=1, 2 and 3. Such percentages can be shown easily in Figure 4.13 as well. As we can see, d_1 perform better and better compared to $d_3 \equiv d_7$ across k.

Table 4.12 Percentage of Reduction

	Percentage of Reduction in		
	AT	AD	AMEV
k=1	4.5	10.3	30.0
k=2	12.5	31.7	39.4
k=3	21.3*	50.6	43.9

*21.3=(AT of d_3 -AT of d_1)/(AT of d_3)

Figure 4.1 Percentage of Reduction



***k* =4, 5, 6**

When *k*=4, 5, and 6, we find designs $d_j, j = 1,\ldots,7$ may only successfully fit some Res III plus *k* models. Table 4.13 shows the number of estimable Res III plus *k* models for each design. Design d_1 performs the best having the estimability for all the Res III plus 4 models, all the Res III plus 5 models, and 185 Res III plus 6 models. Note d_1 is the optimal design D14 that is presented in Ghosh and Tian (2006) for *m*=5, *n*=12, Res III plus 5 models. They stated that D14 is optimal with respect to AT, AD, AMCR (i.e., AMER), GT, GD, and GMCR six optimality criteria.

Table 4.13 Number of estimable Res III plus *k* models for $d_j, j=1,\ldots,7$ when *k*=4, 5, 6

Design	*k*=4 210 models	*k*=5 252 models	*k*=6 210 models
d_1	210	252	185
d_2	195	162	0
$d_3 \equiv d_7$	195	162	0
d_4	195	162	0
d_5	195	162	0
d_6	200	192	80

Comparison of designs $d_1^{(t)},\ldots,d_7^{(t)}$ for *t* =2, 3, and 4

For each design $d_j, j=1,\ldots,7$, projecting 5 columns onto their *t* columns yields one isomorphic design $d_j^{(t)}, t=2,3,4$ from $\binom{5}{t}$ possible projections. For a fixed *t*, we assume a model so that its parameters can be estimated with the seven $d_j^{(t)}$ designs. Some of them may fail to fit the model due to poor projection property. For those that are capable of fitting the model, we further examine their optimality property.

***t* =2**

Each $d_j^{(2)}$ could be considered as a design for a 2^2 factorial experiment. These designs permit the estimation of the general mean, main effects and the two-factor interaction. The $d_3^{(2)}, d_6^{(2)}$, and $d_7^{(2)}$ designs are identical and consist of all four possible treatment combinations that being replicated three times. Table 4.14 demonstrates that design $d_3^{(2)} = d_6^{(2)} = d_7^{(2)}$ performs the best fitting this full factorial model. When projecting onto 2 factors, we denote treatment combinations (-1,-1) as 0, (1,-1) as 1, (-1,1) as 2, and (1,1) as 12.

Table 4.14 Comparison of $d_j^{(2)}$, j=1,…,7 for full factorial model

Design	Frequency	$(\boldsymbol{X}'\boldsymbol{X})^{-1}_{4\times 4}$		
	0,1,2,12	Tr	Det $\times 10^5$	MEV
$d_1^{(2)}$	4,3,3,2	0.354	5.43	0.125
$d_2^{(2)}$	5,3,3,1	0.467	8.68	0.25
$d_3^{(2)} = d_6^{(2)} = d_7^{(2)}$	**3,3,3,3**	**0.333**	**4.82**	**0.083**
$d_4^{(2)}$	4,2,2,4	0.375	6.10	0.125
$d_5^{(2)}$	5,2,2,3	0.383	6.51	0.125

The designs $d_j, j = 1,...,7$ are all BTAs of projectivity 2, as their projection onto every subset of 2 factors contain a complete 2^2 factorial design. But their projection doesn't necessarily contain equal replication of all the 2^2 treatment combinations.

***t* =3**

Each $d_j^{(3)}$ could be considered as a design for a 2^3 factorial experiment. These designs, except $d_2^{(3)}$, permit the estimation of the general mean, main effects, two-factor interactions and the three-factor interaction. Note that design $d_2^{(3)}$ is not applicable since it doesn't contain a complete 2^3 factorial. The $d_3^{(3)}, d_6^{(3)}$, and $d_7^{(3)}$ designs are isomorphic and one representative design $d_3^{(3)} \equiv d_6^{(3)} \equiv d_7^{(3)}$ consists of a complete 2^3 factorial plus a half-replicate of 2^3 with defining relation I=ABC (Cheng (1995)). Also note that design $d_1^{(3)}$ consists of a complete 2^3 factorial plus a half-replicate of 2^3 with no defining relation exists. Designs $d_4^{(3)}$ and $d_5^{(3)}$ contain a complete 2^3 factorial but the remaining four runs is not a half-replicate of 2^3 . Table 4.15 demonstrates designs $d_1^{(3)}, d_3^{(3)} \equiv d_6^{(3)} \equiv d_7^{(3)}$ perform the best. When projecting onto 3 factors, we denote treatment combinations (-1,-1, -1) as 0, (1,-1,-1) as 1, (-1,1,-1) as 2, (-1,-1,1) as 3, (1,1,-1) as 12, (1,-1,1) as 13, (-1,1,1) as 23, (1,1,1) as 123. The designs $d_j, j = 1,3,...,7$ are all BTAs of projectivity 3.

Table 4.15 Comparison of $d_j^{(3)}$, j=1,3,…,7 for full factorial model

Design	Frequency	$(\boldsymbol{X}'\boldsymbol{X})^{-1}_{8\times 8}$		
	0,1,2,3,12,13,23,123	Tr	Det $\times 10^9$	MEV
$d_1^{(3)}$	**2,2,2,2,1,1,1,1**	**0.75**	**3.73**	**0.125**
$d_3^{(3)} \equiv d_6^{(3)} \equiv d_7^{(3)}$	**1,2,2,2,1,1,1,2**	**0.75**	**3.73**	**0.125**
$d_4^{(3)}$	3,1,1,1,1,1,1,3	0.833	6.62	0.125
$d_5^{(3)}$	4,1,1,1,1,1,1,2	0.844	7.45	0.125

***t* =4**

Each $d_j^{(4)}$ could be considered as a design for a 2^4 factorial experiment. These designs, except $d_4^{(4)}$ and $d_5^{(4)}$, permit the estimation of parameters in a Res V model. Note that designs $d_4^{(4)}$ and $d_5^{(4)}$ are not applicable since they only contain 10 distinct runs, which is not enough to estimate 11 parameters. The designs $d_j^{(4)}$ obtained from d_3, d_6 and d_7 are all isomorphic so we denote one representative design as $d_3^{(4)} \equiv d_6^{(4)} \equiv d_7^{(4)}$. Table 4.16 demonstrates that design $d_1^{(4)}$ performs the best for fitting Res V model.

Table 4.16 Comparison of $d_j^{(4)}$, j=1,2,3,6,7 for Res V model

Design	$(X'X)^{-1}_{11\times 11}$		
	Tr	Det $\times 10^{11}$	MEV
$d_1^{(4)}$	**1.31**	**0.73**	**0.25**
$d_2^{(4)}$	3.94	11.6	2.76
$d_3^{(4)} \equiv d_6^{(4)} \equiv d_7^{(4)}$	1.44	1.29	0.25

Note that $d_1^{(4)}$ is the optimal balanced fractional factorial design in Srivastava and Chopra (1971) for *m*=4, *n*=12, Res V model with respect to A-optimality criterion.

Conclusions

Table 4.7 shows that design d_1 is the only design that could fit all the models. Design d_1 has ability to fit the Res III, all Res III plus *k* (*k*=1,2,3) models and it even has ability to fit all Res III plus *k* (*k*=4,5) models. Moreover, its performances for fitting these models are very good. In terms of projection, d_1 is the second best for *t*=2, is the best with $d_j, j = 3,6,7$ for *t*=3, becomes even better than $d_j, j = 3,6,7$ for *t*=4.

Chapter 5

Optimal Resolution V Designs for *m*=4

5.0 Main Results

In this chapter, we exhaustively search for optimal resolution V designs of 2^m series for m=4 and n=11, 12, 13, 14, 15, and 16. Our designs outperform the designs given by Srivastava and Chopra (1971). For each value of n, the optimal designs will be presented with their A-, D- and E-optimality criteria values. It is observed that these designs are equally optimal. Therefore, we go further to study the relations among them and obtain some insightful results.

5.1 Optimal Resolution V Designs for *n*=11

As we know, a 2^4 full factorial design, denoted by T_0, contains 16 distinct runs as shown in Table 5.1. In order to fit a resolution V model for the factorial, a minimum 11 runs are required. Therefore, there are $\binom{16}{11} = 4368$ possible choices of runs in total. A very nature question is which choice will provide the smallest A-, D- and E-optimality criteria.

Specially, we randomly select 11 runs and eliminate the rest 5 runs from T_0 to construct the 11-run design. Among all the 4368 designs, we identify all the resolution V plans and compute their corresponding A-, D- and E-optimality criteria values, accordingly to which all the resolution V plans are divided into several classes as shown in Table 5.2. We show the number of designs included, a representative design and the optimality criteria values in each class. We denote S_0: the run which has -1 in all positions; $S_1^{(u)}$: the run of S_1 which has 1 in the positions u; $S_2^{(u,v)}$: the run of S_2 which has 1 in the positions u and v; $S_3^{(u,v,z)}$: the run of S_3 which has 1 in the positions u,v and z.

Table 5.1 Full 2^4 factorial design T_0

S0	1	-	-	-	-
S1	2	+	-	-	-
	3	-	+	-	-
	4	-	-	+	-
	5	-	-	-	+
S2	6	+	+	-	-
	7	-	-	+	+
	8	+	-	+	-
	9	-	+	-	+
	10	+	-	-	+
	11	-	+	+	-
S3	12	-	+	+	+
	13	+	-	+	+
	14	+	+	-	+
	15	+	+	+	-
S4	16	+	+	+	+

Table 5.2 All possible 11-run designs when fit parameters in res V model

Classes	Number	A design	Res	Det $\times 10^{11}$	Tr	MEV
T_1	16	$S_1 \cup S_2 \cup S_4$	V	2.59	1.49	0.25
T_2	320	$S_1^{(u)} \cup S_2^{(u,v)} \cup S_3 \cup S_4$ $u \neq 1; (u,v) \neq (2,3),(2,4),(3,4)$	V	5.82	2.13	1.00
T_3	192	$S_1^{(u)} \cup S_2^{(u,v)} \cup S_3 \cup S_4$ $u \neq \{1,2\}; (u,v) \neq (1,3),(2,4)$	V	23.3	2.88	1.00
T_4	960	$S_1^{(4)} \cup S_2^{(u,v)} \cup S_3 \cup S_4$ $(u,v) \neq (3,4)$	V	23.3	3.38	1.74
T_5	960	$S_1^{(u)} \cup S_2^{(u,v)} \cup S_3 \cup S_4$ $u \neq \{1,2\}; (u,v) \neq (1,3),(3,4)$	V	23.3	3.88	2.43
T_6	80	$S_2 \cup S_3 \cup S_4$	V	23.3	4.38	3.17
T_7	480	$S_1^{(u)} \cup S_2^{(u,v)} \cup S_3^{(u,v,z)} \cup S_4$ $u \neq \{1,2\}; (u,v) \neq (3,4); (u,v,z) \neq (1,2,4)$	V	23.3	5.38	4.25
T_8	1360	$S_1^{(4)} \cup S_2^{(u,v)} \cup S_3 \cup S_4$ $(u,v) \neq (1,2)$	IV	-	-	-
Total	4368	-	-	-	-	-

From Table 5.2, we can see that among all the 4368 designs, 1360 are not resolution V plans. The remaining 3008 designs can be divided into seven classes where the optimal class T_1 consists of 16 designs shown in Table 5.3. It deserves to point out that the projected designs when projecting the 12-run PB design onto any of its 4 factors, after eliminating the replicated run, belong to T_1. Further, it is observed that given any one design from T_1, the other 15 designs can be obtained by multiplying (-1) to its any one or two or three or four factor columns. Therefore, these 16 designs are isomorphic to each other.

Table 5.3 The 16 optimal designs in $\mathbf{T_1}$

Design	Run # delete from T_0	Design	Run # delete from T_0	Design	Run # delete from T_0	Design	Run # delete from T_0
1	1,6,8,10,12	2	1,6,9,11,13	3	1,7,8,11,14	4	1,7,9,10,15
5	1,12,13,14,15	6	2,3,4,5,16	7	2,3,7,14,15	8	2,4,9,13,15
9	2,5,11,13,14	10	4,5,6,12,13	11	3,4,10,12,15	12	3,5,8,12,14
13	3,7,8,10,16	14	2,7,9,11,16	15	4,6,9,10,16	16	5,6,8,11,16

5.2 Optimal Resolution V Designs for *n*=12

Consider the full 2^4 factorial design T_0, we randomly select 12 runs and eliminate 4 runs from it to construct 12-run design. Therefore, there are $\binom{16}{12} = 1820$ possible designs. Among all the 1820 designs, we identify all the resolution V plans and compute their corresponding A-, D- and E-optimality criteria values, accordingly to which all the resolution V plans are divided into several classes as shown in Table 5.4. We show the number of designs included, a representative design and the optimality criteria values in each class.

Table 5.4 All possible 12-run designs when fit parameters in res V model

Classes	Number	A particular design	Res	Det $\times 10^{11}$	Tr	MEV
K_1	120	$S_0 \cup S_1 \cup S_2 \cup S_4$	V	0.73	1.31	0.25
K_2	480	$S_1^{(u)} \cup S_2^{(u,v)} \cup S_3 \cup S_4$ $u \neq 1; (u,v) \neq (2,4),(3,4)$	V	1.46	1.81	0.85
K_3	480	$S_1^{(u)} \cup S_2^{(u,v)} \cup S_3 \cup S_4$ $u \neq \{1,2\}; (u,v) \neq (1,3)$	V	2.91	2.18	0.85
K_4	160	$S_1^{(4)} \cup S_2 \cup S_3 \cup S_4$	V	2.91	2.43	1.41
K_5	480	$S_1^{(u)} \cup S_2^{(u,v)} \cup S_3 \cup S_4$ $u \neq \{1,2\}; (u,v) \neq (3,4)$	V	2.91	2.69	1.72
K_6	100	$S_1^{(u)} \cup S_2^{(u,v)} \cup S_3 \cup S_4$ $u \neq \{1,2\}; (u,v) \neq (1,2)$	IV	-	-	-
Total	1820	-	-	-	-	-

It is tedious to list all the 120 optimal designs in class K_1. But thanks to the following three propositions, we can characterize the intrinsic relations among these designs. In fact, we will demonstrate in the rest of this section that they all have the same A-, D-, E-optimality criteria.

Proposition 5.1: It's obvious that permutation of rows of design will not change the design, thus will not change the A-, D- and E-optimality criterion values for fitting any model.

