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# Generalized Lie symmetries and almost regular Lagrangians: a link between symmetry and dynamics

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## PAPER

## Generalized Lie symmetries and almost regular Lagrangians: a link between symmetry and dynamics

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Department of Physics, University of California, Berkeley, CA 94720 United States of America<sup>1</sup><sup>1</sup> Also at Physical Science and Engineering Division, Diablo Valley College, Pleasant Hill, CA 94523, USA.E-mail: [ads@berkeley.edu](mailto:ads@berkeley.edu)**Keywords:** constrained dynamics, singular Lagrangian dynamics, generalized Lie symmetries, local gauge symmetries, symmetries and dynamics, almost regular Lagrangians**Abstract**

The generalized Lie symmetries of almost regular Lagrangians are studied, and their impact on the evolution of dynamical systems is determined. It is found that if the action has a generalized Lie symmetry, then the Lagrangian is necessarily singular; the converse is not true, as we show with a specific example. It is also found that the generalized Lie symmetry of the action is a Lie subgroup of the generalized Lie symmetry of the Euler–Lagrange equations of motion. The converse is once again not true, and there are systems for which the Euler–Lagrange equations of motion have a generalized Lie symmetry while the action does not, as we once again show through a specific example. Most importantly, it is shown that each generalized Lie symmetry of the action contributes one arbitrary function to the evolution of the dynamical system. The number of such symmetries gives a lower bound to the dimensionality of the family of curves emanating from any set of allowed initial data in the Lagrangian phase space. Moreover, if second- or higher-order Lagrangian constraints are introduced during the application of the Lagrangian constraint algorithm, these additional constraints could not have been due to the generalized Lie symmetry of the action.

**1. Introduction**

The symmetries of the Euler–Lagrange equations of motion were recently used to study the constrained dynamics of singular Lagrangians [1]. The focus was on almost regular Lagrangians [2–5], and it was found that for these Lagrangians the Euler–Lagrange equations of motion admit a generalized Lie symmetry (also known as a local gauge symmetry). The generators  $\text{Sym}$  of this symmetry group  $\text{Gr}_{\text{Sym}}$  were determined in the Lagrangian phase space approach to Lagrangian mechanics, and were found to lie in the kernel of the Lagrangian two-form  $\Omega_L$ . While it is well-known that the solutions  $\mathbf{X}_E$  of the energy equation,

$$0 = dE - i_{\mathbf{X}_E} \Omega_L, \quad (1)$$

is not unique for almost regular Lagrangians, it was shown in [1] that the action of  $\text{Sym}$  on a general solution to this equation—and in particular, on the second-order, Lagrangian vector field (SOELVF)—will result in a vector field that is no longer a solution of equation (1). Thus, not all solutions of the energy equation have  $\text{Gr}_{\text{Sym}}$  as a symmetry group. It is, however, possible to construct solutions to equation (1) for whom  $\text{Sym}$  does generate a group of symmetry transformations [1]. These vector fields are called second-order, Euler–Lagrange vector fields (SOELVFs). As the evolution of the dynamical system for singular Lagrangians must lie on Lagrangian constraint surfaces [5], a Lagrangian constraint algorithm for SOELVFs was also introduced in [1] to construct such solutions to the energy equation. It was then shown that these SOELVFs, along with the dynamical structures in the Lagrangian phase space needed to describe and determine the motion of the dynamical system, are projectable to the Hamiltonian phase space. In particular, the primary Hamiltonian constraints can be constructed from vectors that lie in the kernel of  $\Omega_L$ , and the Lagrangian constraint algorithm for the SOELVF is equivalent to the stability analysis of the total Hamiltonian (we follow the terminology found in [6]; see also

[7–9]) obtained using constrained Hamiltonian mechanics. Importantly, the end result of this stability analysis gives a Hamiltonian vector field that is the projection of the SOELVF obtained from the Lagrangian constraint algorithm. The Lagrangian and Hamiltonian formulations of mechanics for almost regular Lagrangians were thereby shown to be equivalent.

While [1] focused on the generalized Lie symmetries of the Euler–Lagrange equations of motion and whether the dynamical structures constructed in the Lagrangian phase space are projectable to the Hamiltonian phase space, in this paper the focus is on the symmetries of the action itself and the impact these symmetries have on the evolution of dynamical systems. This impact is found to be quite broad, surprisingly restrictive, and unexpectedly subtle. Indeed, even the seemingly reasonable expectation that any generalized Lie symmetry of the Euler–Lagrange equations of motion should be a reflection of the symmetries of the action itself is not borne out.

We find that if the action has a generalized Lie symmetry, then its Lagrangian is necessarily singular; the converse need not be true, as we show through a specific example. We also find that the generators of the generalized Lie symmetry of the action form a Lie sub-algebra of the generators of the generalized Lie symmetry of the Euler–Lagrange equation of motion; once again, the converse is not true. We give an example of a dynamical system for which the Euler–Lagrange equations of motion has a generalized Lie symmetry, while its action does not. Most importantly, for systems where the Lagrangian is almost regular and for which its Lagrangian two-form  $\Omega_L$  has constant rank, we show that each generalized Lie symmetry of the action contributes one arbitrary constant to the SOELVF. The dimensionality of the space of solutions to the energy equation that have  $\text{Gr}_{\text{Sym}}$  as a symmetry group is thus at least as large as the number of generalized Lie symmetries of the action. Moreover, if second- or higher-order Lagrangian constraints are introduced during the application of the Lagrangian constraint algorithm, these additional constraints cannot be due to the generalized Lie symmetry of the action.

Symmetries of Lagrangian systems have been studied before. However, such analyses have been focused on time-dependent Lagrangians [10–17]; on systems of first-order evolution equations [18–22]; or on general solutions of equation (1) [23] (see also [24]). Importantly, the great majority of these studies have been done using first-order prolongations on first-order jet bundles with a focus on the Lie symmetries of first-order evolution equations. Our interest is in the symmetries of the action, which naturally leads us to consider generalized Lie symmetries and second-order prolongations. To our knowledge, such symmetry analysis of the action has not been done before. (The framework for  $k^{\text{th}}$ -order prolongations on  $k^{\text{th}}$ -order jet bundles have been introduced before [16, 17, 23, 25, 26], but they were not applied to the action or to the Euler–Lagrange equations of motion).

The rest of the paper is arranged as follows. In section 2 the conditions under which the action for a dynamical system, and the conditions under which the Euler–Lagrange equations of motion for this action, have a generalized Lie symmetry are determined. To compare the conditions for each, the analysis for the two are done separately, with each self-contained. In section 3 properties of the Lagrangian phase space are reviewed, and the notation used here established. The generators of the generalized Lie symmetry group for the Euler–Lagrange equations of motion were determined in [1], and a summary of the results found therein that are needed here is given. In section 4 the generators of the generalized Lie symmetry group for the action is found within the Lagrangian phase space approach, and their relation to the generators for the symmetry group of the Euler–Lagrange equations of motion is determined. The impact of the symmetries of the action on the SOELVF is then analyzed by applying the Lagrangian constraint algorithm introduced in [1] to these SOELVF. The results obtained in this paper is then applied to three different dynamical systems in section 5. In particular, an example of a dynamical system that has no generalized Lie symmetries and yet is still singular, and another example where the action has no symmetries and yet the Euler–Lagrange equations of motion do, are given. Concluding remarks can be found in section 6.

## 2. Generalized lie symmetries and Lagrangian mechanics

In this section we determine the conditions under which the action of a dynamical system, and the conditions under which the Euler–Lagrange equations of motion for this system, has a generalized Lie symmetry. While the determination for both is done within Lagrangian mechanics, the analysis for the action is completed separately from that of the equations of motion—with each self-contained—so that the two conditions can be compared. We will later show that every generator of the generalized Lie symmetry of the action is a generator of a generalized Lie symmetry of the Euler–Lagrange equations of motion. Interestingly, the converse is not true.

### 2.1. Symmetries of the action

We begin with Lagrangian mechanics, and an analysis of the generalized Lie symmetry [27] of the action

$$S := \int_{t_1}^{t_2} L(q(t), \dot{q}(t)) dt,$$

for a dynamical system on a  $D$ -dimensional configuration space  $\mathbb{Q}$ . Here,  $L(q(t), \dot{q}(t))$  is the Lagrangian along a path  $q(t) = (q^1(t), \dots, q^D(t))$  on  $\mathbb{Q}$  with end points given by  $Q_1 := q(t_1), Q_2 := q(t_2)$ . These points are chosen at the same time the choice of  $S$  is made, and are fixed.

As  $L(q(t), \dot{q}(t))$  depends on both the position  $q(t)$  and the velocity  $\dot{q}(t)$  of the path, we consider a generalized Lie symmetry that is generated by

$$\mathbf{g}_L := \rho_L(q, \dot{q}) \cdot \frac{\partial}{\partial q},$$

where  $\rho_L(q, \dot{q})$  does not depend explicitly on time. Evolution along the path gives the total time derivative

$$\frac{\mathbf{d}}{\mathbf{d}t} := \dot{q} \cdot \frac{\partial}{\partial q} + \ddot{q} \cdot \frac{\partial}{\partial \dot{q}}. \tag{2}$$

This in turn gives  $\dot{\rho}_L := \mathbf{d}\rho_L/\mathbf{d}t$ , and the second-order prolongation vector [27],

$$\mathbf{pr} \mathbf{g}_L := \rho_L \cdot \frac{\partial}{\partial q} + \dot{\rho}_L \cdot \frac{\partial}{\partial \dot{q}} + \ddot{\rho}_L \cdot \frac{\partial}{\partial \ddot{q}}, \tag{3}$$

on the second-order jet space  $\mathbb{M}^{(2)} = \{(t, q, \dot{q}, \ddot{q})\}$  where this  $\mathbf{pr} \mathbf{g}_L \in \mathbf{TM}^{(2)}$ .

Under this generalized Lie symmetry, the action varies by

$$\delta S = \int_{t_1}^{t_2} \mathbf{pr} \mathbf{g}_L[L(q(t), \dot{q}(t))] dt,$$

with the requirement that  $\rho_L(q(t_1), \dot{q}(t_1)) = 0 = \rho_L(q(t_2), \dot{q}(t_2))$ . Then after an integration by parts,

$$\delta S = \int_{t_1}^{t_2} \rho_L \cdot \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right] dt. \tag{4}$$

It is important to realize that the action may be evaluated along any path on  $\mathbb{Q}$ . As such, if  $\mathbf{g}_L$  generates a symmetry of the action, then equation (4) must vanish for *all* paths  $q(t)$  on  $\mathbb{Q}$ , and not just for those that minimize the action.

To make connection with the Lagrangian phase space approach used in the rest of the paper, we make use of

$$E(q, \dot{q}) := \dot{q}^a \frac{\partial L(q, \dot{q})}{\partial \dot{q}^a} - L(q, \dot{q}),$$

along with

$$M_{ab}(q, \dot{q}) := \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^a \partial \dot{q}^b}, \quad \text{and} \quad F_{ab}(q, \dot{q}) := \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^a \partial q^b} - \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^b \partial q^a},$$

to express equation (4) as

$$\delta S = - \int_{t_1}^{t_2} \rho_L^a \left( \frac{\partial E}{\partial q^a} + F_{ab}(q, \dot{q}) \dot{q}^b + M_{ab}(q, \dot{q}) \dot{q}^b \right) dt. \tag{5}$$

Here, Latin indices run from 1 to  $D$ , and Einstein's summation convention is used. We then arrive at our first result.

**Lemma 1.** *An action  $S$  of a dynamical system has a generalized Lie symmetry generated by  $\mathbf{g}_L$  if and only if there exists a  $\rho_L \in \ker M_{ab}$  such that*

$$0 = \rho_L^a(q, \dot{q}) \left( \frac{\partial E}{\partial q^a} + F_{ab}(q, \dot{q}) \dot{q}^b \right), \tag{6}$$

on  $\mathbf{T}\mathbb{Q}$ .

**Proof.** If  $\mathbf{g}_L$  generates a generalized Lie symmetry of  $S$ , then equation (5) must vanish for all paths on  $\mathbb{Q}$ . For an arbitrary path on  $\mathbb{Q}$  the curvature of the path  $\ddot{q}$  will not depend on either the  $q(t)$  or the  $\dot{q}(t)$  for the path, however. As such, for  $\delta S = 0$  it must be that  $\rho_L^a M_{ab} \dot{q}^b = 0$  for any choice of  $\dot{q}$ , and thus  $\rho_L^a \in \ker M_{ab}$ . The remaining terms in equation (5) gives the condition equation (6). □

The set of all vector fields  $\mathbf{g}_L$  that satisfy lemma 1 is denoted by  $\mathfrak{g}_L$ , while  $\mathbf{pr} \mathfrak{g}_L := \{\mathbf{pr} \mathbf{g}_L \mid \mathbf{g}_L \in \mathfrak{g}_L\}$  is the set of their prolongations. This  $\mathbf{pr} \mathfrak{g}_L$  is involutive [27], and the conditions under which  $\mathbf{pr} \mathfrak{g}_L$  generates a generalized Lie symmetry group are given in [27].

We see from lemma 1 that if the action has a generalized Lie symmetry, then the Lagrangian is necessarily singular, and as such the Lagrangian two-form  $\Omega_L$  will not have maximum rank. It is also important to note that while equations of the form equation (6) often appear in the Lagrangian phase space description of mechanics [1], they appear as Lagrangian constraints, conditions that must be imposed for evolution under the Euler–Lagrange equations to be well defined. Here, equation (6) is not a constraint. Rather, because the action must have this symmetry for *all* possible paths on  $\mathbb{Q}$ , and since the set of all possible paths cover  $\mathbb{Q}$ , equation (6) is a condition on  $\rho_L$  that must be satisfied identically on *all* of  $\mathbb{T}\mathbb{Q}$ —and thus, on the Lagrangian phase space—for  $\mathbf{g}_L$  to be a generator of the symmetry group. We will see that not all the vectors in  $\ker M_{ab}$  satisfy the identity equation (6), however, and thus not all of these vectors will generate a generalized Lie symmetry of the action.

