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Polynomiality properties of tropical refined invariants

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Abstract. Tropical refined invariants of toric surfaces constitute a fascinating interpolation between real and complex enumerative geometries via tropical geometry. They were originally introduced by Block and Göttsche, and further extended by Göttsche and Schroeter in the case of rational curves.

In this paper, we study the polynomial behavior of coefficients of these tropical refined invariants. We prove that coefficients of small codegree are polynomials in the Newton polygon of the curves under enumeration, when one fixes the genus of the latter. This provides a surprising reappearance, in a dual setting, of the so-called node polynomials and the Göttsche conjecture. Our methods, based on floor diagrams introduced by Mikhalkin and the first author, are entirely combinatorial. Although the combinatorial treatment needed here is different, we follow the overall strategy designed by Fomin and Mikhalkin and further developed by Ardila and Block. Hence our results may suggest phenomena in complex enumerative geometry that have not been studied yet.

In the particular case of rational curves, we extend our polynomiality results by including the extra parameter s recording the number of ψ classes. Contrary to the polynomiality with respect to Δ, the one with respect to s may be expected from considerations on Welschinger invariants in real enumerative geometry. This pleads in particular in favor of a geometric definition of Göttsche-Schroeter invariants.

Keywords. Tropical refined invariants, enumerative geometry, Welschinger invariants, Gromov–Witten invariants, floor diagrams

Mathematics Subject Classifications. Primary 14T15, 14T90, 05A15; Secondary 14N10, 52B20

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1. Introduction

1.1. Results

Tropical refined invariants of toric surfaces have been introduced in [BG16b] and further explored in several directions since then, see for example [IM13, FS15, BG16a, GK16, Mik17, NPS18, Shu18, BS19, Bou19b, GS19, Bou19a, Blo19, Bru20, Blo20b, Blo20a]. In this paper, we study the polynomial behavior of the coefficients of these tropical refined invariants, in connection with node polynomials and the Göttsche conjecture on one hand, and with Welschinger invariants on the other hand. Our methods are entirely combinatorial and do not require any specific knowledge in complex or real enumerative geometry, nor in tropical, algebraic or symplectic geometry. Nevertheless our work probably only gains meaning in the light of these frameworks, so we briefly indicate below how tropical refined invariants arose from enumerative geometry considerations, and present some further connections in Section 1.2. We also provide in Section 1.3 a few explicit computations in genus 0 that are interesting to interpret in the light of Section 1.2.

Given a convex integer polygon $\Delta \subset \mathbb{R}^2$, i.e. the convex hull of finitely many points in $\mathbb{Z}^2$, Block and Göttsche proposed in [BG16b] to enumerate irreducible tropical curves with Newton polygon Δ and genus g as proposed in [Mik05], but replacing Mikhalkin's complex multiplicity with its quantum analog. Itenberg and Mikhalkin proved in [IM13] that the resulting symmetric Laurent polynomial in the variable q does not depend on the configuration of points chosen to define it. This Laurent polynomial is called a *tropical refined invariant* and is denoted by $G_\Delta(g)$. As a main feature, tropical refined invariants interpolate between Gromov–Witten invariants (for $q = 1$) and tropical Welschinger invariants (for $q = -1$) of the toric surface X_Δ defined by the polygon Δ. They are also conjectured to agree with the χ_y-refinement of Severi degrees of X_Δ introduced in [GS14].

Göttsche and Schroeter extended the work of [BG16b] in the case when $g = 0$. They defined in [GS19] some tropical refined descendant invariants, denoted by $G_\Delta(0; s)$, depending now on an additional integer parameter $s \in \mathbb{Z}_{\geqslant 0}$. On the complex side, the value at $q = 1$ of $G_\Delta(0; s)$ recovers some genus 0 relative Gromov–Witten invariants (or some descendant invariants) of X_Δ. On the real side and when X_Δ is an unnodal del Pezzo surface, plugging $q = -1$ in $G_\Delta(0; s)$ recovers Welschinger invariants counting real algebraic (or symplectic) rational curves passing through a generic real configuration of $\mathrm{Card}(\partial\Delta \cap \mathbb{Z}^2) - 1$ points in X_Δ containing exactly s pairs of complex conjugated points. The case when $s = 0$ corresponds to tropical Welschinger invariants, and $G_\Delta(0; 0) = G_\Delta(0)$ for any polygon Δ.

For the sake of brevity, we do not recall the definition of tropical refined invariants in this paper. Nevertheless we provide in Theorems 2.7 and 2.13 a combinatorial recipe that computes them when Δ is an h-*transverse* polygon, via the so-called *floor diagrams* introduced by Mikhalkin and the first author in [BM07, BM08]. Since the present work in entirely based on these floor diagram computations, the reader unfamiliar with the invariants $G_\Delta(g)$ and $G_\Delta(0; s)$ may take Theorems 2.7 and 2.13 as definitions rather than statements.

Denoting by ι_Δ the number of integer points contained in the interior of Δ, the invariant $G_\Delta(g)$ is non-zero if and only if $g \in \{0, 1, \cdots, \iota_\Delta\}$. It is known furthermore, see for example

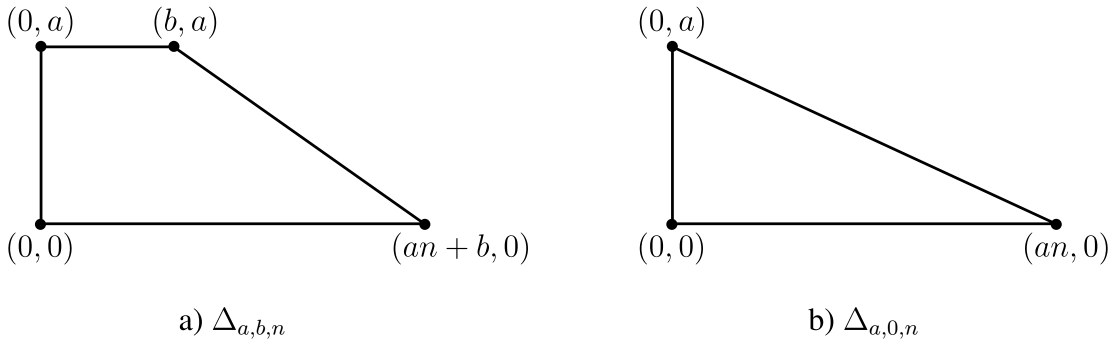


Figure 1.1

[IM13, Proposition 2.11], that in this case $G_\Delta(g)$ has degree[1] $\iota_\Delta - g$. In this paper we establish that coefficients of small *codegree* of $G_{\Delta_{a,b,n}}(g)$ and $G_{\Delta_{a,b,n}}(0;s)$ are asymptotically polynomials in a, b, n, and s, where $\Delta_{a,b,n}$ is the convex polygon depicted in Figure 1.1. By definition the coefficient of codegree i of a Laurent polynomial $P(q)$ of degree d is its coefficient of degree $d-i$, and is denoted by $\langle P\rangle_i$.

Theorem 1.1. *For any $i, g \in \mathbb{Z}_{\geqslant 0}$, the function*

$$\begin{array}{ccc} \mathbb{Z}^3_{\geqslant 0} & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\ (a,b,n) & \longmapsto & \left\langle G_{\Delta_{a,b,n}}(g)\right\rangle_i \end{array}$$

is polynomial on the set $\mathcal{U}_{i,g}$ defined by

$$\begin{cases} n \geqslant 1 \\ b > i \\ b+n > (g+2)i+g \\ a \geqslant i+2g+2 \end{cases}$$

and has degree $i+g$ in each of the variables b and n, and degree $i+2g$ in the variable a.

Theorem 1.1 requires n to be positive, and has the following version for $n=0$.

Theorem 1.2. *For any $i, g \in \mathbb{Z}_{\geqslant 0}$, the function*

$$\begin{array}{ccc} \mathbb{Z}^2_{\geqslant 0} & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\ (a,b) & \longmapsto & \left\langle G_{\Delta_{a,b,0}}(g)\right\rangle_i \end{array}$$

is polynomial on the set defined by

$$\begin{cases} b > (g+2)i+g \\ a \geqslant i+2g+2 \end{cases}$$

and has degree $i+g$ in each of the variables a and b.

[1] As for polynomials, the degree of a Laurent polynomial $\sum_{j=-m}^{n} a_j q^j$ with $a_n \neq 0$ is defined to be n.

In connection to the Göttsche conjecture (see Section 1.2), one may also be interested in fixing $b = 0$ and $n \geqslant 1$, and varying a. Theorem 1.1 can be adapted in this case.

Theorem 1.3. *For any $i, g \in \mathbb{Z}_{\geqslant 0}$, and $n \in \mathbb{Z}_{>0}$, the function*

$$\begin{array}{ccc} \mathbb{Z}_{\geqslant 0} & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\ a & \longmapsto & \left\langle G_{\Delta_{a,0,n}}(g) \right\rangle_i \end{array}$$

is polynomial of degree $i + 2g$ for $a \geqslant i + 2g + 2$.

Example 1.4. Theorem 1.1 may be seen as a partial generalisation of the fact that for any convex integer polygon Δ, one has

$$\langle G_\Delta(g) \rangle_0 = \binom{\iota_\Delta}{g}$$

(see [IM13, Proposition 2.11] and [BG16b, Proposition 4.10]). Indeed, when $\Delta = \Delta_{a,b,n}$, this identity can be rewritten as

$$\left\langle G_{\Delta_{a,b,n}}(g) \right\rangle_0 = \binom{\frac{a^2 n + 2ab - (n+2)a - 2b + 2}{2}}{g},$$

which is a polynomial of degree g in the variables b and n, and of degree g or $2g$ in the variable a depending on whether $n = 0$ or not.