Proposition 5.2: Given any one design from class T_1 (or K_1), by multiplying (-1) to any one or two or three or four columns, the resulting 16 designs will have same A-, D-, and E-optimality criteria values for fitting a resolution V model (see the proof in Appendix).

Proposition 5.3: Given any one design from class T_1 (or K_1), by relabeling the columns with different permutation of factors in all possible ways, the resulting 4!=24 designs will have same A-, D-, and E-optimality criteria values for fitting a resolution V model (see the proof in Appendix).

By Proposition 5.1, 5.2 and 5.3, the 120 optimal designs in class K_1 can be divided into four non-isomorphic groups as shown in Table 5.5. Within each group, any two designs can be transformed from one to another by either multiplying -1 onto one or more factors or relabeling the factors or both. Therefore, within each group, all the designs have the same A-, D-, E-optimality criteria values.

Table 5.5 120 designs include 4 groups of designs

Group	Number	A design	Run # delete from T_0
I	16	t_1	15 + + + - 12 - + + + 13 + - + + 14 + + - +
II	64	t_2	1 - - - - 12 - + + + 13 + - + + 14 + + - +
III	16	t_3	1 - - - - 11 - + + - 8 + - + - 6 + + - -
IV	24	t_4	1 - - - - 11 - + + - 13 + - + + 14 + + - +
Total number	120		-

A question has not been answered yet is that whether the designs from different groups have the same A-, D-, and E-optimality criteria values. Therefore, we continue to investigate the designs from different groups, i.e., design t_1 for group I, design t_2 for group II, design t_3 for group III, and design t_4 for group IV shown in Table 5.5. And we prove in the following that the four designs have the same optimality criteria values.

Connect t_1 and t_2

Let's first consider the designs t_1 and t_2. The design matrix $\boldsymbol{X}$ for t_2 corresponding to main effects and two-factor interactions is

A	B	C	D	AB	AC	AD	BC	BD	CD
1	-1	-1	-1	-1	-1	-1	1	1	1
-1	1	-1	-1	-1	1	1	-1	-1	1
-1	-1	1	-1	1	-1	1	-1	1	-1
-1	-1	-1	1	1	1	-1	1	-1	-1
1	1	-1	-1	1	-1	-1	-1	-1	1
-1	-1	1	1	1	-1	-1	-1	-1	1
1	-1	1	-1	-1	1	-1	-1	1	-1
-1	1	-1	1	-1	1	-1	-1	1	-1
1	-1	-1	1	-1	-1	1	1	-1	-1
-1	1	1	-1	-1	-1	1	1	-1	-1
1	1	1	-1	1	1	-1	1	-1	-1
1	1	1	1	1	1	1	1	1	1

Now we rename factors of t_2 by Table 5.6. The "O" stands for "Original (t_2)" and the "R" stands for "Renamed (t_2)". If we determine the renaming of A, B, C, and D, then the rest will follow.

Table 5.6 Rename t_2

O	**R**
D	A
AD	B
BD	C
CD	D
A	AB
B	AC
C	AD
AB	BC
AC	BD
BC	CD

Now by using Table 5.6, we can get columns of design matrix $\boldsymbol{X}$ for renamed t_2 corresponding to main effects and two-factor interactions.

A	B	C	D	AB	AC	AD	BC	BD	CD
-1	-1	1	1	1	-1	-1	-1	-1	1
-1	1	-1	1	-1	1	-1	-1	1	-1
-1	1	1	-1	-1	-1	1	1	-1	-1
1	-1	-1	-1	-1	-1	-1	1	1	1
-1	-1	-1	1	1	1	-1	1	-1	-1
1	-1	-1	1	-1	-1	1	1	-1	-1
-1	-1	1	-1	1	-1	1	-1	1	-1
1	-1	1	-1	-1	1	-1	-1	1	-1
1	1	-1	-1	1	-1	-1	-1	-1	1
-1	1	-1	-1	-1	1	1	-1	-1	1
-1	-1	-1	-1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1

The first four columns of the above matrix give t_1. Next we calculate $\boldsymbol{X'X}$ of renamed t_2. We permute its columns and rows in a way to make them look nicer and present it in Table 5.7. This matrix is identical to $\boldsymbol{X'X}$ of t_1. So the $\boldsymbol{X'X}$ of original t_2 is exactly the same as that of t_1 after renaming.

Table 5.7 *X'X*

$\boldsymbol{X'X}$ of renamed t_2 is identical to $\boldsymbol{X'X}$ of t_1

	μ	AB	AC	AD	BC	BD	CD	A	B	C	D
μ	12	0	0	0	0	0	0	-2	-2	-2	-2
AB	0	12	0	0	0	0	4	-2	-2	2	2
AC	0	0	12	0	0	4	0	-2	2	-2	2
AD	0	0	0	12	4	0	0	-2	2	2	-2
BC	0	0	0	4	12	0	0	2	-2	-2	2
BD	0	0	4	0	0	12	0	2	-2	2	-2
CD	0	4	0	0	0	0	12	2	2	-2	-2
A	-2	-2	-2	-2	2	2	2	12	0	0	0

B	-2	-2	2	2	-2	-2	2	0	12	0	0
C	-2	2	-2	2	-2	2	-2	0	0	12	0
D	-2	2	2	-2	2	-2	-2	0	0	0	12

Connect t_3 and t_4

Following the same logic, we can also prove that the designs t_3 and t_4 have the same optimality criteria values.

Connect t_1 and t_3

First, we try to connect t_1 and t_3 by mapping (i.e., renaming the factors of one design to make the renamed factors have same treatment combination in another design). In the following, we recall the columns of design matrix of t_1 and columns of design matrix of t_3 corresponding to main effects and two-factor interactions.

Design matrix of t_1

	A	B	C	D	AB	AC	AD	BC	BD	CD
	-1	-1	-1	-1	1	1	1	1	1	1
	1	-1	-1	-1	-1	-1	-1	1	1	1
	-1	1	-1	-1	-1	1	1	-1	-1	1
	-1	-1	1	-1	1	-1	1	-1	1	-1
	-1	-1	-1	1	1	1	-1	1	-1	-1
	1	1	-1	-1	1	-1	-1	-1	-1	1
	-1	-1	1	1	1	-1	-1	-1	-1	1
	1	-1	1	-1	-1	1	-1	-1	1	-1
	-1	1	-1	1	-1	1	-1	-1	1	-1
	1	-1	-1	1	-1	-1	1	1	-1	-1
	-1	1	1	-1	-1	-1	1	1	-1	-1
	1	1	1	1	1	1	1	1	1	1
Number of (-1)'s	7	7	7	7	6	6	6	6	6	6

Design matrix of t_3

	A	B	C	D	AB	AC	AD	BC	BD	CD
	1	-1	-1	-1	-1	-1	-1	1	1	1
	-1	1	-1	-1	-1	1	1	-1	-1	1
	-1	-1	1	-1	1	-1	1	-1	1	-1
	-1	-1	-1	1	1	1	-1	1	-1	-1
	-1	-1	1	1	1	-1	-1	-1	-1	1
	-1	1	-1	1	-1	1	-1	-1	1	-1
	1	-1	-1	1	-1	-1	1	1	-1	-1
	-1	1	1	1	-1	-1	-1	1	1	1
	1	-1	1	1	-1	1	1	-1	-1	1
	1	1	-1	1	1	-1	1	-1	1	-1
	1	1	1	-1	1	1	-1	1	-1	-1
	1	1	1	1	1	1	1	1	1	1
Number of (-1)'s	6	6	6	4	6	6	6	6	6	6

In addition, we sum the number of (-1)'s for each main effect and two-factor interaction. We cannot map t_1 to t_3 as there is no way we can rename a factor of t_1 to make it has four (-1)'s. If we cannot map t_1 to t_3, then we cannot map t_3 to t_1 either. Although mapping is not allowed between t_1 and t_3, we still want to find a way to explain why these two designs have same A-, D-, and E-optimality criteria values. Next, instead of mapping designs, we turn to mapping their ***X'X*** matrices. We recall ***X'X*** of t_1 and t_3 which are shown as follows:

$X'X$ of t_1											
	μ	AB	AC	AD	BC	BD	CD	A	B	C	D
μ	12	0	0	0	0	0	0	**-2**	**-2**	**-2**	**-2**
AB	0	12	0	0	0	0	4	**-2**	**-2**	**2**	**2**
AC	0	0	12	0	0	4	0	**-2**	**2**	**-2**	**2**
AD	0	0	0	12	4	0	0	**-2**	**2**	**2**	**-2**
BC	0	0	0	4	12	0	0	**2**	**-2**	**-2**	**2**
BD	0	0	4	0	0	12	0	**2**	**-2**	**2**	**-2**
CD	0	4	0	0	0	0	12	**2**	**2**	**-2**	**-2**
A	**-2**	**-2**	**-2**	**-2**	**2**	**2**	**2**	12	0	0	0
B	**-2**	**-2**	**2**	**2**	**-2**	**-2**	**2**	0	12	0	0
C	**-2**	**2**	**-2**	**2**	**-2**	**2**	**-2**	0	0	12	0
D	**-2**	**2**	**2**	**-2**	**2**	**-2**	**-2**	0	0	0	12

$X'X$ of t_3											
	μ	AB	AC	AD	BC	BD	CD	A	B	C	D
μ	12	0	0	0	0	0	0	**4**	**0**	**0**	**0**
AB	0	12	0	0	0	0	4	**0**	**4**	**0**	**0**
AC	0	0	12	0	0	4	0	**0**	**0**	**4**	**0**
AD	0	0	0	12	4	0	0	**0**	**0**	**0**	**4**
BC	0	0	0	4	12	0	0	**0**	**0**	**0**	**-4**
BD	0	0	4	0	0	12	0	**0**	**0**	**-4**	**0**
CD	0	4	0	0	0	0	12	**0**	**-4**	**0**	**0**
A	**4**	**0**	**0**	**0**	**0**	**0**	**0**	12	0	0	0
B	**0**	**4**	**0**	**0**	**0**	**0**	**-4**	0	12	0	0
C	**0**	**0**	**4**	**0**	**0**	**-4**	**0**	0	0	12	0
D	**0**	**0**	**0**	**4**	**-4**	**0**	**0**	0	0	0	12

Both $X'X$ of t_1 and t_3 have a nice structure. We partition these two matrices into four parts and denote each part with a new name.

<table>
<tr><td colspan="12">$(\boldsymbol{X}'\boldsymbol{X})_{t_i}, i = 1,3$</td></tr>
<tr><td></td><td>μ</td><td>AB</td><td>AC</td><td>AD</td><td>BC</td><td>BD</td><td>CD</td><td>A</td><td>B</td><td>C</td><td>D</td></tr>
<tr><td>μ</td><td colspan="7" rowspan="7">$\boldsymbol{L}$</td><td colspan="4" rowspan="7">$\boldsymbol{M}_{t_i}$</td></tr>
<tr><td>AB</td></tr>
<tr><td>AC</td></tr>
<tr><td>AD</td></tr>
<tr><td>BC</td></tr>
<tr><td>BD</td></tr>
<tr><td>CD</td></tr>
<tr><td>A</td><td colspan="7" rowspan="4">$\boldsymbol{M}_{t_i}'$</td><td colspan="4" rowspan="4">$12\boldsymbol{I}_4$</td></tr>
<tr><td>B</td></tr>
<tr><td>C</td></tr>
<tr><td>D</td></tr>
</table>

Define matrix ***P*** and ***U*** to be

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{I}_7 & 0 \\ 0 & \boldsymbol{U} \end{pmatrix}, \qquad \boldsymbol{U} = \begin{pmatrix} -0.5 & -0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \end{pmatrix},$$

where ***PP'***=***P'P***=***I***, ***UU'***=***U'U***=***I***, so that ***P*** and ***U*** are both orthogonal matrices. Then

$$\boldsymbol{P}(\boldsymbol{X}'\boldsymbol{X})_{t_1}\boldsymbol{P}' = \begin{pmatrix} \boldsymbol{I}_7 & 0 \\ 0 & \boldsymbol{U} \end{pmatrix}\begin{pmatrix} \boldsymbol{L} & \boldsymbol{M}_{t_1} \\ \boldsymbol{M}_{t_1}' & 12\boldsymbol{I}_4 \end{pmatrix}\begin{pmatrix} \boldsymbol{I}_7 & 0 \\ 0 & \boldsymbol{U}' \end{pmatrix} = \begin{pmatrix} \boldsymbol{L} & \boldsymbol{M}_{t_1} \\ \boldsymbol{U}\boldsymbol{M}_{t_1}' & 12\boldsymbol{U} \end{pmatrix}\begin{pmatrix} \boldsymbol{I}_7 & 0 \\ 0 & \boldsymbol{U}' \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{L} & \boldsymbol{M}_{t_1}\boldsymbol{U}' \\ \boldsymbol{U}\boldsymbol{M}_{t_1}' & 12\boldsymbol{U}\boldsymbol{U}' \end{pmatrix} = \begin{pmatrix} \boldsymbol{L} & \boldsymbol{M}_{t_3} \\ \boldsymbol{M}_{t_3}' & 12\boldsymbol{I}_4 \end{pmatrix} = (\boldsymbol{X}'\boldsymbol{X})_{t_3}.$$

And hence,

$$\left|(\boldsymbol{X}'\boldsymbol{X})_{t_3} - \lambda \boldsymbol{I}\right| = \left|\boldsymbol{P}(\boldsymbol{X}'\boldsymbol{X})_{t_1}\boldsymbol{P}' - \lambda \boldsymbol{I}\right| = \left|\boldsymbol{P}((\boldsymbol{X}'\boldsymbol{X})_{t_1} - \lambda \boldsymbol{I})\boldsymbol{P}'\right| = \left|\boldsymbol{P}\right|\left|(\boldsymbol{X}'\boldsymbol{X})_{t_1} - \lambda \boldsymbol{I}\right|\left|\boldsymbol{P}'\right|$$
$$= 1 * \left|(\boldsymbol{X}'\boldsymbol{X})_{t_1} - \lambda \boldsymbol{I}\right| * 1 = \left|(\boldsymbol{X}'\boldsymbol{X})_{t_1} - \lambda \boldsymbol{I}\right|$$

Therefore, $(\boldsymbol{X}'\boldsymbol{X})_{t_1}$ and $(\boldsymbol{X}'\boldsymbol{X})_{t_3}$ have same Eigenvalues. $(\boldsymbol{X}'\boldsymbol{X})_{t_1}^{-1}$ and $(\boldsymbol{X}'\boldsymbol{X})_{t_3}^{-1}$ have same Eigenvalues. So the optimal criteria values are all same for the designs t_1 and t_3.

