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Ramified lifts of Galois representations and dimension of ordinary deformation rings

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by

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ABSTRACT OF THE DISSERTATION

Ramified lifts of Galois representations
and dimension of ordinary deformation rings

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Professor Chandrashekhar Khare, Chair

Let F be a totally real number field, k a finite field of characteristic p and $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to GL_n(k)$ a continuous Galois representation. Under some technical hypotheses on $\overline{\rho}$ we extend the method of Khare and Ramakrishna of constructing ramified characteristic zero lifts of $\overline{\rho}$ from the setting of GL_2 to GL_n. As an application of this method we prove the existence of closed points $x \in \mathrm{Spec}(R^{\mathrm{ord}}[1/p])$, on a certain nearly ordinary deformation ring R^{ord}, such that Mazur's dimension conjecture is true locally at x. In the process we obtain examples of ordinary Galois representations $\rho : G_{F,S\cup Q} \to GL_n(\mathcal{O})$, where $\mathcal{O}$ is a finite extension of $\mathbb{Z}_p$, such that its adjoint Selmer group $H^1_{\mathcal{L}}(G_{F,S\cup Q}, ad^0\,\rho \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is finite. We then determine the exact size of these Selmer groups in terms of the ramification index of the primes in the auxiliary set.

The dissertation of Mohammedzuhair Mullath Mohammed Sherief is approved.

Abeer Alwan

Don Blasius

Haruzo Hida

Chandrashekhar Khare, Committee Chair

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To my Parents

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VITA

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CHAPTER 1

Introduction

A fundamental conjecture in Galois deformation theory is a conjecture attributed to Barry Mazur on the dimension of deformation rings. It states that the Krull dimension of a deformation ring of a global residual Galois representation is no larger than the one predicted by cohomological constraints. There is an analogous conjecture for deformation rings parameterizing deformations with prescribed local behaviour. In dimension one, as Mazur himself pointed out in his foundational article [Maz89], this conjecture is equivalent to the Leopoldt's conjecture and therefore the general conjecture is, conceivably, of vexing difficulty. But in the special case when the residual representation is unobstructed the conjecture follows trivially. On the other hand, a method of Ramakrishna (suitably generalized to GL_n following [CHT08]) shows that for many residual Galois representations unramified outside a finite set of primes S, we can find an auxiliary set of primes Q such that a new deformation problem, consisting of lifts which are of a prescribed type at primes in Q, is unobstructed.

In this thesis we show that, under several assumptions on the residual representation $\overline{\rho}$, we can find an auxiliary set Q of Ramakrishna primes such that the resulting universal deformation is ramified at all primes in $S \cup Q$. This is done by adapting a *forcing ramification* technique of [KR03]. Such a generalization has also been obtained by Patrikis [Pat16b]. While the focus of [Pat16a] and [Pat16b] is the construction of Galois representation with exceptional monodromy groups, our focus is on the structure deformation rings.

As a consequence of our method we show the existence of closed points on the deformation ring $R^{no}_{S\cup Q}$, parameterizing nearly ordinary deformations of $\overline{\rho}$ unramified

outside $S \cup Q$, which are formally smooth of the correct dimension. This gives some empirical evidence in favor of the dimension conjecture for the generic fiber of these nearly ordinary deformation rings as stated in [Til96, §7.5] .

Here is one of our main results from which we later draw a number of consequences.

Theorem 1. *Let F be a totally real number field, k a finite field of characteristic l and $\overline{\rho}: G_F \to GL_n(k) = GL(V)$ a continuous Galois representation, with $n \geq 2$. Let S be a finite set of places of F containing all the places above l, ∞, and the primes at which $\overline{\rho}$ is ramified. Let $\chi : G_F \to W(k)^\times$ be a lift of $\det \overline{\rho}$.*

Assume the following:

1) $l \geq 2(n+1)$.

2) The image of $\overline{\rho}$ contains $SL_n(\mathbb{F}_l)$.

3) $[F(\overline{\rho})(\zeta_l) : F(\overline{\rho})] \geq 3$, where $F(\overline{\rho})$ denotes the splitting field of $\overline{\rho}$.

4) $\overline{\rho}$ is totally odd. (i.e. for every choice of complex conjugation c_v of F, $tr(\overline{\rho}(c_v)) = 0$ or ± 1).

5) $\overline{\rho}$ is ordinary at each $v|l$ and satisfies the conditions (REG) and (REG), in the sense described below.*

Then there exists a finite set of Ramakrishna primes Q such that taking $\mathcal{D}_v$ to be the nearly ordinary deformation condition at each $v|l$, and the Ramakrishna deformation condition at each $v \in Q$, the global deformation problem $\mathcal{S}^{n.o}(\mathcal{Q}) = (\overline{\rho}, \chi, S \cup Q, \{\mathcal{D}_v\}_{v \in S \cup Q})$ is unobstructed. Furthermore, we can choose Q such that the universal deformation corresponding to $\mathcal{S}^{n.o}(\mathcal{Q})$ is ramified at all places in $S \cup Q$.

Theorem 1 is proved in two stages. First in §3.1 we show the existence of an auxiliary set following [CHT08, §2.6]. The method amounts to choosing primes such that a certain dual Selmer group vanishes. The original method of killing dual Selmer group is due to Ramakrishna [Ram99]. He considered the case $n = 2$. His methods have been

generalized and axiomatized in [Tay03], [CHT08] and [Pat16a]. In the second stage, in §3.2 we adapt the forcing ramification method of [KR03] to get a ramified auxiliary set from an auxiliary set and thus prove Theorem 1.

This theorem has some immediate consequences: finiteness of a Selmer group and existence of points of deformation rings whose formal neighborhoods are smooth of the correct dimension (see Theorem 3 and 4 of Chapter 4).

We make a few remarks about the assumptions on $\overline{\rho}$ and l in the above theorem. The first condition on the characteristic being large with respect to the dimension is so that in junction with absolute irreducibility of $\overline{\rho}$ (which is ensured by the second condition) we have that the pair $(\overline{\rho}(G_F), V)$ is adequate in the sense of [GHT12]; this, in particular, ensures that $H^1(\mathrm{Gal}(F(\overline{\rho})/F), ad^0\overline{\rho}) = 0$, which is crucial for Ramakrishna's method [Ram99]. The second condition on the image ensures that lifts of $\overline{\rho}$ to the artinian rings $W(k)/p^n$ still have large image, so that the method of forcing ramification of [KR03] goes through. The third assumption guarantees the independence of certain Chebotarev conditions associated to Selmer and dual Selmer classes which we are trying to annihilate. The fourth condition on the oddness of $\overline{\rho}$ is not essential to the method; it is imposed so that the numerics in Greenberg-Wiles formula works out neatly. Finally, the last condition of regularity is imposed so that the relevant local deformation rings are formally smooth.

This thesis is organized as follows.

Chapter 2 covers preliminary material. In it we recall the basic notions of Galois deformation theory. We review the natural identification of the tangent spaces of deformation rings with certain Galois cohomolgy groups and the definition of obstruction class. The definition of Selmer and dual Selmer groups and the presentation of deformation rings as a quotient of a power series ring over $W(k)$ are also discussed. The last section is about a criterion of Greenberg about the surjectivity of the global to local map defining Selmer groups of characteristic zero representations.

In Chapter 3 we prove theorem 1 about the existence of ramified auxiliary sets for

$\overline{\rho}$. First we prove the existence of an auxiliary sets Q by showing that a relevant dual Selmer group vanishes. The resulting universal deformation is not guaranteed to ramify at all the primes in Q. We then modify this Q to obtain a ramified auxiliary set. The method of modification is same as that of [KR03].

In Chapter 4 we deduce some consequences of theorem 1. We show the existence of closed points x on the nearly ordinary deformation ring $R^{no}_{S\cup Q}$ such that the localization completion of $R^{no}_{S\cup Q}$ at x is formally smooth of the correct dimension. We deduce that at such point $H^2(G_F, ad^0\rho_x)$ vanishes. Then we turn to showing that the Selmer group associated to a natural map of deformation rings $R^{\mathcal{S}\cup Q} \to R^{\mathcal{S}(\mathcal{Q})} \simeq W(k)$ is finite and compute the exact size of this Selmer group in terms of the ramification index of the primes in the (ramified) auxiliary set Q. Here $\mathcal{S}(\mathcal{Q})$ is a balanced deformation problem, with Selmer and dual Selmer groups of equal size, with Ramakrishna deformation condition at primes in Q. And $\mathcal{S} \cup Q$ is the deformation problem where the local deformation condition at places in S are the ones specified by $\mathcal{S}$ and at places in Q we impose no restrictions on lifts. The theorem we prove (see Theorem 4 and 5 of Chapter 4) is the following:

Theorem. For the canonical map $\pi_Q : R^{\mathcal{S}\cup Q} \to R^{\mathcal{S}(\mathcal{Q})}$ of deformation rings we have

$$\mathrm{Hom}\left(\ker \pi_Q/(\ker \pi_Q)^2,\, \mathbb{Q}_p/\mathbb{Z}_p\right) \cong \oplus_{v\in Q}\, W(k)/(p^{m_v}),$$

where m_v is the ramification index at v, defined in §4.2.

In the setting of GL_2, the isomorphism above is stated as a conjecture in [KR03] (conjecture 23).

The chapter ends with some speculations about the complete intersection property of $R^{\mathcal{S}\cup Q}$.

CHAPTER 2

Preliminaries

2.1 Deformations of Galois representations

2.1.1 Generalities

In this section we review the basic notions of Galois deformation theory.

Throughout this chapter k denotes a finite field of characteristic p, $W = W(k)$ denotes its ring of Witt vectors and K the fraction field of W.

Let Γ be a profinte group. We suppose that Γ satisfies Mazur's p finiteness property: for every open subgroup $\Delta \subseteq \Gamma$

$$|\mathrm{Hom}_{cts}(\Delta, \mathbb{F}_p)| < \infty.$$

This condition is satisfied when $\Gamma = \mathrm{Gal}(F_S/F)$, the Galois group of the maximal extension of a number field F unramified outside a finite set of places S, or when $\Gamma = \mathrm{Gal}(\overline{F}_v/F)$ the local Galois group of F for any place v of F. (This property implies hypothesis (H_3) of Schlessinger's criteria for representability holds true for various functors considered below.)

Let $\overline{\rho} : \Gamma \to GL_n(k)$ be a continuous homomorphism. We shall denote by $V_{\overline{\rho}}$ the underlying k-vector space carrying the Γ action. And by $ad\,\overline{\rho}$ we mean the adjoint representation of Γ on $M_n(k)$ obtained by composing $\overline{\rho}$ with the conjugation action of $GL_n(k)$ on $M_n(k)$. Also, $ad^0\,\overline{\rho}$ denotes its subrepresentation on $M_n^0(k)$–the trace zero matrices in $M_n(k)$.

We are interested in a systematic study of deforming $\overline{\rho}$ into larger rings. We shall work in the category $\mathcal{C}_W$. It is the category of complete noetherian local W-algebras

R for which the structure map $W \to R$ induces an isomorphism of residue fields. The morphisms in $\mathcal{C}_W$ are local W-algebra homomorphisms. Objects in $\mathcal{C}_W$ carry natural topology - the adic topology defined by the maximal ideal.

We consider the functor $\mathrm{Lift}_{\overline{\rho}}$ on $\mathcal{C}_W$:

$$\mathrm{Lift}_{\overline{\rho}}(R) = \{\text{continuos homomorphims } \rho : \Gamma \to GL_n(R) : \rho \bmod \mathfrak{m}_R = \overline{\rho}\}.$$

$\mathrm{Lift}_{\overline{\rho}}$ is representable; we denote the representing object by $R^{\square}_{\overline{\rho}}$ and call it the *universal lifting ring* of $\overline{\rho}$.

Two lifts $\rho_1, \rho_2 \in \mathrm{Lift}_{\overline{\rho}}$ are said to be strictly equivalent if there exists $g \in \ker(GL_n(R) \to GL_n(k))$ such that $\rho_2 = g\rho_1 g^{-1}$. A strict equivalence class of lifts in $\mathrm{Lift}_{\overline{\rho}}(R)$ is called a *deformation* of $\overline{\rho}$ to R. The corresponding functor is denoted by $\mathrm{Def}_{\overline{\rho}}$.

$\mathrm{Def}_{\overline{\rho}}$ is not always representable. When the centralizer of $\overline{\rho}(\Gamma)$ in $GL_n(k)$ consists of only the scalar matrices, i.e. when $\mathrm{Hom}_k(V_{\overline{\rho}}, V_{\overline{\rho}})^{\Gamma} = k$, $\mathrm{Def}_{\overline{\rho}}$ is representable. In such cases, we denote the representing object by $R_{\overline{\rho}}$ and call it the *universal deformation ring.*

Definition 2.1. A *deformation condition* is a subfunctor $\mathcal{D} \subset \mathrm{Lift}_{\overline{\rho}}$ satisfying the following conditions:

- $\mathcal{D}$ is represented by quotient $R^{\square,\mathcal{D}}$ of $R^{\square}_{\overline{\rho}}$.
- For all $R \in \mathcal{C}_W$, $g \in \ker(GL_n(R) \to GL_n(k))$ and $\rho \in \mathcal{D}(R)$, we have $g\rho g^{-1} \in \mathcal{D}(R)$.

For a deformation condition $\mathcal{D}$ let $I(\mathcal{D})$ denote the ideal defining it, i.e., $R^{\square,\mathcal{D}} = R^{\square}_{\overline{\rho}}/I(\mathcal{D})$.

By universality, conjugation of the universal lift to $R^{\square}_{\overline{\rho}}$ by an element of $h \in \ker(GL_n(R^{\square}_{\overline{\rho}}) \to GL_n(k))$ induces a morphism $\phi_h : R^{\square}_{\overline{\rho}} \to R^{\square}_{\overline{\rho}}$. The second part of the above definition implies that $I(\mathcal{D})$ is invariant under ϕ_h for all $h \in \ker((GL_n(R^{\square}_{\overline{\rho}}) \to GL_n(k))$. Conversely, any ideal $I \subset R^{\square}_{\overline{\rho}}$ with this property defines a deformation condition $\mathcal{D}(I)$, represented by $R^{\square}_{\overline{\rho}}/I$. This is a nullstellensatz type correspondence [CHT08, lemma 2.3.3].

Fixing the determinant of lifts defines a deformation condition: if $\chi : \Gamma \to W^\times$ is a continuous lift of $det(\overline{\rho})$ we denote the subfunctor $\mathrm{Lift}_{\overline{\rho}}^{\chi} \subset \mathrm{Lift}_{\overline{\rho}}$ of lifts $\rho : \Gamma \to GL_n(R)$ such that $\det(\rho) = \chi$. (Here we view χ as a character valued in $R^\times$ via the structural map $W \to R$). If $\overline{\rho}$ is representable then we can consider deformations with fixed determinant χ and it is represented by a quotient $R_{\overline{\rho}}^{\chi}$ of $R_{\overline{\rho}}$. Throughout this dissertation we shall be working with a fixed choice of determinant and we will usually ignore to make this explicit in the notation.

For any functor $\mathcal{F} : \mathcal{C}_W \to \mathrm{Sets}$ such that $\mathcal{F}(k) = *$ we define the tangent space of $\mathcal{F}$ as

$$t_{\mathcal{F}} = \mathcal{F}(k[\epsilon]),$$

where $k[\epsilon] = k[X]/(X^2)$ is the dual numbers of k. The tangent space of a representable functor is canonically isomorphic to the tangent space of the representing object; there are canonical isomorphisms

$$\mathrm{Lift}_{\overline{\rho}}(k[\epsilon]) \cong \mathrm{Hom}_{\mathcal{C}_W}(R_{\overline{\rho}}^{\square}, k[\epsilon]) \cong \mathrm{Hom}_k\left(\mathfrak{m}_{R_{\overline{\rho}}^{\square}}/(\mathfrak{m}_{R_{\overline{\rho}}^{\square}}^2, p), k\right).$$

There is also a canonical isomorphism

$$\mathrm{Lift}_{\overline{\rho}}(k[\epsilon]) \cong Z^1(\Gamma, ad\,\overline{\rho}), \tag{2.1}$$

given by sending a cocycle f to the lift $g \mapsto (1 + \epsilon f(g))\overline{\rho}(g) \in GL_n(k[\epsilon])$. Thus we get canonical isomorphisms

$$\mathrm{Lift}_{\overline{\rho}}(k[\epsilon]) \cong \mathrm{Hom}_k\left(\mathfrak{m}_{R_{\overline{\rho}}^{\square}}/(\mathfrak{m}_{R_{\overline{\rho}}^{\square}}^2, p), k\right) \cong Z^1(\Gamma, ad\,\overline{\rho}). \tag{2.2}$$

The isomorphism (2.1) determines an isomorphism

$$\mathrm{Def}_{\overline{\rho}}(k[\epsilon]) \cong H^1(\Gamma, ad\,\overline{\rho}), \tag{2.3}$$

and if $\mathrm{Def}_{\overline{\rho}}$ is representable we also have

$$\mathrm{Def}_{\overline{\rho}}(k[\epsilon]) \cong \mathrm{Hom}_k\left(\mathfrak{m}_{R_{\overline{\rho}}}/(\mathfrak{m}_{R_{\overline{\rho}}}^2, p), k\right) \cong H^1(\Gamma, ad\,\overline{\rho}). \tag{2.4}$$

If $\mathcal{D} \subset \mathrm{Lift}_{\overline{\rho}}$ is a deformation condition then

$$\mathcal{D}(k[\epsilon]) \cong \mathrm{Hom}_{\mathcal{C}_W}(R_{\overline{\rho}}^{\square}/I(\mathcal{D}), k[\epsilon]) \cong \mathrm{Hom}_k\left(\mathfrak{m}_{R_{\overline{\rho}}^{\square}}/(\mathfrak{m}_{R_{\overline{\rho}}^{\square}}^2, p, I(\mathcal{D})), k\right),$$

which under the isomorphism (2.2) identifies $\mathcal{D}(k[\epsilon]) \subset \mathrm{Lift}_{\overline{\rho}}(k[\epsilon])$ as a subspace $\widetilde{L}(\mathcal{D}) \subseteq Z^1(\Gamma, ad\,\overline{\rho})$. Because the ideal defining $\mathcal{D}$ is invariant under $\ker(GL_n(R^{\square}_{\overline{\rho}}) \to GL_n(k)) = 1 + M_n(\mathfrak{m}_{R^{\square}_{\overline{\rho}}})$, $\widetilde{L}(\mathcal{D})$ is the full preimage of a subspace of $H^1(\Gamma, ad\,\rho)$. We denote this subspace by $L(\mathcal{D})$ and refer to it as the tangent space of $\mathcal{D}$. Thus, we have an exact sequence

$$0 \to (ad\,\overline{\rho})^{\Gamma} \to ad\,\overline{\rho} \to \widetilde{L}(\mathcal{D}) \to L(\mathcal{D}) \to 0.$$

Remark. If $\mathcal{D}^{\chi}$ denotes the deformation condition $\mathcal{D} \cap \mathrm{Lift}^{\chi}_{\overline{\rho}}$, statements analogous to above hold true with $ad\,\overline{\rho}$ replaced with $ad^0\,\overline{\rho}$.