The particular case $g = 0$ is much simpler to deal with, and the three theorems above can be made more precise. Let us define

$$\eta(\Delta) = \operatorname{Card}(\partial \Delta \cap \mathbb{Z}^2) - 1.$$

Since there is the additional parameter s in this case, one may also study polynomiality with respect to s. Note that the invariant $G_\Delta(0; s)$ is non-zero if and only if

$$s \in \left\{ 0, \cdots, \left[\frac{\eta(\Delta)}{2} \right] \right\},$$

in which case it has degree ι_Δ.

Theorem 1.5. *For any $i \in \mathbb{Z}_{\geqslant 0}$, the function*

$$\begin{array}{ccc} \mathbb{Z}^4 & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\ (a, b, n, s) & \longmapsto & \left\langle G_{\Delta_{a,b,n}}(0; s) \right\rangle_i \end{array}$$

is polynomial on the set $\mathcal{U}_i$ defined by

$$\left\{ \begin{array}{l} an + b \geqslant i + 2s \\ b > i \\ a > i \end{array} \right. .$$

Furthermore it has degree i in each of the variables a, b, n, and s.

Theorem 1.5 is an easy-to-state version of Theorem 4.3 where we also provide an explicit expression for $\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_i$. As in the higher genus case, Theorem 1.5 can be adapted to the case when $b = 0$ and n is fixed.

Theorem 1.6. *For any $(i, n) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{>0}$, the function*

$$\begin{array}{ccc} \mathbb{Z}^2_{\geqslant 0} & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\ (a, s) & \longmapsto & \left\langle G_{\Delta_{a,0,n}}(0;s)\right\rangle_i \end{array}$$

is polynomial on the set defined by

$$\begin{cases} an \geqslant i + 2s \\ a \geqslant i + 2 \end{cases}$$

Furthermore it has degree i in each of the variables a and s.

As mentioned above, floor diagrams allow the computation of the invariants $G_\Delta(g)$ and $G_\Delta(0;s)$ when Δ is an h-transverse polygons. The polygons $\Delta_{a,b,n}$ are h-transverse, but the converse may not be true. We do not see any difficulty other than technical to generalize all the statements above to the case of h-transverse polygons, in the spirit of [AB13, BG16a]. Since this paper is already quite long and technical, we have restricted ourselves to the case of polygons $\Delta_{a,b,n}$. From an algebro-geometric perspective, these polygons correspond to the toric surfaces $\mathbb{C}P^2$, the n-th Hirzebruch surface $\mathbb{F}_n$, and the weighted projective plane $\mathbb{C}P^2(1, 1, n)$.

It emerges from Section 1.2 that polynomiality with respect to s deserves a separate study from polynomiality with respect to Δ. Clearly, the values $\langle G_\Delta(0;0)\rangle_i, \cdots, \langle G_\Delta(0;s_{max})\rangle_i$ are interpolated by a polynomial of degree at most s_{max}, where

$$s_{max} = \left[\frac{\eta(\Delta)}{2}\right].$$

It is nevertheless reasonable to expect, at least for "simple" polygons, this interpolation polynomial to be of degree $\min(i, s_{max})$. The next Theorem states that this is indeed the case for small values of i. Given a convex integer polygon $\Delta \subset \mathbb{R}^2$, we denote by $d_b(\Delta)$ the length of the bottom horizontal edge of Δ. Note that $d_b(\Delta) = 0$ if this edge is reduced to a point.

Theorem 1.7. *Let Δ be an h-transverse polygon in $\mathbb{R}^2$. If $2i \leqslant d_b(\Delta) + 1$ and $i \leqslant \iota_\Delta$, then the values $\langle G_\Delta(0;0)\rangle_i, \cdots, \langle G_\Delta(0;s_{max})\rangle_i$ are interpolated by a polynomial of degree i, whose leading coefficient is $\frac{(-2)^i}{i!}$. If $\Delta = \Delta_{a,b,n}$, then the result holds also for $2i = d_b(\Delta) + 2$.*

Observe that even when $\Delta = \Delta_{a,b,n}$, Theorem 1.7 cannot be deduced from Theorems 1.5 or from Theorems 1.6. Since the proof of Theorem 1.7 does not seem easier when restricting to polygons $\Delta_{a,b,n}$ for $2i \leqslant d_b(\Delta) + 1$, we provide a proof valid for any h-transverse polygon. We expect that the upper bounds $2i \leqslant d_b(\Delta) + 1$ and $2i \leqslant d_b(\Delta) + 2$ can be weakened; nevertheless the proof via floor diagrams becomes more and more intricate as i grows, as is visible in our proof of Theorem 1.7.

1.2. Connection to complex and real enumerative geometry

Let $N^{\delta}_{\mathbb{C}P^2}(d)$ be the number of irreducible algebraic curves of degree d, with δ nodes, and passing through a generic configuration of $\frac{d(d+3)}{2}-\delta$ points in $\mathbb{C}P^2$. For a fixed $\delta\in\mathbb{Z}_{\geqslant 0}$, this number is polynomial in d of degree 2δ for $d\geqslant\delta+2$. For example, one has

$$\begin{array}{l}\forall d\geqslant 1,\ N^{0}_{\mathbb{C}P^2}(d)=1\\ \forall d\geqslant 3,\ N^{1}_{\mathbb{C}P^2}(d)=3(d-1)^2\\ \forall d\geqslant 4,\ N^{2}_{\mathbb{C}P^2}(d)=\dfrac{3}{2}(d-1)(d-2)(3d^2-3d-11)\end{array}$$

These *node polynomials* have a long history. After some computations for small values of δ, they were conjectured to exist for any δ by Di Francesco and Itzykson in [DFI95]. By around 2000, they were computed up to $\delta=8$, see [KP04] and references therein for an historical account. Göttsche proposed in [Gö98] a more general conjecture: given a non-singular complex algebraic surface X, a non-negative integer δ, and a line bundle $\mathcal{L}$ on X that is sufficiently ample with respect to δ, the number $N^{\delta}_{X}(\mathcal{L})$ of irreducible algebraic curves in the linear system $|\mathcal{L}|$, with δ nodes, and passing through a generic configuration of $\frac{\mathcal{L}^2+c_1(X)\cdot\mathcal{L}}{2}-\delta$ points in X equals $P_\delta(\mathcal{L}^2,c_1(X)\cdot\mathcal{L},c_1(X)^2,c_2(X))$, with $P_\delta(x,y,z,t)$ a universal polynomial depending only on δ.

The Göttsche conjecture was proved in full generality by Tzeng in [Tze12], and an alternative proof was proposed shortly thereafter in [KST11]. Both proofs use algebro-geometric methods. Fomin and Mikhalkin gave in [FM10] a combinatorial proof of the Di Francesco–Itzykson conjecture by mean of floor diagrams. This was generalized by Ardila and Block in [AB13] to a proof of the Göttsche conjecture restricted to the case when X is the toric surface associated to an h-transverse polygon. Ardila and Block's work contains an interesting outcome: combinatorics allows one to transcend from the original realm of the Göttsche conjecture, and to consider algebraic surfaces with mild singularities as well. We are not aware of any algebro-geometric approach to the Göttsche conjecture in the case of singular surfaces.

Motivated by the paper [KST11], Göttsche and Shende defined in [GS14] a χ_y-refined version of the numbers $N^{\delta}_{X}(\mathcal{L})$. In the case when X is the toric surface X_Δ associated to the polygon Δ, these refined invariants are conjecturally equal to the refined tropical invariants $G_\Delta(\frac{\mathcal{L}^2-c_1(X_\Delta)\cdot\mathcal{L}+2}{2}-\delta)$ that were simultaneously defined by Block and Göttsche in [BG16b]. In light of the Göttsche conjecture, it is reasonable to expect the coefficients of $G_\Delta(\frac{\mathcal{L}^2-c_1(X_\Delta)\cdot\mathcal{L}+2}{2}-\delta)$ to be asymptotically polynomial with respect to Δ. Block and Göttsche adapted in [BG16b] the methods from [FM10, AB13] to show that this is indeed the case.

In all the story above, the parameter δ is fixed and the line bundle $\mathcal{L}$ varies. In other words, we are enumerating algebraic curves with a fixed number of nodes in a varying linear system. In particular, the genus of the curves under enumeration in the linear system $d\mathcal{L}$ grows quadratically with respect to d. In a kind of dual setup, one may fix the genus of curves under enumeration. For example one may consider the numbers $N^{\frac{(d-1)(d-2)}{2}-g}_{\mathbb{C}P^2}(d)$ in the case of $\mathbb{C}P^2$, and let d vary. However in this case it seems hopeless to seek for any polynomiality behavior. Indeed, the sequence $N^{\frac{(d-1)(d-2)}{2}-g}_{\mathbb{C}P^2}(d)$ tends to infinity more than exponentially fast. This has been proved by Di Francesco and Itzykson in [DFI95] when $g=0$, and the general case can be obtained for example by an easy adaptation of the proof of Di Francesco and Itzykson's result via floor diagrams proposed in [BM08, Bru08].

Nevertheless, our results can be interpreted as a reappearance of the Göttsche conjecture at the refined level: coefficients of small codegrees of $G_{\Delta_{a,b,n}}(g)$ behave polynomially asymptotically with respect to (a, b, n). It is somewhat reminiscent of the Itenberg–Kharlamov–Shustin conjecture [IKS04, Conjecture 6]: although this conjecture has been shown to be wrong in [Wel07, ABLdM11], its refined version turned out to be true by [Bru20, Corollary 4.5] and Corollary 2.17 below. Anyhow, it may be interesting to understand further this reappearance of the Göttsche conjecture.