5.3 Optimal Resolution V Designs for *n*=13, 14, 15, 16

In this section, we search for optimal resolution V designs for *n*=13, 14, 15, and 16. The results for *n*=13, 14 and 15 are shown in Table 5.8, Table 5.9 and Table 5.10 respectively. For *n*=16, it's known that T_0 is optimal with the A-, D- and E-optimality values being 0.69, 5.68×10^{-14} and 0.09 respectively. For each value of *n*, a design in the optimal class is given as an example.

Table 5.8 All possible 13-run designs for res V model

Class	Number	A design	Tr	Det $\times10^{12}$	MEV
L_1	160	$S_0 \cup S_1 \cup S_2 \cup S_3^{(2,3,4)} \cup S_4$	1.14	2.08	0.25
L_2	240		1.5	3.64	0.70
L_3	160		1.67	4.85	0.81
Total	560	-	-	-	-

In Table 5.8, we further observe that there are two groups of isomorphic designs in the optimal class L_1. One is characterized by having one factor column of 5 (-1)'s or 8 (-1)'s, and the other three factor columns each of which has 6 (-1)'s or 7 (-1)'s. The other is characterized by having all four factor columns each of which has 6 (-1)'s or 7 (-1)'s. The former group consists of 64 designs, and the latter group consists of 96 designs. The designs in L_1 perform better than the balanced design (*n*=13) given by Srivastava and Chopra (1971) for fitting a resolution V model.

Table 5.9 All possible 14-run designs for res V model

Class	Number	A design	Tr	Det $\times 10^{13}$	MEV
F_1	80	$S_1 \cup S_2 \cup S_3^{(u,v,z)} \cup S_4$ $(u,v,z) \neq (1,2,3)$	0.98	6.06	0.25
F_2	40		1.19	9.09	0.5
Total	120	-	-	-	-

In Table 5.9, we further observe that there are two groups of isomorphic designs in the optimal class F_1. One is characterized by having even number of (-1)'s in each factor column. The other is characterized by having odd number of (-1)'s in each factor column. The former group consists of 48 designs, and the latter group consists of 32 designs. The designs in F_1 perform better than the balanced design (*n*=14) given in Srivastava and Chopra (1971) for fitting a resolution V model.

Table 5.10 All possible 15-run designs for res V model

Class	Number	A design	Tr	Det $\times 10^{13}$	MEV
B_1	16	Delete any run from T_0	0.83	1.82	0.2
Total	16				

All the 16 designs are isomorphic to each other, and have the same A-, D-, E-optimality criteria values. Therefore, they are equally optimal for fitting resolution V model.

Chapter 6

Up-Resolution V Designs for *m*=5

6.0 Main Results

Unlike m=4 shown in Chapter 5, the exhaustive search for optimal resolution V designs of 2^m series for m=5 becomes computationally extensive. Srivastava and Chopra (1971) have focused on the class of all balanced fractional factorial designs and searched the optimal design only among this class, which is known as the balanced optimal fractional factorial design (BOFFD). However, the BOFFD sometimes could be worse than some unbalanced designs. Therefore, in this chapter, we propose a method to construct Up-Res V designs that is not limited to balanced designs, and show that the designs can perform slightly better than the BOFFD with respect to A-, D-, and E-optimality criteria. For a given n, all our designs are isomorphic having same optimality properties.

6.1 Construction Method

For m=5 and a given practical value of n, the simplest method to identify the optimal designs for fitting a resolution V model is to search among all possible designs balanced or unbalanced. For example, when n=18, there are $\binom{2^5}{18} = 471435600$ possible designs, among which we can identify the optimal designs to be the one having smallest A-, D- and E-optimality criteria values. However, this method is extremely computationally expensive. Therefore, instead of doing so, in the following we give an alternative approach which significantly saves a huge amount of computation time. Although this approach may not find the optimal designs from the whole class of possible designs, the Up-Res V designs we constructed are slightly better than BOFFDs, possess neat properties, and are easy to obtain.

Given the value of n with $16 \le n \le 32$, the method works in two steps as follows. (1) Add i=n-16 runs randomly selected from the 2^{5-1} fractional factorial design with defining relation I= -ABCDE to the 2^{5-1} fractional factorial design with defining relation I= ABCDE to obtain a set of all the $\binom{16}{i}$ possible designs. (2) Identify the optimal designs from the set defined in (1) to be the designs that have the smallest A-, D-, and E-optimality criteria values. We call these designs Up-Res V designs.

When n=16, we don't need to add any run to the 2^{5-1} fractional factorial design with defining relation I=ABCDE, since it is known to be optimal. Obviously, this method works for any general value n and m as well as shown in next chapter.

In this chapter, we denote $\boldsymbol{X}(16\times 16)$ to be the design matrix corresponding to the regular 2^{5-1} fractional factorial design with I=ABCDE under Resolution V model. Likewise, we denote $\boldsymbol{X}^*(16\times 16)$ to be the design matrix corresponding to the regular 2^{5-1} fractional factorial design with I= -ABCDE under Resolution V model. Let $\boldsymbol{X}_i(i\times 16)$ be a submatrix consisting of i ($0\le i\le 16$) rows of $\boldsymbol{X}^*$. Define $\underset{(16+i)\times 16}{\boldsymbol{X}_{(i)}} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{X}_i \end{pmatrix}$. Therefore, $\boldsymbol{X}_{(i)}$ is the design matrix of the constructed n (n=16+i) run design for a choice of i rows in $\boldsymbol{X}^*$. Obviously, $\boldsymbol{X}'\boldsymbol{X} = \boldsymbol{X}^{*\prime}\boldsymbol{X}^* = 16I_{16}$, $\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)} = \boldsymbol{X}'\boldsymbol{X} + \boldsymbol{X}_i'\boldsymbol{X}_i = 16\boldsymbol{I}_{16} + \boldsymbol{X}_i'\boldsymbol{X}_i$, and $\boldsymbol{X}_i\boldsymbol{X}_i' = 16\boldsymbol{I}_i$.

6.2 Up-Resolution V designs for $16 \le n \le 32$

The A-, D-, and E-optimality criteria values of our Up-Res V designs with m=5 are given in Table 6.1. The same criteria values of the BOFFDDs in Srivastava and Chopra (1971) are given in Table 6.2. These designs are referred as to S-C designs.

Comparing Table 6.1 and Table 6.2, we can see that the criteria values of some Up-Res V designs are slightly less than that of the S-C designs. Moreover, for any particular value of n, the Up-Res V designs we obtained are all isomorphic with respect to A-, D-, and E-optimality criteria, which means that no matter which i (i fixed) rows selected from X^*, the resulting designs share the same optimality criteria values. We refer this property as to "isomorphism of Up designs". Moreover, as n increases by 1, the Trace value is decreased by .03125, and the Determinant value is decreased by half. The Maximum Eigenvalue is .0625 for $16 \leq n \leq 31$.

Table 6.1 Our designs

n	Tr	Det $\times 10^{20}$	MEV
16	1	5.42	.0625
17	.96875	2.71	.0625
18	**.9375**	**1.355**	.0625
19	**.90625**	**.6776**	.0625
20	**.875**	**.3388**	.0625
21	.84375	.1694	.0625
22	.81250	.0847	.0625
23	**.78125**	**.0423**	.0625
24	**.75000**	**.0212**	.0625
25	**.71875**	**.0106**	.0625
26	.68750	.0053	.0625
27	.65625	.0026	.0625
28	**.62500**	**.0013**	.0625
29	**.59375**	**.00066**	.0625
30	**.56250**	**.00033**	**.0625**
31	.53125	.00017	.0625
32	.50000	.00008	.03125

Table 6.2 S-C designs

n	Tr	Det $\times 10^{20}$	MEV
16	1	5.42	.0625
17	.96875	2.71	.0625
18	**.9398**	**1.404**	.0625
19	**.9296**	**.9479**	.0625
20	**.9194**	**.6425**	.0625
21	.84375	.1694	.0625
22	.81250	.0847	.0625
23	**.7979**	**.0537**	.0625
24	**.7881**	**.0362**	.0625
25	**.7815**	**.0268**	.0625
26	.68750	.0053	.0625
27	.65625	.0026	.0625
28	**.6300**	**.0014**	.0625
29	**.6199**	**.00097**	.0625
30	**.5830**	**.00039**	**.1**
31	.53125	.00017	.0625
32	.50000	.00008	.03125

6.3 Characterization by Determinants

In this section, we propose Theorem 6.1 to mathematically prove that all the Up-Res V designs of the same value of n have the same Determinant.

***Theorem 6.1*:** For $0 \le i \le 16$, $|(\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)})^{-1}| = \dfrac{1}{16^{16} \times 2^{i}}$, $\forall\, \boldsymbol{X}_i \subset \boldsymbol{X}^*$.

***Proof*:**

For $0 \le i \le 16$,

$$
\begin{aligned}
|\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)}| &= |\boldsymbol{X}'\boldsymbol{X} + \boldsymbol{X}_i'\boldsymbol{X}_i| \\
&= |\boldsymbol{X}'\boldsymbol{X}||\boldsymbol{I}_i + \boldsymbol{X}_i(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}_i'| \\
&= 16^{16}\,|\boldsymbol{I}_i + \frac{1}{16}\boldsymbol{X}_i\boldsymbol{X}_i'| \\
&= 16^{16}\,|\boldsymbol{I}_i + \frac{16}{16}\boldsymbol{I}_i| \\
&= 16^{16}\,|2\boldsymbol{I}_i| \\
&= 16^{16} \times 2^{i} \qquad\qquad \forall\, \boldsymbol{X}_i \subset \boldsymbol{X}^*.
\end{aligned}
$$

Therefore,

$$
|(\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)})^{-1}| = \frac{1}{16^{16} \times 2^{i}}, \qquad \forall\, \boldsymbol{X}_i \subset \boldsymbol{X}^*.
$$

Proof is completed.

6.4 Characterization by Eigenvalues

In this section, we propose Theorem 6.2 to mathematically prove that all the Up-Res V designs of the same value of *n* have the same Maximum Eigenvalue.

Theorem 6.2: For all $\underline{\boldsymbol{a}} \neq \underline{0}$, $\dfrac{\underline{\boldsymbol{a}}'((\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)})^{-1})\underline{\boldsymbol{a}}}{\underline{\boldsymbol{a}}'\underline{\boldsymbol{a}}} \begin{cases} \leq \dfrac{1}{16} & \text{for } \boldsymbol{i} < 16 \\ = \dfrac{1}{32} & \text{for } \boldsymbol{i} = 16 \end{cases}$.

Proof:

For all $\underline{\boldsymbol{a}} \neq \underline{0}$,

$$\frac{\underline{\boldsymbol{a}}'(\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)})\underline{\boldsymbol{a}}}{\underline{\boldsymbol{a}}'\underline{\boldsymbol{a}}} = \frac{\underline{\boldsymbol{a}}'(\boldsymbol{X}'\boldsymbol{X})\underline{\boldsymbol{a}}}{\underline{\boldsymbol{a}}'\underline{\boldsymbol{a}}} + \frac{\underline{\boldsymbol{a}}'(\boldsymbol{X}_i'\boldsymbol{X}_i)\underline{\boldsymbol{a}}}{\underline{\boldsymbol{a}}'\underline{\boldsymbol{a}}}$$

$$= 16 + \frac{(\boldsymbol{X}_i\underline{\boldsymbol{a}})'(\boldsymbol{X}_i\underline{\boldsymbol{a}})}{\underline{\boldsymbol{a}}'\underline{\boldsymbol{a}}}$$

$$\geq 16.$$

The equality ("$=$") holds when $\boldsymbol{X}_i\underline{\boldsymbol{a}} = \underline{0}$. If $i < 16$, then there is always an $\underline{\boldsymbol{a}}$ satisfying $\boldsymbol{X}_i\underline{\boldsymbol{a}} = \underline{0}$, given that the 16 rows in $\boldsymbol{X}^*$ are all orthogonal to each other. If $i = 16$, then $\boldsymbol{X}_i = \boldsymbol{X}^*$ and $\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)} = 16\boldsymbol{I}_{16} + 16\boldsymbol{I}_{16} = 32\boldsymbol{I}_{16}$. Thus, ***Theorem 6.2*** will follow.

6.5 Characterization by Traces

In this section, we propose Theorem 6.3 to mathematically prove that all the Up-Res V designs of the same value of *n* have the same Trace.

Theorem 6.3: For $0 \leq i \leq 16$, $Tr(\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)})^{-1} = 1 - \dfrac{i}{32}$, $\forall\ \boldsymbol{X}_i \subset \boldsymbol{X}^*$.

***Proof*:** For $0 \le i \le 16$,

$$\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)} = 16\boldsymbol{I} + \boldsymbol{X}_i'\boldsymbol{X}_i$$

So,

$$\begin{aligned}(\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)})^{-1} &= \frac{1}{16}\boldsymbol{I} - \frac{1}{16^2}\boldsymbol{X}_i'(\frac{1}{16}\boldsymbol{X}_i\boldsymbol{X}_i' + \boldsymbol{I}_i)^{-1}\boldsymbol{X}_i \\ &= \frac{1}{16}\boldsymbol{I} - \frac{1}{16^2}\boldsymbol{X}_i'(2\boldsymbol{I}_i)^{-1}\boldsymbol{X}_i \\ &= \frac{1}{16}\boldsymbol{I} - \frac{1}{2\times 16^2}\boldsymbol{X}_i'\boldsymbol{X}_i.\end{aligned}$$

Therefore,

$$\begin{aligned}Tr[(\boldsymbol{X}_{(i)}'\boldsymbol{X}_{(i)})^{-1}] &= 1 - \frac{1}{2\times 16^2}Tr(\boldsymbol{X}_i'\boldsymbol{X}_i) \\ &= 1 - \frac{1}{2\times 16^2}Tr(\boldsymbol{X}_i\boldsymbol{X}_i') \\ &= 1 - \frac{1}{2\times 16^2}Tr(16\boldsymbol{I}_i) \\ &= 1 - \frac{i}{2\times 16} \\ &= 1 - \frac{i}{32}.\end{aligned}$$

Proof is completed.

Chapter 7

Isomorphism of Up designs in 2^m Factorial Experiments

7.0 Main Results

In Chapter 6, we proposed a novel method to construct a series of Up-Res V designs with *m*=5. The designs were found out isomorphic to each other by having same optimality properties for a given *n*. However, such property might not always hold for any value of *m* and for fitting any factorial effects model. Instead, it only holds under certain conditions. Therefore, in this chapter, we first follow Section 6.1 to describe the design construction method for general values of *m* in Section 7.1. In Section 7.2, the conditions are stated and explained mathematically for this isomorphic property to hold in general situations. Special situations are considered in Section 7.3 where the conditions can be significantly simplified. Meanwhile, examples are given for understanding these conditions.

