Title
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Publication Date
1985

Peer reviewed

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Technical Report 85-25
Department of Information and Computer Science
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September 1985

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Keywords: Partition problem, probabilistic analysis, second moment method

The work of the first and second authors was supported by National Science Foundation Grant MCS 81-05217 at the University of California at Berkeley. The work of the third author was supported by National Science Foundation Grants MCS 79-04997 and DCR 85-09667 at the University of California at Irvine.

Probabilistic Analysis of Optimum Partitioning

ABSTRACT

Given a set of n items with real-valued sizes, the Optimum Partition problem asks how it can be partitioned into two subsets so that the absolute value of the difference of the sums of the sizes over the two subsets is minimized. We present bounds on the probability distribution of this minimum under the assumption that the sizes are independent random variables drawn from a common distribution. For a large class of distributions, we determine the asymptotic behavior of the median of this minimum, and show that it is exponentially small.

1. Introduction

Given n real numbers $x_1, x_2, \ldots, x_n$, the *Partition Problem* asks whether it is possible to partition the indices $1, 2, \ldots, n$ into two subsets $A \cup B = 1, 2, \ldots, n$, $A \cap B = \emptyset$, such that

$$\sum_{j \in A} x_j = \sum_{j \in B} x_j.$$

A related question, which we call the Optimum Partition Problem, is to try to determine how nearly one can achieve such a partition, i.e., to find a partition which minimizes

$$\left| \sum_{j \in A} x_j - \sum_{j \in B} x_j \right|.$$

Both of these problems are known to be NP-complete [GJ79, Ka72]. Considerable work has been done on the expected behavior of simple heuristics which approximate the optimum solution for this or closely related problems (see for example [AP80, Bo84, BD82, CFlL84, CFrL84, FR83, FR84, Lo82]). In [KK82] it is shown that a certain linear-time algorithm achieves, for a broad class of distributions, a difference of $O(n^{-\alpha \log n})$, for some $\alpha > 0$, with probability approaching 1 as $n \to \infty$. In this paper we investigate the behavior of the optimum solution; the third and fourth authors wish to thank Mike Steele [St] for suggesting this investigation. While we are unable to determine its expectation, under a broad class of distributions we determine the asymptotic behavior of its median, and show that it is exponentially small.

The proof method involves examination of the first and second moments of the number $Y = Y(x_1, x_2, \ldots, x_n)$ of partitions (A, B) which cause $|\sum_{j \in A} x_j - \sum_{j \in B} x_j|$ to lie in some neighborhood. A similar approach was used in [Lu82] to analyze the problem of selecting a subset of the x_j which have a sum close to some target value; this proved to be useful in the probabilistic analysis of approximation algorithms for the 0-1 Knapsack Problem [Lu82, GM84]. In general, it one obtains easily from Chebyshev's inequality that

$$\Pr\{Y=0\} \le \frac{\mathsf{E}(Y^2)-\mathsf{E}(Y)^2}{\mathsf{E}(Y)^2}. \tag{1.1}$$

The use of this inequality to show the existence of some combinatorial event has been called the *second moment method*; see [ES74] for more information on this method. Shepp [Sh81] has pointed out to us that a stronger inequality can easily be demonstrated. Let χ be 0 if Y is 0, and 1 otherwise. Then $Y = \chi Y$, so by Schwarz's inequality

$$\mathsf{E}(Y)^2 = \mathsf{E}(\chi Y)^2 \le \mathsf{E}(\chi^2)\,\mathsf{E}(Y^2) = \Pr\{Y \ne 0\}\,\mathsf{E}(Y^2),$$

so

$$\Pr\{Y=0\} \le \frac{\mathsf{E}(Y^2)-\mathsf{E}(Y)^2}{\mathsf{E}(Y^2)}. \tag{1.2}$$

Shepp has used this in [Sh72a, Sh72b] and attributes the idea to Billard and Kahane [see Ka68]. The inequality of [GM75, Lemma, Section 2] can be viewed as an instance of (1.2). As we will see below, this inequality can provide much sharper results than (1.1) in some cases.

An application of the second moment method has been used in [Mo70] to prove the following nonprobabilistic result related to the topic of this paper: if $x_1, x_2, \ldots, x_n$ are real values with $\sum_{j=1}^n x_j^2 = 1$, then for any $k > 1$ there are integers δ_j not all 0 and with $|\delta_j| < k$ such that

$$|\delta_1 x_1 + \cdots + \delta_n x_n| \le \sqrt{\frac{k^2-1}{k^{2n}-1}}.$$

It is worth noting that a simple proof based on the pigeon-hole principle easily gives a somewhat weaker bound. Since $\sum_{j=1}^n x_j^2 = 1$, we have $\sum_{j=1}^n |x_j| \le \sqrt{n}$. Hence all sums of the form $\sum_{j=1}^n \delta_j x_j$, with $\delta_j \in \{0, 1, 2, \ldots, (k-1)\}$, must lie in some closed interval I of length $(k-1)\sqrt{n}$. We can cover this interval with $k^n - 1$ closed subintervals of length $(k-1)\sqrt{n}/(k^n-1)$, and by the pigeon-hole principle there must then be two of these sums lying in the same subinterval. Their difference cannot exceed the length of the subinterval, so there exist δ_j chosen from $\{0, \pm 1, \pm 2, \ldots, \pm(k-1)\}$ for which

$$|\delta_1 x_1 + \cdots + \delta_n x_n| \le \frac{(k-1)\sqrt{n}}{(k^n-1)}.$$

This sort of pigeon-hole argument does not apply to the problem investigated in the current paper, since we are allowing the δ_j to be ± 1 but not 0. Intuitively, we might try reason as follows. The sum of n draws from a given distribution is likely to be within $\Theta(\sqrt{n})$ of the origin, and there are 2^n ways of partitioning them into two sets. Thus we might expect to be reasonably likely to find a partition giving a value within $\Theta(\sqrt{n}/2^n)$ of the origin. Of course, this is not a rigorous argument, but we will see later that the estimate it provides is a good one.

The discussion in sections 2 and 3 will require the mention of the *characteristic function* φ of a random variable X. If X has distribution F, this is

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} F\{dx\}.$$

The results in this paper are obtained by using inequalities (1.1) and (1.2), together with asymptotic analysis of sums of real random variables. In section 2 we consider a restricted version of the problem in which we require that the partition be *even*, i.e., that it satisfy $|A| = |B|$. In section 3 we drop this restriction. Since (when n is even) only $\Theta(n^{-1/2})$ of all partitions are even, one might anticipate that we are likely to find a partition with a difference smaller by a factor of $\Theta(\sqrt{n})$ in section 3 than in section 2; this is in fact the case. Section 4 indicates how the results of section 3 can be generalized, and suggests lines for further work.