In the same range of ideas, it may be worthwhile to investigate the existence of universal polynomials giving asymptotic values of $\langle G_{\Delta_{a,b,n}}(g)\rangle_i$. It follows from Examples 1.8 and 1.9 that the polynomials whose existence is attested in Theorems 1.1 and 1.3 are not equal. Nevertheless, we do not know whether there exists a universal polynomial $Q_{g,i}(x, y, z, t)$ such that, under the assumption that the toric surface $X_{\Delta_{a,b,n}}$ is non-singular, the equality

$$\langle G_{\Delta_{a,b,n}}(g)\rangle_i = Q_{g,i}(\mathcal{L}^2_{a,b,n}, c_1(X_{\Delta_{a,b,n}}) \cdot \mathcal{L}_{a,b,n}, c_1(X_{\Delta_{a,b,n}})^2, c_2(X_{\Delta_{a,b,n}}))$$

holds in each of the three regions described in Theorems 1.1, 1.2, and 1.3. In the expression above $\mathcal{L}_{a,b,n}$ denotes the line bundle on $X_{\Delta_{a,b,n}}$ defined by $\Delta_{a,b,n}$. As explained in [AB13, Section 1.3], it is unclear what should generalize the four intersection numbers in the formula above when $X_{\Delta_{a,b,n}}$ is singular. Recall that the surface $X_{\Delta_{a,b,n}}$ is non-singular precisely when $b \neq 0$ or $n = 1$, in which case one has

$$\mathcal{L}^2_{a,b,n} = a^2 n + 2ab, \qquad c_1(X_{\Delta_{a,b,n}}) \cdot \mathcal{L}_{a,b,n} = (n+2)a + 2b,$$

and

$$c_1(X_{\Delta_{a,b,n}})^2 = 8 \text{ and } c_2(X_{\Delta_{a,b,n}}) = 4 \quad \text{if } b \neq 0, \qquad c_1(X_{\Delta_{a,0,1}})^2 = 9 \text{ and } c_2(X_{\Delta_{a,0,1}}) = 3.$$

It follows from the adjunction formula combined with Pick's formula that

$$\iota_{\Delta_{a,b,n}} = \frac{\mathcal{L}^2_{a,b,n} - c_1(X_{\Delta_{a,b,n}}) \cdot \mathcal{L}_{a,b,n} + 2}{2}.$$

As a consequence, for $i = 0$, the universal polynomials $Q_{g,0}$ exist and are given by

$$Q_{g,0}(x, y, z, t) = \binom{\frac{x-y+2}{2}}{g}.$$

At the other extreme, Examples 1.8 and 1.9 suggest that $Q_{0,i}$ may not depend on x.

If this kind of "dual" Göttsche conjecture phenomenon may come as a surprise, polynomiality with respect to s of $\langle G_{\Delta_{a,b,n}}(0; s)\rangle_i$ is quite expected. It is also related to complex and real enumerative geometry, and pleads in favor of a more geometric definition of refined tropical invariants as conjectured, for example, in [GS14]. Given a real projective algebraic surface X, we denote by $W_X(d; s)$ the Welschinger invariant of X counting (with signs) real J-holomorphic rational curves realizing the class $d \in H_2(X;\mathbb{Z})$, and passing through a generic real configuration of $c_1(X)\cdot d - 1$ points in X containing exactly s pairs of complex conjugated points (see [Wel05, Bru20]). Welschinger exhibited in [Wel05, Theorem 3.2] a very simple relation between Welschinger invariants of a real algebraic surface X and its blow-up $\widetilde{X}$ at a real point, with exceptional divisor E:

$$W_X(d;s+1) = W_X(d;s) - 2W_{\widetilde{X}}(d-2[E];s). \tag{1.1}$$

This equation is also obtained in [Bru20, Corollary 2.4] as a special case of a formula relating Welschinger invariants of real surfaces differing by a surgery along a real Lagrangian sphere. As suggested in [Bru20, Section 4], it is reasonable to expect that such formulas admit a refinement. The refined Abramovich–Bertram formula [Bou19a, Corollary 5.1], proving [Bru20, Conjecture 4.6], provides a piece of evidence for such expectation. Hence one may expect that a refinement of formula (1.1) holds both for tropical refined invariants from [BG16b, GS19] and for χ_y-refined invariants from [GS14].

As mentioned earlier, one has

$$G_\Delta(0;s)(-1) = W_{X_\Delta}(\mathcal{L}_\Delta;s)$$

when X_Δ is an unnodal del Pezzo surface. In particular [Bru20, Proposition 4.3] and Proposition 2.19 below state precisely that the refinement of formula (1.1) holds true in the tropical set-up when both X_Δ and $\widetilde{X_\Delta}$ are unnodal toric del Pezzo surfaces.

In any event, reducing inductively to $s=0$, one sees easily that $\langle G_\Delta(d,0;s)\rangle_i$ is polynomial of degree i in s if one takes for granted that

- tropical refined invariants $G_\Delta(0;s)$ generalize to some χ_y-refined tropical invariants $G_{X,\mathcal{L}}(0;s)$, where X is an arbitrary projective surface and $\mathcal{L} \in Pic(X)$ is a line bundle;
- $G_{X,\mathcal{L}}(0;s)$ is a symmetric Laurent series of degree $\frac{\mathcal{L}^2 - c_1(X)\cdot\mathcal{L}+2}{2}$ with leading coefficient equal to 1;
- a refined version of formula (1.1) holds for refined invariants $G_{X,\mathcal{L}}(0;s)$.

Since none of the last three conditions are established yet, Theorem 1.7 may be seen as an evidence that these conditions actually hold.

To end this section, note that all the mentioned asymptotical problems require one to fix either the number δ of nodes of the curves under enumeration, or their genus g. These two numbers are related by the adjunction formula

$$g + \delta = \frac{\mathcal{L}^2 - c_1(X)\cdot\mathcal{L} + 2}{2}.$$

One may wonder whether these asymptotical results generalize when both g and δ are allowed to vary, as long as they satisfy the equation above.

1.3. Some explicit computations in genus 0

Here we present a few computations that illustrates Theorems 4.3, 1.6, and 1.7, and which, in the light of Section 1.2, may point towards interesting directions.

Example 1.8. Theorem 4.3 allows one to compute $\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_i$ for small values of i. For example one computes easily that (recall that the sets $\mathcal{U}_i$ are defined in the statement of Theorem 1.5)

$$\forall (a,b,n)\in\mathcal{U}_1,\ \left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_1 = (n+2)a+2b+2-2s.$$

In relation to the Göttsche conjecture, one may try to express $\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_i$ in terms of topological numbers related to the linear system $\mathcal{L}_{a,b,n}$ defined by the polygon $\Delta_{a,b,n}$ in the Hirzebruch surface $X_{\Delta_{a,b,n}} = \mathbb{F}_n$. Surprisingly, the values of $\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_i$ we compute can be expressed in terms of $c_1(\mathbb{F}_n)\cdot\mathcal{L}_{a,b,n} = (n+2)a+2b$ and s only. Furthermore expressing these values in terms of the number of real points rather than in terms of the number s of pairs of complex conjugated points simplifies even further the final expressions. More precisely, setting $y=(n+2)a+2b$ and $t=y-1-2s$, we obtain

$$\begin{aligned}
\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_0 &= 1\\
\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_1 &= t+3\\
\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_2 &= \frac{t^2+6t+y+19}{2}\\
\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_3 &= \frac{t^3+9t^2+(3y+59)t+9y+147}{3!}\\
\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_4 &= \frac{t^4+12t^3+(6y+122)t^2+(36y+612)t+3y^2+120y+1437}{4!}\\
\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_5 &= \frac{1}{5!}\times\big(t^5+15t^4+(10y+210)t^3+(90y+1590)t^2\\
&\qquad +(15y^2+620y+7589)t+45y^2+1560y+16035\big)\\
\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_6 &= \frac{1}{6!}\times\big(t^6+18t^5+(15y+325)t^4+(180y+3300)t^3\\
&\qquad +(45y^2+1920y+24019)t^2+(270y^2+9720y+102522)t\big)\\
&\qquad +15y^3+945y^2+23385y+207495\big)
\end{aligned}$$

for any (a,b,n,s) in the corresponding $\mathcal{U}_i$. It appears from these computations that the polynomial $\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_i$ has total degree i if t has degree 1 and y and degree 2. In addition, its coefficients seem to be all positive and to also have some polynomial behavior with respect to i:

$$\begin{aligned}
i!\times\langle G_{\Delta_d}(0;s)\rangle_i = t^i + 3it^{i-1} &+ \frac{i(i-1)}{6}(3y+2i+53)\,t^{i-2}\\
&+\frac{i(i-1)(i-2)}{2}(3y+2i+43)\,t^{i-3}+\cdots
\end{aligned}$$

It could be interesting to study further these observations.

Example 1.9. Throughout the text, we use the more common notation Δ_d rather than $\Delta_{d,0,1}$. It follows from Theorem 1.6 combined with Examples 2.15 and 2.18 that

$$\forall d \geqslant 3,\ \langle G_{\Delta_d}(0;s)\rangle_1 = 3d + 1 - 2s.$$

Further computations allow one to compute $\langle G_{\Delta_d}(0;s)\rangle_i$ for the first values of i. Similarly to Example 1.8, it is interesting to express $\langle G_{\Delta_d}(0)\rangle_i$ in terms of $y = 3d = c_1(\mathbb{C}P^2)\cdot d\mathcal{L}_1$ and $t = y - 1 - 2s$:

$$\begin{aligned}
&\forall d \geqslant 3,\ \langle G_{\Delta_d}(0;s)\rangle_1 = t + 2\\
&\forall d \geqslant 4,\ \langle G_{\Delta_d}(0;s)\rangle_2 = \frac{t^2 + 4t + y + 11}{2}\\
&\forall d \geqslant 5,\ \langle G_{\Delta_d}(0;s)\rangle_3 = \frac{t^3 + 6t^2 + (3y+35)t + 6y + 72}{3!}\\
&\forall d \geqslant 6,\ \langle G_{\Delta_d}(0;s)\rangle_4 = \frac{t^4 + 8t^3 + (6y+74)t^2 + (24y+304)t + 3y^2 + 72y + 621}{4!}\\
&\forall d \geqslant 7,\ \langle G_{\Delta_d}(0;s)\rangle_5 = \frac{1}{5!}\times\big(t^5 + 10t^4 + (10y+130)t^3 + (60y+800)t^2\\
&\qquad\qquad + (15y^2 + 380y + 3349)t + 30y^2 + 780y + 6030\big)
\end{aligned}$$

We observe the same phenomenon for the coefficients of the polynomial $\langle G_{\Delta_d}(0;s)\rangle_i$ as in Example 1.8. In particular they seem to have some polynomial behavior with respect to i:

$$\begin{aligned}
i!\times\langle G_{\Delta_d}(0;s)\rangle_i = t^i + 2it^{i-1} &+ \frac{i(i-1)}{6}\,(3y + 2i + 29)\,t^{i-2}\\
&+ \frac{i(i-1)(i-2)}{3}\,(3y + 2i + 30)\,t^{i-3} + \cdots
\end{aligned}$$

Example 1.10. For $n \geqslant 2$, one computes easily that

$$\big\langle G_{\Delta_{2,0,n}}(0;s)\big\rangle_1 = 2n + 2 - 2s = c_1(\mathbb{F}_n)\cdot\mathcal{L}_{2,0} - 2s.$$

In particular, one notes a discrepancy with the case of $\mathbb{C}P^2$, i.e. when $n = 1$. This originates probably from the fact that the toric complex algebraic surface $X_{\Delta_{a,0,n}}$ is singular as soon as $n \geqslant 2$.