7.1 Construction Method

Consider in a 2^m factorial experiment, we want to fit a model M with p_1 parameters. Suppose we have a base design that consists of n_1 runs from the 2^m possible runs ($n_1 \geq p_1$), and is capable of fitting such model. Denote this base design to be T_1, and its corresponding design matrix to be $\boldsymbol{X}_1(n_1 \times p_1)$. Denote the complement of T_1 from the 2^m factorial to be T_1^*, and its corresponding design matrix to be $\boldsymbol{X}_1^*((2^m - n_1) \times p_1)$. Given a practical value of n with $n_1 \le n \le 2^m$ and let $i = n - n_1$, the optimal design for fitting model M can be constructed in the following two steps.

(1) Add $i = n - n_1$ runs randomly selected from T_1^* to T_1 to obtain a set of all $\begin{pmatrix} 2^m - n_1 \\ i \end{pmatrix}$ possible augmented designs. The augmented design can be expressed as $T_{2ij} = \begin{pmatrix} T_1 \\ T_{1ij}^* \end{pmatrix}$ with the corresponding design matrix $\boldsymbol{X}_{2ij} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_{1ij}^* \end{pmatrix}_{n_2 \times p_1}$, where $j = 1,\ 2, \ldots, \begin{pmatrix} 2^m - n_1 \\ i \end{pmatrix}$ indexes j^{th} the selection of i run(s) from T_1^*. Notice that i can take all positive integers from 1 to $2^m - n_1$.

(2) Identify the optimal designs from the set obtained in (1) to be the designs that have the smallest A-, D-, and E-optimality criteria values.

It is of interest to find whether the designs in (1) have the isomorphic property or not. More specifically, we want to know that no matter which i runs are added, whether the resulting T_{2ij}'s will have the same A-, D- and E-optimality criteria values. In the next section, we identify the conditions for the determinant of $(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1}$ to be independent of j, and also the conditions for the trace of $(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1}$ to be independent of j.

7.2 Isomorphism of Up designs Property in General Situations

This following two theorems specify the conditions for the isomorphism of Up designs property (A- and D-optimality criteria) to hold. The E-optimality criterion is too complex to consider here.

Theorem 7.1: The determinant of $(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1}$ is independent of j if and only if $|\boldsymbol{I}_i + \boldsymbol{X}^*_{1ij}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}^*_{1ij}{}'|$ is independent of j.

Theorem 7.2: The trace of $(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1}$ is independent of j if and only if

$Tr((\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}^*_{1ij}{}'(\boldsymbol{I}_i + \boldsymbol{X}^*_{1ij}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}^*_{1ij}{}')^{-1}\boldsymbol{X}^*_{1ij}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1})$ is independent of j.

When $i=1$, $(\boldsymbol{X}_{21j}'\boldsymbol{X}_{21j})^{-1} = (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} - \dfrac{(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}^*_{11j}{}'\boldsymbol{X}^*_{11j}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}}{(1+\boldsymbol{X}^*_{11j}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}^*_{11j}{}')}$, where

$(1+\boldsymbol{X}^*_{11j}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}^*_{11j}{}')$ is a scalar. Hence,

$$Tr(\boldsymbol{X}_{21j}'\boldsymbol{X}_{21j})^{-1} = Tr(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} - \frac{Tr(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}^*_{11j}{}'\boldsymbol{X}^*_{11j}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}}{(1+\boldsymbol{X}^*_{11j}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}^*_{11j}{}')}.$$

Therefore, the condition in Theorem 7.2 becomes

$$\frac{Tr(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_{11j}^{*}{}'\boldsymbol{X}_{11j}^{*}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}}{(1+\boldsymbol{X}_{11j}^{*}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_{11j}^{*}{}')} \text{ is independent of } j.$$

When $i > 1$,

$$\begin{aligned} Tr(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1} &= Tr(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} - Tr((\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_{1ij}^{*}{}'(\boldsymbol{I}_i + \boldsymbol{X}_{1ij}^{*}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_{1ij}^{*}{}')^{-1}\boldsymbol{X}_{1ij}^{*}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}) \\ &= Tr(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} - Tr(\boldsymbol{X}_{1ij}^{*}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-2}\boldsymbol{X}_{1ij}^{*}{}'(\boldsymbol{I}_i + \boldsymbol{X}_{1ij}^{*}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_{1ij}^{*}{}')^{-1}). \end{aligned}$$

Therefore, if both $Tr(\boldsymbol{X}_{1ij}^{*}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-2}\boldsymbol{X}_{1ij}^{*}{}'$ and $\boldsymbol{X}_{1ij}^{*}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_{1ij}^{*}{}'$ are independent of *j*, then the trace of $(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1}$ is independent of *j*. Under such conditions, the determinant of $(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1}$ is also independent of *j*.

In the rest of this section, we illustrate the usage of the above two theorems by the following example. Let's consider a 2^4 factorial experiment under resolution V model. Let T_1 consist of all the 11 runs in $S_1 \cup S_2 \cup S_4$, and its complement design T_1^* consist of all the other 5 runs in $S_0 \cup S_3$. Then for any value of *i* with $1 \le i \le 5$, we can follow the method in Section 7.1 to construct T_{2ij}'s and compute their A-, D-, E-optimality criteria values as shown in Table 7.1. From this table, we can see that all T_{2ij}'s have the same criteria values for the same value of *i*. Therefore, the isomorphic property holds.

Table 7.1 Isomorphic Property of T_{2ij} under Res V model

i	n	Number of designs	Tr	Det $\times 10^{11}$	MEV
0	11	1	1.49	2.59	.25
1	12	5	1.31	.73	.25
2	13	10	1.14	.21	.25
3	14	10	.98	.061	.25
4	15	5	.83	.018	.2
5	16	1	.69	.0057	.06

Theorem 7.1 and Theorem 7.2 can verify the isomorphic property as follows.

When $i = 1$ and $n = 12$, $\forall j$: $\boldsymbol{X}^*_{11j}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}^*_{11j}' = 2.556, \boldsymbol{X}^*_{11j}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-2}\boldsymbol{X}^*_{11j}' = .6173.$

Therefore, by Theorem 7.1 and 7.2, the determinant and the trace of $(\boldsymbol{X}_{21j}'\boldsymbol{X}_{21j})^{-1}$ are independent of ***j*.**

When $i = 2$ and $n = 13$, $\forall j$:

$$\boldsymbol{X}^*_{12j}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}^*_{12j}' = \begin{pmatrix} 2.556 & -.444 \\ -.444 & 2.556 \end{pmatrix}, \boldsymbol{X}^*_{12j}(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-2}\boldsymbol{X}^*_{12j}' = \begin{pmatrix} .6173 & -.1327 \\ -.1327 & .6173 \end{pmatrix}.$$

Therefore, by Theorem 7.1 and 7.2, the determinant and the trace of $(\boldsymbol{X}_{22j}'\boldsymbol{X}_{22j})^{-1}$ are independent of ***j*.**

When i =3, 4, and 5, the proof is similar.

7.3 Isomorphism of Up designs Property in Special Cases

The conditions in Theorem 7.1 and Theorem 7.2 have quite complicated mathematical forms. In practice, a special yet reasonable situation we might expect is that the design matrix of base design, $\boldsymbol{X}_1$ is orthogonal column-wise. That is, $\boldsymbol{X}_1'\boldsymbol{X}_1 = n_1\boldsymbol{I}_{p_1}$ and $(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} = \frac{1}{n_1}\boldsymbol{I}_{p_1}$. Under such situation, the conditions can be significantly simplified as shown in Corollary 7.1 and Corollary 7.2.

***Corollary 7.1*:** Given $\boldsymbol{X}_1'\boldsymbol{X}_1 = n_1\boldsymbol{I}_{p_1}$ and $(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} = \frac{1}{n_1}\boldsymbol{I}_{p_1}$, the determinant of $(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1}$ is independent of *j* if and only if $|\boldsymbol{I}_i + \frac{1}{n_1}\boldsymbol{X}^*_{1ij}\boldsymbol{X}^*_{1ij}{}'|$ is independent of *j*.

***Corollary 7.2*:** Given $\boldsymbol{X}_1'\boldsymbol{X}_1 = n_1\boldsymbol{I}_{p_1}$ and $(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} = \frac{1}{n_1}\boldsymbol{I}_{p_1}$, the trace of $(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1}$ is independent of *j* if and only if $Tr(\boldsymbol{I}_i + \frac{1}{n_1}\boldsymbol{X}^*_{1ij}\boldsymbol{X}^*_{1ij}{}')^{-1}$ is independent of *j*. (See Proof in the Appendix)

When *i*=1, $\forall\, j$:

$$\boldsymbol{X}^*_{11j}\boldsymbol{X}^*_{11j}{}' = p_1,\ Tr(\boldsymbol{X}_{21j}'\boldsymbol{X}_{21j})^{-1} = \frac{p_1 - 1}{n_1} + \frac{1}{n_1}Tr(\frac{1}{n_1}p_1 + 1)^{-1} = \frac{p_1(p_1 + n_1 - 1)}{n_1(p_1 + n_1)}.$$

Therefore, by Corollary 7.1 and 7.2, the determinant and the trace of $(\boldsymbol{X}_{21j}'\boldsymbol{X}_{21j})^{-1}$ are always independent of j.

When i=2, denote $\boldsymbol{X}^*_{12j}\boldsymbol{X}^*_{12j}{}' = \begin{pmatrix} p_1 & q_j \\ q_j & p_1 \end{pmatrix}$, then

$$|I_2 + \frac{1}{n_1}\boldsymbol{X}^*_{12j}\boldsymbol{X}^*_{12j}{}'| = \begin{vmatrix} \frac{p_1}{n_1}+1 & \frac{q_j}{n_1} \\ \frac{q_j}{n_1} & \frac{p_1}{n_1}+1 \end{vmatrix} = \frac{p_1^2 + n_1^2 + 2p_1n_1 - q_j^2}{n_1^2}.$$

By Corollary 7.1, the determinant of $(\boldsymbol{X}_{22j}'\boldsymbol{X}_{22j})^{-1}$ is independent of j if and only if q_j^2 is a constant $\forall\, j$.

Also,

$$Tr(\frac{1}{n_1}\boldsymbol{X}^*_{12j}\boldsymbol{X}^*_{12j}{}' + I_2)^{-1} = Tr\begin{pmatrix} \frac{p_1}{n_1}+1 & \frac{q_j}{n_1} \\ \frac{q_j}{n_1} & \frac{p_1}{n_1}+1 \end{pmatrix}^{-1} = \frac{2n_1(p_1+n_1)}{(p_1+n_1)^2 - q_j^2}.$$

Hence,

$$Tr(\boldsymbol{X}_{22j}'\boldsymbol{X}_{22j})^{-1} = \frac{p_1-2}{n_1} + \frac{2(p_1+n_1)}{(p_1+n_1)^2 - q_j^2}.$$

By Corollary 7.2, the trace of $(\boldsymbol{X}_{22j}'\boldsymbol{X}_{22j})^{-1}$ is independent of *j* if and only if

q_j^2 is a constant $\forall\, j$.

For example, let's consider a 2^4 factorial experiment under the model *M* that contains general mean, all main effects and two 2-factor interactions AB and AC, the base design T_1 is chosen to have defining relation I=ABCD. Therefore, its complement design T_1^* has defining relation I= -ABCD. Then for any value of *i* ($0 \le i \le 8$), we can follow the method in Section 7.1 to construct designs T_{2ij} and compute their A-, D-, and E-optimality criteria values. The results are shown in Table 7.2. From this table, we can see that for any fixed *i*, T_{2ij}'s share the same criteria values. Therefore, the isomorphic property holds.

Table 7.2 Isomorphic Property of T_{2ij} under model *M*

i	***n***	**Number of designs**	**Tr**	**Det**$\times 10^7$	**MEV**
0	8	1	.875	4.768	.125
1	9	8	.81667	2.543	.125
2	10	28	.75893	1.362	.125
3	11	56	.70192	.7336	.125
4	12	70	.64583	.3974	.125
5	13	56	.59091	.2167	.125
6	14	28	.5735	.1192	.125
7	15	8	.48611	.0662	.1111
8	16	1	.4375	.0373	.0625

Corollary 7.1 and Corollary 7.2 can verify the isomorphic property as follows. First, we can easily check that $\boldsymbol{X}_1'\boldsymbol{X}_1 = 8\boldsymbol{I}_7, (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} = \frac{1}{8}\boldsymbol{I}_7$. Therefore, $\boldsymbol{X}_1$ is orthogonal column-wise.

When i=1, by Corollary 7.1 and 7.2, the determinant and the trace of $(\boldsymbol{X}_{21j}'\boldsymbol{X}_{21j})^{-1}$ are always independent of j.

When i=2, we can prove that for any j, $q_j = \pm 1, |I_2 + \frac{1}{n_1}\boldsymbol{X}_{12j}^*\boldsymbol{X}_{12j}^*{}'| = \frac{p_1^2 + n_1^2 + 2p_1n_1 - q_j^2}{n_1^2} = .375$.

$Tr(\boldsymbol{X}_{22j}'\boldsymbol{X}_{22j})^{-1} = \frac{p_1 - 2}{n_1} + \frac{2(p_1 + n_1)}{(p_1 + n_1)^2 - q_j^2} = .75893$. Therefore, by Corollary 7.1 and 7.2, the determinant and the trace of $(\boldsymbol{X}_{22j}'\boldsymbol{X}_{22j})^{-1}$ are independent of j.

When i =3,…,8, the proof is similar.

Chapter 8

Efficient Up-Down Resolution III Designs

8.0 Main Results

In this chapter, we propose a general Up-Down method to search for efficient 2^m fractional factorial designs in fitting a class of models when the number of factors is *m*, and the number of runs is *n*. We present the efficient resolution III designs obtained by the Up-Down method for $3 \le m \le 10$ and a range of practical values of *n*. While many of these designs are found to be the global optimal resolution III designs by exhaustive computer search, the other designs are near global optimal designs. For *m*=4 and 5, we compare our designs with the optimal resolution III+*k* (*k*=0,1,2,…) designs in Ghosh and Tian (2006).

8.1 Up-Down Method

In practice, resolution III plans are very popular when experimenters want to study the main effects of the factors in factorial experiments. For a fixed value of *m*, we already know the optimal resolution III plans are the full factorial design or regular fractional factorial design for *n*=2^m, 2^{m-1}, etc. due to their orthogonality property. For the same reason, 12-run PB design is optimal for *n*=12 and $m \le 11$. However, when *n* takes other values, the optimal resolution III designs become unknown. In this section, we propose an Up-Down method to construct efficient resolution III designs for $3 \le m \le 10$ and a range of practical values of n, which works in four steps as follows.

