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STATIONARY STATUS OF DISCRETE AND CONTINUOUS AGE-STRUCTURED POPULATION MODELS

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Abstract

From Leonhard Euler to Alfred Lotka and in recent years understanding the stationary process of the human population has been of central interest to scientists. Population reproductive measure NRR (net reproductive rate) has been widely associated with measuring the status of population stationarity and it is also included as one of the measures in the millennium development goals.This article argues how the partition theorem-based approach provides more up-to-date and timely measures to find the status of the population stationarity of a country better than the NRR-based approach. We question the timeliness of the value of NRR in deciding the stationary process of the country. We prove associated theorems on discrete and continuous age distributions and derive measurable functional properties. The partitioning metric captures the underlying age structure dynamic of populations at or near stationarity. As the population growth rates for an ever-increasing number of countries trend towards replacement levels and below, new demographic concepts and metrics are needed to better characterize this emerging global demography.

Keywords

Convergence to stationary level; Lebesgue measure; partitioning; net-reproduction rate; 92D25; 03E75; 91D20

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Authors contributions: Both the authors contributed in writing. ASRS Rao designed the study, conceptualization, overall project management, developed the methods, collected data, performed mathematical and data analysis, computing, wrote the first draft. JRC designed the study, contributed in writing, performed analysis, editing and writing the draft.

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1. Introduction

The value of the net reproduction rate (NRR) of the population of a country computed for a year is used to understand the degree of population replacement level for that country [1]. There are several limitations of using NRR as a measure to understand the population replacement level of a country or a region [2]. In technical demography literature, the replacement level of a population means the level at which a generation of a population will be replaced by a new population that is produced by the original population [3, 4, 5]. Alfred Lotka [4, 5] associated the population replacement level of a population with the reproduction rate of that population. The data used in NRR computation is of female births only and hence it is generally regarded as the average number of girls replaced by a woman before she successfully completes her replacement age [2]. The quantity NRR is denoted by the symbol R_0 by Alfred Lotka [4, 5]. There were advancements in understanding net reproduction functions within a framework of population growth, epidemic spread [6, 7, 8, 9, 10, 11], structured populations [12, 13, 14, 15, 16, 17], and continuous structured populations [18, 19, 20]. Our focus in this article is net reproductive functions within the classical demography perspective. When $R_0 = 1$ is attained for a country in a year we say the population of that country reached the exact replacement level [1]. When $R_0 > 1$ we say the population is at above replacement level and if $R_0 < 1$ we say the population is below replacement level or sub-replacement level. Reaching the exact replacement level of a country could have happened from that country being above the replacement level or being below the replacement level (see Figure 1.1).

Before we move our discussion on measuring replacement levels through R_0 let us understand the technicalities in R_0. The quantity R_0 in demography [4, 5, 9] is defined as

$$R_0 = \int_{x \in Rep} f(x, G) g(x, G) dx, \tag{1.1}$$

where $f(x, G)$ is the age-specific fertility rates (only girl children born) of women of age x in the year t, $g(x, G)$ is the survival probability function for the women to live up to the age x in the year t. In (1.1), $x \in Rep$ indicates that we are integrating with respect to x over all the reproductive ages (Rep) of women. The length of generation L in population theories using R_0 (demography) is defined as

$$L = \frac{1}{R_0}\left[\int_{x \in Rep} x f(x, G) g(x, G) dx\right]. \tag{1.2}$$

The function $f(x, G)$ could be computed from the observed data either from Census records or from a large population survey. The function $g(x, G)$ is obtained from life tables (either 5-year abridged or complete life tables). The product $f(x, G)g(x, G)$ gives us a product of fertility rates and survival probability of the newborn girl babies surviving up to each age x, and integrating this product gives us the number of new girls that will be produced by a cohort of women in their reproductive life time. Hence, R_0 is a measure of the average rate of replacement of a woman by girl babies who will survive up to her reproductive age by combining our current knowledge of fertility rates to produce girl babies and current age specific mortality rates of women in the reproductive ages. Since the sex ratio at birth is close to 1 (approximately 1.05), R_0 can also be treated as an approximate measure to understand the average replacement rate for men in their reproductive lifetime [21, 22, 23]. Please note that much before in epidemiology the notation of R_0 was used in demographic literature [24, 25].

The technicalities of gross and net reproductive rates and the data used in their computations were well discussed in other articles, for example [24, 25, 26, 27, 28, 29]. Although there were some concerns regarding limitations of R_0 in terms of interpretations due to using of life table survival rates [2], hardly there were any concerns whether R_0 is able to predict the stationarity of actual populations (non-life table populations). The generations and time sensitives of the generations in computation of NRR and the associated lags were discussed in earlier studies [30, 31, 32, 33, 34, 35, 36, 37, 38]. Moreover, the concerns raised on the usage of life table-based survival rates in the computation of R_0 did not find much interest among demographers and there are no evidences of any alternate strategies. The quantity R_0, in general, is accepted widely for capturing the true rate of natural increase of a population whose fertility and mortality data for a period is used in the computation. Mortality data and fertility data are used for an age range, and survival probabilities are adopted from the corresponding life tables. The value of R_0 can be used to obtain the rate of natural increase, r by the relation $r = L^{-1}\ln R$. We see that $r = 0$ if, and only if, $R_0 = 1$. However, the timing of the data used in the computation of R_0 is very crucial. One of the limitations of R_0, especially, in its utility with respect to timeliness is that the value of the replacement rate equal to 1 would not be effective for the period in which it was computed [2]. That is, although the value of R_0 is equal to one is generally perceived as the population attained stationary, however, population do not attain stationary in the same period for which the data was used. To overcome the the limitations of conventional NRR formula, three new formulae were introduced in [2] which were denoted by Q_0, $Q_0^{'}$, and $Q_0^{''}$. We describe them briefly below:

The quantity Q_0 was defined as

$$Q_0 = \int_{\alpha}^{\beta} p_1(x,t)p_2(x,t)dx, \tag{1.3}$$

where $p_1(x,t) = B_x^t / W_x^t$ and $p_2(x,t) = F_x^{t+x} / B_0^t$. Here B_x^t is the total number of female children born to the women of age x for the year t, W_x^t is the effective number of women of age x for the year t, and $B_0^t = \sum_{\alpha}^{\beta} B_x^t$. The quantities α and β are the reproductive age range of women W_x^t. The survivors of B_0^t after several years would be part of the set, F_α^β for $F_\alpha^\beta = \left\{F_\alpha^{t+\alpha}, F_{\alpha+1}^{t+\alpha+1}), \ldots, F_\beta^{t+\beta}\right\}$, where F_x^{t+x} is the size of the female population of age x in the year $t+x$ who was born in the year t, for $x = \alpha, \alpha+1, \ldots, \beta$. where F_x^{t+x} is the size of the female population of age x in the year $t+x$ who was born in the year t, for $j = \alpha, \alpha+1, \ldots, \beta$. The probability of actual survival of girl babies to age x who were born in the year t will be $p_2(x, t)$ for $x = \alpha, \alpha + 1, \ldots, \beta$. The quantity Q_0 in (1.3) will be equal to R_0 under the stable population [39, 40, 41, 42, 38, 43, 44, 45, 46]. The formulae in (1.3), is a continuous version of the discrete formula introduced in [2].

The quantity $Q_0^{'}$ in [2] was defined as

$$Q_0^{'} = \frac{\sum_{s=t}^{s=t+\beta} B\left(s : \sum_{\alpha}^{\beta} W_x^t\right)}{\sum_{\alpha}^{\beta} W_x^t}, \tag{1.4}$$

where $B\left(s : \sum_{\alpha}^{\beta} W_x^t\right)$ is the total number of girl babies born for survivors of the women $\sum_{\alpha}^{\beta} W_x^t$ until these women completed their childbearing period.

The quantity $Q_0^{''}$ in [2] was defined as

$$Q_0^{''} = \frac{\sum_{c=\alpha}^{c=\beta} B_c^t}{\sum_{c=\alpha}^{c=\beta} W_c^t} \tag{1.5}$$

where B_c^t is the total number of children born to the women who completed their childbearing at the age c for $c \in [\alpha, \beta]$. The quantity W_c^t is the number of women in the year t who have completed their childbearing at the age c. Both the definitions in (1.4) and (1.5) are newly introduced in [2] with an aim to over the limitations of NRR described above.