### 2.2. Symmetries of the Euler–Lagrange equations of motion

While in section 2.1 the focus was on arbitrary paths on the configuration space  $\mathbb{Q}$  and the symmetries of the action, in this section the focus is on the trajectories that minimize the action and the generalized Lie symmetries of them. These trajectories are solutions of the Euler–Lagrange equations of motion, and for almost regular Lagrangians such solutions form a family of curves. It is, in fact, the presence of this family of curves that gives rise to the generalized Lie symmetry. The treatment here follows closely to that given in [1].

For almost regular Lagrangians the solutions of the Euler–Lagrange equations of motion

$$M_{ab}(q, \dot{q})\ddot{q}^b = -\frac{\partial E}{\partial q^a} - F_{ab}(q, \dot{q})\dot{q}^b, \tag{7}$$

are not unique. While for these Lagrangians the rank of  $M_{ab}(q, \dot{q}) = D - N_0$ —with  $N_0 = \dim(\ker M_{ab}(q, \dot{q}))$ —is constant, this rank is not maximal, and thus equation (7) does not have a unique solution for  $\ddot{q}$ . Instead, for a chosen set of initial data ( $q_0 = q(t_0)$ ,  $\dot{q}_0 = \dot{q}(t_0)$ ), the solution to equation (7) results in a family of solutions that evolve from this  $(q_0, \dot{q}_0)$ . As with the paths in section 2.1, these solutions are related to one another through a generalized Lie symmetry [27].

Following [27], the collection of functions

$$\Delta_a(q, \dot{q}, \ddot{q}) := \frac{\partial E(q, \dot{q})}{\partial q^a} + F_{ab}(q, \dot{q})\dot{q}^b + M_{ab}(q, \dot{q})\ddot{q}^b, \tag{8}$$

defines a set of surfaces  $\Delta_a(q, \dot{q}, \ddot{q}) = 0$  on  $\mathbb{M}^{(2)}$ , while the family of solutions to equation (7)

$$\mathcal{O}(q_0, \dot{q}_0) := \{q(t) \mid \Delta_a(q, \dot{q}, \ddot{q}) = 0 \text{ with } q(t_0) = q_0, \dot{q}(t_0) = \dot{q}_0\},$$

that evolve from the same initial data  $(q_0, \dot{q}_0)$  gives the collection of trajectories that lie on these surfaces. Indeed, for any two such solutions  $q^a(t)$  and  $\tilde{q}^a(t)$  there exists a  $\mathfrak{z}(q, \dot{q}) \in \ker M_{ab}(q, \dot{q})$  such that  $\ddot{Q}^a - \ddot{\tilde{q}}^a = \mathfrak{z}^a$ . Importantly, because  $\mathfrak{z}^a$  depends on both  $q$  and  $\dot{q}$ , the symmetry group that maps one member of  $\mathcal{O}$  to another must be a generalized Lie symmetry. We therefore take the generator of this symmetry group to be

$$\mathbf{g} := \rho(q, \dot{q}) \cdot \frac{\partial}{\partial q},$$

with the corresponding the second-order prolongation vector for  $\mathbf{g}$  being,

$$\mathbf{pr} \mathbf{g} := \rho \cdot \frac{\partial}{\partial q} + \dot{\rho} \cdot \frac{\partial}{\partial \dot{q}} + \ddot{\rho} \cdot \frac{\partial}{\partial \ddot{q}},$$

with this  $\mathbf{pr} \mathbf{g} \in \mathbb{T}\mathbb{M}^{(2)}$ . As with the above, the total time derivative is given by equation (2), but unlike the analysis in section 2.1, the evolution of the path—and indeed, for all the trajectories in  $\mathcal{O}(q_0, \dot{q}_0)$ —here is given by the Euler–Lagrange equations of motion.

The action of this prolongation on  $\Delta_a$  on the  $\Delta_a = 0$  surface gives,

$$\mathbf{pr} \mathbf{g}[\Delta_a(q, \dot{q}, \ddot{q})] = -\frac{\partial \dot{q}^b}{\partial q^a} M_{bc}(q, \dot{q})\rho^c + \frac{d}{dt}[F_{ab}(q, \dot{q})\rho^b + M_{ab}(q, \dot{q})\dot{\rho}^b].$$

Since  $N_0 > 0$ ,  $\ddot{q}$  is not unique on this surface, and yet  $\mathbf{g}$  must generate the same symmetry group for all the trajectories in  $\mathcal{O}(q_0, \dot{q}_0)$ . Necessarily,  $\rho(q, \dot{q}) \in \ker M_{ab}(q, \dot{q})$ . It then follows that  $\mathbf{pr} \mathbf{g}[\Delta_a(q, \dot{q}, \ddot{q})] = 0$  if and only if (iff) there are constants  $b_a$  such that  $b_a = F_{ab}\rho^b + M_{ab}\dot{\rho}^b$ . The solutions in  $\mathcal{O}(q_0, \dot{q}_0)$  all have the same initial data, however, and thus necessarily  $\rho(q_0, \dot{q}_0) = 0 = \dot{\rho}(q_0, \dot{q}_0)$ . We conclude that  $b_a = 0$ . The following result, first proved in [1], then follows.

**Lemma 2.** *If  $\mathbf{g}$  is a generalized infinitesimal symmetry of  $\Delta_a$ , then  $\rho^a(q, \dot{q}) \in \ker M_{ab}(q, \dot{q})$ , and  $\dot{\rho}^a(q, \dot{q})$  is a solution of*

$$0 = F_{ab}(q, \dot{q})\rho^b(q, \dot{q}) + M_{ab}(q, \dot{q})\dot{\rho}^b(q, \dot{q}). \tag{9}$$

As before, we denote the set of all vector fields  $\mathbf{g}$  that satisfy lemma 2 by  $\mathfrak{g}$ , while  $\mathbf{pr} \mathfrak{g} := \{\mathbf{pr} \mathbf{g} \mid \mathbf{g} \in \mathfrak{g}\}$  is the set of their prolongations. Once again  $\mathbf{pr} \mathfrak{g}$  is involutive, and the conditions under which  $\mathbf{pr} \mathfrak{g}$  generates a generalized Lie symmetry group are given in [27]. Note, however, that while  $\rho = 0$  and  $\dot{\rho} = \mathfrak{z}$  for any  $\mathfrak{z} \in \ker M_{ab}(q, \dot{q})$  is a solution of equation (9), we require that  $\dot{\rho} = \mathbf{d}\rho/\mathbf{d}t$ ; these solutions cannot be generators of the generalized Lie symmetry. Next, if  $\dot{\rho}$  is a solution of equation (9), then  $\dot{\rho}^a + \mathfrak{z}$  is a solution of equation (9) as well, and thus these solutions are not unique. This, along with the previous observation, leads us to generators that are constructed from equivalence classes of prolongations. Finally, equation (8) gives for any  $\mathfrak{z} \in \ker M_{ab}(q, \dot{q})$ ,

$$0 = \mathfrak{z}^a \left( \frac{\partial E}{\partial q^a} + F_{ab}(q, \dot{q})\dot{q}^b \right), \tag{10}$$

on the solution surface  $\Delta_a(q, \dot{q}, \ddot{q}) = 0$ . If equation (10) does not hold identically, it must be imposed, leading to Lagrangian constraints [5]. More importantly, because each  $q(t) \in \mathcal{O}(u_0)$  must lie on the Lagrangian constraint submanifold, any symmetry transformation of  $q(t)$  generated by  $\mathbf{pr} \mathfrak{g}$  must give a path  $Q(t)$  that also lies on the constraint submanifold.

Not all vectors in  $\mathbf{pr} \mathfrak{g}$  will be generators of the generalized Lie symmetry group for  $\mathcal{O}(u_0)$ . Determining which of these vectors are, and the relationship between the generators of symmetries of the Euler–Lagrange equations of motion and those of the action, is best done within the Lagrangian phase space framework. To accomplish this, we will need the following generalization of lemma 2.

Consider the vector

$$\mathbf{k} := c \cdot \frac{\partial}{\partial q} + \dot{c} \cdot \frac{\partial}{\partial \dot{q}},$$

with a  $c \in \ker M_{ab}(q, \dot{q})$  along with the quantity

$$l_a := F_{ab}c^b(q, \dot{q}) + M_{ab}\dot{c}^b(q, \dot{q}).$$

After an integration by parts,

$$\begin{aligned} l_a &= c^b(q, \dot{q}) \left\{ F_{ab}(q, \dot{q}) - \frac{\mathbf{d}}{\mathbf{d}t} \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b} \right\}, \\ &= c^b(q, \dot{q}) \left\{ F_{ab}(q, \dot{q}) - \left[ \frac{\mathbf{d}}{\mathbf{d}t}, \frac{\partial}{\partial \dot{q}^a} \right] \frac{\partial L}{\partial \dot{q}^b} - \frac{\partial}{\partial \dot{q}^a} \left( \frac{\mathbf{d}}{\mathbf{d}t} \frac{\partial L}{\partial \dot{q}^b} \right) \right\}. \end{aligned}$$

Using equation (2) we have

$$\left[ \frac{\mathbf{d}}{\mathbf{d}t}, \frac{\partial}{\partial \dot{q}^a} \right] \frac{\partial L}{\partial \dot{q}^b} = - \frac{\partial^2 L}{\partial q^a \partial \dot{q}^b} - \frac{\partial \dot{q}^c}{\partial \dot{q}^a} \frac{\partial^2 L}{\partial \dot{q}^c \partial \dot{q}^b}.$$

As  $q(t)$  is a solution of the Euler–Lagrange equations of motion, we find that

$$l_a = c^b(q, \dot{q}) \left\{ F_{ab}(q, \dot{q}) + \frac{\partial^2 L}{\partial q^a \partial \dot{q}^b} - \frac{\partial^2 L}{\partial \dot{q}^a \partial q_b} + \frac{\partial \dot{q}^c}{\partial \dot{q}^a} \frac{\partial^2 L}{\partial \dot{q}^c \partial \dot{q}^b} \right\}.$$

This last expression vanishes after the definition of  $F_{ab}(q, \dot{q})$  is used along with the requirement that  $c \in \ker M_{ab}(q, \dot{q})$ . We then have the following result.

**Lemma 3.** For any vector

$$\mathbf{k} = c \cdot \frac{\partial}{\partial q} + \dot{c} \cdot \frac{\partial}{\partial \dot{q}},$$

such that  $c \in \ker M_{ab}$ ,

$$0 = F_{ab}c^b(q, \dot{q}) + M_{ab}\dot{c}^b(q, \dot{q}).$$

### 3. Generators of the generalized lie symmetry for the Euler–Lagrange equations of motion

The generators of the generalized Lie symmetry for both the Euler–Lagrange equations of motion and the action are best found using the Lagrangian phase space approach to mechanics. This phase space and its concomitant mathematical structure provide the tools needed to determine both the generators of the symmetry and the solutions to the energy equation on which they act. For the Euler–Lagrange equations of motion this determination was done in [1]. In this section we will review the Lagrangian phase space approach, establish the

notation used in this paper, and summarize the results obtained in [1] that are needed here. (We will also take the opportunity to correct typographical errors made in [1].) Proofs of the majority of the assertions listed in this section will not be given; the reader is instead referred to [1] where the proofs and the context of their development can be found.

### 3.1. The Lagrangian phase space

For a configuration space  $\mathbb{Q}$  the Lagrangian phase space  $\mathbb{P}_L$  is the tangent space  $\mathbb{P}_L = \mathbf{T}\mathbb{Q}$ , with the coordinates on  $\mathbb{P}_L$  denoted as  $\mathbf{u} = (q^1, \dots, q^D, v^1, \dots, v^D)$ . Integral flows on  $\mathbb{P}_L$ ,  $t \in [t_0, \infty) \rightarrow \mathbf{u}(t) \in \mathbb{P}_L$  [28], for a set of initial data  $\mathbf{u}_0 = (q_0, v_0)$  are given as solutions to

$$\frac{d\mathbf{u}}{dt} := \mathbf{X}(\mathbf{u}),$$

where  $\mathbf{X}$  is a smooth vector field in  $\mathbf{T}\mathbb{P}_L = \mathbf{T}(\mathbf{T}\mathbb{Q})$ . The two tangent spaces  $\mathbf{T}\mathbb{Q}$  and  $\mathbf{T}\mathbb{P}_L$  have the bundle projections:  $\tau_{\mathbb{Q}}: \mathbf{T}\mathbb{Q} \rightarrow \mathbb{Q}$  and  $\tau_{\mathbf{T}\mathbb{Q}}: \mathbf{T}(\mathbf{T}\mathbb{Q}) \rightarrow \mathbf{T}\mathbb{Q}$ . They can be used to construct two other projection maps:  $\tau_{\mathbb{Q}} \circ \tau_{\mathbf{T}\mathbb{Q}}: \mathbf{T}(\mathbf{T}\mathbb{Q}) \rightarrow \mathbb{Q}$  and the prolongation of  $\tau_{\mathbf{T}\mathbb{Q}}$  to  $\mathbf{T}(\mathbf{T}\mathbb{Q})$  (see [3] and [28]). This prolongation is the map  $\mathbf{T}\tau_{\mathbf{T}\mathbb{Q}}: \mathbf{T}(\mathbf{T}\mathbb{Q}) \rightarrow \mathbf{T}\mathbb{Q}$ , and is defined by requiring that the two maps  $\tau_{\mathbb{Q}} \circ \tau_{\mathbf{T}\mathbb{Q}}$  and  $\tau_{\mathbb{Q}} \circ \mathbf{T}\tau_{\mathbf{T}\mathbb{Q}}$  map any point in  $\mathbf{T}(\mathbf{T}\mathbb{Q})$  to the same point in  $\mathbb{Q}$ . The vertical subbundle  $[\mathbf{T}\mathbb{P}_L]^v$  of  $\mathbf{T}(\mathbf{T}\mathbb{Q})$  is  $[\mathbf{T}\mathbb{P}_L]^v = \ker \mathbf{T}\tau_{\mathbf{T}\mathbb{Q}}$  [3]; a  $\mathbf{X}^v \in [\mathbf{T}_u\mathbb{P}_L]^v$  above a point  $\mathbf{u} \in \mathbb{P}_L$  is called a vertical vector field. The horizontal subbundle  $[\mathbf{T}\mathbb{P}_L]^h$  of  $\mathbf{T}(\mathbf{T}\mathbb{Q})$  is  $[\mathbf{T}\mathbb{P}_L]^h = \text{Image } \mathbf{T}\tau_{\mathbf{T}\mathbb{Q}}$ ; a  $\mathbf{X}^h \in [\mathbf{T}_u\mathbb{P}_L]^h$  is called a horizontal vector field. Consequently, each  $\mathbf{X} \in \mathbf{T}_u\mathbb{P}_L$  consists of a  $\mathbf{X}^h \in [\mathbf{T}_u\mathbb{P}_L]^h$  and a  $\mathbf{X}^v \in [\mathbf{T}_u\mathbb{P}_L]^v$  with  $\mathbf{X} = \mathbf{X}^h + \mathbf{X}^v$ . In terms of local coordinates,

$$\mathbf{X}^h := X^{qa} \frac{\partial}{\partial q^a}, \quad \text{and} \quad \mathbf{X}^v := X^{va} \frac{\partial}{\partial v^a}.$$

