Title
Unreachability of Pointclasses in L(R)

Author
Levinson, Derek James

Publication Date
2023

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA

Los Angeles

Unreachability of Pointclasses in $L(\mathbb{R})$

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Derek James Levinson

2023

ABSTRACT OF THE DISSERTATION

Unreachability of Pointclasses in $L(\mathbb{R})$

by

Derek James Levinson

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2023

Professor Itay Neeman, Chair

This dissertation is a contribution to the genre of applications of inner model theory to descriptive set theory. Applying assumptions of determinacy, we investigate the possible lengths of sequences of distinct sets of reals from a fixed pointclass $\boldsymbol{\Gamma}$.

Substantial work has been done on this question in the case that $\boldsymbol{\Gamma}$ is a level of the projective hierarchy. In [1], Hjorth shows from ZF + AD + DC that there is no sequence of distinct $\boldsymbol{\Sigma^1_2}$ sets of length $\boldsymbol{\delta^1_2}$. Sargsyan extended Hjorth's technique to prove an analogous result for every even level of the projective hierarchy (see [2]).

We show from ZF + AD + DC + V = $L(\mathbb{R})$ that for every inductive-like pointclass $\boldsymbol{\Gamma}$ in $L(\mathbb{R})$, there is no sequence of distinct $\boldsymbol{\Gamma}$ sets of length $(\boldsymbol{\delta_\Gamma})^+$. This is the optimal result for inductive-like $\boldsymbol{\Gamma}$. An essential tool for the proof is Woodin and Steel's computation of $HOD^{L(\mathbb{R})}$ in terms of the direct limit of the system of countable iterates of $M_\omega^\#$. We adapt their method to analyze the direct limit of the system of countable iterates of some Γ-suitable mouse. This allows us code each set in some sequence $\langle A_\alpha : \alpha < \lambda \rangle \subset \boldsymbol{\Gamma}$ by a set of conditions in Woodin's extender algebra at the least Woodin cardinal of this direct limit. The coding sets are contained in the direct limit up to $\boldsymbol{\delta_\Gamma}$, bounding $|\lambda|$ by the successor of $\boldsymbol{\delta_\Gamma}$ in the direct limit. Our approach also gives a new proof of Sargsyan's theorem.

Chapter 1 surveys prior work in this area. Chapter 2 covers background necessary for the

proof of our main result, including some of the descriptive set theory of $L(\mathbb{R})$ and a hasty review of inner model theory. Our main result on inductive-like pointclasses is proven in Chapter 3. Chapter 4 briefly examines how one might apply the techniques of Chapter 3 to obtain analogous results for some projective-like pointclasses in $L(\mathbb{R})$.

The dissertation of Derek James Levinson is approved.

Artem Chernikov

Andrew Scott Marks

Donald A. Martin

Itay Neeman, Committee Chair

University of California, Los Angeles

2023

To Bob Thurston and Jon Mormino.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

Thanks go first and foremost to my advisors Itay Neeman and Grigor Sargsyan. Itay was endlessly generous with his time, meeting with me every week through holidays and paternity leave. He was as patient with me as his toddlers. Grigor suggested the problem tackled in this thesis, and was no less instrumental in solving it. No one could be more generous sharing ideas, credit, or advice.

Thanks also to the other members of my thesis committee: Artem Chernikov, Andrew Marks, and Tony Martin. Artem was the first professor I met at UCLA. I was excited to learn his creative applications of model theory and that it was acceptable to wear shorts. Thanks to Andrew for his efforts in making the logic group such a great community, as well as being one of the best lecturers I have had.

Thanks to all of the graduate students and postdocs I overlapped with at UCLA, who all contributed to my wonderful experience in graduate school. Special thanks go to Riley Thornton, to whose patience I owe answers to my dumbest questions, and Clark Lyons, whose sense of humor was subjected to my dumbest jokes.

Thanks to Matt for believing in me and Nat for doubting me. Of course I owe the most to my family. My father was surely the first person to spark my interest in math. I was scarcely less lucky in my big sister, who seemed to think she was cool because she was the head of the math club, rather than in spite of it. My mother cannot add fractions, but was nonetheless an excellent parent. Finally, to Brenna, who sacrificed more than I did to see this through, and knows where commas are supposed to go.

VITA

2022–2023	Research Assistant, Mathematics Department, UCLA.
2017–2022	Teaching Assistant, Mathematics Department, UCLA.
2017	B.S. (Mathematics), Brown University.

CHAPTER 1

Introduction

Definition 1.0.1. *For a boldface pointclass $\mathbf{\Gamma}$, we say λ is $\mathbf{\Gamma}$-reachable if there is a sequence of distinct $\mathbf{\Gamma}$ sets of length λ and λ is $\mathbf{\Gamma}$-unreachable if λ is not $\mathbf{\Gamma}$-reachable.*

The problem of unreachability is to determine the minimal λ which is $\mathbf{\Gamma}$-unreachable for each pointclass $\mathbf{\Gamma}$. As this problem is trivial assuming the axiom of choice, unreachability is exclusively studied under determinacy assumptions. Under AD, unreachability yields an interesting measure of the complexity of a pointclass. An early result in this area is Harrington's theorem that there is no injection of ω_1 into any pointclass strictly below the pointclass of Borel sets in the Wadge hierarchy (see [3]).

Theorem 1.0.2 (Harrington)**.** *If $\beta < \omega_1$, then ω_1 is $\mathbf{\Pi}^0_{\boldsymbol{\beta}}$-unreachable.*

A recent application of Harrington's theorem was the resolution of the decomposability conjecture by Marks and Day (see [4]).

Prior work on unreachability has focused on levels of the projective hierarchy. Kechris gave a lower bound on the complexity of the pointclass needed to reach $\boldsymbol{\delta}^1_{2n+2}$ (see [5]).

Theorem 1.0.3 (Kechris)**.** *Assume $ZF + AD + DC$. Then $\boldsymbol{\delta}^1_{2n+2}$ is $\mathbf{\Delta}^1_{2n+1}$-unreachable.*

In [5], Kechris conjectured his own result could be strengthened to $\boldsymbol{\delta}^1_{2n+2}$ is $\mathbf{\Delta}^1_{2n+2}$-unreachable. He also makes a second, stronger conjecture that $\boldsymbol{\delta}^1_{2n+2}$ is $\mathbf{\Sigma}^1_{2n+2}$-unreachable. Jackson proved the former in [6].

Theorem 1.0.4 (Jackson)**.** *Assume $ZF + AD + DC$. Then $\boldsymbol{\delta}^1_{2n+2}$ is $\mathbf{\Delta}^1_{2n+2}$-unreachable.*

But the resolution of Kechris's second conjecture eluded the traditional techniques of descriptive set theory. Hjorth proved one case of the conjecture in [1] with the use of inner model theory and an application of the Kechris-Martin theorem.

Theorem 1.0.5 (Hjorth)**.** *Assume $ZF + AD + DC$. Then $\boldsymbol{\delta}^1_2$ is $\boldsymbol{\Sigma}^1_2$-unreachable.*

Kechris also pointed out the following corollary of Hjorth's result.

Corollary 1.0.6. *A $\boldsymbol{\Pi}^1_2$ equivalence relation has either $2^{\aleph_0}$ or $\leq \aleph_1$ equivalence classes.*

Hjorth's use of the Kechris-Martin Theorem in [1] precluded an easy generalization of his technique to other projective pointclasses. The rest of Kechris's second conjecture survived another two decades, until Sargsyan found a modification of Hjorth's proof which generalized to the rest of the projective hierarchy (see [2]).

Theorem 1.0.7 (Sargsyan)**.** *Assume $ZF + AD + DC$. Then $\boldsymbol{\delta}^1_{2n+2}$ is $\boldsymbol{\Sigma}^1_{2n+2}$-unreachable.*

Corollary 1.0.8. *Assume $ZF + AD + DC$. $\boldsymbol{\delta}^1_{2n+2}$ is the least cardinal which is $\boldsymbol{\Sigma}^1_{2n+1}$-unreachable.*

Sargsyan's theorem solves the problem of unreachability for every level of the projective hierarchy. He conjectured an analogous result holds for every regular Suslin pointclass.

Conjecture 1.0.9 (Sargsyan)**.** *Assume AD^+. Suppose κ is a regular Suslin cardinal. Then κ^+ is $S(\kappa)$-unreachable.*

Below, we prove part of Conjecture 1.0.9.

Theorem 1.0.10 (L., Neeman, Sargsyan)**.** *Assume $ZF + AD + DC + V = L(\mathbb{R})$. Suppose $\boldsymbol{\Gamma}$ is an inductive-like pointclass. Then $\boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$ is $\boldsymbol{\Gamma}$-unreachable.*

Theorem 1.0.10 is a special case of Conjecture 1.0.9, since $ZF + AD + DC + V = L(\mathbb{R})$ implies both AD^+ and any inductive-like pointclass is of the form $S(\kappa)$ for some regular Suslin cardinal κ.

Our proof of 1.0.10 extends the inner model theory approach pioneered in [1]. Our technique also gives an alternative proof of Theorem 1.0.7.

We conclude our introduction with another perspective on Conjecture 1.0.9. For λ which are $\mathbf{\Gamma}$-reachable, one might ask how sequences of distinct $\mathbf{\Gamma}$ sets of length λ arise. An obvious example of a sequence of distinct $\mathbf{\Gamma}$ sets is a sequence of strictly increasing (or strictly decreasing) $\mathbf{\Gamma}$ sets. And to a descriptive set theorist, a natural way to procure a sequence of strictly increasing $\mathbf{\Gamma}$ sets is from a norm. This suggests other questions. What is the least cardinal λ such that there is no strictly increasing (or strictly decreasing) sequence of $\mathbf{\Gamma}$ sets of length λ. Is λ the length of a norm whose levels are in $\mathbf{\Gamma}$? And does λ coincide with the least $\mathbf{\Gamma}$-unreachable ordinal?

Significant progress has been made on the first two questions.

Theorem 1.0.11 (Kechris)**.** *Assume $ZF + AD + DC$. Suppose κ is a Suslin cardinal. Then there is a strictly increasing sequence $\langle A_\alpha : \alpha < \kappa \rangle$ contained in $S(\kappa)$.*

In fact the sequence $\langle A_\alpha \rangle$ in Theorem 1.0.11 can be taken to be $A_\alpha = \{x \in \mathbb{R} : \phi_0(x) = \alpha\}$, where $\langle \phi_n : n < \omega \rangle$ is a scale witnessing κ is reliable (see Lemma 3.4 of [7]). In [6], Jackson showed Theorem 1.0.11 is optimal if κ is regular.

Theorem 1.0.12 (Jackson)**.** *Assume $ZF + AD + DC$. Suppose κ is a Suslin cardinal, and κ is either a successor or a regular limit cardinal. Then there is no strictly increasing (or strictly decreasing) sequence $\langle A_\alpha : \alpha < \kappa^+ \rangle$ contained in $S(\kappa)$.*

Theorem 1.0.11 also shows Conjecture 1.0.9 is optimal. The conjecture can be viewed as a partial generalization of 1.0.12 as well as an extension of 1.0.7.

CHAPTER 2

Background

2.1 Some Descriptive Set Theory

We first introduce some basics of descriptive set theory. A good reference for most of this material is [8]. For our purposes, a pointset A will refer to a subset of $\omega^n \times (\omega^\omega)^m$ for some $n, m \in \omega$. Since each $\omega^n \times (\omega^\omega)^m$ is Borel isomorphic to $\mathbb{R}$, we will often abuse notation and consider pointsets as just sets of reals. For pointsets A and B, we write $A \leq_w B$ if there is a continuous function f such that $x \in A \iff f(x) \in B$. Clearly $\leq_w$ is reflexive and transitive. A Wadge degree is an equivalence class of pointsets under the preorder $\leq_w$.

Theorem 2.1.1. $(ZF + AD)$ *The relation* $<_w$ *is wellfounded. If* A *and* B *are pointsets, then either* $A \leq_w B$ *or* $B \leq_w A^c$.

The structure of the Wadge degrees is called the Wadge hierarchy. Theorem 2.1.1 tells us that the Wadge hierarchy is nearly wellordered by $\leq_w$ — any pointset incomparable with A under $\leq_w$ is in the equivalence class of A^c.

We will consider lightface pointclasses to be sets of pointsets closed under preimages by computable functions. We will consider boldface pointclasses to be sets of pointsets closed under preimages by continuous functions. Equivalently, a boldface pointclass is a set of pointsets closed downward under $\leq_w$. Each lightface pointclass Γ has an associated boldface pointclass $\boldsymbol{\Gamma}$ consisting of all pointsets which are preimages of some pointset in Γ by a continuous function.

We say a pointclass Γ is selfdual if for any $A \in \Gamma$, A^c is also in Γ. Γ is closed under projection if for any $A \in \Gamma$, $\exists^{\mathbb{R}} A \in \Gamma$. Γ is closed under coprojection if for any $A \in \Gamma$,

$\forall^{\mathbb{R}} A \in \Gamma$. If A and B are pointsets, we say A is projective in B if A is in the smallest (lightface) pointclass containing B which is closed under recursive substitution, number quantification, projection and coprojection.

A prewellordering $\leq$ is a binary relation which is transitive, reflexive, connected, and wellfounded. A norm on a pointset A is a function $\phi : A \to ON$. For a pointclass Γ, we say ϕ is Γ-norm if there are relations $\leq_{\Gamma}^{\phi}$ and $\leq_{\Gamma^c}^{\phi}$ in Γ and Γ^c, respectively, such that whenever $y \in A$,

$$(x \in A \wedge \phi(x) \leq \phi(y)) \iff x \leq_{\Gamma}^{\phi} y \iff x \leq_{\Gamma^c}^{\phi} y.$$

A scale on A is a sequence $\langle \phi_n : n < \omega \rangle$ of norms on A such that whenever $\langle x_i : i < \omega \rangle \subseteq A$, $lim_{i \to \infty} x_i = x$, and $lim_{i \to \infty} \phi_n(x_i)$ exists for all n, $x \in A$ and $\phi_n(x) \leq lim_{i \to \infty} \phi_n(x_i)$. We say $\langle \phi_n : n < \omega \rangle$ is a Γ-scale if each norm ϕ_n is a Γ-norm (uniformly). Γ is scaled if every pointset in Γ admits a Γ-scale.

For any pointclass Γ, we define

$$\begin{aligned} &\Delta_\Gamma = \Gamma \cap \Gamma^c \text{ and} \\ &\delta_\Gamma = sup\{| \leq^* | : \leq^* \text{ is a prewellordering in } \Delta_\Gamma\}. \end{aligned}$$

. If $\boldsymbol{\Gamma}$ is a boldface pointclass, we will write $\boldsymbol{\Delta_\Gamma}$ and $\boldsymbol{\delta_\Gamma}$ in bold to emphasize this. Let $\Theta = \boldsymbol{\delta_{P(\mathbb{R})}}$. Equivalently, $\Theta = sup\{\alpha : \text{exists a surjection of } \mathbb{R} \text{ onto } \alpha\}$.

For $x \in \mathbb{R}$ and Γ a lightface pointclass, we set

$$C_\Gamma(x) = \{y \in R : y \text{ is } \Delta_\Gamma \text{ in some countable ordinal}\}.$$

. We can extend this definition to $a \in HC$ by setting

$$\begin{aligned} C_\Gamma(a) &= \{b \subseteq a : \text{ for all } x \in \mathbb{R} \text{ coding } a,\, b_x \in C_\Gamma(x)\} \\ &= \{b \subseteq a : \text{ for comeager many } x \in \mathbb{R} \text{ coding } a,\, b_x \in C_\Gamma(x)\}, \end{aligned}$$

where b_x is the real representing b in the coding induced by x. See [9] for the equivalence of these two definitions of $C_\Gamma(a)$.

For a cardinal κ, we say $A \subseteq \omega^\omega$ is κ-Suslin if there is a tree $\mathcal{T} \subset (\omega \times \kappa)^{<\omega}$ such that $A = \rho[\mathcal{T}] = \{x \in \omega^\omega : (\exists f : \omega \to \kappa)(\forall n < \omega)((x(0), f(0)), ..., (x(n), f(n))) \in \mathcal{T}\}$. A pointclass Γ is Suslin if there is a cardinal κ such that Γ is the collection of all κ-Suslin sets.

2.2 The Pointclasses of $L(\mathbb{R})$

We will assume for this section $ZF + DC + AD + V = L(\mathbb{R})$. All of the results in this section are due to Steel and are proven outright or else implicit in [10].

The boldface pointclasses we are interested in all appear in a hierarchy we will now define. If $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$ are non-selfdual pointclasses, say $\{\boldsymbol{\Gamma}, \boldsymbol{\Gamma^c}\} <_w \{\boldsymbol{\Lambda}, \boldsymbol{\Lambda^c}\}$ if $\boldsymbol{\Gamma} \subset \boldsymbol{\Lambda} \cap \boldsymbol{\Lambda^c}$. This is a wellordering by Theorem 2.1.1. For $\alpha < \Theta$, consider the αth pair $\{\boldsymbol{\Gamma}, \boldsymbol{\Gamma^c}\}$ in this wellordering such that $\boldsymbol{\Gamma}$ or $\boldsymbol{\Gamma^c}$ is closed under projection. Let $\boldsymbol{\Sigma^1_\alpha}$ denote whichever of the two is closed under projection — if both are, $\boldsymbol{\Sigma^1_\alpha}$ denotes whichever has the separation property. Let $\boldsymbol{\Pi^1_\alpha} = (\boldsymbol{\Sigma^1_\alpha})^c$. Let $\boldsymbol{\delta^1_\alpha} = \boldsymbol{\delta}_{\boldsymbol{\Sigma^1_\alpha}}$.

The pointclasses $\{\boldsymbol{\Sigma^1_n} : n \in \omega\}$ and $\{\boldsymbol{\Pi^1_n} : n \in \omega\}$ are the usual levels of the projective hierarchy. We will refer to the collection of pointclasses $\{\boldsymbol{\Sigma^1_\alpha} : \alpha \in ON\} \cup \{\boldsymbol{\Pi^1_\alpha} : \alpha \in ON\}$ as the extended projective hierarchy.

We now define a hierarchy slightly coarser than the one above. If $n \in \omega$ and $\alpha \in ON$, we say a pointset A is in the pointclass $\boldsymbol{\Sigma_n}(J_\alpha(\mathbb{R}))$ if there is a Σ_n formula ϕ with real parameters such that $A = \{x : J_\alpha(\mathbb{R}) \models \phi[x]\}$. $\boldsymbol{\Pi_n}(J_\alpha(\mathbb{R}))$ is defined analogously with Π_n-formulas.[1] The Levy hierarchy consists of all pointclasses of the form $\boldsymbol{\Sigma_n}(J_\alpha(\mathbb{R}))$ or $\boldsymbol{\Pi_n}(J_\alpha(\mathbb{R}))$ for some n and α. It is clear any pointclass in the Levy hierarchy equals $\boldsymbol{\Sigma^1_\alpha}$ or $\boldsymbol{\Pi^1_\alpha}$ for some α, but the converse is false.

We write $\boldsymbol{\Sigma^2_1}$ for the pointclass $\boldsymbol{\Sigma_1}(J_\Theta(\mathbb{R}))$ (or equivalently, the pointclass of sets definable by a Σ_1-formula in $L(\mathbb{R})$ from real parameters). Let $\boldsymbol{\delta^2_1} = \boldsymbol{\delta}_{\boldsymbol{\Sigma^2_1}}$.

In this section, we will classify the scaled pointclasses within the Levy hierarchy, relate

[1]See [10] for the definition of $J_\alpha(\mathbb{R})$. Alternatively, the reader will not lose too much of importance by pretending $J_\alpha(\mathbb{R}) = L_\alpha(\mathbb{R})$.

the Levy hierarchy to the extended projective hierarchy, and classify the regular Suslin pointclasses.

2.2.0.1 Classification of Scaled Pointclasses

A Σ_1-gap is a maximal interval $[\alpha, \beta]$ such that for any real x, the Σ_1-theory of x is the same in $J_\alpha(\mathbb{R})$ and $J_\beta(\mathbb{R})$.

We say the gap $[\alpha, \beta]$ is admissible if $J_\alpha(\mathbb{R}) \models KP$, equivalently, if the pointclass $\boldsymbol{\Sigma_1}(J_\alpha(\mathbb{R}))$ is closed under coprojection. Suppose $[\alpha, \beta]$ is an admissible gap. Let $n_\beta \in \omega$ be least such that the pointclass $\boldsymbol{\Sigma_{n_\beta}}(J_\beta(\mathbb{R}))$ is not contained in $J_\beta(\mathbb{R})$. We say $[\alpha, \beta]$ is a strong gap if for any $b \in J_\beta(\mathbb{R})$, there is $\beta' < \beta$ and $b' \in J_{\beta'}(\mathbb{R})$ such that the Σ_{n_β} and Π_{n_β} theories of b' in $J_{\beta'}(\mathbb{R})$ are the same as the Σ_{n_β} and Π_{n_β} theories of b in $J_\beta(\mathbb{R})$. Otherwise, we say $[\alpha, \beta]$ is weak.