This timeliness aspect of R_0 in demographic studies has been ignored so far, perhaps, the computation involves generation time which usually is of several years. As long as we use life table-based survival rates as a multiplier, we will be unable to overcome this limitation of timeliness. On the other hand, by using period life table population computed from the mortality data and then comparing the remaining life as described in the partition Theorem in [47] and in the current article could avoid timeliness limitation.

For the reasons above, R_0 value is often treated as a population replacement measure rather than measure to understand female replacement level only. Although, $R_0 = 1$ in demography indicates exact replacement of a generation of population, but that replacement phase is not going to complete in the year in which two functions $f(x, G)$ and $g(x, G)$ are computed [47]. Further, $R_0 = 1$ in the year t for a population does not guarantee that this population is a stationary population in the year t, because there is no guarantee that the schedules of mortality and fertility that went into computation of $f(x, G)$ and $g(x, G)$ will remain the same during the years after t. So the question of what is the current level of replacement (or stationary status) of a population in the year of t based on two functions $f(x, G)$ and $g(x, G)$ computed in the year is still unanswered by computing R_0.

Through this article, we have advanced the methods and arguments built on recent advancements in stationary population theory and its role in understanding the true status of population replacement levels in a country or in a region [2, 47, 48, 49, 50, 51, 52]. Our methods proposed could lead to a proper quantitative analysis of population replacement levels in the world and in countries/regions which is one of the important annual country-level population exercises conducted by the United Nations [53].

In section 2, we prove two sets of theorems for the conditions of population attaining stationary. One set of theorems is proved by considering the age of individuals in a population as discrete elements of a set, and the second set of theorems is proved by considering age distribution in the population as continuous age. In section 3, we describe the advantages of new measures and compare them with the traditional NRR of populations. In section 4, we demonstrate results for nine selected countries that are at different demographic population transitions. We conclude the article in section 5.

2. Theories on Population Attaining Stationary Status

To address the limitations of NRR discussed in the previous section in bringing timely conclusions on population replacement level we have taken the notation and terminology from the recent articles and then we built on their theories. The partition theorem [47] was used as a tool to decide if a given population in a country or a region attained stationary status. We state the theorem in [47] below:

Theorem 1.

Statement of Partition Theorem of Population [47]*: Stationary Population Identity (SPI) partitions a randomly selected large population into stationary and non-stationary components (if such components are non-empty).*

Partition theorem uses life table identity or stationary population identity [47] as a tool to decide for which of the single-year ages, the fraction of the population who are at age x in the year t is exactly equal to the fraction of the people in the life table constructed for the year t who have x years remaining. They called the set of all ages x for which the above identity satisfied as the stationary component of the population and other ages (say y's) as the non-stationary component of the population. Sum of the stationary and non-stationary components would become total population. Let S and N be stationary and non-stationary components of the population described above whose elements consist of ages x_i's and $y_j^{'}$s, respectively, i.e.

$$\begin{aligned} S &= \{x_i : i = 1, 2, \ldots \omega_1\} \\ N &= \{y_j : j = 1, 2, \ldots \omega_2\} \end{aligned} \tag{2.1}$$

The stationary population identity (or the life table identity) is satisfied for the ages x_i's and it is not satisfied for the ages $y_j^{'}$s. See Figure 2.1. When for all the ages the above identity is satisfied, we call the population is at replacement level (stationary), i.e. when $N = \phi$ (empty set), the total population is in the stationary state. Even if $N \neq \phi$, still there are ages for which life-table identity satisfies. When $|S| > |N|$ in (2.1) i.e. the size of the set S is larger than the size of the set N, the stationary component is more than the non-stationary component and vice-versa when $|S| < |N|$. The concept of partition theorem is helpful for instant measurement of population replacement in a year in which life table identity or stationary population identity is constructed. This partitioning principle can be repeated using every year's population data and using the corresponding life table constructed for that year's data to check the status of stationary in the population.

Suppose the total number of births in a year $b(x)$ born to women $w(x)$ of age x is divided into female births $b(x, G)$ and male births $b(x, B)$, respectively. Let $g(x)$ be the survival function corresponding to $b(x)$. **In general for all the countries' data, the inequality** $b(x) < w(x)$ **is satisfied, so we assume** $\frac{b(x)}{(w)} < 1$. Then we will have

$$\int_{x \in Rep} \frac{b(x)}{w(x)} g(x) dx > \int_{x \in Rep} \frac{b(x, G)}{w(x)} g(x, G) dx \tag{2.2}$$

If $R_0 = 1$, then

$$\left(\int_{x \in Rep} \frac{b(x)}{w(x)} g(x) dx \right)^{-1} < 1.$$

Even if $R_0 \neq 1$, the age-specific fertility rates are less than the survival rates in most of the situations, which gives us the inequality

$$\left(\int_{x \in Rep} g(x) dx \right)^{-1} > \left(\int_{x \in Rep} \frac{b(x)}{w(x)} g(x) dx \right)^{-1} > \left(\int_{x \in Rep} \frac{b(x)}{w(x)} dx \right)^{-1}. \tag{2.3}$$

Similarly, we can obtain another inequality

$$\left(\int_{x \in Rep} g(x, G) dx \right)^{-1} > \left(\int_{x \in Rep} \frac{b(x, G)}{w(x)} g(x, G) dx \right)^{-1} > \left(\int_{x \in Rep} \frac{b(x, G)}{w(x)} dx \right)^{-1} \tag{2.4}$$

.

Note that the function $g(x, G)$ provides the survival probability of the newborn girl babies surviving up to each age x and obtained from life tables (either 5-year abridged or complete life tables). The quantity $b(x)$ is the total births (female and male combined) born to women $w(x)$ of age x such that $b(x) = b(x, G) + b(x, B)$. For this reason, we have defined $g(x)$ and $g(x, G)$ separately. The function $g(x)$ accounts for the total births $b(x)$ and the function $g(x, G)$ accounts for only female births born to $w(x)$.

2.1. Age as discrete elements in a set.

Let $L(x, t)$ and $P(x, t)$ be the life table and observed populations in the age x and at time t. Let

$$\left\{\frac{L(x,t)}{\sum_{x=0}^{\omega} L(x,t)}, \frac{P(x,t)}{\sum_{x=0}^{\omega} P(x,t)}\right\} \tag{2.5}$$

be the pair of values computed for each $x \in A = \{0, 1, 2, \ldots, \omega\}$ at time t. We define two sets E_t and $N_t^{\dagger}$ as below

$$E_t = \left\{x \in A: \frac{L(x,t)}{\sum_{x=0}^{\omega} L(x,t)} = \frac{P(x,t)}{\sum_{x=0}^{\omega} P(x,t)}\right\} \tag{2.6}$$

$$N_t = \left\{x \in A: \frac{L(x,t)}{\sum_{x=0}^{\omega} L(x,t)} \neq \frac{P(x,t)}{\sum_{x=0}^{\omega} P(x,t)}\right\} \tag{2.7}$$

When $|E_t| > |N_t|$ we consider the partition of stationary component in a population (or a country or a region) is larger than the partition of non-stationary component. $|E_t|$ and $|N_t|$ represent the cardinalities of two sets E_t and N_t, respectively. If $|E_t| < |N_t|$, then, we consider the partition of non-stationary component in a population is larger than the partition of stationary component. We have,

$$E_t \cup N_t = A \text{ and } E_t \cap N_t = \phi \text{ at each time } t,$$

and

$$A = \begin{cases} E_t \text{ if } N_t = \phi \\ N_t \text{ if } E_t = \phi, \end{cases} \tag{2.8}$$

where ϕ is the empty set. Also, $|E_t| \leq |A|$ and $|N_t| \leq |A|$. Since $A = E_t \cup N_t$, so the elements of A are partitioned into E_t and N_t, so we can write equivalence of these sets as

$$A \sim \begin{cases} E_t \text{ if } N_t = \phi \\ N_t \text{ if } E_t = \phi, \end{cases} \tag{2.9}$$

At any given time t, one of the inequalities $|E_t| > |N_t|$ or $|E_t| \leq |N_t|$ is true. The relation (2.9) also holds because $E_t \cap N_t = \phi$. This gives us

$$|E_t| < |N_t| < |A|$$

or

$$|N_t| < |E_t| < |A|, \tag{2.10}$$

$$|E_t| + |N_t| = |E_t + N_t|. \tag{2.11}$$

The elements in the set N_t are mapped to a quantity which could be either positive or negative. If we define two functions, f_1 and f_2 such that

$$f_1 : N_t \to \left(\frac{L(x,t)}{\sum_{x=0}^{\omega} L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega} P(x,t)} \right), \tag{2.12}$$

$$f_2 : E_t \to \left(\frac{L(x,t)}{\sum_{x=0}^{\omega} L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega} P(x,t)} \right). \tag{2.13}$$

The function f_1 is bijective and f_2 is not because f_2 is a constant function taking only singleton 0. In the next following theorems, we will study further properties of f_1 by studying the quantity

$$\left(\frac{L(x,t)}{\sum_{x=0}^{\omega} L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega} P(x,t)} \right)$$

for elements in the set N_t.