A *small surjection* in $\mathcal{C}_W$ is a morphism $R \to R/I$ such that $\mathfrak{m}_R.I = 0$. We then view I as a k vector space, necessarily finite dimensional. Associated to such a small surjection we have an exact sequence

$$1 \to 1 + M_n(I) \to GL_n(R) \to GL_n(R/I) \to 1. \tag{2.5}$$

Whether or not a $\rho \in \mathrm{Lift}_{\overline{\rho}}(R/I)$ admits a lift to R is determined by an element $\mathrm{Obs}(\rho) \in H^2(\Gamma, ad\,\rho) \otimes_k I$ being zero or not. $\mathrm{Obs}(\rho)$, called the *obstruction class* of ρ depends only on the deformation class of ρ and is defined as follows. Choose a continuous (set theoretic) section $\sigma : GL_n(R/I) \to GL_n(R)$ of (2.5). Consider the map $\widetilde{\rho} = \sigma \circ \rho : \Gamma \to GL_n(R)$ and define a map $c_\sigma : \Gamma \times \Gamma \to ad\,\rho \otimes_k I$ as

$$(\alpha, \beta) \mapsto \widetilde{\rho}(\alpha\beta)\widetilde{\rho}(\alpha)^{-1}\widetilde{\rho}(\beta)^{-1} - 1 \in M_n(I) \cong ad\,\rho \otimes_k I.$$

It is easily verified that c_σ defines a 2-cocycle in $Z^2(\Gamma, ad\,\rho \otimes_k I) = Z^2(\Gamma, ad\,\rho) \otimes_k I$ and one checks that the corresponding cocyle in $H^2(\Gamma, ad\,\rho) \otimes_k I$ is independent of the section σ and depends only on the deformation class of ρ [Til96, lemma 5.4]; the class of c_σ is $\mathrm{Obs}(\rho)$. If $\mathrm{Obs}(\rho) = 0$, i.e. if $\mathrm{Obs}(\rho) = \delta\phi$ for $\phi \in Z^1(\Gamma, ad\,\rho) \otimes_k I$ then viewing $1 + \phi$ as taking values in $1 + M_n(I)$,

$$\hat{\rho} = (1 + \phi)\widetilde{\rho} : \Gamma \to GL_n(R)$$

defines a homorphism which lifts ρ. Conversely, if ρ does admit a lift τ to R then $g \mapsto \tau(g)\widetilde{\rho}(g)^{-1} - 1$ defines a cocycle ψ such that $\delta\psi = \mathrm{Obs}(\rho)$. In summary, we have:

Proposition 2.1. *For a small surjection $\varphi : R \to R/I$ the image of $\varphi : \mathrm{Lift}_{\overline{\rho}}(R) \to \mathrm{Lift}_{\overline{\rho}}(R/I)$ are those $\rho \in \mathrm{Lift}_{\overline{\rho}}(R/I)$ such that $\mathrm{Obs}(\rho) = 0$*

Definition 2.2. A deformation condition $\mathcal{D}$ is said to be *liftable* if for all small surjections $R \to R/I$, the induced map

$$\mathcal{D}(R) \longrightarrow \mathcal{D}(R/I)$$

is surjective. This is equivalent to $R^{\square,\mathcal{D}}$ being isomorphic to a power series ring over W in $\dim_k \widetilde{L}(\mathcal{D})$ variables.

For a small surjection $R \to R/I$ while $H^2(\Gamma, ad\,\rho)$ controls the image of $\varphi : \mathrm{Lift}_{\overline{\rho}}(R) \to \mathrm{Lift}_{\overline{\rho}}(R/I)$, $H^1(\Gamma, ad\,\rho)$ controls the fibers of φ.

Proposition 2.2. *For a small surjection $\varphi : R \to R/I$ the fibers of $\varphi : \mathrm{Lift}_{\overline{\rho}}(R) \to \mathrm{Lift}_{\overline{\rho}}(R/I)$ are torsors for $Z^1(\Gamma, ad\,\rho) \otimes_k I$. And the fibers of $\varphi : \mathcal{D}(R) \to \mathcal{D}(R/I)$ are torsors for $\widetilde{L}(\mathcal{D}) \otimes_k I$.*

Proof. Let $\rho \in \mathrm{Lift}_{\overline{\rho}}(R/I)$ and suppose $\widetilde{\rho} \in \mathrm{Lift}_{\overline{\rho}}(R)$ maps to ρ under φ. Then for any $f \otimes \eta \in Z^1(\Gamma, ad\,\rho) \otimes_k I$, $(1 + f \otimes \eta)\widetilde{\rho}$ gives another lift of ρ in $\mathrm{Lift}_{\overline{\rho}}(R)$. Conversely, if $\tau \in \mathrm{Lift}_{\overline{\rho}}(R)$ is another lift of ρ then the map

$$g \mapsto \widetilde{\rho}(g)\tau^{-1}(g) - 1$$

defines a cocycle in $Z^1(\Gamma, ad\,\rho) \otimes_k I$. Clearly, these maps are inverses of each other.

Now, suppose $\rho \in \mathcal{D}(R/I)$. The second part follows from above and the following statement: if $\widetilde{\rho} \in \mathcal{D}(R)$ is a lift of ρ then for any $\phi \in Z^1(\Gamma, ad\,\rho) \otimes_k I$

$$(1 + \phi)\widetilde{\rho} \in \mathcal{D}(R) \iff \phi \in \widetilde{L}(\mathcal{D}) \otimes_k I.$$

In other words, under the the simply transitive action of $Z^1(\Gamma, ad\,\rho) \otimes_k I$ on the fiber $\varphi^{-1}(\rho)$ (supposing it is not empty) $\widetilde{L}(\mathcal{D}) \otimes_k I$ is precisely the subspace that preserves the subset of $\varphi^{-1}(\rho)$ of type $\mathcal{D}$ lifts. This follows in a straight forward way from the definition on $\widetilde{L}(\mathcal{D})$ [CHT08, lemma 2.2.5]. Suppose $\rho \in \mathcal{D}(R/I)$ corresponds to a homomorphism

$\alpha : R_{\overline{\rho}}^{\square} \to R$ (which factors through $R^{\square,\mathcal{D}}$). And suppose $\widehat{\rho} = (1+\phi)\widetilde{\rho} \in \mathrm{Lift}_{\overline{\rho}}(R)$ corresponds to a homomorphism $\beta : R_{\overline{\rho}}^{\square} \to R$. α and β agree mod I and α vanishes on $I(\mathcal{D})$. Set

$$f = \beta - \alpha : R_{\overline{\rho}}^{\square} \to I;$$

it has the following properties :

$f(x+y) = f(x) + f(y)$,

$f(xy) = f(x)\alpha(y) + \alpha(x)f(y)$,

$f|_W = 0$.

Therefore, f is determined by its restriction to $\mathfrak{m}_{R_{\overline{\rho}}^{\square}}$ and by the second and third properties it vanishes on $(\mathfrak{m}^2_{R_{\overline{\rho}}^{\square}}, p)$. Thus f gives rise to (and is determined by) a W-linear map

$$f \in \mathrm{Hom}_W(\mathfrak{m}_{R_{\overline{\rho}}^{\square}}/(\mathfrak{m}^2_{R_{\overline{\rho}}^{\square}}, p), I) \cong \mathrm{Hom}_k(\mathfrak{m}_{R_{\overline{\rho}}^{\square}}/(\mathfrak{m}^2_{R_{\overline{\rho}}^{\square}}, p), I),$$

which under the isomorphism (2.2) maps to ϕ. Now, β will factor through $R^{\square,\mathcal{D}}$ if and only if f vanishes on $I(\mathcal{D})$ if and only if $\phi \in \widetilde{L}(\mathcal{D})$. $\square$

Remark. It could happen that $\rho \in \mathrm{Lift}_{\overline{\rho}}(R/I)$ admits no lifts to R in which case the torsors are empty. This happens precisely when $\mathrm{Obs}(\rho) \neq 0$. On the other hand if ρ does admit a lift R then the propsition implies that it admits a lift of type $\mathcal{D}$ to R.

It is also true that the fibers of $\varphi : \mathrm{Def}_{\overline{\rho}}(R) \to \mathrm{Def}_{\overline{\rho}}(R/I)$ are $H^1(\Gamma, ad\,\rho) \otimes_k I$-torsors. This follows from the above proposition, by checking that two cocycles ϕ and ψ define the same deformation of ρ to R if and only if $\phi - \psi$ is a coboundary.

2.1.2 Relative deformations

For an object $A \in \mathcal{C}_W$ we denote by $A[\epsilon]$ its ring of dual numbers $A[X]/(X^2)$. The natural surjection $A[\epsilon] \to A$ is canonically split.

For any $A \in \mathcal{C}_W$ we denote by $\mathcal{C}_W(A)$ the category whose objects are diagrams $h : B \to A$ in $\mathcal{C}_W$. Morphisms in $\mathcal{C}_W(A)$ are defined the obvious way. (Note that $\mathcal{C}_W(k) = \mathcal{C}_W$).

For any functor $\mathcal{F} : \mathcal{C}_W(A) \to \text{Sets}$ such that $\mathcal{F}(A) = *$ we define the tangent space of $\mathcal{F}$ as

$$t_{\mathcal{F}} = \mathcal{F}(A[\epsilon]).$$

Given $\rho \in \text{Lift}_{\overline{\rho}}(A)$ we define the functor of relative lifts and deformations of ρ on the category $\mathcal{C}_W(A)$ as follows:

$$\begin{aligned}\text{Lift}_{\rho}(B) &= \{\tau \in \text{Lift}_{\overline{\rho}}(B) : h \circ \tau = \rho\} \\ &\cong \text{Hom}_{\mathcal{C}_W(A)}(R^{\square}_{\overline{\rho}}, B); \\ \text{Def}_{\rho}(B) &= \{\xi \in \text{Def}_{\overline{\rho}}(B) : h \circ \xi = [\rho]\}.\end{aligned}$$

Similarly, for any deformation condition $\mathcal{D}$ and $\rho \in \mathcal{D}(A)$ we can define the functor $\mathcal{D}_{\rho}$ of (relative) lifts of ρ of type $\mathcal{D}$ on $\mathcal{C}_W(A)$ as:

$$\mathcal{D}_{\rho}(B) = \{\tau \in \mathcal{D}(B) : h \circ \tau = \rho\} = \mathcal{D}(B) \cap \text{Lift}_{\rho}(B).$$

We note that we still have a cohomological interpretation for the tangent spaces of the relative functors [Maz97, §21]. Denoting the underlying free A-module for the representation ρ by V,

$$\begin{aligned}t_{\text{Lift}_{\rho}} &\cong Z^1(\Gamma, \text{End}_A(V)) \cong \text{Hom}_A\left(\Omega_{R^{\square}_{\overline{\rho}}/W} \hat{\otimes}_{R^{\square}_{\overline{\rho}}} A, A\right), \\ t_{\text{Def}_{\rho}} &\cong H^1(\Gamma, \text{End}_A(V)) \cong \text{Hom}_A\left(\Omega_{R_{\overline{\rho}}/W} \hat{\otimes}_{R_{\overline{\rho}}} A, A\right),\end{aligned}$$

where the last isomorphism holds when $\text{Def}_{\overline{\rho}}$ is representable. We recall that when $A = k$, we have a canonical isomorphism

$$\Omega_{R^{\square}_{\overline{\rho}}/W} \hat{\otimes}_{R^{\square}_{\overline{\rho}}} k \cong \mathfrak{m}_{R^{\square}_{\overline{\rho}}}/(\mathfrak{m}^2_{R^{\square}_{\overline{\rho}}}, p),$$

so that the above isomorphisms recovers (2.2) and (2.4). Naturally, we also have,

$$t_{\mathcal{D}_{\rho}} \cong \text{Hom}_A\left(\Omega_{R^{\square,\mathcal{D}}/W} \hat{\otimes}_{R^{\square,\mathcal{D}}} A, A\right).$$

$t_{\mathcal{D}_{\rho}}$ is naturally an sub A-module of $Z^1(\Gamma, End_A(V))$ which, like in the residual case, is the full preimage of a sub A-module of $H^1(\Gamma, \text{End}_A(V))$. We have the following exact sequence:

$$0 \to (\text{End}_A(V))^{\Gamma} \to \text{End}_A(V) \to \widetilde{L}(\mathcal{D}_{\rho}) \to L(\mathcal{D}_{\rho}) \to 0,$$

which is useful for computing the A-module $L(\mathcal{D}_\rho)$ (or computing its cardinality when A is artinian).

A case of special interest to us is when we start with a $\rho : R^{\square}_{\overline{\rho}} \to W$ that factors through $R^{\square,\mathcal{D}}$. In this case we can reduce ρ mod p^i for any $i \geq 1$ and we denote the reduction by ρ_i and the underlying free $W_i := W/p^iW$ module by V_i. We may view V_i's as a directed system of A-modules,

$$\cdots \to V_i \xrightarrow{i_n} V_{i+1} \to \cdots$$

where i_n is the injective map induced from multiplication by p viewed as an endomorphism of V_{n+1} (i.e. $i_n(x) = p\widetilde{x}$, $\widetilde{x}$ being a lift of x in V_{i+1}). This also gives us a directed system of Γ modules

$$\cdots \to \operatorname{End}_{W_i} V_i \xrightarrow{j_n} \operatorname{End}_{W_{i+1}} V_{i+1} \to \cdots$$

where, for $f \in \operatorname{End}_{W_i} V_i$

$$j_n(f) : V_{i+1} \twoheadrightarrow V_i \xrightarrow{f} V_i \xrightarrow{i_n} V_{i+1}.$$

This gives a directed system of cohomology groups

$$\cdots \longrightarrow H^1(\Gamma, \operatorname{End}_{W_i} V_i) \xrightarrow{j_n} H^1(\Gamma, \operatorname{End}_{W_{i+1}} V_{i+1}) \longrightarrow \cdots \tag{2.6}$$

the direct limit of which is

$$H^1(\Gamma, ad\,\rho) \otimes_W K/W.$$

For each $i \geq 1$ we also a subspace

$$t_{\mathcal{D},\rho_i} \subseteq H^1(\Gamma, \operatorname{End}_{W_i} V_i), \tag{2.7}$$

which is the tangent space of relative deformations of type $\mathcal{D}$ of ρ_i (to $V_i[\epsilon]$). One checks that (see [Maz97, §27]) the inclusions (2.7) are compatible with (2.6); by taking direct limit we get a subspace

$$\mathcal{L}_{\mathcal{D}}(\rho) \subseteq H^1(\Gamma, ad\,\rho) \otimes_W K/W.$$

Such subspaces and calculation of Selmer groups associated to such subspaces will play an important role in Chapter 4.

In proposition 2.3 we shall give yet another description of $\mathcal{L}_{\mathcal{D}}(\rho)$.

2.1.3 Global deformations with local conditions

Through out the rest of this dissertation we shall work with a fixed number field F and representations of $\mathrm{Gal}(\overline{F}/F)$ factoring through quotients $\mathrm{Gal}(F_\Sigma/F)$ for various finite set of places Σ of F.