Example 1.11. Here we illustrate Theorem 1.7 in the case of Δ_4. According to Example 2.18 and setting $t = 11 - 2s$, one has

$$\begin{aligned}
G_{\Delta_4}(0;s) = q^{-3} &+ (2+t)q^{-2} + \frac{t^2+4t+23}{2}q^{-1} + \frac{t^3+3t^2+59t+81}{6} +\\
q^3 &+ (2+t)q^2 + \frac{t^2+4t+23}{2}q\,.
\end{aligned}$$

In particular one has

$$\langle G_{\Delta_4}(0;s)\rangle_3 = \frac{t^3+3t^2+59t+81}{6} \neq \frac{t^3+6t^2+(3\times 12+35)t+6\times 12+72}{6}$$

meaning that the threshold $d \geqslant i+2$ in Example 1.9 is sharp.

1.4. Method and outline of the paper

Fomin and Mikhalkin were the first to use floor diagrams to address the Göttsche conjecture in [FM10]. The basic strategy, further developed by Ardila and Block in [AB13], is to decompose floor diagrams into elementary building blocks, called *templates*, that are suitable for a combinatorial treatment. Even though the basic idea to prove Theorems 1.1, 1.2, and 1.3, is to follow this overall strategy, the building blocks and their combinatorial treatment we need here differ from those used in [FM10, AB13].

However in the special case when $g = 0$, the situation simplifies drastically, and there is no need of the templates machinery to prove Theorems 1.5 and 1.6. Indeed, one can easily describe all floor diagrams coming into play, and perform a combinatorial study by hand. In particular, we are able to provide an explicit expression for $\langle G_{\Delta_{a,b,n}}(0; s)\rangle_i$ in Theorem 4.3.

On the other hand, we use a different strategy than the one from [FM10, AB13] to tackle polynomiality with respect to s when Δ is fixed. We prove Theorem 1.7 by establishing that the sequence $(\langle G_\Delta(0; s)\rangle_i)_s$ is interpolated by a polynomial whose ith discrete derivative (or ith difference) is constant.

The remaining part of this paper is organized as follows. We start by recalling the definition of floor diagrams in Section 2, and how to use them to compute tropical refined invariants of h-transverse polygons. In particular, Theorems 2.7 and 2.13 may be considered as definitions of these invariants for readers unfamiliar with tropical geometry. We collect some general facts about codegrees that will be used throughout the text in Section 3. In Section 4, we prove polynomiality results for tropical refined invariants in genus 0. We first treat the very explicit case when $\Delta = \Delta_{a,b,n}$ with $b \neq 0$, before turning to the slightly more technical situation when b vanishes. We end this section by proving polynomiality with respect to s alone with the help of discrete derivatives. Lastly, Section 5 is devoted to higher genus and becomes more technical. We define a suitable notion of templates, and adapt the proofs from Section 4 in this more general situation. Some well-known or easy identities on quantum numbers are recast in Appendix A in order to ease the reading of the text.

2. Floor diagrams

2.1. h-transverse polygons

The class of h-transverse polygons enlarges slightly the class of polygons $\Delta_{a,b,n}$.

Definition 2.1. A convex integer polygon Δ is called *h-transverse* if every edge contained in its boundary $\partial\Delta$ is either horizontal, vertical, or has slope $\frac{1}{k}$, with $k \in \mathbb{Z}$.

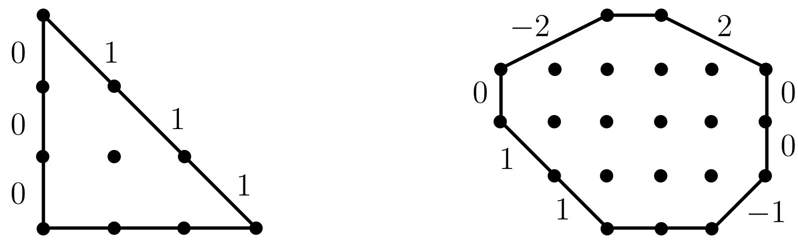
Given an h-transverse polygon Δ, we use the following notation:

- $\partial_l\Delta$ and $\partial_r\Delta$ denote the sets of edges $e \subset \partial\Delta$ with an external normal vector having negative and positive first coordinate, respectively;
- $d_l\Delta$ and $d_r\Delta$ denote the unordered lists of integers k appearing $j \in \mathbb{Z}_{>0}$ times, such that the vector $(jk, -j)$ belongs to $\partial_l\Delta$ and $\partial_r\Delta$, respectively, with j maximal;

- $d_b\Delta$ and $d_t\Delta$ denote the lengths of the horizontal edges at the bottom and top, respectively, of Δ.

Note that both sets $d_l\Delta$ and $d_r\Delta$ have the integer height of Δ for cardinality.

Example 2.2. As said above, all polygons $\Delta_{a,b,n}$ are h-transverse. Recall that we use the notation Δ_d instead of $\Delta_{d,0,1}$. We depict in Figure 2.1 two examples of h-transverse polygons, where the integer close to a segment in $\partial_l\Delta$ or $\partial_r\Delta$ denotes its contribution to $d_l\Delta$ or $d_r\Delta$, respectively.



a) $d_l\Delta_3 = \{0,0,0\}$, $d_r\Delta_3 = \{1,1,1\}$, $d_b\Delta_3 = 3$, $d_t\Delta_3 = 0$.

b) $d_l\Delta = \{-2,0,1,1\}$, $d_r\Delta = \{2,0,0,-1\}$, $d_b\Delta = 2$, $d_t\Delta = 1$.

Figure 2.1: Examples of h-transverse polygons.

2.2. Block-Göttsche refined invariants via floor diagrams

In this text, an oriented multigraph Γ consists of a set of vertices $V(\Gamma)$, a collection $E^0(\Gamma)$ of oriented bivalent edges in $V(\Gamma)\times V(\Gamma)$ and two collections of monovalent edges: a collection of sources $E^{-\infty}(\Gamma)$, and a collection of sinks $E^{+\infty}(\Gamma)$. A source adjacent to the vertex v is oriented towards v, and a sink adjacent to the vertex v is oriented away from v. Given such an oriented graph, we define the set of all edges of Γ by

$$E(\Gamma) = E^0(\Gamma)\cup E^{-\infty}(\Gamma)\cup E^{+\infty}(\Gamma).$$

We use the notation $\underset{\longrightarrow v}{e}$ and $\underset{v\longrightarrow}{e}$ if the edge e is oriented toward the vertex v and away from v, respectively.

A *weighted* oriented graph (Γ,ω) is an oriented graph endowed with a function $\omega : E(\Gamma)\to\mathbb{Z}_{>0}$. The *divergence* of a vertex v of a weighted oriented graph is defined as

$$\operatorname{div}(v) = \sum_{\underset{\longrightarrow v}{e}}\omega(e) - \sum_{\underset{v\longrightarrow}{e}}\omega(e).$$

Definition 2.3. A *floor diagram* $\mathcal{D}$ with Newton polygon Δ is a quadruple $\mathcal{D} = (\Gamma,\omega,l,r)$ such that

1. Γ is a connected weighted acyclic oriented graph with $\mathrm{Card}(d_l\Delta)$ vertices, with $d_b\Delta$ sources and $d_t\Delta$ sinks;
2. all sources and sinks have weight 1;
3. $l : V(\Gamma) \longrightarrow d_l\Delta$ and $r : V(\Gamma) \longrightarrow d_r\Delta$ are bijections such that for every vertex $v \in V(\Gamma)$, one has $\mathrm{div}(v) = r(v) - l(v)$.

By a slight abuse of notation, we will not distinguish in this text between a floor diagram $\mathcal{D}$ and its underlying graph Γ. The first Betti number of $\mathcal{D}$ is called the *genus* of the floor diagram $\mathcal{D}$. The vertices of a floor diagram are called its *floors*, and its edges are called *elevators*. The *degree* of a floor diagram $\mathcal{D}$ is defined as

$$\deg(\mathcal{D}) = \sum_{e\in E(\mathcal{D})} (\omega(e) - 1).$$

Given an integer $k \in \mathbb{Z}$, the quantum integer $[k](q)$ is defined by

$$[k](q) = \frac{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{\frac{k-1}{2}} + q^{\frac{k-3}{2}} + \cdots + q^{-\frac{k-3}{2}} + q^{-\frac{k-1}{2}}.$$

For the reader's convenience, we collect some easy or well-known properties of quantum integers in Appendix A.