Theorem 2.—$\sum_{x \in A} \left| \frac{L(x,t)}{\sum_{x=0}^{\omega} L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega} P(x,t)} \right| = 0$, *if, and only if,*

$$\frac{L(x,t)}{\sum_{x=0}^{\omega} L(x,t)} = \frac{P(x,t)}{\sum_{x=0}^{\omega} P(x,t)}$$

for all $x \in A$.

<u>Proof.:</u> We omit the proof as one can easily see the result is true. □

Consider an unordered age set A_1, where

$$A_1 = \left\{ x_1, x_2, \ldots, x_{k_1}, y_1, y_2, \ldots, y_{k_2}, z_1, z_2, \ldots, z_{k_3} \right\}$$

such that A_1 A, i.e. A_1 and A have $1-1$ correspondence. Further, A_1 is written as union of unordered sets X, Y and Z such that

$$A_1 = X \cup Y \cup Z, \tag{2.14}$$

where

$$X = \left\{ x_1, x_2, \ldots, x_{k_1} \right\},$$

$$Y = \left\{ y_1, y_2, \ldots, y_{k_2} \right\},$$

$$Z = \{z_1, z_2, \ldots, z_{k3}\},$$

such that

$$\sum_{x \in X}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) > 0, \tag{2.15}$$

$$\sum_{x \in Y}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) < 0, \tag{2.16}$$

$$\sum_{x \in Z}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) = 0. \tag{2.17}$$

Theorem 3.—$\sum_{x \in A}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) = 0$, *if, and only if,*

$$\sum_{x \in X}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) = -\Sigma_{x \in Y}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)$$

for all $x \in A$.

Proof.

Suppose

$$\sum_{x \in A}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) = 0. \tag{2.18}$$

Then, one trivial possibility would be

$$\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} = \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)} \text{ for all } x \in A.$$

This gives us

$$\sum_{x \in X}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) = -\sum_{x \in Y}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right). \tag{2.19}$$

Let us now consider a non-trivial situation. Suppose

$$\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} \neq \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)} \tag{2.20}$$

holds for some values of x within the set A. Then, there must be a minimum of two such values of x in A such that

$$\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) > 0 \tag{2.21}$$

for exactly one values of x in X (say, $x = a_1$), and

$$\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) < 0 \tag{2.22}$$

for exactly one values of x in Y (say, $x = a_2$) and simultaneously (2.17) also holds. This implies, (2.18) is true for one value of x in X and one value of y in Y. Therefore

$$\left(\frac{L(a_1,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(a_1,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) = -\left(\frac{L(a_2,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(a_2,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right). \tag{2.23}$$

Suppose, there are three values of x in A for which (2.20) holds and suppose for the rest of the x in A values (2.17) holds. Then, there arises two following cases:

case (i): two values of x in $X \subset A$ satisfy (2.15) and one value of y in $Y \subset A$ satisfy (2.16).

case (ii): one value of x in $X \subset A$ satisfy (2.15) and two values of y in $Y \subset A$ satisfy (2.16).

Under case (i), (2.15) holds for the two values of x in X, say $x = a_1, a_2$ and (2.16) holds for one value of y in Y, say, $x = a_3$. This implies (2.17) is true for three values of x in A (two values of X, one value of Y and rest from the set Z). Therefore

$$\begin{aligned}&\left(\frac{L(a_1,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(a_1,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) + \left(\frac{L(a_2,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(a_2,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)\\ &= -\left(\frac{L(a_3,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(a_3,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right).\end{aligned} \tag{2.24}$$

Under case (ii), (2.15) holds for the one values of x in X, say $x = b_1$ and (2.16) holds for two values of y in Y, say, $x = b_2, b_3$. This implies (2.17) is true for three values of x in A (one values of X, one value of Y and rest from the set Z). Therefore

$$\begin{aligned}&\left(\frac{L(b_1,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(b_1,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)\\ &= -\left(\frac{L(b_2,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(b_2,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)\end{aligned} \tag{2.25}$$

$$-\left(\frac{L(b_3,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(b_3,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) \tag{2.26}$$

The forward part of the proof is done for the situation of three values of x in A.

case (iii): n values of x in A. Suppose, there are $n(n > 3)$ values of x in A for which (2.20) holds and suppose for the rest of the x in A values (2.17) holds. The combination of the number of values of x in A and y in Y are listed in Table 1.

For each of the combinations of numbers of x and y values in X and Y in Table 1, (2.15) and (2.16) holds. Similar to the three-element situation explained above, we can see that

$$\begin{aligned}&\sum_{i=1}^{n-1}\left(\frac{L(b_i,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(b_i,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)\\&=-\left(\frac{L(b_n,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(b_n,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)\end{aligned} \tag{2.27}$$

where $x = b_i \in X$ for $i = 1, 2, \ldots, n - 1$, and $y = b_n \in Y$, and

$$\begin{aligned}&\left(\frac{L(b_1,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(b_1,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)\\&=\sum_{n}^{i=2}\left(\frac{L(b_i,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(b_i,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)\end{aligned} \tag{2.28}$$

where $x = b_1 \in X$ and $y = b_i \in Y$ for $i = 2, \ldots, n$. All other combinations of X and Y in Table 1 can be expressed. With this, we proved (2.19) in general and the forward part of the theorem is proved. Conversely, suppose (2.19) is given. We need to prove (2.18). Since (2.19) holds, we have

$$\sum_{x\in X}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)+\sum_{x\in Y}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)=0. \tag{2.29}$$

Adding the quantity

$$\sum_{x\in Z}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)$$

on both sides of (2.29), we get

$$\begin{aligned}&\sum_{x\in X}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)+\sum_{x\in Y}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)\\&+\sum_{x\in Z}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)\\&=\sum_{x\in Z}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)}-\frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right)\end{aligned} \tag{2.30}$$

The equation (2.30) becomes

$$\sum_{x \in A}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) = 0\,. \tag{2.31}$$

Corollary 4.—$\sum_{x \in A}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) \neq 0$ *whenever*

$$\sum_{x \in X}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) \neq -\sum_{x \in Y}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right). \tag{2.32}$$

Proof.: When (2.32) holds, then

$$\sum_{x \in A}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) < 0,$$

or

$$\sum_{x \in A}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) > 0.$$

Theorem 5.—*(a)* $\sum_{x \in A}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) > 0$ *if, only if,*

$$\sum_{x \in X}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) > -\sum_{x \in Y}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right). \tag{2.33}$$

(b) $\sum_{x \in A}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) < 0$ *if, only if,*

$$\sum_{x \in X}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) < -\sum_{x \in Y}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right). \tag{2.34}$$

Proof.: (a) Forward part. Suppose

$$\sum_{x \in A}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) > 0. \tag{2.35}$$

This imples,

$$\sum_{x \in X}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) + \sum_{x \in Y}\left(\frac{L(x,t)}{\sum_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\sum_{x=0}^{\omega}P(x,t)}\right) > 0, \tag{2.36}$$

and (2.33) follows. For the ‘only if’ part, suppose (2.33) is true, then (2.36) follows, and

$$\sum_{x \in X}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) + \sum_{x \in Y}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) + \sum_{x \in Z}\left(\frac{L(x,t)}{\Sigma_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\Sigma_{x=0}^{\omega}P(x,t)}\right) > 0, \tag{2.37}$$

and, it follows the required inquality (2.35).