Now suppose $\Gamma = G_{F,S}$ and we start with an absolutely irreducible $\overline{\rho} : G_{F,S} \to GL_n(k)$. For each $v \in S$ we have fixed a choice of decomposition group G_v at v inside $G_{F,S}$. By restricting, we get homomorhisms $\overline{\rho}_v := \overline{\rho}|_{G_v} : G_v \to GL_n(k)$ and all the associated functors defined in §2.1.1. Thus we have functors $\mathrm{Lift}_{\overline{\rho}_v}$ and $\mathrm{Der}_{\overline{\rho}_v}$ and we can define subfunctors $\mathcal{D}_v \subset \mathrm{Lift}_{\overline{\rho}_v}$ which are deformation conditions as in definition 2.1.

Definition 2.3. A *global deformation problem* is a tuple

$$\mathcal{S} = (S, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S}),$$

where

- S is a finite set of places of F,
- $\overline{\rho} : G_{F,S} \to GL_n(k)$ is a continuous homomorphism,
- $\chi : G_{F,S} \to W(k)^\times$ is a continuous lift of $\det \overline{\rho}$.
- for each $v \in S$, $\mathcal{D}_v \subset \mathrm{Lift}_{\overline{\rho}_v}$ is a deformation condition.

Associated to such a global deformation problem $\mathcal{S} = (S, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S})$ is the functor of *deformations of type* $\mathcal{S}$, $\mathrm{Def}_{\mathcal{S}} \subset \mathrm{Def}_{\overline{\rho}}$ defined on $\mathcal{C}_W$ as follows:

$$\mathrm{Def}_{\mathcal{S}}(A) = \{[\rho] \in \mathrm{Def}^{\chi}_{\overline{\rho}}(A) : \text{for all } v \in S,\ \rho|_{G_v} \in \mathcal{D}^{\chi}_v(A)\}.$$

Let us remark that this definition make sense because if one lift $\rho \in [\rho]$ is of type $\mathcal{D}_v$ at v then any other lift in $[\rho]$ is also of type $\mathcal{D}_v$ at v because of the second condition in definition 2.1.

$\mathrm{Def}_{\mathcal{S}}$ is representable whenever $\mathrm{Def}_{\overline{\rho}}$ is [CHT08, proposition 2.2.9]. When $\mathrm{Def}_{\mathcal{S}}$ is representable we denote the representing object by $R^{\mathcal{S}}$. From now on we shall further assume that S contains all the primes at which $\overline{\rho}$ is ramified, all the places above l and the infinite places of F. This assumption makes $\mathrm{Def}_{\mathcal{S}}$ 'better behaved'.

The tangent space $\mathrm{Def}_{\mathcal{S}}(k[\epsilon])$ of $\mathrm{Def}_{\mathcal{S}}$ is a subspace of $H^1(G_{F,S}, ad^0\,\overline{\rho})$ which is determined by the local tangent spaces

$$L_v := L(\mathcal{D}_v^{\chi}) \subseteq H^1(G_v, ad^0\,\overline{\rho}).$$

Let us see how. Put $\widetilde{L}_v = \widetilde{L}_v(\mathcal{D}_v^{\chi})$–the preimage of L_v in $Z^1(G_{F,S}, ad^0\,\overline{\rho})$. We shall make use of certain complexes as defined in [CHT08, §2.2] whose cohomology groups controls the presentation of $R^{\mathcal{S}}$ as a quotient of a power series ring over W. We do not recall the definition of this complex whose definition can be found in loc.cit. Taking the cohomology of the complex gives an exact sequence :

$$\begin{aligned}
0 \longrightarrow H^1_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho}) &\longrightarrow H^1(G_{F,S}, ad^0\,\overline{\rho}) \longrightarrow \oplus_{v\in S} H^1(G_v, ad^0\,\overline{\rho})/L_v \\
\longrightarrow H^2_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho}) &\longrightarrow H^2(G_{F,S}, ad^0\,\overline{\rho}) \longrightarrow \oplus_{v\in S} H^2(G_v, ad^0\,\overline{\rho}) \\
\longrightarrow H^3_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho}) &\longrightarrow H^3(G_{F,S}, ad^0\,\overline{\rho}) \longrightarrow \cdots
\end{aligned} \tag{2.8}$$

The cohomology group

$$H^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho})) = \ker\Big(H^1(G_{F,S}, ad^0\,\overline{\rho}) \longrightarrow \oplus_{v\in S}\, \frac{H^1(G_v, ad^0\overline{\rho})}{L_v}\Big)$$

is called the (residual) Selmer group associated to $\mathcal{S}$.

Lemma 2.1. *The tangent space* $\mathrm{Def}_{\mathcal{S}}(k[\epsilon])$ *is canonically isomorphic to* $H^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho}))$. *Hence* $R^{\mathcal{S}}$ *is a quotient of a power series ring in* $\dim_k H^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho})$ *many variables over* W. *Furthermore, if each* $\mathcal{D}_v$ *is liftable it is sufficient to quotient out by at most* $\dim_k H^2_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho})$ *many relations.*

Proof. For $\phi \in Z^1(G_{F,S}, ad^0\,\overline{\rho})$, it follows from the proof of proposition 2.2 that the lift

$$\widetilde{\rho} = (1+\epsilon\phi)\overline{\rho} : G_{F,S} \to GL_n(k[\epsilon])$$

will be of type $\mathcal{D}_v$ at v if and only if

$$\phi|_{G_v} \in \widetilde{L}_v.$$

The first part of the lemma follows from this. Thus we have natural isomorphisms:

$$\mathrm{Hom}_{\mathcal{C}_W}(R^{\mathcal{S}}, k[\epsilon]) \cong \mathrm{Hom}_k(\mathfrak{m}_{R^{\mathcal{S}}}/(\mathfrak{m}^2_{R^{\mathcal{S}}}, p), k) \cong H^1_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho}).$$

As a result we can obtain a surjection in $\mathcal{C}_W$

$$\pi : W[[X_1, \ldots, X_g]] \to R^{\mathcal{S}}, \qquad g = \dim_k H^1_{\mathcal{S}}(G_{F,S}, ad^0 \overline{\rho}) \tag{2.9}$$

by sending $X_1, \ldots, X_g$ to elements $s_1, \ldots, s_g \in \mathfrak{m}_{R^{\mathcal{S}}}$ which generate the g-dimensional k-vector space $\mathfrak{m}_{R^{\mathcal{S}}}/(\mathfrak{m}^2_{R^{\mathcal{S}}}, p)$.

The second part of the lemma is equivalent to $J = \ker \pi$ being generated by at most $\dim_k H^2_{\mathcal{S}}(G_{F,S}, ad^0 \overline{\rho})$ many elements. This is done by first showing the existence of an obstruction class $\mathrm{Obs}_{\mathcal{S}}$ in $H^2_{\mathcal{S}}(G_{F,S}, ad^0 \overline{\rho})$ which governs the image of $\mathrm{Def}_{\mathcal{S}}(R) \to \mathrm{Def}_{\mathcal{S}}(R/I)$ for any small surjection $R \to R/I$. The definition of this obstruction class, though similar to the one in §2.1.1 uses the liftability of local deformation conditions [CHT08, lemma 2.2.11]. After this one proceeds as in [Maz89, proposition 2]. □

Corollary 2.1. *If $\mathcal{D}_v$ is liftable for all $v \in S$ and $H^2_{\mathcal{S}}(G_{F,S}, ad^0 \overline{\rho}) = 0$ then $R^{\mathcal{S}}$ is isomorphic to a power series ring over W in $\dim_k H^1_{\mathcal{S}}(G_{F,S}, ad^0 \overline{\rho})$ many variables.*

Now suppose that we are given a homomorphism $\alpha : R^{\mathcal{S}} \to W$. Denote the corresponding deformation to W by ρ_α. As described in §2.1.2 we can construct a subspace $\mathcal{L}_v(\alpha) \subseteq H^1(G_v, ad^0 \rho_\alpha \otimes K/W)$ by taking direct limits of $\mathcal{L}_{v,i}(\alpha)$. Thus we can define a Selmer group associated to α:

$$H^1_{\mathcal{S}}(G_{F,S}, ad^0 \rho_\alpha \otimes K/W) = \ker\Big(H^1(G_{F,S}, ad^0 \rho_\alpha \otimes K/W) \to \oplus_{v\in S} \frac{H^1(G_v, ad^0 \rho_\alpha \otimes K/W)}{\mathcal{L}_v(\alpha)}\Big).$$

Proposition 2.3. *We have the following isomorphism for the Pontryagin dual of* $\ker \alpha/(\ker \alpha)^2$

$$\mathrm{Hom}_W((\ker \alpha)/(\ker \alpha)^2, K/W) \cong H^1_{\mathcal{S}}(G_{F,S}, ad^0 \rho_\alpha \otimes K/W). \tag{2.10}$$

Proof. Denote by α_n the composition $\alpha : R^{\mathcal{S}} \to W \to W/p^n$. Consider the topological W algebra $W \oplus \epsilon K/W$ where $\epsilon^2 = 0$. It is not an object in C_W but is the direct limit of $W \oplus \epsilon\, p^{-i} W/W$ which are objects in C_W. Any continuous homomorphism $\widetilde{\alpha} : R^{\mathcal{S}} \to W \oplus \epsilon\, K/W$ that lifts α will actually take values in some $W \oplus \epsilon\, p^{-i} W/W$ because of compactness. The left hand side of (2.10) is the direct limit of

$$\mathrm{Hom}_W((\ker \alpha)/(\ker \alpha)^2, p^{-i} W/W)$$

which correspond to such homomorphisms.

On the other hand, a lift of α to $W \oplus \epsilon p^{-i}W/W$ will be of type $\mathcal{D}_v$ at v if the cohomology class corresponding to the lift in $H^1(G_{F,S}, ad^0\rho_\alpha \otimes p^{-i}W/W)$ on restriction to G_v lies in $\mathcal{L}_{v,i}(\alpha) \subseteq H^1(G_v, ad^0\rho_\alpha \otimes p^{-i}W/W)$. Thus we have a natural identification

$$\mathrm{Hom}_W((\ker\alpha)/(\ker\alpha)^2, p^{-i}W/W) \cong H^1_{\mathcal{S}}(G_{F,S}, ad^0\rho_\alpha \otimes p^{-i}W/W)$$

from which the proposition follows by taking direct limits. □

2.2 Galois Cohomology

2.2.1 Local and global duality

Let us record the statements of duality theorems for local of global fields which play an important role in this dissertation. The main reference here is [Mil06, chapter 1].

For v a finite place of F let $G_v = \mathrm{Gal}(\overline{F}_v/F)$ and let M be a finite G_v module of exponent m (i.e. $mM = 0$). Let $M^* = \mathrm{Hom}(M, \mu_m(\overline{\mathbb{Q}}))$ be the Cartier dual of M. Also let O_v be the valuation ring of F_v. We have the following theorem due to Tate:

Theorem. (a) The groups $H^i(G_v, M)$ are finite for all $i \geq 0$ and vanishes for all $i \geq 3$.

(b) Duality: for $i = 0, 1, 2$, the G_v pairing $M \times M^* \to \mu_m$ induces via cup product a perfect pairing

$$H^i(G_v, M) \times H^{2-i}(G_v, M^*) \to H^2(G_v, \mu_m) \cong \tfrac{1}{m}\mathbb{Z}/\mathbb{Z}$$

(c) If $M \otimes O_v = 0$ (i.e. then the unramified cohomology groups $H^1(G_v/I_v, M^{I_v})$ and $H^1(G_v/I_v, M^{*I_v})$ are annihilators of each other under

$$H^1(G_v, M) \times H^1(G_v, M^*) \to H^2(G_v, \mu_m) \cong \tfrac{1}{m}\mathbb{Z}/\mathbb{Z}$$

(d) local Euler characteristic formula:

$$\frac{\#H^1(G_v, M)}{\#H^0(G_v, M)\#H^2(G_v, M)} = \#(M \otimes O_v).$$

The case of interest to us is $M = ad^0\,\overline{\rho}$. In this case the trace pairing

$$ad^0\,\overline{\rho} \times ad^0\,\overline{\rho} \to k, \quad (A,B) \mapsto \mathrm{Trace}(AB)$$

gives an isomorphism $ad^0\,\overline{\rho} \cong (ad^0\,\overline{\rho})^\vee$ of Galois modules and we see that

$$(ad^0\,\overline{\rho})^* = \mathrm{Hom}(ad^0\,\overline{\rho}, \mu_p) \cong (ad^0\,\overline{\rho})^\vee \otimes \mu_p \cong ad^0\,\overline{\rho} \otimes \mu_p \cong ad^0\,\overline{\rho}(1),$$

as Galois modules. Hence the paring on cohomology induced from $ad^0\,\overline{\rho} \times ad^0\,\overline{\rho}(1) \to k(1)$, $(A,B) \mapsto \mathrm{Trace}(AB)$, via cup product is a perfect pairing by the local Tate duality (b) above:

$$H^1(G_v, ad^0\,\overline{\rho}) \times H^1(G_v, ad^0\,\overline{\rho}(1)) \to H^2(G_v, k(1)) \cong k. \tag{2.11}$$

The isomorphism $H^2(G_v, k(1)) \cong k$ is from local class field theory. For us the choice of this isomorphism will be irrelevant.

Now suppose $\mathcal{S} = (S, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S})$ is a global deformation problem. Associated $\mathcal{D}_v$ we have its tangent space $L_v \subseteq H^1(G_v, ad^0\,\overline{\rho})$. We define a subspace

$$L_v^\perp \subset H^1(G_v, ad^0\,\overline{\rho}(1)),$$

as the annihilator of L_v under the pairing (2.11). Note that $L_v^\perp$ does not depend on the choice of the isomorphism $H^2(G_v, k(1)) \cong k$ in (2.11). We also remark that if we take $L_v = H^1(G_v/I_v, (ad^0\,\overline{\rho})^{I_v})$ then $L_v^\perp = H^1(G_v/I_v, (ad^0\,\overline{\rho}(1))^{I_v})$ by part (c) of Tate's theorem above.

We define the (residual) dual Selmer group associated to $\mathcal{S}$ as

$$H^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1)) = \ker\Big(H^1(G_{F,S}, ad^0\overline{\rho}) \longrightarrow \oplus_{v\in S}\,\frac{H^1(G_v, ad^0\overline{\rho})}{L_v^\perp}\Big).$$

The groups $H^1_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho})$ and $H^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\,\overline{\rho}(1))$ are part of an exact sequence which is obtained by splicing the Poitou-Tate nine term exact sequence of global duality [Mil06, theorem 4.10]:

$$\begin{aligned} 0 &\longrightarrow H^1_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho}) \longrightarrow H^1(G_{F,S}, ad^0\,\overline{\rho}) \longrightarrow \oplus_{v\in S} H^1(G_v, ad^0\,\overline{\rho})/L_v \\ &\longrightarrow H^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\,\overline{\rho}(1))^\vee \longrightarrow H^2(G_{F,S}, ad^0\,\overline{\rho}) \longrightarrow \oplus_{v\in S} H^2(G_v, ad^0\,\overline{\rho}) \\ &\longrightarrow H^0(G_{F,S}, ad^0\,\overline{\rho}(1))^\vee \longrightarrow 0 \end{aligned} \tag{2.12}$$

Lemma 2.2. *(a)* $\dim_k H^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\,\overline{\rho}(1)) = \dim_k H^2_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho})$.

(b) Greenberg-Wiles formula:

$$\frac{\#H^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho})}{\#H^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1))} = \frac{\#H^0(G_{F,S}, ad^0\overline{\rho})}{\#H^0(G_{F,S}, ad^0\overline{\rho}(1))} \prod_{v\in S} \frac{\#L_v}{\#H^0(G_v, ad^0\overline{\rho})}.$$

Proof. First part follows from comparing the exact sequence (2.8) with (2.12). The second part is a consequence of (2.12), local Tate duality and the global Euler characteristic formula for the $G_{F,S}$ module $ad^0\,\overline{\rho}$:

$$\frac{\#H^1(G_{F,S}, ad^0\,\overline{\rho})}{\#H^2(G_{F,S}, ad^0\,\overline{\rho})} = \#H^0(G_{F,S}, ad^0\,\overline{\rho}) \prod_{v\mid\infty} \frac{|ad^0\,\overline{\rho}|}{\#H^0(G_v, ad^0\,\overline{\rho})}.$$

□

Corollary 2.2. *For a global deformation problem* $\mathcal{S} = (S, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S})$ *suppose that each* $\mathcal{D}_v$ *is liftable and* $H^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1)) = 0$*. Then* $R^{\mathcal{S}}$ *is a power series ring over* W *in* $\dim_k H^1_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho})$ *many variables.*

Proof. Immediate from lemma 2.1 and part (a) of lemma 2.2. □

Notation. From now on, without further mention, we shall feel free use $h^i(\bullet)$ to denote $\dim_k H^i(\bullet)$ of various (residual) cohomolgy groups, with additional embellishments that shall be self-explanatory.