2. The optimum solution for even partitions

In this section we look at a version of the problem in which the partitions are constrained to be such that $|A| = |B|$. For this version we use inequality (1.2) to obtain rather tight bounds on the probability of achieving a partition with a difference of at most ϵ.

A bit of notation is useful. Let $N = \{1, 2, \ldots, n\}$. For any subset A of N, let $\overline{A} = N - A$. The pair $(A, \overline{A})$ will be called a partition of N; we will consider $(A, \overline{A})$ and $(\overline{A}, A)$ to be distinct partitions. Since specifying A is sufficient to specify the partition $(A, \overline{A})$, we will often abbreviate terminology by calling A a partition. We will say A is an *even partition* of N if $|A| = |\overline{A}|$. A partition A is *admissible* if

$$\left|\sum_{j\in A} x_j - \sum_{j\in \overline{A}} x_j\right| < \epsilon.$$

The conditions we impose on the random variables x_j in this section will require, among other things, that their distribution have a bounded density. In section 3 we will require, among other things, that their characteristic function $\varphi(t)$ decrease reasonably rapidly as $t \to \infty$. These conditions certainly hold for a random variable drawn uniformly from $[0, 1]$ and for many other commonly encountered random variables. They will fail, however, for any random variable that takes on some fixed value with a positive probability. For some such distributions, in fact, Theorem 2.1, as well as Theorem 3.1 of the next section, would not hold. If x_j assumes the values 0 or 1 with probability 1/2 each, for example, then the probability that there will be an odd number of x_j which equal 1 is 1/2 for $n > 1$, and so with probability 1/2 there is no admissible partition for any $\epsilon < 1$. We discuss a way to obtain a partial solution to the problem posed by such phenomena in section 4.

THEOREM 2.1. *Suppose a random variable X has a bounded density f, and variance σ^2; also suppose that its third and fourth moments exist. Let $x_1, x_2, \ldots, x_n$, with n even, be i.i.d. random variables with density f. Let Y_n be the number of distinct partitions $(A, \overline{A})$, with $|A| = |\overline{A}|$, which satisfy*

$$\Big|\sum_{j\in A} x_j - \sum_{j\in \overline{A}} x_j\Big| < \epsilon.$$

Then

$$\Pr\{Y_n \neq 0\} \leq z\big(1 + O(n^{-1})\big) \tag{2.1a}$$

and

$$\Pr\{Y_n \neq 0\} \geq \frac{z}{1+z}\big(1 + O(n^{-1})\big), \tag{2.1b}$$

where

$$z = \frac{2^n \epsilon}{\sigma\pi n}. \tag{2.2}$$

In particular, the median absolute value of the best partition difference is $\Theta(n/2^n)$.

Proof. In the proof we assume that $\sigma = 1$, to minimize notation; the general case follows by a simple scaling. Also, it is convenient to let $k = n/2$, since this value arises frequently during the proof; note that then k is an integer. We may henceforth assume that

$$\epsilon \leq n^2 2^{-n}/4 = k^2 4^{-k}. \tag{2.3}$$

since then the Theorem will follow for larger ϵ by monotonicity. Let the function $Z(A)$ be 1 if A is admissible, and 0 otherwise. Then the number of even admissible partitions is

$$Y_n = \sum_{\substack{A\subseteq N\\ |A|=k}} Z(A).$$

We begin by computing $\mathsf{E}(Y_n)$. Note that

$$\mathsf{E}(Y_n) = \mathsf{E}\Big(\sum_{\substack{A\subseteq N\\ |A|=k}} Z(A)\Big) = \sum_{\substack{A\subseteq N\\ |A|=k}} \mathsf{E}\big(Z(A)\big). \tag{2.4}$$

Now $Z(A)$ has the same expectation for any even partition A. To analyze the behavior of $Z(A)$, a bit more notation will be useful. Let X_0 denote the symmetrization of X, i.e., a random variable distributed as the difference between two independent variables with density $f(x)$; let $f_0(x)$ be the density function of X_0. Next, let f_0^{*m} denote the density function for the sum of m independent variables with density f_0. (Note that the random variable corresponding to f_0^{*m} has mean 0 and variance $2m$.) Then the probability that a fixed A is admissible is $\int_{-\epsilon}^{\epsilon} f_0^{*k}(x)\,dx$. Thus (2.4) becomes

$$\mathsf{E}(Y_n) = \binom{2k}{k} \int_{-\epsilon}^{\epsilon} f_0^{*k}(x)\,dx = \binom{2k}{k} \frac{\epsilon}{\sqrt{\pi k}} (1 + O(k^{-1})), \tag{2.5}$$

where the second equation follows from Lemma 2.1 below. By simple asymptotic analysis,

$$\binom{2k}{k} \frac{\epsilon}{\sqrt{\pi k}} (1 + O(k^{-1})) = \frac{4^k \epsilon}{\pi k} (1 + O(k^{-1})) = 2z(1 + O(k^{-1})), \tag{2.6}$$

where we have used (2.2) and our assumption that $\sigma = 1$. Now since we are counting $(A, \overline{A})$ and $(\overline{A}, A)$ as distinct partitions, Y_n is guaranteed to be even, so using (2.5) and (2.6) we obtain

$$\Pr\{Y_n \neq 0\} \leq \tfrac{1}{2}\,\mathsf{E}(Y_n) = z(1 + O(k^{-1})),$$

giving (2.1a).

To obtain (2.1b), we use Lemma 2.2, proved below, which states that

$$\mathsf{E}(Y_n^2) = \binom{2k}{k} \left(\frac{2\epsilon}{\sqrt{\pi k}} + \binom{2k}{k} \frac{\epsilon^2}{\pi k} \right) (1 + O(k^{-1})).$$

Hence, by an invocation of (1.2),

$$\begin{aligned}
\Pr\{Y_n \neq 0\} &\geq \frac{\mathsf{E}(Y_n)^2}{\mathsf{E}(Y_n^2)} \\
&= \frac{\binom{2k}{k}^2 \frac{\epsilon^2}{\pi k}}{\binom{2k}{k} \left(\frac{2\epsilon}{\sqrt{\pi k}} + \binom{2k}{k} \frac{\epsilon^2}{\pi k} \right)} (1 + O(k^{-1})) \\
&= \frac{\binom{2k}{k} \frac{\epsilon}{\sqrt{\pi k}}}{2 + \binom{2k}{k} \frac{\epsilon}{\sqrt{\pi k}}} (1 + O(k^{-1})) \\
&= \frac{z}{1+z} (1 + O(k^{-1})),
\end{aligned}$$

where we have used (2.6) in the last step. ■

Note that if z is small and n is large, (2.1a) and (2.1b) provide a rather tight bound on the behavior of $\Pr\{Y_n \neq 0\}$. Had we based our bound on (1.1) instead of (1.2), we would have obtained the bound

$$\Pr\{Y_n \neq 0\} \geq 1 - \frac{1}{z} + O(n^{-1})$$

instead of (2.1b); this would not be useful for $z \le 1$.