(b) Forward part and the ‘only if’ part are proved similar to proof in (a). ☐

2.2. Continuous age interval.

Let $B = [0, \omega]$ be the interval of all ages on which we are interested to understand the role of the partition theorem (Theorem 1). Here the age variable x is represented as a continuous value such that $0 \leq x \leq \omega$ and

$$\int_0^{\omega}\left|\frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}\right|dx = 0 \tag{2.38}$$

implies the population at time t attains stationary. There exits two continuous mappings, one mapping from

$$[0, \omega] \text{ to } \frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx}, \tag{2.39}$$

and second mapping from

$$[0, \omega] \text{ to } \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}. \tag{2.40}$$

Let us denote the two mappings (2.39) and (2.40) by f_3 and f_4, respectively, such that

$$f_3: [0, \omega] \to \frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx}, \tag{2.41}$$

$$f_4: [0, \omega] \to \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}. \tag{2.42}$$

Suppose the interval $[0, \omega]$ is partitioned into three components U, V, W such that,

$$\begin{aligned} B &= U \cup V \cup W \\ U \cap V \cap W &= \phi \end{aligned} \tag{2.43}$$

The set U consists of all the elements of B such that

$$\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} > \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx}, \tag{2.44}$$

the set V consists of all the elements of B such that

$$\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} < \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx}, \tag{2.45}$$

and the set W consists of all the elements of B such that

$$\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} = \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx}. \tag{2.46}$$

The functions f_3 and f_4 have the same domain $[0,\omega]$. We can define a new function $\psi(a,b)$ on the domain $[0,\omega] \times [0,\omega]$ for each $b \in [0,\omega]$. The function defined on $[0,\omega]$ is a measurable on $[0,\omega]$. The function $\psi(a,t)$ as $t \to b$ for a fixed $b \in [0,\omega]$ is almost everywhere converges on $[0,\omega]$. **Here** $\psi(a,t) : [0,\omega] \times [0,\omega] \to \mathbb{R}$.

Theorem 6.— *The Lebesgue integral* $\int_{[0,\omega]} \psi(a,b)da$ *exists for each* $b \in [0,\omega]$.

Proof.: Let us consider $f_3(x,t)$ and $f_4(x,t)$ at time t. Then, for an arbitrary $b \in [0,\omega]$,

we have

$$\lim_{t \to b} \int_{[0,\omega]} \psi(a,b)da = \int_{[0,\omega]} \lim_{t \to b} \psi(a,b)da. \tag{2.47}$$

Since $f_3(x,t)$ and $f_4(x,t)$ re bounded functions, $\psi(a,b)$ is a Lebesgue integrable. Suppose $(b_n)_{n \geq 1}$ be a sequence of points on $[0,\omega]$. Then,

$$\int_{[0,\omega]} \psi(a,b_n)da \tag{2.48}$$

exists, and

$$\lim_{n \to \infty} \phi(b_n) = \phi(b), \tag{2.49}$$

where

$$\lim_{n \to \infty} \phi(b_n) = \int_{[0,\omega]} \psi(a,b)da. \tag{2.50}$$

Suppose $\Delta\varphi(x) = \hat{\varphi}(x) - \breve{\varphi}(x)$, where

$$\widehat{\varphi}(x) = \sup\left|\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx}\right| \text{ for } x \in \boldsymbol{B}, \tag{2.51}$$

$$\breve{\varphi}(x) = \inf\left|\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx}\right| \text{ for } x \in \boldsymbol{B}. \tag{2.52}$$

Both $\widehat{\varphi}(x)$ and $\breve{\varphi}(x)$ exist and $\Delta\varphi(x)$ is bounded. Two PDEs that govern the dynamics over age within $[0, \omega]$ are

$$\frac{\partial P(x,t)}{\partial x} = r_1 \frac{\partial P(x,t)}{\partial t}, \tag{2.53}$$

$$\frac{\partial L(x,t)}{\partial x} = -r_2 \frac{\partial L(x,t)}{\partial t}, \tag{2.54}$$

where r_1 is defined as

$$r_1 = \int_{x \in B} x\left(\frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx}\right)dx, \tag{2.55}$$

where r_2 is defined as

$$r_2 = \int_{x \in B} x\left(\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx}\right)dx\,. \tag{2.56}$$

Theorem 7.— f_3. f_4 is Lebesgue integrable on $[0, \omega]$.

Proof.: f_3 is measurable and f_4 is measurable and both f_3 and f_4 are boinded on $[0, \omega]$.

So, f_3 is a Lebesgue integrable on $[0, \omega]$ and f_4 is a Lebesgue integrable on $[0, \omega]$. Also, $|f_3(x)f_4(x)|$ is bounded for $x \in [0, \omega]$ (because $(f_3, f_4) \le \|f_3\|\|f_4\|$). Since $b_n \to b$ for some $b \in [0, \omega]$, we have

$$\begin{aligned} \int_{[0,\omega]} f_3(x) f_4(x) db(x) &= |f_5(x_1, x_2)| \int_{[0,\omega]} |f_3(x_1, x_3)||f_4(x_2, x_3)| db(x) \\ &\le |f_5(x_1, x_2)| \left(\int_{[0,\omega]} |f_3(x_1, x_3)|^2 db(x) \right)^{1/2}. \end{aligned} \tag{2.57}$$

$$\left(\int_{[0,\omega]} |f_4(x_2, x_3)| db(x) \right)^{1/2}, \tag{2.58}$$

where x_1, x_2, and x_3 are in $[0, \omega]$, and f_5 could be equal to either f_3 or f_4. ☐

Theorem 8.— $\int_{x \in B} \left(\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx} \right) dx = 0$, *if, and only, if,*

$$\begin{aligned} &\int_{x \in U} \left(\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx} \right) dx = \\ &- \int_{x \in V} \left(\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx} \right) dx. \end{aligned} \tag{2.59}$$

Proof.: We first prove the forward part. Suppose

$$\int_{x \in B} \left(\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx} \right) dx = 0 \tag{2.60}$$

is true. The two elements $\{0\}$, $\{\omega\}$ could be in any of the sets U, V, W and all possible combinations of their locations are listed in Table 2.

Let us consider the case (i) $0 \in U$ and $\omega \in U$ in Table 2. Since $0 \in U$, we have

$$\frac{L(0,t)}{\int_{x=0}^{\omega} L(x,t)dx} > \frac{P(0,t)}{\int_{x=0}^{\omega} P(x,t)dx}. \tag{2.61}$$

This implies f_4 touches f_3 from below at some value of the set B. Refer to Figure 2.1 for an overall concept. Let z_1 be that point in B and $z_1 \in W$ because at z_1 the condition (2.46) is satisfied. The function f_4 either crosses f_3 such that (2.45) is true for $x > z_1$ or the function f_4 satisfies (2.44) for $x > z_1$ or the function f_4 satisfies (2.42) for $x > z_1$. If f_4 crosses at z_1 to satisfy (2.45) then f_4 must touch f_3 from the above at least once because (2.60) is true. Suppose it touches once from the above, say it touches at $x = z_2$, and $f_4 < f_3$ for $x \in [z_2, \omega]$ then

$$\begin{aligned} &\int_0^{z_1} \left(\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx} \right) dx + \int_{z_2}^{\omega} \left(\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx} \right) dx \\ &= - \int_{z_1}^{z_2} \left(\frac{L(x,t)}{\int_{x=0}^{\omega} L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega} P(x,t)dx} \right) dx \end{aligned} \tag{2.62}$$

holds (since (2.60) is true). Refer to Figure 2.2 for understanding (2.62).

If $f_4 < f_3$ holds only for $x \in [z_2, z_3]$, where (2.46) is satisfied at z_3, then f_4 at $x > z_3$ will have three possibilities those were discussed above. Let us say f_4 touches f_3 from above at $x = z_4$, and $f_4 < f_3$ for $x \in [z_2, \omega]$, then

$$\int_0^{z_1}\left(\frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}\right)dx + \int_{z_2}^{z_3}\left(\frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}\right)dx$$
$$\int_{z_4}^{\omega}\left(\frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)} - \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}\right)dx = -\int_{z_1}^{z_2}\left(\frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}\right)dx$$
$$-\int_{z_3}^{z_4}\left(\frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}\right)dx \quad (2.63)$$

Refer to Figure 2.3 for understanding (2.63). Suppose after $f_3 = f_4$ at $x = z_1$, after $f_3 = f_4$ is satisfied for $x \in [z_1, z_2]$, $f_3 > f_4$ is satisfied for $x \in [z_2, z_3]$, $f_3 > f_4$ is satisfied for $x \in [z_3, \omega]$, then

$$\int_0^{z_1}\left(\frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}\right)dx + \int_{z_3}^{\omega}\left(\frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}\right)dx = -\int_{z_2}^{z_3}\left(\frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}\right)dx \quad (2.64)$$

Refer to Figure 2.4 for understanding (2.64). Instead of the above situation, suppose after $f_3 = f_4$ at $x = z_1$, f_4 starts satisfying (2.60) for $x > z_1$, then after some x for $x = z_1$, f_4 must cross f_3, (say at z_2) from below to satisfy (2.45) and crosses f_3 again from above, because (2.60) is true. If f_4 does not cross from above after $x > z_1$, then condition (2.60) is not satisfied and there arrives a contradiction for $\omega \in U$ in case (i) of Table 2. We could build several other possibilities of positions of f_3 and f_4 after $x = z_1$ under the case (i). For all other combinations of locations of ages $\{0\}$ and $\{\omega\}$ in Table 2 we can construct equations of types (2.62) – (2.64).