If $H^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1)) \neq 0$, but still assuming that all local deformation problems are liftable, lemma 2.1 and lemma 2.2 implies that the relative dimension of $R^{\mathcal{S}}$ over W is at least

$$\begin{aligned} h^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho}) - h^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1)) = {} & h^0(G_F, ad^0\overline{\rho}) - h^0(G_F, ad^0\overline{\rho}(1)) \\ & + \sum_{v\in S\setminus S_\infty} \big(\dim_k L_v - h^0(G_v, ad^0\overline{\rho})\big) \\ & - \sum_{v\mid\infty} h^0(G_v, ad^0\,\overline{\rho}). \end{aligned} \tag{2.13}$$

Equation (2.13) also implies the following:

Corollary 2.3. *Let Q be a finte set of primes of F disjoint from S and let $(\mathcal{D}_v)_{v\in Q}$ be a collection of deformation conditions at these primes for $\overline{\rho}$. For the deformation problem*

$$\mathcal{S}(\mathcal{Q}) = (S \cup Q, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S\cup Q})$$

we have

$$h^1_{\mathcal{S}(\mathcal{Q})}(G_{F,S\cup Q}, ad^0\,\overline{\rho}) - h^1_{\mathcal{S}(\mathcal{Q})^\perp}(G_{F,S\cup Q}, ad^0\,\overline{\rho}(1)) = h^1_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho}) - h^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\,\overline{\rho}(1)) + \sum_{v\in Q}(\dim_k L_v - h^0(G_v, ad^0\,\overline{\rho}). \quad (2.14)$$

Definition 2.4. A a global deformation problem $\mathcal{S} = (S, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S})$ is said to be *unobstructed* if

$$h^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\,\overline{\rho}(1)) = 0.$$

$\mathcal{S}$ is said to be *balanced* if

$$h^1_{\mathcal{S}}(G_{F,S}, ad^0\,\overline{\rho}) = h^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\,\overline{\rho}(1)).$$

Remark. For a balanced deformation problem $\mathcal{S}$ it is expected that the deformation ring $R^{\mathcal{S}}$ is a finite flat complete intersection over W.

2.2.2 Selmer groups: definition

In this section we shall follow [Gre10] in defining Selmer groups in very general contexts.

As in the previous sections, F denotes a number field and S a finite set of places of F containing all the archimedean places of F and all the places above p. For each place $v \in S$ we shall fix a decomposition group G_v of v inside $G_{F,S}$.

Let us start with a continuous Galois representation $G_{F,S} \to GL_n(R)$ where R is a noetherian complete local W-algebra with residue field k. Denote the underlying free R-module on which $G_{F,S}$ acts by $\mathcal{T}$. Define $\mathcal{D} = \mathcal{T} \otimes_R \widehat{R}$, where $\widehat{R} = \mathrm{Hom}(R, \mathbb{Q}_p/\mathbb{Z}_p)$ is the Pontryagin dual of R carrying the trivial $G_{F,S}$ action. $G_{F,S}$ acts on $\mathcal{D}$ via its action on $\mathcal{T}$. Thus, $\mathcal{D}$ is a discrete R module isomorphic to $\widehat{R}^n$ with a continuous R-linear $G_{F,S}$ action.

The Galois cohomology group $H^1(G_{F,S}, \mathcal{D})$ can be considered as a discrete R-module and it is cofinitely generated in the sense that its Pontryagin dual is a finitely generated R-module.

A *Selmer specification* for $\mathcal{D}$ is collection of R-submodules $L_v(\mathcal{D}) \subseteq H^1(G_v, \mathcal{D})$ for each $v \in S$. We denoted it by $\mathcal{L}(\mathcal{D})$ or simply by $\mathcal{L}$. Define

$$P(\mathcal{D}) = \prod_{v \in S} H^1(G_v, \mathcal{D}) \quad \text{and} \quad L(\mathcal{D}) = \prod_{v \in S} L_v(\mathcal{D}).$$

The natural global-to-local restriction maps for $H^1(\cdot\,, \mathcal{D})$ induce a map

$$\Phi_{\mathcal{L}} : H^1(G_{F,S}, \mathcal{D}) \longrightarrow P(\mathcal{D})/L(\mathcal{D}).$$

The *Selmer group* of $\mathcal{D}$ for the specification $\mathcal{L}$ is defined to be $\ker(\Phi_{\mathcal{L}})$. We denote it by $H^1_{\mathcal{L}}(G_{F,S}, \mathcal{D})$.

It is clear that $H^1_{\mathcal{L}}(G_{F,S}, \mathcal{D})$ is an R-submodule of $H^1_{\mathcal{L}}(G_{F,S}, \mathcal{D})$ and hence is a discrete, cofinitely generated R-module.

Now suppose $\mathcal{D}$ is a discrete, cofinitely generated W-module with a continuous W-linear $G_{F,S}$ action. Define

$$\mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_{p\infty}).$$

It is a compact, finitely generated W-module. Then we can define the (continuous) cohomology groups $H^1(G_{F,S}, \mathcal{T}^*)$ and $H^1(G_v, \mathcal{T}^*)$ as in [NSW00, Chapter I]. These are compact, finitely generated W-modules.

The perfect pairing $\mathcal{D} \times \mathcal{T}^* \to \mu_{p^\infty}$ induces a non degenerate W-pairing [Gre10, §2]

$$\langle \cdot, \cdot \rangle_v : H^1(G_v, \mathcal{D}) \times H^1(G_v, \mathcal{T}^*) \to \mathbb{Q}_p/\mathbb{Z}_p \tag{2.15}$$

for each $v \in S$. Let $L_v(\mathcal{T}^*)$ denote the orthogonal complement of $L_v(\mathcal{D})$ under the above pairing. Denote by $\mathcal{L}(\mathcal{T}^*)$ the Selmer specification for $\mathcal{T}^*$ consisting of $L_v(T^*)$. Define $P(\mathcal{T}^*)$ and $L(\mathcal{T}^*)$ as above. The Selmer group for $\mathcal{T}^*$ associated to the specification $\mathcal{L}^* = \mathcal{L}(\mathcal{T}^*)$ is defined as the kernel of

$$\Phi_{\mathcal{L}^*} : H^1(G_{F,S}, \mathcal{T}^*) \longrightarrow P(\mathcal{T}^*)/L(\mathcal{T}^*).$$

We denote it by $H^1_{\mathcal{L}^*}(G_{F,S}, \mathcal{T}^*)$; it is a W-submodule of $H^1(G_{F,S}, \mathcal{T}^*)$. (It is also called the dual Selmer group of $H^1_{\mathcal{L}}(G_{F,S}, \mathcal{D})$ in the literature).

In §4.2 we shall be interested in proving that a certain $\Phi_{\mathcal{L}}$ is surjective. The condition for the surjectivity of $\Phi_{\mathcal{L}}$ will also involve the cohomology groups of $\mathcal{T}^*$. For this purpose let us define

$$Ш^1(S, \mathcal{T}^*) = \ker\big(H^1(G_{F,S}, \mathcal{T}^*) \longrightarrow \bigoplus_{v \in S} H^1(G_v, \mathcal{T}^*)\big). \tag{2.16}$$

Obviously, $Ш^1(S, \mathcal{T}^*)$ is a W-submodule of $H^1_{\mathcal{L}^*}(G_{F,S}, \mathcal{T}^*)$ for any $\mathcal{L}^*$.

The cokernel of $\Phi_{\mathcal{L}}$ has been studied by Greenberg in [Gre10] and we shall be using results from his paper.

Proposition 2.4. *For the Pontryagin dual of* $\operatorname{coker}(\Phi_{\mathcal{L}})$*, we have an isomorphism of* W*-modules :*

$$\widehat{\operatorname{coker}(\Phi_{\mathcal{L}})} \cong H^1_{\mathcal{L}^*}(G_{F,S}, \mathcal{T}^*)/Ш^1(S, \mathcal{T}^*).$$

In particular, if $H^1_{\mathcal{L}^*}(G_{F,S}, \mathcal{T}^*) = 0$*, then* $\Phi_{\mathcal{L}}$ *is surjective.*

Proof. This is [Gre10, Proposition 3.1.1]. It follows from duality theorems of Poitou and Tate. □

Thus we are lead to finding conditions under which the dual Selmer group $H^1_{\mathcal{L}^*}(G_{F,S}, \mathcal{T}^*)$ vanishes. Using long exact sequence of Poitou-Tate the size of $H^1_{\mathcal{L}^*}(G_{F,S}, \mathcal{T}^*)$ can be controlled using the size of $H^1_{\mathcal{L}}(G_{F,S}, \mathcal{D}))$ and the size of the local conditions $\mathcal{L}_v$'s. To make this precise, let us define for a discrete cofinitely generated W-module $\mathcal{M}$ the W-corank of $\mathcal{M}$ as $\operatorname{corank}_W(\mathcal{M}) = \operatorname{rank}_W(\widehat{\mathcal{M}})$.

The quantity $\operatorname{corank}_W(H^1_{\mathcal{L}}(G_{F,S}, \mathcal{D})) - \operatorname{rank}_W(H^1_{\mathcal{L}^*}(G_{F,S}, \mathcal{T}^*))$ is of importance to our situation. Note that both these numbers are finite in our situation.

Proposition 2.5. *Assume* $\mathcal{D}$ *is discrete and cofinitely generated as a* W*-module. We have*

$$\begin{aligned} &\operatorname{corank}_W(H^1_{\mathcal{L}}(G_{F,S}, \mathcal{D})) - \operatorname{rank}_W(H^1_{\mathcal{L}^*}(G_{F,S}, \mathcal{T}^*)) \\ &= \operatorname{corank}_W(H^0(G_{F,S}, \mathcal{D})) - \operatorname{rank}_W(H^0(G_{F,S}, \mathcal{T}^*)) + \sum_{v \in S} \operatorname{corank}_W(\mathcal{L}_v) - \operatorname{corank}_W(H^0(G_v, \mathcal{D})) \end{aligned}$$

Proof. For finite modules M the analogous formula was obtained by Wiles [DDT94, Theroem 2.18] by splicing the Poitou-Tate nine term long exact sequence for M and its dual M^* [Mil06, Theorem. 4.10]. We can deduce the above proposition (see [Lun16, Lemma 2.6], for example) by considering $M_i = \mathcal{D}[p^i]$ and M_i^* and taking direct and inverse limits, since we have

$$\mathcal{D} = \varinjlim_i M_i \quad \text{and} \quad \mathcal{T}^* = \varprojlim_i M_i^*.$$

□

Even if we could prove $\mathrm{rank}_W(H^1_{\mathcal{L}^*}(G_{F,S}, \mathcal{T}^*)) = 0$, it may still happen that $H^1_{\mathcal{L}^*}(G_{F,S}, \mathcal{T}^*)$ is non zero because it has torsion. The following proposition gives a condition under which this can be ruled out.

First, note that $\mathcal{D}[p]$, the p torsion in $\mathcal{D}$, is a finite dimensional k vector space with a $G_{F,S}$ action.

Proposition 2.6. *Assume that $\mathcal{D}$ is divisible as a W-module and that $\mathcal{D}[p]$ has no subquotient isomorphic to μ_p for the action of $G_{F,S}$. Then $H^1(G_{F,S}, \mathcal{T}^*)$ is torsion free as a W-module.*

Proof. This is [Gre10, Proposition 2.2.1]. □

CHAPTER 3

Ramified Lifts

3.1 Existence of auxiliary sets

Let $\mathcal{S} = (S, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S})$ be a global deformation problem for $\overline{\rho}$.

Theorem 2. *Keeping the previous notation, suppose $\mathcal{S}$ satisfies the following assumptions:*

1) $l \geq 2(n+1)$.

2) The image of $\overline{\rho}$ contains $SL_n(\mathbb{F}_l)$.

3) $[F(\overline{\rho})(\zeta_l) : F(\overline{\rho})] \geq 3$, where $F(\overline{\rho})$ denotes the splitting field of $\overline{\rho}$.

4) For each $v \in S$ the deformation condition $\mathcal{D}_v$ is liftable and

$$h := h^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho}) - h^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1)) \geq 0.$$

Then there exists an auxiliary set Q of Ramakrishna primes of size $h^1_{\mathcal{S}^\perp}$ such that taking $\mathcal{D}_v$ Ramakrishna deformation condition at each $v \in Q$ the global deformation problem $\mathcal{S}(\mathcal{Q}) = (\overline{\rho}, \chi, S\cup Q, \{\mathcal{D}_v\}_{v\in S\cup Q})$ is unobstructed. i.e.,

$$R_{\mathcal{S}(\mathcal{Q})} \cong W(k)[\![X_1, \ldots, X_h]\!].$$

In particular there is a lifting $\rho : G_F \to GL_n(W(k))$ of $\overline{\rho}$ unramified outside $S \cup Q$, with determinant χ and such that $\rho|_{G_v}$ lies in $\mathcal{D}_v$ for all $v \in S$.

The rest of the section is devoted to the proof of theorem 2. First, let us note some consequences of our assumptions (1), (2) and (3) on $\overline{\rho}$. As the image of $\overline{\rho}$ contains

$SL_n(\mathbb{F}_l)$, $\overline{\rho}$ is absolutely irreducible. As $\mathrm{M}_n^0(k)$ is an irreducible $SL_n(\mathbb{F}_l)$ module, we also have that $ad^0\overline{\rho}$ (and hence $ad^0\overline{\rho}(1)$) is an irreducible $k[G_F]$ module.

Proposition 3.1. *Under the above assumptions $ad^0\overline{\rho}$ and $ad^0\overline{\rho}(1)$ are non-isomorphic $k[G_F]$ modules.*

Proof. Let ϵ_l denote the cyclotomic character and χ the character of the adjoint representation , i.e. the composite

$$G_F \longrightarrow GL(ad^0\overline{\rho}) \xrightarrow{\mathrm{Tr}} k.$$

It suffices to show that $\chi(g) \neq \chi(g)\epsilon_l(g)$ for some $g \in G_F$. But this follows from the the assumption that $F(\zeta_l)$ is not contained in $F(ad^0\overline{\rho})$. Take any $g \in G_F$ which act trivially on $ad^0\overline{\rho}$ and non-trivially on ζ_l. Then $\chi(g) = n^2 - 1 \in k$ is non-zero by our assumption that $l \geq 2(n+1)$, and we are done. $\square$

Proposition 3.2. *Let K be the compositum of $F(ad^0\overline{\rho})$ and $F(\zeta_l)$. Then,*

$$H^1(\mathrm{Gal}(K/F), ad^0\overline{\rho}) = 0.$$

Proof. Let L be the compositum of $F(\overline{\rho})$ and $F(\zeta_l)$. Let E denote the intersection of $F(\overline{\rho})$ and $F(\zeta_l)$. Since the inflation map with respect to the normal subgroup $\mathrm{Gal}(L/K) \subset \mathrm{Gal}(L/F)$ is injective, it suffices to prove that $H^1(\mathrm{Gal}(L/F), ad^0\overline{\rho}) = 0$. Under the assumption that $l \geq 2(n+1)$, any subgroup of $GL_n(k)$ which acts absolutely irreducibly on V is adequate (see [GHT12], theorem 9). In particular $H^1(\mathrm{Gal}(F(\overline{\rho})/F), ad^0\overline{\rho}) = 0$.

$$\begin{tikzcd}
 & L \arrow[dl, dash] \arrow[dr, dash] & \\
F(\overline{\rho}) \arrow[dr, dash] & & F(\zeta_l) \arrow[dl, dash] \\
 & E \arrow[d, dash] & \\
 & F &
\end{tikzcd}$$

Now, consider the inflation restriction sequence

$$0 \to H^1(\mathrm{Gal}(F(\overline{\rho})/F), ad^0\overline{\rho}) \to H^1(\mathrm{Gal}(L/F), ad^0\overline{\rho}) \to H^1(\mathrm{Gal}(L/F(\overline{\rho})), ad^0\overline{\rho})^{\mathrm{Gal}(L/F)}.$$

The rightmost term is zero since $[L : F(\overline{\rho})] = [F(\zeta_l) : E]$ is prime to l. This proves that $H^1(\mathrm{Gal}(L/F), ad^0\overline{\rho}) = 0$ and hence $H^1(\mathrm{Gal}(K/F), ad^0\overline{\rho}) = 0$. $\square$

Proposition 3.3. *We also have* $H^1(\mathrm{Gal}(K/F), ad^0\overline{\rho}(1)) = 0$.

Proof. Consider the inflation restriction sequence

$$0 \to H^1(\mathrm{Gal}(F(ad^0\overline{\rho})/F), ad^0\overline{\rho}(1)^{\mathrm{Gal}(K/F(ad^0\overline{\rho}))}) \to H^1(\mathrm{Gal}(K/F), ad^0\overline{\rho}(1)) \\ \to H^1(\mathrm{Gal}(K/F(ad^0\overline{\rho})), ad^0\overline{\rho}(1))^{\mathrm{Gal}(K/F)}.$$

As before, the right most term is zero. Therefore, if $H^1(\mathrm{Gal}(K/F), ad^0\overline{\rho}(1))$ is non zero, so is $H^1(\mathrm{Gal}(F(ad^0\overline{\rho})/F), ad^0\overline{\rho}(1)^{\mathrm{Gal}(K/F(ad^0\overline{\rho}))})$. But $ad^0\overline{\rho}(1)^{\mathrm{Gal}(K/F(ad^0\overline{\rho}))}$ is trivial unless $F(\zeta_l) \subset F(ad^0\overline{\rho})$, which is against our assumption. $\square$

3.1.1 Ramakrishna deformations

We describe the deformation conditions that we shall use at auxiliary primes.

Let v be a finite place of F not in S such that

- $Nv \not\equiv 1 \bmod l$
- $\overline{\rho}|_{G_v} = \overline{\psi}_v\chi_l \oplus \overline{\psi}_v \oplus \overline{s_v}$, where $\dim_k \overline{\psi}_v = 1$, and $\overline{s}_v$ contains neither $\overline{\psi}_v$ nor $\overline{\psi}_v\chi_l$ as a sub-quotient.