Of special interest is the second order Lagrangian vector field  $\mathbf{X}_L$ . This vector field is the particular solution of equation (1) for which  $\mathbf{T}\tau_{\mathbf{T}\mathbb{Q}} \circ \mathbf{X}_L$  is the identity on  $\mathbf{T}\mathbb{Q}$  (see [28]). In terms of local coordinates

$$\mathbf{X}_L = v^a \frac{\partial}{\partial q^a} + X^{va} \frac{\partial}{\partial v^a}.$$

The space of one-forms on  $\mathbb{P}_L$  is the cotangent space  $\mathbf{T}^*\mathbb{P}_L$ . For a one-form  $\alpha \in \mathbf{T}_u^*\mathbb{P}_L$ , and a vector field  $\mathbf{X} \in \mathbf{T}_u\mathbb{P}_L$ , the dual prolongation map  $\mathbf{T}^*\tau_{\mathbf{T}\mathbb{Q}}$  is defined as

$$\langle \alpha | \mathbf{T}\tau_{\mathbf{T}\mathbb{Q}} \mathbf{X} \rangle = \langle \mathbf{T}^*\tau_{\mathbf{T}\mathbb{Q}} \alpha | \mathbf{X} \rangle,$$

after a useful adaptation of Dirac's bra and ket notation. In addition, for a general  $k$ -form  $\omega$  in the  $k$ -form bundle  $\Lambda^k(\mathbb{P}_L)$ ,

$$\omega(x): \mathbf{Y}_1 \otimes \dots \otimes \mathbf{Y}_k \rightarrow \langle \omega(x) | \mathbf{Y}_1 \otimes \dots \otimes \mathbf{Y}_k \rangle \in \mathbb{R},$$

with  $\mathbf{Y}_j \in \mathbf{T}_u\mathbb{P}_L$  for  $j = 1, \dots, k$ . The vertical one-form subbundle  $[\mathbf{T}^*\mathbb{P}_L]^v$  of  $\mathbf{T}^*\mathbb{P}_L$  is  $[\mathbf{T}^*\mathbb{P}_L]^v := \ker \mathbf{T}^*\tau_{\mathbf{T}\mathbb{Q}}$ ; a  $\alpha_v \in [\mathbf{T}_u^*\mathbb{P}_L]^v$  is called a vertical one-form. The horizontal one-form subbundle  $[\mathbf{T}^*\mathbb{P}_L]^h$  of  $\mathbf{T}^*\mathbb{P}_L$  is  $[\mathbf{T}^*\mathbb{P}_L]^h = \text{Image } \mathbf{T}^*\tau_{\mathbf{T}\mathbb{Q}}$ ; a  $\alpha_q \in [\mathbf{T}_u^*\mathbb{Q}]^h$  is called a horizontal one-form. Each one-form  $\varphi \in \mathbf{T}_u^*\mathbb{P}_L$  consists of a  $\varphi_q \in [\mathbf{T}_u^*\mathbb{Q}]^h$  and a  $\varphi_v \in [\mathbf{T}_u^*\mathbb{P}_L]^v$  such that  $\varphi = \varphi_q + \varphi_v$ . In terms of local coordinates  $\varphi_q := \varphi_{qa} \mathbf{d}q^a$  and  $\varphi_v := \varphi_{va} \mathbf{d}v^a$ .

Following [3, 4], the Lagrangian two-form is defined as  $\Omega_L := -\mathbf{d}\mathbf{d}_j L$ , where  $\mathbf{d}_j$  is the vertical derivative (see [3]). This two-form can be expressed as  $\Omega_L := \Omega_F + \Omega_M$  such that for any  $\mathbf{X}, \mathbf{Y} \in \mathbf{T}_u\mathbb{P}_L$ .

$$\Omega_F(\mathbf{X}, \mathbf{Y}) := \Omega_L(\mathbf{T}\tau_{\mathbf{T}\mathbb{Q}} \mathbf{X}, \mathbf{T}\tau_{\mathbf{T}\mathbb{Q}} \mathbf{Y}),$$

and is thus the horizontal two-form of  $\Omega_L$ . As  $\Omega_M(\mathbf{X}, \mathbf{Y}) = \Omega_L(\mathbf{X}, \mathbf{Y}) - \Omega_F(\mathbf{X}, \mathbf{Y})$ ,  $\Omega_M$  is then a mixed two-form of  $\Omega_L$ . In terms of local coordinates,

$$\Omega_L = -\mathbf{d}\theta_L, \quad \text{where } \theta_L := \frac{\partial L}{\partial v^a} \mathbf{d}q^a,$$

while

$$\Omega_F := \frac{1}{2} F_{ab} \mathbf{d}q^a \wedge \mathbf{d}q^b, \quad \text{and} \quad \Omega_M := M_{ab} \mathbf{d}q^a \wedge \mathbf{d}v^b.$$

For regular Lagrangians  $\mathbf{X}_L$  is the unique solution of equation (1). For almost regular Lagrangians, on the other hand, this solution is not unique, but instead depends on

$$\ker \Omega_L(\mathbf{u}) := \{\mathbf{K} \in \mathbf{T}_u\mathbb{P}_L \mid i_{\mathbf{K}} \Omega_L = 0\}.$$

From section 2 we expect this kernel to play a role in determining the generators of the generalized Lie symmetry of both the Euler–Lagrange equations of motion and the action. Indeed, consider the natural isomorphism  $iso: (t, q, \dot{q}, \ddot{q}) \in \mathbb{M}^{(2)} \rightarrow (t, q, v, X^{va})$  defined in [1], and the prolongation  $\mathbf{pr} \mathbf{g}$  of a generator  $\mathbf{g} \in \mathfrak{g}$  of a generalized Lie symmetry of the Euler–Lagrange equations of motion. This  $\mathbf{pr} \mathbf{g}$  contains the vector



$$\mathbf{k} = \rho \cdot \frac{\partial}{\partial q} + \dot{\rho} \cdot \frac{\partial}{\partial \dot{q}}.$$

The collection of all such vectors has been shown to be involutive (see [1]). The isomorphism maps  $iso: \mathbf{k} \rightarrow \mathbf{k}'$  where

$$\mathbf{k}' = \rho \cdot \frac{\partial}{\partial q} + \dot{\rho} \cdot \frac{\partial}{\partial v}.$$

Then  $\mathbf{k}' \in \mathbb{T}\mathbb{P}_L$ , and from lemma 2,  $\mathbf{k}' \in \ker \Omega_L(\mathbf{u})$  as well. A similar result holds for the generators in  $\mathfrak{g}_L$  after lemma 1 and lemma 3 are used.

The two-form  $\Omega_L$  gives the lowering map  $\Omega_L^b: \mathbb{T}_u\mathbb{P}_L \rightarrow \mathbb{T}_u^*\mathbb{P}_L$ , with  $\Omega_L^b \mathbf{X} := i_{\mathbf{X}} \Omega_L$ . This map consists of  $\Omega_L^b = \Omega_F^b + \Omega_M^{vb} + \Omega_M^{qb}$ , with  $\Omega_F^b: \mathbf{X} \in \mathbb{T}_u\mathbb{P}_L \rightarrow [\mathbb{T}_u^*\mathbb{P}_L]^q$ ;  $\Omega_M^{qb}: \mathbf{X} \in \mathbb{T}_u\mathbb{P}_L \rightarrow [\mathbb{T}_u^*\mathbb{P}_L]^q$ ; and  $\Omega_M^{vb}: \mathbf{X} \in \mathbb{T}_u\mathbb{P}_L \rightarrow [\mathbb{T}_u^*\mathbb{P}_L]^v$ . In terms of local coordinates,  $\Omega_F^b \mathbf{X} = F_{ab} X^{qa} \mathbf{d}q^b$ ,  $\Omega_M^{qb} \mathbf{X} = -M_{ab} X^{va} \mathbf{d}q^b$ , and  $\Omega_M^{vb} \mathbf{X} = M_{ab} X^{qa} \mathbf{d}v^b$ .

For almost regular Lagrangians  $\ker \Omega_M^{vb} = \mathcal{C} \oplus [\mathbb{T}_u\mathbb{P}_L]^v$  while  $\ker \Omega_M^{qb} = [\mathbb{T}_u\mathbb{P}_L]^q \oplus \mathcal{G}$ . Here

$$\mathcal{C} := \{ \mathbf{C} \in [\mathbb{T}_q\mathbb{P}_L]^q \mid i_{\mathbf{C}} \Omega_M = 0 \},$$

and

$$\mathcal{G} := \{ \mathbf{G} \in [\mathbb{T}_q\mathbb{P}_L]^v \mid i_{\mathbf{G}} \Omega_M = 0 \}.$$

As  $M_{ab}(\mathbf{u})$  has constant rank on  $\mathbb{P}_L$ , there exists a basis,

$$\{ \mathfrak{z}_{(n)}(\mathbf{u}) = (\mathfrak{z}_{(n)}^1(\mathbf{u}), \dots, \mathfrak{z}_{(n)}^D(\mathbf{u})) \mid M_{ab}(\mathbf{u}) \mathfrak{z}_{(n)}^b(\mathbf{u}) = 0, n = 1, \dots, N_0 \},$$

for  $\ker M_{ab}(\mathbf{u})$  at each  $\mathbf{u} \in \mathbb{P}_L$ . Spans of both

$$\mathcal{C} = \text{span} \left\{ \mathbf{U}_{(n)}^q = \mathfrak{z}_{(n)} \cdot \frac{\partial}{\partial q}, n = 1, \dots, N_0 \right\}, \text{ and}$$

$$\mathcal{G} = \text{span} \left\{ \mathbf{U}_{(n)}^v = \mathfrak{z}_{(n)} \cdot \frac{\partial}{\partial v}, n = 1, \dots, N_0 \right\},$$

can then be constructed. Importantly,  $\mathcal{G}$  is involutive [5], and when the rank of  $\Omega_L(\mathbf{u})$  is constant on  $\mathbb{P}_L$ ,  $\ker \Omega_L(\mathbf{u})$  is involutive as well.

Corresponding to  $\mathbf{U}_{(n)}^q$  and  $\mathbf{U}_{(n)}^v$  we have the one-forms  $\Theta_q^{(m)}$  and  $\Theta_v^{(m)}$  where  $\langle \Theta_q^{(m)} \mid \mathbf{U}_{(n)}^q \rangle = \delta_{(n)}^{(m)}$  and  $\langle \Theta_v^{(m)} \mid \mathbf{U}_{(n)}^v \rangle = \delta_{(n)}^{(m)}$ . Then  $[\mathbb{T}_u\mathbb{P}_L]^q = \mathcal{C} \oplus \mathcal{C}_{\perp}$  and  $[\mathbb{T}_u\mathbb{P}_L]^v = \mathcal{G} \oplus \mathcal{G}_{\perp}$ , where

$$\mathcal{C}_{\perp} := \{ \mathbf{X} \in [\mathbb{T}_u\mathbb{P}_L]^q \mid \langle \Theta_q^{(n)} \mid \mathbf{X} \rangle = 0, n = 1, \dots, N_0 \}, \text{ and}$$

$$\mathcal{G}_{\perp} := \{ \mathbf{X} \in [\mathbb{T}_u\mathbb{P}_L]^v \mid \langle \Theta_v^{(n)} \mid \mathbf{X} \rangle = 0, n = 1, \dots, N_0 \}.$$

The vectors that lie in  $\ker \Omega_L(\mathbf{u})$  can be determined by using the reduced matrix  $F_{nm} := \mathfrak{z}_{(n)}^a F_{ab} \mathfrak{z}_{(m)}^b$  to define

$$\bar{\mathcal{C}} := \left\{ \bar{\mathbf{C}} \in \mathcal{C} \mid \sum_{m=1}^{N_0} \bar{F}_{nm} \bar{\mathbf{C}}^{(m)} = 0 \right\} \subset \mathcal{C}.$$

Then,

**Theorem 4.** The vectors  $\mathbf{K} = \mathbf{K}^q + \mathbf{K}^v \in \ker \Omega_L$  are given by,

$$\mathbf{K}^q = \bar{\mathbf{C}}, \quad \mathbf{K}^v = \mathbf{G} + \hat{\mathbf{C}},$$

where,  $\bar{\mathbf{C}} \in \bar{\mathcal{C}}$ ,  $\mathbf{G} \in \mathcal{G}$ , and  $\hat{\mathbf{C}} \in \mathcal{G}_{\perp}$  is the unique solution of  $M_{ab} \hat{\mathbf{C}}^b = -F_{ab} \bar{\mathbf{C}}^b$ .