(1) Given the values of m and n, find the nearest lower value n_L and nearest upper value n_U for which the corresponding optimal resolution III designs are available. For some small value n_L, a resolution III design might be unavailable due to run limitation; we would use some regular fractional factorial design to substitute. Denote these two designs to be d_L and d_U.

(2) Add $n - n_L$ runs randomly selected from the remaining $2^m - n_L$ runs to d_L to form all the $\binom{2^m - n_L}{n - n_L}$ Up designs. Calculate their A-, D-, and E-optimality criteria values for fitting resolution III model.

(3) Eliminate $n_U - n$ runs from d_U to form all the $\binom{n_U}{n_U - n}$ Down designs. Calculate their A-, D-, and E-optimality criteria values for fitting resolution III model.

(4) Among all the Up (Down) designs, identify the best Up (Down) designs, denoted by $Z_n^{\uparrow}$ ($Z_n^{\downarrow}$), to the ones that have the smallest optimality criteria values. The better one of $Z_n^{\uparrow}$ and $Z_n^{\downarrow}$ is chosen to be our efficient resolution III designs.

Most of the time, the runs in d_L and d_U are distinctive. However, this is not always the case. For example, when *m*=4, *n*=12, the PB design with 4 columns is an optimal resolution III plan that has two replicated runs. Under such case, steps (2) and (3) in the Up-Down method might need to be changed slightly. When d_L contains replicated runs, then in step (2), we add $n - n_L$ runs randomly selected from the remaining $2^m - n_L^*$ runs to d_L to form an Up design, where n_L^* is the number of distinctive runs in d_L. When d_U contains replicated runs, actually it will not cause problems, as the way we do in step (3) will cover all the Down designs.

Let's illustrate the Up-Down method by using *m*=5 as an example. In Section 8.4, we present the efficient resolution III designs among Up and Down designs that we construct for *n*=6, 7, 9, 10, 11, 13, 14, and 15. For *n*=8, 12, and 16, we already know that the existing orthogonal arrays of strength 2 are the optimal resolution III plans. Also, for *n*=4, a resolution III design is impossible due to run limitation, but we believe that the design with defining relation I=ABC=CDE=AD should be an efficient design d_L. Thus, for *n*=6, 7, we build Up designs by adding runs to the above 4-run design d_L and build Down designs by eliminating runs from 8-run design d_U. For *n*=9, 10, 11, we build Up and Down designs based on 8-run and 12-run designs. For *n*=13, 14, 15, we build Up and Down designs based on 12-run and 16-run designs. For a particular *n*, we search out the best resolution III designs among the Up and Down designs, and present them in the table.

8.2 *m*=3

The efficient resolution III designs for *m*=3, *n*=4, 5, 6, 7, and 8 are presented in Table 8.1. For *n*=4, the regular fractional factorial design Z_4 with defining relation I=ABC is optimal. For *n*=8, the full factorial design Z_8 is optimal. For *n*=5,6,7, we add *n*-4 runs to Z_4 to construct the Up designs and eliminate 8- *n* runs from Z_8 to construct the Down designs. Since the best UP designs $Z_n^{\uparrow}$ and the best DOWN designs $Z_n^{\downarrow}$ are equally efficient, so we list both of them in the table. In the last column in Table 8.1, we present the runs of one representative design. We refer the Up-Down designs to as G-Z designs.

Table 8.1 *m*=3, Up-Down G-Z res III designs

Designs	*n*	Tr	Det $\times 10^4$	MEV	Optimality criteria	Runs
Z_4	4	1	39.1	.25	A,D,E	*1,2,3,123
$Z_5^{\uparrow}, Z_5^{\downarrow}$	5	.875	19.5	.25	A,D,E	1,2,3,123, any one run from the remaining
$Z_6^{\uparrow}, Z_6^{\downarrow}$	6	.75	9.77	.25	A,D,E	1,2,3,123, any two runs from the remaining
$Z_7^{\uparrow}, Z_7^{\downarrow}$	7	.625	4.88	.25	A,D,E	1,2,3,123, any three runs from the remaining
Z_8	8	.5	2.44	.125	A,D,E	**0,1,2,3,12,13,23,123

* *n*=4, I=ABC. ** *n*=8, full factorial.

The case considered in this section is so simple that we can use computer to search out the global optimal resolution III designs as shown in Table 8.2, where Dn in the first column denotes the optimal designs for *n*-run designs. The “global” is defined in the way that computer searches out all possible $\binom{2^m}{n}$ designs that with no replicated runs and identifies the optimal ones.

Table 8.2 *m*=3, optimal res III plans by computer search

Designs	n	Tr	Det $\times 10^4$	MEV	Optimality criteria
D4	4	1	39.1	.25	A,D,E
D5	5	.875	19.5	.25	A,D,E
D6	6	.75	9.77	.25	A,D,E
D7	7	.625	4.88	.25	A,D,E
D8	8	.5	2.44	.125	A,D,E

From Table 8.2, we can see that Designs D4 to D8 are equivalent to our Up-Down designs in Table 8.1 since they have the same A, D, and E-optimality criteria values. Therefore, our Up-Down designs in this case are not just efficient, but globally optimal.

8.3 *m*=4

The efficient resolution III designs for m=4, n=5,…,12 are presented in Table 8.3. For n=5, 6, 7, we construct Up designs by adding n-4 runs to the regular fractional factorial design with defining relation I=ABC= BCD=AD. For n=7, one class of Up designs $Z_{7.1}^{\uparrow}$ is efficient with respect to A- and D-optimality criteria, one class of Up designs $Z_{7.2}^{\uparrow}$ is efficient with respect to E-optimality criterion. For n=12, Z_{12} is the design obtained by projecting the PB design onto its four columns. It is known to be optimal resolution III design.

Table 8.3 *m*=4, Up-Down G-Z res III designs

Designs	n	Tr	Det $\times 10^6$	MEV	Optimality criteria	Runs
$Z_5^{\uparrow}, Z_5^{\downarrow}$	5	1.75	977	1	A,D,E	2,3,14,1234,1
$Z_6^{\uparrow}$	6	.975	195	.25	A,D,E	2,3,14,1234,1,4
$Z_{7.1}^{\uparrow}, Z_7^{\downarrow}$	7	.833	81.4	.333	A,D	2,3,14,1234,1,34,24
$Z_{7.2}^{\uparrow}$	7	.841	88.8	.25	E	2,3,14,1234,1,4,24
Z_8	8	.625	30.5	.125	A,D,E	*0,12,13,14,23,24,34,1234
$Z_9^{\uparrow}$	9	.577	18.8	.125	A,D,E	0,12,13,14,23,24,34,1234, any one run from remaining
$Z_{10}^{\uparrow}$	10	.530	11.6	.125	A,D,E	0,12,13,14,23,24,34,1234,1,134
$Z_{11}^{\downarrow}$	11	.476	6.89	.143	A,D	take out any one run from 0,1,3,4,12,23,24,123,124,134,134,234
$Z_{11}^{\uparrow}$	11	.483	7.23	.125	E	0,12,13,14,23,24,34,1234,1,134,3
Z_{12}	12	.417	4.02	.083	A,D,E	**0,1,3,4,12,23,24,123,124,134,134,234

* n=8, I=ABCD. ** n=12, PB4(col 1 to 4).

Again, we use computer to search global optimal resolution III designs Dn's as shown in Table 8.4.

Table 8.4 m=4, optimal res III designs by computer search

Design	n	Tr	Det $\times10^6$	MEV	Optimality criteria	#
D5	5	1.11	434	.25	A,D,E	16
D6	6	.975	195	.25	A,D,E	160
D7.1	7	.833	81.4	.333	A,D	80
D7.2	7	.841	88.8	.25	E	480
D8	8	.625	30.5	.125	A,D,E	10
D9	9	.577	18.8	.125	A,D,E	80
D10	10	.530	11.6	.125	A,D,E	240
D11.1	11	.476	6.89	.143	A,D	16
D11.2	11	.483	7.23	.125	E	320
D12	12	.438	4.52	.125	A,D,E	120

From Table 8.4, we can see that Designs D6 to D11.2 are equivalent to our Up-Down designs in Table 8.3. Therefore, our Up-Down designs in this case are not only efficient but also optimal. For n=12, design Z_{12} outperforms D12. This finding implies that a design with replicates sometimes can be better than a design with all distinct runs depending on the model.

Ghosh and Tian (2006) considered a set of models M_k with parameters as the general mean, main effects, and k two-factor interactions for a factorial experiment with m factors. They considered a class of fractional factorial designs with n runs permitting the unbiased estimation of the factorial effects under each model in the set of models considered. They presented the optimal designs from this class of designs satisfying some optimality criterion functions for m=4, (n,k)=(5,0), (6,1), (7,1), (8,2), (9,3), (10,5), (11,6); m=5, (n,k)=(6,0), (7,1), (8,1), (9,2), (10,3), (11,3), (12,5), (13,5), (14,7), (15,9), (16,10). In Table 8.5, we take their designs T_{n} of m=4 to fit resolution III model, and present their

A-, D-, and E-optimality criteria values. In Table 8.6, we compare our G-Z designs with their G-T designs.

Table 8.5 *m*=4, G-T designs under res III model

Design	n	Tr	Det $\times 10^6$	MEV
T_5	5	1.111	434	.25
T_6	6	.975	195	.25
T_7	7	.833	81.4	.333
$T_{8.1}$	8	.708	40.7	.25
$T_{8.2}$	8	1.125	122	.427
T_9	9	.577	18.8	.125
T_{10}	10	.542	12.1	.167
T_{11}	11	.476	6.89	.143

Table 8.6 Comparisons of G-Z and G-T designs for *m*=4 under res III model

n	Tr		Det $\times 10^6$		MEV	
	G-Z	G-T	G-Z	G-T	G-Z	G-T
5	1.75	1.111	977	434	1	.25
6	.975	.975	195	195	.25	.25
7	.833	.833	81.4	81.4	.333	.333
	.841		88.8		**.25**	
8	**.625**	.708	**30.5**	40.7	**.125**	.25
		1.125		122		.427
9	.577	.577	18.8	18.8	.125	.125
10	**.530**	.542	**11.6**	12.1	**.125**	.167
11	.483	.476	7.23	6.89	**.125**	.143
	.476		6.89		.143	

From Table 8.6, we can see that when n=7, 8, 10 and 11, our designs are better than G-T designs. When n=6 and 9, our designs are equivalent to G-T designs. When n=5, G-T designs are better.

In the following sections, we show similar tables for *m*=5, 6, 7, 8, 9, 10. The tables are quite self-explanatory.

8.4 *m*=5

Table 8.7 *m*=5, Up-Down G-Z res III designs

Design	*n*	Tr	Det $\times 10^7$	MEV	Optimality criteria	Runs
$Z_6^{\uparrow}, Z_6^{\downarrow}$	6	1.5	610	.5	A,D,E	*3,14,25,12345+24,345,
$Z_{7.1}^{\uparrow}$	7	1.058	163	.25	A,E	3,14,25,12345,24,12,5
$Z_{7.2}^{\uparrow}$	7	1.063	153	.35	D	3,14,25,12345,24,12,15
Z_8	8	.75	38.1	.125	A,D,E	**I=ABC=CDE=ABDE
$Z_9^{\uparrow}$	9	.696	21.8	.125	A,D,E	I=ABC=CDE=ABDE, any one run from remaining
$Z_{10}^{\uparrow}$	10	.643	12.5	.125	A,D,E	I=ABC=CDE=ABDE, 1,34
$Z_{11}^{\downarrow}$	11	.583	6.697	.167	A,D	PB5.2 takes out any one run or PB5.1 takes out one replicate
$Z_{11}^{\uparrow}$	11	.592	7.27	.125	E	I=ABC=CDE=ABDE, 1,34,1234
Z_{12}	12	.5	3.34	.083	A,D,E	***PB5.1 or PB5.2
$Z_{13}^{\uparrow}$	13	.472	2.23	.083	A,D,E	PB5.2, any one run from remaining
$Z_{14}^{\uparrow}$	14	.444	1.49	.083	A,D,E	PB5.2, 1,23
$Z_{15}^{\downarrow}$	15	.413	.95	.1	A,D	I=ABCDE take out any one run
$Z_{15}^{\uparrow}$	15	.418	1.00	.083	E	PB5.2, 1,23,24
Z_{16}	16	.375	.596	.0625	A,D,E	I=ABCDE

* *n*=4, I=ABC=CDE=AD: 3,14,25,12345.

** *n*=8, I=ABC=CDE.

*** *n*=12, PB5.1(col 1 to 4 and 10): 0,12,15,23,24,35,45,134,134,1235,1245,2345;

PB5.2(col 1 to 5): 0,4,12,15,35,123,134,234,235,245,1245,1345.

In Table 8.8, we take G-T designs T_n of m=5 to fit resolution III model, and present their A-, D-, and E-optimality criteria values. The design T_6 is the optimal design for resolution III model obtained by Ghosh and Tian (2006). In Table 8.9, we compare our G-Z designs with their G-T designs.

Table 8.8 *m*=5, G-T designs under res III model

Design	n	Tr	Det $\times 10^7$	MEV
T_6	6	1.20	391	.25
$T_{7.1}$	7	1.21	203	.5
$T_{7.2}$	7	1.16	244	.25
$T_{7.3}$	7	1.058	163	.25
T_8	8	.92	69.8	.25
T_9	9	.828	35.4	.25
T_{10}	10	.656	13.6	.125
$T_{11.1}$	11	.606	7.93	.125
$T_{11.2}$	11	.637	9.84	.125
T_{12}	12	.527	3.89	.133
T_{13}	13	.484	2.38	.113
T_{14}	14	.475	1.80	.125
T_{15}	15	.413	.95	.1
T_{16}	16	.375	.596	.0625

Table 8.9 Comparisons of G-Z and G-T designs for *m*=5 under res III model

n	Tr		Det$\times10^7$		MEV	
	G-Z	G-T	G-Z	G-T	G-Z	G-T
6	1.50	1.20	610	391	.5	.25
7	1.058 ($Z^{\uparrow}_{7.1}$)	1.21 ($T_{7.1}$)	163 ($Z^{\uparrow}_{7.1}$)	203 ($T_{7.1}$)	.25 ($Z^{\uparrow}_{7.1}$)	.5 ($T_{7.1}$)
	1.063 ($Z^{\uparrow}_{7.2}$)	1.16 ($T_{7.2}$)	153 ($Z^{\uparrow}_{7.2}$)	244 ($T_{7.2}$)	.35 ($Z^{\uparrow}_{7.2}$)	.25 ($T_{7.2}$)
		1.058 ($T_{7.3}$)		163 ($T_{7.3}$)		.25 ($T_{7.3}$)
8	**.75**	.92	**38.1**	69.8	**.125**	.25
9	**.696**	.828	**21.8**	35.4	**.125**	.25
10	**.643**	.656	**12.5**	13.6	.125	.125
11	**.583 ($Z^{\downarrow}_{11}$)**	.606 ($T_{11.1}$)	**6.70 ($Z^{\downarrow}_{11}$)**	7.93 ($T_{11.1}$)	.167 ($Z^{\downarrow}_{11}$)	.125 ($T_{11.1}$)
	.592 ($Z^{\uparrow}_{11}$)	.637 ($T_{11.2}$)	7.27 ($Z^{\uparrow}_{11}$)	9.84 ($T_{11.2}$)	.125 ($Z^{\uparrow}_{11}$)	.125 ($T_{11.2}$)
12	**.5**	.527	**3.34**	3.89	**.083**	.133
13	**.472**	.484	**2.23**	2.38	**.083**	.113
14	**.444**	.475	**1.49**	1.80	**.083**	.125
15	.413 ($Z^{\downarrow}_{15}$)	.413	.95 ($Z^{\downarrow}_{15}$)	.95	.1 ($Z^{\downarrow}_{15}$)	.1
	.418 ($Z^{\uparrow}_{15}$)		1.00 ($Z^{\uparrow}_{15}$)		.083 ($Z^{\uparrow}_{15}$)	
16	.375	.375	.596	.596	.0625	.0625

From Table 8.9, we can see that when *n*=8, 9, 10, 11, 12, 13 and 14, our designs are better than G-T designs. When *n*=7, 15 and 16, our designs are equivalent to G-T designs. When *n*=6, G-T designs are better.