Let us call such a place of F a *Ramakrishna prime*, or simply an R-prime. For the moment, we assume the existence of such primes for $\overline{\rho}$. The local deformation condition $\mathcal{D}_v^{\mathrm{Ram}}$ at an R-prime v consists of lifts $\rho : G_{F_v} \to GL_n(A)$ of $\overline{\rho}|_{G_{F_v}}$ which are $(1+M_n(\mathfrak{m}_A))$-conjugate to a lift of the form

$$\begin{pmatrix} \psi\chi_l & * & 0 \\ 0 & \psi & 0 \\ 0 & 0 & s \end{pmatrix},$$

where ψ an unramified lift of $\overline{\psi}_v$ and s an unramified lift of $\overline{s}_v$. Note that the corresponding tangent space L_v^{Ram} consists of cohomology classes consisting of cocycles

$G_v \to M_n^0(k)$ of the form

$$\begin{pmatrix} \phi & f & 0 \\ 0 & \phi & 0 \\ 0 & 0 & \sigma \end{pmatrix},$$

where ϕ and σ are unramified cocycles.

Proposition 3.4. *$\mathcal{D}_v^{Ram}$ is liftable and*

$$dim_k\, L_v^{Ram} = dim_k\, H^0(G_v, ad^0\overline{\rho}).$$

Proof. We only have to lift the cocycle f. See [CHT08, §2.4.7]. □

If Q is a finite set of R-primes for $\overline{\rho}$ and $\mathcal{S}(\mathcal{Q}) = (S \cup Q, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v \in S \cup Q})$ denotes the deformation problem where for $v \in Q$ we take $\mathcal{D}_v = \mathcal{D}_v^{\mathrm{Ram}}$, then from (2.14) we see that

$$h^1_{\mathcal{S}(\mathcal{Q})} - h^1_{\mathcal{S}(\mathcal{Q})^\perp} = h^1_{\mathcal{S}} - h^1_{\mathcal{S}^\perp}. \tag{3.1}$$

3.1.2 An auxiliary deformation condition

We also need an auxiliary deformation condition $\mathcal{D}_v^{\mathrm{aux}}$ at an R-prime v [CHT08, 2.4.8]. These consists of lifts $\rho : G_v \to GL_n(A)$ of $\overline{\rho}|_{G_v}$, which are $(1 + M_n(\mathfrak{m}_A))$-conjugate to a lift of the of the form

$$\begin{pmatrix} \psi_1 & * & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & s \end{pmatrix},$$

where ψ_1 is an unramified lift of $\overline{\psi}_v\chi_l$, ψ_2 is an uramified lift of $\overline{\psi}_v$ and s an unramified lift of $\overline{s}_v$. Let $\mathcal{S}^{\mathrm{aux}}(\{v\})$ denote the global deformation problem where the deformation condition at v is $\mathcal{D}_v^{\mathrm{aux}}$.

The tangent space of $\mathcal{D}_v^{\mathrm{aux}}$, denoted by L_v^{aux}, consists of cohomology classes corresponding to cocycles $G_v \to M_n^0(k)$ of the form

$$\begin{pmatrix} \phi_1 & f & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \sigma \end{pmatrix},$$

where ϕ_1, ϕ_2 and σ are unramified cocycles. It is clear that L_v^{aux} contains L_v^{Ram} as a subspace of codimension 1. We also note that L_v^{aux} also contains the unramified space $H^1(G_v/I_v, ad^0\overline{\rho})$ as a subspace of codimension 1. This follows from our assumption that $\overline{s}_v$ contains neither $\overline{\psi}_v\chi_l$ nor $\overline{\psi}_v$ as a sub-quotient, so we have that

$$H^1(G_v/I_v, ad^0\overline{\rho}) = H^1(G_v/I_v, (ad\,\overline{t} \oplus ad\,\overline{s}) \cap ad^0\overline{\rho}),$$

where $\overline{t} = \overline{\psi}_v\chi_l \oplus \overline{\psi}_v$.

Now suppose that for each $v \in S$, the local deformation problems $\mathcal{D}_v$ is liftable. Then, if we can find a finite set Q of Ramakrishna primes such that the dual-Selmer group $H^1_{\mathcal{S}(\mathcal{Q})^\perp}(G_{F,S\cup Q}, ad^0\overline{\rho}(1))$ vanishes, this would mean by corollary 2.2 that the deformation ring $R^{\mathcal{S}(\mathcal{Q})}$ representing the global deformation problem $\mathcal{S}(\mathcal{Q}) = (S \cup Q, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v \in S\cup Q})$ is isomorphic to $W(k)[\![X_1, \ldots, X_h]\!]$, (where $h = h^1_{\mathcal{S}(\mathcal{Q})}$). That such an auxiliary set exists when $h^1_{\mathcal{S}} - h^1_{\mathcal{S}^\perp} \geq 0$ and under the assumptions on $\overline{\rho}$ made above is proved below, following [CHT08] theorem 2.6.3 and its proof.

3.1.3 Proof of theorem 2

We begin with proof of theorem 2. Recall the assumptions on the global deformation problem $\mathcal{S} = (\overline{\rho}, \chi, S, \{\mathcal{D}_v\}_{v\in S})$

1) $l \geq 2(n+1)$.

2) The image of $\overline{\rho}$ contains $SL_n(\mathbb{F}_l)$.

3) $[F(\overline{\rho})(\zeta_l) : F(\overline{\rho})] \geq 3$, where $F(\overline{\rho})$ denotes the splitting field of $\overline{\rho}$.

4) $\mathcal{D}_v$ is liftable for each $v \in S$ and $h^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho}) - h^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1)) \geq 0$.

If $h^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1)) = 0$ then the theorem follows from corollary 2.2 by taking $Q = \emptyset$. So assume $h^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1)) \geq 1$. By assumption 4 this implies $h^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho}) \geq 1$ as well. Suppose we have a systematic method to find a Ramakrishna prime v such that dimension of the dual Selmer group decreases by one, i.e.,

$$h^1_{\mathcal{S}(\{v\})^\perp} = h^1_{\mathcal{S}^\perp} - 1.$$

Then we can inductively use this method to get a Ramakrishna-set Q as in the theorem (this would work because the new deformation problem $\mathcal{S}(\{v\})$ also satisfies assumption 4).

Let i_1 (resp. π_1) denote the G_v-equivariant inclusion $\overline{\psi_v}\chi_l \hookrightarrow \overline{\rho}$ (resp. projection $\overline{\rho} \twoheadrightarrow \overline{\psi_v}\chi_l$). Similarly let i_2 and π_2 denote the ones corresponding to $\overline{\psi_v}$. From the above remarks, we have the following obvious exact sequences

$$0 \to H^1_{\mathcal{S}}(S, ad^0\overline{\rho}) \to H^1_{\mathcal{S}^{\mathrm{aux}}(\{v\})}(S \cup \{v\}, ad^0\overline{\rho}) \to \frac{L_v^{\mathrm{aux}}}{L_v^{\mathrm{un}}},$$

$$0 \to H^1_{\mathcal{S}(\{v\})}(S \cup \{v\}, ad^0\overline{\rho}) \to H^1_{\mathcal{S}^{\mathrm{aux}}(\{v\})}(S \cup \{v\}, ad^0\overline{\rho}) \to \frac{L_v^{\mathrm{aux}}}{L_v^{\mathrm{Ram}}},$$

and

$$0 \to H^1_{\mathcal{S}^{\mathrm{aux}}(\{v\})^{\perp}}(S \cup \{v\}, ad^0\overline{\rho}(1)) \to H^1_{\mathcal{S}^{\perp}}(S, ad^0\overline{\rho}(1)) \to \frac{\left(L_v^{\mathrm{un}}\right)^{\perp}}{\left(L_v^{\mathrm{aux}}\right)^{\perp}}.$$

We note that

$$\frac{L_v^{\mathrm{aux}}}{L_v^{\mathrm{un}}} \cong H^1(I_v, k(i_1\pi_2)), \tag{3.2}$$

and

$$\frac{L_v^{\mathrm{aux}}}{L_v^{\mathrm{Ram}}} \cong H^1(G_v/I_v, k(i_1\pi_1 - i_2\pi_2)). \tag{3.3}$$

Also, by a local computation at v using the trace pairing, we also get that

$$\begin{aligned}\frac{\left(L_v^{\mathrm{un}}\right)^{\perp}}{\left(L_v^{\mathrm{aux}}\right)^{\perp}} \cong \frac{\left(L_v^{\mathrm{un}}\right)^{\perp}}{\left(L_v^{\mathrm{un}} + L_v^{\mathrm{Ram}}\right)^{\perp}} &\cong \frac{\left(L_v^{\mathrm{Ram}}\right)^{\perp}}{\left(L_v^{\mathrm{un}}\right)^{\perp} \cap \left(L_v^{\mathrm{Ram}}\right)^{\perp}} \\ &\cong H^1(G_v/I_v, (ad^0\,\overline{t}/k(i_1\pi_2))(1)),\end{aligned} \tag{3.4}$$

where $\overline{t} = \overline{\psi_v}\chi_l \oplus \overline{\psi_v}$, as before.

Let $[\phi] \in H^1_{\mathcal{S}^{\perp}}(G_{F,S}, ad^0\overline{\rho}(1))$ and $[\psi] \in H^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho})$ be non-zero cohomology classes. Suppose

$$\begin{gathered}[\phi] \text{ maps non-trivially under} \\ H^1_{\mathcal{S}^{\perp}}(S, ad^0\overline{\rho}(1)) \to H^1(G_v/I_v, (ad^0\,\overline{t}/k(i_1\pi_2))(1))\end{gathered} \tag{3.5}$$

and

$$\begin{gathered}[\psi] \text{ maps non-trivially under} \\ H^1_{\mathcal{S}}(S, ad^0\overline{\rho}) \hookrightarrow H^1_{\mathcal{S}^{\mathrm{aux}}(\{v\})}(S \cup \{v\}, ad^0\overline{\rho}) \to H^1(G_v/I_v, k(i_1\pi_1 - i_2\pi_2)).\end{gathered} \tag{3.6}$$

Since we have by (2.13) that

$$h^1_{\mathcal{S}^{\mathrm{aux}}(\{v\})} - h^1_{\mathcal{S}^{\mathrm{aux}}(\{v\})^\perp} = 1 + h^1_{\mathcal{S}} - h^1_{\mathcal{S}^\perp},$$

and since

$$h^1_{\mathcal{S}(\{v\})} - h^1_{\mathcal{S}(\{v\})^\perp} = h^1_{\mathcal{S}} - h^1_{\mathcal{S}^\perp}$$

because of (3.1), we could then conclude that

$$h^1_{\mathcal{S}^\perp} - h^1_{\mathcal{S}(\{v\})^\perp} = 1 = h^1_{\mathcal{S}} - h^1_{\mathcal{S}(\{v\})},$$

and the theorem would follow from the remarks above.

We now explain how to find such an v. Recall the notation from above. K denotes the composite of $F(ad^0\overline{\rho})$ and $F(\zeta_l)$, D their intersection, L denotes the composite of $F(\overline{\rho})$ and $F(\zeta_l)$ and E the intersection of $F(\overline{\rho})$ and$F(\zeta_l)$. Identify $\mathrm{Gal}(F(\zeta_l)/E)$ as a subgroup of $\mathbb{F}_l^\times$. Let $x \in \mathrm{Gal}(F(\zeta_l)/E)$ be such that $x^2 \neq 1$. Since the extension E/F is abelian, $\mathrm{Gal}(F(\overline{\rho})/E)$ contains the commutator subgroup of $\mathrm{Gal}(F(\overline{\rho})/F)$ and in particular, by assumption (2) the commutator subgroup of $SL_n(\mathbb{F}_l)$. But $[SL_n(\mathbb{F}_l), SL_n(\mathbb{F}_l)] = SL_n(\mathbb{F}_l)$ as $l \geq 7$. Consider the element $\mathrm{diag}(x, x^{-1}, 1, \ldots, 1) \in \mathrm{Gal}(F(\overline{\rho})/E)$ and let $\tilde{x}$ denote its projection to $\mathrm{Gal}(F(ad^0\overline{\rho})/D)$. Let $\sigma = (\tilde{x}, x) \in \mathrm{Gal}(K/D) \subset \mathrm{Gal}(K/F)$. Let v be a place of F not in S such that $Frob_v = \sigma \in \mathrm{Gal}(K/F)$; by construction v is an R-prime for $\overline{\rho}$.

Put $\beta = x^{-1} \in k$ and $\alpha = x = \beta\chi(\sigma)$. Under the action of σ, V has a decomposition

$$V = V_{\sigma,\alpha} \oplus V_{\sigma,\beta} \oplus S,$$

where $V_{\sigma,\alpha}$ and $V_{\sigma,\beta}$ are the α and β generalized eigenspaces of $\overline{\rho}(\sigma)$ respectively; both are one dimensional by construction. Let π_α, π_β and i_α, i_β denote the σ-equivariant projections respectively inclusions to the subspaces.

To ensure v will satisfy conditions (3.5) and (3.6), we impose the corresponding conditions on v by working over a larger field. Since G_K acts trivially on $ad^0\overline{\rho}$, $\psi|_{G_K} : G_K \to ad^0\overline{\rho}$ is a homomorphism. Moreover by looking at the inflation restriction sequence and using proposition 3.2, we see that $\psi|_{G_K}$ is a non-zero $\mathrm{Gal}(K/F)$-equivariant

homomorphism. Let K_ψ/K be its splitting field. Note that ψ induces an isomorphism of $\mathrm{Gal}(K_\psi/K)$ into a $\mathbb{F}_l[G_F]$ submodule of $ad^0\overline{\rho}$. Since $ad^0\overline{\rho}$ is an irreducible $k[G_F]$ module the k span of this subspace must be all of $ad^0\overline{\rho}$. Consider the following property for elements of $f \in ad^0\overline{\rho}$:

$$\pi_\alpha \circ f \circ i_\alpha \neq \pi_\beta \circ f \circ i_\beta;$$

note that for a subset $W \subset ad^0\overline{\rho}$, there is an element in the k span of W with the above property if and only if there is an element in the $\mathbb{F}_l$ span W with the above property. Therefore from the remarks above we conclude that there is an $\eta \in \mathrm{Gal}(K_\psi/K)$ such that

$$\pi_\alpha \circ \psi(\eta) \circ i_\alpha \neq \pi_\beta \circ \psi(\eta) \circ i_\beta. \tag{3.7}$$

Similarly using proposition 3.3, we see that $\phi|_{G_K} : G_K \to ad^0\overline{\rho}(1)$ is a non-zero $\mathrm{Gal}(K/F)$-equivariant homomorphism. Let K_ϕ/K be its splitting field. Arguing as above, we can find an element $\theta \in \mathrm{Gal}(K_\phi/K)$ such that

$$\pi_\beta \circ \phi(\theta) \circ i_\alpha \neq 0. \tag{3.8}$$

By proposition 3.1 K_ψ and K_ϕ are linearly disjoint over K. Let M be their compositum. Let

$$\begin{aligned}\tau \in \mathrm{Gal}(M/F) &\simeq \mathrm{Gal}(M/K) \rtimes \mathrm{Gal}(K/F)\\ &\simeq \Big(\mathrm{Gal}(K_\psi/K) \times \mathrm{Gal}(K_\phi/K)\Big) \rtimes \mathrm{Gal}(K/F)\end{aligned}$$

be the element $(\eta, \theta) \rtimes \sigma$. Let v be a prime of F unramified in M and not in S such that $Frob_v = \tau$. There are infinitely such primes by the Chebotarev density theorem. Property (3.8) implies that assumption (3.5) holds true and property (3.7) implies that assumption (3.6) holds true and we are done. This completes the proof of theorem 2.

3.2 Existence of ramified auxiliary lifts

3.2.1 Ordinary and nearly ordinary deformations

In this section we shall describe some variants of the ordinary deformation problem which we use as local conditions at primes v above l. Let E be a finite extension of $\mathbb{Q}_l$. Denote $\mathrm{Gal}(\overline{E}/E)$ by G_E and its inertia subgroup by I_E. A representation $\overline{\rho} : G_E \to GL_n(k)$ is said to be ordinary if $\overline{\rho}(G_E) \subset B(k)$ for a Borel subgroup B of GL_n. Let us recall the definition of the *nearly ordinary deformation* functor associated to such a $\overline{\rho}$.

A lifting $\rho : G_E \to GL_n(A)$ of $\overline{\rho}$ to a complete Noetherian W algebra A is nearly ordinary if there exists $g \in \ker(GL_n(A) \to GL_n(k))$ such that ${}^g\rho(G_E) \subset B(A)$. Let $\mathfrak{g}$ (resp. $\mathfrak{b}$) denote the Lie algebra of GL_n (resp. B). We consider them as $k[G_E]$ modules via the adjoint action through $\overline{\rho}$. The following conditions on $\overline{\rho}$ ensures that the nearly ordinary deformation functor (and its variant described below) is a liftable deformation condition (see [Til96, chapter6]):

$$\text{(REG)} \qquad H^0(G_E, \mathfrak{g}/\mathfrak{b}) = 0.$$

$$\text{(REG}^*\text{)} \qquad H^0(G_E, (\mathfrak{g}/\mathfrak{b})(1)) = 0.$$

Let N denote the unipotent radical of B and let T be a quotient torus of the torus B/N. We can push forward $\overline{\rho}$ to a homomorphism

$$\overline{\rho}_T : G_E \to (B/N)(k) \to T(k).$$

Fix a lift $\chi_T : I_E \to T(W)$ of $\overline{\rho}_T|_{I_E}$.