Definition 2.4. The refined multiplicity of a floor diagram $\mathcal{D}$ is the Laurent polynomial defined by

$$\mu(\mathcal{D})(q) = \prod_{e\in E(\mathcal{D})} [\omega(e)(q)]^2.$$

Note that $\mu(\mathcal{D})(q)$ is in $\mathbb{Z}_{>0}[q^{\pm 1}]$, is symmetric, and has degree $\deg(\mathcal{D})$.

Example 2.5. Examples of floor diagrams together with their refined multiplicities are depicted in Figure 2.2. Conventionally, floors and elevators are represented by ellipses and vertical lines, respectively. Orientation on elevators is understood from bottom to top and will not be depicted; neither will be the weight on elevators of weight 1. All floor diagrams with Newton polygon Δ_3 are depicted in Figures 2.2a), b), c), and d). Since both functions l and d are trivial in this case, we do not specify them on the picture. An example of floor diagram with Newton polygon depicted in Figure 2.1b) is depicted in Figure 2.2e). We specify the value of l and r at each floor by an integer on the left and on the right in the corresponding ellipse, respectively.

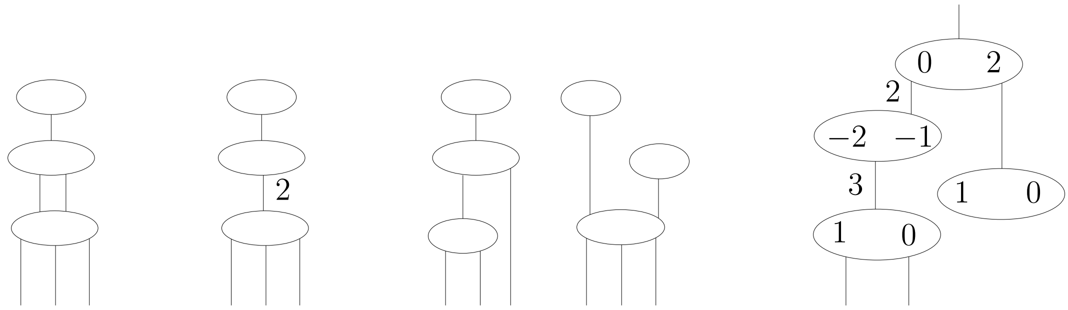
For a floor diagram $\mathcal{D}$ with Newton polygon Δ and genus g, we define

$$\eta(\mathcal{D}) = \eta(\Delta) + g.$$

Note that, by a simple Euler characteristic computation, we also have

$$\eta(\mathcal{D}) = \mathrm{Card}(V(\mathcal{D})) + \mathrm{Card}(E(\mathcal{D})).$$

The orientation of $\mathcal{D}$ induces a partial ordering on $\mathcal{D}$, that we denote by $\preccurlyeq$. A map $m : A \subset \mathbb{Z} \to V(\mathcal{D}) \cup E(\mathcal{D})$ is said to be *increasing* if $i \leqslant j$ whenever $m(i) \preccurlyeq m(j)$.



a) $\mu = 1$ b) $\mu = q + 2 + q^{-1}$ c) $\mu = 1$ d) $\mu = 1$ e) $\mu = q^3 + 4q^2 + 8q + 10 + q^{-3} + 4q^{-2} + 8q^{-1}$

Figure 2.2: Example of floor diagrams with their refined multiplicities.

Definition 2.6. A *marking* of a floor diagram $\mathcal{D}$ with Newton polygon Δ is an increasing bijection

$$m\colon \{1, 2, \ldots, \eta(\mathcal{D})\} \longrightarrow V(\mathcal{D}) \cup E(\mathcal{D}).$$

Two marked floor diagrams $(\mathcal{D}, m)$, $(\mathcal{D}', m')$ with Newton polygon Δ are said to be *isomorphic* if there exists an isomorphism of weighted oriented graphs $\varphi : \mathcal{D} \longrightarrow \mathcal{D}'$ such that $l = l' \circ \varphi$, $r = r' \circ \varphi$, and $m' = \varphi \circ m$.

The next theorem is a slight generalisation of [BM08, Theorem 3.6].

Theorem 2.7 ([BG16a, Theorem 4.3])**.** *Let Δ be an h-transverse polygon in $\mathbb{R}^2$, and $g \geqslant 0$ an integer. Then one has*

$$G_\Delta(g)(q) = \sum_{(\mathcal{D},m)} \mu(\mathcal{D})(q),$$

where the sum runs over all isomorphism classes of marked floor diagrams with Newton polygon Δ and genus g.

Example 2.8. Using Figures 2.2a), b), c), and d) one obtains

$$G_{\Delta_3}(1)(q) = 1 \qquad \text{and} \qquad G_{\Delta_3}(0)(q) = q + 10 + q^{-1}.$$

Example 2.9. Combining Theorem 2.7 with Figures 2.3, 2.4, and 2.5, where all floor diagrams with Newton polygon Δ_4 are depicted, one obtains:

$$G_{\Delta_4}(3) = 1, \qquad G_{\Delta_4}(2) = 3q^{-1} + 21 + 3q, \qquad G_{\Delta_4}(1) = 3q^{-2} + 33q^{-1} + 153 + 33q + 3q^2,$$

$$G_{\Delta_4}(0) = q^{-3} + 13q^{-2} + 94q^{-1} + 404 + 94q + 13q^2 + q^3.$$

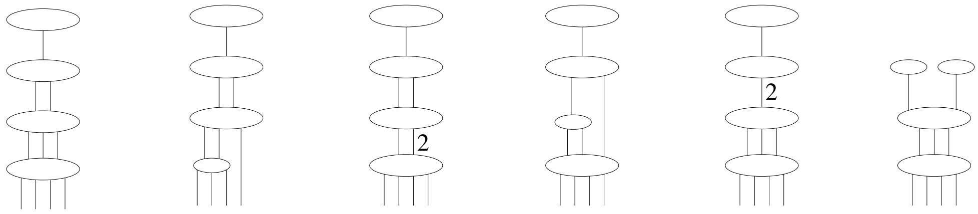


Figure 2.3: Floor diagrams of genus 3 and 2 with Newton polygon Δ_4.

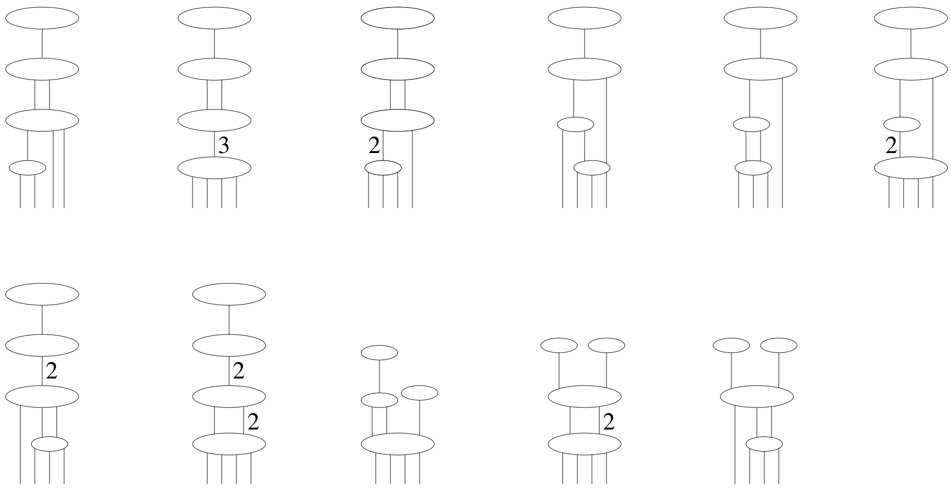


Figure 2.4: Floor diagrams of genus 1 with Newton polygon Δ_4.

2.3. Göttsche-Schroeter refined invariants via floor diagrams

In the case when $g = 0$, one can tweak the notion of marking of a floor diagram in order to compute Göttsche-Schroeter refined invariants.

Definition 2.10. A *pairing of order* s of the set $\mathcal{P} = \{1, \cdots, n\}$ is a set S of s disjoint pairs $\{i, i+1\} \subset \mathcal{P}$.

Given a floor diagram $\mathcal{D}$ and a pairing S of the set $\{1, \cdots, \eta(\mathcal{D})\}$, a marking m of $\mathcal{D}$ is said to be *compatible with* S if for any $\{i, i+1\} \in S$, the set $\{m(i), m(i+1)\}$ consists of one of the following sets (see Figure 2.6):

- an elevator and an adjacent floor;
- two elevators that have a common adjacent floor, from which both are emanating or both are ending.

We generalize the refined multiplicity of a marked floor diagram in the presence of a pairing. Given $(\mathcal{D}, m)$ a marked floor diagram compatible with a pairing S, we define the following sets

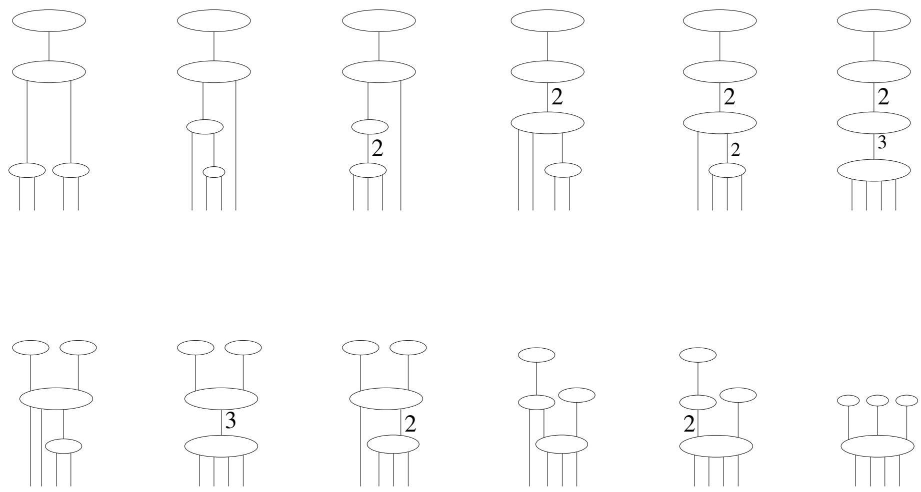


Figure 2.5: Floor diagrams of genus 0 with Newton polygon Δ_4.