We found in [1] that  $\dim(\ker \Omega_L(\mathbf{u})) = N_0 + \bar{D}$ , where  $\bar{D} := \dim \bar{\mathcal{C}} \leq N_0$  (see [1] for proof). However, the results of lemma 3 show that we can construct from any vector  $\mathbf{U}^q \in \mathcal{C}$  a vector that lies in the  $\ker \Omega_L(\mathbf{u})$ , and as  $\dim(\mathcal{C}) = N_0$ , it follows that  $\dim(\ker \Omega_L(\mathbf{u})) = 2N_0$ .

### 3.2. First-order Lagrangian constraints

For singular Lagrangians solutions of the energy equation  $\mathbf{X}_E$  are not unique. It is well known that they also do not, in general, exist throughout  $\mathbb{P}_L$ , but are instead confined to a submanifold of the space given by Lagrangian constraints.

With  $\mathbf{X}_E = \mathbf{X}_E^q + \mathbf{X}_E^v$ , it is convenient to use the one form  $\Psi$

$$\Omega_M^{qb} \mathbf{X}_E^v = \Psi.$$

constructed from the energy equation. The first-order constraint functions are then  $\gamma_n^{[1]} := \langle \Psi \mid \mathbf{U}_{(n)}^q \rangle = 0$  for  $n = 1, \dots, N_0$ . In terms of local coordinates,

$$\gamma_n^{[1]} = U_{(n)}^{qa} \left( \frac{\partial E}{\partial q^a} + F_{ab} v^b \right).$$

They may also be expressed [3, 4] as  $\gamma_n^{[1]} = \langle \mathbf{d}E | \mathbf{P}_{(n)} \rangle = \mathbf{P}_{(n)} E$  for any basis  $\{\mathbf{P}_{(n)}\}$  of  $\ker \Omega_L(\mathbf{u})$  for which  $\langle \Theta_q^{(m)} | \mathbf{P}_{(n)} \rangle = \delta_{(n)}^{(m)}$ . In general,  $\gamma_n^{[1]} \neq 0$  on  $\mathbb{P}_L$ . Instead, the condition  $\gamma_n^{[1]} = 0$  must be imposed, and this in turn defines a set of submanifolds of  $\mathbb{P}_L$  given by the collection  $C_L^{[1]} := \{\gamma_1^{[1]}, \dots, \gamma_{N_0}^{[1]}\}$ . The collection of these surfaces,  $\mathbb{P}_L^{[1]} := \{\mathbf{u} \in \mathbb{P}_L \mid \gamma_n^{[1]}(\mathbf{u}) = 0, n = 1, \dots, N_0\}$  is called the first-order Lagrangian constraint submanifold, and has  $\dim \mathbb{P}_L^{[1]} = 2D - I_{[1]}$ . Here  $I_{[1]}$  is the number of independent functions in  $C_L^{[1]}$  with  $I_{[1]} = \text{rank}\{\mathbf{d}\gamma_n^{[1]}\} \leq N_0$ .

The constraint one-form

$$\beta[\mathbf{X}_E] := \mathbf{d}E - i_{\mathbf{X}_E} \Omega_L,$$

was introduced in [1] with the condition  $\beta[\mathbf{X}_E] = 0$  giving both the solution of the energy equation and the submanifold  $\mathbb{P}_L^{[1]}$ . As  $\langle \beta | \mathbf{U}_{(n)}^q \rangle = \gamma_n^{[1]}$ , this  $\beta[\mathbf{X}_E]$  can also be expressed as

$$\beta[\mathbf{X}_E] = \sum_{n=1}^{N_0} \gamma_n^{[1]} \Theta_q^{(n)}. \tag{11}$$

### 3.3. The generalized lie symmetry group for the Euler–Lagrange equations of motion

The generalized Lie symmetry group for  $\mathcal{O}(\mathbf{u}_0)$  is determined using

$$\ker \bar{\Omega}_L(\mathbf{u}) := \left\{ \mathbf{P} \in \ker \Omega_L(\mathbf{u}) \mid [\mathbf{G}, \mathbf{P}] \in [\mathbf{T}_u \mathbb{P}_L]^v \ \forall \ \mathbf{G} \in \mathcal{G} \right\}, \tag{12}$$

along with the following collection of functions on  $\mathbb{P}_L$ ,

$$\bar{\mathcal{F}} := \{f \in C^\infty \text{ on } \mathbb{P}_L \mid \mathbf{G}f = 0 \ \forall \ \mathbf{G} \in \mathcal{G}\}.$$

This  $\overline{\ker \Omega_L(\mathbf{u})}$  is also involutive.

The following results were proved in [1].

**Lemma 5.** *Let  $\mathbf{X} \in \mathbf{T}_u \mathbb{P}_L$  and  $\mathbf{G} \in \mathcal{G}$  such that  $[\mathbf{G}, \mathbf{X}] \in \ker \Omega_L(\mathbf{u})$ . Then  $[\mathbf{G}, \mathbf{X}] \in [\mathbf{T}_u \mathbb{P}_L]^v$  iff  $[\mathbf{G}, \mathbf{X}] \in \mathcal{G}$ .*

It then follows that  $[\mathbf{G}, \mathbf{P}] \in \mathcal{G}$  for all  $\mathbf{P} \in \overline{\ker \Omega_L(\mathbf{u})}$ . As  $\mathcal{G}$  is involutive and as  $\mathcal{G} \subset \ker \Omega_L(\mathbf{u})$ ,  $\mathcal{G} \subset \ker \Omega_L(\mathbf{u})$  as well, and thus  $\mathcal{G}$  is an ideal of  $\ker \Omega_L(\mathbf{u})$ .

**Lemma 6.** *There exists a choice of basis for  $\ker \Omega_L(\mathbf{u})$  that is also a basis of  $\overline{\ker \Omega_L(\mathbf{u})}$ .*

As  $\mathcal{G}$  is an ideal of  $\overline{\ker \Omega_L(\mathbf{u})}$ , we may define for any  $\mathbf{P}_1, \mathbf{P}_2 \in \overline{\ker \Omega_L(\mathbf{u})}$  the equivalence relation:  $\mathbf{P}_1 \sim \mathbf{P}_2$  iff  $\mathbf{P}_1 - \mathbf{P}_2 \in \mathcal{G}$ . The equivalence class,

$$[\mathbf{P}] := \{\mathbf{Y} \in \overline{\ker \Omega_L(\mathbf{u})} \mid \mathbf{Y} \sim \mathbf{P}\}, \tag{13}$$

can be constructed along with the quotient space  $\ker \Omega_L(\mathbf{u}) / \mathcal{G}$ . (For the sake of notational clarity we will suppress the square brackets for equivalence classes when there is no risk of confusion.) This space is a collection of vectors that lie in the kernel of  $\Omega_L$ , but with the vectors in  $\mathcal{G}$  removed;  $\ker \Omega_L(\mathbf{u}) / \mathcal{G}$  thereby addresses the first two observations listed at the end of section 2.2.

We now turn our attention to the third observation. Because the integral flow  $u_{\mathbf{X}}(t)$  of any solution  $\mathbf{X}$  of the energy equation must lie on  $\mathbb{P}_L^{[1]}$ , a symmetry transformation of  $u_{\mathbf{X}}(t)$  must result in an integral flow  $u_{\mathbf{Y}}(t)$  of another solution  $\mathbf{Y}$  of the energy equation, which must also lie on  $\mathbb{P}_L^{[1]}$ . Implementing this condition is done through  $\beta[\mathbf{X}_E]$ .

As  $\langle \beta[\mathbf{X}_E] | \mathbf{G} \rangle = \langle \mathbf{d}E | \mathbf{G} \rangle = \mathbf{G}E = 0$  for all  $\mathbf{G} \in \mathcal{G}$  on  $\mathbb{P}_L^{[1]}$ , the Lie derivative  $\mathcal{L}_{\mathbf{G}} \beta$  of  $\beta$  along  $\mathbf{G}$  is,

$$\mathcal{L}_{\mathbf{G}} \beta[\mathbf{X}_E] = \sum_{n=1}^{N_0} (\mathbf{G} \gamma_n^{[1]}) \Theta_q^{(n)}.$$

Given a  $\mathbf{P}_{(n)} \in \overline{\ker \Omega_L(\mathbf{u})}$  such that  $\mathbf{P}_{(n)} = \mathbf{U}_{(n)}^q + \widehat{\mathbf{U}}_{(n)} + \mathbf{G}'$  with  $\mathbf{G}' \in \mathcal{G}$ ,  $\mathbf{G} \gamma_n^{[1]} = [\mathbf{G}, \mathbf{P}_{(n)}]E + \mathbf{P}_{(n)} \mathbf{G}E$ . But  $\mathcal{G}$  is an ideal of  $\overline{\ker \Omega_L(\mathbf{u})}$ , and thus  $\mathbf{G} \gamma_n^{[1]} = 0$  on the first-order constraint manifold. It follows that  $\mathcal{L}_{\mathbf{G}} \beta = 0$  on  $\mathbb{P}_L^{[1]}$ . The collection of vectors,

$$\text{Sym} := \{\mathbf{P} \in \overline{\ker \Omega_L(\mathbf{u})} / \mathcal{G} \mid \mathcal{L}_{\mathbf{P}} \beta[\mathbf{X}_E] = \mathbf{d} \langle \beta[\mathbf{X}_E] | \mathbf{P} \rangle \text{ on } \mathbb{P}_L^{[1]}\},$$

is therefore well defined, and is involutive. It follows that  $\mathbf{P} \in \text{Sym}$  iff  $\langle \mathbf{d} \beta[\mathbf{X}_E] | \mathbf{P} \otimes \mathbf{X} \rangle = 0$  for all  $\mathbf{X} \in \mathbf{T} \mathbb{P}_L$ . We are then able to construct from each  $\mathbf{P} \in \text{Sym}$  a one-parameter subgroup  $\sigma_{\mathbf{P}}(\epsilon, \mathbf{x})$  defined as the solution to

$$\frac{d\sigma_{\mathbf{P}}}{d\epsilon} := \mathbf{P}(\sigma_{\mathbf{P}}),$$

where  $\sigma_{\mathbf{P}}(0, \mathbf{u}) = \mathbf{u}$  for  $\mathbf{u} \in \mathbb{P}_L$ . The collection of such subgroups will give the Lie group  $\text{Gr}_{\text{Sym}}$ .

### 3.4. Euler–Lagrange solutions of the energy equation

We denote the set of general solutions to the energy equation as

$$\text{Sol} := \{\mathbf{X}_E \in \mathbf{T}_u\mathbb{P}_L \mid i_{\mathbf{X}_E}\Omega_L = \mathbf{d}E \text{ on } \mathbb{P}_L^{[1]}\}.$$

If  $\mathbf{u}(t)$  is the integral flow of a vector in  $\text{Sol}$  whose projection onto  $\mathbb{Q}$  corresponds to a solution of the Euler–Lagrange equations of motion, then  $\text{Gr}_{\text{Sym}}$  must map one of such flows into another one. However, while  $\mathcal{L}_{\mathbf{G}}\mathbf{X}_L = [\mathbf{G}, \mathbf{X}_L] \in \ker \Omega_L(\mathbf{u})$ , in general  $\mathcal{L}_{\mathbf{G}}\mathbf{X}_L \notin \mathcal{G}$ . The action of  $\sigma_{\mathbf{P}}$  on the flow  $\mathbf{u}_{\mathbf{X}_L}$  will in general result in a flow  $\mathbf{u}_{\mathbf{Y}}$  generated by a  $\mathbf{Y}$  that is *not* a SOLVF. It need not even be a solution of the energy equation. By necessity, general solutions of the energy equation must be considered, leading us to consider the collection of solutions

$$\overline{\text{Sol}} := \{\overline{\mathbf{X}}_{EL} \in \text{Sol} \mid [\mathbf{G}, \overline{\mathbf{X}}_{EL}] \in [\mathbf{T}_u\mathbb{P}_L]^\nu \quad \forall \mathbf{G} \in \mathcal{G}\}.$$

This collection generates the family of integral flows

$$\mathcal{O}_{EL}(\mathbf{u}_0) := \left\{ \mathbf{u}(t) \mid \frac{d\mathbf{u}}{dt} = \overline{\mathbf{X}}_{EL}(\mathbf{u}), \overline{\mathbf{X}}_{EL} \in \overline{\text{Sol}}, \text{ and } \mathbf{u}(t_0) = \mathbf{u}_0 \right\}.$$

Importantly, if  $\mathbf{P} \in \text{Sym}$ , then

$$i_{[\mathbf{X}_E, \mathbf{P}]} \Omega_L = i_{\mathbf{P}} \mathbf{d}\beta[\mathbf{X}_E] = 0.$$

As such, we find that

**Lemma 7.**  $[\overline{\mathbf{X}}_{EL}, \mathbf{P}] \in \overline{\ker \Omega_L(\mathbf{u})}$  for all  $\mathbf{P} \in \text{Sym}$ .

It then follows that

**Theorem 8.**  $\text{Gr}_{\text{Sym}}$  forms a group of symmetry transformations of  $\mathcal{O}_{EL}(\mathbf{u}_0)$ .

Proof of both assertions can be found in [1].