Definition 3.1. Let $\rho : G_E \to GL_n(A)$ be a lift of $\overline{\rho}$. We shall say ρ is (B, T, χ_T)-ordinary if

- there exists $g \in \ker(GL_n(A) \to GL_n(k))$ such that ${}^g\rho(G_E) \subset B(A)$.
- the restriction to inertia of the push-forward

  $$({}^g\rho)_T|_{I_E} : I_E \to B(A) \to T(A)$$

  is equal to χ_T.

Denote the subfunctor of $\mathrm{Lift}_{\overline{\rho}}$ consisting of (B, T, χ_T)-ordinary lifts by $\mathrm{Lift}_{\overline{\rho}}^{B,\chi_T}$.

We note that if we take T to be the trivial torus, then we recover the definition of the nearly ordinary deformation functor above. On the other hand, if we take $T = B/N$ then we get what is usually called the ordinary deformation functor.

Lemma 3.1. *Assume $\overline{\rho}$ satisfies (REG) and (REG*) and that $\zeta_l \notin E$. Then $\mathrm{Lift}_{\overline{\rho}}^{B,\chi_T}$ is a liftable local deformation condition of tangent space dimension*

$$(dim_k\, \mathfrak{b} - dim_k\, \mathfrak{t})[E : \mathbb{Q}_l] + dim_k\, H^0(G_E, ad\,\overline{\rho}).$$

Proof. The proof is similar to the proof of proposition 4.4 of [Pat16a]. □

Remark. An analogous result holds if we fix the determinant of the lifts. Let $\mu : G_E \to W^\times$ be a lift of $\det(\overline{\rho})$. Let $\chi_T : I_E \to T(W)$ be a lift of $\overline{\rho}_T|_{I_E}$ such that $\det(\chi_T) = \mu|_{I_E}$. Let $\mathrm{Lift}_{\overline{\rho}}^{\mu,B,\chi_T}$ be the subfunctor of $\mathrm{Lift}_{\overline{\rho}}^{B,\chi_T}$ consisting of (B, T, χ_T)-ordinary lifts with determinant μ. Then $\mathrm{Lift}_{\overline{\rho}}^{\mu,B,\chi_T}$ is a liftable deformation condition of tangent space dimension

$$(\dim_k \mathfrak{b} - \dim_k \mathfrak{t})[E : \mathbb{Q}_l] + \dim_k H^0(G_E, ad^0\,\overline{\rho}).$$

3.2.2 A balancing act

We return to the global setting as in theorem 1. To recall the notations, F is a totally real number field, which we assume to be Galois over $\mathbb{Q}$ for simplicity, k a finite field of characteristic l and $\overline{\rho} : G_F \to GL_n(k) = GL(V)$ a continuous Galois representation, with $n \geq 2$. S denotes a finite set of places of F containing all the places above l, ∞, and the primes at which $\overline{\rho}$ is ramified. And $\chi : G_F \to W(k)^\times$ be a lift of $\det \overline{\rho}$. Assume from now on that $\overline{\rho}$ and l satisfies hypotheses (1)-(5) of theorem 1.

The purpose of this section is to demonstrate the existence of liftable local deformation conditions $\{\mathcal{D}_v\}_{v\in S}$ such that the corresponding global deformation problem $\mathcal{S} = (S, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S})$ is balanced, i.e.the size of the Selmer group is equal to the size of the dual Selmer group. To get a balanced deformation problem, we shall choose

ordinary deformation conditions of the type defined above such that $h^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho}) = h^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1))$.

At places $v \in S$ not dividing $l.\infty$ we can always choose liftable deformation conditions $\mathcal{D}_v$ with tangent space dimension

$$dim_k L_v = \dim_k H^0(G_v, ad^0\,\overline{\rho}).$$

For example, we can use *minimal deformations*; see [CHT08, §2.4.4]. Having chosen such local deformation problems, only the places above l and ∞ contribute to the Greenberg-Wiles formula for the difference $h^1_{\mathcal{S}}(G_{F,S}, ad^0\overline{\rho}) - h^1_{\mathcal{S}^\perp}(G_{F,S}, ad^0\overline{\rho}(1))$. Also, assumption (4) of theorem 1, of total oddness of $\overline{\rho}$, implies that for every $v|\infty$

$$h^0(G_v, ad^0\overline{\rho}) = \begin{cases} (n^2-2)/2, \text{ for } n \text{ even} \\ (n^2-1)/2, \text{ for } n \text{ odd} \end{cases}$$

Therefore the contribution from the Archimedean places is

$$\sum_{v|\infty} h^0(G_v, ad^0\overline{\rho}) = \begin{cases} \frac{(n^2-2)}{2}[F:\mathbb{Q}], \text{ for } n \text{ even} \\ \frac{(n^2-1)}{2}[F:\mathbb{Q}], \text{ for } n \text{ odd.} \end{cases}$$

For each place $\lambda|l$, let $(B_\lambda, T_\lambda, (\chi_T)_\lambda)$ denote a choice of a Borel subgroup, a quotient torus, and an inertial weight as in definition 3.1. The contribution from places above l in the Greenberg-Wiles formula is

$$\sum_{\lambda|l} (\dim_k \mathfrak{b}_\lambda - \dim_k \mathfrak{t}_\lambda)[F_\lambda : \mathbb{Q}_l].$$

Since we have assumed F is Galois over $\mathbb{Q}$ each $[F_\lambda : \mathbb{Q}_l]$ is the same and it is clear that we can choose tori T_λ so as to get a balanced deformation problem $\mathcal{S} = (S, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S})$.

3.2.3 Proof of theorem 1

Let $\mathcal{S}^{no}$ denote the nearly ordinary problem for $\overline{\rho} : G_{F,S} \to GL_n(k)$. In this section we demonstrate the existence of a ramified auxiliary set Q for $\mathcal{S}^{no}$, i.e., we shall find

a finite set of Ramakrishna primes Q disjoint from S such that the new deformation problem $\mathcal{S}^{no}(\mathcal{Q})$ has trivial dual Selmer group and the universal deformation is ramified at all places $v \in S \cup Q$. The method is a modification of the one due to Khare and Ramakrishna for GL_2, [KR03].

We recall the two local deformation conditions we shall use at an auxiliary prime q. One is the Ramkrishna condition ([CHT08, 2.4.7]) with tangents space by L_q^{Ram}. The second one is an auxiliary deformation condition ([CHT08, 2.4.8]) which includes all unramified lifts and all Ramakrishna lifts. Its tangent space L_q^{aux} contains L_q^{Ram} as a subspace of codimension one.

From now on we shall fix a choice of $\mathcal{S} = (S, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v \in S})$ which is a balanced deformation condition for $\overline{\rho}$, whose existence is demonstrated above. Observe that a ramified auxiliary set Q for $\mathcal{S}$ is also for $\mathcal{S}^{no}$. (This is because

$$h^1_{\mathcal{S}(\mathcal{Q})^\perp} = 0 \quad \Rightarrow \quad h^1_{\mathcal{S}^{no}(\mathcal{Q})^\perp} = 0.)$$

Now we start with an auxiliary set Q for $\mathcal{S}$, existence of which is assured by theorem 2, and describe how to get a ramified one . Let $R^{\mathcal{S}(\mathcal{Q})}$ denote the corresponding universal deformation ring.

$$R^{\mathcal{S}(\mathcal{Q})} \cong W(k).$$

Also, let

$$\rho^{\mathcal{S}(\mathcal{Q})} : G_{F,S \cup Q} \to GL_n(R^{\mathcal{S}(\mathcal{Q})})$$

be the universal deformation. If $\rho^{\mathcal{S}(\mathcal{Q})}$ is ramified at all $v \in Q$ we are done. Otherwise, write $Q = Q_{ram} \cup Q_{un}$ a disjoint union into ramified primes and unramified primes. Also, discard any primes in Q_{un} such that resulting set along with Q_{ram} is auxiliary. The idea in [KR03] is to inductively replace each $q \in Q_{un}$ with two new R-primes q_1, q_2 such that $Q'' := Q \setminus \{q\} \cup \{q_1, q_2\}$ is auxiliary for $\mathcal{S}$ and

$$\rho^{\mathcal{S}(\mathcal{Q}'')} : G_{F,S \cup Q''} \to GL_n(R^{\mathcal{S}(\mathcal{Q}'')})$$

is ramified at $Q_{ram} \cup \{q_1, q_2\}$.

For $r \geq 0$, we put $\rho_r^{\mathcal{S}(\mathcal{Q})} : G_{F,S\cup Q} \to GL_n(W(k)/l^r))$ for the reduction mod l^r of $\rho^{\mathcal{S}(\mathcal{Q})}$. Let m be an integer such that $\rho_{m-1}^{\mathcal{S}(\mathcal{Q})}$ is ramified at all primes in Q_{ram}. The primes q_1 and q_2 will be chosen so that $\rho_{m-1}^{\mathcal{S}(\mathcal{Q}'')}$ will be **equal** to $\rho_{m-1}^{\mathcal{S}(\mathcal{Q})}$; therefore $\rho^{\mathcal{S}(\mathcal{Q}'')}$ will be ramified at all primes of Q_{ram}. We shall impose additional conditions on $\{q_1, q_2\}$ so that $\rho^{\mathcal{S}(\mathcal{Q}'')}$ will be ramified at q_1 and q_2 as well.

Let $Q' = Q \backslash \{q\}$, where $q \in Q_{un}$. Then $H^1_{\mathcal{S}(\mathcal{Q}')}$ and $H^1_{\mathcal{S}(\mathcal{Q}')^\perp}$ are both one dimensional. Let ψ and φ span these spaces respectively. Let q_1 be an R-prime not in $S \cup Q$ and put $Q'_1 = Q' \cup \{q_1\}$. We have the following exact sequences:

$$0 \longrightarrow H^1_{\mathcal{S}(\mathcal{Q}')} \longrightarrow H^1_{\mathcal{S}^{\text{aux}}(\mathcal{Q}'_1)} \longrightarrow L^{\text{aux}}_{q_1}/L^{\text{un}}_{q_1},$$

$$0 \longrightarrow H^1_{\mathcal{S}(\mathcal{Q}'_1)} \longrightarrow H^1_{\mathcal{S}^{\text{aux}}(\mathcal{Q}'_1)} \longrightarrow L^{\text{aux}}_{q_1}/L^{\text{Ram}}_{q_1},$$

and

$$0 \longrightarrow H^1_{\mathcal{S}^{\text{aux}}(\mathcal{Q}'_1)^\perp} \longrightarrow H^1_{\mathcal{S}(\mathcal{Q}')^\perp} \longrightarrow \left(L^{\text{un}}_{q_1}\right)^\perp/\left(L^{\text{aux}}_{q_1}\right)^\perp.$$

Suppose q_1 is chosen such that it satisfies the following hypotheses:

(a) $\varphi|_{G_{F_{q_1}}} \neq 0$.

(b) $\psi|_{G_{F_{q_1}}} = 0$.

(c) $\rho_{m-1}^{\mathcal{S}(\mathcal{Q})}$ is of Ramakrishna type at q_1 but $\rho_m^{\mathcal{S}(\mathcal{Q})}$ is not.

Conditions (a) and (b) imply, by the Greenberg-Wiles formula, that $H^1_{\mathcal{S}(\mathcal{Q}'_1)}$ and $H^1_{\mathcal{S}(\mathcal{Q}'_1)^\perp}$ are both one dimensional. $H^1_{\mathcal{S}(\mathcal{Q}'_1)}$ is spanned by the inflation of ψ. Let $\tilde{\varphi}$ span $H^1_{\mathcal{S}(\mathcal{Q}'_1)^\perp}$.

Now, suppose we can also find an R-prime q_2 not in $S \cup Q \cup \{q_1\}$, such that

(d) $\psi|_{G_{F_{q_2}}}$ is non-zero in $L^{un}_{q_2}$.

(e) $\tilde{\varphi}|_{G_{F_{q_2}}} \neq 0$.

(f) $\varphi|_{G_{F_{q_2}}} \neq 0$.

(g) $\rho_{m-1}^{\mathcal{S}(\mathcal{Q})}$ is of Ramakrishna type at q_2.

As before, we have the following exact sequences:

$$0 \longrightarrow H^1_{\mathcal{S}(\mathcal{Q}_1')} \longrightarrow H^1_{\mathcal{S}^{\mathrm{aux}}(\mathcal{Q}'')} \longrightarrow L_{q_2}^{\mathrm{aux}}/L_{q_2}^{\mathrm{un}},$$

$$0 \longrightarrow H^1_{\mathcal{S}(\mathcal{Q}'')} \longrightarrow H^1_{\mathcal{S}^{\mathrm{aux}}(\mathcal{Q}'')} \longrightarrow L_{q_2}^{\mathrm{aux}}/L_{q_2}^{\mathrm{Ram}},$$

and

$$0 \longrightarrow H^1_{\mathcal{S}^{\mathrm{aux}}(\mathcal{Q}'')^\perp} \longrightarrow H^1_{\mathcal{S}(\mathcal{Q}_1')^\perp} \longrightarrow \left(L_{q_2}^{\mathrm{un}}\right)^\perp / \left(L_{q_2}^{\mathrm{aux}}\right)^\perp.$$

Put $Q_2' = Q' \cup \{q_2\}$. The first two assumptions imply, again by the Greenberg-Wiles formula, that $Q'' = Q' \cup \{q_1, q_2\}$ is auxiliary for $\mathcal{S}$. The third condition implies that $H^1_{\mathcal{S}^{\mathrm{aux}}(\mathcal{Q}_2')}$ is one dimensional (and that $S \cup Q_2'$ is auxiliary for $\mathcal{S}$). The other assumptions on q_1 and q_2 imply the following:

- $\rho_{m-1}^{\mathcal{S}(\mathcal{Q})}$ is of type $\mathcal{S}$ for all $v \in S$, is of Ramkrishna type at all $\nu \in Q''$, and unramified outside $S \cup Q''$. Obviously, $\rho_{m-1}^{\mathcal{S}(\mathcal{Q}'')}$ also has these properties. Since $R^{\mathcal{S}(\mathcal{Q})}$ and $R^{\mathcal{S}(\mathcal{Q}'')}$ are both isomorphic to $W(k)$, we must have that $\rho_{m-1}^{\mathcal{S}(\mathcal{Q})}$ and $\rho_{m-1}^{\mathcal{S}(\mathcal{Q}'')}$ are equal (as deformations of $\overline{\rho}$ to $W(k)/l^{m-1}$).

- $\rho_m^{\mathcal{S}(\mathcal{Q})}$ is unramified at q and is not of Ramakrishna type at q_1.

We regard both $\rho_m^{\mathcal{S}(\mathcal{Q})}$ and $\rho_m^{\mathcal{S}(\mathcal{Q}'')}$ as homomorphisms $G_{F,S\cup Q''} \to GL_n(W(k)/l^m)$. Note that they are both equal mod l^{m-1}. It follows that there is an $h \in H^1(G_{F,S\cup Q''}, ad^0\overline{\rho})$ such that

$$\rho_m^{\mathcal{S}(\mathcal{Q}'')} = (1 + l^{m-1}h)\rho_m^{\mathcal{S}(\mathcal{Q})}.$$

Proposition 3.5. *([KR03, proposition 14]) $\rho^{\mathcal{S}(\mathcal{Q}'')}$ is ramified at both q_1 and q_2.*

Proof. We have

$$\rho_m^{\mathcal{S}(\mathcal{Q}'')} = (1 + l^{m-1}h)\rho_m^{\mathcal{S}(\mathcal{Q})}. \tag{3.9}$$

Note that $h|_{G_{F_v}} \in L_v$ for all $v \in S \cup Q'$ since both sides are the mod l^m reductions of elements of L_v for these v.

Suppose $\rho_m^{\mathcal{S}(\mathcal{Q}'')}$ is unramified at q_i for $i = 1$ or 2. Then h inflates from $H^1(G_{F_{S\cup Q''\setminus\{q_i\}}}, ad^0\overline{\rho})$. Since we also have $h|_{G_{F_v}} \in L_v$ for all $v \in S \cup Q'$, it follows that $h \in H^1_{\mathcal{S}^{\mathrm{aux}}(\mathcal{Q}_i')}$. But $H^1_{\mathcal{S}^{\mathrm{aux}}(\mathcal{Q}_i')}$ are one dimensional, spanned by the inflation of ψ, for $i = 1, 2$. By our choice, ψ is trivial at q_1 and $\rho_m^{\mathcal{S}(\mathcal{Q})}$ is not of Ramakrishna type at q_1. Therefore the right hand side of (3.9) is not of Ramakrishna type. But the left hand side is of Ramakrishna type at q_1. This contradiction proves the proposition. $\square$

We now prove that we can find q_1 and q_2 satisfying the above seven conditions (a)-(g).

As before, let K denote the compositum $F(ad^0\overline{\rho}, \mu_l)$. The cohomology classes ψ and ϕ when restricted to G_K becomes homomorphisms. Let K_ψ and K_ϕ denote the field extensions of K cut out by these respectively. They are Galois extensions of F. Denote by P_m the fixed field of $\rho_n^{\mathcal{S}(\mathcal{Q})}|_{G_K}$.