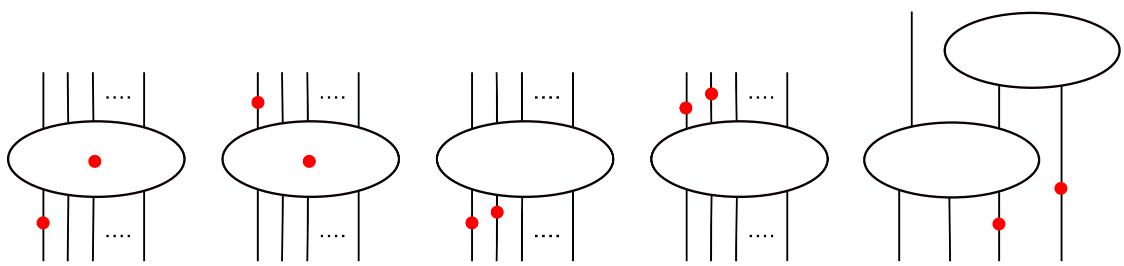


Figure 2.6: Marking and pairing; the red dots corresponds to the image of i and $i+1$.

of elevators of $\mathcal{D}$:

$$E_0 = \{e \in E(\mathcal{D}) \mid e \notin m(S)\};$$
$$E_1 = \{e \in E(\mathcal{D}) \mid \{e, v\} = m(\{i, i+1\}) \text{ with } v \in V(\mathcal{D}) \text{ and } \{i, i+1\} \in S\};$$
$$E_2 = \{\{e, e'\} \subset E(\mathcal{D}) \mid \{e, e'\} = m(\{i, i+1\}) \text{ with } \{i, i+1\} \in S\}.$$

Definition 2.11. The refined S-multiplicity of a marked floor diagram $(\mathcal{D}, m)$ is defined by

$$\mu_S(\mathcal{D}, m)(q) = \prod_{e \in E_0} [\omega(e)]^2(q) \prod_{e \in E_1} [\omega(e)](q^2) \prod_{\{e,e'\} \in E_2} \frac{[\omega(e)] \times [\omega(e')] \times [\omega(e) + \omega(e')]}{[2]}(q)$$

if $(\mathcal{D}, m)$ is compatible with S, and by

$$\mu_S(\mathcal{D}, m)(q) = 0$$

otherwise.

Clearly $\mu_S(\mathcal{D},m)(q)$ is symmetric in $q^{\frac{1}{2}}$, but more can be said.

Lemma 2.12. *For any marked floor diagram $(\mathcal{D},m)$ compatible with a pairing S, one has*

$$\mu_S(\mathcal{D},m)(q) \in \mathbb{Z}_{\geqslant 0}[q^{\pm 1}].$$

Furthermore $\mu_S(\mathcal{D},m)(q)$ has degree $\deg(\mathcal{D})$.

Proof. The degree of $\mu_S(\mathcal{D},m)(q)$ is clear. Next, the factors of $\mu_S(\mathcal{D},m)(q)$ coming from elevators in E_0 and E_1 are clearly in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1}]$. Given a pair $\{e,e'\}$ in E_3, one of the integers $\omega(e),\omega(e')$ or $\omega(e)+\omega(e')$ is even, and the remaining two terms have the same parity. Hence it follows from Lemmas A.1 and A.3 that

$$\frac{[\omega(e)]\times[\omega(e')]\times[\omega(e)+\omega(e')]}{[2]}(q) \in \mathbb{Z}_{\geqslant 0}[q^{\pm 1}],$$

and the lemma is proved. □

Recall that we use the notation $\eta(\Delta)=\mathrm{Card}(\partial\Delta\cap\mathbb{Z}^2)-1$.

Theorem 2.13. *Let Δ be an h-transverse polygon in $\mathbb{R}^2$, and let s be a non-negative integer. Then for any pairing S of order s of $\{1,\cdots,\eta(\Delta)\}$, one has*

$$G_\Delta(0;s)(q) = \sum_{(\mathcal{D},m)} \mu_S(\mathcal{D},m)(q),$$

where the sum runs over all isomorphism classes of marked floor diagrams with Newton polygon Δ and genus 0.

Proof. Given a marked floor diagram $(\mathcal{D},m)$ with Newton polygon Δ, of genus 0, and compatible with S, we construct a marked Psi-floor diagram of type $(\eta(\mathcal{D})-2s,s)$ with a fixed order induced by S on the Psi-powers of the vertices (in the terminology of [BGM12, Definition 4.1 and Remark 4.6]), as depicted in Figure 2.7 and its symmetry with respect to the horizontal axis. This construction clearly establishes a surjection Ψ from the first set of floor diagrams to the second one. Furthermore, given a marked Psi-floor diagram $(\mathcal{D},m')$, all marked floor diagrams such that $\Psi(\mathcal{D},m)=\Psi(\mathcal{D},m')$ are described by the two conditions:

1. $m(\{i,i+1\})=m'(\{i,i+1\})$ if $\{i,i+1\}\in S$;
2. $m(i)=m'(i)$ if i does not belong to any pair in S.

Substituting the integer multiplicity of a Psi-floor diagram from [BGM12] of type (k_0,k_1) by the refined multiplicity from [GS19, Definition 3.1], see also [BS19, Section 2.1], one obtains the result. □

Remark 2.14. Theorem 2.13 implies that the right-hand side term only depends on s, and not on a particular choice of S. This does not look immediate to us. It may be interesting to have a proof of this independence with respect to S which does not go through tropical geometry as in [GS19].

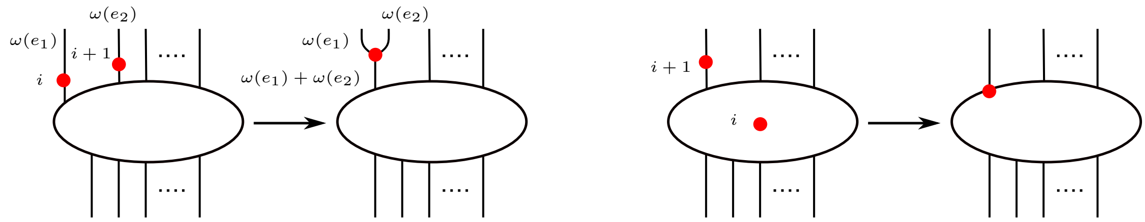


Figure 2.7: From marked floor diagrams to Psi-floor diagrams ($\{i, i+1\} \in S$).

Another type of pairing and multiplicities has been proposed in [BM08] to compute Welschinger invariants $W_{X_\Delta}(\mathcal{L}_\Delta; s)$, when X_Δ is a del Pezzo surface. Note that the multiplicities from [BM08] do not coincide with the refined S-multiplicities defined in Definition 2.11 evaluated at $q=-1$.

Example 2.15. We continue Examples 2.5 and 2.8. All marked floor diagrams of genus 0 and Newton polygon Δ_3 are depicted in Table 2.1. Below each of them, we write the multiplicity μ and the multiplicities μ_{S_i} for $S_i = \{(9-2i, 10-2i), \cdots, (7,8)\}$. The first floor diagram has an elevator of weight 2, but we do not mention it in the picture to avoid confusion. According to Theorem 2.13 we find $G_{\Delta_3}(0; s) = q + 10 - 2s + q^{-1}$. It is interesting to compare this computation with [BM08, Example 3.10].

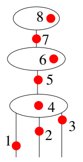
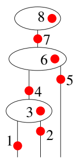
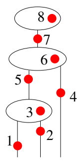
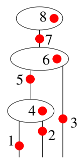
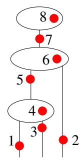
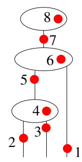
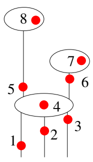
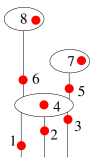
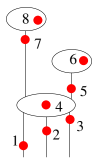
									
μ	$q+2+q^{-1}$	1	1	1	1	1	1	1	1
μ_{S_1}	$q+2+q^{-1}$	1	1	1	1	1	0	0	1
μ_{S_2}	$q+q^{-1}$	1	1	1	1	1	0	0	1
μ_{S_3}	$q+q^{-1}$	1	0	0	1	1	0	0	1
μ_{S_4}	$q+q^{-1}$	1	0	0	0	0	0	0	1

Table 2.1: Computation of $G_{\Delta_3}(0; s)$.

The following proposition states that the decreasing of $\mu_S(\mathcal{D}, m)$ with respect to S that one observes in Table 2.1 is actually a general phenomenon. Given two elements $f, g \in \mathbb{Z}_{\geqslant 0}[q^{\pm 1}]$, we write $f \geqslant g$ if $f - g \in \mathbb{Z}_{\geqslant 0}[q^{\pm 1}]$.