The generators of the generalized Lie symmetry for  $\mathcal{O}_{EL}(\mathbf{u}_0)$  are thus given by  $\text{Sym}$ . The corresponding solutions to the Euler–Lagrange equations that have this symmetry are given by  $\overline{\text{Sol}}$ , and a vector  $\overline{\mathbf{X}}_{EL} \in \overline{\text{Sol}}$  is called a second-order, Euler–Lagrange vector field (SOELVF). It has the general form,

$$\overline{\mathbf{X}}_{EL} = \overline{\mathbf{X}}_L + \sum_{m=1}^{N_0} u^m(\mathbf{u})[\mathbf{P}_{(m)}], \tag{14}$$

where  $u^m(\mathbf{u}) \in \overline{\mathcal{F}}$  and  $\{\mathbf{P}_{(m)}, n = 1, \dots, N_0\}$  is a choice of basis for  $\overline{\ker \Omega_L(\mathbf{u})}/\mathcal{G}$ . The vector field  $\overline{\mathbf{X}}_L$  is constructed from the second order Lagrangian vector field  $\mathbf{X}_L$  and vectors in  $\ker \Omega_L(\mathbf{u})$  by requiring  $\overline{\mathbf{X}}_L \in \overline{\text{Sol}}$ . This construction is described in [1]; we will only need the existence of such a vector field in this paper.

## 4. Generalized lie symmetries of the action and its impact on dynamics

We now turn our attention to the generators of the generalized Lie symmetry of the action, and the impact this symmetry has on the evolution of dynamical systems.

### 4.1. The generalized lie symmetry of the action

In determining the conditions (as listed in lemma 1) under which the action admits a generalized Lie symmetry, the understanding that the action must have this symmetry for all possible paths on  $\mathbb{Q}$  played an essential role. By necessity, these conditions could only be placed on  $\rho_L$ , and not on  $\dot{\rho}_L$ ; unlike  $\rho_L$ ,  $\dot{\rho}_L$  depends explicitly on the evolution of a particular path, while the symmetry must hold for all paths. We note, however, that the family  $\mathcal{O}_{EL}$  of trajectories determined by the Euler–Lagrange equations of motion also consists of paths on  $\mathbb{Q}$ , and as such the generalized Lie symmetry of the action is also a symmetry of  $\mathcal{O}_{EL}$ . Importantly, how these trajectories evolve with time is known, and as such, the  $\dot{\rho}_L$  for a given  $\rho_L$  is also known for these trajectories. With this understanding, and after comparing lemma 1 and the results of lemma 3 with lemma 2, we conclude that the generators of the generalized Lie symmetry of the action must also be generators of the generalized Lie symmetry of the Euler–Lagrange equations of motion. This leads us to consider the following collection of vectors.

$$\text{Sym}\mathcal{L} = \{\mathbf{P} \in \overline{\ker \Omega_L(\mathbf{u})} / \mathcal{G} \mid \gamma_{\mathbf{P}}^{[1]} = \langle \beta | \mathbf{P} \rangle = 0 \text{ on } \mathbb{P}_L\}.$$

We will also need  $N_{\text{Sym}\mathcal{L}} = \dim(\text{Sym}\mathcal{L})$  in the following.

**Lemma 8.**  $\text{Sym}\mathcal{L} \subset \text{Sym}$ .

**Proof.** Let  $\{\mathbf{P}_{(l)}, l = 1, \dots, N_0\}$  be a basis of  $\overline{\ker \Omega_L(\mathbf{u})} / \mathcal{G}$  such that  $\mathbf{P}_{(l)} \in \text{Sym}\mathcal{L}$  for  $l = 1, \dots, N_{\text{Sym}\mathcal{L}}$ . We may choose the basis of  $\mathcal{C}$  such that  $\langle \Theta_q^{(m)} | \mathbf{P}_{(l)} \rangle = \delta_{(l)}^{(m)}$ . Then for any  $\mathbf{P}_{(n)} \in \text{Sym}$ , we see from equation (11) that,

$$\langle \mathbf{d}\beta | \mathbf{P}_{(n)} \otimes \mathbf{Y} \rangle = \sum_{m=1}^{N_0} (\langle \mathbf{d}\gamma_m^{[1]} | \mathbf{P}_{(n)} \rangle \langle \Theta_q^{(m)} | \mathbf{Y} \rangle - \langle \mathbf{d}\gamma_m^{[1]} | \mathbf{Y} \rangle \langle \Theta_q^{(m)} | \mathbf{P}_{(n)} \rangle + \gamma_m^{[1]} \langle \mathbf{d}\Theta_q^{(m)} | \mathbf{P}_{(n)} \otimes \mathbf{Y} \rangle),$$

for any  $\mathbf{Y} \in \mathbb{T}\mathbb{P}_L$ . The last term vanishes on the first-order constraint manifold  $\mathbb{P}_L^{[1]}$ , while for the second term,  $\langle \mathbf{d}\gamma_m^{[1]} | \mathbf{Y} \rangle \langle \Theta_q^{(m)} | \mathbf{P}_{(n)} \rangle = \langle \mathbf{d}\gamma_n^{[1]} | \mathbf{Y} \rangle \delta_{(n)}^{(m)}$ . But as  $\mathbf{P}_{(n)} \in \text{Sym}\mathcal{L}$ ,  $\gamma_n^{[1]} = 0$  on  $\mathbb{P}_L$ , and this term vanishes as well.

Finally, for the first term,  $\langle \mathbf{d}\gamma_m^{[1]} | \mathbf{P}_{(n)} \rangle = \mathbf{P}_{(n)} \mathbf{P}_{(m)} E = [\mathbf{P}_{(n)}, \mathbf{P}_{(m)}] E + \mathbf{P}_{(m)} \mathbf{P}_{(n)} E$ . But  $\gamma_n^{[1]} = \mathbf{P}_{(n)} E = 0$  on  $\mathbb{P}_L$ , while  $\overline{\ker \Omega(\mathbf{u})}$  is involutive. There then exists a  $\mathbf{P}_{(nm)} \in \overline{\ker \Omega(\mathbf{u})}$  such that  $\mathbf{P}_{(nm)} = [\mathbf{P}_{(n)}, \mathbf{P}_{(m)}]$ . As

$\mathbf{P}_{(nm)} E := \gamma_{(nm)}^{[1]}$ , this  $\gamma_{(nm)}^{[1]}$  must be a linear combination of first-order constraint functions, and they also vanish on  $\mathbb{P}_L^{[1]}$ . It then follows that  $\langle \mathbf{d}\beta | \mathbf{P}_{(n)} \otimes \mathbf{Y} \rangle = 0$  on  $\mathbb{P}_L^{[1]}$ , and  $\mathbf{P}_{(n)} \in \text{Sym}$ . □

If  $\mathbf{P}_1, \mathbf{P}_2 \in \text{Sym}\mathcal{L}$ , then  $\gamma_{[\mathbf{P}_1, \mathbf{P}_2]}^{[1]} = \mathbf{P}_1 \mathbf{P}_2 E - \mathbf{P}_2 \mathbf{P}_1 E = \mathbf{P}_1 \gamma_{\mathbf{P}_2} - \mathbf{P}_2 \gamma_{\mathbf{P}_1} = 0$ , and thus  $\text{Sym}\mathcal{L}$  is involutive. Then for each  $\mathbf{P} \in \text{Sym}\mathcal{L}$  we once again have the one-parameter subgroup  $\sigma_{\mathbf{P}}^{\text{Sym}\mathcal{L}}(\epsilon, \mathbf{u})$  define as the integral flow of

$$\frac{\mathbf{d}\sigma_{\mathbf{P}}^{\text{Sym}\mathcal{L}}}{\mathbf{d}\epsilon} := \mathbf{P},$$

with  $\sigma_{\mathbf{P}}^{\text{Sym}\mathcal{L}}(0, \mathbf{u}) = \mathbf{u}$  for  $\mathbf{u} \in \mathbb{P}_L$ . The collection of such subgroups gives the Lie group  $\text{Gr}_{\text{Sym}\mathcal{L}}$ . As  $\text{Sym}\mathcal{L} \subset \text{Sym}$ ,  $\text{Gr}_{\text{Sym}\mathcal{L}}$  is a Lie subgroup of  $\text{Gr}_{\text{Sym}}$ . It then follows from theorem 8 that  $\text{Gr}_{\text{Sym}\mathcal{L}}$  also forms a group of symmetry transformations of  $\mathcal{O}_{EL}(\mathbf{u}_0)$ . As the family  $\mathcal{O}_{EL}(\mathbf{u}_0)$  of trajectories are paths on  $\mathbb{Q}$ , and as the symmetry transformation of the action must be the same for all paths on  $\mathbb{Q}$ , it also follows that,

**Theorem 10.**  $\text{Gr}_{\text{Sym}\mathcal{L}}$  forms the group of symmetry transformations of the action  $S$ .

### 4.2. Symmetries and dynamics

While  $\mathcal{O}_{EL}(\mathbf{u}_0)$  gives the family of integral flows on which both  $\text{Gr}_{\text{Sym}}$  and  $\text{Gr}_{\text{Sym}\mathcal{L}}$  act, a general flow in  $\mathcal{O}_{EL}(\mathbf{u}_0)$  need not be confined to  $\mathbb{P}_L^{[1]}$ , and yet this is the submanifold on which the solutions  $\overline{\mathbf{X}}_{EL} \in \overline{\text{Sol}}$  of the energy equations exist. In such cases it is necessary to jointly choose a SOELVF  $\overline{\mathbf{X}}_{EL}$  and a submanifold of  $\mathbb{P}_L^{[1]}$  on which the resultant flow  $\mathbf{u}_{\overline{\mathbf{X}}_{EL}}$  will be confined. This is done through the implementation of a constraint algorithm, one of which was proposed in [1]. In that paper the product of this algorithm was the most that could be said about the general structure of SOELVFs that have integral flow fields which lie on  $\mathbb{P}_L^{[1]}$ . Here, with the results obtained in section 4.1, we can say much more, and we will see that the presence of a generalized Lie symmetry of the action greatly restricts the structure of the SOELVFs that such systems can have.

Following [1], we introduce for a  $\overline{\mathbf{X}}_{EL} \in \overline{\text{Sol}}$  the notation

$$\overline{\mathbf{X}}_{EL}^{[1]} := \overline{\mathbf{X}}_{EL}, \quad \overline{\mathbf{X}}_L^{[1]} := \overline{\mathbf{X}}_L, \quad \mathbf{P}_{(n)}^{[1]} := \mathbf{P}_{(n)}, \quad u_{[1]}^m := u^m, \quad N_0^{[1]} := N_0,$$

when the constraint algorithm is implemented, with the superscript [1] denoting the first iteration of this algorithm. (This notation is only used in this section.) In addition, we choose  $\mathbf{P}_{(n)}^{[1]} \in \text{Sym}\mathcal{L}$  for  $n = 1, \dots, N_{\text{Sym}\mathcal{L}}$ .

For the integral flow field of  $\overline{\mathbf{X}}_{EL}$  to lie on  $\mathbb{P}_L^{[1]}$ ,

$$\mathcal{L}_{\overline{\mathbf{X}}_{EL}} \beta = 0, \tag{15}$$

which reduces to  $\mathcal{L}_{\overline{\mathbf{X}}_{EL}} \gamma_n^{[1]} = 0$  on  $\mathbb{P}_L^{[1]}$ . This is called the constraint condition. As both  $u_{[1]}^n, \gamma_n^{[1]} \in \overline{\mathcal{F}}$ ,  $[\mathbf{P}_{(n)}^{[1]}] \gamma_m^{[1]} = \mathbf{P}_{(n)} \gamma_m^{[1]}$ , and after making use of the general form of a SOELVF given in equation (14), equation (15) reduces to

$$\sum_{m=1}^{N_0} \Gamma_{nm}^{[1]} u_{[1]}^m = -\langle \mathbf{d}\gamma_n^{[1]} | \overline{\mathbf{X}}_L^{[1]} \rangle, \quad \text{with} \quad \Gamma_{nm}^{[1]} := \langle \mathbf{d}\gamma_n^{[1]} | \mathbf{P}_{(m)}^{[1]} \rangle. \tag{16}$$

Since  $\langle \mathbf{d}\gamma_n^{[1]} | \mathbf{P}_{(m)}^{[1]} \rangle = \mathbf{P}_{(m)}^{[1]} \mathbf{P}_{(n)}^{[1]} E = [\mathbf{P}_{(m)}^{[1]}, \mathbf{P}_{(n)}^{[1]}] E + \Gamma_{nm}^{[1]}$ . But  $\overline{\ker \Omega_L(\mathbf{u})}$  is involutive, and thus  $[\mathbf{P}_{(m)}^{[1]}, \mathbf{P}_{(n)}^{[1]}] E$  is a linear combination of first-order Lagrangian constraints. As these constraints vanishes on  $\mathbb{P}_L^{[1]}$ ,  $\Gamma_{nm}^{[1]} = \Gamma_{mn}^{[1]}$  on the first-order constraint manifold.