Let us consider the 'only if' part of Theorem 8. That is, assume (2.59) is true. Note that

$$\int_{x \in W}\left(\frac{L(x,t)}{\int_{x=0}^{\omega}L(x,t)dx} - \frac{P(x,t)}{\int_{x=0}^{\omega}P(x,t)dx}\right)dx = 0. \quad (2.65)$$

Adding (2.59) to (2.65), we get (2.60). That proves the theorem. □

Remark 9.—The quantity $R_0 = 1$ for an year does not mean $N = \phi$ will hold for the same year.

Proof.: When $R_0 = 1$ then replacement level in the population is not realized immediately for the reasons described in the section II. Where as $N = \phi$ Indicates stationary population identity is realized in the population in the same year in which $N = \phi$ is observed. Hence R_0 based approach is not an instant measure for stationary population model in the population.

The objective of the stable stationary population model we present here is identical to that outlined in [21, 54, 55, 56]. See [21, 55] for stable population models more generally—to trace the dynamic characteristics of a population that starts off with an arbitrary age structure and is submitted from that moment on to a specified demographic regime. The regime for the stationary model assumes a closed population, one sex, fixed death rates and constant, replacement-level births.

The assumptions of stable models including the stationary model reveal the limitations of its application to real-world populations of humans as well as of non-human species—i.e., actual populations are seldom closed, most consist of two sexes, and vital rates are virtually never fixed for long periods. None-the-less we believe that understanding the stationary population model as a special case of stable population theory is important in both basic and practical contexts. First, the stationary model unites cohorts and populations both conceptually and mathematically since stationary populations and life table populations are identical. Thus per capita birth and death rates in cohorts equal these same rates in a replacement population. Second, the stationary model provides demographic baselines for use in comparing the properties of the subset of actual populations throughout the world that are presently at or near replacement levels (e.g., Italy; Japan). Thus historical and current age structures of countries can be compared with each of their respective stationary (theoretical) distributions using period life tables corresponding to the year. Third, the demographic "distance" from stationarity as defined by stable theory can be estimated from projections for different populations using period life tables and replacement-level fertility. The results can provide an idealized framework for considering the world population as a series of subpopulations (countries) each of which is converging to a hypothetical stable stationary age distribution.

3. Advantages of New Approach

New ideas proposed using the partition theorem uses the age-structured data from life tables and actual population census. Traditional NRR is obtained through the equation (3.1),

$$NRR = \frac{1}{2.05} \sum_{x=15}^{49} {}_1L_x^t f_x^t \tag{3.1}$$

where, 2.05 is generally taken as the sex ratio at birth, i.e. on an average 1 female child is born per 1.05 male children, ${}_1L_x^t$ is the life table population of women at age x obtained from single-age life tables constructed for the year t, f_x^t is the age-specific fertility rates computed for the year t. When $NRR = 1$ for the year t, we say that the population is stationary in the year t. See [23, 32, 30, 34] for more discussion on the formula for NRR and see [2] for the limitations of NRR. The differences between the data that is required for computing NRR, and new NRR formulas suggeted in (1.3) – (1.5) are thoroughly discussed in [2]. See Table 3 for data requirements in the traditional NRR and the partition theorem approach of this article.

New measure of stationary is not a replacement for the measure to understand whether a population is stationary through traditional NRR approach. However, the new measure can provide more timely measure of stationary status because momentum generated through the equation (3.1) is avoided in the new approach. Another key distinction is that by the population theorem approach, stationary component of the population is instantly identified which does not have any momentum in the component.

4. Results

We have plotted the proportion of the population at each age within the age-group [0, 100] of single-age life table in 2017 for nine countries for which we have readymade life tables are available [57]. These countries in alphabetical order are, namely, Australia, Canada, Germany, Greece, Hong Kong, Italy, Poland, South Korea, and the U.S.A. We have also plotted the corresponding year's proportion of the population from the census data for all these countries (See Table 4). Below we provide analysis of the partition theorem-based status of stationary population component and NRR-based status of a stationary population.

We illustrate comparative aspects of population partitioning using summary information contained in two figures. The age frequency distributions for the 2017 populations of nine countries and their hypothetical life table (stationary) populations based on 2017 period life tables are jointly plotted in Figure 4.1. The red-filled areas in these plots depict the magnitude of the differences at each age. These nine sets of plots can be grouped into three subsets by row based on these age distributions. The bottom-most row of plots for Greece, Germany and Italy show that, relative to their respective stationary cases, the age distributions of their 2017 populations have a dearth of individuals under ages 30 or 40, an excess of individuals from around 40 to 60 or 70 and nearly equal proportions of individuals after age 70. The set of plots in the middle row include those for South Korea, Australia and USA with all showing an excess of individuals at ages less than around 60 relative to their hypothetical stationary populations, and a dearth of individuals at all ages beyond. The set of plots in the top row include those for Canada, Hong Kong and Poland. The partitioning for these countries is similar to that for those in the bottom row in that they show a relative dearth of individuals at younger ages followed by an excess at middle ages. However, their age distributions relative to their stationary cases differ from the countries in the bottom row in that, unlike Greece, Germany and Italy which have nearly identical age distributions as the stationary case, these countries have a dearth of individuals. Along with each country's 2017 net reproductive rates, the sum totals of the proportional age distribution differences between the hypothetical stationary populations for each country and their 2017 population (i.e., red-shaded areas of plots in Figure 3) are presented in Figure 4.2. Three aspects of these relationships merit comment. First, the age distributions of the 2017 populations of Hong Kong and South Korea depart from their respective stationary cases far greater than any of the other countries but each due to substantially different age distributions as was described above. Whereas Hong Kong's large value is due to excesses of middle aged and deaths for early and late ages, South Korea's is due to excesses of individuals at young and middle ages and a dearth at older ages. Also noteworthy is that the net reproductive rate of South Korea is substantially higher than Hong Kong's NRR. Second, the USA has the lowest value for the partitioning parameter but one of the highest NRR's. This leads to the third point which is that there is virtually no correlation between NRR and the partitioning parameter.

We anticipate the stationary proportion to reach close to 100% as the NRR values reduced further. The relation between NRR values below replacement and proportion of stationary component in a population follows a pattern, but a direct quantifiable association was not seen among the countries considered in Table 4. For example, between Canada, Australia, and Hong Kong, the NRR values are declining but the stationary component of the current population did not show a similar trend (Figure 4.2). In general, for all the countries considered in Figure 4, the proportion of people who are in the stationary component are within the range of 0.8–0.90 whereas the range of NRR values for these countries range within 0.62–0.88.

5. Conclusions

Understanding population replacement levels are one of the millennium development goals (MDG) of the United Nations [59]. Despite such important annual exercises to understand the population replacement levels and stationary status of populations, there are several limitations in their approaches discussed in our article. We argue that a proper measure needs to be adopted by the UNDP that can compute the annual level of population replacement. We are not discussing through our article other MDG goals, like global disease burden, poverty measures, etc that heavily depend on accurate population estimates [60, 61].

Deeper investigations are needed to identify the criteria required for specifying population stationarity than are currently used with the inherently static measure of NRR. This requirement is because of the paradox that a population is inherently dynamic due, not only to its age structure but also to the trajectories of birth and death rates. Thus conducting repeated comparisons of the age structure proposed in the article and NRR (through partition theorem-based) over several decades will shed important light on the dynamic component of stationarity in a country.

One of the highly intuitive results that we theoretically proved was Theorem 3. The idea was that when the sum of the difference between proportions of the lifetable population and the actual population at the age x for all such age x in a population becomes zero then the sum of the difference between proportions of populations for a set of age x values greater than zero will be exactly equal to the negative sum for all those values of age x below zero. The vice-versa of the previous statement was also proved here. Theorem 3 brings an innovative implication of the partition Theorem of [47]. Such implications and interpretations of population stationary were not seen earlier. A continuous version of Theorem 3 was stated in proved through Theorem 8. The implications of Theorem 3 and Theorem 8 was shown in Figure 2.1 as well.