Theorem 2.2.1. *Suppose $\boldsymbol{\Gamma}$ is a pointclass in the Levy hierarchy. If $\boldsymbol{\Gamma}$ is scaled, then one of the following holds.*

1. $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}_{2k+1}(J_\alpha(\mathbb{R}))$ *for some $k \in \omega$ and some α beginning an inadmissible gap.*
2. $\boldsymbol{\Gamma} = \boldsymbol{\Pi}_{2k+2}(J_\alpha(\mathbb{R}))$ *for some $k \in \omega$ and some α beginning an inadmissible gap.*
3. $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ *for some α beginning an admissible gap.*
4. $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}_{n_\beta+2k}(J_\beta(\mathbb{R}))$ *for some $k \in \omega$ and some β ending a weak gap.*
5. $\boldsymbol{\Gamma} = \boldsymbol{\Pi}_{n_\beta+2k+1}(J_\beta(\mathbb{R}))$ *for some $k \in \omega$ and some β ending a weak gap.*

Definition 2.2.2. *A self-justifying system (sjs) is a countable set $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ which is closed under complements and has the property that every $B \in \mathcal{B}$ admits a scale $\vec{\psi}$ such that $\leq_{\psi_n} \in \mathcal{B}$ for all n.*

Definition 2.2.3. *Let $z \in \mathbb{R}$ and $\gamma \in ON$. $OD^{<\gamma}(z)$ is the set of $x \in \mathbb{R}$ such that x is ordinal definable from the parameter z in $J_\xi(\mathbb{R})$ for some $\xi < \gamma$. $OD^{<\gamma}$ denotes $OD^{<\gamma}(0)$.*

The proof of Theorem 2.2.1 also gives:

Theorem 2.2.4. *Suppose $[\alpha, \beta]$ is an admissible gap. Let β' be the least ordinal such that there is a scale for a universal $\mathbf{\Pi_1}(J_\alpha(\mathbb{R}))$-set definable over $J_{\beta'}(\mathbb{R})$. Then there is $z \in \mathbb{R}$ and a sjs $\mathcal{B} \subset OD^{<\beta'}(z)$ such that a universal $\mathbf{\Pi_1}(J_\alpha(\mathbb{R}))$-set is in $\mathcal{B}$ and either*

1. *$[\alpha, \beta]$ is weak and $\beta' = \beta$ or*

2. *$[\alpha, \beta]$ is strong and $\beta' = \beta + 1$.*

We say a boldface pointclass $\mathbf{\Gamma}$ is inductive-like if $\mathbf{\Gamma}$ is $\mathbb{R}$-parameterized, has the scale property, and is closed under $\wedge$, $\vee$, $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$, and continuous preimages. A lightface pointclass Γ is inductive-like if Γ is ω-parameterized, has the scale property, and is closed under $\wedge$, $\vee$, $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$, and computable preimages.

Remark 2.2.5. *Suppose $\mathbf{\Gamma}$ is a boldface inductive-like pointclass in $L(\mathbb{R})$. Then*

1. *$\mathbf{\Gamma} = \mathbf{\Sigma_1}(J_\alpha(\mathbb{R}))$ for some α beginning an admissible gap,*

2. *there is $x \in \mathbb{R}$ such that letting Γ be the class of pointsets which are Σ_1-definable over $J_\alpha(\mathbb{R})$ from the parameter x, $\mathbf{\Gamma}$ is the closure of Γ under preimages by continuous functions, and*

3. *$\mathbf{\Gamma} = (\mathbf{\Sigma^2_1})^{\mathbf{\Delta_\Gamma}}$.*

2.2.0.2 Relationship between the Levy Hierarchy and the Extended Projective Hierarchy

Definition 2.2.6. *Suppose $\lambda < \Theta$ is a limit ordinal. We say*

- *λ is type I if $\mathbf{\Sigma^1_\lambda}$ is closed under finite intersection but not countable intersection,*

- *λ is type II if $\mathbf{\Sigma^1_\lambda}$ is not closed under finite intersection,*

- *λ is type III if $\mathbf{\Sigma^1_\lambda}$ is closed under countable intersection but not coprojection, and*

- *λ is type IV if $\boldsymbol{\Sigma}^1_{\boldsymbol{\lambda}}$ is closed under coprojection.*

Let $\langle \delta_\alpha : \alpha < \Theta \rangle$ enumerate the ordinals δ such that $(J_{\delta+1}(\mathbb{R}) \cap P(\mathbb{R})) \backslash (J_\delta(\mathbb{R}) \cap P(\mathbb{R})) \neq \emptyset$. Let n_α be minimal such that $\boldsymbol{\Sigma}_{\boldsymbol{n_\alpha}}(J_{\delta_\alpha}(\mathbb{R})) \not\subset J_{\delta_\alpha}(\mathbb{R})$.

Theorem 2.2.7. *Suppose $\alpha < \Theta$.*

1. *If $\omega\alpha$ is type I, then $\boldsymbol{\Sigma}^1_{\boldsymbol{\omega\alpha+k}} = \boldsymbol{\Sigma}_{\boldsymbol{n_\alpha+k}}(J_{\delta_\alpha}(\mathbb{R}))$ for all $k \in \omega$.*
2. *If $\omega\alpha$ is type II or III, then $\boldsymbol{\Sigma}^1_{\boldsymbol{\omega\alpha+k+1}} = \boldsymbol{\Sigma}_{\boldsymbol{n_\alpha+k}}(J_{\delta_\alpha}(\mathbb{R}))$ for all $k \in \omega$.*
3. *If $\omega\alpha$ is type IV, then $\boldsymbol{\Pi}^1_{\boldsymbol{\omega\alpha}} = \boldsymbol{\Sigma}_{\boldsymbol{n_\alpha}}(J_{\delta_\alpha}(\mathbb{R}))$ and $\boldsymbol{\Sigma}^1_{\boldsymbol{\omega\alpha+k+1}} = \boldsymbol{\Sigma}_{\boldsymbol{n_\alpha+k}}(J_{\delta_\alpha}(\mathbb{R}))$ for all $k \in \omega \backslash \{0\}$.*

2.2.0.3 Classification of Suslin Pointclasses

There is a related classification of the Suslin pointclasses. For $\alpha < \Theta$, let κ_α be the αth Suslin cardinal. Let ν_α be the αth ordinal ν such that $\boldsymbol{\Sigma}^1_{\boldsymbol{\nu}}$ or $\boldsymbol{\Pi}^1_{\boldsymbol{\nu}}$ is scaled.

Theorem 2.2.8. *Let $\lambda < \boldsymbol{\delta}^2_1$ be a limit cardinal and $\nu = sup\{\nu_\alpha : \alpha < \lambda\}$.*

1. *If ν is type I, then for all $k \in \omega$*
    - *$\boldsymbol{\Sigma}^1_{\boldsymbol{\nu+2k}}$ and $\boldsymbol{\Pi}^1_{\boldsymbol{\nu+2k+1}}$ are scaled,*
    - *$S(\kappa_{\lambda+k}) = \boldsymbol{\Sigma}^1_{\boldsymbol{\nu+k+1}}$,*
    - *$\kappa_{\lambda+2k+1} = \boldsymbol{\delta}^1_{\boldsymbol{\nu+2k+1}} = (\kappa_{\lambda+2k})^+$, and*
    - *$cof(\kappa_{\lambda+2k}) = \omega$.*
2. *If ν is type I or III, then for all $k \in \omega$*
    - *$\boldsymbol{\Sigma}^1_{\boldsymbol{\nu+2k+1}}$ and $\boldsymbol{\Pi}^1_{\boldsymbol{\nu+2k}}$ are scaled,*
    - *$S(\kappa_{\lambda+k}) = \boldsymbol{\Sigma}^1_{\boldsymbol{\nu+k+1}}$,*
    - *$\kappa_{\lambda+2k+2} = \boldsymbol{\delta}^1_{\boldsymbol{\nu+2k+2}} = (\kappa_{\lambda+2k+1})^+$, and*

- *$cof(\kappa_{\lambda+2k+1}) = \omega$.*

3. *If ν is type IV, then $\boldsymbol{\Pi^1_\nu}$ is scaled, $S(\kappa_\lambda) = \boldsymbol{\Pi^1_\nu}$, and for all $k \in \omega$, letting $\mu = \nu_{\lambda+1}$,*

   - *$\boldsymbol{\Sigma^1_{\mu+2k}}$ and $\boldsymbol{\Pi^1_{\mu+2k+1}}$ are scaled,*
   - *$S(\kappa_{\lambda+k+1}) = \boldsymbol{\Sigma^1_{\mu+k+1}}$,*
   - *$\kappa_{\lambda+2k+2} = \boldsymbol{\delta^1_{\mu+2k+1}} = (\kappa_{\lambda+2k+1})^+$, and*
   - *$cof(\kappa_{\lambda+2k+1}) = \omega$.*

Corollary 2.2.9. *Suppose $\boldsymbol{\Gamma} = S(\kappa)$ for a regular Suslin cardinal $\kappa \leq \boldsymbol{\delta^2_1}$. Then one of the following holds.*

1. *$\boldsymbol{\Gamma} = \boldsymbol{\Sigma}_{2k+1}(J_\alpha(\mathbb{R}))$ for some $k \in \omega$ and some α beginning an inadmissible gap.*
2. *$\boldsymbol{\Gamma} = \boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ for some α beginning an admissible gap.*
3. *$\boldsymbol{\Gamma} = \boldsymbol{\Sigma}_{n_\beta+2k}(J_\beta(\mathbb{R}))$ for some $k \in \omega$ and some β ending a weak gap.*

Corollary 2.2.10. *If $\boldsymbol{\Gamma} = S(\kappa)$ for a regular Suslin cardinal $\kappa \leq \boldsymbol{\delta^2_1}$, then $\boldsymbol{\Gamma}$ is scaled.*

Corollary 2.2.10 is not true for any Suslin pointclass. In general it may be that the complement of $\boldsymbol{\Gamma}$ is scaled instead.

Theorem 2.2.11. *If $\boldsymbol{\Gamma} = S(\kappa)$ for some $\kappa \leq \boldsymbol{\delta^2_1}$, then either $\boldsymbol{\Gamma}$ or $\boldsymbol{\Gamma^c}$ is scaled.*

2.3 Iteration Strategies on Premice

We first review the notation of premice and iteration trees. We closely follow the presentation of [11], though in significantly less detail.

A potential premouse M is a structure of the form $M = (J^{\vec{E}}_\alpha, \in, \vec{E} \restriction \alpha, E_\alpha)$ for a fine extender sequence $\vec{E} = \langle E_\eta : \eta \leq \alpha \rangle$. We say α is the height of M. E_α is called the top extender of M. We say M is active if $E_\alpha \neq \emptyset$. If $\beta \leq \alpha$, $M|\beta$ represents the premouse

$(J_\beta^{\vec{E}}, \in, \vec{E} \restriction \beta, E_\beta)$. We say N is an initial segment of M and write $N \trianglelefteq M$ if $N = M|\beta$ for some $\beta \leq \alpha$. If also $N \neq M$, we say N is a proper initial segment of M and write $N \triangleleft M$.

Let $\mathcal{C}_0(M)$ be the Σ_0 code of M.[2] The first projectum of M, $\rho_1(M)$, is defined to be the least ordinal α such that there is $A \subset \alpha$ which is not in $\mathcal{C}_0(M)$, but is definable over $\mathcal{C}_0(M)$ by a Σ_1 formula (possibly with parameters). The first standard parameter of M, $p_1(M)$, is the lexicographically least sequence of ordinals in M such that there is a set of ordinals A definable over $\mathcal{C}_0(M)$ from parameters in $p_1(M)$ such that $A \cap \rho_1(M) \notin \mathcal{C}_0(M)$.

For a structure M and $X \subset M$,

$$Hull^M(X) = \{x \in M : x \text{ is definable in } M \text{ from parameters in } X\} \text{ and}$$
$$Hull_1^M(X) = \{x \in M : x \text{ is definable in } M \text{ from parameters in } X \text{ by a } \Sigma_1 \text{ formula}\}.$$

Define the first core of M, $\mathcal{C}_1(M)$, to be the transitive collapse of $Hull_1^{\mathcal{C}_0(M)}(\rho_1(X) \cup \{p_1(M)\})$. We say M is 1-sound if $\mathcal{C}_1(M) = \mathcal{C}_0(M)$. For $n < \omega$, the objects $\rho_n(M)$, $p_n(M)$, and $\mathcal{C}_n(M)$, and the property of n-soundness, can be defined inductively. For example, $\rho_{n+1}(M)$ is roughly the least ordinal α such that there is $A \subset \alpha$ which is not in $\mathcal{C}_n(M)$, but is definable over $\mathcal{C}_n(M)$ by a Σ_{n+1} formula.[3] $p_{n+1}(M)$ and $\mathcal{C}_{n+1}(M)$ are defined analogously. And a premouse is $n+1$-sound if M is n-sound and $\mathcal{C}_{n+1}(M) = \mathcal{C}_n(M)$. For $\eta \in M$, we will say M projects to η if there is $n \in \omega$ such that M is n-sound and $\rho_n(M) \leq \eta$.

We say M is ω-sound if M is n-sound for all $n < \omega$. $\rho_\omega(M)$ is the eventual value of $\rho_n(M)$ and $\mathcal{C}_\omega(M)$ is the eventual value of $\mathcal{C}_n(M)$. M is a premouse if any $N \triangleleft M$ is ω-sound.

Fix $k \leq \omega$ and let M be a k-sound premouse M. We define a game $\mathcal{G}_k(M, \Theta)$ of at most Θ moves. As the game progresses, we define a tree order T on Θ, a sequence of premice $\langle M_\xi^{\mathcal{T}} \rangle$, a sequence of extenders $\langle F_\xi^{\mathcal{T}} \rangle$ used in the game, a set of ordinals $\mathcal{D}^{\mathcal{T}}$, a function $deg^{\mathcal{T}}$ from ordinals into $\omega \cup \{\omega\}$, and a matrix of iteration embeddings $\langle i_{\xi,\zeta} \rangle$. Suppose $\alpha+1$ moves have been played thus far and we have defined $T \restriction \alpha+1$, $\langle M_\xi^{\mathcal{T}} : \xi \leq \alpha \rangle$, $\langle F_\xi^{\mathcal{T}} : \xi < \alpha \rangle$,

[2] The reader who is not interested in the quiddities and quillities of fine structure may pretend $\mathcal{C}_0(M) = M$. The distinction is necessary to prove some theorems in this section, but happily will not appear explicitly in what follows.

[3] Σ_{n+1} is not quite the correct notion of definability for this, but gives the right intuition for our purposes.

$\mathcal{D}^{\mathcal{T}} \cap \alpha + 1$, and $i_{\xi,\zeta}$ for select $\xi, \zeta \leq \alpha$. Player I picks an extender $F_\alpha^{\mathcal{T}}$ from the extender sequence of $M_\alpha^{\mathcal{T}}$. We form $M_{\alpha+1}^{\mathcal{T}} = Ult_n(N, F_\alpha^{\mathcal{T}})$, where $n \leq \omega$ and $N \trianglelefteq M_\beta^{\mathcal{T}}$ for some $\beta \leq \alpha$.[4] Set $deg^{\mathcal{T}}(\alpha+1) = n$. We say $\mathcal{T}$ drops at $\alpha+1$ if $N \triangleleft M_\beta^{\mathcal{T}}$ or $deg^{\mathcal{T}}(\alpha+1) < deg^{\mathcal{T}}(\beta)$. Add $\eta T(\alpha+1)$ to the tree order T for every η such that $\eta = \alpha+1$ or $\eta T \beta$. Put $\alpha+1$ in $\mathcal{D}^{\mathcal{T}}$ if $N \triangleleft M_\beta^{\mathcal{T}}$. Let $i_{\beta,\alpha+1}^{\mathcal{T}} : \mathcal{C}_0(N) \to \mathcal{C}_0(M_{\alpha+1}^{\mathcal{T}})$ be the ultrapower embedding induced by $F_\alpha^{\mathcal{T}}$ and if $\eta T \beta$ and $\mathcal{D}^{\mathcal{T}} \cap (\eta, \alpha+1]_T = \emptyset$, let $i_{\eta,\alpha+1}^{\mathcal{T}} = i_{\beta,\alpha+1}^{\mathcal{T}} \circ i_{\eta,\beta}^{\mathcal{T}}$.

Now suppose λ moves have been played so far for a limit ordinal λ and we have constructed $T \restriction \lambda$, $\mathcal{D}^{\mathcal{T}} \cap \lambda$, $deg^{\mathcal{T}} \restriction \lambda$, and the aforementioned sequences up to λ. Player II picks a cofinal branch b through the tree order constructed thus far such that $\mathcal{D}^{\mathcal{T}} \cap b$ is finite. Let $M_\lambda^{\mathcal{T}}$ be the premouse coded by the direct limit of the system of Σ_0 codes

$$\langle \mathcal{C}_0(M_\alpha^{\mathcal{T}}) : \alpha \in b \wedge \mathcal{D}^{\mathcal{T}} \cap (\alpha, \lambda)_T = \emptyset \rangle$$

and embeddings

$$\langle i_{\alpha,\beta}^{\mathcal{T}} : \alpha, \beta \in b \wedge \alpha T \beta \wedge \mathcal{D}^{\mathcal{T}} \cap (\alpha, \lambda)_T = \emptyset \rangle$$

Add $\beta T \lambda$ to the tree order for all $\beta \in b$. Set $deg^{\mathcal{T}}(\lambda)$ equal to the eventual value of $deg^{\mathcal{T}}(\alpha)$ for $\alpha \in b$. For any β such that $\beta T \lambda$ and $\mathcal{D}^{\mathcal{T}} \cap (\beta, \lambda)_T = \emptyset$, let $i_{\beta,\lambda}^{\mathcal{T}}$ be the natural direct limit embedding. Player II loses the game $\mathcal{G}_k(M, \Theta)$ if he ever fails to pick a valid branch or any model on the sequence $\langle M_\alpha^{\mathcal{T}} \rangle$ is illfounded. Otherwise Player II wins.

A (normal) iteration tree $\mathcal{T}$ is a partial play of the game $\mathcal{G}_k(\Theta)$. We write $lh(\mathcal{T})$ for the ordinal number of moves in the game constructing $\mathcal{T}$. $\mathcal{T}$ is determined by the tree order T, sequence of premice $\langle M_\alpha^{\mathcal{T}} : \alpha < lh(\mathcal{T}) \rangle$, sequence of extenders $\langle F_\alpha^{\mathcal{T}} : \alpha < lh(\mathcal{T}) \rangle$, set $\mathcal{D}^{\mathcal{T}} \subset lh(\mathcal{T})$, function $deg^{\mathcal{T}}$, and matrix of iteration embeddings $\langle i_{\alpha,\beta}^{\mathcal{T}} : \alpha T \beta \wedge \mathcal{D}^{\mathcal{T}} \cap (\alpha, \beta]_T = \emptyset \rangle$. We will omit the superscripts when there is no chance of confusion. If $\mathcal{T}$ has successor length, we call $M_{lh(\mathcal{T})-1}^{\mathcal{T}}$ the "last model" of $\mathcal{T}$. Although the maps $i_{\alpha,\beta}^{\mathcal{T}}$ are technically between the Σ_0 codes of our premice, they induce embeddings between the premice themselves, and we

[4] N are n are chosen to be maximal such that $Ult_n(N, F_\alpha)$ makes sense, unless $D^{\mathcal{T}} \cap [0, \beta]_T = \emptyset$ and $N = M_\beta^{\mathcal{T}}$, in which case we require $n \leq k$. $Ult_n(N, F_\alpha)$ is defined similarly to the usual ultrapower, but using functions which are $r\Sigma_n$-definable over $\mathcal{C}_0(N)$ instead of just functions in N to form the ultrapower.

will ignore the distinction. Suppose $\eta \in M_0^{\mathcal{T}}$. We say that $\mathcal{T}$ is below η if $\mathcal{T}$ can be regarded as an iteration tree on $M_0^{\mathcal{T}}|\eta$. $\mathcal{T}$ is above η if for each $\alpha < lh(\mathcal{T})$, the extender $E_\alpha^{\mathcal{T}}$ has critical point above the image of η under the iteration from $M_0^{\mathcal{T}}$ to $M_\alpha^{\mathcal{T}}$.

A strategy for Player II in the game $G_k(M, \Theta)$ is a partial function Σ which, given an iteration tree $\mathcal{T}$ on M, outputs a maximal branch through $\mathcal{T}$. We require the domain of Σ to be all iteration trees $\mathcal{T}$ on M such that if $\lambda < lh(\mathcal{T})$ is a limit ordinal, then $\Sigma(\mathcal{T} \restriction \lambda) = \{\alpha : \alpha T \lambda\}$.

A (k, Θ)-iteration strategy for M is a winning strategy for Player II in the game $G_k(M, \Theta)$. We say an iteration tree $\mathcal{T}$ is according to a strategy Σ if every branch b picked by Player II in forming $\mathcal{T}$ is the branch chosen by Σ. We may also consider the game $G_k(M, \alpha, \Theta)$. This consists of α rounds. Each round looks like a play of some game $G_l(N, \Theta)$, where $N = M$ in round 0, in round $\beta + 1$, N is an initial segment chosen by Player I of the last model constructed in round β, and if β is a limit, then N is an initial segment chosen by Player I of the direct limit of the last models of the prior rounds. l is also chosen by Player I, subject to some constraints. Player II wins the game so long as he does not lose any individual round. A partial play of $G_k(M, \alpha, \Theta)$ produces a stack of trees $\vec{\mathcal{T}}$. A (k, α, Θ)-iteration strategy is a winning strategy for Player II in the game $G_k(M, \alpha, \Theta)$. We will typically suppress the parameter k in what follows. All the premice which we are concerned with will have an (ω_1, ω_1)-iteration strategy. So we will say a premouse M is iterable if it has such a strategy, although we will typically only need the first round of the game. An iterable premouse is called a mouse.