Next, when  $n = 1, \dots, N_{\text{Sym}\mathcal{L}}$ ,  $\mathbf{P}_{(n)}^{[1]} \in \text{Sym}\mathcal{L}$ , and  $\gamma_n^{[1]} = 0$ . Thus,  $\Gamma_{nm}^{[1]} = 0$  when  $n = 1, \dots, N_{\text{Sym}\mathcal{L}}$ , and as  $\Gamma_{nm}^{[1]}$  is a symmetric matrix on  $\mathbb{P}_L^{[1]}$ ,  $\Gamma_{nm}^{[1]} = 0$  for these values of  $n$  as well. Thus while  $\Gamma_{nm}^{[1]}$  is a  $N_0 \times N_0$  matrix, the only nonzero components of this matrix lie in the  $(N_0 - N_{\text{Sym}\mathcal{L}}) \times (N_0 - N_{\text{Sym}\mathcal{L}})$  submatrix

$\Gamma_{\bar{n}\bar{m}}^{[1]} := \langle \mathbf{d}\gamma_{\bar{n}+N_{\text{Sym}\mathcal{L}}}^{[1]} | \mathbf{P}_{\bar{m}+N_{\text{Sym}\mathcal{L}}}^{[1]} \rangle$  where  $\bar{n}, \bar{m} = 1, \dots, N_0 - N_{\text{Sym}\mathcal{L}}$ . As  $\langle \mathbf{d}\gamma_n^{[1]} | \bar{\mathbf{X}}_L^{[1]} \rangle = 0$  as well when  $n = 1, \dots, N_{\text{Sym}\mathcal{L}}$ , equation (16) reduces to

$$\sum_{\bar{m}=1}^{N_0-N_{\text{Sym}\mathcal{L}}} \Gamma_{\bar{n}\bar{m}}^{[1]} u_{[1]}^{\bar{m}+N_{\text{Sym}\mathcal{L}}} = -\langle \mathbf{d}\gamma_{\bar{n}+N_{\text{Sym}\mathcal{L}}}^{[1]} | \bar{\mathbf{X}}_L^{[1]} \rangle. \tag{17}$$

It is then readily apparent that the  $N_{\text{Sym}\mathcal{L}}$  arbitrary functions  $u_{[1]}^m$  for  $m = 1, \dots, N_{\text{Sym}\mathcal{L}}$  are not determined at this iteration, while  $r^{[1]} = \text{rank} \bar{\Gamma}_{\bar{n}\bar{m}}^{[1]}$  of the  $u_{[1]}^m$  for  $m > N_{\text{Sym}\mathcal{L}}$  are. There are then  $N_0^{[2]} := N_0^{[1]} - r^{[1]}$  second-order Lagrangian constraint functions

$$\gamma_{n_{[2]}}^{[2]} := \langle \mathbf{d}\gamma_{n_{[2]}}^{[1]} | \bar{\mathbf{X}}_L^{[1]} \rangle, \quad n_{[2]} = 1, \dots, N_0^{[2]},$$

with the conditions  $\gamma_{n_{[2]}}^{[2]} = 0$  imposed if necessary. In general there will be  $I_{[2]} := \text{rank} \left\{ \mathbf{d}\gamma_{n_{[1]}}^{[1]}, \mathbf{d}\gamma_{n_{[2]}}^{[2]} \right\}$

independent functions in  $C_L^{[2]} := C_L^{[1]} \cup \left\{ \gamma_{n_{[2]}}^{[2]} \mid n_{[2]} = 1, \dots, N_0^{[2]} \right\}$ , and  $\mathbb{P}_L^{[1]}$  is reduced to the second-order constraint submanifold,

$$\mathbb{P}_L^{[2]} := \left\{ \mathbf{u} \in \mathbb{P}_L^{[1]} \mid \gamma_{n_{[2]}}^{[2]}(\mathbf{u}) = 0, \quad n_{[2]} = 1, \dots, N_0^{[2]} \right\},$$

where  $\dim \mathbb{P}_L^{[2]} = 2D - I_{[2]}$ . At this point, there are two possibilities. If  $I_{[2]}=I_{[1]}$  or  $I_{[2]}=2D$ , the iterative process stops, and no new Lagrangian constraints are introduced. If not, the process continues.

For the second iteration in the constraint algorithm, we choose a basis  $\{\mathbf{P}_{(n)}^{[2]}\}$  for  $\overline{\ker \Omega_L(\mathbf{u})} / \mathcal{G}$  and the arbitrary functions  $\{u_{[2]}^m\}$  such that for  $m = 1, \dots, N_0^{[2]}$ ,  $u_{[2]}^m$  are linear combinations of  $u_{[1]}^m$  that lie in the kernel  $\Gamma_{nm}^{[1]}$ . We once again require that  $\mathbf{P}_{(n)}^{[2]} \in \text{Sym}\mathcal{L}$  for  $n = 1, \dots, N_{\text{Sym}\mathcal{L}}$ . Then

$$\bar{\mathbf{X}}_{EL}^{[2]} = \bar{\mathbf{X}}_L^{[2]} + \sum_{m=1}^{N_0^{[2]}} u_{[2]}^m [\mathbf{P}_{(m)}^{[2]}],$$

with

$$\bar{\mathbf{X}}_L^{[2]} = \bar{\mathbf{X}}_L^{[1]} + \sum_{m=N_0^{[2]}+1}^{N_0^{[1]}} u_{[2]}^m [\mathbf{P}_{(m)}^{[2]}].$$

Here, the functions  $u_{[2]}^m$  for  $m = N_0^{[2]} + 1, \dots, N_0^{[1]}$  have been determined through the constraint analysis of  $\gamma_n^{[1]}$ .

As shown in [1],  $\mathbf{G}u_{[1]}^m = 0$ . Similarly,  $\mathbf{G}\gamma_n^{[2]} = \mathcal{L}_{[\mathbf{G}, \bar{\mathbf{X}}_{EL}^{[2]}} \mathbf{d}\gamma_n^{[2]} = 0$ . Clearly  $\gamma_n^{[2]} \in \overline{\mathcal{F}}$  and we may require  $u_{[2]}^m \in \overline{\mathcal{F}}$  as well. It then follows that  $[\mathbf{P}_{(n)}^{[2]}] \gamma_m^{[2]} = \mathbf{P}_{(n)}^{[2]} \gamma_m^{[2]}$ , and imposing equation (15) on  $\gamma_n^{[2]}$ , gives

$$\sum_{m=1}^{N_0^{[2]}} \Gamma_{nm}^{[2]} u_{[2]}^m = -\langle \mathbf{d}\gamma_n^{[2]} | \bar{\mathbf{X}}_L^{[2]} \rangle, \quad \text{where} \quad \Gamma_{nm}^{[2]} := \langle \mathbf{d}\gamma_n^{[2]} | \mathbf{P}_{(m)}^{[2]} \rangle, \quad n = 1, \dots, N_0^{[2]}. \tag{18}$$

Once again,  $\Gamma_{nm}^{[2]} = \Gamma_{mn}^{[2]}$ , but now on the constraint manifold  $\mathbb{P}_L^{[2]}$ . Moreover, since  $\gamma_n^{[2]} = \gamma_n^{[1]} = 0$  for  $n = 1, \dots, N_{\text{Sym}\mathcal{L}}$ ,  $\Gamma_{nm}^{[2]} = 0 = \Gamma_{mn}^{[2]}$ , and  $\langle \mathbf{d}\gamma_n^{[2]} | \bar{\mathbf{X}}_L^{[2]} \rangle = 0$ . There is once again a reduction of equation (18), and we are left with

$$\sum_{\bar{m}=1}^{N_0^{[2]}-N_{\text{Sym}\mathcal{L}}} \Gamma_{n\bar{m}}^{[2]} u_{[2]}^{\bar{m}+N_{\text{Sym}\mathcal{L}}} = -\langle \mathbf{d}\gamma_n^{[2]} | \bar{\mathbf{X}}_L^{[2]} \rangle.$$

where  $\bar{\Gamma}_{\bar{n}\bar{m}}^{[2]} := \langle \mathbf{d}\gamma_{\bar{n}+N_{\text{Sym}\mathcal{L}}}^{[2]} | \mathbf{P}_{(\bar{m}+N_{\text{Sym}\mathcal{L}})}^{[2]} \rangle$ . As before, the  $N_{\text{Sym}\mathcal{L}}$  arbitrary functions  $u_{[2]}^m$  are not determined, while  $r^{[2]} := \text{rank} \bar{\Gamma}_{\bar{n}\bar{m}}^{[2]}$  of the remaining  $u_{[2]}^m$  for  $m > N_{\text{Sym}\mathcal{L}}$  are. There are now  $N_0^{[3]} = N_0^{[2]} - r^{[2]}$  third-order Lagrangian constraint functions,

$$\gamma_{n_{[3]}}^{[3]} = \langle \mathbf{d}\gamma_{n_{[3]}}^{[2]} | \bar{\mathbf{X}}_L^{[2]} \rangle, \quad n_{[3]} = 1, \dots, N_0^{[3]},$$

with the conditions  $\gamma_{n_{[3]}}^{[3]} = 0$  imposed if necessary. With

$$I_{[3]} := \text{rank} \left\{ \mathbf{d}\gamma_{n_{[1]}}^{[1]}, \mathbf{d}\gamma_{n_{[2]}}^{[2]}, \mathbf{d}\gamma_{n_{[3]}}^{[3]} \right\},$$

independent functions in  $C_L^{[3]} := C_L^{[2]} \cup \left\{ \gamma_{n_{[3]}}^{[3]}, \quad n_{[3]} = 1, \dots, N_0^{[3]} \right\}$ , we now have the third-order constraint submanifold,

$$\mathbb{P}_L^{[3]} := \left\{ \mathbf{u} \in \mathbb{P}_L^{[2]} \mid \gamma_{n_{[3]}}^{[3]}(\mathbf{u}) = 0, \quad n_{[3]} = 1, \dots, N_0^{[3]} \right\}.$$

Once again, the process stops when  $I_{[3]} = I_{[2]}$  or  $I_{[3]} = 2D$ . However, if  $I_{[2]} < I_{[3]} < 2D$ , the process continues until at the  $n_F$ -iteration when either  $I_{[n_F]} = I_{[n_F-1]}$  or  $I_{[n_F]} = 2D$ .

Following [1], the end result of this algorithm is

**Table 1.** A summary of the symmetries of the three examples considered in this paper. With the exception of the  $I_{[1]}$  column, the numerical entries are the dimensionality of the vector spaces listed along the first row. Notice the case where the Euler–Lagrange equations of motion has a generalized Lie symmetry while the action itself does not. In all three examples,  $\dim(\text{Sym}\mathcal{L}) = \dim(\overline{\text{Sol}}_{\mathbb{P}_L^{[n_F]}})$ .

Action	Potential	$\overline{\ker \Omega_L} / \mathcal{G}$	Sym	Sym $\mathcal{L}$	$I_{[1]}$	$\overline{\text{Sol}}_{\mathbb{P}_L^{[n_F]}}$
$S_1$	$V_{AS}(\hat{q}^a)$	1	1	1	0	1
	$V_{sph}( q ) + V_{AS}(\hat{q}^a)$	1	1	0	1	0
	$V(q^a)$	1	0	0	1	0
$S_2$	$\frac{\lambda}{2} \left[ \frac{q_1^a}{ q_2 } \frac{d}{dt} \left( \frac{q_{2a}}{ q_1 } \right) - \frac{q_2^a}{ q_1 } \frac{d}{dt} \left( \frac{q_{1a}}{ q_2 } \right) \right]$	2	1	1	1	1
$S_3$	0	2	2	2	0	2

1. A submanifold  $\mathbb{P}_L^{[n_F]} \subset \mathbb{P}_L$  on which dynamics takes place.
2. A collection  $C_L^{[n_F]} \subset \overline{\mathcal{F}}$  of constraint functions of order 1 to  $n_F$ .
3. A second-order, Euler–Lagrange vector field

$$\overline{\mathbf{X}}_{EL}^{[n_F]} = \overline{\mathbf{X}}_L^{[n_F]} + \sum_{m=1}^{N_0^{[n_F]}} u_{[n_F]}^m(\mathbf{u}) [\mathbf{P}_{(m)}^{[n_F]}],$$

with  $N_0^{[n_F]} \geq N_{\text{Sym}\mathcal{L}}$  arbitrary functions  $u_{[n_F]}^m(\mathbf{u}) \in \overline{\mathcal{F}}$  for  $m = 1, \dots, N_0^{[n_F]}$ , and

$$\overline{\mathbf{X}}_L^{[n_F]} = \overline{\mathbf{X}}_L^{[1]} + \sum_{m=N_0^{[n_F]}+1}^{N_0^{[1]}} u_{[n_F]}^m(\mathbf{u}) [\mathbf{P}_{(m)}^{[n_F]}],$$

where the  $N_0^{[1]} - N_0^{[n_F]}$  functions  $u_{[n_F]}^m(\mathbf{u}) \in \overline{\mathcal{F}}$ ,  $m = N_0^{[n_F]} + 1, \dots, N_0^{[1]}$ , have been uniquely determined through the constraint algorithm.

We assume that the rank of  $\Gamma_{mm}^{[l]}$  is constant on  $\mathbb{P}_L$  for each  $l = 1, \dots, n_F$ , and that  $\mathbb{P}_L^{[n_F]}$  is non-empty.

The end result of the constraint algorithm  $\overline{\mathbf{X}}_{EL}^{[n_F]}$  is still a SOELVF, and we define the collection of such vector fields as

$$\overline{\text{Sol}}_{\mathbb{P}_L^{[n_F]}} := \{ \overline{\mathbf{X}}_{EL} \in \overline{\text{Sol}} \mid \mathcal{L}_{\overline{\mathbf{X}}_{EL}} \beta = 0 \}.$$

Importantly,  $\dim \overline{\text{Sol}}_{\mathbb{P}_L^{[n_F]}} \geq N_{\text{Sym}\mathcal{L}}$ .

### 5. The generalized lie symmetries of three dynamical systems

Three examples of dynamical systems with almost regular Lagrangians were introduced in [1]. In that paper the focus of these examples was on the explicit construction of the dynamical structures needed to describe and predict motion in the Lagrangian phase space, and to show that these structures are projectable to the Hamiltonian phase space. We return to these examples here, but with the focus now being on the generalized Lie symmetries of each, and the application of the results we have found in this paper. In particular, we are interested in the dimensionality of the symmetry groups for each of the systems as compared to the dimensionality of  $\overline{\text{Sol}}_{\mathbb{P}_L^{[n_F]}}$  of each. A summary of our results can be found in table 1.