Lemma 3.2. *Under the assumption (2) of theorem 1, that the image of $\overline{\rho}$ contains $SL_n(\mathbb{F}_l)$, the field extensions K_ψ, K_ϕ, P_m and $K_{\zeta_{l^m}}$ are strongly linearly disjoint over K for all $m \geq 0$. (i.e. each of them is linearly disjoint over K with the compositum of the other three).*

Proof. This is essentially [Pat16b, Lemma 3.13] $\square$

Lemma 3.3. *Under the assumption that the image of $\overline{\rho}$ contains $SL_n(\mathbb{F}_l)$, given a balanced deformation problem $\mathcal{S}$ and an Ramakrishna set Q for $\mathcal{S}$, we can find q_1 and q_2 satisfying conditions (a)-(g).*

Proof. The proof is similar to the case of GL_2, [KR03, Lemma 8]. We refer to [Pat16b, Proposition 3.14]. $\square$

CHAPTER 4

Consequences of ramified lifts

4.1 Formal neighborhoods of ordinary deformation rings

In this section we shall denote by $\mathcal{S} \cup Q$ (resp. $\mathcal{S}^{no} \cup Q$) the deformation problem (resp. nearly ordinary deformation problem) where we take the local deformation condition at an auxiliary primes $v \in Q$ to be unrestricted (i.e., all lifts of $\overline{\rho}|_{G_v}$) and at $v \in S$ to be the one prescribed by $\mathcal{S}$.

For a balanced deformation problem $\mathcal{S}$ as above, it is proved in Theorem 1 that there is an auxiliary set Q such that the new deformation problem $\mathcal{S}(\mathcal{Q})$ is unobstructed and the corresponding deformation ring $R^{\mathcal{S}(\mathcal{Q})} \simeq W(k)$. This easily implies that the corresponding nearly ordinary problem $\mathcal{S}^{no}(\mathcal{Q})$ is also unobstructed and the corresponding deformation ring $R^{\mathcal{S}^{no}(\mathcal{Q})}$ is isomorphic to $W(k)[[X_1, \cdots, X_h]]$ where $h = ([n/2]+1)[F : \mathbb{Q}]$. This still leaves the structure of $R^{\mathcal{S} \cup Q}$ and $R^{\mathcal{S}^{no} \cup Q}$ mysterious. We have the natural maps $R^{\mathcal{S}^{no} \cup Q} \to R^{\mathcal{S}^{no}(\mathcal{Q})}$ and $R^{\mathcal{S} \cup Q} \to R^{\mathcal{S}(\mathcal{Q})}$.

Theorem 1 also further assures us that we can further choose the auxiliary set set Q so that for all places $v \in Q$, the corresponding universal representation $\rho^{\mathcal{S}(Q)} : G_F \to GL_n(W(k))$ is ramified at all the primes in Q and similarly for $\rho^{\mathcal{S}^{no}(Q)} : G_F \to GL_n(R^{\mathcal{S}^{no}(\mathcal{Q})})$. This implies that the corresponding universal representation $\rho^{\mathcal{S}^{no}(Q)} : G_F \to GL_n(W(k)[[X_1, \cdots, X_h]])$ is ramified for a Zariski dense set of specializations $\rho_x^{\mathcal{S}^{no}(\mathcal{Q})}$ corresponding to closed points $x \in \operatorname{Spec}\left(R^{\mathcal{S}^{no}(\mathcal{Q})}[1/p]\right)$. This is proved in lemma 4.2. We denote the pull back of these morphisms x to $R^{\mathcal{S}^{no} \cup Q}$ by the same symbol.

Theorem 3. *For any of the Zariski dense set of specializations x as above at which*

$\rho_x = \rho_x^{\mathcal{S}^{no}(Q)}$ *is ramified at all the places in* Q*, we have an isomorphism*

$$\left(R^{\mathcal{S}^{no}\cup Q}\right)_x^\wedge \simeq \left(R^{\mathcal{S}^{no}(\mathcal{Q})}\right)_x^\wedge . \tag{4.1}$$

Thus the dimension conjecture is true locally at such an x*.*

Proof. For the first part we shall use a result of Kisin about the generic fiber of deformation rings which relates deformation rings of char p deformations and ones in char 0. Let $R^{\mathcal{D}}$ be the universal deformation ring corresponding to a deformation problem $\mathcal{D} = (\Sigma, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in\Sigma})$. Then it is known (see [Gro66, §10.5], or [Tay08, lemma 2.6]) that maximal ideals are dense in $\mathrm{Spec}(R^{\mathcal{D}}[1/p])$. In fact $R^{\mathcal{D}}[1/p]$ is what is called a *Jacobson ring* and any closed point $x : R^{\mathcal{D}}[1/p] \to \overline{\mathbb{Q}_p}$ defines a finite extension $L/\mathbb{Q}_p$, i.e., $R^{\mathcal{D}}[1/p]/\mathfrak{m}_x = L$. Moreover, x maps $R^{\mathcal{D}}$ into the valuation ring $\mathcal{O}_L$ of L, and the map $R^{\mathcal{D}} \to \mathcal{O}_L$ is a local map. (Thus x determines a prime ideal $\mathfrak{p}_x$ of $R^{\mathcal{D}}$ such that $R^{\mathcal{D}}/\mathfrak{p}_x = \mathcal{O}_L$.)

Let

$$\rho_x : G_{F,S} \to GL_n(R^{\mathcal{D}}) \to GL_n(R^{\mathcal{D}}[1/p]) \xrightarrow{x} GL_n(L)$$

be the specialization of the universal deformation to L and

$$\widetilde{\rho}_x : G_{F,S} \to GL_n((R^{\mathcal{D}}[1/p])_x^\wedge)$$

be induced from the natural map $R^{\mathcal{D}} \to (R^{\mathcal{D}}[1/p])_x^\wedge$. Note that $(R^{\mathcal{D}}[1/p])_x^\wedge \cong (R^{\mathcal{D}})_x^\wedge$, on abusing notation by writing $(R^{\mathcal{D}})_x$ for $(R^{\mathcal{D}})_{\mathfrak{p}_x}$. Of course, we have the commutative diagram

$$\begin{tikzcd} G_{F,S} \arrow[r, "\widetilde{\rho}_x"] \arrow[dr, "\rho_x"'] & GL_n((R^{\mathcal{D}})_x^\wedge) \arrow[d] \\ & GL_n(L) \end{tikzcd}$$

Lemma 4.1. $\widetilde{\rho}_x$ *is universal among deformations of* ρ_x *of type* $\mathcal{D}$*. More precisely, let* $\mathcal{C}_L$ *be the category of complete local Artinian* L*-algebras with with residue field* L*, then* $(R^{\mathcal{D}})_x^\wedge$ *(pro) represents the functor on* $\mathcal{C}_L$ *which takes objects* $A \in \mathcal{C}_L$ *to the set of deformations of* ρ_x *of type* $\mathcal{D}$ *to* A*, which are continuous for the* p*-adic topology on* A *(on viewing* A *as finite dimension vector spaces over* L*).*

Proof. This is a consequence of a theorem of Kisin. See [Kis09, Lemma 2.3.3 and Proposition 2.3.5]. $\square$

In view of this lemma, to prove theorem 3 we have to show that any deformation ρ_A of ρ_x to an Artinian L-algebra A which satisfies the deformation condition $\mathcal{S}^{no} \cup Q$, actually satisfies the more stringent condition $\mathcal{S}^{no}(\mathcal{Q})$.

Proposition 4.1. *ρ_A restricted to a decomposition group D_v for a place v in Q is of the form:*

$$\begin{pmatrix} \psi_1 & f & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & s \end{pmatrix},$$

where ψ_i are unramified lifts of the corresponding characters for ρ_x with f when restricted to I_v a homomorphism. Also, the representation $\rho_A|_{D_v}$ factors through the tame quotient of D_v.

Proof. This is because the universal deformation $\widetilde{\rho}_x$ of ρ_x to $\left(R^{\mathcal{S}^{no}\cup Q}\right)^{\wedge}_x$, which is obtained by specializing along $R^{\mathcal{S}^{no}\cup Q} \to \left(R^{\mathcal{S}^{no}\cup Q}\right)^{\wedge}_x$ itself has such a form, by definition.

To see tame ramification of $\rho_A|_{D_v}$ note that $\rho^{\mathcal{S}^{no}\cup Q}$ maps I_v to

$$\ker\left(GL_n(R^{\mathcal{S}^{no}\cup Q}) \to GL_n(k)\right) \cong 1 + M_n\left(\mathfrak{m}_{R^{\mathcal{S}^{no}\cup Q}}\right)$$

which is a pro-p group. Hence the wild inertia in I_v, which is the maximal pro-l subgroup of I_v, maps to zero (recall for $v \in Q$, $p \nmid Nv = q_v$). As $\widetilde{\rho}_x$ is obtained by specializing $\rho^{\mathcal{S}^{no}\cup Q}$, it too is tamely ramified at v, and hence so is ρ_A. $\square$

The tame decomposition group G_v has a presentation of the form $\langle \sigma_v, \tau_v | \sigma_v \tau_v \sigma_v^{-1} = \tau_v^{Nv} \rangle$, where σ_v is a lift of Frobenius $\in D_v/I_v \simeq \mathrm{Gal}(\overline{k}_v/k_v)$ and τ_v is a generator of tame inertia. The presentation forces the relation $\rho_A(\sigma_v)\rho_A(\tau_v)\rho_A(\sigma_v)^{-1} = \rho_A(\tau_v)^{Nv}$, i.e.,

$$\begin{pmatrix}\psi_1(\sigma_v) & f(\sigma_v) & 0\\ 0 & \psi_2(\sigma_v) & 0\\ 0 & 0 & s(\sigma_v)\end{pmatrix}\begin{pmatrix}1 & f(\tau_v) & 0\\ 0 & 1 & 0\\ 0 & 0 & s(\tau_v)\end{pmatrix}\begin{pmatrix}\psi_1(\sigma_v) & f(\sigma_v) & 0\\ 0 & \psi_2(\sigma_v) & 0\\ 0 & 0 & s(\sigma_v)\end{pmatrix}^{-1}$$
$$= \begin{pmatrix}1 & f(\tau_v) & 0\\ 0 & 1 & 0\\ 0 & 0 & s(\tau_v)\end{pmatrix}^{Nv}$$

This implies $\psi_1(\sigma_v)\psi_2^{-1}(\sigma_v)f(\tau_v) = Nvf(\tau_v) = \epsilon_l(\sigma_v)f(\tau_v)$, i.e.,

$$\left(\psi_1(\sigma_v)\psi_2^{-1}(\sigma_v) - \epsilon_l(\sigma_v)\right) f(\tau_v) = 0.$$

But the residual representation ρ_x of ρ_A is ramified at each $v \in Q$, so we must have $f(\tau_v) \in A^\times$. Hence $\psi_1(\sigma_v)\psi_2^{-1}(\sigma_v) - \epsilon_l(\sigma_v) = 0$; meaning ρ_A is of type $\mathcal{S}^{no}(\mathcal{Q})$. Finally, the following lemma completes the proof of theorem 3.

□

Lemma 4.2. *Given a ramified auxiliary set Q there exists a Zariski open set of points $x \in \operatorname{Spec}\left(R^{\mathcal{S}^{no}(\mathcal{Q})}[1/p]\right)$ such that the specialization $\rho_x^{\mathcal{S}^{no}(\mathcal{Q})}$ is ramified at all places $v \in Q$. The pullback of such points is Zariski dense in* $\operatorname{Spec}\left(R^{\mathcal{S}^{no}\cup Q}[1/p]\right)$.

Proof. By definition $\rho^{\mathcal{S}^{no}(\mathcal{Q})} \to GL_n(R^{\mathcal{S}^{no}(\mathcal{Q})})$ when restricted to G_v has the form

$$\begin{pmatrix}\psi\epsilon_l & f_v & 0\\ 0 & \psi & 0\\ 0 & 0 & s\end{pmatrix},$$

where ψ and s are unramified and $f_v : I_v \to R^{\mathcal{S}^{no}(\mathcal{Q})}$ is non-zero a homomorphism which factors through the tame quotient I_v^t. Choosing the generator τ_v of I_v^t as above, and denoting $f_v(\tau_v)$ by x_v we observe that $x_v \in R^{\mathcal{S}^{no}(\mathcal{Q})}$ is neither nilpotent nor annihilated by a power of p since $R^{\mathcal{S}^{no}(\mathcal{Q})} \simeq W(k)[[X_1, \cdots, X_h]]$. Therefore the localization $\operatorname{Spec}\left(R^{\mathcal{S}^{no}(\mathcal{Q})}\left[1/p, \left(1/x_v\right)_{v\in Q}\right]\right)$ is Zariski open in $\operatorname{Spec}\left(R^{\mathcal{S}^{no}(\mathcal{Q})}[1/p]\right)$ and the specializations coming from closed points

$$x \in \operatorname{Spec}\left(R^{\mathcal{S}^{no}(\mathcal{Q})}\left[1/p, \left(1/x_v\right)_{v\in Q}\right]\right)$$

will suffice.

□

Corollary 4.1. *For any of the Zariski dense set of specializations of* $x : R^{\mathcal{S}^{no} \cup Q}[1/p] \to \overline{\mathbb{Q}}_p$ *as in theorem 3, we have that*

$$H^2(G_{F,S\cup Q}, ad^0\, \rho_x) = 0.$$

Proof. Using the standard obstruction theory [Maz89, §1.6] $H^2(G_{F,S\cup Q}, ad^0\, \rho_x)$ controls obstructions to lifting ρ_x to Artinian rings in $\mathcal{C}_{K_x}$. The deformation functor for ρ_x is represented by $\left(R^{\mathcal{S}^{no} \cup Q}\right)^{\wedge}_{x}$. But for such an x, by the isomorphism (4.1) from theorem 3, this ring is formally smooth, so there are no obstructions to lifting ρ_x and the lemma follows. □

4.2 Selmer groups

Now, we draw some consequences from the existence of ramified auxiliary sets Q for the balanced deformation problem $\mathcal{S}$.

For $v \in Q$, the restriction of $\rho^{\mathcal{S}(Q)} : G_F \to GL_n(W(k))$ to G_v is of the form

$$\begin{pmatrix} \psi\epsilon_l & f_v & 0 \\ 0 & \psi & 0 \\ 0 & 0 & s \end{pmatrix},$$

for $f_v : I_v \to W(k)$ a homomorphism. We denote the ideal of $W(k)$ generated by $f_v(I_v)$ by (p^{m_v}) and call m_v the *ramification index* of $\rho^{\mathcal{S}(Q)}$ at v.

Theorem 4. *The kernel of the morphism* $\eta : R^{\mathcal{S}\cup Q} \to R^{\mathcal{S}(Q)} (= W(k))$ *is a minimal prime ideal, and* $\ker \eta / (\ker \eta)^2$ *embeds into* $\oplus_{v \in Q} W(k)/(p^{m_v})$*, and in particular is finite.*

Proof. The proof closely follows the proof of Lemma 16 of [KR03]. Namely we shall first realize $\ker \eta/(\ker \eta)^2$ as the dual of a Selmer group which, by a computation, we shall prove is finite.

Suppose we are given a global deformation condition $\mathcal{D}_\Sigma = (\overline{\rho}, \chi, \Sigma, \{\mathcal{D}_v\}_{v\in\Sigma})$ parametrized by universal deformation ring $R^{\mathcal{D}_\Sigma}$ and homomorphism $\mu : R^{\mathcal{D}_\Sigma} \to W$ in $\mathcal{C}_W$. Let ρ_μ be the associated deformation $G_{F,\Sigma} \to GL_n(W)$ and (for each $i \geq 1$) $\rho_{\mu,i} : G_{F,\Sigma} \to GL_n(W/p^i)$ be its reduction mod p^i. Then by the formalism discussed in §2.1.2 for each $v \in \Sigma$ we get a subspace

$$\mathcal{L}_{v,i} \subseteq H^1(G_v, ad^0\rho_{\mu,i}) = H^1(G_v, ad^0\rho_\mu \otimes p^{-i}W/W),$$

which is the tangent (sub)space of the relative deformations of $\rho_{\mu,i}$ of type $\mathcal{D}_v$ inside the tangent space of all relative deformations of $\rho_{\mu,i}$. Specifically,

$$\mathcal{L}_{v,i} = \mathcal{D}_v^{\rho_{\mu,i}}(W/p^i[\epsilon]) \subseteq \mathrm{Def}^{\rho_{\mu,i}}(W/p^i[\epsilon]) = H^1(G_v, ad^0\rho_{\mu,i}).$$

(In this notation, the tangent spaces $L_v = \mathcal{D}_v(k[\epsilon])$ that appear in §2.1 in the definition of residual Selmer groups are nothing but $\mathcal{L}_{v,1}$). Taking direct limit over all i and using the fact that cohomology commutes direct limit of discrete modules we get a subspace

$$\mathcal{L}_v \subseteq H^1(G_v, ad^0\rho_\mu \otimes K/W).$$

Here K is the fraction field of W. For this collection of local conditions $\mathcal{L} = \{\mathcal{L}_v\}_{v\in\Sigma}$ we can define the Selmer group

$$H^1_{\mathcal{L}}(G_{F,\Sigma}, ad^0\rho_\mu \otimes K/W) = \ker(\Phi_{\mathcal{L}}),$$

where

$$\Phi_{\mathcal{L}} : H^1(G_{F,\Sigma}, ad^0\rho_\mu \otimes K/W) \to \bigoplus_{v\in\Sigma} \frac{H^1(G_v, ad^0\rho_\mu \otimes K/W)}{\mathcal{L}_v} \tag{4.2}$$

is the global to local map obtained by restricting to the decomposition groups G_v.