If $lh(\mathcal{T})$ is a limit ordinal, we set

$$\delta(\mathcal{T}) = sup\{lh(E_\alpha^{\mathcal{T}}) : \alpha < lh(\mathcal{T})\}$$

and

$$\mathcal{M}(\mathcal{T}) = \bigcup_{\alpha < lh(\mathcal{T})} M_\alpha^{\mathcal{T}}|lh(E_\alpha^{\mathcal{T}})$$

If b is a cofinal branch through T such that $\mathcal{D}^{\mathcal{T}} \cap b$ is finite, then $M_b^{\mathcal{T}}$ refers to the direct limit of the models on the branch b. If also $\mathcal{D}^{\mathcal{T}} \cap b = \emptyset$, we will write $i_b^{\mathcal{T}} : M_0^{\mathcal{T}} \to M_b^{\mathcal{T}}$ for the direct limit embedding. We say b is wellfounded if $M_b^{\mathcal{T}}$ is wellfounded. We say b does not drop if there is no $\alpha \in b$ such that $\mathcal{T}$ drops at α.

Suppose M is a premouse with iteration strategy Σ. We say N is an iterate of M if there is an iteration tree $\mathcal{T}$ according to Σ of successor length on M such that N is the last model of $\mathcal{T}$. If in addition there are no drops on the branch $(0, lh(\mathcal{T})-1]$, we say N is a complete iterate of M. We sometimes say "N is a complete iterate of M" when we should also specify the iteration realizing this is according to the relevant iteration strategy for M.

An essential tool in the study of mice is comparison. The coiteration of two (or more) premice M and N is a simultaneous iteration of M and N, constructing iteration trees $\mathcal{T}$ and $\mathcal{U}$ on M and N, respectively. Suppose thus far the coiteration has constructed $\mathcal{T} \restriction \lambda$ and $\mathcal{U} \restriction \lambda$.

If $\lambda = \alpha + 1$, let γ be the least ordinal such that $M_\alpha^{\mathcal{T}}|\gamma \neq M_\alpha^{\mathcal{U}}|\gamma$ (if no such γ exists below the height of $M_\alpha^{\mathcal{T}}$ and $M_\alpha^{\mathcal{U}}$, the coiteration terminates). Letting F_1 be the top extender of $M_\alpha^{\mathcal{T}}|\gamma$ and F_2 be the top extender of $M_\alpha^{\mathcal{U}}|\gamma$, $F_1 \neq \emptyset$ or $F_2 \neq \emptyset$ and $F_1 \neq F_2$. We construct $\mathcal{T} \restriction \lambda + 1$ and $\mathcal{U} \restriction \lambda + 1$ by letting Player I pick the extenders F_1 and F_2 in the next move of the iteration games forming $\mathcal{T} \restriction \lambda$ and $\mathcal{U} \restriction \lambda$, respectively.[5]

If λ is a limit ordinal, let b_1 and b_2 be the branches through $\mathcal{T} \restriction \lambda$ and $\mathcal{U} \restriction \lambda$ chosen by (fixed) iteration strategies for M and N, respectively. Then construct $\mathcal{T} \restriction \lambda + 1$ and $\mathcal{U} \restriction \lambda + 1$ by letting Player II pick b_1 and b_2 in the next move of the iteration games forming $\mathcal{T} \restriction \lambda$ and $\mathcal{U} \restriction \lambda$, respectively.

If M and N are sufficiently iterable, the coiteration terminates with the last model of $\mathcal{T}$ an initial segment of the last model of $\mathcal{U}$, or vice versa. In particular, we have the following theorem.

Theorem 2.3.1 (Comparison Lemma)**.** *Suppose M and N are k-sound premice of size $\leq \theta$ with $(k, \theta^+ + 1)$-iteration strategies. Then the coiteration of M and N produces iteration*

[5] Technically, this does not quite fit our definition on an iteration tree, since it is possible $F_1 = \emptyset$ (or $F_2 = \emptyset$). It is not difficult to extend our definition of iteration tree to include this possibility: We set $M_{\alpha+1}^{\mathcal{T}} = M_\alpha^{\mathcal{T}}$ and let $i_{\alpha,\alpha+1} : M_\alpha^{\mathcal{T}} \to M_{\alpha+1}^{\mathcal{T}}$ be the identity. The other modifications are both obvious and irrelevant in what follows.

trees $\mathcal{T}$ and $\mathcal{U}$ on M and N with last models $M_\alpha^{\mathcal{T}}$ and $M_\alpha^{\mathcal{U}}$, respectively, such that $\alpha < \Theta^+$ and at least one of the following holds:

1. *The branch $(0, \alpha]_T$ does not drop and $M_\alpha^{\mathcal{T}} \trianglelefteq M_\alpha^{\mathcal{U}}$.*
2. *The branch $(0, \alpha]_U$ does not drop and $M_\alpha^{\mathcal{U}} \trianglelefteq M_\alpha^{\mathcal{T}}$.*

We say N outiterates M if 2 does not hold and M outiterates N if 1 does not hold.

Remark 2.3.2. *Suppose M and N are premice, M is k-sound, $\rho_k(M) = \omega$, and N outiterates M. Then $M \triangleleft N$.*

Theorem 2.3.1 and Remark 2.3.2 allow the following definition.

Definition 2.3.3. *Let $n \leq \omega$ and suppose there exists an active mouse satisfying "There exist n Woodins." Then $M_n^\#$ is the minimal active, $\omega_1 + 1$-iterable premouse such that $M \models$ "There exist n Woodins" and there is $k < \omega$ such that M is k-sound and $\rho_k(M_n^\#) = \omega$. Here "minimal" means if N is any active, ω-sound, $\omega_1 + 1$-iterable premouse and $N \models$ "There exist n Woodins," then $M_n^\# \trianglelefteq N$.*

Fact 2.3.4. *AD implies ω_1 is measurable, and thus if M is a premouse with an ω_1-iteration strategy Σ, then Σ extends to an $\omega_1 + 1$-iteration strategy.*

We only need Fact 2.3.4 so that we can apply Theorem 2.3.1 to mice with ω_1-iteration strategies.

All of the mice we shall consider will have the Dodd-Jensen property, which establishes the uniqueness of iteration embeddings.

Definition 2.3.5. *Let M be a premouse with iteration strategy Σ. We say Σ has the Dodd-Jensen property if whenever $\mathcal{T}$ is an iteration tree according to Σ with last model $M_\alpha^{\mathcal{T}}$, $N \trianglelefteq M_\alpha^{\mathcal{T}}$, and $\pi : M \to N$ is a (fine-structural) embedding, then*

1. *$N = M_\alpha^{\mathcal{T}}$,*

2. $[0, \alpha]_{\mathcal{T}}$ *does not drop, and*

3. $i^{\mathcal{T}}_{0,\alpha}(x) \leq_L \pi(x)$ *for any* $x \in M$*, where* $\leq_L$ *is the constructability order on* $M^{\mathcal{T}}_{\alpha}$*.*

We mention one important application of the above theorems. Suppose M is a countable mouse with an $(\omega_1, \omega_1 + 1)$-iteration strategy satisfying the Dodd-Jensen property and N_1 and N_2 are countable, complete iterates of M. Then N_1 and N_2 iterate to the same mouse P by a countable coiteration. Moreover, if $\pi_1 : M \to P$ is the iteration map induced by iterating M to P via N_1 and $\pi_2 : M \to P$ is the analogous map induced by the iteration via N_2, then $\pi_1 = \pi_2$. It follows that the set of all countable, complete iterates of M, together with the iteration maps between them, forms a directed system. A striking example of the power of inner model theory is Woodin and Steel's analysis of $HOD^{L(\mathbb{R})}$ as the direct limit of the system of countable iterates of $M_\omega^{\#}$, developed in [12]. Their techniques are essential for our proof of Theorem 1.0.10.

The following theorem will be used to establish that certain Skolem hulls of mice collapse to initial segments of the mouse.

Theorem 2.3.6 (Condensation)**.** *Let* M *be an* ω*-sound mouse which is* $(\omega, \omega_1, \omega_1 + 1)$*-iterable. Suppose* $\pi : N \to M$ *is elementary and the critical point of* π *is* $\rho_\omega(N)$*. Then either*

1. $N \trianglelefteq M$ *or*

2. There is an extender E *in* M *such that the length of* E *is* $\rho_\omega(N)$ *and* N *is a proper initial segment of* $Ult_0(M, E)$*.*

We can also consider premice built over hereditarily countable sets. We say $a \in HC$ is self-wellordered if there is a wellorder of a which is in $J_1(a)$. Fix some $a \in HC$ which is self-wellordered (for our purposes, we only need to consider the cases $a = \emptyset$, $a \in \mathbb{R}$, or a is itself a premouse). A potential a-premouse $M = (J^{\vec{E}}_\alpha(a), \in, \vec{E}, E_\alpha, a)$ is constructed just like the potential premice above, but is built over a and includes a predicate for a. We also require extenders on the fine extender sequence of M to have critical point above $ON \cap a$. All of the definitions of this section can be relativized to a-premice. It is important that for a-premice we only consider behavior "over" a. For example, $\rho_1(M)$ should be defined as the least ordinal $\alpha \geq ON \cap a$ such that there is $A \subset \alpha$ which is not in $\mathcal{C}_0(M)$, but is definable over $\mathcal{C}_0(M)$ by a Σ_1 formula (possibly with parameters). Similarly, $\mathcal{C}_1(M) = Hull_1^{\mathcal{C}_0(M)}(a \cup \rho_1(M) \cup p_1(M))$. If M is k-sound and $\rho_k(M) = ON \cap a$ for some k, we will say M projects to a.

Premice not built over some $a \in HC$ can equivalently be considered as 0-premice. Note if a is itself a premouse, the universe of an M-premouse is also that of a 0-premouse. So we may analyze some model M as an a-premouse at some times, and as a 0-premouse at others.

2.4 Woodin Cardinals and Iterations

The existence of Woodin cardinals in a mouse is a necessary and sufficient condition for iteration trees with multiple cofinal branches. The existence of Woodins allows us to construct iteration trees which make whichever reals we like generic over a mouse. On the other hand, the absence of Woodins is our best guide to defining iteration strategies; in many cases the lack of Woodinness guarantees an iteration strategy must pick the unique cofinal branch through an iteration tree.

For a model M, let δ_M denote the least Woodin cardinal of M (if one exists) and Ea_M denote Woodin's extender algebra in M at δ_M. Let κ_M be the least cardinal of M which is $< \delta_M$-strong in M. ea will refer to the generic over Ea_M. When considering the product extender algebra $Ea_M \times Ea_M$, we will write $ea_l \times ea_r$ for the generic. ea_r will typically code a pair which we shall write (ea_r^1, ea_r^2). For posets of the form $Col(\omega, X)$, $\dot{g}$ denotes a name for the generic.

Theorem 2.4.1. *Suppose M is a countable mouse and $M \models$ "There is a Woodin cardinal." Then Ea_M is a δ_M-c.c. Boolean algebra and for any $x \in \mathbb{R}$, there is a countable, complete iterate N of M such that x is Ea_N-generic over N.*

Corollary 2.4.2. *Suppose M is a countable mouse and $M \models$ "There is a Woodin cardinal."*

Then there is a countable, complete iterate N of M such that x is $Col(\omega, \delta_N)$-generic over N.

See Section 7.2 of [11] for a proof of Theorem 2.4.1 and its corollary.

Theorem 2.4.3. *Suppose b and c are distinct wellfounded branches of a normal iteration tree $\mathcal{T}$ and $A \subseteq \delta(\mathcal{T})$ is in $M_b^{\mathcal{T}} \cap M_c^{\mathcal{T}}$. Then there is $\kappa < \delta(\mathcal{T})$ such that $M_b^{\mathcal{T}} \models$ "κ is A-reflecting in $\delta(\mathcal{T})$," and this is witnessed by an extender on the sequence of $\mathcal{M}(\mathcal{T})$.*

See 6.10 of [11] for the proof of Theorem 2.4.3. The theorem justifies the following definitions.

Definition 2.4.4. *Suppose b is a wellfounded branch through a normal iteration tree $\mathcal{T}$. Let $\mathcal{Q}(b, \mathcal{T})$ be the least initial segment of $M_b^{\mathcal{T}}$ extending $\mathcal{M}(\mathcal{T})$ such that there is $A \subset \delta(\mathcal{T})$ which is definable over $\mathcal{Q}(b, \mathcal{T})$ and realizes $\delta(\mathcal{T})$ is not Woodin via extenders in $\mathcal{M}(\mathcal{T})$, if such an initial segment exists.*

Definition 2.4.5. *Suppose M is a premouse and $\eta \in M$. We say η is a cutpoint of M if there is no extender on the fine extender sequence of M with critical point less than η and length greater than η. η is a strong cutpoint if there is no extender on the extender sequence of M with critical point less than or equal to η and length greater than η.*

Definition 2.4.6. *Suppose $\mathcal{T}$ is a normal iteration tree. Let $\mathcal{Q}(\mathcal{T})$ be the least $\delta(\mathcal{T})$-sound mouse extending $\mathcal{M}(\mathcal{T})$ and projecting to $\delta(\mathcal{T})$ such that $\delta(\mathcal{T})$ is a strong cutpoint of $\mathcal{Q}(\mathcal{T})$ and there is $A \subset \delta(\mathcal{T})$ which is definable over $\mathcal{Q}(\mathcal{T})$ and realizes $\delta(\mathcal{T})$ is not Woodin via extenders in $\mathcal{M}(\mathcal{T})$, if one exists.*

If follows from Theorem 2.4.3 that there is at most one wellfounded branch b through $\mathcal{T}$ such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq M_b^{\mathcal{T}}$. In many cases, we will be able to locate the branch a strategy Σ chooses as the unique branch which absorbs $\mathcal{Q}(\mathcal{T})$ in this sense.

Note an ω_1-iteration strategy on a countable premouse can be coded by a set of reals.

For $a \in HC$ and a pointclass Γ, this allows us to define

$$Lp^{\Gamma}(a) = \bigcup\{N : N \text{ is an } \omega\text{-sound } a\text{-premouse projecting to } a$$
$$\text{with an } \omega_1\text{-iteration strategy in } \mathbf{\Delta_\Gamma}\}.$$

$Lp^{\Gamma}(a)$ can be reorganized as an a-premouse, which is what we will typically use $Lp^{\Gamma}(a)$ to refer to.

Theorem 2.4.7. *Assume $AD^{L(\mathbb{R})}$. Suppose Γ is a (lightface) inductive-like pointclass in $L(\mathbb{R})$ and $a \in HC$. Then $C_\Gamma(a) = Lp^{\Gamma}(a) \cap P(a)$.*

Remark 2.4.8. *Suppose a and b are countable, transitive sets and $a \in b$. It is easy to see from the definition of C_Γ that $C_\Gamma(a) \subseteq C_\Gamma(b)$. This, and the theorem above, implies $Lp^{\Gamma}(a) \subseteq Lp^{\Gamma}(b)$.*

2.5 The Mitchell-Steel Construction

We shall require several methods of building an a-premouse inside a premouse M which contains a. Our main tool for this purpose is the fully backgrounded Mitchell-Steel construction developed in [13]. This section reviews the construction and its properties.

We say a premouse M is reliable if $\mathcal{C}_\omega(M)$ exists and is universal and solid. As we shall see in a moment, we will end the Mitchell-Steel construction if we reach a premouse which is not reliable. [13] defines reliable to include the stronger property that $\mathcal{C}_\omega(M)$ is iterable. But the weaker properties of universality and solidity are enough to propagate the construction, and our weaker requirement ensures the construction does not end prematurely when performed inside a mouse. The definitions of universality and solidity can be found in [11]. In all of the cases relevant to us, universality and solidity are guaranteed and the reader will lose little by taking on faith that the construction does not end.

For the moment we will work in V and assume ZFC. Fix $z \in \mathbb{R}$. Define a sequence of z-premice $\langle \mathcal{M}_\xi : \xi \in On \rangle$ inductively as follows.

1. $\mathcal{M}_0 = (V_\omega, \in, \emptyset, \emptyset, z)$

2. Suppose we have constructed $\mathcal{M}_\xi = (J_\alpha^{\vec{E}}, \in, \vec{E}, \emptyset, z)$. Note $\mathcal{M}_\xi$ is a passive premouse. Suppose also there is an extender F^* over V, an extender F over $\mathcal{M}_\xi$, and $\nu < \alpha$ such that

   (a) $V_{\nu+\omega} \subset Ult(V, F^*)$,

   (b) ν is the support of F,

   (c) $F \restriction \nu = F^* \cap ([\nu]^{<\omega} \times \mathcal{M}_\xi)$, and

   (d) $\mathcal{N}_{\xi+1} = (J_\alpha^{\vec{E}}, \in, \vec{E}, F, z)$ is a premouse.

   If $\mathcal{N}_{\xi+1}$ is reliable, let $\mathcal{M}_{\xi+1} = \mathcal{C}_\omega(\mathcal{N}_{\xi+1})$. Otherwise, the construction ends. If there are multiple such F^*, we pick one which minimizes the support of F. We say F^* is the extender used as a background at step $\xi + 1$.

3. Suppose we have constructed $\mathcal{M}_\xi = (J_\alpha^{\vec{E}}, \in, \vec{E}, E_\alpha, z)$ and either $\mathcal{M}_\xi$ is active or $\mathcal{M}_\xi$ is passive and there is no extender F^* as above. Let $\mathcal{N}_{\xi+1} = (J_{\alpha+1}^{\vec{E}^\frown E_\alpha}, \in, \vec{E}^\frown E_\alpha, \emptyset, z)$. If $\mathcal{N}_{\xi+1}$ is reliable, let $\mathcal{M}_{\xi+1} = \mathcal{C}_\omega(\mathcal{N}_{\xi+1})$. Otherwise, the construction ends.

4. Suppose we have constructed $\langle \mathcal{M}_\xi : \xi < \lambda \rangle$ for λ a limit ordinal. Let $\eta = lim\,inf_{\xi<\lambda}(\rho_\omega(\mathcal{M}_\xi)^+)^{\mathcal{M}_\xi}$. Let $\mathcal{N}_\lambda$ be the passive premouse of height η such that $\mathcal{N}_\lambda|\beta = lim_{\xi<\lambda}\,\mathcal{M}_\xi|\beta$ for all $\beta < \eta$. If $\mathcal{N}_\lambda$ is reliable, let $\mathcal{M}_\lambda = \mathcal{C}_\omega(\mathcal{N}_\lambda)$. Otherwise, the construction ends.

Suppose the construction never breaks down. That is, $\mathcal{M}_\xi$ is defined for all $\xi \in On$.

Theorem 2.5.1. *Suppose ζ_0 and ξ are ordinals such that $\zeta_0 < \xi$ and $\kappa = \rho_\omega(\mathcal{M}_\xi) \leq \rho_\omega(\mathcal{M}_\zeta)$ for all $\zeta \geq \zeta_0$. Then $\mathcal{M}_\xi \trianglelefteq \mathcal{M}_\eta$ for all $\eta \geq \xi$. Moreover, $\mathcal{M}_{\xi+1} \models$ "every set has cardinality at most κ."*

Let $\mathcal{M}$ be the class-sized model such that whenever $\xi \in On$ satisfies $\mathcal{M}_\xi \trianglelefteq \mathcal{M}_\eta$ for all $\eta \geq \xi$, $\mathcal{M}_\xi$ is an initial segment of $\mathcal{M}$. We call $\mathcal{M}$ the output of the Mitchell-Steel construction over z. For $\delta \in On$, we call $\mathcal{M}_\delta$ the output of the Mitchell-Steel construction of length δ over z.

Theorem 2.5.2. *Assume ZFC. Suppose δ is minimal such that δ is Woodin in $L(V_\delta)$. Suppose the Mitchell-Steel construction in V_δ does not break down, and let $\mathcal{M}$ be the output of the construction. Then δ in Woodin in $L(\mathcal{M})$.*

See the proof of Theorem 11.3 of [13].

Theorem 2.5.3 (Universality)**.** *Assume ZFC. Let δ be Woodin and $z \in \mathbb{R}$. Assume the Mitchell-Steel construction of length δ over z does not break down. Let N be the output of the construction. Suppose no initial segment of N satisfies "there is a superstrong cardinal." Let W be a premouse over x of height $\leq \delta$, and suppose P and Q are the final models above W and N, respectively, in a successful coiteration. Then $P \trianglelefteq Q$.*

See Theorem 11.1 of [14].

Theorem 2.5.4. *Suppose M is a mouse with Woodin cardinal δ satisfying enough of ZFC and $z \in M \cap \mathbb{R}$. Then the Mitchell-Steel construction of length δ over z done inside M does not break down. Let N be the output of the construction. Then N is a z-mouse of height δ.*

For a premouse M satisfying enough of ZFC and $z \in M \cap \mathbb{R}$, we write $Le[M, z]$ for the output of the Mitchell-Steel construction in M over z (assuming the construction does not break down). $Le[M]$ will refer to $Le[M, \emptyset]$. $Le[M, z]$ is a z-premouse. If M is iterable, so is $Le[M, z]$.

We are most interested in cases in which M is a mouse with a Woodin cardinal δ, no largest cardinal, and no total extenders above δ. Then $Le[M|\delta, z]$ is equal to the Mitchell-Steel construction of length δ over z, done inside M, and $Le[M, z]$ is an initial segment of $L(Le[M|\delta.z])$.

Remark 2.5.5. *Suppose M is a mouse, $z \in M \cap \mathbb{R}$, and κ is inaccessible in M. Let $\langle \mathcal{M}_\xi : \xi < \kappa \rangle$ be the models of the Mitchell-Steel construction in M of length κ over z.*

Suppose an extender is added at step $\xi+1$ in the construction. Let F^*, F, and ν be as in Case 2 of the construction. Then there is $F' \in M|\kappa$ such that $M \models V_{\nu+\omega} \subset Ult(M, F')$ and $F' \cap ([\nu]^{<\omega} \times \mathcal{M}_\xi) = F \restriction \nu$. So we may assume if F^* is used as a background in the construction of length κ, then $F^* \in M|\kappa$.

In particular, if M is a mouse, $z \in M \cap \mathbb{R}$, and κ is inaccessible in M, then $Le[M|\kappa, z]$ equals the Mitchell-Steel construction of length κ over z, done in M.