#### 5.1. A lagrangian with and without a generalized lie symmetry

Whether the action

$$S_1 := \int \left[ \frac{1}{2} m \left( \frac{d\hat{q}}{dt} \right)^2 - V(q^a) \right] dt,$$

with  $|q| = \sqrt{q^a q_a}$  and  $\hat{q}^a := q^a / |q|$ ,  $a = 1, \dots, D$ , has a generalized Lie symmetry depends on the choice of potential  $V(q)$ . With one choice both the Lagrangian and the Euler–Lagrange equations of motion have a generalized gauge symmetry; with a second choice the equations of motion has a generalized Lie symmetry while the Lagrangian does not; and with a third choice neither the action nor the equations of motion have a symmetry. Irrespective of the choice of  $V(q)$ , however,  $L$  is singular, demonstrating that while all actions with a generalized Lie symmetry have a singular Lagrangian, not all singular Lagrangians have a generalized Lie symmetry.

Defining  $\Pi_{ab}(q) := \delta_{ab} - \hat{q}_a \hat{q}_b$ , we find

$$\Omega_M = \frac{m}{|q|^2} \Pi_{ab}(q) \mathbf{d}q^a \wedge \mathbf{d}v^b, \quad \Omega_F = \frac{m}{|q|^3} (\hat{q} \cdot \mathbf{d}q) \wedge (v \cdot \Pi(q) \cdot \mathbf{d}q).$$

Then  $\mathcal{C}$  and  $\mathcal{G}$  are spanned by  $\mathbf{U}_{(1)}^q = \hat{q} \cdot \partial/\partial q$  and  $\mathbf{U}_{(1)}^v = \hat{q} \cdot \partial/\partial v$ , respectively, while  $\overline{\ker \Omega_L(u)}$  is spanned by  $\mathbf{U}_{(1)}^v$  and

$$P_{(1)} = \hat{q} \cdot \frac{\partial}{\partial q} + \frac{1}{|q|} v \cdot \frac{\partial}{\partial v}.$$

That  $\dim(\overline{\ker \Omega_L(u)}/\mathcal{G}) = 1$  then follows.

The energy is

$$E = \frac{1}{2} \frac{m}{|q|^2} v \cdot \Pi(q) \cdot v + V(q),$$

and there is only one first-order Lagrangian constraint,

$$\gamma^{[1]} = \mathbf{U}_{(1)}^q V, \tag{19}$$

so that  $\beta[\mathbf{X}_{EL}] = \gamma^{[1]} \Theta_q^{(1)}$ , where  $\Theta_q^{(1)} = \hat{q} \cdot \mathbf{d}q$ . Using equation (19),

$$\mathfrak{L}_{\mathbf{P}_{(1)}} \beta = \mathbf{d}[\mathbf{U}_{(1)}^q V] - \frac{1}{|q|^2} \hat{q} \cdot \frac{\partial}{\partial q} \left( \Pi_a^b(q) \frac{\partial V}{\partial \hat{q}^b} \right) \mathbf{d}q^a. \tag{20}$$

Whether or not  $\mathcal{S}ym$  or  $Sym\mathcal{L}$  is empty therefore depends on the symmetries of  $V(q)$ , as we would expect.

It was found in [1] that

$$\overline{\mathbf{X}}_L = v \cdot \Pi(q) \cdot \frac{\partial}{\partial q} + \frac{(\hat{q} \cdot v)}{|q|} v \cdot \frac{\partial}{\partial v} - \frac{|q|^2}{m} \frac{\partial V}{\partial q} \cdot \Pi(q) \cdot \frac{\partial}{\partial v},$$

and a general SOELVF is given by  $\overline{\mathbf{X}}_{EL} = \overline{\mathbf{X}}_L + u(u) \mathbf{P}_{(1)}$ , where  $u(u) \in \overline{\mathcal{F}}$ . As the constraint algorithm gives

$$\mathfrak{L}_{\overline{\mathbf{X}}_{EL}} \gamma^{[1]} = v \cdot \Pi \cdot \frac{\partial \gamma^{[1]}}{\partial q} + u(u) \mathbf{U}_{(1)}^q \gamma^{[1]}, \tag{21}$$

whether or not  $u(u)$  (which in turn determines the dimensionality of  $\overline{\mathcal{S}ol}_{\mathbb{P}_L^{[n]}}$ ) is determined by the constraint condition also depends on the symmetries of  $V(q)$ .

There are three cases to consider.

*The symmetric potential*

For  $\mathbf{P}_{(1)}$  to generate a generalized Lie symmetry of the Euler–Lagrange equations of motion,

$$0 = \frac{1}{|q|^2} \hat{q} \cdot \frac{\partial}{\partial q} \left( \Pi_a^b(q) \frac{\partial V}{\partial \hat{q}^b} \right),$$

and as such the potential must satisfy

$$\frac{\partial V}{\partial \hat{q}^a} = \frac{\partial V_{AS}(\hat{q}^a)}{\partial \hat{q}^a},$$

where  $V_{AS}$  is a function of  $\hat{q}^a$  only. It follows that  $\mathbf{P}_{(1)}$  generates a generalized Lie symmetry iff  $V(q^a) = V_{Sph}(|q|) + V_{AS}(\hat{q}^a)$ , where  $V_{Sph}$  is a function of  $|q|$  only. For this potential,  $\mathcal{S}ym$  is one-dimensional, and is spanned by  $\mathbf{P}_{(1)}$ .

The constraint condition equation (21) for this potential reduces to

$$0 = u(u) \frac{d^2 V_{Sph}(q)}{d|q|^2},$$

which must be satisfied on  $\mathbb{P}_L^{[1]}$ . There are two possibilities.

*Case 1:*  $\frac{d^2 V_{Sph}}{d|q|^2} = 0$ .

Then  $V_{Sph}(|q|) = a|q| + b$ , but since

$$\gamma^{[1]} = \frac{dV_{Sph}}{d|q|} = a,$$

the condition  $\gamma^{[1]} = 0$  requires  $a = 0$ . It then follows that  $\gamma^{[1]} = 0$  on  $\mathbb{P}_L$ , and thus  $Sym\mathcal{L}$  is one-dimensional; it also is spanned by  $\mathbf{P}_{(1)}$ . The potential is then  $V(q) = b + V_{AS}(\hat{q}^a)$ , and the Lagrangian is invariant under the transformation  $q^a \rightarrow \alpha q^a$ , where  $\alpha$  is an arbitrary, nonvanishing function on  $\mathbb{P}_L$ . This Lagrangian therefore has a local conformal symmetry. Importantly, the function  $u(u)$  is not determined, and thus the dynamics of the particle is given only up to an arbitrary function. Then  $\dim(\overline{\mathcal{S}ol}_{\mathbb{P}_L^{[n]}}) = 1$  as well, and is also spanned by  $\mathbf{P}_{(1)}$ .

Case 2:  $\frac{d^2 V_{sph}}{d|q|^2} \neq 0$ .

In this case  $u(u) = 0$ , and the dynamics of the particle is completely determined by its initial data;  $\overline{\text{Sol}}_{\mathbb{P}_L^{[nF]}} = \{\overline{\mathbf{X}}_L\}$ . The first-order Lagrangian constraint  $\gamma^{[1]}$  does not vanish automatically, but instead defines a surface on  $\mathbf{P}_L$ , and it follows that  $\text{Sym}\mathcal{L} = \emptyset$ . Indeed, the action's lack of a local gauge symmetry in this case can be seen explicitly. Equation (19) reduces to

$$0 = \hat{q} \cdot \frac{\partial V_{sph}}{\partial q},$$

and for dynamics to be possible the set of solutions

$$\left\{ R_i \in \mathbb{R} \mid \left. \frac{dV_{sph}}{d|q|} \right|_{R_i} = 0 \right\},$$

must be non-empty. Dynamics are on the surfaces  $|q| - R_i = 0$  where the potential reduces to  $V(q) = V_{sph}(R_i) + V_{AS}(\hat{q}^a)$ . This reduced potential has the same symmetry as the potential  $V_{AS}(\hat{q}^a)$  in Case 1, and it is for this reason that the Euler–Lagrange equations of motion have the same generalized Lie symmetry for the two cases. This is explicitly shown in the appendix.

In Case 1 the action has a local conformal symmetry, while in Case 2 it does not. (In [1] it was erroneously stated that in this case the action has a global rotational symmetry.) The Lagrangian for the two cases do not have the same invariances, resulting in one case dynamics that are determined only up to an arbitrary  $u(u)$ , and in the other case to a  $u(u) = 0$  and dynamics that are instead completely determined by the choice of initial data.

*The asymmetric potential*

For a general  $V$  the second term in equation (20) does not vanish,  $\mathbf{P}_{(1)}$  does not generate a symmetry of the equations of motion, and  $\text{Sym} = \{\emptyset\}$ . As before,  $\gamma^{[1]}$  does not vanish, and thus  $\text{Sym}\mathcal{L} = \{\emptyset\}$  as well. Furthermore, as equation (21) results in

$$\overline{\mathbf{X}}_E = \overline{\mathbf{X}}_L - \frac{v \cdot \Pi \cdot \frac{\partial \gamma^{[1]}}{\partial q}}{q^2 \mathbf{U}_{(1)}^q \gamma^{[1]}} [\mathbf{P}_{(1)}],$$

the dynamics of the particle is uniquely determined by its initial data, and  $\overline{\text{Sol}}_{\mathbb{P}_L^{[nF]}} = \{\overline{\mathbf{X}}_{EL}\}$  once again consists of a single point.

**5.2. A lagrangian with local conformal symmetry**

The action,

$$S_2 := \int \left\{ \frac{1}{2} m \left( \frac{d\hat{q}_1}{dt} \right)^2 + \frac{1}{2} m \left( \frac{d\hat{q}_2}{dt} \right)^2 + \frac{\lambda}{2} \left[ \frac{q_1^a}{|q_2|} \frac{d}{dt} \left( \frac{q_{2a}}{|q_1|} \right) - \frac{q_2^a}{|q_1|} \frac{d}{dt} \left( \frac{q_{1a}}{|q_2|} \right) \right] \right\} dt,$$

where  $a = 1, \dots, d, D = 2d$ , describes an interacting, two particle system that is invariant under the local conformal transformation  $q_1^a \rightarrow \alpha(u)q_1^a$  and  $q_2^a \rightarrow \alpha(u)q_2^a$ .

With

$$\begin{aligned} \Omega_M &= \frac{m}{|q_1|^2} \Pi_{ab}(q_1) \mathbf{d}q_1^a \wedge \mathbf{d}v_1^b + \frac{m}{|q_2|^2} \Pi_{ab}(q_2) \mathbf{d}q_2^a \wedge \mathbf{d}v_2^b, \text{ and} \\ \Omega_F &= \frac{m}{|q_1|^3} (\hat{q}_1 \cdot \mathbf{d}q_1) \wedge (v_1 \cdot \Pi(q_1) \cdot \mathbf{d}q_1) + \frac{m}{|q_2|^3} (\hat{q}_2 \cdot \mathbf{d}q_2) \wedge (v_2 \cdot \Pi(q_2) \cdot \mathbf{d}q_2) \\ &\quad - \frac{\lambda}{|q_1||q_2|} [\mathbf{d}q_1^a \wedge (\Pi(q_2) \cdot \mathbf{d}q_2)_a + (\Pi(q_1) \cdot \mathbf{d}q_1)_a \wedge \mathbf{d}q_2^a - (\Pi(q_1) \cdot \mathbf{d}q_1)^a \wedge (\Pi(q_2) \cdot \mathbf{d}q_2)_a] \\ &\quad - \frac{\lambda}{|q_1|^2} (\hat{q}_1 \cdot \mathbf{d}q_1) \wedge (\hat{q}_2 \cdot \Pi(q_1) \cdot \mathbf{d}q_1) + \frac{\lambda}{|q_2|^2} (\hat{q}_2 \cdot \mathbf{d}q_2) \wedge (\hat{q}_1 \cdot \Pi(q_2) \cdot \mathbf{d}q_2), \end{aligned}$$

$\mathcal{C}$  and  $\mathcal{G}$  are two-dimensional, and are spanned by

$$\mathbf{U}_{(1)}^q = \hat{q}_1 \cdot \frac{\partial}{\partial q_1}, \quad \mathbf{U}_{(2)}^q = \hat{q}_2 \cdot \frac{\partial}{\partial q_2}, \quad \text{and} \quad \mathbf{U}_{(1)}^v = \hat{q}_1 \cdot \frac{\partial}{\partial v_1}, \quad \mathbf{U}_{(2)}^v = \hat{q}_2 \cdot \frac{\partial}{\partial v_2},$$

respectively. The reduced  $\overline{F} = 0$ , and  $\overline{\ker \Omega_L(u)}$  is spanned by  $\mathbf{U}_{(1)}^v, \mathbf{U}_{(2)}^v$ ,

$$\mathbf{P}_{(+)} = q_1 \cdot \frac{\partial}{\partial q_1} + q_2 \cdot \frac{\partial}{\partial q_2} + v_1 \cdot \frac{\partial}{\partial v_1} + v_2 \cdot \frac{\partial}{\partial v_2},$$



and

$$\mathbf{P}_{(-)} = q_1 \cdot \frac{\partial}{\partial q_1} - q_2 \cdot \frac{\partial}{\partial q_2} + v_1 \cdot \frac{\partial}{\partial v_1} - v_2 \cdot \frac{\partial}{\partial v_2} - \frac{2\lambda}{m} \left[ \frac{|q_1|}{|q_2|} q_2 \cdot \frac{\partial}{\partial v_1} + \frac{|q_2|}{|q_1|} q_1 \cdot \frac{\partial}{\partial v_2} \right].$$

As such,  $\dim(\overline{\ker \Omega_L})/\mathcal{G} = 2$ .