Proposition 2.16. *Let $(\mathcal{D}, m)$ be a marked floor diagram of genus 0, and $S_1 \subset S_2$ be two pairings of the set $\{1, \cdots, \eta(\mathcal{D})\}$. Then one has*

$$\mu_{S_1}(D, m)(q) \geqslant \mu_{S_2}(D, m)(q).$$

Proof. Since $\mu_{S_1}(D, m) \in \mathbb{Z}_{\geqslant 0}[q^{\pm 1}]$, the result obviously holds if $\mu_{S_2}(D, m) = 0$. If $\mu_{S_2}(D, m) \neq 0$, then the result follows from Corollary A.4, and from the inequality

$$[k](q^2) \leqslant [2k-1](q) \leqslant [k]^2(q),$$

the last inequality holding by Lemma A.1. □

The next corollary generalizes [Bru20, Corollary 4.5] to arbitrary h-transverse polygon. Recall that we use the notation

$$s_{max} = \left[\frac{\eta(\Delta)}{2}\right].$$

Corollary 2.17. *For any h-transverse polygon Δ in $\mathbb{R}^2$ and any $i \in \mathbb{Z}_{\geqslant 0}$, one has*

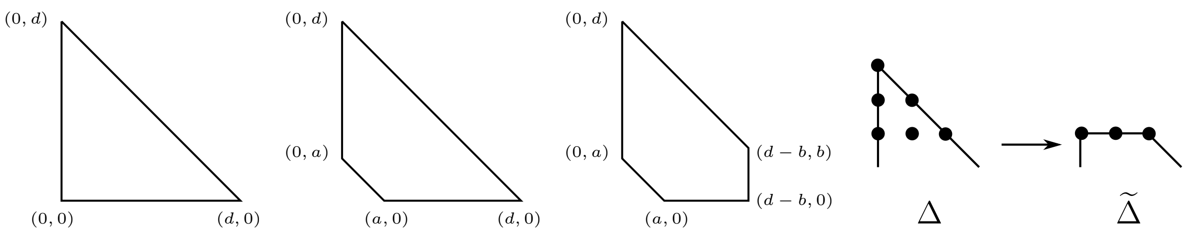
$$\langle G_\Delta(0;0)\rangle_i \geqslant \langle G_\Delta(0;1)\rangle_i \geqslant \langle G_\Delta(0;2)\rangle_i \geqslant \cdots \geqslant \langle G_\Delta(0;s_{max})\rangle_i \geqslant 0.$$

Proof. Since $\mu_S(D,m) \in \mathbb{Z}_{\geqslant 0}[q^{\pm 1}]$ for any marked floor diagram $(\mathcal{D},m)$ and any pairing S, we have that $\langle G_\Delta(0;s)\rangle_i \geqslant 0$ for any s. The decreasing of the sequence $(\langle G_\Delta(0;s)\rangle_i)_s$ is a direct consequence of Proposition 2.16 and Theorem 2.13. □

Example 2.18. Thanks to Figure 2.5, one can compute:

$$\begin{array}{lclclclclclclcl}
G_{\Delta_4}(0;0) & = & q^{-3} & + & 13q^{-2} & + & 94q^{-1} & + & 404 & + & 94q & + & 13q^2 & + & q^3\\
G_{\Delta_4}(0;1) & = & q^{-3} & + & 11q^{-2} & + & 70q^{-1} & + & 264 & + & 70q & + & 11q^2 & + & q^3\\
G_{\Delta_4}(0;2) & = & q^{-3} & + & 9q^{-2} & + & 50q^{-1} & + & 164 & + & 50q & + & 9q^2 & + & q^3\\
G_{\Delta_4}(0;3) & = & q^{-3} & + & 7q^{-2} & + & 34q^{-1} & + & 96 & + & 34q & + & 7q^2 & + & q^3\\
G_{\Delta_4}(0;4) & = & q^{-3} & + & 5q^{-2} & + & 22q^{-1} & + & 52 & + & 22q & + & 5q^2 & + & q^3\\
G_{\Delta_4}(0;5) & = & q^{-3} & + & 3q^{-2} & + & 14q^{-1} & + & 24 & + & 14q & + & 3q^2 & + & q^3
\end{array}$$

A particular case of Corollary 2.17 has first been proved in [Bru20] using the next proposition. For the sake of brevity, the proof of [Bru20, Proposition 4.3] has been omitted there. We close this gap here.



a) $d \geqslant 2$ b) $d-a \geqslant 2$ c) $d-\max(a,b) \geqslant 2$ d)

Figure 2.8

Proposition 2.19 ([Bru20, Proposition 4.3])**.** *Let Δ be one of the integer polygons depicted in Figures 2.8a),b), or c), and let $\widetilde{\Delta}$ be the integer polygon obtained by chopping off the top of Δ as depicted in Figure 2.8d). If $2s \leqslant \eta(\Delta) - 2$, then one has*

$$G_\Delta(0;s+1) = G_\Delta(0;s) - 2G_{\widetilde{\Delta}}(0;s).$$

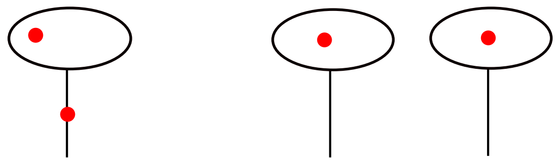


Figure 2.9: A partition of the set of marked floor diagrams, the red dots represent $m(\eta(\Delta)-1)$ and $m(\eta(\Delta))$.

Proof. Let S be a pairing of order s of the set $\{1,\cdots,\eta(\Delta)-2\}$, and let $\widetilde{S}=S\cup\{\eta(\Delta)-1,\eta(\Delta)\}$. Let $A\sqcup B$ be the partition of the set of marked floor diagrams $(\mathcal{D},m)$ with Newton polygon Δ and genus 0 such that

$$(\mathcal{D},m)\in B \Longleftrightarrow m(\{\eta(\Delta)-1,\eta(\Delta)\})\subset V(\mathcal{D}),$$

see Figure 2.9. Then by Theorem 2.13, one has

$$G_\Delta(0;s)(q)=\sum_{(\mathcal{D},m)\in A\cup B}\mu_S(\mathcal{D},m)(q) \qquad \text{and} \qquad G_\Delta(0;s+1)(q)=\sum_{(\mathcal{D},m)\in A}\mu_{\widetilde{S}}(\mathcal{D},m)(q).$$

By chopping off the two top floors of the floor diagrams from the set B, it follows again from Theorem 2.13 that

$$G_{\widetilde{\Delta}}(0;s)(q)=\frac{1}{2}\sum_{(\mathcal{D},m)\in B}\mu_S(\mathcal{D},m)(q).$$

Since the divergence of any top floor of $\mathcal{D}$ is 1, we have that $\mu_{\widetilde{S}}(\mathcal{D},m)(q)=\mu_S(\mathcal{D},m)(q)$ for any marked floor diagram $(\mathcal{D},m)\in A$. Hence we have

$$\begin{aligned}G_\Delta(0;s+1)(q)&=\sum_{(\mathcal{D},m)\in A}\mu_S(\mathcal{D},m)(q)\\&=\sum_{(\mathcal{D},m)\in A\cup B}\mu_S(\mathcal{D},m)(q)-\sum_{(\mathcal{D},m)\in B}\mu_S(\mathcal{D},m)(q),\end{aligned}$$

which concludes the proof. □

3. Codegrees

3.1. Codegree of a floor diagram

Recall that we use the notation

$$\iota_\Delta=\mathrm{Card}(\mathrm{Int}(\Delta)\cap\mathbb{Z}^2).$$

We define the *codegree* of a floor diagram $\mathcal{D}$ of genus g with Newton polygon Δ by

$$\mathrm{codeg}(\mathcal{D})=\iota_\Delta-g-\deg(\mathcal{D}).$$

By [IM13, Proposition 2.11], one has $\deg(\mathcal{D})\leqslant\iota_\Delta-g$, and so $\mathrm{codeg}(\mathcal{D})\geqslant 0$. Furthermore, the codegree of $\mathcal{D}$ is zero if and only if

- the order $\preccurlyeq$ is total on the set of floors of $\mathcal{D}$;
- any edge in $E^0(\mathcal{D})$ connects two consecutive vertices;
- elevators in $E^{-\infty}(\mathcal{D})$ are all adjacent to the minimal floor of $\mathcal{D}$, and elevators in $E^{+\infty}(\mathcal{D})$ are all adjacent to the maximal floor of $\mathcal{D}$;
- the function $l : V(\mathcal{D}) \to d_l$ is decreasing, and the function $r : V(\mathcal{D}) \to d_r$ is increasing.

With this characterization, one sees easily that there exists exactly $\binom{\iota_\Delta}{g}$ marked floor diagrams of genus g with Newton polygon Δ and codegree 0 (see [IM13, Proposition 2.11] and [BG16b, Proposition 4.10]).

Example 3.1. Figures 2.2a) and b) depict the only floor diagrams of codegree 0 with Newton polygon Δ_3 and genus 1 and 0, respectively. The only codegree 0 floor diagram with Newton polygon depicted in Figure 2.1b) and genus 0 is depicted in Figure 3.1a). All codegree 0 floor diagrams with Newton polygon depicted in Figure 2.1b) and genus 1 are depicted in Figures 3.1b), c), d), e), f), and g). Note that the floor diagram depicted in Figure 3.1b) admits a single marking, while the floor diagrams depicted in Figures 3.1c), d), e), f), and g) admit exactly two different markings.