2.6 S-constructions

Below we outline the S-construction (introduced as the P-construction in [15]).

Suppose $M = (J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma, E_\gamma, a)$ is a countable a-premouse and $\delta \in M$ is a cardinal and cutpoint of M. Suppose $ON \cap \bar{S} = \delta + \omega$, δ is a Woodin cardinal of $\bar{S}$, $\bar{S}$ is definable over M, and there is a generic G (for the version of Woodin's extender algebra with δ propositional letters) such that $\bar{S}[G] = M|\delta+1$. Inductively define a sequence $\langle S_\alpha : \delta + 1 \leq \alpha \leq \gamma \rangle$ as follows. $S_{\delta+1}$ is set to be $\bar{S}$. At a limit λ, $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$. If $M|\lambda$ is active, add a predicate for $E_\lambda \cap S_\lambda$ to S_λ. For the successor step, we define $S_{\alpha+1}$ by constructing one more level over S_α. The construction proceeds until we construct S_γ, or we reach some S_α such that δ is not Woodin in S_α. We refer to S_γ as the maximal S-construction in M over $\bar{S}$ if the construction reaches γ. We are primarily interested in cases where δ is Woodin in M, in which case the construction is guaranteed to reach γ.

Lemma 2.6.1. *Suppose $M, \bar{S}, \delta, \gamma$, and G are as above. Assume also M is iterable, ω-sound, and $\rho_\omega(M) \geq \delta$. If the construction reaches γ, then for each α such that $\delta+1 \leq \alpha \leq \gamma$, S_α is an $\bar{S}$-mouse and $S_\alpha[G] = M|\alpha$. If also $\alpha < \gamma$, or $\alpha = \gamma$ and δ is definably Woodin over S_α, then $\rho_n(S_\alpha) = \rho_n(M|\alpha)$ for all n and S_α is ω-sound.*

Lemma 1.5 of [15] gives everything in Lemma 2.6.1 except the iterability of S_γ. The iteration strategy for S_γ in Lemma 2.6.1 comes from lifting an iteration tree on S_γ to iteration trees on M above δ. In particular, we have:

Fact 2.6.2. *Suppose $M, \bar{S}, \delta, \gamma$, and G are as in Lemma 2.6.1. Then the iteration strategy for S_γ (as an $\bar{S}$-premouse) is projective in the iteration strategy for M restricted to iteration trees above δ.*

The S-construction serves two purposes in what follows. It allows us to "undo" generic extensions from Woodin's extender algebra. And combined with the fully-backgrounded Mitchell-Steel construction, it provides an inner model of a premouse with convenient properties.

Definition 2.6.3. *Let M be a mouse with a Woodin cardinal and $z \in M \cap \mathbb{R}$. Let $\bar{S}$ be the result of constructing one level of the $\mathcal{J}$-hierarchy over $Le[M|\delta_M, z]$. Let $StrLe[M, z]$ denote the maximal S-construction in M over $\bar{S}$.*

CHAPTER 3

The Inductive-Like Case

In this section we will prove Theorem 1.0.10. We now assume $ZF + AD + DC + V = L(\mathbb{R})$ and fix a boldface inductive-like pointclass $\boldsymbol{\Gamma}$. Let $\boldsymbol{\Delta} = \boldsymbol{\Delta_\Gamma}$. By a reflection argument, we may assume $\boldsymbol{\Gamma} \neq \boldsymbol{\Sigma^2_1}$. We then have $\boldsymbol{\Gamma} = \boldsymbol{\Sigma_1}(J_{\alpha_0}(\mathbb{R}))$ for some α_0 beginning an admissible Σ_1-gap $[\alpha_0, \beta_0]$. Fix a lightface pointclass Γ as in Remark 2.2.5 such that $\boldsymbol{\Gamma}$ is the closure of Γ under preimages by continuous functions.

Section 3.1 reviews some results we will need from the core model induction, most importantly the analysis of iteration strategies of suitable mice. In Sections 3.2 through 3.4 we analyze the directed system of iterates of a suitable mouse and show the directed system can be approximated inside a larger suitable mouse. Section 3.5 covers some lemmas about the StrLe construction inside a suitable mouse. Section 3.6 contains a lemma we will use to obtain witnesses for Σ_1 statements inside an initial segment of a suitable mouse. Finally, Theorem 1.0.10 is proven in Section 3.7.

One of the key ideas to our proof of 1.0.10 is a different coding than the one used in [1] and [2]. In [2], $\boldsymbol{\Sigma^1_{2n+2}}$ sets are coded by conditions in the extender algebra at the least Woodin of some complete iterate N of $M^{\#}_{2n+1}$. The reflection argument from [1] ensures a code for each $\boldsymbol{\Sigma^1_{2n+2}}$ set appears below the least $< \delta_N$-strong cardinal κ_N of some iterate N (in fact it gives a uniform bound below κ_N). But this reflection argument depends upon the pointclass $\boldsymbol{\Sigma^1_{2n+2}}$ not being closed under coprojection.

Our proof of 1.0.10 instead codes $\boldsymbol{\Gamma}$-sets by sets of conditions in the extender algebra of some Γ-suitable mouse N. A weaker reflection argument than the reflection in [1] is used to contain each code in $N|\kappa_N$. This weaker reflection is sufficient for the proof. This technique also applies to the projective pointclasses, so our proof of 1.0.10 also gives a new proof of Theorem 1.0.7.

3.1 Suitable Mice

We now review some results from the core model induction. Most of the concepts below are from [16], with some minor additions. We need to work with mice with an inaccessible cardinal above a Woodin, so in Definition 3.1.2 we introduce a modification of the standard notion of a suitable premouse. [16] proves the existence of terms in suitable mice capturing certain sets of reals. We will need analogous lemmas for our modified definition. In fact we require more than is stated in [16] — it is essential for our purposes that there is a canonical term capturing each set. Fortunately, this stronger claim is already implicit in the proofs of [16].

Definition 3.1.1. *Suppose $x \in HC$. Say an x-premouse N is Γ-suitable if N is countable and*

1. *$N \models$ there is exactly one Woodin cardinal δ_N.*
2. *Letting $N_0 = Lp^\Gamma(N|\delta_N)$ and $N_{i+1} = Lp^\Gamma(N_i)$, we have that $N = \bigcup_{i<\omega} N_i$.*
3. *If $\xi < \delta_N$, then $Lp^\Gamma(N|\xi) \models \xi$ is not Woodin.*

Definition 3.1.2. *Suppose $x \in HC$. Say an x-premouse N is Γ-super-suitable (Γ-ss) if N is countable and*

1. *$N \models$ There is exactly one Woodin cardinal δ_N.*
2. *$N \models$ There is exactly one inaccessible cardinal above δ_N. We denote this inaccessible by ν_N.*
3. *Letting $N_0 = Lp^\Gamma(N|\nu_N)$ and $N_{i+1} = Lp^\Gamma(N_i)$, we have that $N = \bigcup_{i<\omega} N_i$.*
4. *For each $\xi \geq \delta_N$, $N|(\xi^+)^N = Lp^\Gamma(N|\xi)$.*

5. *If $\xi < \delta_N$, then $Lp^\Gamma(N|\xi) \models \xi$ is not Woodin.*

Definition 3.1.3. *Let N be a mouse and $\delta \in N$. We say δ is a Γ-Woodin of N if δ is Woodin in $Lp^\Gamma(N|\delta)$.*

A Γ-suitable premouse is a minimal premouse with a Γ-Woodin cardinal which is closed under Lp^Γ, in that none of its initial segments have this property. Similarly, a Γ-ss premouse can be considered a minimal premouse with a Γ-Woodin which is closed under Lp^Γ and has an inaccessible cardinal above its Γ-Woodin.

Definition 3.1.4. *Let $A \subseteq \mathbb{R}$, N a countable premouse, η an uncountable cardinal of N, and $\tau \in N^{Col(\omega,\eta)}$. We say that η weakly captures A over N if whenever g is $Col(\omega,\eta)$-generic over N, $\tau[g] = A \cap N[g]$.*

Lemma 3.1.5. *Suppose $\mathcal{B}$ is a self-justifying system and N and M are transitive models of enough of ZFC such that $N \in M$. Let $\mathcal{C}$ be a comeager set of $Col(\omega, N)$ generics over M and suppose for each $B \in \mathcal{B}$ there is a term $\tau_B \in M$ such that if $g \in \mathcal{C}$, then $\tau_B[g] = B \cap M[g]$. Let $\pi : \bar{M} \to M$ be elementary with $\{N\} \cup \{\tau_B : B \in \mathcal{B}\} \subset ran(\pi)$. Let $(N, \tau_B) = \pi(\bar{N}, \bar{\tau}_B)$. Then whenever g is $Col(\omega, \bar{N})$-generic over $\bar{M}$, $\bar{\tau}_B[g] = B \cap \bar{M}[g]$.*

See Lemma 3.7.2 of [16].

Let β' be the least ordinal greater than α_0 such that there is a scale for a universal $\mathbf{\Pi_1}(J_{\alpha_0}(\mathbb{R}))$ set definable over $J_{\beta'}(\mathbb{R})$. By Theorem 2.2.4, $\beta' = \beta_0$ or $\beta' = \beta_0 + 1$ and there is a self-justifying system $\mathcal{G} = \{G_n : n \in \omega\}$ such that

$$\begin{aligned} G_0 = \{(x, y) : &\ x \text{ codes some transitive set } a \text{ and } y \text{ codes an } \omega\text{-sound } a\text{-premouse} \\ &R \text{ such that } R \text{ projects to } a \text{ and } R \text{ has an } \omega_1\text{-iteration strategy in } \mathbf{\Delta}\}, \end{aligned}$$

G_1 is a universal $\mathbf{\Sigma_1}(J_{\alpha_0}(\mathbb{R}))$-set, and $\mathcal{G}$ is contained in $OD^{<\beta'}(z)$ for some $z \in \mathbb{R}$ (Note $G_0 \in \mathbf{\Gamma}$, by part 3 of Remark 2.2.5). For ease of notation, assume $\mathcal{G} \subset OD^{<\beta'}$.

Definition 3.1.6. *Suppose $B \subset \mathbb{R}$, N is a premouse, and η is a cardinal of N. Let $\tau^N_{B,\eta}$ be the set of pairs $(\sigma, p) \in N$ such that*

1. *σ is a $Col(\omega, \eta)$-standard term for a real,*

2. *$p \in Col(\omega, \eta)$, and*

3. *for comeager many $g \subset Col(\omega, \eta)$ which are $Col(\omega, \eta)$-generic over N such that $p \in g$, $\sigma[g] \in B$.*

For $n \in \omega$, let $\tau^N_{n,\eta} = \tau^N_{G_n,\eta}$ and if N has a Woodin cardinal let $\tau^N_n = \tau^N_{n,\delta_N}$.

Lemma 3.1.7. *Suppose N is a Γ-suitable or Γ-ss premouse, $z \in N$, $B \in OD^{<\beta'}(z)$, and η is a cardinal of N. Then $\tau^N_{B,\eta}$ is in N.*

See the proof of Lemma 3.7.5 of [16]. In Lemma 5.4.3 of [16], Lemma 3.1.5 is used to show:

Lemma 3.1.8 (Woodin)**.** *Suppose $z \in \mathbb{R}$, N is a Γ-suitable (or Γ-ss) z-premouse, and $\mathcal{B}$ is a sjs containing a universal $\boldsymbol{\Sigma_1}(J_{\alpha_0}(\mathbb{R}))$-set such that each $B \in \mathcal{B}$ is $OD^{<\beta'}(z)$. Suppose $\pi : M \to N$ is Σ_1-elementary and for every $B \in \mathcal{B}$ and $\eta \geq \delta_N$, $\tau^N_{B,\eta} \in range(\pi)$. Then*

1. *M is Γ-suitable (Γ-ss) and*

2. *$\pi(\tau^M_{B,\bar{\eta}}) = \tau^N_{B,\eta}$, where $\bar{\eta}$ is such that $\pi(\bar{\eta}) = \eta$.*

As a result of Lemmas 3.1.5 and 3.1.7 we have:

Corollary 3.1.9. *If N is Γ-suitable or Γ-ss and η is an uncountable cardinal of N, then $\tau^N_{n,\eta}$ weakly captures G_n.*

Definition 3.1.10. *Let $\mathcal{T}$ be a normal iteration tree on a Γ-suitable (or Γ-ss) premouse N. Suppose also $\mathcal{T}$ is below δ_N. Say $\mathcal{T}$ is Γ-short if for all limit $\xi \leq lh(\mathcal{T})$, $Lp^\Gamma(\mathcal{M}(\mathcal{T} \restriction \xi)) \models \delta(\mathcal{T} \restriction \xi)$ is not Woodin. Otherwise, say $\mathcal{T}$ is Γ-maximal.*

Definition 3.1.11. *Let N be a Γ-suitable (Γ-ss) premouse and Σ an (ω_1, ω_1)-iteration strategy for N. Say Σ is fullness-preserving if whenever P is an iterate of N by Σ via an iteration below δ_N, then*

1. *if the branch to P does not drop, then P is Γ-suitable (Γ-ss), and*

2. *if the branch to P does drop, then P has an ω_1-iteration strategy in $J_{\alpha_0}(\mathbb{R})$.*

Remark 3.1.12. *Let N be a Γ-suitable (or Γ-ss) mouse with a fullness-preserving iteration strategy Σ. Suppose $P \triangleleft N|\delta_N$, and Σ' is the iteration strategy for P given by restricting the domain of Σ to trees on P. Suppose $\mathcal{T}$ is an iteration tree on P according to Σ'. Then the branch b through $\mathcal{T}$ chosen by Σ' can be determined from $\mathcal{Q}(\mathcal{T})$. And $\mathcal{Q}(\mathcal{T})$ is the unique $\mathcal{M}(\mathcal{T})$-mouse projecting to ω with an iteration strategy in $\boldsymbol{\Delta}$. It follows from Remark* 2.2.5 *and the uniqueness of $\mathcal{Q}(\mathcal{T})$ that Σ' is coded by a set in $\boldsymbol{\Delta}$.*

Definition 3.1.13. *Let $\mathcal{T}$ be a Γ-maximal iteration tree on a Γ-suitable (or Γ-ss) premouse N and let b be a cofinal branch through $\mathcal{T}$. Say b respects $\vec{G}_n$ if $i_b(\tau^N_{k,\eta}) = \tau^{M_b(\mathcal{T})}_{k,i_b(\eta)}$ for all $k < n$ and every cardinal η of N above δ_N.*

Definition 3.1.14. *Let N be a Γ-suitable (or Γ-ss) premouse and Σ a fullness-preserving iteration strategy for N. Say Σ is guided by $\mathcal{G}$ if whenever $\mathcal{T}$ is an iteration tree according to Σ of limit length and $b = \Sigma(\mathcal{T})$, then*

1. *if $\mathcal{T}$ is Γ-short, then $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{Q}(b, \mathcal{T}) \in Lp^{\Gamma}(\mathcal{M}(\mathcal{T}))$, and*

2. *if T is Γ-maximal, then $\Sigma(b)$ respects $\vec{G}_n$ for all $n \in \omega$.*

Lemma 3.1.15. *If N is Γ-suitable (or Γ-ss) and Σ is an ω_1-iteration strategy for N which is guided by $\mathcal{G}$, then Σ is not in $\boldsymbol{\Gamma}$.*

Proof. There is $n \in \omega$ such that G_n is a universal $\boldsymbol{\Gamma^c}$-set. Then $y \in G_n$ if and only if there exists a countable, complete iterate N^* of N according to Σ such that y is Ea_{N^*}-generic over N^* and $y \in \tau_n^{N^*}[g]$. Since $\boldsymbol{\Gamma}$ is closed under projection, if Σ were in $\boldsymbol{\Gamma}$, G_n would also be in $\boldsymbol{\Gamma}$. □

Theorem 3.1.16 (Woodin)**.** *For any $x \in HC$, there is a (unique) ω-sound, Γ-suitable x-mouse W_x projecting to x with a (unique) iteration strategy that is fullness-preserving,* *condenses well,*[1] *and is guided by $\mathcal{G}$. Similarly, there is a (unique) ω-sound, Γ-ss x-mouse M_x projecting to x with a (unique) iteration strategy that is fullness-preserving, condenses well, and is guided by $\mathcal{G}$.*

Chapter 5 of [16] demonstrates the existence of such a Γ-suitable mouse. It is not difficult to see this gives the existence of the required Γ-ss mouse as well.

For any Γ-suitable (or Γ-ss) premouse N and any $n \in \omega$, let

$$\gamma_n^N = Hull^N(\{\tau_i^N : i < n\}) \cap \delta_N.$$

The regularity of δ_N in N implies each γ_n^N is an ordinal. Lemma 3.1.8 can be used to show

Fact 3.1.17. *$\langle \gamma_n^N : n \in \omega \rangle$ is cofinal in δ_N.*

Lemma 3.1.18. *Let $\mathcal{T}$ be a normal iteration tree on a Γ-suitable (or Γ-ss) premouse N and let b and c be branches through $\mathcal{T}$ which respect $\vec{G}_n$. Then $i_b \restriction \gamma_n^N = i_c \restriction \gamma_n^N$. Moreover, if b and c both respect $\vec{G}_n$ for all n, then $b = c$.*

See Lemma 6.25 of [12].

Lemma 3.1.18 implies if b is the branch chosen by the nice iteration strategy for a Γ-suitable premouse given by Theorem 3.1.16 and c is any branch respecting $\vec{G}_n$, then the direct limit maps given by b and c agree up to γ_n^N. In particular, to track the iteration of a Γ-suitable mouse up to some point below its least Woodin, it is sufficient to know finitely many of the sets in $\mathcal{G}$.

[17] presents work of Steel and Woodin analyzing the direct limit of all countable iterates of $M_\omega^\#$. This direct limit cut to its least Woodin is $(HOD||\Theta)^{L(\mathbb{R})}$. [12] goes further in showing that the entire class $HOD^{L(\mathbb{R})}$ is a strategy mouse. The iteration maps through trees on $M_\omega^\#$ are approximated using indiscernibles, analogously to the use of terms in Lemma 3.1.18. These approximations are merged to give an ordinal definable definition of the direct

[1] *In the sense of Definition 5.3.7 of [16].*

limit in $L(\mathbb{R})$. In particular, initial segments of the direct limit maps are definable from finitely many indiscernibles. $M_\omega^\#$ is the mouse corresponding to the pointclass $\boldsymbol{\Sigma^2_1}$. [12] also demonstrates that if the Σ_1-gap $[\alpha_0, \beta_0]$ is of a nice form, a good stand-in for $M_\omega^\#$ is the minimal mouse with ω Woodins whose derived model satisfies a new Σ_1-sentence realized at $J_{\beta_0+1}(\mathbb{R})$. This mouse corresponds to the pointclass $\boldsymbol{\Gamma} = \boldsymbol{\Sigma_1}(J_{\alpha_0}(\mathbb{R}))$. A similar analysis can be performed to internalize the direct limit of this mouse to $J_{\beta_0}(\mathbb{R})$.

Unfortunately, the same technique does not apply to every inductive-like pointclass. Instead we shall analyze the direct limit of a Γ-suitable mouse and prove that portions of the direct limit maps are definable within a Γ-ss mouse.

Our task is simpler in that we only need to reach up to $\boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$, which we show in section 3.2 is below the least Woodin of our direct limit. So a single approximation using only finitely many sets from $\mathcal{G}$ will suffice. Another advantage we have is that there is no harm in working over a real parameter, so we can work in a Γ-ss mouse over a real which codes W_0. On the other hand, we will have some extra work to do in Section 3.3 ensuring enough information about Γ and $\mathcal{G}$ is definable in a Γ-ss mouse before we internalize the directed system in Section 3.4.

[12] also makes use of the fact that the derived model of $M_\omega^\#$ is essentially $L(\mathbb{R})$. So for $x \in M_\omega^\# \cap \mathbb{R}$, a Σ^2_1 statement about x is true if and only if it holds in the derived model of $M_\omega^\#$. In other words, there is a natural way to ask about Σ^2_1 truth inside of $M_\omega^\#$. A second, though minor, inconvenience of having to use a Γ-suitable mouse is we cannot talk about its derived model, since it only has one Woodin. Instead we will use the fine-structural witness condition of [16].

Remark 3.1.19. *We can associate to any Σ_1-formula ϕ a sequence of formulas $\langle \phi^k : k < \omega \rangle$ such that for any ordinal γ and any real z, $J_{\gamma+1}(\mathbb{R}) \models \phi[z] \iff (\exists k) J_\gamma(\mathbb{R}) \models \phi^k[z]$. Moreover, the map $\phi \to \langle \phi^k : k < \omega \rangle$ is recursive.*

Definition 3.1.20. *Suppose $\phi(v)$ is a Σ_1-formula and $z \in \mathbb{R}$. A $\langle \phi, z \rangle$-witness is an ω-sound z-mouse N in which there are $\delta_0 < \ldots < \delta_9$, $\mathcal{S}$, and $\mathcal{T}$ such that N satisfies the formulae* *expressing*

1. *ZFC,*
2. $\delta_0 < \ldots < \delta_9$ *are Woodin,*
3. $\mathcal{S}$ *and* $\mathcal{T}$ *are trees on some* $\omega \times \eta$ *which are absolutely complementing in* $V^{Col(\omega,\delta_9)}$*, and*
4. *For some* $k < \omega$, $\rho[T]$ *is the* Σ_{k+3}*-theory (in the language with names for each real) of* $J_\gamma(\mathbb{R})$*, where* γ *is least such that* $J_\gamma(\mathbb{R}) \models \phi^k[z]$*.*

Other than iterability, the rest of the properties of being a $\langle \phi, z \rangle$-witness are first order. The following two lemmas illustrate the usefulness of this definition.