8.5 *m*=6, 7, 8, 9, 10

Table 8.10 *m*=6, Up-Down G-Z designs under Res III model

Design	n	Tr	Det $\times 10^8$	MEV	Optimality criteria	Runs
$Z_7^{\uparrow}$	7	1.438	381	.512	A,D,E	*12,34,56,123456+35,136,236
Z_8	8	.875	47.7	.125	A,D,E	**12,34,56,123456,135,146,236,245
$Z_9^{\uparrow}$	9	.817	25.4	.125	A,D,E	12,34,56,123456,135,146,236,245+any one run from the remaining
$Z_{10}^{\uparrow}$	10	.759	13.6	.125	A,D,E	12,34,56,123456,135,146,236,245+0,2356
$Z_{11}^{\downarrow}$	11	.7	6.70	.2	A,D	take out any one run from PB6.1 (or PB6.2)
$Z_{11}^{\uparrow}$	11	.702	7.34	.125	E	12,34,56,123456,135,146,236,245+0,2356,1456
Z_{12}	12	.583	2.79	.083	A,D,E	***PB6.1 or PB6.2
$Z_{13}^{\uparrow}$	13	.553	1.76	.083	A,D,E	PB6.1 (or PB6.2), any one run from the remaining
$Z_{14}^{\uparrow}$	14	.522	1.12	.083	A,D,E	PB 6.1, 1,12356
$Z_{15}^{\downarrow}$	15	.486	.662	.111	A,D	take out any one run from 12,34,56,123456,135,146,236,245,0,136,145,235,246,3456,1256,1234
$Z_{15}^{\uparrow}$	15	.492	.709	.083	E	PB 6.1, 1,12356,345
Z_{16}	16	.438	.373	.063	A,D,E	****12,34,56,123456,135,146,236,245,0,136,145,235,246,3456,1256,1234

**n*=4, I=ABCD=CDEF=ACE=ABCDEF: 12,34,56,123456.

***n*=8, I=ABCD=CDEF=ACE.

****n*=12, PB6.1(col 1 to 6): 12456,2356,1346,245,356,46,15,126,123,234,1345,0;

PB6.2(col 1 to 5 and 7): 1245,2356,1346,2456,35,46,156,12,1236,234,1345,0.

*****n*=16, I=ABCD=CDEF.

Table 8.11 *m*=7, Up-Down G-Z designs under Res III model

Design	n	Tr	Det $\times10^{10}$	MEV	Optimality criteria	Runs
Z_8	8	1	596	.125	A,D,E	*127,135,146,236,245,347,567,1234567
$Z_9^{\uparrow}$	9	.9375	298	.125	A,D,E	127,135,146,236,245,347,567,1234567, any one run from the remaining
$Z_{10}^{\uparrow}$	10	.875	149	.125	A,D,E	127,135,146,236,245,347,567,1234567, +0,3467
$Z_{11}^{\downarrow}$	11	.833	69.8	.25	D	take out any one run from PB7
$Z_{11}^{\uparrow}$	11	.8125	74.5	.125	A,E	127,135,146,236,245,347,567,1234567, +0,3467,2456
Z_{12}	12	.667	23.3	.083	A,D,E	*PB7
$Z_{13}^{\uparrow}$	13	.633	14.0	.083	A,D,E	PB7, any one run from the remaining
$Z_{14}^{\uparrow}$	14	.60	8.37	.083	A,D,E	PB7, 1,13567
$Z_{15}^{\downarrow}$	15	.5625	4.66	.125	A,D	take out any one run from 127,135,146,236,245,347,567,1234567,7,136,145,235,246,34567,12567,12347
$Z_{15}^{\uparrow}$	15	.567	5.02	.083	E	PB7, 1,13567,267
Z_{16}	16	.5	2.33	.0625	A,D,E	***127,135,146,236,245,347,567,1234567,7,136,145,235,246,34567,12567,12347

*n=8, I=ABCD=ABEF=CDG=ADF.

**n=12, PB7 (col 1 to 7): 0,126,157,234,356,467,1237,1345,2457,12456,13467,23567.

***n=16, I=ABCD=ABEF=CDG.

Table 8.12 *m*=8, Up-Down G-Z designs under Res III model

Design	n	Tr	Det $\times 10^{11}$	MEV	Optimality criteria	Runs
$Z_9^{\downarrow}$	9	2	2093	1	A,D,E	take out 0,23567,24578 from PB8
$Z_{10}^{\downarrow}$	10	1.33	349	.50	A,D,E	take out 23567,24578 from PB8
$Z_{11}^{\downarrow}$	11	1	77.5	.33	A,D	take out any one run from PB8
$Z_{11}^{\uparrow}$	11	1.11	181	.25	E	*12,56,2367,1357,1468,2458,3478,12345678 + 4,38,78
Z_{12}	12	.667	19.4	.083	A,D,E	**PB8
$Z_{13}^{\uparrow}$	13	.714	11.1	.083	A,D,E	0,467,1237,1268,1345,1578,2348,3568,12456,23567,24578,134678, any one run from the remaining
$Z_{14}^{\uparrow}$	14	.678	6.34	.083	A,D,E	0,467,1237,1268,1345,1578,2348,3568,12456,23567,24578,134678, 1,12378
$Z_{15}^{\downarrow}$	15	.643	3.33	.14	A,D	take out any one run from 12,56,2367,1357,1468,2458,3478,12345678,3456,2467,1457,1234,78,1368,2358,125678
$Z_{15}^{\uparrow}$	15	.643	3.64	.083	A,E	0,467,1237,1268,1345,1578,2348,3568,12456,23567,24578,134678, 1,12378,357
Z_{16}	16	.5	1.46	.063	A,D,E	***12,56,2367,1357,1468,2458,3478,12345678,3456,2467,1457,1234,78,1368,2358,125678

*n=8, I=ABCD=CDEF =EFGH =AEG =ADF: 12,56,2367,1357,1468,2458,3478,12345678.

**n=12, PB8(col 1 to 8): 0,467,1237,1268,1345,1578,2348,3568,12456,23567,24578,134678.

***n=16, I=ABCD=CDEF =EFGH =AEG.

Table 8.13 *m*=9, Up-Down G-Z designs under Res III model

Design	*n*	Tr	Det $\times 10^{11}$	MEV	Optimality criteria	Runs
$Z_{10}^{\downarrow}$	10	1.667	58.14	.5	A,D,E	take out 1578, 23567 from PB9
$Z_{11}^{\downarrow}$	11	1.25	9.69	.5	A,D,E	take out any one run from PB9
Z_{12}	12	.83	1.62	.083	A,D,E	*PB9
$Z_{13}^{\uparrow}$	13	.795	.88	.083	A,D,E	PB9 plus any one run from remaining
$Z_{14}^{\uparrow}$	14	.758	.48	.083	A,D,E	PB9 plus 3, 1278
$Z_{15}^{\uparrow}$	15	.721	.26	.083	A,E	$Z_{14}^{\uparrow}$ plus 12569
$Z_{15}^{\downarrow}$	15	.729	.24	.167	D	take out any one run from Z_{16}
Z_{16}	16	.625	.09	.063	A,D,E	**see below

**n*=12, PB9(col 1 to 9)

***n*=16, I=ABCE=BCDF=ACDG=ABDH=ABCDJ.

For *n*=10,11,15, we only consider Down designs because of the computation difficulty on Up designs. However, for *n*=15, we consider all possible Up designs by adding a run to $Z_{14}^{\uparrow}$.

Table 8.14 *m*=10, Up-Down G-Z designs under Res III model

Design	n	Tr	Det $\times 10^{12}$	MEV	Optimality criteria	Runs
$Z_{11}^{\downarrow}$	11	1.833	16.15	1	A,D,E	take out any one run from PB10
Z_{12}	12	.917	1.35	.083	A,D,E	*PB10
$Z_{13}^{\uparrow}$	13	.877	.70	.083	A,D,E	PB10 plus any one run from remaining
$Z_{14}^{\uparrow}$	14	.837	.37	.083	A,D,E	PB10 plus 1, 12678910
$Z_{15}^{\uparrow}$	15	.798	.19	.083	A,E	$Z_{14}^{\uparrow}$ plus 2567
$Z_{15}^{\downarrow}$	15	.825	.18	.2	D	take out any one run from Z_{16}
Z_{16}	16	.688	.06	.063	A,D,E	**see below

*n=12, PB10(col 1 to 10).

**n=16, I=ABCE=BCDF=ACDG=ABDH=ABCDJ=ABK.

For n=11 and 15, we only consider Down designs because of the computation difficulty on Up designs. However, for n=15, we consider all possible Up designs by adding a run to $Z_{14}^{\uparrow}$.

Chapter 9

Variance Property

9.0 Main Results

In this chapter, we study the variance-covariance matrix of the parameter estimates under resolution III+k models, specifically for k=1, for designs $d_1,...,d_7$ introduced in Section 4.2. The seven designs are divided into two groups with respect to the variance-covariance matrix. The first group consists of designs $d_1,...,d_5,d_7$. And the second group is just design d_6. This study is valuable to study the design performances for fitting the models assumed.

9.1 Variance-Covariance matrix for designs $\boldsymbol{d}_1,...,\boldsymbol{d}_5,\boldsymbol{d}_7$

To study the dependence of a response variable on m factors $A_1, A_2,..., A_m$, each at two levels, we consider the following resolution III+1 models

$$M_{i_1 i_2}: \begin{cases} E(\boldsymbol{y}) = \boldsymbol{j}_n\beta_0 + \boldsymbol{A}_1\beta_1 + ... + \boldsymbol{A}_m\beta_m + \boldsymbol{A}_{i_1}\boldsymbol{A}_{i_2}\beta_{i_1 i_2},\ 1 \le i_1 < i_2 \le m \\ \mathrm{var}(\boldsymbol{y}) = \sigma^2 \boldsymbol{I} \end{cases} \tag{9.1}$$

where $\boldsymbol{y}(n \times 1)$ is a column vector of n observations on the response variable; $\boldsymbol{j}_n$ $(n \times 1)$ is a unit column vector; β_0 is an unknown parameter corresponding to general mean; $\boldsymbol{A}_1,...,\boldsymbol{A}_m$, and $\boldsymbol{A}_{i_1}\boldsymbol{A}_{i_2}$ are known $(n \times 1)$ column vectors from the design; $\beta_1,...,\beta_m$ are the unknown parameters corresponding to the main effects; $\beta_{i_1 i_2}$ is an unknown parameter corresponding to the $A_{i_1}A_{i_2}$ two-factor interaction; and σ^2 is an unknown parameter.

Under the designs $d_1,...,d_7$, n=12 and m=5, the models in (9.1) can be expressed as

$$M_{i_1 i_2}: \begin{cases} E(\boldsymbol{y}) = \boldsymbol{X}_{(i_1 i_2)}\boldsymbol{\beta}_{(i_1 i_2)},\ 1 \le i_1 < i_2 \le 5 \\ \text{var}(\boldsymbol{y}) = \boldsymbol{\sigma}^2 \boldsymbol{I} \end{cases} \tag{9.2}$$

where $\boldsymbol{\beta}_{(i_1 i_2)} = (\beta_0, \beta_1, ..., \beta_5, \beta_{i_1 i_2})'$ and $\boldsymbol{X}_{(i_1 i_2)} = [\boldsymbol{j}_n, \boldsymbol{A}_1, ..., \boldsymbol{A}_5, \boldsymbol{A}_{i_1}\boldsymbol{A}_{i_2}]$. The least squares estimator of $\boldsymbol{\beta}_{(i_1 i_2)}$ and its variance are $\hat{\boldsymbol{\beta}}_{(i_1 i_2)} = (\boldsymbol{X}_{(i_1 i_2)}'\boldsymbol{X}_{(i_1 i_2)})^{-1}\boldsymbol{X}_{(i_1 i_2)}'\boldsymbol{y}$, $\text{var}(\hat{\boldsymbol{\beta}}_{(i_1 i_2)}) = \boldsymbol{\sigma}^2(\boldsymbol{X}_{(i_1 i_2)}'\boldsymbol{X}_{(i_1 i_2)})^{-1}$.

In the following, we identify the variance-covariance matrix $\text{var}(\hat{\boldsymbol{\beta}}_{(i_1 i_2)})$ under resolution III+1 model for designs $d_1,...,d_5,d_7$. This matrix can be partitioned into three components.