Apart from the theoretical gains, the construction of sets of populations A and representing it as the union of various sets X and Y based on certain properties that they obey brings an easier and more holistic approach in handling population data sets.

A summary Figure 2.1 is central to all the novel ideas of the article. We provide first-time country-specific analysis for selected under the newly proposed measure. The traditional survival analysis-based or life table-based approaches are applicable for shorter-time projections, and what we propose through this article provides population replacement levels in a given year based on the data available around that year [62, 63, 64, 65, 66]. Our article not only adds to the recent developments in stationary population models but brings fresh thoughts on the interpretation of stationary populations and more meaningful population analysis involving both stationary and non-stationary populations. The theoretical gains of the article were enormous.

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Highlights

- The traditional formula of NRR in Demography is questioned.
- Proposed an alternative measure for the population replacement levels.
- New theories were developed on measuring instant population replacement levels.

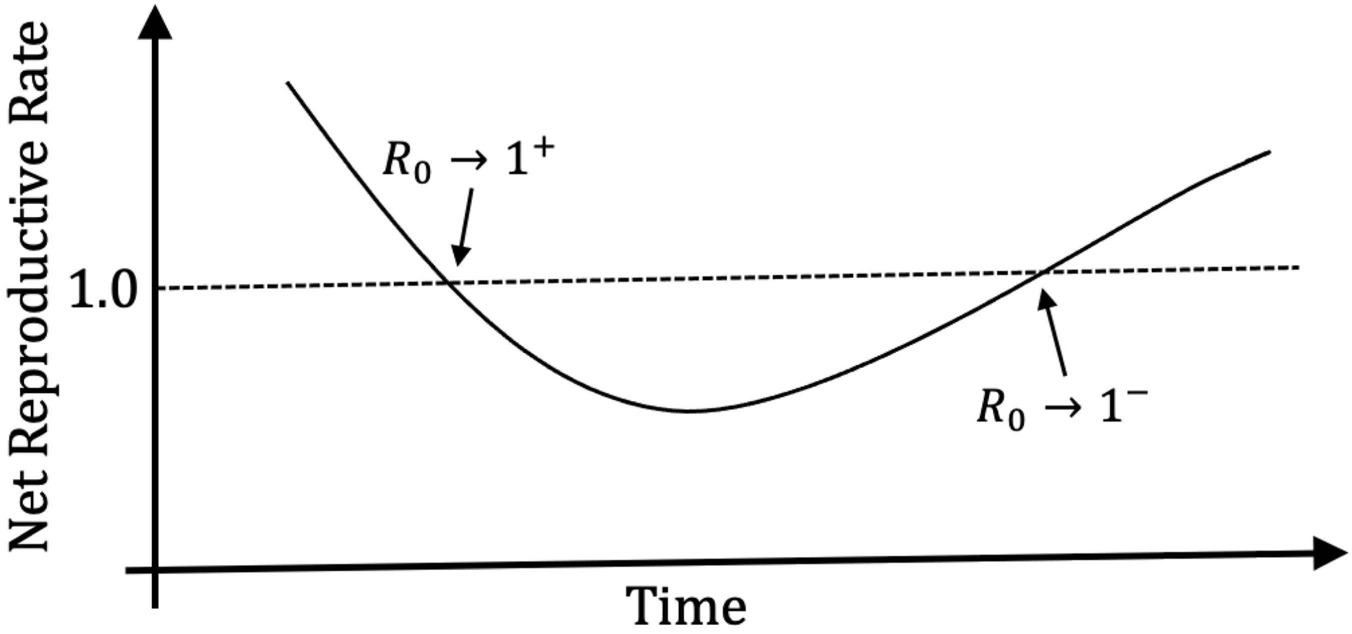

Figure 1.1.
Trend line of Net Reproductive Rate (NRR or R_0) and replacement direction. R_0 value of 1.0 of a country could be reached from the above replacement level, a situation we depict here as $R_0 \to 1^+$ or it could be reached from the below replacement level, a situation we depict here as $R_0 \to 1^-$.

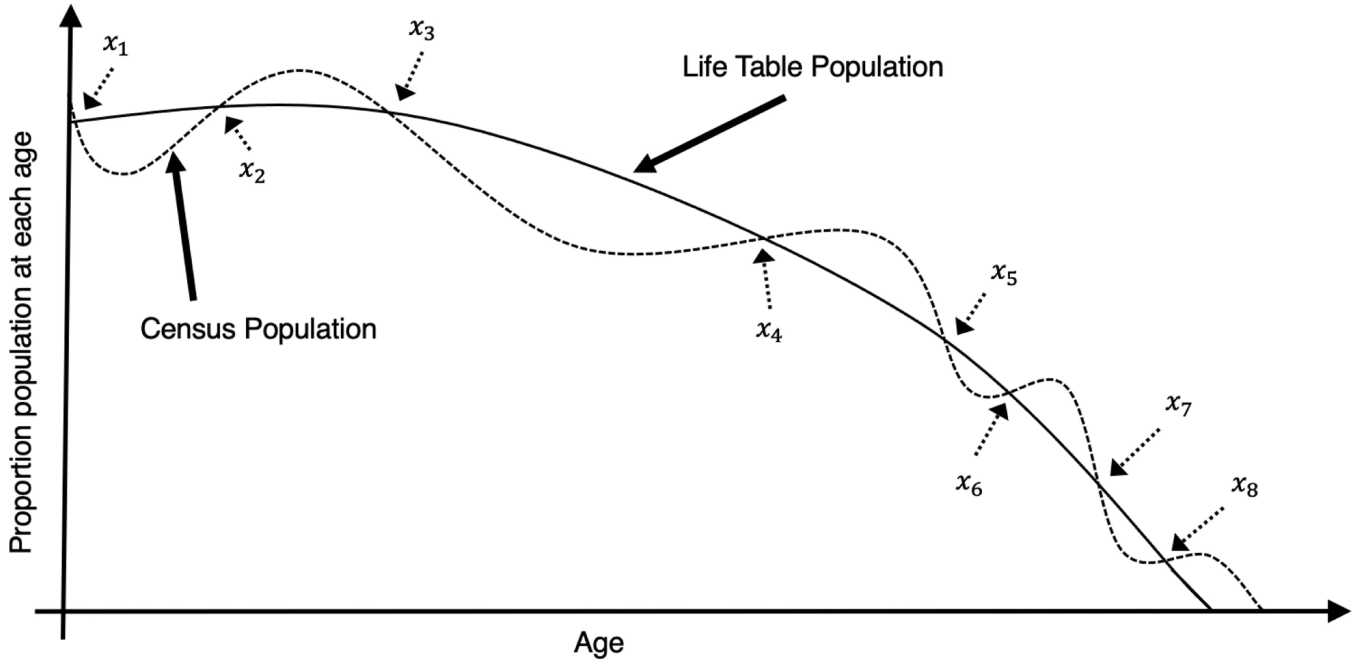

Figure 2.1.
Partitioning of stationary and non-stationary populations. Proportion population obtained from the Census data for a year is compared with the same year's life table population. The values $x_1, x_2, \ldots, x_8$ forms the stationary component set S.