Lemma 3.1.21. *If there is a* $\langle \phi, z \rangle$*-witness, then* $L(\mathbb{R}) \models \phi[z]$*.*

Lemma 3.1.22. *Suppose* ϕ *is a* Σ_1*-formula,* $z \in \mathbb{R}$*,* γ *is a limit ordinal, and* $J_\gamma(\mathbb{R}) \models \phi[z]$*. Then there is a* $\langle \phi, z \rangle$*-witness* N *such that the iteration strategy for* N *restricted to countable trees is in* $J_\gamma(\mathbb{R})$*. By taking a Skolem hull, we can also ensure* $\rho_\omega(N) = \omega$*.*

3.2 The Direct Limit

Let $W = W_0$ and let $\mathcal{I}$ be the directed system of countable, complete iterates of W according to its (ω_1, ω_1)-iteration strategy. Let M_∞ be the direct limit of $\mathcal{I}$. For $M, N \in \mathcal{I}$ and N an iterate of M, let $\pi_{M,N} : M \to N$ be the iteration map and $\pi_{M,\infty} : M \to M_\infty$ the direct limit map. Here we demonstrate a few properties of M_∞. The proofs of this section are generalizations of arguments in [11] and [12] giving analogous properties of the direct limit of all countable, complete iterates of $M_\omega^\#$.

Lemma 3.2.1. $\kappa_{M_\infty} \leq \boldsymbol{\delta_\Gamma}$ [2]

Proof. Suppose $\xi < \kappa_{M_\infty}$. Let $M \in \mathcal{I}$ and $\bar{\xi} \in M$ be such that $\pi_{M,\infty}(\bar{\xi}) = \xi$. Let P be an initial segment of M such that $\bar{\xi} \in P$ and the largest cardinal of P is a cutpoint of M.

[2] *In fact* $\kappa_{M_\infty} = \boldsymbol{\delta_\Gamma}$*, but we don't need this.*

The iteration strategy Σ for P is in $\mathbf{\Delta}$ by Remark 3.1.12. Let $\mathcal{I}_P$ be the directed system of countable, complete iterates of P by Σ. Then $\bar{\xi}$ is sent to ξ by the direct limit map of this system, since the largest cardinal of P is a cutpoint of M. So a prewellordering of height ξ is projective in Σ and therefore $\boldsymbol{\delta}_{\mathbf{\Gamma}} > \xi$. $\square$

Lemma 3.2.2. $\delta_{M_\infty} > (\boldsymbol{\delta}_{\mathbf{\Gamma}})^+$

Proof. Let Σ be the (ω_1, ω_1)-iteration strategy for W. Recall Σ is not in $\mathbf{\Gamma}$. We will show Σ is in $S(\delta_{M_\infty})\backslash S(\boldsymbol{\delta}_{\mathbf{\Gamma}}^+)$.

Claim 3.2.3. *Σ is δ_{M_∞}-Suslin.*

Proof. Let $\mathcal{T}$ be a tree on $(\omega \times \omega) \times \delta_{M_\infty}$ such that $(x, y, f) \in [\mathcal{T}]$ if and only if x codes a countable iteration tree $\mathcal{S}$ on W of limit length, y codes a cofinal, wellfounded branch b through $\mathcal{S}$, and f codes an embedding $\pi : M_b^{\mathcal{S}} \to M_\infty$ such that $\pi \circ i_b^{\mathcal{S}} = \pi_{W,\infty}$. Let $\Sigma' = \rho[\mathcal{T}]$.

If $(x, y) \in \Sigma$, then x codes an iteration tree $\mathcal{S}$ on W according to Σ and y codes the cofinal, wellfounded branch b through $\mathcal{S}$ chosen by Σ. And $\pi_{M_b^{\mathcal{S}},\infty} \circ i_b^{\mathcal{S}} = \pi_{W,\infty}$. So if $f : \omega \to \delta_{M_\infty}$ codes the embedding $\pi_{M_b^{\mathcal{S}},\infty}$, then $(x, y, f) \in [\mathcal{T}]$. Thus $(x, y) \in \Sigma'$.

On the other hand, suppose $(x, y) \in \Sigma'$ and x codes an iteration tree $\mathcal{S}$ according to Σ. Fix $f : \omega \to \delta_{M_\infty}$ such that $(x, y, f) \in [\mathcal{T}]$. Let b be the branch coded by y and π the embedding coded by f.

Subclaim 3.2.4. *For all n, $\pi_b^{\mathcal{S}}(\tau_n^W) = \tau_n^{M_b^{\mathcal{S}}}$.*

Proof. Let $Q \in \mathcal{I}$ be such that $range(\pi) \subseteq range(\pi_{Q,\infty})$. Let $\pi' = \pi_{Q,\infty}^{-1} \circ \pi$. Then $\pi' : M_b^{\mathcal{S}} \to Q$ and $\pi'(i_b^{\mathcal{S}}(\tau_n^W)) = \tau_n^Q$. Then by Lemma 3.1.5, $i_b^{\mathcal{S}}(\tau_n^W)$ weakly captures G_n for all n. In fact, $i_b^{\mathcal{S}}(\tau_n^W) = \tau_n^{M_b^{\mathcal{S}}}$. $\square$

From the subclaim and the last part of Lemma 3.1.18, we have that $(x, y) \in \Sigma$.

We can now characterize Σ as the set of $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that

1. x codes an iteration tree $\mathcal{S}$ on W of limit length,

2. y codes a cofinal, wellfounded branch through $\mathcal{S}$,

3. $(x, y) \in \Sigma'$, and

4. for any $(x_0, y_0) \leq_T x$ such that x_0 codes a proper initial segment $\mathcal{S}_0$ of $\mathcal{S}$ of limit length and y_0 codes the branch through $\mathcal{S}_0$ determined by $\mathcal{S}$, $(x_0, y_0) \in \Sigma'$.

Condition 4 is just to guarantee $\mathcal{S}$ is in the domain of Σ. It does so because any proper initial segment $\mathcal{S}_0$ of $\mathcal{S}$ is coded by some real computable from x. From this, and the preceding paragraphs, it is clear these conditions characterize Σ. Since Σ' is δ_{M_∞}-Suslin, this characterization of Σ makes plain that Σ is also δ_{M_∞}-Suslin.

$\square$

Claim 3.2.5. $\boldsymbol{\Gamma} = S(\boldsymbol{\delta_\Gamma})$.

Proof. First, let's establish $\boldsymbol{\Gamma}$ is Suslin. Let

$$\Omega = \{\boldsymbol{\Sigma_1}(J_\gamma(\mathbb{R})) : \gamma < \alpha_0 \text{ and } \gamma \text{ begins a } \Sigma_1\text{-gap}\}.$$

It follows from Theorem 2.2.7 that $\boldsymbol{\Gamma}$ is the minimal non-selfdual pointclass closed under projection which contains every pointclass in Ω. Let

$$\Psi = \{\boldsymbol{\Sigma_1}(J_\gamma(\mathbb{R})) \in \Omega : \boldsymbol{\Sigma_1}(J_\gamma(\mathbb{R})) \text{ is Suslin}\}.$$

By Theorem 2.2.8, Ψ is cofinal in Ω. But the minimal Suslin pointclass larger than any element of Ψ is just the minimal non-selfdual pointclass closed under projection which contains every pointclass in Ω (by part 3 of Theorem 2.2.8). Since Ψ is cofinal in Ω, this is $\boldsymbol{\Gamma}$.

So $\boldsymbol{\Gamma} = S(\lambda)$ for some cardinal λ. By the Kunen-Martin Theorem, there is a prewellordering of length λ in $\boldsymbol{\Gamma}$ but no prewellordering of length λ^+. The latter implies that $\lambda \geq \boldsymbol{\delta_\Gamma}$, since $\boldsymbol{\delta_\Gamma}$ is a limit cardinal,[3] and since there are prewellorderings of length α in $\boldsymbol{\Gamma}$ for all $\alpha < \boldsymbol{\delta_\Gamma}$. The former implies that $\lambda \leq \boldsymbol{\delta_\Gamma}$, since there is no prewellordering of length $\boldsymbol{\delta_\Gamma^+}$ in $\boldsymbol{\Gamma}$ (Otherwise a proper initial segment of this prewellordering would be of length $\boldsymbol{\delta_\Gamma}$, giving a prewellordering of length $\boldsymbol{\delta_\Gamma}$ in $\boldsymbol{\Delta}$). So $\boldsymbol{\delta_\Gamma} = \lambda$. $\square$

[3]See Theorem 7D.8 of [8].

By the previous two claims, $\Sigma \in S(\delta_{M_\infty})\backslash S(\boldsymbol{\delta}_{\boldsymbol{\Gamma}})$. In particular, $\delta_{M_\infty} \geq \lambda'$ where λ' is the next Suslin cardinal after $\boldsymbol{\delta}_{\boldsymbol{\Gamma}}$.[4] But $cof(\lambda') = \omega$ by part 3 of Theorem 2.2.8, so $\delta_{M_\infty} \geq \lambda' > \boldsymbol{\delta}_{\boldsymbol{\Gamma}}^+$. □

Lemma 3.2.6. *Suppose $\mu < \delta_{M_\infty}$ is a regular cardinal of M_∞. Then μ is not measurable in M_∞ if and only if μ has cofinality ω in $L(\mathbb{R})$.*

Proof. Suppose μ is not measurable in M_∞. Fix $M \in \mathcal{I}$ and $\bar{\mu}$ such that $\pi_{M,\infty}(\bar{\mu}) = \mu$. Then $\bar{\mu}$ is regular but not measurable in M. Since M is countable, there is a sequence of ordinals $\langle \bar{\xi}_n : n < \omega \rangle$ cofinal in $\bar{\mu}$. Let $\xi_n = \pi_{M,\infty}(\bar{\xi}_n)$. Since $\bar{\mu}$ is regular and not measurable in M, $\pi_{M,\infty}$ is continuous at $\bar{\mu}$ (This is because $\pi_{M,\infty}$ is essentially an iteration embedding — in fact it is an iteration embedding in $V^{Col(\omega,\mathbb{R})}$. And any iteration embedding is continuous at a cardinal which is regular but not measurable, since ultrapower embeddings are continuous at such cardinals). So $\langle \xi_n : n < \omega \rangle$ is cofinal in μ.

Now suppose μ has cofinality ω in $L(\mathbb{R})$. Let $\langle \xi_n : n < \omega \rangle$ be cofinal in μ. Fix $M \in \mathcal{I}$ such that there is $\bar{\mu} \in M$ and $\langle \bar{\xi}_n : n < \omega \rangle \subset M$ with $\pi_{M,\infty}(\bar{\mu}) = \mu$ and $\pi_{M,\infty}(\bar{\xi}_n) = \xi_n$. If μ is measurable in M_∞, then there is a total extender F on the fine extender sequence of M with critical point $\bar{\mu}$. Let M' be the ultrapower of M by F and $j : M \to M'$ the embedding induced by F. Then for any $n < \omega$,

$$\xi_n = \pi_{M,\infty}(\bar{\xi}_n) = \pi_{M',\infty} \circ j(\bar{\xi}_n) = \pi_{M',\infty}(\bar{\xi}_n) < \pi_{M',\infty}(\bar{\mu}) < \pi_{M',\infty} \circ j(\bar{\mu}) = \mu.$$

So $\pi_{M',\infty}(\bar{\mu})$ is an upper bound for $\bar{\xi}_n$ below μ, a contradiction. □

3.3 Definability in Suitable Mice

Lemma 3.3.1. *Suppose N is a premouse satisfying enough of ZFC, ν is a cardinal of N, $Lp^\Gamma(a) \subset N$ for each $a \in N|\nu$, and $\tau \in N^{Col(\omega,\nu)}$ weakly captures G_0. Then the map with domain $N|\nu$ defined by $a \mapsto Lp^\Gamma(a)$ is definable in N from τ.*

[4] In fact $\delta_{M_\infty} = \lambda'$, but we don't need this.

Proof. Recall

$$G_0 = \{(x, y) : x \text{ codes some transitive set } a \text{ and } y \text{ codes an } \omega\text{-sound } a\text{-premouse}$$
$$R \text{ such that } R \text{ projects to } a \text{ and } R \text{ has an } \omega_1\text{-iteration strategy in } \mathbf{\Delta}\}.$$

Fix $a \in N|\nu$. If R is any set in $N|\nu$ and g is any $Col(\omega, \nu)$-generic over N, then there are reals x and y in $N[g]$ coding a and R, respectively. It is easy to see from this that

$$Lp^{\Gamma}(a) = \bigcup\{R \in N : \emptyset \Vdash^{N}_{Col(\omega,\nu)} (\exists x, y)[(x, y) \in \tau \wedge x \text{ codes } a \wedge y \text{ codes } R]\}.$$

□

Corollary 3.3.2. *If P is Γ-ss, then the map with domain $P|\nu_P$ defined by $a \mapsto Lp^{\Gamma}(a)$ is definable in P from τ^{P}_{0,ν_P}.*

Proof. It is clear from Remark 2.4.8 and Corollary 3.1.9 that P and τ^{P}_{0,ν_P} satisfy the conditions of Lemma 3.3.1. □

Lemma 3.3.3. *Suppose P is Γ-ss and $N \in P|\nu_P$ is Γ-suitable. Then $\{\tau^{N}_{n,\mu} : \mu$ is an uncountable cardinal of $N\}$ is definable in P from N and τ^{P}_{n,ν_P} (uniformly in P and N).*

Proof. Let μ be an uncountable cardinal of N.

Note if g is $Col(\omega, \nu_P)$-generic over P and $f \in P$ is a surjection of ν_P onto μ, then $f \circ g$ is P-generic for $Col(\omega, \mu)$. In particular, $f \circ g$ is N-generic for $Col(\omega, \mu)$. Fix such an f which is minimal in the constructibility order of P. Let

$$\tau_{n,\mu} = \{(\sigma, p) : \sigma \text{ is a } Col(\omega, \mu)\text{-standard term for a real}, p \in Col(\omega, \mu),$$
$$\text{and } \emptyset \Vdash^{P}_{Col(\omega,\nu_P)} (\check{p} \in \check{f} \circ \dot{g} \rightarrow \check{\sigma}[\check{f} \circ \dot{g}] \in \tau^{P}_{n,\nu_P})\}$$

It is clear that $\tau_{n,\mu}$ is definable in P from N, μ, and τ^{P}_{n,ν_P}. It suffices to show $\tau_{n,\mu} = \tau^{N}_{n,\mu}$.

$\tau_{n,\mu} \subseteq \tau^{N}_{n,\mu}$ by Definition 3.1.6 and that comeager many $h \subset Col(\omega, \mu)$ which are generic over N are of the form $f \circ g$ for some g which is $Col(\omega, \nu_P)$-generic over P.

On the other hand, suppose $(\sigma, p) \in \tau^N_{n,\mu}$. By Corollary 3.1.9, $\sigma[h] \in G_n$ for any h which is $Col(\omega, \mu)$-generic over N such that $p \in h$. In particular, $\sigma[f \circ g] \in \tau^P_{n,\nu_P}[g]$ for any g which is $Col(\omega, \nu_P)$-generic over P such that $p \in f \circ g$. Thus $(\sigma, p) \in \tau_{n,\mu}$. □

We will also need versions of Corollary 3.3.2 and Lemma 3.3.3 in generic extensions of Γ-ss mice.

Lemma 3.3.4. *Suppose $B \subseteq \mathbb{R}$, P is a premouse, δ is Woodin in P, $\mu \geq \delta$, $\tau \in P^{Col(\omega,\mu)}$ weakly captures B over P, and y is Ea_P-generic over P. Then there is $\tau' \in P[y]^{Col(\omega,\mu)}$ which weakly captures B over $P[y]$. Moreover, τ' is definable in $P[y]$ from τ and y (uniformly).*

Proof. $Col(\omega, \mu)$ is universal for pointclasses of size μ. So there is a complete embedding $\Phi : Ea_p \times Col(\omega, \mu) \to Col(\omega, \mu)$.[5] If g is $Col(\omega, \mu)$-generic over P, let (y_g, f_g) be the $Ea_P \times Col(\omega, \mu)$-generic consisting of all conditions $(p, q) \in Ea_P \times Col(\omega, \mu)$ such that $\Phi((p, q)) \in g$ (see Chapter 7, Theorem 7.5 of [18]). Let

$$\begin{aligned}\tau^* = \{(\sigma, (p, q)) :\ & \sigma \text{ is an } Ea_P\text{-term for a } Col(\omega, \mu)\text{-standard term for a real,} \\ & (p, q) \in Ea_P \times Col(\omega, \mu), \text{ and } \Phi((p, q)) \Vdash^P_{Col(\omega,\mu)} \sigma[y_g][f_g] \in \tau[g]\}.\end{aligned}$$

Claim 3.3.5. *For any (y, f) which is $Ea_P \times Col(\omega, \mu)$-generic over P, $\tau^*[y][f] = B \cap P[y][f]$.*

Proof. Suppose $x \in \tau^*[y][f]$. $x = \sigma[y][f]$ for some $(\sigma, (p, q)) \in \tau^*$ such that $p \in y$ and $q \in f$. Let g be $Col(\omega, \mu)$-generic such that $y_g = y$ and $f_g = f$. In particular, $\Phi((p, q)) \in g$. Then $P[g] \models \sigma[y_g][f_g] \in \tau[g]$. Since $x = \sigma[y_g][f_g]$ and $\tau[g] = B \cap P[g]$, $x \in B \cap P[g]$.

Now suppose $x \in B \cap P[y][f]$. Let σ be an Ea_P-term for a $Col(\omega, \mu)$-standard term for a real such that $x = \sigma[y][f]$.

$\bigcup \Phi''\{(p, q) : (p, q) \in y \times g\}$ is a function $g_1 : S \to \mu$ for some $S \subseteq \omega$. Let

$$\mathbb{Q} = \{r \in Col(\omega, \mu) : domain(r) \cap S = \emptyset\}$$

[5] In the sense of Definition 7.1 of Chapter 7 of [18].

($\mathbb{Q}$ is the quotient of $Col(\omega, \mu)$ by g_1). Let g_2 be $\mathbb{Q}$-generic over $P[g_1]$. Then $g = g_1 \cup g_2$ is $Col(\omega, \mu)$-generic over P.

We have $x \in \tau[g]$. Pick $s \in g$ such that $s \Vdash^P_{Col(\omega,\mu)} \sigma[y_g][f_g] \in \tau[g]$. $s = r_1 \cup r_2$ for some $r_1 \in g_1$ and $r_2 \in g_2$.

Subclaim 3.3.6. $r_1 \Vdash^P_{Col(\omega,\mu)} \sigma[y_g][f_g] \in \tau$.

Proof. Suppose not. Then there is g_2' which is $\mathbb{Q}$-generic over $P[g_1]$ such that, letting $g' = g_1 \cup g_2'$, $\sigma[y_{g'}][f_{g'}] \notin \tau[g']$. $\sigma[y_{g'}][f_{g'}] = x$, since y_g and f_g depend only on $g \restriction S$. But then $x \in (B \cap P[g'])\backslash\tau[g']$, contradicting that τ weakly captures B. $\square$

Pick $p \in y$ and $q \in f$ such that $\Phi((p,q))$ extends r_1. Then $(\sigma,(p,q)) \in \tau^*$. So $x \in \tau^*[y][f]$. $\square$

Let

$$\tau' = \{(\sigma[y], q) : \exists p \in y \text{ such that } (\sigma,(p,q)) \in \tau^*\}.$$

τ' is definable in $P[y]$ from τ and y. It is clear from Claim 3.3.5 that τ' weakly captures B over $P[y]$. $\square$

Lemma 3.3.7. *Let P be Γ-ss and y be Ea_P-generic over P. Then for any $a \in P[y]$, $Lp^\Gamma(a) \subset P[y]$.*

Proof. Let N be a Γ-suitable mouse built over P. N has a Woodin cardinal δ_N above δ_P. The iteration strategy for any proper initial segment of $N|\delta_N$ restricted to trees above δ_P is in $\boldsymbol{\Delta}$. And no initial segment of N above δ_N projects strictly below δ_N. It follows that any cardinal of P remains a cardinal in N. In particular, δ_P remains Woodin in N and y is also Ea_P-generic over N.

Suppose R is an ω-sound a-premouse with an ω_1-iteration strategy in $\boldsymbol{\Delta}$ such that R projects to a. It suffices to show $R \in P[y]$.

Let α be the height of R. Iterating N above R if necessary, we may assume there is a real g which is $Col(\omega, \delta_N)$-generic over N such that some real in $N[y][g]$ codes R. By Lemma 3.3.4, there is a $Col(\omega, \delta_N)$-term τ in $N[y]$ which weakly captures G_0. Then R is the unique premouse in $N[y][g]$ of height α such that if x_a codes a and x_R codes R, then $(x_a, x_R) \in \tau[g]$. By homogeneity of the forcing, for any g' which is $Col(\omega, \delta_N)$-generic over N, there is a premouse $R' \in N[y][g']$ of height α and reals x_a and $x_{R'}$ in $N[y][g']$ coding a and R', respectively, such that $(x_a, x_{R'}) \in \tau[g']$. The uniqueness of R implies $R \in N[y]$. Since R is coded by a subset of a, $R \in P[y]$. $\square$

Corollary 3.3.8. *If P is Γ-ss and y is Ea_P-generic over P, then the map with domain $P[y]|\nu_P$ defined by $a \mapsto Lp^{\Gamma}(a)$ is definable in $P[y]$ from τ^P_{0,ν_P} and y (uniformly in P and y).*

Proof. $P[y]$ is Lp^{Γ}-closed by Lemma 3.3.7. Then by Lemma 3.3.1, the map $a \to Lp^{\Gamma}(a)$ with domain $P[y]|\nu_P$ is definable from any term $\tau \in P[y]^{Col(\omega,\nu_P)}$ which weakly captures G_0 over $P[y]$.