LEMMA 2.1. *If* $x \in [-\epsilon, \epsilon]$, *then*

$$f_0^{*m}(x) = \frac{1 + O(m^{-1})}{\sqrt{4\pi m}}.$$

Proof. Note that X_0 has mean 0 and variance 2. Since f_0 is symmetric, the third moment μ_3 of X_0 is 0. Moreover, its fourth moment exists since the fourth moment of X exists. Finally, if we let φ be the characteristic function of X, then by [Fe71, Section XV.1, Corollary to Lemma 2], the characteristic function of X_0 is $|\varphi|^2$, and then by [Fe71, Section XV.3, Corollary to Theorem 3], we can deduce that $|\varphi|^2$ is integrable over $(-\infty, \infty)$. Thus by [Fe71, Chapter XVI.2, Theorem 2], with $r = 4$, we may write

$$f_0^{*m}(x) = \Big(1 + m^{-1} P_4\big(\frac{x}{\sqrt{2m}}\big)\Big)\mathfrak{n}^{*2m}(x) + o(m^{-3/2}),$$

where P_4 is a polynomial of degree 4 whose coefficients depend only on f_0, and $\mathfrak{n}^{*2m}$ is the normal density function with mean 0 and variance $2m$; note that we have scaled x and the densities differently from [Fe71] by a factor of $\sqrt{2m}$. Now by (2.3) and the hypothesis of this Lemma, $|x| \le \epsilon \le k^2 4^{-k}$, so

$$\mathfrak{n}^{*2m}(x) = \frac{1 + o(m^{-1})}{\sqrt{4\pi m}}$$

and

$$P_4\Big(\frac{x}{2m}\Big) \sim P_4(0) = O(1),$$

from which the Lemma follows. ∎

LEMMA 2.2. $\mathsf{E}(Y_n^2) = \binom{2k}{k}\left(\frac{2\epsilon}{\sqrt{\pi k}} + \binom{2k}{k}\frac{\epsilon^2}{\pi k}\right)(1 + O(k^{-1}))$.

Proof. Note that

$$\mathsf{E}(Y_n^2) = \mathsf{E}\Big(\sum_{\substack{A \subseteq N \\ |A| = k}} Z(A) \sum_{\substack{B \subseteq N \\ |B| = k}} Z(B)\Big) = \sum_{\substack{A \subseteq N \\ |A| = k}} \sum_{\substack{B \subseteq N \\ |B| = k}} \mathsf{E}\Big(Z(A)Z(B)\Big). \tag{2.7}$$

It is useful to reorganize the sum according to the number of elements which A and B share, since once we know this number we can specify the value of $\mathsf{E}(Z(A)Z(B))$; let I_{km} be the value of $\mathsf{E}(Z(A)Z(B))$ in the case where $|A \cap B| = m$. (Since we are assuming in this section that the partitions are even, $0 \le m \le k$.) Then (2.7) becomes

$$\mathsf{E}(Y_n^2) = \binom{2k}{k} \sum_{m=0}^{k} \binom{k}{m}\binom{k}{k-m} I_{km} = \binom{2k}{k} \sum_{m=0}^{k} \binom{k}{m}^2 I_{km}. \tag{2.8}$$

It is useful at this point to investigate I_{km}; assume m is neither 0 nor n. Suppose we fix two subsets A and B of N. Let

$$V_1 = \sum_{j \in A \cap B} x_j - \sum_{j \in \overline{A} \cap \overline{B}} x_j,$$

$$V_2 = \sum_{j \in A \cap \overline{B}} x_j - \sum_{j \in \overline{A} \cap B} x_j.$$

Then the set A is admissible if and only if $V_1 + V_2 \in [-\epsilon, \epsilon]$, and B is admissible if and only if $V_1 - V_2 \in [-\epsilon, \epsilon]$. Note that

$$|A \cap B| = |\overline{A} \cap \overline{B}| = m, \quad \text{and} \quad |A \cap \overline{B}| = |\overline{A} \cap B| = k - m,$$

and that V_1 and V_2 are independent, so the probability that both A and B are admissible is

$$I_{km} = \Pr\{V_1 + V_2 \in [-\epsilon, \epsilon] \text{ and } V_1 - V_2 \in [-\epsilon, \epsilon]\} = \iint_{A_\epsilon} f_0^{*m}(x)\, f_0^{*(k-m)}(y)\, dx\, dy, \quad (2.9)$$

where A_ϵ is the region formed by the x and y such that $|x \pm y| \in [-\epsilon, \epsilon]$. Note that this region has area $2\epsilon^2$ and lies within a radius ϵ of the origin. From Lemma 2.1 it follows that

$$\begin{aligned} I_{km} &= \frac{2\epsilon^2}{\sqrt{4\pi m}\sqrt{4\pi(k-m)}} \left(1 + O\left(\frac{1}{m} + \frac{1}{k-m}\right)\right) \\ &= \frac{\epsilon^2}{2\pi\sqrt{m(k-m)}} \left(1 + O\left(\frac{1}{m} + \frac{1}{k-m}\right)\right). \end{aligned} \quad (2.10)$$

We now estimate the value of the sum on the right of (2.8), by breaking the range of the summation into three parts, as follows.

a) $k/4 \le m \le 3k/4$. Then by (2.10) and asymptotic analysis, the contribution from this region of the summation is

$$\sum_{m=\lceil k/4 \rceil}^{\lfloor 3k/4 \rfloor} \binom{k}{m}^2 \frac{\epsilon^2}{2\pi\sqrt{m(k-m)}} (1 + O(k^{-1})) = \binom{2k}{k} \frac{\epsilon^2}{\pi k} (1 + O(k^{-1})).$$

b) m is outside $\{0\} \cup \{k\} \cup [k/4, 3k/4]$. Then by a bit of asymptotic analysis,

$$\binom{k}{m}^2 \le \binom{k}{\lceil k/4 \rceil}^2 = \binom{2k}{k} O(e^{-\alpha k}),$$

for some $\alpha > 0$. Also, by (2.9) and the fact that f (and hence f_0) is bounded, for these m we have

$$I_{km} \leq C\epsilon^2$$

for some constant C. Thus each summand on the right of (2.8) has a value which is $O\left(\binom{2k}{k}\epsilon^2 e^{-\alpha k}\right)$, so summing over this range gives a value of $O\left(k\binom{2k}{k}\epsilon^2 e^{-\alpha k}\right)$; hence the sum over this range is swallowed by that over the range considered in part (a).

c) $m = 0$ or $m = k$. We consider first the case $m = 0$. Then A and B are identical, so using Lemma 2.1 we conclude that

$$I_{0k} = \int_{-\epsilon}^{\epsilon} f_0^{*k}(x)\,dx = \frac{\epsilon}{\sqrt{\pi k}}(1 + O(k^{-1})).$$