The energy is

$$E = \frac{1}{2} \frac{m}{|q_1|^2} v_1 \cdot \Pi(q_1) \cdot v_1 + \frac{1}{2} \frac{m}{|q_2|^2} v_2 \cdot \Pi(q_2) \cdot v_2.$$

We find that  $\gamma_{(+)}^{[1]} = 0$  while

$$\gamma_{(-)}^{[1]} = -\frac{2\lambda}{|q_1||q_2|} (q_2 \cdot \Pi(q_1) \cdot v_1 + q_1 \cdot \Pi(q_2) \cdot v_2),$$

giving,

$$\beta[\mathbf{X}_{EL}] = \frac{1}{2} \gamma_{(-)}^{[1]} \left( \frac{\Theta_q^{(1)}}{|q_1|} - \frac{\Theta_q^{(2)}}{|q_2|} \right).$$

Then  $\text{Sym}\mathcal{L}$  is one-dimensional and spanned by  $\mathbf{P}_{(+)}$ . As expected,  $\mathcal{L}_{\mathbf{P}_{(+)}}\beta = 0$ . Because

$$\mathcal{L}_{\mathbf{P}_{(-)}}\beta = -\frac{4\lambda}{m} [1 - (\hat{q}_1 \cdot \hat{q}_2)^2] \left( \frac{\Theta_q^{(1)}}{|q_1|} - \frac{\Theta_q^{(2)}}{|q_2|} \right),$$

Sym is also one-dimensional, and is also spanned by  $\mathbf{P}_{(+)}$ .

A general SOELVF is

$$\overline{\mathbf{X}}_{EL} = \overline{\mathbf{X}}_L - \frac{m}{8\lambda^2} \frac{\overline{\mathbf{X}}_L \gamma_{(-)}^{[1]}}{[1 - (\hat{q}_1 \cdot \hat{q}_2)]} [\mathbf{P}_{(-)}] + u^{(+)}(u) [\mathbf{P}_{(+)}], \tag{22}$$

where  $u^{(+)}(u) \in \overline{\mathcal{F}}$ , and from [1],

$$\begin{aligned} \overline{\mathbf{X}}_L &= v_1 \cdot \Pi(q_1) \cdot \frac{\partial}{\partial q_1} + v_2 \cdot \Pi(q_2) \cdot \frac{\partial}{\partial q_2} \\ &+ \left( \frac{\hat{q}_1 \cdot v_1}{|q_1|} \right) v_1 \cdot \Pi(q_1) \cdot \frac{\partial}{\partial v_1} + \left( \frac{\hat{q}_2 \cdot v_2}{|q_2|} \right) v_2 \cdot \Pi(q_2) \cdot \frac{\partial}{\partial v_2} \\ &+ \frac{\lambda}{m} \left( \frac{|q_1|}{|q_2|} v_2 \cdot \Pi(q_2) \cdot \Pi(q_1) \cdot \frac{\partial}{\partial v_1} - \frac{|q_2|}{|q_1|} v_1 \cdot \Pi(q_1) \cdot \Pi(q_2) \cdot \frac{\partial}{\partial v_2} \right), \end{aligned}$$

after the constraint algorithm is applied. Equation (22) is a consequence of the identity  $\langle \mathbf{d}\gamma_{(+)}^{[1]} | \overline{\mathbf{X}}_L \rangle = 0$  and

$$\begin{aligned} -\frac{1}{2\lambda} \langle \mathbf{d}\gamma_{(-)}^{[1]} | \overline{\mathbf{X}}_L \rangle &= -2(\hat{q}_1 \cdot \hat{q}_2) \frac{E}{m} + \frac{2}{|q_1||q_2|} v_1 \cdot \Pi(q_1) \cdot \Pi(q_2) \cdot v_2 \\ &- \frac{\lambda}{m} (\hat{q}_1 \cdot \hat{q}_2) \left[ \frac{v_2}{|q_2|} \cdot \Pi(q_2) \cdot \hat{q}_1 - \frac{v_1}{|q_1|} \cdot \Pi(q_1) \cdot \hat{q}_2 \right]. \end{aligned}$$

We see that  $\overline{\text{Sol}}_{\mathbb{P}_L^{[1]}}$  is also one-dimensional, and is also spanned by  $\mathbf{P}_{(+)}$ .

### 5.3. A lagrangian with local conformal and time-reparametization invariance

The action

$$S_3 := sm \int \left[ s \left( \frac{d\hat{q}}{dt} \right)^2 \right]^{1/2} dt,$$

where  $s = \pm 1$ , is invariant under both the local conformal transformations,  $q^a \rightarrow \alpha(u)q^a$ , and the reparametization of time  $t \rightarrow \tau(t)$  where  $\tau$  is a monotonically increasing function of  $t$ . Then

$$\Omega_L = \frac{m}{|q|} \frac{P_{ab}(u)}{\sqrt{sv \cdot \Pi(q) \cdot v}} \mathbf{d}q^a \wedge \mathbf{d}v^b,$$

and  $\Omega_F = 0$ . Here,  $a = 1, \dots, D$ ,

$$u_a = \frac{\Pi_{ab}(q)v^b}{\sqrt{sv \cdot \Pi(q) \cdot v}},$$

so that  $u^2 = s$ , while  $P_{ab}(u) = \Pi_{ab}(q) - su_a u_b$ . As such,  $\ker \Omega_L(u) = \ker \Omega_M(u)$ . Both  $\mathcal{C}$  and  $\mathcal{G}$  are two-dimensional, and are spanned by

$$\mathbf{U}_{(1)}^q = \hat{q} \cdot \frac{\partial}{\partial q}, \quad \mathbf{U}_{(2)}^q = u \cdot \frac{\partial}{\partial q}, \quad \text{and} \quad \mathbf{U}_{(1)}^v = \hat{q} \cdot \frac{\partial}{\partial v}, \quad \mathbf{U}_{(2)}^v = u \cdot \frac{\partial}{\partial v},$$

respectively. It follows that  $\dim(\overline{\ker \Omega_L} / \mathcal{G}) = 2$ .

Because this system is fully constrained,  $E = 0$ . As  $\Omega_F = 0$  as well, there are no Lagrangian constraints. It follows that  $\text{Sym}\mathcal{L}$  is two dimensional and spanned by  $\mathbf{U}_{(1)}^q$  and  $\mathbf{U}_{(2)}^q$ . As  $\beta = 0$  as well,  $\text{Sym}$  is also two dimensional, and is also spanned by  $\mathbf{U}_{(1)}^q$  and  $\mathbf{U}_{(2)}^q$ .

We found in [1] that  $\overline{\mathbf{X}}_L = 0$ . A general SOELVF is then  $\overline{\mathbf{X}}_{EL} = u^1(u)[\mathbf{U}_{(1)}^q] + u^2(u)[\mathbf{U}_{(2)}^q]$ , with  $u^n(u) \in \overline{\mathcal{F}}$  for  $n = 1, 2$ . It follows that  $\overline{\text{Sol}}_{\mathbb{P}_L^{[mF]}}$  is also two-dimensional, and is spanned by  $\mathbf{U}_{(1)}^q$  and  $\mathbf{U}_{(2)}^q$ , as well.

### 6. Concluding remarks

That each generalized Lie symmetry of the action contributes one arbitrary function to the SOELVF for a dynamical system is known anecdotally, and is a result expected on physical grounds. For almost regular Lagrangians, the appearance in physics of a generalized Lie symmetry is due to a local gauge symmetry of the dynamical system, and thus to the absence of a gauge—the length of vectors for local conformal invariance, or a measure for time for time-reparametrization invariance—for some dynamical property of the system. As the generalized Lie symmetries of the action for an almost regular Lagrangian would have  $N_{\text{Sym}\mathcal{L}}$  of these gauge freedoms, it is reasonable that the absence of these gauges will result in an equal number of arbitrary functions in the SOELVF. An equal number of terms to fix these gauges would then be needed to determine the dynamics of the system uniquely. But while these expectations are reasonable, up to now they have been fulfilled only on a case-by-case basis. This is in great part because the analysis of dynamical systems with a local gauge symmetry has traditionally been done using constrained Hamiltonian mechanics. Such analysis relies on the canonical Hamiltonian, however, and the connection between the canonical Hamiltonian and the symmetries of the Lagrangian is indirect at best, in contrast to the Lagrangian approach followed here. Moreover, the process of determining the total Hamiltonian for the system is often prescriptive, with results that are specific to the system at hand. By focusing on the Lagrangian and on the Lagrangian phase space, we have been able to show for all systems with an almost regular Lagrangian that has a constant rank Lagrangian two-form, a direct link between local gauge symmetries and its dynamics. In particular, it establishes a link between the number of gauge symmetries of the action and the number of arbitrary functions that naturally appear in the evolution of such dynamical systems.

As  $\gamma_{\mathbf{P}}^{[1]} = 0$  for any choice of  $\mathbf{P} \in \text{Sym}\mathcal{L}$ , the vectors in  $\text{Sym}\mathcal{L}$  do not contribute to the first-order constraint manifold  $\mathbb{P}_L^{[1]}$ , and as such do not contribute to the Lagrangian constraint algorithm at this order, or at any higher orders. It is for this reason that the  $N_{\text{Sym}\mathcal{L}}$  arbitrary functions  $u_{[1]}^m$  are not determined by the algorithm, and why these functions will still contribute to  $\overline{\mathbf{X}}_{EL}$  even after the algorithm has been completed. It also means that if second- and higher-order Lagrangian constraints are introduced, they are accidental and cannot be due to the local gauge symmetries of the action. Interestingly, we have yet to find a dynamical system with a Lagrangian that is both almost-regular and has a Lagrangian two-form with constant rank where second- or higher-order Lagrangian constraints are introduced.

This impact of generalized Lie symmetries on the dynamics of particles illustrates the inherent differences between the analysis of the symmetries of regular Lagrangians and that of almost regular Lagrangians. For regular Lagrangians, the generator of the generalized Lie symmetry (at times referred to as a global symmetry) gives rise to a prolongation vector, and the action of this prolongation on the Lagrangian gives the variation of the action,  $\delta S$ , under this symmetry. When the Euler–Lagrange equations of motion are then imposed, the conserved quantity for this symmetry along the path given by the solution of these equations of motion is then obtained. While the generator of the generalized Lie symmetry for the almost regular Lagrangian  $\mathbf{g}_L$  does give a prolongation vector  $\mathbf{pr} \mathbf{g}_L$  equation (3), and while the action of  $\mathbf{pr} \mathbf{g}_L$  on  $L$  does give  $\delta S$ , imposing the Euler–Lagrange equations of motion on  $\delta S$  in equation (4) gives the vacuous statement  $\delta S = 0$ . Instead, the requirement that  $\delta S = 0$  for all paths on  $\mathbb{Q}$  gives the conditions that the generators of the symmetry must satisfy. This in turn shows that the existence of these generators is due solely to the Lagrangian being singular. These conditions then affect the dynamics of the system through  $\gamma_{\mathbf{P}}^{[1]} = 0$ , and in doing so, sets a lower bound to the dimensionality of  $\overline{\text{Sol}}_{\mathbb{P}_L^{[mF]}}$ .

We have found it quite difficult to construct more than one example of a dynamical system that has an almost regular Lagrangian with both a generalized Lie symmetry and a Lagrangian two-form with constant rank on  $\mathbb{P}_L$ . We have, on the other hand, found it quite easy to construct examples of dynamical systems that have an almost regular Lagrangian with a generalized Lie symmetry and a Lagrangian two-form whose rank varies across  $\mathbb{P}_L$ . Indeed, it is the latter case that is the more prevalent one, and yet much of the results of this paper and a good portion of the results of our previous one [1] relies on the condition that the rank of the Lagrangian two-form be constant on  $\mathbb{P}_L$ . This is even more concerning when we realize that these more prevalent systems are expected, by their nature, to have much richer dynamics and mathematical structures (indeed, we have found that such systems often require the introduction of second- or higher-order Lagrangian constraints), and yet it is not known which of the results that have been shown to hold for systems with constant rank Lagrangian two-forms will still hold when the rank varies across  $\mathbb{P}_L$ . Determining the generalized Lie symmetries of these systems; showing that the passage from the Lagrangian to the Hamiltonian phase space is possible; and finding the links between symmetry and dynamics is a necessity for future research.

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## Data availability statement

No new data were created or analysed in this study.

## Appendix

The Euler-Lagrangian equations of motion for the action  $S_1$  is

$$0 = \frac{m}{|q|^3} \Pi_{ab}(q) \dot{q}^b - \frac{2m}{|q|^3} (\hat{q} \cdot \dot{q}) \Pi_{ab}(q) \dot{q}^b + \frac{\partial V}{\partial q^a}. \quad (\text{A.1})$$

Contracting both sides of this equation with  $\hat{q}$  results in the first-order Lagrangian constraint equation (19), and it is clear that dynamics is only possible on this constraint surface. Acting on equation (A.1) with  $\Pi_{ab}(q)$  gives

$$0 = \frac{m}{|q|^3} \Pi_{ab}(q) \dot{q}^b - \frac{2m}{|q|^3} (\hat{q} \cdot \dot{q}) \Pi_{ab}(q) \dot{q}^b + \Pi_a^b(q) \frac{\partial V}{\partial q^b}, \quad (\text{A.2})$$

since  $\Pi_{ac}(q) \Pi_b^c(q) = \Pi_{ab}(q)$ . But in this case  $V(q^a) = V_{sph}(|q|) + V_{AS}(\hat{q})$ , and as

$$\Pi_a^b(q) \frac{\partial V_{sph}}{\partial q^b} = \Pi_a^b(q) \frac{\partial |q|}{\partial q^b} V'_{sph}(|q|) = 0,$$

while the identity

$$\frac{\partial \hat{q}^a}{\partial q^b} = \Pi_b^a(q),$$

ensures that

$$\Pi_a^b(q) \frac{\partial V_{AS}}{\partial q^b} = \frac{\partial V_{AS}}{\partial q^a},$$

equation (A.2) thereby reduces to the same equations of motion for the system as found for *Case 1*. It is for this reason that the two cases have same generalized Lie symmetry.

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