Lemma 3.3.4 shows there is a term $\tau \in P[y]^{Col(\omega,\mu)}$ which weakly captures G_0 over $P[y]$ and is definable from τ^P_{0,ν_P} and y in $P[y]$. $\square$

Corollary 3.3.9. *Suppose P is Γ-ss, y is Ea_P-generic over P, and $N \in P[y]|\nu_P$ is Γ-suitable. Then $\{\tau^N_{n,\mu} : \mu$ is an uncountable cardinal of $N\}$ is definable in $P[y]$ from N, y, and τ^P_{n,ν_P} (uniformly in P, y, and N).*

Proof. This is by the proof of Lemma 3.3.3, using from Lemma 3.3.4 that there is a term in P which weakly captures G_n over $P[y]$ and is definable from τ^P_{n,ν_P} and y. $\square$

3.4 Internalizing the Direct Limit

Let $x_0 \in \mathbb{R}$ be any real which is Turing above some real coding W and consider some M which is a countable, complete iterate of M_{x_0}. For elements of $M|\nu_M$, being a Γ-suitable premouse, a Γ-short iteration tree, or a Γ-maximal iteration tree is definable over M from

τ^M_{0,ν_M} (This follows easily from Corollary 3.3.2). Let

$$\mathcal{I}^M = \{P \in M|\nu_M : P \in \mathcal{I}\}.$$

Lemma 3.4.1. *Let $\mathcal{T} \in M|\nu_M$ be a Γ-short tree on some Γ-suitable $P \in M$. Then the branch b picked by the iteration strategy for P is in M and b is definable in M from $\mathcal{T}$ and τ^M_{0,ν_M} (uniformly). In particular, $M^{\mathcal{T}}_b$ and the iteration map $i^{\mathcal{T}}_b : P \to M^{\mathcal{T}}_b$ are definable in M from $\mathcal{T}$ and τ^M_{0,ν_M}.*

Proof. Let g be $Col(\omega, \nu_M)$-generic over M. Note b is the unique branch through $\mathcal{T}$ which absorbs $\mathcal{Q}(\mathcal{T})$. So by Shoenfield absoluteness, $b \in M[g]$ (in $M[g]$ the existence of such a branch is a Σ^1_2 statement about reals). But b is independent of the generic g, so $b \in M$.

It then follows from Corollary 3.3.2 that b, and therefore also $M^{\mathcal{T}}_b$ and $i^{\mathcal{T}}_b$, are definable in M from τ^M_{0,ν_M}. □

Corollary 3.4.2. *Suppose $P \in \mathcal{I}^M$ and Σ is the iteration strategy for P. Suppose also $\mathcal{T} \in M|\nu_M$ is a normal iteration tree on P below δ_P of limit length. Whether $\mathcal{T}$ is according to Σ is definable in M from parameter τ^M_{0,ν_M} by a formula independent of $\mathcal{T}$ and the choice of Γ-ss mouse M.*

Lemma 3.4.3. *Suppose $P, Q \in \mathcal{I}^M$. Then there is $R \in \mathcal{I}^M$ such that R is a complete iterate of both P and Q by normal iteration trees in $M|\nu_M$. Moreover, R is definable in M from P, Q, and τ^M_{0,ν_M} (uniformly).*

Proof. We perform a coiteration of P and Q inside M. Suppose so far from the coiteration we have obtained iteration trees $\mathcal{T}$ and $\mathcal{U}$ on P and Q, respectively.

Suppose $\mathcal{T}$ and $\mathcal{U}$ have successor length. Let P' and Q' be the last models of $\mathcal{T}$ and $\mathcal{U}$, respectively. First consider the case $P' \trianglelefteq Q'$ or $P' \trianglelefteq Q'$. If either is a proper initial segment of the other, or there are any drops on the branches to P' or Q', we have violated the Dodd-Jensen property. So $P' = Q'$ and P' is a common complete iterate of P and Q. Otherwise, we continue the coiteration as usual by applying the extender at the least point of disagreement between the last models of $\mathcal{T}$ and $\mathcal{U}$, respectively.

Now suppose $\mathcal{T}$ and $\mathcal{U}$ are of limit length. In this case $\mathcal{M}(\mathcal{T}) = \mathcal{M}(\mathcal{U})$. If $\mathcal{T}$ is Γ-short, so is $\mathcal{U}$, and by Lemma 3.4.1 M can identify the branches the iteration strategies for P and Q pick through $\mathcal{T}$ and $\mathcal{U}$, respectively. So the coiteration can be continued inside M. Otherwise, $\mathcal{T}$ and $\mathcal{U}$ are Γ-maximal. In this case let R be the unique Γ-suitable mouse extending $\mathcal{M}(\mathcal{T})$. R is just the result of applying Lp^{Γ} to $\mathcal{M}(\mathcal{T})$ ω times, so M can identify R by Lemma 3.3.1. Then R is a complete iterate of P and Q.

It only remains to show that when the coiteration terminates trees $\mathcal{T}$ and $\mathcal{U}$ have length less than ν_M. We cannot directly apply Theorem 2.3.1 — M cannot locate branches through Γ-maximal trees, so P and Q are not iterable in M. However, the proof of 2.3.1 still gives $\mathcal{T}$ and $\mathcal{U}$ have length less than the cardinal successor of $max\{|P|^M, |Q|^M\}$ in M. Since ν_M is inaccessible in M, this implies $lh(\mathcal{T})$ and $lh(\mathcal{U})$ are less than ν_M. $\square$

The lemma implies $\mathcal{I}^M$ is a directed system. $\mathcal{I}^M$ is countable and contained in $\mathcal{I}$, so we may define the direct limit $\mathcal{H}^M$ of $\mathcal{I}^M$, and $\mathcal{H}^M \in \mathcal{I}$. Let

$$\begin{aligned}\tilde{\mathcal{I}}^M = \{P \in \mathcal{I}^M : &\ P \text{ is realized to be a complete iterate of } W \\ &\text{by a tree in } M|\nu_M\}.\end{aligned}$$

. $\tilde{\mathcal{I}}^M$ is definable in M by Corollary 3.4.2.

Lemma 3.4.4. *$\tilde{\mathcal{I}}^M$ is cofinal in $\mathcal{I}^M$. In particular, the direct limit of $\tilde{\mathcal{I}}^M$ is $\mathcal{H}^M$.*

Proof. Suppose $P \in \mathcal{I}^M$. By Lemma 3.4.3, there is $R \in \mathcal{I}^M$ which is a common, complete iterate of both P and W by trees in M. Then R is below P in $\mathcal{I}^M$ and $R \in \tilde{\mathcal{I}}^M$. $\square$

Lemma 3.4.5. *Suppose $P \in \mathcal{I}^M$. Let Σ be the (unique) iteration strategy for P. Suppose $\mathcal{T} \in M|\nu_M$ is an iteration tree on P according to Σ. Let $b = \Sigma(\mathcal{T})$ and let $Q = M_b(\mathcal{T})$. Then Q is definable in M from $\mathcal{T}$ and τ^M_{0,ν_M}. And $\pi_{P,Q} \upharpoonright \gamma^P_n$ is definable in M from $\mathcal{T}$ and $\langle \tau^M_{k,\nu_M} : k < n\rangle$ (uniformly).*

Proof. If $\mathcal{T}$ is Γ-short this is by Lemma 3.4.1.

Suppose $\mathcal{T}$ is Γ-maximal. Then $Q = \bigcup_{i<\omega} Q_i$, where $Q_0 = \mathcal{M}(\mathcal{T})$ and $Q_{i+1} = Lp^{\Gamma}(Q_i)$. So Q is definable from $\mathcal{M}(\mathcal{T})$ and τ^M_{0,ν_M} by Corollary 3.3.2. And $\pi_{P,Q} \restriction \gamma^P_n = \pi_c \restriction \gamma^P_n$, where c is any branch through $\mathcal{T}$ respecting $\vec{G}_n$. The argument of Lemma 3.4.1 shows there is a branch c in M respecting $\vec{G}_n$. Then $\pi_{P,Q} \restriction \gamma^P_n = \pi_c \restriction \gamma^P_n$ for any wellfounded branch $c \in M$ through $\mathcal{T}$ such that $\pi_c(\langle \tau^P_k : k < n\rangle) = \langle \tau^Q_k : k < n\rangle$. $\langle \tau^P_k : k < n\rangle$ and $\langle \tau^Q_k : k < n\rangle$ are definable in M from P, Q, and $\langle \tau^M_{k,\nu_M} : k < n\rangle$ by Lemma 3.3.3. So $\pi_{P,Q} \restriction \gamma^P_n$ is definable in M from $\mathcal{T}$ and $\langle \tau^M_{k,\nu_M} : k < n\rangle$. $\square$

It follows from the previous lemma that for any $P \in \mathcal{I}^M$, $\pi_{P,\mathcal{H}^M} \restriction \gamma^P_n$ is definable in M from P and $\langle \tau^M_{k,\nu_M} : k < n\rangle$ (uniformly in M). The same lemmas hold in $M[y]$ for y Ea_M-generic over M. In particular, we have:

Lemma 3.4.6. *Suppose y is Ea_M-generic over M and $P \in \mathcal{I} \cap M[y]|\nu_M$. Let Σ be the (unique) iteration strategy for P. Suppose $\mathcal{T} \in M[y]|\nu_M$ is an iteration tree on P according to Σ. Let $b = \Sigma(\mathcal{T})$ and let $Q = M_b(\mathcal{T})$. Then Q is definable in $M[y]$ from $\mathcal{T}$ and τ^M_{0,ν_M}. And $\pi_{P,Q} \restriction \gamma^P_n$ is definable in $M[y]$ from $\mathcal{T}$ and $\langle \tau^M_{k,\nu_M} : k < n\rangle$ (uniformly). Moreover, the definition is independent not just of the choice of Γ-ss mouse M, but also of the generic y.*

Lemma 3.4.7. *Suppose $p \in Ea_M$ and $\dot{S}$ is an Ea_M-name in $M|\nu_M$ such that $p \Vdash_{Ea_M}$ "$\dot{S}$ is a complete iterate of W." Then there is $R \in \tilde{I}^M$ such that R is a complete iterate of $S[y]$ for every $y \in \mathbb{R}$ which is Ea_M-generic over M. Moreover, we can pick R such that R is (uniformly) definable in M from parameters $\dot{S}$ and p.*

Proof. Let $\mathbb{P}$ be the finite support product $\Pi_{j<\omega}\mathbb{P}_j$, where each $\mathbb{P}_j$ is a copy of the part of Ea_M below p. Let H be $\mathbb{P}$-generic over M. We can represent H as $\Pi_{j<\omega}H_j$, where H_j is $\mathbb{P}_j$-generic over M. Let $\dot{S}_j$ be a $\mathbb{P}$-name for $\dot{S}[H_j]$. Let $S_j = \dot{S}_j[H]$ for $j \in \omega$ and $S_{-1} = W$.

Lemma 3.4.6 tells us that $M[H]$ can perform the simultaneous coiteration of all of the S_j for $j \in [-1, \omega)$. The proof of Lemma 2.3.1 gives that this coiteration terminates after fewer than ν_M steps. Let R_j be the last model of the iteration tree on S_j produced by the coiteration. Since each S_j is a complete iterate of W, the Dodd-Jensen property implies there are no drops on the branches from S_j to R_j and $R_j = R_i$ for all $i, j \in [-1, \omega)$. Let $R = R_j$ for some (equivalently all) $j \in [-1, \omega)$. Then R is a complete iterate of M and R is a complete iterate of S_j for each $j \in \omega$. Let $\mathcal{U}$ be the iteration tree on W from the coiteration.

Claim 3.4.8. *R is independent of the choice of generic H.*

Proof. Code R by a set of ordinals X contained in ν_M. Let $\dot{X}$ be a name for X. If R is not independent of H, then there is $\alpha < \nu_M$ and $q_1, q_2 \in \mathbb{P}$ such that $q_1 \Vdash \check{\alpha} \in \dot{X}$ and $q_2 \Vdash \check{\alpha} \notin \dot{X}$.

Let $N > max(support(q_2))$. Let $\bar{q}_1$ be the condition q_1 shifted over by N — that is, $support(\bar{q}_1) = \{j \in [N, \omega) : j - N \in support(q_1)\}$ and for $j \in support(\bar{q}_1)$, $\bar{q}_1(j) = q_1(j - N)$. So $\bar{q}_1$ is compatible with q_2 and by symmetry, $\bar{q}_1 \Vdash \check{\alpha} \in \dot{X}$. But then there is $r \leq q_2, \bar{q}_1$ which forces both $\check{\alpha} \in \dot{X}$ and $\check{\alpha} \notin \dot{X}$. $\square$

Claim 3.4.9. *$\mathcal{U}$ is independent of the choice of generic H.*

Proof. The same proof as in Claim 3.4.8 works. $\square$

Claim 3.4.8 implies $R \in M|\nu_M$ and R is a complete iterate of $S[y]$ for any y which is Ea_M-generic over M. Claim 3.4.9 gives that $\mathcal{U} \in M$ and thus $R \in \tilde{\mathcal{I}}^M$. $\square$

3.5 The StrLe Construction

Recall the mouse operator $x \to M_x$ defined in Section 3.1. In the following lemmas let $z, x \in \mathbb{R}$ be such that $z \in M_x$ and let $M = M_x$.

Lemma 3.5.1. *Suppose $P = StrLe[M, z]$. Then P is Γ-ss and $\delta_P = \delta_M$.*

Proof. Let $\delta = \delta_M$. By Lemma 2.6.1, the cardinals of P above δ are the same as the cardinals of M and ν_M is inaccessible in P. Any inaccessible of P above δ is inaccessible in M, since M is a generic extension of P by a δ-c.c. forcing. In particular, ν_M is the unique inaccessible of P above δ. Then it suffices to show the following claim.

Claim 3.5.2. *(a) If $\eta < \delta_P$, then $Lp^\Gamma(P|\eta) \triangleleft P$.*

(b) δ is a Γ-Woodin of P. That is, δ is Woodin in $Lp^\Gamma(P|\delta)$.

(c) If $\eta \in P$ and $\eta \geq \delta$, then $P|(\eta^+)^P \trianglelefteq Lp^\Gamma(P|\eta)$.

(d) $P \models \delta$ is Woodin.

(e) If $\eta < \delta$, η is not Woodin in $Lp^\Gamma(P|\eta)$.

(f) If $\eta \in P$ and $\delta_P \leq \eta$, then $Lp^\Gamma(P|\eta) \subseteq P$.

Proof. To prove (a), it suffices to show if $\eta < \delta_P$, $R \triangleleft Lp^\Gamma(P|\eta)$, and $\rho_\omega(R) = \eta$, then $R \triangleleft P$. Coiterate R against $Le[M,z]$. Suppose $\mathcal{T}$ and $\mathcal{U}$ are the iteration trees on R and $Le[M,z]$, respectively, from the coiteration. $\mathcal{T}$ is above η because $Le[M,z]|\eta = R|\eta$ and η is a cutpoint of R. Let $\lambda < lh(\mathcal{T})$ be a limit ordinal and $Q = \mathcal{Q}(\mathcal{T}) = \mathcal{Q}(\mathcal{U})$. Since $R \in Lp^\Gamma(P|\eta)$ and $\mathcal{T}$ is above η, $Q \in Lp^\Gamma(\mathcal{M}(\mathcal{T}))$. $[0,\lambda]_T$ and $[0,\lambda]_U$ are the unique branches through $\mathcal{T}$ and $\mathcal{U}$, respectively, which absorb Q. By Corollary 3.3.2, these branches can be identified in M. In particular, the coiteration of R and $Le[M,z]$ can be performed in M. Theorem 2.5.3 gives that R cannot outiterate $Le[M,z]$. Then since R is ω-sound, R projects to η, and $Le[M,z]$ does not project to η, R is a proper initial segment of $Le[M,z]|(\eta^+)^{Le[M,z]}$. $Le[M,z]$ agrees with P up to δ, so $R \triangleleft P$.

(b) is by the proof of Theorem 11.3 of [13]. For (c), the iteration strategies for initial segments of $P|(\eta^+)^P$ restricted to iteration trees above δ are in $\mathbf{\Delta}$ by Fact 2.6.2. (d) is immediate from (b) and (c). See Sublemma 7.4 of [12] for a proof of (e).

Towards (f), let $Q = Lp^\Gamma(P|\eta)$. Let $\mathbb{P}$ be the extender algebra in P at δ with δ generators. $M|\delta$ is $\mathbb{P}$-generic over P. Note δ is Woodin in Q by (b). In particular, $\mathbb{P}$ is also δ-c.c. in Q, so any antichain of $\mathbb{P}$ in Q is also in P and $M|\delta$ is also $\mathbb{P}$-generic over Q.

Let $B \in Lp^\Gamma(P|\eta)$. B is in $M = P[M|\delta_M]$ since $P|\eta$ is in M and M is closed under Lp^Γ. So let $\dot{B}$ be a $\mathbb{P}$-name in P such that $\dot{B}[M|\delta] = B$.

Choose $p \in \mathbb{P}$ such that $p \Vdash^Q_{\mathbb{P}} \dot{B} = \check{B}$.

Any G which is $\mathbb{P}$-generic over P is also $\mathbb{P}$-generic over Q. So for any G which is $\mathbb{P}$-generic over P such that $p \in G$, $\dot{B}[G] = B$. But then B is in P, since $B = \{\xi < \delta : p \Vdash^P_{\mathbb{P}} \check{\xi} \in \dot{B}\}$.[6] $\square$

$\square$

Lemma 3.5.3. *Suppose $P = StrLe[M, z]$. Let $\mu \geq \delta_P$ be a cardinal of P. $\tau^P_{n,\mu}$ exists and is definable in M from $\tau^M_{n,\mu}$ and z.*

Proof. This is similar to the proof of 3.3.3. Let

$$\sigma = \{\dot{y} \in P : \dot{y} \text{ is a } Col(\omega, \mu)\text{-standard term for a real and } \emptyset \Vdash^M_{Col(\omega,\mu)} \dot{y} \in P[\dot{g}] \cap \tau^M_{n,\mu}\}.$$

Since $\tau^M_{n,\mu} \in M$ and P is definable over M from z, $\sigma \in M$ and is definable from $\tau^M_{n,\mu}$ and z.

Claim 3.5.4. *$\sigma[g] = P[g] \cap G_n$ for any g which is $Col(\omega, \mu)$-generic over M.*

Proof. First, we have $\sigma[g] \subseteq P[g] \cap \tau^M_{n,\mu}[g] = P[g] \cap (G_n \cap M[g]) = P[g] \cap G_n$.

Now suppose $y \in P[g] \cap G_n$. Let $\dot{y}$ be a name for y in P. Let $w \in P \cap G_n$ and let $r \in Col(\omega, \mu)$ be such that $r \Vdash^M_{Col(\omega,\mu)} \dot{y} \in \tau^M_{n,\mu}$ and $r \in g$. In P, there is a name $\dot{y}'$ such that whenever h is $Col(\omega, \mu)$-generic over P, $r \in h$ implies $\dot{y}'[h] = \dot{y}[h]$ and $r \notin h$ implies $\dot{y}'[h] = w$. Then $\dot{y}' \in \sigma$ and $\dot{y}'[g] = y$. So $y \in \sigma[g]$. $\square$

A comeager set of $Col(\omega, \mu)$-generics over M are also generic over P. So by the previous claim, and Lemma 3.1.5, $\sigma[g] = P[g] \cap G_n$ for any g which is $Col(\omega, \mu)$-generic over M.

It is not hard to see that $\tau^P_{n,\mu}$ is definable in M from σ. So $\tau^P_{n,\mu}$ is definable in M from $\tau^M_{n,\mu}$ and z. $\square$

Lemma 3.5.5. *Suppose $P = StrLe[M, z]$. The iteration strategy for P is fullness-preserving and guided by $\mathcal{G}$.*

[6] Viewing B as a subset of δ.

Proof. Let Σ be the (unique) iteration strategy for P. The proof of Theorem 2.5.4 gives that Σ is determined by lifting an iteration on P to one on M. More precisely, if $\mathcal{T}$ is a non-dropping iteration tree on $P_0 = P$ with $\langle P_\alpha \rangle$ the models of the iteration and $i_{\beta,\alpha}$ the associated iteration maps for $\beta <_T \alpha$, then we maintain an iteration tree $\mathcal{T}^*$ on $M_0 = M$ with models $\langle M_\alpha \rangle$ and associated iteration embeddings $i^*_{\beta,\alpha}$. We also maintain embeddings $\pi_\alpha : P_\alpha \to StrLe[M_\alpha, z]$ such that $\pi_\alpha \circ i_{\beta,\alpha} = i^*_{\beta,\alpha} \circ \pi_\beta$ and $\pi_0 = id$. In particular, $\pi_\alpha \circ i_{0,\alpha} = i^*_{0,\alpha}$.

Let μ be a cardinal of P above δ_P. By Lemma 3.5.3, $i^*_{0,\alpha}(\tau^P_{n,\mu}) = \tau^{StrLe[M_\alpha,z]}_{n,\mu}$ for each $n < \omega$. Then $\pi_\alpha \circ i_{0,\alpha}(\tau^P_{n,\mu}) = \tau^{StrLe[M_\alpha,z]}_{n,\mu}$. Then by Lemma 3.1.8, P_α is Γ-ss and $\pi_\alpha(\tau^{P_\alpha}_{n,\mu}) = \tau^{StrLe[M_\alpha,z]}_{n,\mu}$. This gives Σ is fullness-preserving. A second application of 3.1.8 gives $i_{0,\alpha}(\tau^P_{n,\mu}) = \tau^{P_\alpha}_{n,\mu}$. So Σ is guided by $\mathcal{G}$. □

Corollary 3.5.6. *Suppose $P = StrLe[M, z]$. Then the ω_1-iteration strategy for P is not in $\boldsymbol{\Gamma}$.*

Proof. Immediate from Lemmas 3.1.15 and 3.5.5. □

Lemma 3.5.7. *Suppose $x, z \in \mathbb{R}$ and x codes a mouse N which is a complete iterate of M_z. Let $P = StrLe[M_x, z]$. Then P is a complete iterate of N below δ_N.*

Proof. Coiterate N and P. Let $\mathcal{T}$ and $\mathcal{U}$ be the iteration trees on N and P, respectively, from the coiteration. Let N^* and P^* be the last models of $\mathcal{T}$ and $\mathcal{U}$, respectively.