A similar argument holds for I_{kk} so the total contribution to the sum is

$$\frac{2\epsilon}{\sqrt{\pi k}}(1 + O(k^{-1})).$$

We may finally state the behavior of the sum on the right of (2.8) by combining the results of (a), (b), and (c) above:

$$\sum_{m=0}^{k} \binom{k}{m}^2 I_{km} = \left(\binom{2k}{k}\frac{\epsilon^2}{\pi k} + \frac{2\epsilon}{\sqrt{\pi k}}\right)(1 + O(k^{-1})).$$

Hence, by (2.8),

$$\mathsf{E}(Y_n^2) = \binom{2k}{k}\left(\frac{2\epsilon}{\sqrt{\pi k}} + \binom{2k}{k}\frac{\epsilon^2}{\pi k}\right)(1 + O(k^{-1})). \qquad \blacksquare$$

3. The optimum solution for unconstrained partitions

In this section we prove a theorem dealing with the case in which the partitions are unconstrained, i.e., we drop the constraint that $|A| = |B|$. As one might expect, this decreases the optimum by a factor of $\Theta(\sqrt{n})$. The theorem presented here is also more general in that it deals with the case in which we are trying to achieve a specified difference between the sums over A and B, rather than trying to make these sums equal. The proof will illustrate a different method for estimating the expectations, which uses characteristic functions directly rather than the central limit theorem.

THEOREM 3.1. *Let $x_1, x_2, \ldots, x_n$ be i.i.d. random variables. Suppose that the distribution function of the x_j is such that*

$$\mathsf{E}(x_j^4) < \infty \tag{3.1}$$

and that if

$$\phi(t) = \mathsf{E}(\cos t x_j)$$

then for some $\gamma > 0$

$$\phi(t) \le \frac{1}{1+|t|^\gamma}. \tag{3.2}$$

Also suppose that $\beta = o(\sqrt{n}/\log n)$. *Then*

a) the probability that there is a partition $(A, \overline{A})$ *of* $\{1, 2, \ldots, n\}$ *such that*

$$\Big|\sum_{j\in A} x_j - \sum_{j\in \overline{A}} x_j - \beta\Big| \le \epsilon \tag{3.3}$$

is $O\big(2^n n^{-1/2}\epsilon\big)$, *and*

b) the probability that there is no such partition is $O\Big(\frac{\sqrt{n}}{2^n\epsilon} + \frac{1}{n} + \frac{\beta^4}{n^2}\Big)$,

as $n \to \infty$. *In particular, the median value of the minimum achievable on the left of (3.3) is* $\Theta(\sqrt{n}/2^n)$.

Before giving the proof we pause to give some motivation. We are interested in sums of the form

$$\sum_{j=1}^{n} \delta_j x_j,$$

where the δ_j are chosen to be ± 1. We seek to show that at least one such sum lies in $[\beta - \epsilon, \beta + \epsilon]$. If Σ were a sum under consideration, and we were to let

$$\omega_0(y) = \begin{cases} 1, & \text{if } |y| \le 1, \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

then we would have

$$\omega_0\big(\epsilon(\Sigma - \beta)\big) = 1$$

if and only if Σ were within ϵ of β. The proof incorporates some of the ideas from the proof of the central limit theorem appearing in [Fe71, XV.5, Theorem 2]; we use characteristic functions to facilitate the analysis of the distribution of sums of random variables. In this application, however, we are concerned with the number of sums lying in a given range, rather than the distribution of the sum itself. The sharp edges on the function ω_0, which was used to select a given range, would cause problems for the transform methods used in the proof, so instead we define a function which is similar but lies in C^∞.

Proof of Theorem 3.1. We begin by recording for later use some observations about ϕ. First, by (3.1),

$$\phi(t) = \exp\big(-Ct^2 + O(t^4)\big), \quad \text{as } t \to 0, \quad \text{where } C = \tfrac{1}{2}\,\mathsf{E}(x_j^2). \tag{3.4}$$

Also, using (3.1) and (3.2), we note that we can choose t_0 and c so that

$$t_0^{-\gamma} \le 1/5, \tag{3.5a}$$

and

$$|\phi(t)| \le \begin{cases} 1 - ct^2, & \text{if } t \le 2t_0; \\ |t|^{-\gamma}, & \text{for all } t, \end{cases} \tag{3.5b}$$

Now choose a function $\omega(y) \in C^\infty(-\infty, \infty)$ such that $0 \le \omega(y) \le 2\pi$, $\omega(y) = \omega(-y)$, $\omega(y) = 0$ for $|y| \ge 1$, $\omega(y) \ge 1$ for $|y| \le 1/2$, and $\int_{-\infty}^{\infty} \omega(y)dy = 2\pi$. Then we can write

$$\omega(y) = \int_{-\infty}^{\infty} w(t)e^{ity}\,dt, \tag{3.6}$$

where w is a real-valued function with $w(0) = 1$, $w(t) = w(-t)$, $|w(t)| \le 1$, and

$$\int_{-\infty}^{\infty} t^k |w(t)|\,dt \quad \text{converges for any } k \ge 0. \tag{3.7}$$

Define

$$W = \sum_{\delta_j = \pm 1} \omega\Big(\epsilon^{-1} \sum_{j=1}^{n} \delta_j x_j - \epsilon^{-1}\beta\Big),$$

where the outer sum is over all 2^n choices of $\delta_j = \pm 1$, for $1 \le j \le n$. Then since ω vanishes outside $(-1, 1)$ we observe that W is nonzero only if some partition satisfies (3.3). By (3.6),

$$\begin{aligned} W &= \sum_{\delta_j=\pm 1} \int_{-\infty}^{\infty} w(t) \exp\Big(it\epsilon^{-1} \sum_{j=1}^{n} \delta_j x_j - it\epsilon^{-1}\beta\Big)\,dt \\ &= \int_{-\infty}^{\infty} w(t)\,dt \sum_{\delta_j=\pm 1} \exp\Big(it\epsilon^{-1} \sum_{j=1}^{n} \delta_j x_j - it\epsilon^{-1}\beta\Big) \\ &= 2^n \int_{-\infty}^{\infty} w(t)\,dt \exp(-it\epsilon^{-1}\beta) \prod_{j=1}^{n} \frac{e^{i\epsilon^{-1}tx_j} + e^{-i\epsilon^{-1}tx_j}}{2} \\ &= 2^n \int_{-\infty}^{\infty} w(t)\,dt \exp(-it\epsilon^{-1}\beta) \prod_{j=1}^{n} \cos(\epsilon^{-1}tx_j) \\ &= 2^n \int_{-\infty}^{\infty} w(t)\,dt \cos(\epsilon^{-1}t\beta) \prod_{j=1}^{n} \cos(\epsilon^{-1}tx_j) \end{aligned}$$