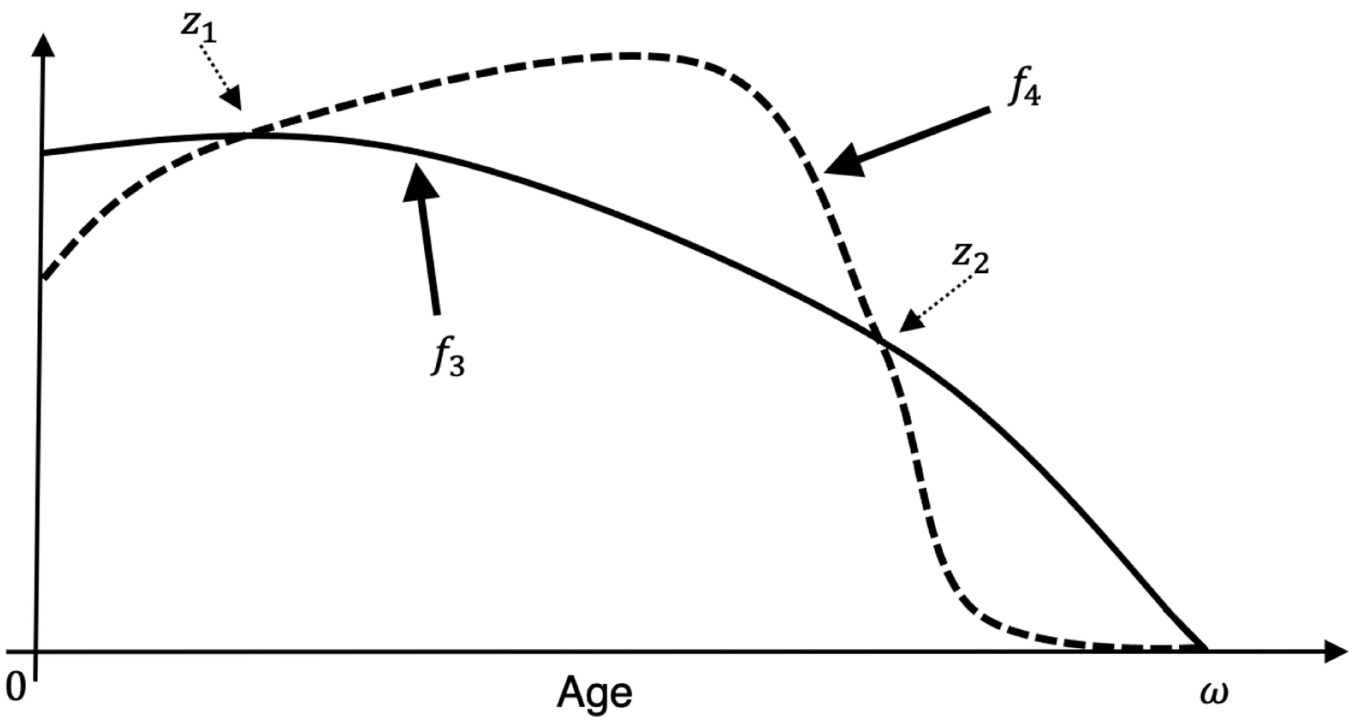

Figure 2.2.
Relationship between f_3 and f_4 associated with the equation (2.62).

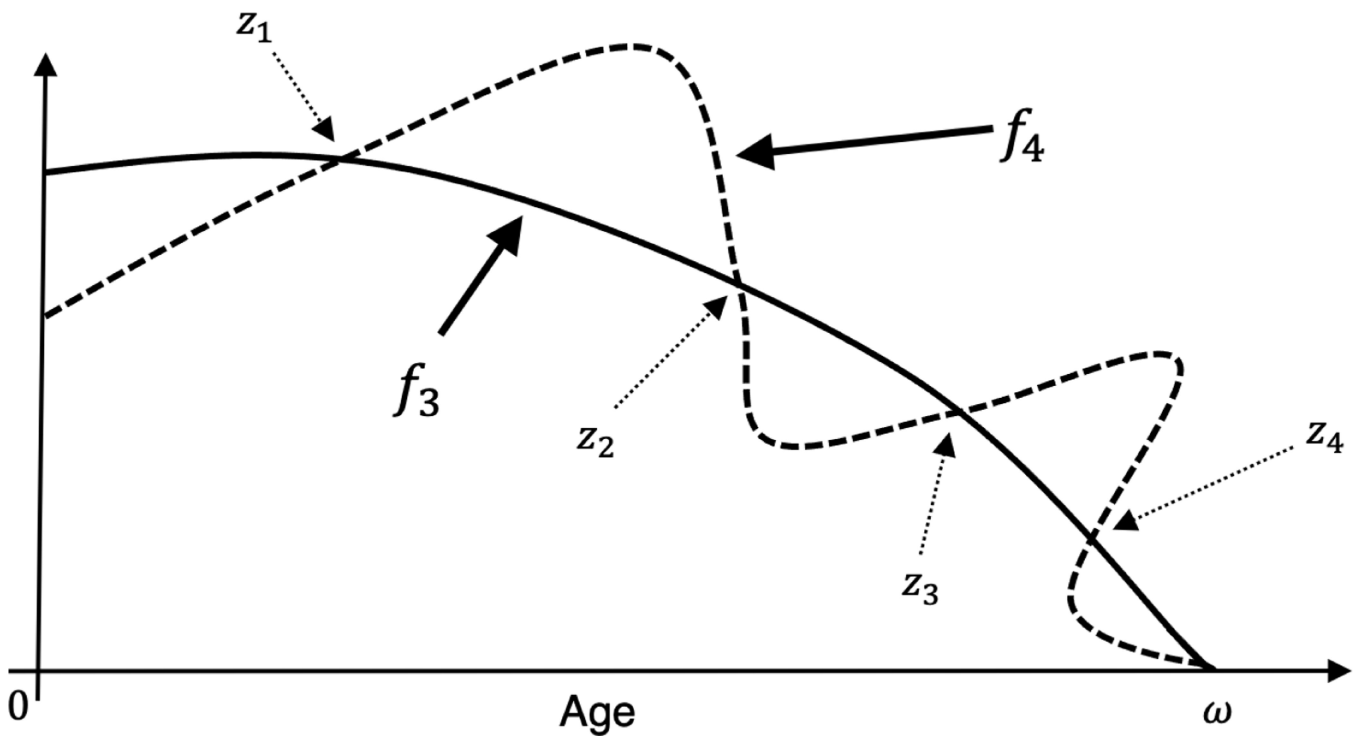

Figure 2.3.
Relationship between f_3 and f_4 associated with the equation (2.63).

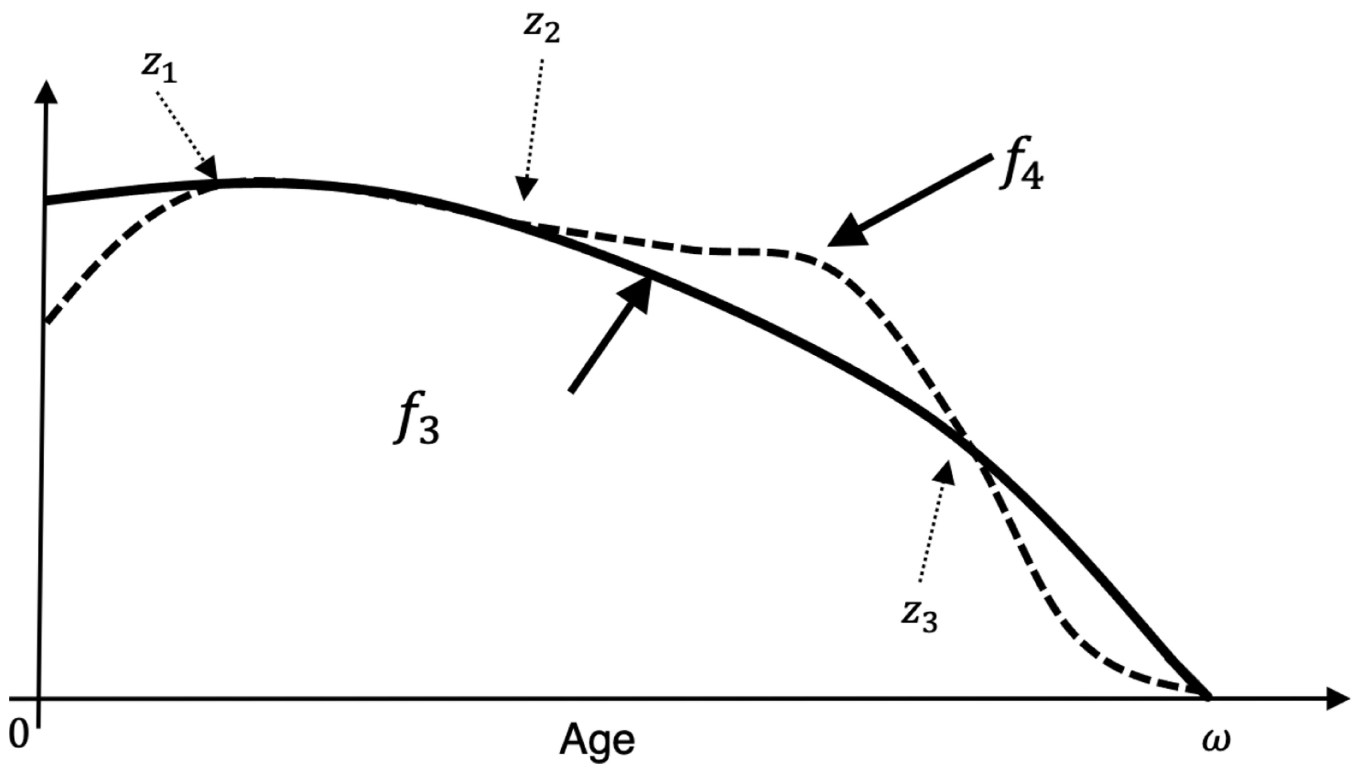

Figure 2.4.
Relationship between f_3 and f_4 associated with the equation (2.64).