Suppose P outiterates N. One possibility is that there is a drop on the branch of $\mathcal{T}$ from P to P^*. Since the iteration strategy for P is fullness-preserving by Lemma 3.5.5, P^* has an ω_1-iteration strategy in $\boldsymbol{\Delta}$. But the strategy for N is fullness-preserving and guided by $\mathcal{G}$. So N^* cannot have an iteration strategy in $\boldsymbol{\Delta}$, contradicting that $N^* \trianglelefteq P^*$.

If there is no drop between P and P^*, then $N^* \triangleleft P$. Since neither side of the coiteration drops, N^* and P^* are both Γ-ss. But no Γ-ss mouse can have a proper initial segment which is Γ-ss.

An identical argument shows N cannot outiterate P. Thus $N^* = P^*$ and $\mathcal{T}$ and $\mathcal{U}$ realize N^* and P^* are complete iterates of N and P, respectively. Since there are no total extenders on N above δ_N, $\mathcal{T}$ is below δ_N. Similarly, $\mathcal{U}$ is below δ_P. Then stationarity of the Mitchell-Steel construction[7] implies that $P^* = P$. So $\mathcal{T}$ realizes that P is a complete iterate of N. $\square$

3.6 A Reflection Lemma

In this section we prove a lemma that any Σ_1 statement true in M_x also holds in some $N \triangleleft M_x|\kappa_{M_x}$ with the property that $StrLe[N] \triangleleft StrLe[M]$. A thorough reader not already familiar with the fully-backgrounded Mitchell-Steel construction may wish to review Section 2.5 before proceeding. A lazy one may read the statement of Lemma 3.6.4 and skip to Section 3.7.

First, we need to show M_x can compute the iteration strategies of its own initial segments below its Woodin cardinal. More precisely, we have:

Lemma 3.6.1. *Let $x \in \mathbb{R}$, $N \triangleleft M_x|\delta_{M_x}$ and $\mathcal{T} \in M_x$ be an iteration tree on N of limit length $< \delta_{M_x}$, according to the (unique) iteration strategy for N. The cofinal branch b through $\mathcal{T}$ determined by the iteration strategy for N is definable in M_x (uniformly in N and $\mathcal{T}$, from the parameter $\tau^{M_x}_{0,\nu_{M_x}}$).*

Proof. Let $M = M_x$. By Corollary 3.3.2, the function $a \mapsto Lp^\Gamma(a)$ with domain $M|\delta_M$ is definable in M from the parameter τ^M_{0,ν_M}.

Let N and $\mathcal{T}$ be as in the statement of the lemma. Let $S = \mathcal{M}(\mathcal{T})$. Clearly S is definable from $\mathcal{T}$. Let $Q = \mathcal{Q}(\mathcal{T})$. Q is an initial segment of $Lp^\Gamma(S)$. The previous paragraph implies Q is definable in M from S and τ^M_{0,ν_M}. The branch b through $\mathcal{T}$ chosen by the iteration strategy for N is the unique branch which absorbs Q.

It remains to show b is in M. Iterate M to M' well above where $\mathcal{T}$ is constructed to

[7] See e.g. 3.23 of [19].

make some g generic over $Ea^{M'}_{\delta_{M'}}$ so that g codes b. $M'[g]$ satisfies that b is the unique branch which absorbs Q. Since b is in fact the unique such branch in V, symmetry of the forcing gives b is in M'. But the iteration from M to M' does not add any subsets of $lh(\mathcal{T})$, so in fact b is in M. $\square$

We need to put down a few more properties of the Mitchell-Steel construction before proving the main lemma of this section.

Lemma 3.6.2. *Suppose N is a mouse with a Woodin cardinal δ_N. Let $z \in N \cap \mathbb{R}$. There is a club C of $\tau < \delta_N$ such that $Le[N|\delta_N, z]|\tau = \mathcal{M}_\tau$, where $\mathcal{M}_\tau$ is the Mitchell-Steel construction of length τ in $N|\delta_N$. Moreover, we can take C to be definable in N.*

Proof. Let $\langle \mathcal{M}_\xi : \xi < \delta_N \rangle$ be the models from the Mitchell-Steel construction of length δ_N over z, done inside $N|\delta_N$. Let $C' \subset \delta_N$ be the set of $\tau < \delta_N$ such that $\mathcal{M}_\tau$ has height τ and $\rho_\omega(\mathcal{M}_\xi) \geq \tau$ whenever ξ is between τ and the height of N. It is not hard to see from the material in Section 2.5 that C is a club and if $\tau \in C$, then $\mathcal{M}_\tau = Le[N|\delta_N, z]|\tau$. $\square$

Corollary 3.6.3. *Let N, z, and C be as in Lemma 3.6.2. Let S be the set of inaccessible cardinals of N below δ_N. Then $Le[N|\delta_N, z] = \bigcup_{\tau \in C \cap S} Le[N|\tau, z]$.*

Proof. Since δ_N is Woodin in N, $N \models$ "S is stationary." And C is definable in N, so $C \cap S$ is cofinal in δ_N. Since $Le[N|\delta_N, z]$ has height δ_N, $Le[N|\delta_N, z] = \bigcup_{\tau \in C \cap S} Le[N|\delta_N, z]|\tau$. So it suffices to show if $\tau \in C \cap S$, then $Le[N, z]|\tau = Le[N|\tau, z]$.

Let $\langle \mathcal{M}_\xi : \xi < \delta_N \rangle$ be the models from the Mitchell-Steel construction of length δ_N over z, done inside N. $\tau \in C$ guarantees $Le[N, z]|\tau = \mathcal{M}_\tau$. And by Remark 2.5.5, $\tau \in S$ gives $\mathcal{M}_\tau = Le[N|\tau, z]$. So $Le[N, z]|\tau = Le[N|\tau, z]$ for $\tau \in C \cap S$. $\square$

Lemma 3.6.4. *Suppose $M_x \models \phi[\vec{a}]$ for some Σ_1 formula ϕ, $z \in \mathbb{R} \cap M_x$, and $\vec{a} \in \mathbb{R}^{|\vec{a}|} \cap M_x$. Then there exists $N \triangleleft M_x|\kappa_{M_x}$ such that*

(a) N has one Woodin cardinal,

(b) δ_N is an inaccessible cardinal of M_x,

(c) $N \models \phi[\vec{a}]$, *and*

(d) $StrLe[N, z] \triangleleft StrLe[M_x, z]$.

Proof. Denote M_x by M. For ease of notation we will assume $z = 0$. Let μ be a cardinal of M above δ_M such that $M|\mu \models \phi[\vec{a}]$.

Claim 3.6.5. *There is a stationary set of $\tau < \delta_M$ such that τ is inaccessible in M and if $\tau \leq \zeta < \delta_M$, then ζ is not definable in $M|\mu$ from parameters below τ.*

Proof. Work in M. Let S be the set of inaccessible cardinals below δ_M. Since δ_M is Woodin, S is stationary. Define $f : S \to \delta_M$ by setting $f(\zeta)$ to be the least η such that there is $\zeta \leq \iota < \delta_M$ definable in $M|\mu$ from parameters in η. If the claim is false, then f is regressive on a stationary set. Then by Fodor's Lemma, there is a stationary set S_0 and $\eta < \delta_M$ such that $f''S_0 = \{\eta\}$. But $cof(\delta_M) > |\eta^{<\omega}| \times \aleph_0$, so we cannot have cofinally many elements of δ_M defined by some formula and parameters from η. $\square$

Fix τ as in Lemma 3.6.2 and Claim 3.6.5. Let $H = Hull^{M|\mu}(\tau)$. Let N be the transitive collapse of H and $\pi : N \to M|\mu$ the anti-collapse map.

By Theorem 2.3.6, $N \triangleleft M|\mu$. Clearly $N \triangleleft M|\delta_M$, $N \models \phi[\vec{a}]$, τ is the unique Woodin of N, τ is inaccessible in M, and $\rho_\omega(N) = \tau$.

Claim 3.6.6. $Le[N|\tau] \triangleleft Le[M|\delta_M]$

Proof. For $\zeta < \tau$, $Le[N|\zeta] \triangleleft Le[N|\tau] \iff Le[M|\zeta] \triangleleft Le[M|\delta_M]$ by elementarity. But $Le[N|\zeta] = Le[M|\zeta]$ for $\zeta < \tau$. So if $Le[N|\zeta]$ is an initial segment of $Le[N|\tau]$, then it is also an initial segment of $Le[M|\delta_M]$. But this implies $Le[N|\tau] \triangleleft Le[M|\delta_M]$, since by Corollary 3.6.3, $Le[N|\tau]$ is a union of mice of the form $Le[N|\zeta]$ for $\zeta < \tau$. $\square$

We have found $N \triangleleft M|\delta_M$ satisfying (a), (b), (c), $\rho_\omega(N) = \delta_N$, and $Le[N|\delta_N] \triangleleft Le[M|\delta_M]$ (since $\delta_N = \tau$). Our next step is to reflect this below κ_M. Let F be a total extender in M such that the strength of F is greater than $On \cap N$. In particular, we have $N \triangleleft Ult(M|\delta_M, F)$.

Claim 3.6.7. $Le[N|\tau] \triangleleft Le[Ult(M|\delta_M, F)]$.

Proof. τ is inaccessible in $Ult(M|\delta_M, F)$. So by Remark 2.5.5, $Le[N|\tau]$ equals the Mitchell-Steel construction of length τ in $Ult(M|\delta_M, F)$.

Suppose the claim fails. Then there is a mouse Q built during the Mitchell-Steel construction in $Ult(M|\delta_M, F)$ after $Le[N|\tau]$ is constructed, such that Q projects to some $\beta < \tau$. Pick such a Q which minimizes β. By Lemma 3.6.1, any initial segment of M below δ_M is iterable in M. Then M has iteration strategies for $Ult(P, F)$ for any $P \triangleleft M|\delta_M$. Q is a mouse built during the Mitchell-Steel construction in $Ult(P, F)$ for some $P \triangleleft M|\delta_M$, so Q is also iterable in M. Let $Q' = \mathcal{C}_\omega(Q)$. Then Q' is an ω-sound mouse over $Le[N|\tau]||\beta$ projecting to β which is iterable in M. It follows from Theorem 2.5.3 that $Le[M|\delta_M]$ outiterates Q'. Since both extend $Le[N|\tau]||\beta$, and Q' is ω-sound and projects to β, $Q' \triangleleft Le[M|\delta_M]$. But then since τ is inaccessible in M, $Le[M|\tau]$ has height τ, and $Le[M|\tau] \triangleleft Le[M|\delta_M]$, Q' is in $Le[M|\tau]$. This is a contradiction, since a subset of β which is not in $Le[M|\tau]$ is definable over Q'. $\square$

By elementarity of i_F, there exists $N \triangleleft M|\kappa_M$ satisfying (a), (b), (c), $\rho_\omega(N) = \delta_N$, and $Le[N|\delta_N] \triangleleft Le[M|\delta_M]$. It remains to prove the following claim.

Claim 3.6.8. $StrLe[N] \triangleleft StrLe[M]$.

Proof. Since N projects to δ_N, so does $StrLe[N]$ (by Lemma 2.6.1). And $StrLe[N]$ agrees with $StrLe[M]$ up to δ_N since $Le[N|\delta_N] \triangleleft Le[M|\delta_M]$. So it suffices to show $StrLe[M]$ outiterates $StrLe[N]$. But $StrLe[N]$ has an iteration strategy in $\mathbf{\Gamma}$, and $StrLe[M]$ cannot by Lemma 3.5.6. $\square$

$\square$

3.7 Main Theorem

We are ready to prove Theorem 1.0.10. Suppose for contradiction $\langle A_\alpha | \alpha < \boldsymbol{\delta}^+_{\mathbf{\Gamma}} \rangle$ is a sequence of distinct $\mathbf{\Gamma}$ sets. Let $U \subset \mathbb{R} \times \mathbb{R}$ be a universal $\mathbf{\Gamma}$ set.

Let $\mathcal{J} = \{(P, \xi) : P \in \mathcal{I} \wedge \xi < \delta_P\}$. Say $(P, \xi) \leq_* (Q, \zeta)$ if $(P, \xi), (Q, \zeta) \in \mathcal{J}$ and whenever S is a complete iterate of both P and Q, $\pi_{P,S}(\xi) \leq \pi_{Q,S}(\zeta)$. By Lemma 3.2.2, the relation $\leq_*$ has length $> \boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$. Fix n such that for some (equivalently any) $P \in \mathcal{I}$, $\pi_{P,\infty}(\gamma^P_n) > \boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$. Let $\leq'_*$ be $\leq_*$ restricted to pairs (P, ξ) such that $\xi < \gamma^P_n$. Then $\leq'_*$ has length $\geq \boldsymbol{\delta}_{\boldsymbol{\Gamma}}$ and $\leq'_*$ is in $J_{\beta'}(\mathbb{R})$.[8] Let $B_\alpha = \{y : U_y = A_\alpha\}$. By the Coding Lemma there is a set D in $J_{\beta'}(\mathbb{R})$ such that $(x, y) \in D$ implies x codes a pair in the domain of $\leq'_*$ and $y \in B_{|x|_{\leq'_*}}$, and D_x is nonempty for all x in the domain of $\leq'_*$.

Let $z_0 \in \mathbb{R}$ be such that z_0 codes W and $D \in OD^{<\beta'}(z_0)$. Let $\mathcal{I}'$ be the directed system of all countable, complete iterates of M_{z_0}. Let M'_∞ be the direct limit of $\mathcal{I}'$. For $M, N \in \mathcal{I}'$ and N an iterate of M, let $\pi_{M,N} : M \to N$ be the iteration map and $\pi_{M,\infty} : M \to M'_\infty$ the direct limit map (We also used $\pi_{M,N}$ and $\pi_{M,\infty}$ for $M, N \in \mathcal{I}$, but this should not cause any confusion).

For $M \in \mathcal{I}'$, let $\tau^M = \tau^M_{D,\delta_M}$. There is a slight issue in that our current definitions do not obviously guarantee that τ^M is moved correctly. That is, we might have a complete iterate N of M such that $\pi_{M,N}(\tau^M) \neq \tau^N$. This can happen because we defined the operator $x \to M_x$ so that M_x is guided by $\mathcal{G}$, but it is possible $D \notin \mathcal{G}$. There is no real issue here, since we can expand $\mathcal{G}$ to a larger self-justifying system $\mathcal{G}'$ such that $D \in \mathcal{G}'$ and require M_x be guided by $\mathcal{G}'$. However, we should leave the operator $x \to W_x$ as is, otherwise we risk altering our construction of D. This raises another minor complication, because in Sections 3.3 and 3.4 we assumed our Γ-ss mouse M was guided by the same self-justifying system as our Γ-suitable mouse W. Fortunately, the results of those sections remain true so long as $\mathcal{G} \subseteq \mathcal{G}'$, modulo increasing the number of terms required as parameters in some of the lemmas. For simplicity, in what follows we will just assume τ^M is moved correctly.

Definition 3.7.1. *Say $M \in \mathcal{I}'$ is locally α-stable if there is $\xi \in M$ such that $\pi_{\mathcal{H}^M,\infty}(\xi) = \alpha$. Write α_M for this ordinal ξ.*

Definition 3.7.2. *Say $M \in \mathcal{I}'$ is α-stable if M is locally α-stable and whenever $N \in \mathcal{I}'$ is a complete iterate of M, $\pi_{M,N}(\alpha_M) = \alpha_N$.*

[8] This is done by similar arguments to those in Section 3.4.

Lemma 3.7.3. *For any $\alpha < \boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$, there is an α-stable $M \in \mathcal{I}'$.*

Proof. This is essentially the same as the proof of the analogous lemma in [2]. We will show for any $P \in \mathcal{I}'$, there is an iterate of P which is α-stable.

Claim 3.7.4. *For any $P \in \mathcal{I}'$, there is a countable, complete iterate R of P which is locally α-stable*

Proof. Fix $S \in \mathcal{I}$ and $\zeta \in S$ such that $\pi_{S,\infty}(\zeta) = \alpha$. Let R be a countable, complete iterate of P such that S is Ea_R-generic over R.

Let $\dot{S}$ be an Ea_R-name for S such that $\emptyset \Vdash^R_{Ea_R}$ "$\dot{S}$ is a complete iterate of W." Applying Lemma 3.4.7 yields $S' \in \mathcal{I}^R$ which is a complete iterate of S. Then

$$\begin{aligned} \pi_{\mathcal{H}^R,\infty} \circ \pi_{S',\mathcal{H}^R} \circ \pi_{S,S'}(\zeta) &= \pi_{S,\infty}(\zeta) \\ &= \alpha. \end{aligned}$$

In particular, $\alpha \in range(\pi_{\mathcal{H}^R,\infty})$. □

Now suppose no $M \in \mathcal{I}'$ is α-stable. Let $\langle R_j : j < \omega \rangle$ be a sequence in $\mathcal{I}'$ such that for all j, R_j is locally α-stable and R_{j+1} is an iterate of R_j, but $\pi_{R_j,R_{j+1}}(\alpha_{R_j}) \neq \alpha_{R_{j+1}}$.

Claim 3.7.5. $\pi_{R_j,R_{j+1}}(\alpha_{R_j}) \geq \alpha_{R_{j+1}}$

Proof. By elementarity, $\pi_{R_j,R_{j+1}} \restriction \mathcal{H}^{R_j}$ is an embedding of $\mathcal{H}^{R_j}$ into $\mathcal{H}^{R_{j+1}}$. Then the Dodd-Jensen property implies for any common complete iterate Q of $\mathcal{H}^{R_j}$ and $\mathcal{H}^{R_{j+1}}$,

$$\pi_{\mathcal{H}^{R_{j+1}},Q} \circ \pi_{R_j,R_{j+1}}(\alpha_{R_j}) \geq \pi_{\mathcal{H}^{R_j},Q}(\alpha_{R_j}).$$

Then

$$\begin{aligned} \pi_{\mathcal{H}^{R_{j+1}},\infty} \circ \pi_{R_j,R_{j+1}}(\alpha_{R_j}) &\geq \pi_{\mathcal{H}^{R_j},\infty}(\alpha_{R_j}) \\ &= \alpha \\ &= \pi_{\mathcal{H}^{R_{j+1}},\infty}(\alpha_{R_{j+1}}). \end{aligned}$$

So $\pi_{R_j,R_{j+1}}(\alpha_{R_j}) \geq \alpha_{R_{j+1}}$. □

Let R_ω be the direct limit of the sequence $\langle R_j : j < \omega\rangle$. Let $\alpha_j = \pi_{R_j,R_\omega}(\alpha_{R_j})$. Claim 3.7.5 implies $\alpha_{j+1} < \alpha_j$ for all j, contradicting the wellfoundedness of R_ω. □

Let p^M be a maximal condition in Ea_M such that p forces the generic ea is a pair (ea^1, ea^2), where ea^1 codes a pair (R_{ea^1}, ξ_{ea^1}) such that there exists an iteration tree on W (according to the strategy for W) with last model R_{ea^1} and $\xi_{ea^1} < \delta_{R_{ea^1}}$.[9] In fact let p^M be the least such condition in the construction of M, to ensure p^M is definable in M.

Lemma 3.7.6. *There is $Q^M \in \mathcal{I}^M$ such that p^M forces Q^M is a complete iterate of R_{ea^1}. Moreover, Q^M is definable in M from parameter p^M (uniformly in M).*

Proof. Apply Lemma 3.4.7 to the condition p^M and a name for R_{ea^1}. □

Definition 3.7.7. *For α-stable $M \in \mathcal{I}'$, say $p \in Ea_M$ is α-good if p extends p^M and p forces*

1. $\pi_{\check{Q}^M,\mathcal{H}^M} \circ \pi_{R_{ea^1},\check{Q}^M}(\xi_{ea^1}) = \alpha_M$ and

2. $(ea^1, ea^2) \in \tau^M$.

Remark 3.7.8. *If $\alpha < \boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$, being α-good is definable over α-stable $M \in \mathcal{I}'$ from α_M, τ^M, and $\langle \tau^M_{k,\nu_M} : k < n\rangle$ (uniformly in M). This follows from Lemmas* 3.4.5 *and* 3.4.6.

Let p^M_α be the maximal α-good condition in M which is least in the construction of M. Note if M is α-stable and N is a complete iterate of M, then $\pi_{M,N}(p^M_\alpha) = p^N_\alpha$.

For $w \in \mathbb{R} \cap M$ and a Σ_1 formula $\psi(w)$, write $M \models [\psi(w)]$ to mean whenever g is $Col(\omega, \delta_M)$-generic over M, there is a proper initial segment of $M[g]$ which is a $\langle \psi', g\rangle$-witness, where $\psi'(x)$ is a formula expressing "$\psi(f(x))$" for some computable function f such that $f(g) = w$. Note "$M \models [\psi(w)]$" is Σ_1 over M if M is iterable.

For α-stable $M \in \mathcal{I}'$, let S^M_α be the set of conditions q such that there exist $N, r \in M$ satisfying

(a) $N \triangleleft M|\kappa_M$,

[9] This is first order by Corollary 3.3.8 and Lemma 3.4.6.