By the natural identification (2.10) provided by proposition 2.3 we have an isomorphism

$$\mathrm{Hom}_W(\ker\mu/(\ker\mu)^2, K/W) \cong H^1_{\mathcal{L}}(G_{F,\Sigma}, ad^0\rho_\mu \otimes K/W) \tag{4.3}$$

In the case of interest to us, i.e. in the context of the theorem above, $\mathcal{D}^\Sigma$ is the deformations condition $\mathcal{S} \cup Q$ and $\eta : R^{\mathcal{S}\cup Q} \to R^{\mathcal{S}(Q)} \simeq W$ induces $\rho_\eta = \rho^{\mathcal{S}(Q)}$. For

$\mathcal{S} \cup Q$ the local deformation condition at $v \in Q$ is the auxiliary condition of §3.1.2 with tangent space $\mathcal{L}_v^{\mathrm{aux}}$. We have,

$$\begin{aligned}\mathrm{Hom}_W(\ker\eta/(\ker\eta)^2, K/W) &\cong H^1_{\mathcal{S}\cup Q}(G_{F,S\cup Q}, ad^0\rho_\eta \otimes K/W) \\ &= \ker(\Phi_{\mathcal{S}\cup Q}) \\ &= \ker\Big(H^1(G_{F,S\cup Q}, ad^0\rho_\eta \otimes K/W) \to \bigoplus_{v\in S\cup Q} \frac{H^1(G_v, ad^0\rho_\eta \otimes K/W)}{\mathcal{L}_v}\Big)\end{aligned}$$

Also, since $R^{\mathcal{S}(\mathcal{Q})} \simeq W$, we have that

$$\ker(\Phi_{\mathcal{S}(\mathcal{Q})}) = H^1_{\mathcal{S}(\mathcal{Q})}(G_{F,S\cup Q}, ad^0\rho_\eta \otimes K/W) = 0. \tag{4.4}$$

But, it follows from definitions that

$$\ker(\Phi_{\mathcal{S}(\mathcal{Q})}) = \ker\Big(\ker(\Phi_{\mathcal{S}\cup Q}) \to \bigoplus_{v\in Q} \frac{\mathcal{L}_v^{\mathrm{aux}}}{\mathcal{L}_v^{\mathrm{Ram}}}\Big), \text{ i.e.,}$$

$$H^1_{\mathcal{S}(\mathcal{Q})}(G_{F,S\cup Q}, ad^0\rho_\eta \otimes K/W) = \ker\Big(H^1_{\mathcal{S}\cup Q}(G_{F,S\cup Q}, ad^0\rho_\eta \otimes K/W) \to \bigoplus_{v\in Q} \frac{\mathcal{L}_v^{\mathrm{aux}}}{\mathcal{L}_v^{\mathrm{Ram}}}\Big) \tag{4.5}$$

Thus by (4.4) $H^1_{\mathcal{S}\cup Q}(G_{F,S\cup Q}, ad^0\rho_\eta \otimes K/W)$ injects into

$$\bigoplus_{v\in Q} \frac{\mathcal{L}_v^{\mathrm{aux}}}{\mathcal{L}_v^{\mathrm{Ram}}}. \tag{4.6}$$

Hence by (4.3) the dual of $\ker\eta/(\ker\eta)^2$ injects into the same.

We now show (4.6) is a finite group. Recall that for $v \in Q$, $\mathcal{L}_v = \mathcal{L}_v^{\mathrm{Ram}}$, whose definition we now recall. Observe that by definition $\rho_\eta|_{G_v} (= \rho^{\mathcal{S}(\mathcal{Q})}|_{G_v})$ is $(1 + M_n(\mathfrak{m}_{R^{\mathcal{S}(\mathcal{Q})}}))$ conjugate to a lift of the form

$$\begin{pmatrix} \psi\chi_l & * & 0 \\ 0 & \psi & 0 \\ 0 & 0 & s \end{pmatrix},$$

where ψ and s are unramified lifts of the corresponding pieces of $\overline{\rho}|_{G_v}$. Of course, the reduction of $\rho_\eta|_{G_v}$ mod p^i has the same shape. Also, as ρ_η is ramified at v the entry corresponding to $*$ is non zero after some $i \geq m_v$. Let A_i be the underlying free $p^{-i}W/W$

module of rank n on which G_v acts giving rise to $ad^0\,\rho_\eta \otimes_W p^{-i}W/W$ (restricted to G_v). Denoting by $\mathcal{W}_i$ the elements of $\mathrm{End}^0(A_i)$ of the form

$$\begin{pmatrix} \alpha & \phi & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \sigma \end{pmatrix}.$$

Write $\mathcal{W}_i = \mathcal{R}_i \oplus \mathcal{S}_i$, where $\mathcal{R}_i$ is the two dimensional piece, which under the adjoint action of G_v decomposes as $p^{-i}W/W \oplus p^{-i}W/W(1)$. Then

$$\begin{aligned} \mathcal{L}_{v,i}^{\mathrm{Ram}} &= H^1\big(G_v/I_v, \big(p^{-i}W/W\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)^{I_v}\big) \oplus H^1(G_v, p^{-i}W/W(1)) \oplus H^1\left(G_v/I_v, (\mathrm{End}\,\mathcal{S}_i)^{I_v}\right) \\ &\subseteq H^1(G_v, \mathrm{End}^0 A_i) \end{aligned} \tag{4.7}$$

Similarly, denoting $\mathcal{V}_i$ to be the elements of $\mathrm{End}^0(A_i)$ of the form

$$\begin{pmatrix} \alpha & \phi & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \sigma \end{pmatrix},$$

and writing $\mathcal{V}_i = \mathcal{T}_i \oplus S_i$, where $\mathcal{T}_i$ as a G_v module decomposes as $(p^{-i}W/W)^{\oplus 2} \oplus p^{-i}W/W(1)$, we see that

$$\begin{aligned} \mathcal{L}_{v,i}^{\mathrm{aux}} = H^1\big(G_v/I_v, \big(p^{-i}W/W\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)^{I_v}\big) \quad & \oplus H^1\big(G_v/I_v, \big(p^{-i}W/W\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\big)^{I_v}\big) \oplus H^1(G_v, p^{-i}W/W(1)) \\ & \oplus H^1\left(G_v/I_v, (\mathrm{End}\,\mathcal{S}_i)^{I_v}\right) \\ \subseteq H^1(G_v, \mathrm{End}^0 A_i) & \end{aligned}$$

Taking the direct limit over i we see that the cokernel of the inclusion $\mathcal{L}_v^{\mathrm{Ram}} \subseteq \mathcal{L}_v^{\mathrm{aux}}$ is given by direct limit

$$\varinjlim_i H^1\big(G_v/I_v, \big(p^{-i}W/W\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\big)^{I_v}\big).$$

But this cardinality is clearly p^{m_v}. This proves the first part of the theorem 4.

Now we can prove that $\ker\eta$ is a minimal prime ideal of $R^{\mathcal{S}\cup Q}$. Denote $\ker\eta$ by $\mathfrak{p}_Q$. Since $R^{\mathcal{S}\cup Q}/\mathfrak{p}_Q \simeq W(k)$, p cannot belong to $\mathfrak{p}_Q$ therefore p is invertible in the localization

$(R^{\mathcal{S}\cup\mathcal{Q}})_{\mathfrak{p}_Q}$. This implies that maximal ideal $\mathfrak{m} = \mathfrak{p}_Q\,(R^{\mathcal{S}\cup\mathcal{Q}})_{\mathfrak{p}_Q}$ has the property that $\mathfrak{m}/\mathfrak{m}^2 = 0$, because by theorem 4 this group, being p-torsion, is annihilated by p. Since $(R^{\mathcal{S}\cup\mathcal{Q}})_{\mathfrak{p}_Q}$ is noetherian we must have $\mathfrak{m} = 0$; therefore $\mathfrak{p}_Q$ is minimal.

□

Corollary 4.2. *$\eta : R^{\mathcal{S}\cup\mathcal{Q}} \to R^{\mathcal{S}(\mathcal{Q})}$ defines an irreducible component of $R^{\mathcal{S}\cup\mathcal{Q}}$ isomorphic to $W(k)$.*

We sharpen the statement of theorem 4, determining the exact size of Selmer group.

Theorem 5. *For $\eta : R^{\mathcal{S}\cup\mathcal{Q}} \to R^{\mathcal{S}(\mathcal{Q})}$,*

$$(\widehat{\ker\eta/(\ker\eta})^2) \cong \ \oplus_{v\in Q} W(k)/(p^{m_v})$$

Proof. To prove this we shall using the results of §2.2.2. Essentially, we need to prove the map in (4.5)

$$H^1_{\mathcal{S}\cup\mathcal{Q}}(G_{F,S\cup Q}, ad^0\rho_\eta \otimes K/W) \longrightarrow \bigoplus_{v\in Q} \frac{\mathcal{L}_v^{\mathrm{aux}}}{\mathcal{L}_v^{\mathrm{Ram}}},$$

which is injective, is also surjective.

This would in turn follow from the surjectivity of

$$\Phi_{\mathcal{S}(\mathcal{Q})} : H^1(G_{F,S\cup Q}, ad^0\rho_\eta \otimes K/W) \longrightarrow \bigoplus_{v\in S\cup Q} \frac{H^1(G_v, ad^0\rho_\eta \otimes K/W)}{\mathcal{L}_v}, \tag{4.8}$$

which we now prove. The proof proceeds by showing that the dual Selmer group associated to $\mathcal{S}(\mathcal{Q})$, namely $H^1_{\mathcal{S}(\mathcal{Q})^*}(G_{F,S\cup Q}, ad^0\rho_\eta(1))$, also vanishes.

Since under the assumptions of thereom 1 $ad^0\overline{\rho}$ is irreducible, the modules $ad^0\rho_\eta \otimes_W K/W$ and $ad^0\rho_\eta(1)$ are also irreducible. Hence by taking $\mathcal{D} = ad^0\rho_\eta$ in proposition 2.5 and noting that $\mathcal{T}^* \simeq ad^0\rho_\eta(1)$ we get that

$$\mathrm{rank}_W H^1_{\mathcal{S}(\mathcal{Q})^*}(G_{F,S\cup Q}, ad^0\rho_\eta(1)) = \sum_{v\in S\cup Q} \mathrm{corank}_W \mathcal{L}_v - \mathrm{corank}_W H^0(G_v, ad^0\rho_\eta \otimes K/W) \tag{4.9}$$

Since $\mathcal{S}$ is a balanced deformation problem we have that for each $i \geq 1$

$$\sum_{v\in S} \mathrm{length}_W \mathcal{L}_{v,i} = \sum_{v\in S} \mathrm{length}_W H^0(G_v, ad^0\rho_\eta \otimes \pi^{-i}W/W). \tag{4.10}$$

This is clearly true for $i = 1$ by choice of $\mathcal{S}$. But the computations of tangent spaces for $v \in S$ in the residual case can be carried out to determine $\mathcal{L}_{v,i} \subseteq H^1(G_v, ad^0\rho_\eta \otimes \pi^{-i}W/W)$ without change. For example, if $v \in S, v \nmid l$, we insist that the lifts be minimally ramified at v, and for this deformation condition we have

$$\mathcal{L}_{v,i}^{\min} = H^1\left(G_v/I_v, (ad^0\rho_\eta \otimes \pi^{-i}W/W)^{I_v}\right),$$

and therefore, $\text{length}_W \mathcal{L}_{v,i}^{\min} = \text{length}_W H^0(G_v, ad^0\rho_\eta \otimes \pi^{-i}W/W)$ for all $i \geq 1$.

Taking direct limit of (4.10) over all i we get that

$$\sum_{v \in S} \text{corank}_W \mathcal{L}_v - \text{corank}_W H^0(G_v, ad^0\rho_\eta \otimes K/W) = 0.$$

On the other hand, for the Ramakrishna deformation condition we have from the tangent space description as given in (4.7) that $\text{corank}_W \mathcal{L}_v^{\text{Ram}} = \text{corank}_W H^0(G_v, ad^0\rho_\eta \otimes K/W)$. Then, from (4.9) we conclude that $\text{rank}_W H^1_{\mathcal{S}(\mathcal{Q})^*}(G_{F,S\cup Q}, ad^0\rho_\eta(1)) = 0$.

This means that $H^1_{\mathcal{S}(\mathcal{Q})^*}(G_{F,S\cup Q}, ad^0\rho_\eta(1))$ is at most a torsion W-module. But the irreducibility of $ad^0\overline{\rho}$ implies that it has no μ_p subquotient; proposition 2.6 applies and we conclude $H^1_{\mathcal{S}(\mathcal{Q})^*}(G_{F,S\cup Q}, ad^0\rho_\eta(1) = 0$. Finally, we can apply proposition 2.4 to conclude that (4.8) is indeed surjective, completing the proof of theorem 5. □

4.3 Concluding remarks

Fix a lift σ_v of Frobenius to the tame quotient of the decomposition group G_v, define $\alpha_v = \psi_1(\sigma_v)\psi_2^{-1}(\sigma_v) - \epsilon_l(\sigma_v) = \psi_1(\sigma_v)\psi_2^{-1}(\sigma_v) - q_v$, where ψ_1 and ψ_2 are the characters appearing in the universal deformation $\rho^{\mathcal{S}\cup Q} \to GL_n(R^{\mathcal{S}\cup Q})$ when restricted to G_v:

$$\begin{pmatrix} \psi_1 & f_v & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & s \end{pmatrix}.$$

Lemma 4.3. *1. The kernel of the map $R^{\mathcal{S}\cup Q} \to R^{\mathcal{S}(\mathcal{Q})}$ is generated by (α_v) with v running through the places in Q.*

2. *The kernel of the map $R^{\mathcal{S}^{no}\cup Q} \to R^{\mathcal{S}\cup Q}$ is generated by a regular sequence of length h. Thus if $R^{\mathcal{S}\cup Q}$ is finite as a $W(k)$-module, then the dimension conjecture is true for $R^{\mathcal{S}^{no}\cup Q}$.*

Proof. The first part of the lemma follows from presentation results of Böckle [B99, theorem 5.6 and corollary 7.1] by observing that the condition of α_v being zero is precisely what it means for the deformation to be of Ramakrishna type at v .

For the second part observe that we have the following commutating diagram of deformation rings with all maps surjections:

$$\begin{array}{ccc} R^{\mathcal{S}^{no}\cup Q} & \longrightarrow & R^{\mathcal{S}^{no}(\mathcal{Q})} \\ \downarrow & & \downarrow \\ R^{\mathcal{S}\cup Q} & \longrightarrow & R^{\mathcal{S}(\mathcal{Q})} \end{array}$$

Of course, the vertical arrow on the right corresponds to the map $W(k)[[X_1, \ldots, X_h]] \to W(k)$ obtained by sending all X_i's to zero, the kernel of which is obviously a regular sequence of length h.

□

Let us point out that it follows from aforementioned presentation results of Böckle (or from Böckle's appendix to [Kha03]) that the ring $R^{\mathcal{S}\cup Q}$ is well presented; that is, it has a presentations of the form

$$W(k)[[X_1, \ldots, X_g]]/(f_1, \ldots, f_g).$$

Hence $R^{\mathcal{S}\cup Q}$ being finite as a $W(k)$-module is equivalent to it being a finite flat complete intersection over $W(k)$ [DDT94, corollary 5.12]. Wiles's numerical criterion for complete intersection [DDT94, Theorem 5.27] might hold here, even without the prior knowledge of finite flatness of $R^{\mathcal{S}\cup Q}$ over $W(k)$. In fact, this is almost true and we are thankful to Najmuddin for showing us a proof of this generalization of Wiles's criterion under the assumption that $\mathrm{depth}(R^{\mathcal{S}\cup Q}) \geq 1$. Thus, one is lead to study the η-invariant of the composition π_Q of $R^{\mathcal{S}\cup Q} \to R^{\mathcal{S}(\mathcal{Q})}$ with the isomorphism $R^{\mathcal{S}(\mathcal{Q})} \simeq W(k)$. By lemma 4.3

and the result of Najmuddin, proving finiteness of $R^{\mathcal{S}\cup Q}$ as a $W(k)$-module amounts to showing that for the ideal

$$J_Q = \ker(\pi_Q) = (\alpha_v)_{v\in Q} \subset R^{\mathcal{S}\cup Q},$$

we have

$$\pi_Q(\mathrm{Ann}_{R^{\mathcal{S}\cup Q}}(J_Q)) \subseteq \Big(\prod_{v\in Q} p^{m_v}\Big) \qquad \big(\subseteq W(k)\big), \tag{4.11}$$

where m_v is the ramification index defined in §4.2.

In [Kha03], the containment (4.11) was obtained in the setting of GL_2 by proving the analogous containment for a Hecke algebra. One could hope that a similar technique is possible in general, via the cohomology of arithmetic groups and automorphy lifting methods, for GL_n.

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