(since the imaginary part vanishes by symmetry)

$$= 2^n \epsilon \int_{-\infty}^{\infty} w(u\epsilon) du \cos(u\beta) \prod_{j=1}^{n} \cos(ux_j). \tag{3.8}$$

If we now use $\mathsf{E}(W)$ to denote the expected value of W over the choices of the x_j, then, by the independence of the x_j, we find

$$\mathsf{E}(W) = 2^n \epsilon \int_{-\infty}^{\infty} w(u\epsilon) \cos(u\beta)\, \mathsf{E}\Big(\prod_{j=1}^{n} \cos(ux_j)\Big)\, du$$

$$= 2^n \epsilon \int_{-\infty}^{\infty} w(u\epsilon) \phi(u)^n \cos(u\beta)\, du \tag{3.9}$$

We may henceforth assume that

$$\epsilon \leq n^{3/2} 2^{-n}, \tag{3.10}$$

since otherwise part (a) of the Theorem holds trivially, and since part (b) will follow for larger ϵ by monotonicity and the fact that $n^{-1} > n^{1/2} 2^{-n} \epsilon^{-1}$ for $\epsilon > n^{3/2} 2^{-n}$.

Also, $w(v) = 1 + o(|v|)$ as $v \to 0$, so if

$$U = n^{1/2} / \log n, \tag{3.11}$$

say, then using (3.4) and (3.10)

$$\int_{|u| \leq U^{-1}} w(u\epsilon) \phi(u)^n \cos(u\beta)\, du$$

$$= \int_{|u| \leq U^{-1}} \big(1 + o(n^2 2^{-n})\big) \exp\big(-nCu^2 + O(nu^4)\big) \big(1 - o(1)\big)\, du$$

$$\sim \sqrt{\frac{\pi}{nC}} \quad \text{as } n \to \infty \tag{3.12}$$

since $\beta = o(U)$ by the hypotheses of this Theorem. On the other hand, by (3.5) and the fact that $|w(u)| \leq 1$, we have for large n

$$\Big| \int_{|u| > U^{-1}} w(u\epsilon) \phi(u)^n \cos(u\beta)\, du \Big| \leq \int_{|u| > U^{-1}} |\phi(u)|^n\, du$$

$$\leq 2 \int_{\log n / \sqrt{n}}^{t_0} e^{-cnu^2}\, du + 2 \int_{t_0}^{\infty} u^{-\gamma n}\, du$$

$$= O\big(e^{-c(\log n)^2}\big) + O\big(t_0^{-\gamma n + 1}\big)$$

$$= O\big(e^{-c(\log n)^2}\big). \tag{3.13}$$

Combining (3.9), (3.12), and (3.13), we obtain

$$\mathsf{E}(W) \sim 2^n \epsilon \sqrt{\frac{\pi}{nC}} \quad \text{as } n \to \infty. \tag{3.14}$$

Now by the definition of W and the fact that $\omega(y) \geq 1$ for $|y| \leq 1/2$, it follows that $\mathsf{E}(W)$ is an upper bound on the probability that there is a partition with difference within $\epsilon/2$ of β. Thus part (a) of the Theorem follows from (3.14).

We now concentrate on part (b) of the Theorem, which is the more difficult part. We may henceforth assume that $n^{1/2}2^{-n} \leq \epsilon$, since otherwise the bound of part (b) becomes trivial. Note that this, combined with (3.10), means that the term $2^n\epsilon$ which will appear frequently is in the range $[n^{1/2}, n^{3/2}]$.

To prove a lower bound on the probability that good partitions exist, we will make use of the second moment of W. We will show that for appropriate choices of the parameters, $\mathsf{E}(W^2)$ is relatively close to $\mathsf{E}(W)^2$, which will yield the desired result. Now by (3.8),

$$W^2 = 2^{2n}\epsilon^2 \int_{-\infty}^{\infty} w(u\epsilon)\cos(u\beta)\, du \int_{-\infty}^{\infty} w(t\epsilon)\cos(t\beta)\, dt \prod_{j=1}^{n} \{\cos(ux_j)\cos(tx_j)\}. \tag{3.15}$$

Since

$$\cos(ux_j)\cos(tx_j) = \tfrac{1}{2}\cos(u+t)x_j + \tfrac{1}{2}\cos(u-t)x_j,$$

and the x_j are independent,

$$\begin{aligned} \mathsf{E}\Big(\prod_{j=1}^{n}\{\cos(ux_j)\cos(tx_j)\}\Big) &= \prod_{j=1}^{n} \mathsf{E}\Big(\tfrac{1}{2}\cos(u+t)x_j + \tfrac{1}{2}\cos(u-t)x_j\Big) \\ &= \Big(\tfrac{1}{2}\phi(u+t) + \tfrac{1}{2}\phi(u-t)\Big)^n, \end{aligned}$$

so (3.15) gives us

$$\begin{aligned} \mathsf{E}(W^2) = 2^{2n}\epsilon^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & w(u\epsilon)w(t\epsilon) \times \\ & \Big(\tfrac{1}{2}\cos(u+t)\beta + \tfrac{1}{2}\cos(u-t)\beta\Big)\Big(\tfrac{1}{2}\phi(u+t) + \tfrac{1}{2}\phi(u-t)\Big)^n du\, dt. \end{aligned}$$

Comparing this with (3.9) gives

$$\begin{aligned}
&\mathsf{E}(W^2) - \mathsf{E}(W)^2 \\
&\quad = 2^{2n}\epsilon^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(u\epsilon)w(t\epsilon) \times \\
&\qquad \Big\{\Big(\tfrac{1}{2}\cos(u+t)\beta + \tfrac{1}{2}\cos(u-t)\beta\Big)\Big(\tfrac{1}{2}\phi(u+t) + \tfrac{1}{2}\phi(u-t)\Big)^n - \\
&\qquad\qquad \cos(u\beta)\cos(t\beta)\phi(u)^n\phi(t)^n\Big\}\, du\, dt. \qquad (3.16)
\end{aligned}$$

Again we pause to provide some motivation before proceeding. The function $\phi(u)^n$ is the characteristic function of a variable V_1 constructed by choosing δ_j, for $j = 1, \ldots, n$, independently to be ± 1 (with equal probability) and setting $V_1 = \sum_{j=1}^n \delta_j x_j$. The function $\phi(u)^n\phi(t)^n$ is the two-dimensional characteristic function for two variables constructed independently in this fashion. The function $\big(\frac{1}{2}\phi(u+t) + \frac{1}{2}\phi(u-t)\big)^n$ is the two-dimensional characteristic function for a pair (V_1, V_2) of random variables constructed as follows:

a) Draw x_j, for $j = 1, \ldots, n$.

b) Set δ_j, $j = 1, \ldots, n$, independently to be ± 1 with equal probability.

c) Set δ'_j, $j = 1, \ldots, n$, independently to be ± 1 with equal probability.

d) Set $V_1 = \sum_{j=1}^n \delta_j x_j$, and $V_2 = \sum_{j=1}^n \delta'_j x_j$.