(b) N has one Woodin,

(c) δ_N is a cardinal of M,

(d) $q, r \in Ea_N$ and $(q, r) \Vdash^N_{Ea_N \times Ea_N} [U(ea_l, ea^2_r)]$,[10] and

(e) r is compatible with p^M_α.

Let $S_\alpha = \pi_{M,\infty}(S^M_\alpha)$ for some (equivalently any) α-stable $M \in \mathcal{I}'$. S_α can be viewed as an element of $P(\kappa_{M'_\infty})^{M'_\infty}$.

Let A'_α be the set of reals x such that there is an α-stable $M \in \mathcal{I}'$ and $q \in M$ satisfying

1. $q \in S^M_\alpha$,

2. $x \models q$, and

3. x is Ea_M-generic over M.

Lemma 3.7.9. $A'_\alpha = A_\alpha$

It suffices to show Lemma 3.7.9. The lemma implies $\alpha \neq \beta \implies S_\alpha \neq S_\beta$. By the same proof as for M_∞ given in Lemma 3.2.1, $\kappa_{M'_\infty} \leq \boldsymbol{\delta_\Gamma}$. So we have $\boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$ distinct subsets of $\boldsymbol{\delta_\Gamma}$ in M'_∞. Then the successor of $\boldsymbol{\delta_\Gamma}$ in M'_∞ is the successor of $\boldsymbol{\delta_\Gamma}$ in $L(\mathbb{R})$, contradicting the following claim.

Claim 3.7.10. *Let $\eta = \boldsymbol{\delta_\Gamma}$. Then $(\eta^+)^{M'_\infty} < (\eta^+)^{L(\mathbb{R})}$*

Proof. Let $\lambda = (\eta^+)^{M'_\infty}$. Since λ is regular in M'_∞ but not measurable, Lemma 3.2.6 implies λ has cofinality ω in $L(\mathbb{R})$.

Let $f \in L(\mathbb{R})$ be a cofinal function from ω to λ. Let $\langle g_\xi : \xi < \lambda \rangle$ be a sequence of functions in M'_∞ such that $g_\xi : \eta \to \xi$ is a surjection. Such a sequence exists because M'_∞ satisfies AC. Then in $L(\mathbb{R})$ we can construct from f and $\langle g_\xi \rangle$ a surjection from η onto λ. $\square$

[10] Here by U we really mean some fixed Σ_1-formula defining U in $J_{\alpha_0}(\mathbb{R})$.

Proof of Lemma 3.7.9. First suppose $x \in A_\alpha$. Pick $y \in \mathbb{R}$ such that $y = (y^1, y^2)$, $D(y^1, y^2)$ holds, and $|y^1|_{\leq^*} = \alpha$. Pick an α-stable $\bar{M} \in \mathcal{I}'$ such that $\alpha_{\bar{M}}$ exists. Let z be a real coding $\bar{M}$ and let $P = M_{\langle x,y,z\rangle}$. Let $S = StrLe[P, z_0]$.

Claim 3.7.11. *x and y are Ea_S-generic over S.*[11]

Claim 3.7.12. *S is a complete iterate of $\bar{M}$ by an iteration below $\delta_{\bar{M}}$.*

Proof. See Lemma 3.5.7. □

Claim 3.7.13. *There exist conditions $q, r \in Ea_S$ such that $x \models q$, $y \models r$, and $(q, r) \Vdash^S_{Ea_S \times Ea_S}$ $[U(ea_l, ea_r^2)]$.*

Proof. By the choice of y, y satisfies some α-good condition r. Let y_0 be $S[x]$-generic such that $y_0 \models r$. Then by the definition of α-good, $y_0 = (y_0^1, y_0^2)$ where $(y_0^1, y_0^2) \in D$ and $|y_0^1|_{\leq^*} = \alpha$. It follows that $U_{y_0^2} = A_\alpha$. So $x \in U_{y_0^2}$.

Subclaim 3.7.14. $S[x][y_0] \models [U(x, y_0^2)]$.

Proof. Let g be $Col(\omega, \delta_S)$-generic over $S[x][y_0]$. Note $S[x][y_0][g] = S[g]$ is a g-mouse. By the proof of Lemma 3.3.7, $Lp^\Gamma(g)$ is contained in $S[g]$. Let f be a computable function such that $f(g) = (x, y_0^2)$ and let $U'(v)$ be a formula expressing $U(f(v))$ holds. By Lemma 3.1.22, there is a $\langle U', g\rangle$-witness which is sound, projects to ω, and has an iteration strategy in $\mathbf{\Delta}$. Since $Lp^\Gamma(g) \subseteq S[g]$, this witness is an initial segment of $S[g]$. □

We have shown $S[x][y_0] \models [U(x, y_0^2)]$ for any y_0 which satisfies r and is $S[x]$-generic. Thus there is $q \in Ea_S$ such that x satisfies q and $(q, r) \Vdash [U(ea_l, ea_r^2)]$. □

We next would like to find some $N \triangleleft S|\kappa_S$ with the properties of S we obtained above. Note Claims 3.7.11 and 3.7.13 are not first order over S, since x and y are not in S. So a straightforward reflection argument inside S will not suffice. The point of introducing P and obtaining S as a construction inside P is that these claims are first order in P. The next

[11] *This is a standard property of the fully-backgrounded construction - see Section 1.7 of [2].*

claim demonstrates we can perform a reflection in P to obtain the desired initial segment of S.

Claim 3.7.15. *There is $N \triangleleft S|\kappa_S$ such that N has one Woodin, δ_N is an inaccessible cardinal of S, x and y are generic for Ea_N, and there exist $q, r \in Ea_N \times Ea_N$ such that $x \models q$, $y \models r$, and $(q, r) \Vdash [U(ea_l, ea_r^2)]$.*

Proof. By Claims 3.7.11 and 3.7.13, P satisfies

1. x and y are $Ea_{StrLe[P,z_0]}$-generic over $StrLe[P, z_0]$ and
2. there exist conditions $q, r \in StrLe[P, z_0]$ such that $x \models q$, $y \models r$, and $(q, r) \Vdash^{StrLe[P,z_0]}_{Ea_{StrLe[P,z_0]} \times Ea_{StrLe[P,z_0]}} [U(ea_l, ea_r^2)]$.

Both properties are Σ_1 over P in parameters x, y, and z_0. Then we may apply Lemma 3.6.4 to obtain $P' \triangleleft P|\kappa_P$ such that P' has one Woodin cardinal, $\delta_{P'}$ is an inaccessible cardinal of P, $StrLe[P', z_0] \triangleleft S$, and P' satisfies properties 1 and 2.

Let $N = StrLe[P', z_0]$. Note $\delta_N = \delta_{P'}$ is an inaccessible cardinal of S. Then all the properties we required of N are apparent except that $N \triangleleft S|\kappa_S$. Standard properties of the Mitchell-Steel construction imply that $\kappa_S \geq \kappa_P$.[12] Then N has cardinality less than κ_S in P, since N is contained in P'. Since also $N \triangleleft S$, we have $N \triangleleft S|\kappa_S$. $\square$

To get $x \in A'_\alpha$, it remains to show the following claim.

Claim 3.7.16. *r is compatible with p^S_α.*

Proof. Note by choice of y, y^1 codes a pair (R, ξ) such that R is a complete iterate of W, $\pi_{R,\mathcal{H}^S}(\xi) = \alpha_S$, and $D(y^1, y^2)$ holds. Then there is $p \in Ea_S$ such that $y \models p$, p forces $\pi_{\check{Q}^S,\mathcal{H}^S} \circ \pi_{R_{ea^1},\check{Q}^S}(\xi_{ea^1}) = \alpha_S$, and $(ea^1, ea^2) \in \tau^S$. We may assume p extends r. p is α-good, so by maximality p is compatible with p^S_α. Then r is compatible with p^S_α as well. $\square$

[12]Suppose $\lambda < \delta_S = \delta_P$ and E is an extender on the fine extender sequence of S witnessing κ_S is λ-strong in S. Let E^* be the background extender for E on the fine extender sequence of P. Then E^* witnesses κ_S is λ-strong in P.

Now suppose $x \in A'_\alpha$. Let M, q realize this and let N, r realize $q \in S^M_\alpha$. Let y be $M[x]$-generic for Ea_M such that $y \models r \wedge p^M_\alpha$. Since $y \models p^M_\alpha$, $y = (y^1, y^2)$ where $U_{y^2} = A_\alpha$. Since $(x, y) \models (q, r)$, $M[x][y] \models [U(x, y^2)]$. Let $g \subset Col(\omega, \delta_M)$ be $M[x][y]$-generic. Then $M[x][y][g] = M[g]$ has an initial segment R witnessing $U(x, y^2)$. By taking the least such R, we may assume R projects to ω and hence $R \in Lp^\Gamma(g)$. It follows that $x \in U_{y^2} = A_\alpha$. $\square$

CHAPTER 4

Remarks on Some Projective-Like Cases

Here we provide a few brief comments on the problem of unreachability for projective-like cases. Section 4.1 covers the projective pointclasses. In Section 4.2, we discuss what appears to be the main obstacle to proving the rest of the following conjecture.

Conjecture 4.0.1. *Assume $ZF+AD+DC+V=L(\mathbb{R})$. Suppose $\kappa \leq \boldsymbol{\delta}^2_1$ is a Suslin cardinal and κ is either a successor cardinal or a regular limit cardinal. Then κ^+ is $S(\kappa)$-unreachable.*

4.1 The Projective Cases

In the introduction, we discussed Sargsyan's theorem solving the problem of unreachability for the projective pointclasses:

Theorem 4.1.1 (Sargsyan)**.** *Assume $ZF+AD+DC$. Then $\boldsymbol{\delta}^1_{2n+2}$ is $\boldsymbol{\Sigma}^1_{2n+2}$-unreachable.*

Our technique for proving Theorem 1.0.10 gives an alternative proof of Sargsyan's theorem, which we outline below. We will assume $ZF+AD+DC$ for the rest of this section.

Let $W=M^\#_{2n+1}$. Let $\mathcal{I}$ be the directed system of countable, complete iterates of W and let M_∞ be the direct limit of $\mathcal{I}$. Sargsyan performed an analysis of $\mathcal{I}$ in [20].

Fact 4.1.2. *$\kappa_{M_\infty} < \boldsymbol{\delta}^1_{2n+2}$ and $\delta_{M_\infty} > (\boldsymbol{\delta}^1_{2n+2})^+$.*

The iteration strategy for W is guided by indiscernibles, analogously to how our iteration strategies in Chapter 3 were guided by terms for sets in a sjs. [20] covers this analysis of the iteration strategy for W in detail. Also analogously to Sections 3.3 and 3.4 of Chapter 3, inside an iterate M of $M^{\#}_{2n+1}(x_0)$ for some $x_0 \in \mathbb{R}$ coding W, we can form the direct limit $\mathcal{H}^M$ of countable iterates of W in M and approximate the iteration maps from W to $\mathcal{H}^M$. This internalization is covered in [2].

The following fact gives us an analogue of the notion of a $\langle \phi, z \rangle$-witness.

Fact 4.1.3. *There is a computable function which sends a Σ^1_{2n+2}-formula ϕ to a formula $\phi^* = \phi^*(u_0, ..., u_{2n-1}, v)$ in the language of mice such that the following hold:*

1. *If $x \in \mathbb{R}$, M is a countable, $\omega_1 + 1$-iterable x-premouse, $M \models ZFC$, M has $2n$ Woodin cardinals $\delta_0, ..., \delta_{2n-1}$, ϕ is a Σ^1_{2n+2} formula, and $M \models \phi^*[\delta_0, ..., \delta_{2n-1}, x]$, then $\phi(x)$ holds.*

2. *If $x \in \mathbb{R}$, $\delta_0, ..., \delta_{2n-1}$ are the Woodin cardinals of $M^{\#}_{2n}(x)$, ϕ is a Σ^1_{2n+2} formula, and $\phi(x)$ holds, then a proper initial segment of M above δ_{2n-1} satisfies ZFC and $\phi^*[\delta_0, ..., \delta_{2n-1}, x]$.*

With these tools it is not difficult to adapt our proof of Theorem 1.0.10 into a proof of Theorem 4.1.1.

Here is a brief overview of the proof of Theorem 4.1.1 in [2]. The basis of this proof is also studying the directed system $\mathcal{I}'$ of countable iterates of $M^{\#}_{2n+1}(z_0)$ for some $z_0 \in \mathbb{R}$. Suppose $\langle A_\alpha : \alpha < \boldsymbol{\delta^1_{2n+2}} \rangle$ is a sequence of distinct $\boldsymbol{\Sigma^1_{2n+2}}$ sets. Fix a $\Pi^1_{2n+3} \backslash \Sigma^1_{2n+3}$ set $A \subset \omega$. If $n \in A$, this is witnessed in a proper initial segment of any M_{2n+1}-like Π^1_{2n+2}-iterable premouse M. Then there is a Σ^1_{2n+3} set $A' \subset A$ consisting of, roughly speaking, all $n \in \omega$ which are witnessed in such an M before some $x \in A_\alpha$ is witnessed. There is $n_0 \in A' \backslash A$. This is witnessed in some proper initial segment $\bar{N}_M$ of $M | \kappa_M$ for any $M \in \mathcal{I}'$. A coding set S^M is defined analogously to our coding sets in the proof of 1.0.10, but with the additional requirement that the conditions appear below $\bar{N}_M$. The coding sets are used to show a $\boldsymbol{\Sigma^1_{2n+2}}$ code for A_α is small generic over M. The contradiction is obtained from this.

The technique described in the previous paragraph is a stronger argument than the one we used for Theorem 1.0.10, since it gives coding sets which are uniformly bounded below the least strong cardinal. It is not clear whether a similar argument could work for inductive-like pointclasses. There is no obvious analogue of the $\Pi^1_{2n+3}\backslash\Sigma^1_{2n+3}$ set A for an inductive-like pointclass Γ, since there is no universal $\Gamma\backslash\Gamma^c$ set of integers. So the proof from [2] is not applicable to inductive-like pointclasses. On the other hand, the techniques of Chapter 3 are applicable to the projective pointclasses. And this yields a substantially simpler proof of Theorem 4.1.1, since it eliminates the need for a uniform bound on our coding sets.

4.2 Mouse Sets and Open Problems

In this section we discuss the relationship between the problem of unreachability and well-known conjectures on mouse sets. We will assume $ZF + AD + DC + V = L(\mathbb{R})$, although this is overkill for some of the results stated below.

Definition 4.2.1. *$X \subset \mathbb{R}$ is a mouse set if there is an $\omega_1 + 1$-iterable premouse M such that $X = M \cap \mathbb{R}$.*

Theorem 4.2.2 (Steel)**.** *Suppose $\Gamma = \Sigma^1_{n+2}$ for some $n \in \omega$. Then C_Γ is a mouse set.*

Theorem 4.2.3 (Woodin)**.** *Suppose λ is a limit ordinal and let*
$\Gamma = \{A \subseteq \mathbb{R} : A$ is definable in $J_\beta(\mathbb{R})$ for some $\beta < \lambda\}$. Then C_Γ is a mouse set.

See [21] and [9] for proofs of Theorems 4.2.2 and 4.2.3, respectively. [9] also gives the following conjecture.

Conjecture 4.2.4 (Steel)**.** *Suppose Γ is a level of the (lightface) Levy hierarchy. Then C_Γ is a mouse set.*

Conjecture 4.2.4 is a way of asking if there is a mouse corresponding exactly to the pointclass Γ. For each Γ in the Levy hierarchy, the core model induction constructs a mouse which contains C_Γ, but in some cases the mouse constructed is too large. For example, let

J be the mouse operator $J(x) = \bigcup_{n<\omega} M_n^{\#}(x)$. If $\Gamma = \Sigma_{n+2}(J_2(\mathbb{R}))$, then

$$M_n^{J^{\#}} \cap \mathbb{R} \subsetneq C_\Gamma \subsetneq M_{n+1}^{J^{\#}} \cap \mathbb{R}.$$

There are many similar cases in which the mice constructed in [16] skip the (hypothesized) mouse realizing Conjecture 4.2.4. Recent progress has been made towards Conjecture 4.2.4 in [22], which resolves the case $\Gamma = \Sigma_2(J_2(\mathbb{R}))$.

The problem of unreachability is connected to a boldface version of Conjecture 4.2.4.

Conjecture 4.2.5. *Suppose $\alpha \in ON$ and $n \in \omega$. For $x \in \mathbb{R}$, let Γ_x consist of all pointsets A for which there is a Σ_n formula ϕ with parameter x such that $A = \{y : J_\alpha(\mathbb{R}) \models \phi[y]\}$. Then for any $y \in \mathbb{R}$, there is $x \in \mathbb{R}$ such that $y \leq_T x$ and C_{Γ_x} is a mouse set.*

Presumably a proof of Conjecture 4.2.4 would relativize, so a proof of Conjecture 4.2.4 would also resolve Conjecture 4.2.5.

The mouse operator $x \mapsto M_{2k}^{\#}(x)$ realizes Conjecture 4.2.5 holds for $\alpha = 1$ and $n = 2k+2$. To prove $\boldsymbol{\delta}^1_{2n+2}$ is $\boldsymbol{\Sigma}^1_{2n+2}$-unreachable, we studied the direct limit of $M = M_{2n+1}^{\#}(x_0)$ for some $x_0 \in \mathbb{R}$. Note if g is $Col(\omega, \delta_M)$-generic over M, then $M[g] = M_{2n}^{\#}(g)$.

For α admissible, the mouse operator $x \mapsto M_x$ of Theorem 3.1.16 realizes Conjecture 4.2.5 holds in the case $n = 1$. Note if g is $Col(\omega, \delta_{M_x})$-generic over M_x, then $M_x[g] \cap \mathbb{R} = Lp^\Gamma(g) \cap \mathbb{R} = C_\Gamma(g)$. So in the inductive-like case as well we studied the direct limit of a mouse such that collapsing its least Woodin yields a mouse realizing one case of Conjecture 4.2.5.

Thus for each pointclass $\boldsymbol{\Sigma_n}(J_\alpha(\mathbb{R}))$ for which we have proven Conjecture 1.0.9 holds, we used a mouse operator realizing Conjecture 4.2.5 holds for α and n. It seems likely a proof of Conjecture 4.0.1 would involve proving Conjecture 4.2.5 for each α and n such that $\boldsymbol{\Sigma_n}(J_\alpha(\mathbb{R})) = S(\kappa)$ for some Suslin cardinal κ which is a successor cardinal or a regular limit cardinal.

References

[1] Greg Hjorth. Two applications of inner model theory to the study of $\mathbf{\Sigma^1_2}$ sets. *The Bulletin of Symbolic Logic*, 2(1):94–107, 1996.

[2] Grigor Sargsyan. Hjorth's reflection argument, 2021.

[3] Leo Harrington. Analytic determinacy and $0^{\#}$. *The Journal of Symbolic Logic*, 43(4):685–693, 1978.

[4] Adam Day and Andrew Marks. The decomposability conjecture. https://www.youtube.com/watch?v=Nio3PkWBRz4, 2021.

[5] Alexander S. Kechris. On transfinite sequences of projective sets with an application to Σ^1_2 equivalence relations. In Angus Macintyre, Leszek Pacholski, and Jeff Paris, editors, *Logic Colloquium '77*, volume 96 of *Studies in Logic and the Foundations of Mathematics*, pages 155–160. Elsevier, 1978.

[6] Steve Jackson. Partition properties and well-ordered sequences. *Annals of Pure and Applied Logic*, 48(1):81–101, 1990.

[7] Steve Jackson. Structural consequences of AD. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*, pages 1753–1876. Springer Netherlands, Dordrecht, 2010.

[8] Yiannis Moschovakis. *Descriptive Set Theory*. Mathematical surveys and monographs; no. 155. American Mathematical Society, 2009.

[9] John R. Steel. A theorem of woodin on mouse sets. In Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors, *Ordinal Definability and Recursion Theory: The Cabal Seminar, Volume III*, volume 3 of *Lecture Notes in Logic*, page 243–256. Cambridge University Press, 2016.

[10] John R. Steel. Scales in L(R). In Alexander S. Kechris, Donald A. Martin, and Yiannis N. Moschovakis, editors, *Cabal Seminar 79–81*, pages 107–156, Berlin, Heidelberg, 1983. Springer Berlin Heidelberg.

[11] John Steel. An outline of inner model theory. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*, pages 1595–1684. Springer Netherlands, Dordrecht, 2009.

[12] John R. Steel and W. Hugh Woodin. Hod as a core model. In Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors, *Ordinal Definability and Recursion Theory: The Cabal Seminar, Volume III*, volume 3 of *Lecture Notes in Logic*, page 257–346. Cambridge University Press, 2016.

[13] William J. Mitchell and John R. Steel. *Fine Structure and Iteration Trees*. Lecture Notes in Logic. Cambridge University Press, 2017.

[14] John Steel. Derived models associated to mice. *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, 14, 05 2008.

[15] Ralf Schindler and John Steel. The self-iterability of L[E]. *The Journal of Symbolic Logic*, 74(3):751–779, 2009.

[16] Ralf Schindler and John Steel. The core model induction.

[17] John R. Steel. $HOD^{L(R)}$ is a core model below θ. *The Bulletin of Symbolic Logic*, 1(1):75–84, 1995.

[18] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs*. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company, 1983.

[19] Farmer Schlutzenberg and Nam Trang. Scales in hybrid mice over $\mathbb{R}$, 2012.

[20] Grigor Sargsyan. On the prewellorderings associated with the directed systems of mice. *The Journal of Symbolic Logic*, 78(3):735–763, 2013.

[21] J.R. Steel. Projectively well-ordered inner models. *Annals of Pure and Applied Logic*, 74(1):77–104, 1995.

[22] Mitch Rudominer. The mouse set theorem just past projective, 2023.