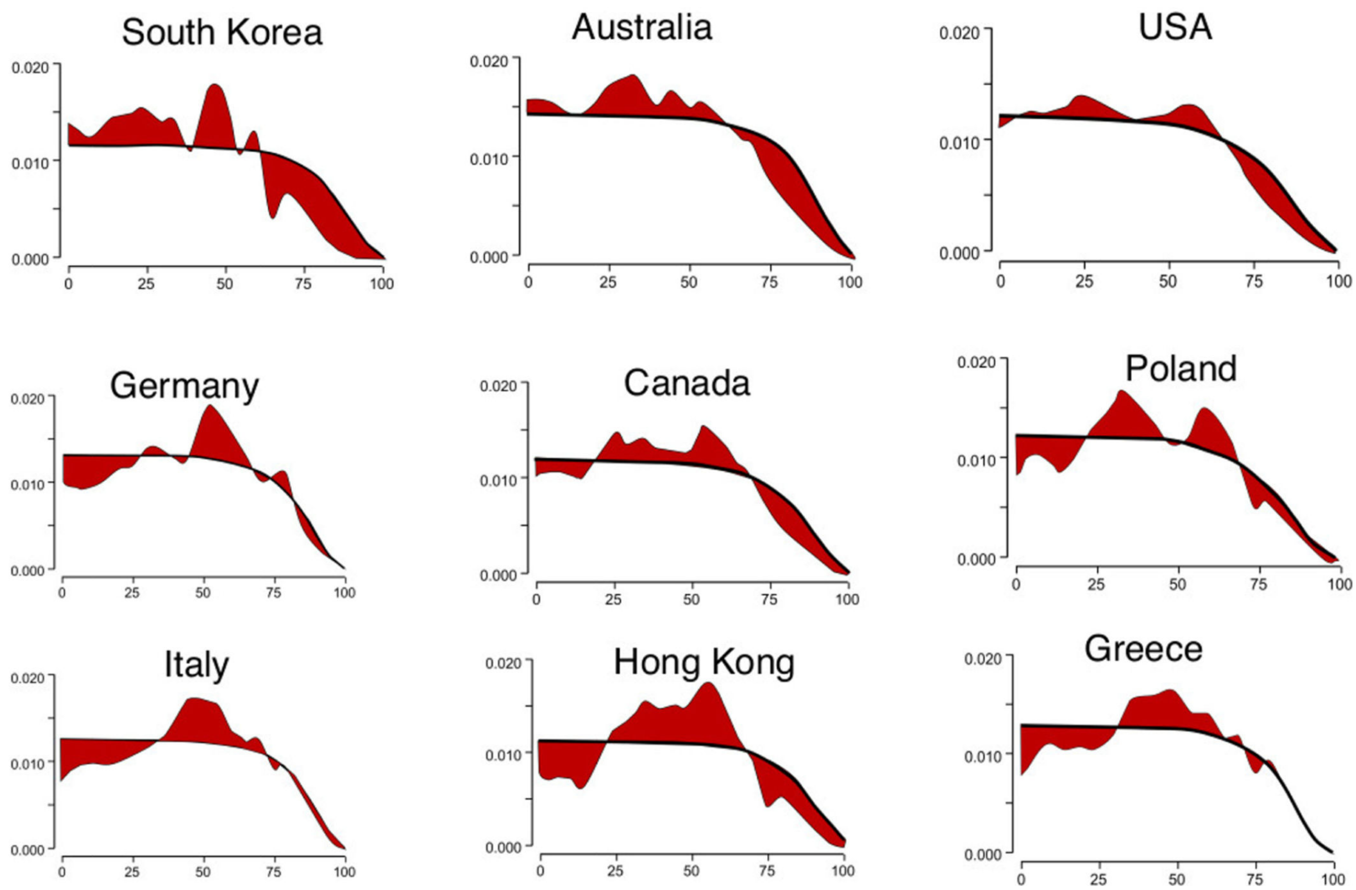

Figure 4.1.

Country-specific comparison of stationary population identity and actual population proportionate age-structure.

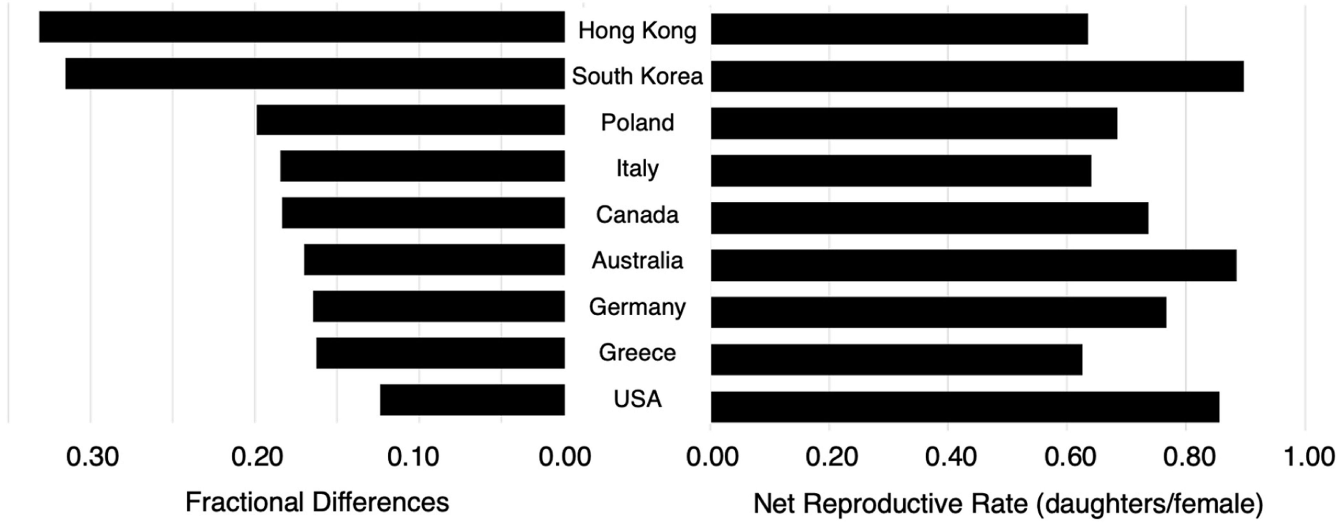

Figure 4.2.

Country-specific proportions of people in the stationary component of actual population and corresponding NRR values.

Table 1.

Possible combinations of number of elements in the sets X and Y.

Number of x values in X	Number of y values in Y
$n-1$	1
$n-2$	2
$\vdots$	$\vdots$
$n-(n-1)$	$n-1$

Table 2.

The list of all possible combinations of locations of 0 and ω

(i) $0 \in U$ and $\omega \in U$
(ii) $0 \in V$ and $\omega \in V$
(iii) $0 \in W$ and $\omega \in W$
(iv) $0 \in U$ and $\omega \in V$
(v) $0 \in U$ and $\omega \in W$
(vi) $0 \in V$ and $\omega \in U$
(vii) $0 \in V$ and $\omega \in W$
($viii$) $0 \in W$ and $\omega \in U$
(ix) $0 \in W$ and $\omega \in V$

Table 3.

Data required to compute NRR and new approach proposed

NRR for year *t*	Population Theorem metric for year *t*
Sex-ratio at birth	Actual population size by single year
Life table for female	Life table for total population
Age-specific fertility rates at single-year ages	–

Table 4.

Population partitioning metrics for nine countries including mean ages of either the 2016 or 2016 population and the stationary population, the percentage differences between the age distributions, and the net reproductive rates (Data sources: United Nations (2019) and Berkeley Mortality Database).

	Mean Age		**Fractional differences**	**NRR**
	2016*/2017 Population	**Stationary Population**		
Poland*	40.7	40.1	0.1990	0.685
Hong Kong	42.8	43	0.3309	0.635
Canada*	40.5	42.1	0.1834	0.736
S. Korea	35.0	42.2	0.3152	0.897
Australia	38.3	42.3	0.1701	0.884
USA	38.5	40.9	0.1239	0.855
Greece	43.5	41.5	0.1628	0.626
Germany	44.0	41.3	0.1644	0.767
Italy	44.6	42.1	0.1843	0.641