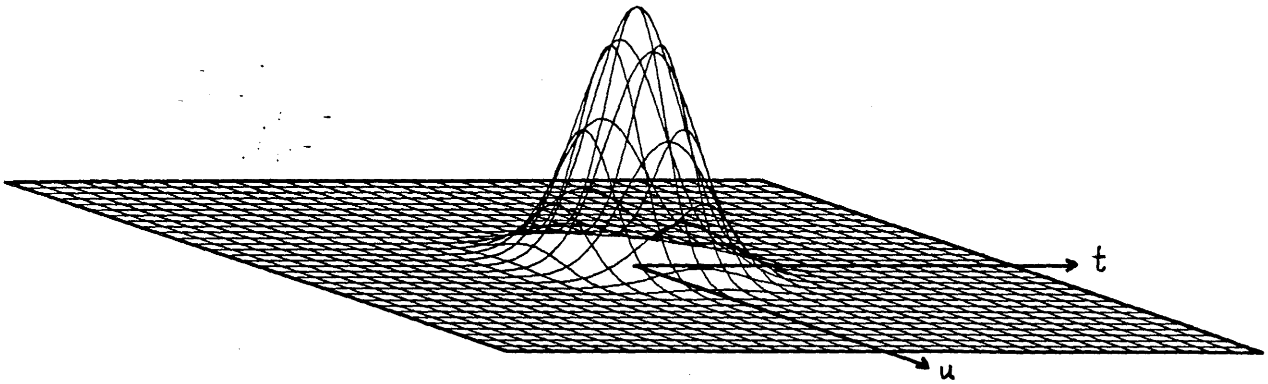
For large n, the function $\phi(u)^n\phi(t)^n$ approaches 0 rapidly as we move away from the origin in any direction, because of (3.2). The function $\big(\frac{1}{2}\phi(u+t) + \frac{1}{2}\phi(u-t)\big)^n$, on the other hand, is bounded below by 2^{-n} where $u = \pm t$; this can be seen clearly as the ridges in Figure 1b. Intuitively, this has a simple interpretation in terms of characteristic functions. There is a probability of 2^{-n} that $\delta_j = \delta'_j$ for $j = 1, \ldots, n$; if we condition (V_1, V_2) on this event, then $V_1 = V_2$, so the corresponding characteristic function is 1 along the line $u = -t$. Similarly, there is also a probability of 2^{-n} that $\delta_j = -\delta'_j$ for $j = 1, \ldots, n$; if we condition (V_1, V_2) on this event, then $V_1 = -V_2$, so the corresponding characteristic function is 1 along the line $u = t$.

Continuing the proof, first consider $|u|, |t| \le U^{-1}$, with U given by (3.11). In that region, by (3.4),

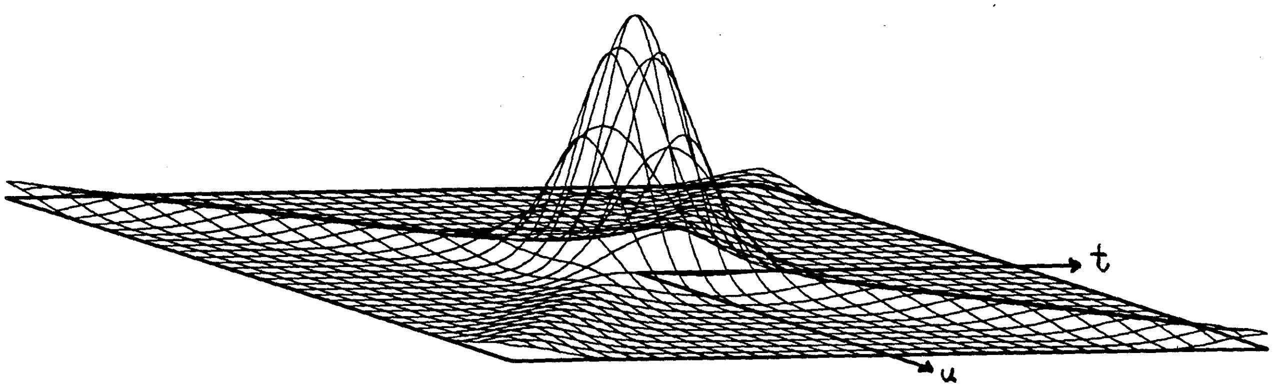
$$\begin{aligned}
&\tfrac{1}{2}\phi(u+t) + \tfrac{1}{2}\phi(u-t) \\
&\quad = \tfrac{1}{2}\exp\big(-C(u+t)^2 + O(u^4+t^4)\big) + \tfrac{1}{2}\exp\big(-C(u-t)^2 + O(u^4+t^4)\big) \\
&\quad = \tfrac{1}{2}\exp\big(-Cu^2 - Ct^2\big)\big(e^{-2Ctu} + e^{2Ctu}\big)\exp\big(O(u^4+t^4)\big) \\
&\quad = \exp\big(-Cu^2 - Ct^2 + O(u^4+t^4)\big),
\end{aligned}$$

and

$$\phi(u)\phi(t) = \exp\big(-Cu^2 - Ct^2 + O(u^4+t^4)\big),$$



1a. The function $\phi(u)^n\phi(t)^n$.



1b. The function $\left(\frac{1}{2}\phi(u+t)+\frac{1}{2}\phi(u-t)\right)^n$.

Figure 1. Plots of the terms from the integrands involving the function ϕ. Here the x_j are assumed to have a normal distribution with zero mean, β is assumed to be 0, and n is 4.

and similarly, for $\beta = o(U)$,

$$\tfrac{1}{2}\cos(u+t)\beta + \tfrac{1}{2}\cos(u-t)\beta = \exp\left(-\tfrac{1}{2}\beta^2(u^2+t^2) + O(\beta^4u^4+\beta^4t^4)\right),$$

and

$$\cos u\beta \cos t\beta = \exp\left(-\tfrac{1}{2}\beta^2(u^2+t^2) + O(\beta^4u^4+\beta^4t^4)\right).$$

Therefore

$$\Big\{\left(\tfrac{1}{2}\cos(u+t)\beta + \tfrac{1}{2}\cos(u-t)\beta\right)\left(\tfrac{1}{2}\phi(u+t)+\tfrac{1}{2}\phi(u-t)\right)^n - \cos(u\beta)\cos(t\beta)\phi(u)^n\phi(t)^n\Big\}$$

$$= \exp\left(-nCu^2 - nCt^2 - \tfrac{1}{2}\beta^2(u^2+t^2)\right) \times$$
$$\left\{\exp\left(O(n+\beta^4)(u^4+t^4)\right) - \exp\left(O(n+\beta^4)(u^4+t^4)\right)\right\}$$
$$= \exp\left(-nCu^2 - nCt^2 - \tfrac{1}{2}\beta^2(u^2+t^2)\right) O\left((n+\beta^4)(u^4+t^4)\right).$$