1. $$\frac{1}{\sigma^2}\operatorname{var}\begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_{i_1}\\ \hat{\beta}_{i_2}\end{pmatrix}=\begin{pmatrix} a_{(1\times1)} & b\underline{\boldsymbol{j}}'_{(1\times2)}\\ b\underline{\boldsymbol{j}}_{(2\times1)} & c\boldsymbol{I}+d\boldsymbol{J}_{(2\times2)}\end{pmatrix},$$

2. $$\frac{1}{\sigma^2}\operatorname{cov}\left(\begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_{i_1}\\ \hat{\beta}_{i_2}\end{pmatrix},\begin{pmatrix}\hat{\beta}_{u_1}\\ \hat{\beta}_{u_2}\\ \hat{\beta}_{u_3}\\ \hat{\beta}_{i_1i_2}\end{pmatrix}\right)=\begin{pmatrix} e\underline{\boldsymbol{j}}'_{(1\times3)} & f_{(1\times1)}\\ k\boldsymbol{J}_{(2\times3)} & g\underline{\boldsymbol{j}}_{(2\times1)}\end{pmatrix},$$

3. $$\frac{1}{\sigma^2}\operatorname{var}\begin{pmatrix}\hat{\beta}_{u_1}\\ \hat{\beta}_{u_2}\\ \hat{\beta}_{u_3}\\ \hat{\beta}_{i_1i_2}\end{pmatrix}=\begin{pmatrix} h\boldsymbol{I}+p\boldsymbol{J}_{(3\times3)} & q\underline{\boldsymbol{j}}_{(3\times1)}\\ q\underline{\boldsymbol{j}}'_{(1\times3)} & s_{(1\times1)}\end{pmatrix},$$

where $1\le u_1<u_2<u_3\le 5$, $u_1,u_2,u_3\ne i_1,i_2$, $\underline{\boldsymbol{j}}$ is a unity vector, $\boldsymbol{I}$ is an identity matrix, $\boldsymbol{J}$ is a square matrix with all elements unity. a,b,c,d,e,f,k,g,h,p,q,s are values indenpendent of i_1,i_2. That implies no matter which two-factor interaction is included in the model, the variace-covariance matrix of the estimates is invariant under a permutation of the factor symbols or rename of the factors. This property is true for balanced design of full strength (Srivastava and Chopra,1971). For example, the variace-covariance matrix when taking product of the first two factor columns as the two-factor interaction A_1A_2 $(i_1=1, i_2=2)$ under a resolution III+1 model for design d_1 is

$$
\frac{1}{\sigma^2}\operatorname{var}\begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\\ \hat{\beta}_4\\ \hat{\beta}_5\\ \hat{\beta}_{12}\end{pmatrix} = \begin{pmatrix} \mu & A_1 & A_2 & A_3 & A_4 & A_5 & A_1A_2 \\ .097 & .016 & .016 & .017 & .017 & .017 & -.003 \\ & .0885 & .005 & 0 & 0 & 0 & .016 \\ & & .0885 & 0 & 0 & 0 & .016 \\ & & & .0889 & .006 & .006 & -.017 \\ & & & & .0889 & .006 & -.017 \\ & & & & & .0889 & -.017 \\ & & & & & & .097 \end{pmatrix},
$$

where elements in the lower diagonal matrix are symmetrical to those in the upper diagonal matrix. It is interesting to observe that we can group the 7 factorial effects estimates into three groups: (1) $\hat{\beta}_1,\hat{\beta}_2$; (2) $\hat{\beta}_3,\hat{\beta}_4,\hat{\beta}_5$; (3) $\hat{\beta}_0,\hat{\beta}_{12}$. The estimates within the same group have same variance. Moreover, the estimates in group (1) and the estimates in group (2) have covariance **0**. Within group (2), any two estimates have covariance .006. The estimates in group (1) and the estimates in group (3) have covariance .016. The estimates in group (2) and the estimate $\hat{\beta}_0$ in group (3) have covariance .017. The estimates in group (2) and the estimate $\hat{\beta}_{12}$ in group (3) have covariance -.017.

9.2 Variance-Covariance matrix for design d_6

In this section, we study the variance-covariance matrix var($\hat{\boldsymbol{\beta}}_{(i_1 i_2)}$) of the parameter estimates under resolution III+1 model for design d_6. The reason for us to consider it separately is that it is not a balanced design of full strength, and therefore its variance-covariance matrix does not necessarily have the invariant property described in

Section 9.1. In fact, in this section, we are going to show that d_6 does not have the property. However, after changing the signs of one or two factors on top of renaming factors, it will have the property.

The variance-covariance matrix $\operatorname{var}(\hat{\boldsymbol{\beta}}_{(12)})$ for $i_1 = 1,\ i_2 = 2$ is

$$\frac{1}{\sigma^2}\operatorname{var}\begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\\ \hat{\beta}_4\\ \hat{\beta}_5\\ \hat{\beta}_{12}\end{pmatrix} = \begin{array}{c} \begin{array}{ccccccc} \mu & A_1 & A_2 & A_3 & A_4 & A_5 & A_1A_2 \end{array} \\ \begin{pmatrix} .083 & 0 & 0 & 0 & 0 & 0 & 0 \\ & .083 & 0 & 0 & 0 & 0 & 0 \\ & & .083 & 0 & 0 & 0 & 0 \\ & & & .097 & .014 & .014 & .042 \\ & & & & .097 & .014 & .042 \\ & & & & & .097 & .042 \\ & & & & & & .125 \end{pmatrix} \end{array}. \tag{9.3}$$

The variance-covariance matrix $\operatorname{var}(\hat{\boldsymbol{\beta}}_{(13)})$ for $i_1 = 1,\ i_2 = 3$ is

$$\frac{1}{\sigma^2}\operatorname{var}\begin{pmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\\ \hat{\beta}_4\\ \hat{\beta}_5\\ \hat{\beta}_{13}\end{pmatrix} = \begin{array}{c} \begin{array}{ccccccc} \mu & A_1 & A_2 & A_3 & A_4 & A_5 & A_1A_3 \end{array} \\ \begin{pmatrix} .083 & 0 & 0 & 0 & 0 & 0 & 0 \\ & .083 & 0 & 0 & 0 & 0 & 0 \\ & & .097 & 0 & -.014 & .014 & .042 \\ & & & .083 & 0 & 0 & 0 \\ & & & & .097 & -.014 & -.042 \\ & & & & & .097 & .042 \\ & & & & & & .125 \end{pmatrix} \end{array}. \tag{9.4}$$

By renaming the factors A_2 as A_3, matrix (9.4) becomes

$$\frac{1}{\sigma^2}\text{var}\begin{pmatrix}\hat{\beta}_0\\\hat{\beta}_1\\\hat{\beta}_2\\\hat{\beta}_3\\\hat{\beta}_4\\\hat{\beta}_5\\\hat{\beta}_{13}\end{pmatrix} = \begin{array}{c} \begin{array}{ccccccc} \mu & A_1 & A_2 & A_3 & A_4 & A_5 & A_1A_2 \end{array} \\ \begin{pmatrix} .083 & 0 & 0 & 0 & 0 & 0 & 0 \\ & .083 & 0 & 0 & 0 & 0 & 0 \\ & & .083 & 0 & 0 & 0 & 0 \\ & & & .097 & -.014 & .014 & .042 \\ & & & & .097 & -.014 & -.042 \\ & & & & & .097 & .042 \\ & & & & & & .125 \end{pmatrix} \end{array}. \qquad (9.5)$$

By changing the sign of entries for factor A_4, matrix (9.5) becomes exactly (9.3) as follows:

$$\frac{1}{\sigma^2}\text{var}\begin{pmatrix}\hat{\beta}_0\\\hat{\beta}_1\\\hat{\beta}_2\\\hat{\beta}_3\\\hat{\beta}_4\\\hat{\beta}_5\\\hat{\beta}_{13}\end{pmatrix} = \begin{array}{c} \begin{array}{ccccccc} \mu & A_1 & A_2 & A_3 & -A_4 & A_5 & A_1A_2 \end{array} \\ \begin{pmatrix} .083 & 0 & 0 & 0 & 0 & 0 & 0 \\ & .083 & 0 & 0 & 0 & 0 & 0 \\ & & .083 & 0 & 0 & 0 & 0 \\ & & & .097 & .014 & .014 & .042 \\ & & & & .097 & .014 & .042 \\ & & & & & .097 & .042 \\ & & & & & & .125 \end{pmatrix} \end{array}. \qquad (9.3)$$

Then, under this mapping $\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ A_1 & A_3 & A_2 & -A_4 & A_5 \end{pmatrix}$, that is, renaming the factors A_2 as A_3, changing the sign of entries for factor A_4; the variance-covariance matrix (9.4) becomes (9.3). The above mapping is from var($\hat{\boldsymbol{\beta}}_{(13)}$) to var($\hat{\boldsymbol{\beta}}_{(12)}$), where the two-factor interactions A_1A_3 and A_1A_2 in two models have one factor in common. Let's look one example that two-factor interactions A_3A_4 and A_1A_2 in two models have no factor in common and see how to map $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(34)})$ to $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(12)})$. The variance-covariance matrix $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(34)})$ for $i_1 = 3,\ i_2 = 4$ is

$$\frac{1}{\sigma^2}\mathrm{var}\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \\ \hat{\beta}_5 \\ \hat{\beta}_{34} \end{pmatrix} = \begin{array}{c} \begin{array}{ccccccc} \mu & A_1 & A_2 & A_3 & A_4 & A_5 & A_3A_4 \end{array} \\ \begin{pmatrix} .083 & 0 & 0 & 0 & 0 & 0 & 0 \\ & .097 & -.014 & 0 & 0 & -.014 & -.042 \\ & & .097 & 0 & 0 & .014 & .042 \\ & & & .083 & 0 & 0 & 0 \\ & & & & .083 & 0 & 0 \\ & & & & & .097 & .042 \\ & & & & & & .125 \end{pmatrix} \end{array}. \tag{9.6}$$

This mapping can take two steps: (1) $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(34)})$ to $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(13)})$; (2) $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(13)})$ to $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(12)})$. Under the mapping $\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ A_4 & A_2 & A_3 & A_1 & A_5 \end{pmatrix}$, matrix (9.6) becomes exactly (9.4) as follows:

$$\frac{1}{\sigma^2}\mathrm{var}\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \\ \hat{\beta}_5 \\ \hat{\beta}_{34} \end{pmatrix} = \begin{array}{c} \begin{array}{ccccccc} \mu & A_1 & A_2 & A_3 & A_4 & A_5 & A_3A_1 \end{array} \\ \begin{pmatrix} .083 & 0 & 0 & 0 & 0 & 0 & 0 \\ & .083 & 0 & 0 & 0 & 0 & 0 \\ & & .097 & 0 & -.014 & .014 & .042 \\ & & & .083 & 0 & 0 & 0 \\ & & & & .097 & -.014 & -.042 \\ & & & & & .097 & .042 \\ & & & & & & .125 \end{pmatrix} \end{array}. \tag{9.4}$$

Therefore, step (1) can be achieved. Under the mapping $\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ A_1 & A_3 & A_2 & -A_4 & A_5 \end{pmatrix}$ we just go over, step (2) can be achieved. Hence, if we combine steps (1) and (2), we can map $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(34)})$ to $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(12)})$ under the mapping $\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ A_4 & A_3 & A_2 & -A_1 & A_5 \end{pmatrix}$.

In fact, the variance-covariance matrix $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(i_1 i_2)})$ for all the $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10$ choices of i_1 and i_2 are the same as $\mathrm{var}(\hat{\boldsymbol{\beta}}_{(12)})$ after renaming the factors and/or changing the sign of one or two factors as shown in Table 9.1.

Table 9.1 Variance-Covariance matrices mapping

A_1A_2	A_1	A_2	A_3	A_4	A_5		A_1A_2	A_1	A_2	A_3	A_4	A_5
A_1A_3	A_1	A_2	A_3	A_4	A_5		A_1A_3	A_1	A_3	A_2	$-A_4$	A_5
A_1A_4	A_1	A_2	A_3	A_4	A_5		A_1A_4	A_1	A_4	$-A_3$	A_2	$-A_5$
A_1A_5	A_1	A_2	A_3	A_4	A_5		A_1A_5	A_1	A_5	A_3	$-A_4$	A_2
A_2A_3	A_1	A_2	A_3	A_4	A_5	$\overrightarrow{mapping}$	A_2A_3	A_3	A_2	A_1	A_4	A_5
A_2A_4	A_1	A_2	A_3	A_4	A_5		A_2A_4	A_4	A_2	A_3	A_1	$-A_5$
A_2A_5	A_1	A_2	A_3	A_4	A_5		A_2A_5	A_5	A_2	A_3	$-A_4$	A_1
A_3A_4	A_1	A_2	A_3	A_4	A_5		A_3A_4	A_4	A_3	A_2	$-A_1$	A_5
A_3A_5	A_1	A_2	A_3	A_4	A_5		A_3A_5	A_5	A_3	A_2	A_4	A_1
A_4A_5	A_1	A_2	A_3	A_4	A_5		A_4A_5	$-A_4$	$-A_5$	A_3	A_1	A_2

Now, let's revisit variance-covariance matrix (9.3). It is interesting to observe that we can group the 7 factorial effects estimates into three groups: (1) $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$; (2)

$\hat{\beta}_3, \hat{\beta}_4, \hat{\beta}_5$; (3) $\hat{\beta}_{12}$. The estimates within the same group have same variance. Moreover, within group (1), any two estimates have covariance 0. The estimates in group (1) and the estimates in group (2) have covariance **0**. Within group (2), any two estimates have covariance .014. The estimates in group (2) and the estimate in group (3) have covariance .042.

Chapter 10

Conclusions

We discuss orthogonal projection and non-orthogonal projection by presenting some existing results and our own findings on projection from balanced arrays. We study the projection of factorial design from the perspective of statistical modeling and characterize the projection property by using linear models.

We investigate the projection properties of PB design when projecting onto its 4 factor columns and 5 factor columns. We examine the estimability of the projected designs when fitting various models, and give some helpful results that are not available in current literature. Moreover, we compare the projected PB design of 5 columns with BAs and obtain many interesting results.

We exhaustively search for optimal resolution V designs of 2^m series for m=4 and n=11, 12, 13, 14, 15, and 16. Unlike m=4 shown in Chapter 5, the exhaustive search for optimal resolution V designs of 2^m series for m=5 becomes computationally extensive. We propose a method to construct Up-Resolution designs that are not limited to balanced designs (Srivastava and Chopra, 1971), and show that the designs perform slightly better than the BOFFDs. For a given n, all our designs are isomorphic having same optimality properties. For general m and n, the conditions are derived for obtaining such isomorphic designs with respect to Trace and Determinant.

We propose a general Up-Down method to search for efficient 2^m fractional factorial designs in fitting a class of models when the number of factors is m, and the number of runs is n. We present the efficient resolution III designs obtained by the Up-Down method for $3 \le m \le 10$ and a range of practical values of n.

We study the variance-covariance matrix of the parameter estimates under resolution III+k models, specifically for k=1, for designs $d_1,\ldots,d_7$ introduced in Section 4.2.

Appendix

Proof 1.1: First, we argue matrix **C** given property P_2, if dimension allows, for every submatrix $\mathbf{C_s}(n\times 3)$ of matrix **C**, there exist one submatrix $\mathbf{D}(3\times 3)$ which satisfies the following three conditions: (Note: We can prove the preceding argument by contradiction: if this kind of submatrix $\mathbf{D}(3\times 3)$ doesn't exist, then the property P_2 cannot be obtained.)

a, all the elements are 1 and -1.

b, three distinct columns and none of the three columns is complement of another column.

c, three distinct rows and none of the three rows is complement of another row.