Hence

$$\int\int_{\substack{|u|\le U^{-1}\\ |t|\le U^{-1}}} w(u\epsilon)w(t\epsilon)\Big\{\left(\tfrac{1}{2}\cos(u+t)\beta + \tfrac{1}{2}\cos(u-t)\beta\right)\left(\tfrac{1}{2}\phi(u+t)+\tfrac{1}{2}\phi(u-t)\right)^n - \cos(u\beta)\cos(t\beta)\phi(u)^n\phi(t)^n\Big\}\,du\,dt$$
$$= O\Big((n+\beta^4)\int\int_{\substack{|u|\le U^{-1}\\ |t|\le U^{-1}}} (u^4+t^4)\exp\left(-(nC+\tfrac{1}{2}\beta^2)(u^2+t^2)\right)du\,dt\Big)$$
$$= O\left(\frac{n+\beta^4}{(n+\beta^2)^3}\right) = O(n^{-2}+\beta^4 n^{-3}). \tag{3.17}$$

As in the proof of (3.14),

$$\Bigg|\int\int_{\substack{|u|\ge U^{-1} \text{ or}\\ |t|\ge U^{-1}}} w(u\epsilon)w(t\epsilon)\phi(u)^n\phi(t)^n\,du\,dt\Bigg| = O(e^{-c(\log n)^2}). \tag{3.18}$$

Finally, by an application of symmetry,

$$\int\int_{\substack{|u|\ge U^{-1} \text{ or}\\ |t|\ge U^{-1}}} \left|w(u\epsilon)w(t\epsilon)\right| \left|\tfrac{1}{2}\phi(u+t)+\tfrac{1}{2}\phi(u-t)\right|^n du\,dt$$
$$\le 8\int_{U^{-1}}^{\infty} |w(u\epsilon)|\,du \int_0^u \left\{\tfrac{1}{2}|\phi(u+t)| + \tfrac{1}{2}|\phi(u-t)|\right\}^n dt.$$

It is convenient to break up the range of integration into several regions, as shown in Figure 2, so that we can select the appropriate case from the bounds in (3.5). If X is any of the regions A, B, C_1, C_2, or $C = C_1 \cup C_2$, let

$$I_X = \iint_X |w(u\epsilon)|\left\{\tfrac{1}{2}|\phi(u+t)| + \tfrac{1}{2}|\phi(u-t)|\right\}^n du\,dt.$$

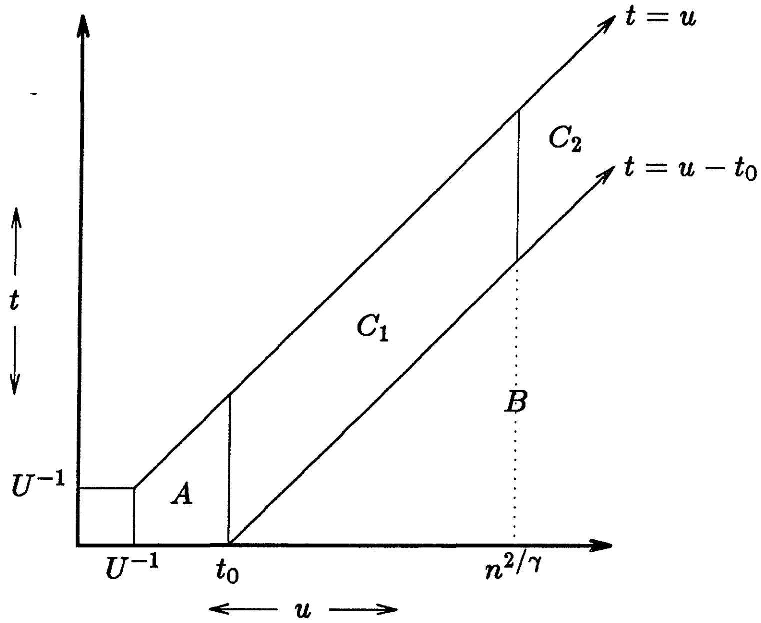


Figure 2. Regions of integration.

Throughout region A, $u+t$, as well as u, lies in the interval $(U^{-1}, 2t_0)$, so from (3.5) and (3.11) we have

$$\begin{aligned}\tfrac{1}{2}|\phi(u+t)| + \tfrac{1}{2}|\phi(u-t)| &\le \tfrac{1}{2}(1-cu^2) + \tfrac{1}{2} \\ &\le 1 - cu^2/2 \\ &\le e^{-cu^2/2} \\ &\le e^{-c(\log n)^2/(2n)}.\end{aligned}$$

Thus since the area of this region is $O(1)$, and $|w|$ is at most 1, we have

$$I_A \le O(e^{-c(\log n)^2/2}).$$

For region B, $u+t$ and $u-t$ are at least t_0, so (3.5) yields

$$\tfrac{1}{2}|\phi(u+t)| + \tfrac{1}{2}|\phi(u-t)| \le \frac{1}{5},$$

and therefore by (3.7) we obtain

$$I_B \le \int_{t_0}^{\infty} |w(u\epsilon)|\, du \int_0^{u-t_0} \frac{1}{5^n}\, dt \le \int_0^{\infty} |w(u\epsilon)| \frac{u}{5^n}\, du = O(5^{-n}\epsilon^{-2}).$$

The integral over region $C = C_1 \cup C_2$ is the most delicate, and corresponds to the ridges noted in Figure 1b. Let $s = u - t$; then, in C, s lies in the range $(0, t_0)$, so using

(3.5) we may write

$$\begin{aligned}\tfrac{1}{2}|\phi(u+t)|+\tfrac{1}{2}|\phi(u-t)| &= \tfrac{1}{2}|\phi(s)|+\tfrac{1}{2}|\phi(2u-s)| \\ &\le \tfrac{1}{2}(e^{-cs^2}+u^{-\gamma}) \\ &\le \tfrac{1}{2}\{(1+u^{-\gamma})e^{-c's^2}\},\end{aligned}$$

for some $c' > 0$. Hence

$$\begin{aligned}I_C &= \int_{t_0}^{\infty} |w(u\epsilon)|\,du \int_0^{t_0} \{\tfrac{1}{2}|\phi(s)|+\tfrac{1}{2}|\phi(2u-s)|\}^n ds \\ &\le \int_{t_0}^{\infty} |w(u\epsilon)|\,\frac{1}{2^n}\exp(nu^{-\gamma})\,du \int_0^{t_0} e^{-c'ns^2} ds \\ &\le \frac{1}{2^n}\sqrt{\frac{\pi}{4nc'}}\int_{t_0}^{\infty} |w(u\epsilon)|\exp(nu^{-\gamma})\,du.\end{aligned}$$