Without loss of generality, we can suppose a matrix **D**, being changed the sign of a row/column and being rearranged the rows/columns, then will have the following form that the first row is all 1's and the first column is (1,-1,1)' (Note: this transformation will not change the property $P_\mathbf{t}$ of matrix **D**):

$$\mathbf{D}=\begin{pmatrix} 1 & 1 & 1 \\ -1 & \mathbf{x}1 & \mathbf{x}3 \\ 1 & \mathbf{x}2 & \mathbf{x}4 \end{pmatrix}$$

Now, in order to make **D** still have the aforementioned three conditions, the matrix **D** can only have the following three stuctures:

$$\mathbf{D}_1=\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \text{or } \mathbf{D}_2=\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \text{ or } \mathbf{D}_3=\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

It's easy to check that the above three **D** matrices all have the property P_3. Hence, we can claim that property P_2 implies property P_3.

Proof Proposition 5.2: Let's denote a design that is randomly chosen from class T_1 as $\boldsymbol{T}_{11\times 4}$, the corresponding design matrix as $\boldsymbol{X}_{11\times 11}$ with columns that are corresponding to general mean (μ), all main effects (A,B,C,D) and all two-way interactions (AB,AC,AD,BC,BD,CD). Denote each of the resulted 16 designs as $\boldsymbol{T}^{(i)}_{11\times 4}$, the corresponding design matrix as $\boldsymbol{X}^{(i)}_{11\times 11}$. So that $\boldsymbol{T}^{(i)}_{11\times 4}$ and $\boldsymbol{X}^{(i)}_{11\times 11}$ can be represented as the following:

$$\boldsymbol{T}^{(i)} = \boldsymbol{T} * \begin{pmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 \\ 0 & 0 & \delta_3 & 0 \\ 0 & 0 & 0 & \delta_4 \end{pmatrix},$$

$$\boldsymbol{X}^{(i)} = \boldsymbol{X} * \boldsymbol{D}^{(i)} = \boldsymbol{X} * \begin{pmatrix}
\delta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta_1\delta_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta_1\delta_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_1\delta_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_2\delta_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_2\delta_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_3\delta_4
\end{pmatrix}_{11\times 11},$$

where $\delta_0 = 1,\ \delta_1, \delta_2, \delta_3, \delta_4 = 1$ or -1. It's obvious that the diagonal elements of $\boldsymbol{D}^{(i)}$ is either 1 or -1, so $\left|\boldsymbol{D}^{(i)}\right| = \left|\boldsymbol{D}^{(i)'}\right| = 1$ or -1.

Therefore,

$$
\begin{aligned}
&\left|\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)}\right| \\
&= \left|(\boldsymbol{X}\boldsymbol{D}^{(i)})'(\boldsymbol{X}\boldsymbol{D}^{(i)})\right| \\
&= \left|\boldsymbol{D}^{(i)'}\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}^{(i)}\right| \\
&= \left|\boldsymbol{D}^{(i)'}\right|\left|\boldsymbol{X}'\boldsymbol{X}\right|\left|\boldsymbol{D}^{(i)}\right| \\
&= \left|\boldsymbol{D}^{(i)}\right|^2\left|\boldsymbol{X}'\boldsymbol{X}\right| \\
&= \left|\boldsymbol{X}'\boldsymbol{X}\right|,
\end{aligned}
$$

so $\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)}$ and $\boldsymbol{X}'\boldsymbol{X}$ have the same determinant which is greater than 0. Then $\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)}$ and $\boldsymbol{X}'\boldsymbol{X}$ are both invertible, $(\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)})^{-1}$ and $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ have the same determinant.

Also,

$$
\begin{aligned}
&\left|\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)} - \lambda\boldsymbol{I}\right| \\
&= \left|\boldsymbol{D}^{(i)'}\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}^{(i)} - \lambda\boldsymbol{D}^{(i)'}\boldsymbol{D}^{(i)}\right| \\
&= \left|\boldsymbol{D}^{(i)'}\left(\boldsymbol{X}'\boldsymbol{X} - \lambda\boldsymbol{I}\right)\boldsymbol{D}^{(i)}\right| \\
&= \left|\boldsymbol{D}^{(i)'}\right|\left|\boldsymbol{X}'\boldsymbol{X} - \lambda\boldsymbol{I}\right|\left|\boldsymbol{D}^{(i)}\right| \\
&= \left|\boldsymbol{D}^{(i)}\right|^2\left|\boldsymbol{X}'\boldsymbol{X} - \lambda\boldsymbol{I}\right| \\
&= \left|\boldsymbol{X}'\boldsymbol{X} - \lambda\boldsymbol{I}\right|,
\end{aligned}
$$

so $\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)}$ and $\boldsymbol{X}^{'}\boldsymbol{X}$ have the same Eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_{11}$. Also, as Then $(\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)})^{-1}$ and $(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}$ have same Eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, ..., \frac{1}{\lambda_{11}}$.Hence, they have the same max Eigenvalue. Also,

$$\mathrm{tr}((\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)})^{-1}) = \sum_{i=1}^{11} \frac{1}{\lambda_i},$$

$$\mathrm{tr}((\boldsymbol{X}^{'}\boldsymbol{X})^{-1}) = \sum_{i=1}^{11} \frac{1}{\lambda_i},$$

$$\left|(\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)})^{-1}\right| = \prod_{i=1}^{11} \frac{1}{\lambda_i},$$

$$\left|(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}\right| = \prod_{i=1}^{11} \frac{1}{\lambda_i}.$$

It’s easy to see that $(\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)})^{-1}$ and $(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}$ have the same A-, D- and E-optimality criteria values. Proof ends.

Proof Proposition 5.3: Let’s denote a design that is randomly chosen from class T_1 as $\boldsymbol{T}_{12\times4}$, and label column 1, 2, 3 and 4 as factor A, B, C and D, respectively. The corresponding design matrix as $\boldsymbol{X}_{12\times11}$ with columns that are corresponding to general mean(μ), main effects (A,B,C,D) and two-way interactions (AB,AC,AD,BC,BD,CD). Now, we relabel the columns of $\boldsymbol{T}_{12\times4}$ with different permutation of factors, and denote the resulting design as $\boldsymbol{T}_{12\times4}^{(i)}$ which has columns corresponding to factor A, B, C and D, the corresponding design matrix as $\boldsymbol{X}^{(i)}_{12\times 11}$ which has columns corresponding to general mean (μ), main effects (A,B,C,D) and two-way interactions (AB,AC,AD,BC,BD,CD).

It's easy to find $\boldsymbol{T}^{(i)}_{12\times 4}$ and $\boldsymbol{X}^{(i)}_{12\times 11}$ can be represented as the following:

$$\boldsymbol{T}^{(i)} = \boldsymbol{T} * \boldsymbol{P}^{(i)}_{\boldsymbol{T}_1},$$

$$\boldsymbol{X}^{(i)} = \boldsymbol{X} * \boldsymbol{P}^{(i)}_{\boldsymbol{X}},$$

where

$$\boldsymbol{P}^{(i)}_{\boldsymbol{X}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \boldsymbol{P}^{(i)}_{\boldsymbol{T}_1} & 0 \\ 0 & 0 & \boldsymbol{P}^{(i)}_{\boldsymbol{T}_2} \end{pmatrix}_{11\times 11},$$

where $\boldsymbol{P}^{(i)}_{\boldsymbol{T}_1}$ is a 4×4 permutation matrix, $\boldsymbol{P}^{(i)}_{\boldsymbol{T}_2}$ is a 6×6 permutation matrix and $\boldsymbol{P}^{(i)}_{\boldsymbol{X}}$ is a 11×11 permutation matrix.

A permutation matrix is a square binary matrix that has exactly one entry 1 in each row and each column and 0s elsewhere. For instance,

$$\boldsymbol{P}^{(i)}_{\boldsymbol{T}_1} = \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{pmatrix},$$

where

$$P_{ij} = 1 \; or \; 0,$$
$$\sum_{i=1}^{4} P_{ij} = 1,$$
$$\sum_{j=1}^{4} P_{ij} = 1.$$

Permutation matrices are orthogonal matrices, i.e., $\boldsymbol{P}_{\boldsymbol{T}_1}^{(i)}\boldsymbol{P}_{\boldsymbol{T}_1}^{(i)'} = \boldsymbol{I}, \boldsymbol{P}_{\boldsymbol{T}_2}^{(i)}\boldsymbol{P}_{\boldsymbol{T}_2}^{(i)'} = \boldsymbol{I}, \boldsymbol{P}_{\boldsymbol{X}}^{(i)}\boldsymbol{P}_{\boldsymbol{X}}^{(i)'} = \boldsymbol{I}$. It's obvious that $\left|\boldsymbol{P}_{\boldsymbol{X}}^{(i)}\right| = \left|\boldsymbol{P}_{\boldsymbol{X}}^{(i)'}\right| = 1$ or -1.

Therefore,

$$\begin{aligned}
&\left|\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)}\right| \\
&= \left|(\boldsymbol{X}\boldsymbol{P}_{\boldsymbol{X}}^{(i)})^{'}(\boldsymbol{X}\boldsymbol{P}_{\boldsymbol{X}}^{(i)})\right| \\
&= \left|\boldsymbol{P}_{\boldsymbol{X}}^{(i)'}\boldsymbol{X}^{'}\boldsymbol{X}\boldsymbol{P}_{\boldsymbol{X}}^{(i)}\right| \\
&= \left|\boldsymbol{P}_{\boldsymbol{X}}^{(i)'}\right|\left|\boldsymbol{X}^{'}\boldsymbol{X}\right|\left|\boldsymbol{P}_{\boldsymbol{X}}^{(i)}\right| \\
&= \left|\boldsymbol{X}^{'}\boldsymbol{X}\right|,
\end{aligned}$$

so $\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)}$ and $\boldsymbol{X}^{'}\boldsymbol{X}$ have the same determinant which is greater than 0. Then $\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)}$ and $\boldsymbol{X}^{'}\boldsymbol{X}$ are both invertible, $(\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)})^{-1}$ and $(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}$ have the same determinant.

Also,

$$
\begin{aligned}
&\left|\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)} - \lambda\boldsymbol{I}\right| \\
&= \left|\boldsymbol{P}_{\boldsymbol{X}}^{(i)'}\boldsymbol{X}^{'}\boldsymbol{X}\boldsymbol{P}_{\boldsymbol{X}}^{(i)} - \lambda\boldsymbol{P}_{\boldsymbol{X}}^{(i)'}\boldsymbol{P}_{\boldsymbol{X}}^{(i)}\right| \\
&= \left|\boldsymbol{P}_{\boldsymbol{X}}^{(i)'}\left(\boldsymbol{X}^{'}\boldsymbol{X} - \lambda\boldsymbol{I}\right)\boldsymbol{P}_{\boldsymbol{X}}^{(i)}\right| \\
&= \left|\boldsymbol{P}_{\boldsymbol{X}}^{(i)'}\right|\left|\boldsymbol{X}^{'}\boldsymbol{X} - \lambda\boldsymbol{I}\right|\left|\boldsymbol{P}_{\boldsymbol{X}}^{(i)}\right| \\
&= \left|\boldsymbol{X}^{'}\boldsymbol{X} - \lambda\boldsymbol{I}\right|,
\end{aligned}
$$

so $\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)}$ and $\boldsymbol{X}^{'}\boldsymbol{X}$ have the same Eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_{11}$. Then $(\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)})^{-1}$ and $(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}$ have same Eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, ..., \frac{1}{\lambda_{11}}$. Hence, they have the same max Eigenvalue.

Also,

$$
\begin{aligned}
&\mathrm{tr}((\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)})^{-1}) = \sum_{\boldsymbol{i}=1}^{11}\frac{1}{\lambda_{\boldsymbol{i}}}, \\
&\mathrm{tr}((\boldsymbol{X}^{'}\boldsymbol{X})^{-1}) = \sum_{\boldsymbol{i}=1}^{11}\frac{1}{\lambda_{\boldsymbol{i}}}, \\
&\left|(\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)})^{-1}\right| = \prod_{\boldsymbol{i}=1}^{11}\frac{1}{\lambda_{\boldsymbol{i}}}, \\
&\left|(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}\right| = \prod_{\boldsymbol{i}=1}^{11}\frac{1}{\lambda_{\boldsymbol{i}}}.
\end{aligned}
$$

It's easy to see that $(\boldsymbol{X}^{(i)'}\boldsymbol{X}^{(i)})^{-1}$ and $(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}$ have the same A-, D-, and E-optimality criteria values. Proof ends.

Proof Corollary 7.2:

Since

$$\boldsymbol{X}_1'\boldsymbol{X}_1 = n_1 I_{p_1},\ (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} = \frac{1}{n_1} I_{p_1},$$

we have

$$(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1} = \frac{1}{n_1}\boldsymbol{I}_{p_1} - \frac{1}{n_1^2}\boldsymbol{X}_{1ij}^{*}{}'(\frac{1}{n_1}\boldsymbol{X}_{1ij}^{*}\boldsymbol{X}_{1ij}^{*}{}' + \boldsymbol{I}_i)^{-1}\boldsymbol{X}_{1ij}^{*}.$$

Therefore,

$$Tr(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1} = \frac{p_1}{n_1} - \frac{1}{n_1^2}Tr(\frac{1}{n_1}\boldsymbol{X}_{1ij}^{*}\boldsymbol{X}_{1ij}^{*}{}' + \boldsymbol{I}_i)^{-1}\boldsymbol{X}_{1ij}^{*}\boldsymbol{X}_{1ij}^{*}{}',$$

And so,

$$\begin{aligned}
n_1 Tr(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1} &= p_1 - Tr(\frac{1}{n_1}\boldsymbol{X}_{1ij}^{*}\boldsymbol{X}_{1ij}^{*}{}' + \boldsymbol{I}_i)^{-1}(\frac{1}{n_1}\boldsymbol{X}_{1ij}^{*}\boldsymbol{X}_{1ij}^{*}{}') \\
&= p_1 - Tr(\boldsymbol{I}_i - (\frac{1}{n_1}\boldsymbol{X}_{1ij}^{*}\boldsymbol{X}_{1ij}^{*}{}' + \boldsymbol{I}_i)^{-1}) \\
&= p_1 - i + Tr(\frac{1}{n_1}\boldsymbol{X}_{1ij}^{*}\boldsymbol{X}_{1ij}^{*}{}' + \boldsymbol{I}_i)^{-1}.
\end{aligned}$$

Therefore,

$$Tr(\boldsymbol{X}_{2ij}'\boldsymbol{X}_{2ij})^{-1} = \frac{p_1 - i}{n_1} + \frac{1}{n_1}Tr(\frac{1}{n_1}\boldsymbol{X}_{1ij}^{*}\boldsymbol{X}_{1ij}^{*}{}' + \boldsymbol{I}_i)^{-1}.$$

The proof is completed.

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