Then, since $u^{-\gamma}$ is at most $1/5$ for $u \ge t_0$ by (3.5), the integral over C_1 is

$$\begin{aligned}I_{C_1} &\le \frac{1}{2^n}\sqrt{\frac{\pi}{4nc'}}\int_{t_0}^{n^{2/\gamma}} \exp(nu^{-\gamma})\,du \\ &\le \frac{1}{2^n}\sqrt{\frac{\pi}{4nc'}}\,n^{2/\gamma}e^{n/5} \\ &\le \frac{1}{2^n}\sqrt{\frac{\pi}{4nc'}}\,n^{2/\gamma}(5/4)^n \\ &= o(0.75^n).\end{aligned}$$

Similarly, the integral over C_2 is at most

$$\begin{aligned}I_{C_2} &\le \frac{1}{2^n}\sqrt{\frac{\pi}{4nc'}}\int_{n^{2/\gamma}}^{\infty} |w(u\epsilon)|\,\exp(nu^{-\gamma})\,du \\ &\le \frac{1}{2^n}\sqrt{\frac{\pi}{4nc'}}\exp\big(n(n^{2/\gamma})^{-\gamma}\big)\int_0^{\infty} |w(u\epsilon)|\,du \\ &\le \frac{1}{2^n}\sqrt{\frac{\pi}{4nc'}}(1+o(1))O(\epsilon^{-1}) \\ &= O\left(\frac{1}{2^n\epsilon\sqrt{n}}\right).\end{aligned}$$

Recalling that $2^n\epsilon$ is of polynomial magnitude, we see that asymptotically only I_{C_2} is significant, so

$$\int\int_{\substack{|u|\geq U^{-1} \text{ or} \\ |t|\geq U^{-1}}} \left|w(u\epsilon)w(t\epsilon)\right| \left|\tfrac{1}{2}\phi(u+t)+\tfrac{1}{2}\phi(u-t)\right|^n du\,dt$$

$$\leq 8(I_A + I_B + I_{C_1} + I_{C_2})$$

$$= O\left(\frac{1}{2^n \epsilon\sqrt{n}}\right). \tag{3.19}$$

Combining (3.16), (3.17), (3.18), and (3.19), we obtain

$$\mathsf{E}(W^2) - \mathsf{E}(W)^2 = 2^{2n}\epsilon^2\, O\Big(\frac{1}{n^2} + \frac{\beta^4}{n^3} + \frac{1}{2^n\sqrt{n}\epsilon}\Big). \tag{3.20}$$

Thus an application of (1.1) using (3.14) and (3.20) yields part (b) of the theorem. ■

4. Remarks

With additional information about the moments of the distribution of the x_j, one can improve on the $1/n$ term in the statement of Theorem 3.1. For example, if the x_j are drawn from the uniform distribution on $[0,1]$, then by looking at

$$\mathsf{E}\Big(\big(W^2 + a\,\mathsf{E}(W)W + b\,\mathsf{E}(W)^2\big)^2\Big),$$

when $b = 3 \pm 2\sqrt{2}$, $a = -1-b$, one can show that for $\beta = 0$, the $O(n^{-1})$ term in the bound of the the theorem can be replaced by $O(n^{-2})$. One cannot obtain this result by simply strengthening the asymptotic analysis in the argument of Section 3, since it can be shown that the $1/n^2$ term is actually present in (3.20).

A significant question which our results leave open is the *expected* value of the difference for the best partition.

One can apply the method of section 3 to partitions into k subsets. Instead of $\delta_j = \pm 1$, we use for the δ_j vectors from $\Re^{k-1}$; each δ_j, $j = 1, 2, \ldots, n$, takes on values from the set

$$\{(-1,-1,\ldots,-1) + ke_l \,|\, 1 \leq l \leq k\},$$

where e_l is the l^{th} coordinate vector $(0,0,\ldots,0,1,0,\ldots,0)$ for $1 \leq l \leq k-1$ and $e_k = (0,0,\ldots,0)$. Finding a combination

$$\sum_{j=1}^{n} \delta_j x_j$$

with a small vector norm corresponds to an equitable partition of the x_j into k subsets, with x_j being assigned to the l^{th} subset, $1 \le l \le k$, if

$$\delta_j = (-1, -1, \dots, -1) + ke_l.$$

To find a vector with small norm, we use

$$W = \sum_{\delta} \omega\Big(\epsilon^{-1} \sum_{j=1}^{n} \delta_j x_j\Big),$$

where ω is now a $(k-1)$-variable function with appropriate smoothness properties.

We also note that one can easily extend Theorem 3.1 to provide results under more general distributions. For example, suppose that the distribution function is

$$F(x) = pF_1(x) + qF_2(x)$$

where $p + q = 1$, $0 < p < 1$, F_1 is a distribution obeying the conditions of Theorem 3.1, and F_2 is, say, the distribution of a variable which assumes the values ± 1 with equal probability. Then the characteristic function of F does not even approach 0 at infinity, so it surely does not obey the requirements of Theorem 3.1. We can still show, however, that we are likely to be able to form a partition of exponentially small size. We can imagine the selection of $x_1, x_2, \dots, x_n$ as a two-step process in which we first randomly color each x_j green (respectively red) with probability p (respectively q), and then draw values for the green (respectively red) variables from F_1 (respectively F_2). Now regardless of the number or values of the red variables, it is clear that they can be partitioned into two subsets whose sums differ by at most 1. Thus, by Theorem 3.1 the probability that we cannot then add the green variables to these subsets so as to achieve a final difference of at most ϵ is bounded by

$$\sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m} \min\Big\{1,\ C_0\Big(\frac{\sqrt{m}}{2^m \epsilon} + \frac{1}{m} + \frac{\beta^4}{m^2}\Big)\Big\},$$

where C_0 is some constant, β is 1, and m is the number of variables colored green. Asymptotic analysis shows that the sum is

$$O\Big(\frac{\sqrt{n}(q + p/2)^n}{\epsilon} + \frac{1}{n}\Big).$$

We leave the detailed investigation of optimum partitioning under pathological distributions for further research.

Another interesting question is the design and analysis of good algorithms for the partition problem. As mentioned in the introduction, the algorithm of [KK82] achieves a difference of only $O(n^{-\alpha \log n})$ with high probability, for some $\alpha > 0$. While this is a very small difference, and was a great improvement over other results, it is still much greater than the optimum difference which is shown in this paper to be likely. It would be very interesting, though quite possibly very difficult, to improve upon that algorithm.

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