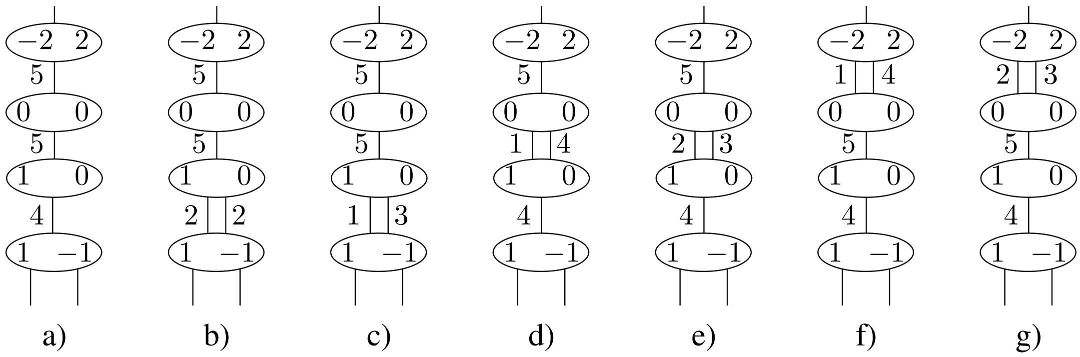


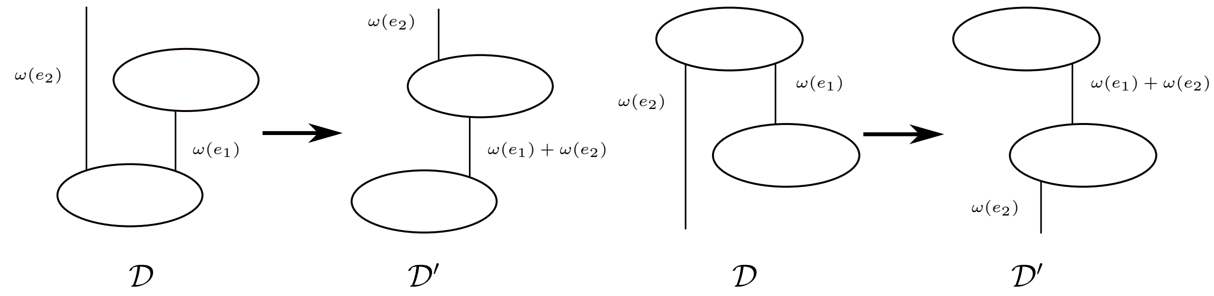
Figure 3.1: Codegree 0 floor diagrams of genus 0 and 1 with Newton polygon from Figure 2.1b).

Throughout the remainder of the text, we will make an extensive use of the following four operations on a floor diagram $\mathcal{D}$:

A^+: Suppose that there exist two floors v_1 and v_2 of $\mathcal{D}$ connected by an elevator e_1 from v_1 to v_2, and an additional elevator e_2 originating from v_1 but not adjacent to v_2. Then construct a new floor diagram $\mathcal{D}'$ out of $\mathcal{D}$ as depicted in Figure 3.2a).

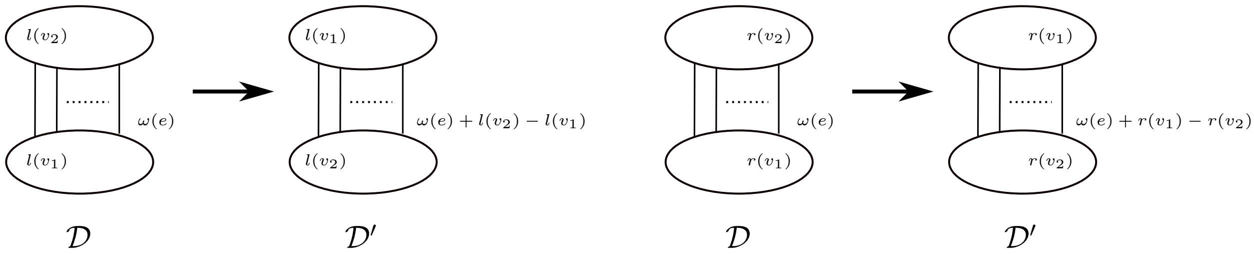
A^-: Suppose that there exist two floors v_1 and v_2 of $\mathcal{D}$ connected by an elevator e_1 from v_1 to v_2, and an additional elevator e_2 ending at v_2 but not adjacent to v_1. Then construct a new floor diagram $\mathcal{D}'$ out of $\mathcal{D}$ as depicted in Figure 3.2b).

B^l: Suppose that there exist two consecutive floors $v_1 \preccurlyeq v_2$ of $\mathcal{D}$ such that $l(v_1) < l(v_2)$. Then construct a new floor diagram $\mathcal{D}'$ out of $\mathcal{D}$ as depicted in Figure 3.3a), where e is any elevator adjacent to v_1 and v_2.



a) Operation A^+ b) Operation A^-

Figure 3.2: Operations A on floor diagrams.



a) Operation B^l b) Operation B^r

Figure 3.3: Operations B on floor diagrams.

B^r: Suppose that there exist two consecutive floors $v_1 \preccurlyeq v_2$ of $\mathcal{D}$ such that $r(v_1) > r(v_2)$. Then construct a new floor diagram $\mathcal{D}'$ out of $\mathcal{D}$ as depicted in Figure 3.3b), where e is any elevator adjacent to v_1 and v_2.

The following lemma is straightforward.

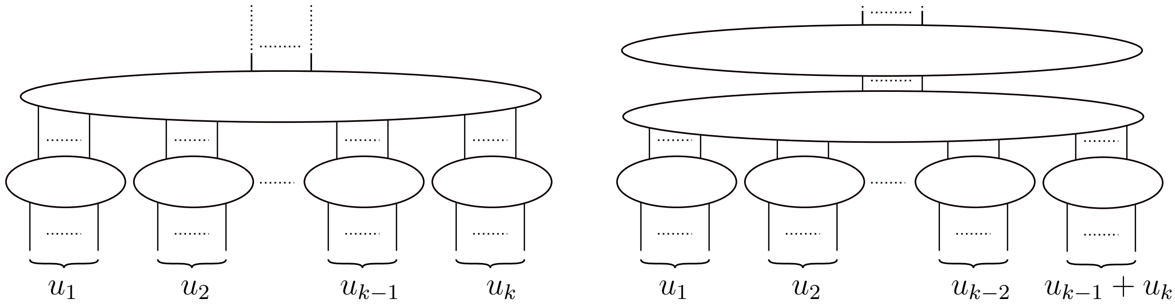
Lemma 3.2. *Genus and Newton polygon are invariant under operations $A^\pm$, B^l, and B^r. Furthermore, the codegree decreases by $w(e_2)$ under operations $A^\pm$, by $l(v_2) - l(v_1)$ under operations B^l, and by $r(v_1) - r(v_2)$ under operations B^r.*

The next lemma is an example of application of Lemma 3.2. For the sake of simplicity, we state and prove it only for floor diagrams with constant divergence. Generalizing it to floor diagrams with any h-transverse Newton polygon presents no difficulties other than technical.

Lemma 3.3. *Let $\mathcal{D}$ be a floor diagram with constant divergence $n \in \mathbb{Z}$. If $\mathcal{D}$ has k minimal floors, then one has that*

$$\operatorname{codeg}(\mathcal{D}) \geqslant (k-1)\left(\operatorname{Card}(E^{-\infty}(\mathcal{D})) - n\frac{k}{2}\right).$$

Proof. Denote by $v_1, \cdots, v_k$ these minimal floors, and by u_i the number of elevators in $E^{-\infty}(\mathcal{D})$ to which v_i is adjacent. By a finite succession of operations A^- and applications of Lemma 3.2,



a) $\mathcal{D}$ b) $\mathcal{D}'$

Figure 3.4

we may assume that

$$\sum_{i=1}^{k} u_i = \mathrm{Card}(E^{-\infty}(\mathcal{D})).$$

Next, by a finite succession of operations $A^{\pm}$ and applications of Lemma 3.2, we may assume that there exists $v \in V(\mathcal{D})$ greater than all floors $v_1, \cdots, v_k$, and such that any elevator in $E(\mathcal{D}) \setminus E^{-\infty}(\mathcal{D})$ adjacent to v_i is also adjacent to v, see Figure 3.4a). This implies in particular that if $e_{i,1}, \cdots, e_{i,k_i}$ are the elevators in $E^0(\mathcal{D})$ adjacent to v_i, then one has

$$\sum_{j=1}^{k_i} \omega(e_{i,j}) = u_i - n.$$

By a finite succession of operations A^- and applications of Lemma 3.2, we now construct a floor diagram $\mathcal{D}'$ with $k-1$ minimal floors and satisfying (see Figure 3.4b)

$$\mathrm{codeg}(\mathcal{D}) = \mathrm{codeg}(\mathcal{D}') + \mathrm{Card}(E^{-\infty}(\mathcal{D})) - n(k-1).$$

Now the result follows by induction on k. □

3.2. Degree of codegree coefficients

Here we prove a couple of intermediate results regarding the degree of codegree i coefficients of some families of Laurent polynomials. Given two integers $k, l \geqslant 0$, we define

$$F(k,l) = \sum_{\substack{i_1+i_2+\cdots+i_k=l \\ i_1,\cdots,i_k \geqslant 1}} \prod_{j=1}^{k} i_j \qquad \text{and} \qquad \Phi_l(k) = F(k, k+l).$$

Example 3.4. One computes easily that

$$\Phi_0(k) = 1 \qquad \text{and} \qquad \Phi_1(k) = 2k.$$

Lemma 3.5. *For any fixed $l \in \mathbb{Z}_{\geqslant 0}$, the function $\Phi_l : k \in \mathbb{Z}_{\geqslant 0} \mapsto \Phi_l(k)$ is polynomial of degree l.*

Proof. The proof goes by induction on l. The case $l = 0$ is covered by Example 3.4. Now suppose that $l \geqslant 1$ and that the lemma holds up to $l-1$. For $l \geqslant k$, one has

$$\begin{aligned} F(k,l) &= \sum_{i_1=1}^{l-k+1} i_1 \sum_{\substack{i_2+\cdots+i_k=l-i_1 \\ i_2,\cdots,i_k\geqslant 1}} \prod_{j=2}^{k} i_j \\ &= \sum_{i_1=1}^{l-k+1} i_1\, F(k-1, l-i_1), \end{aligned}$$

and so

$$\begin{aligned} \Phi_l(k) &= F(k, k+l) \\ &= \sum_{i_1=1}^{l+1} i_1\, F(k-1, k+l-i_1) \\ &= \sum_{i_1=1}^{l+1} i_1\, \Phi_{l-i_1+1}(k-1) \\ &= \Phi_l(k-1) + \sum_{i_1=2}^{l+1} i_1\, \Phi_{l-i_1+1}(k-1). \end{aligned}$$

By induction on l, the function $P_l : k \mapsto \Phi_l(k) - \Phi_l(k-1)$ is then polynomial of degree $l-1$. Since $\Phi_l(0) = F(0,l) = 0$, one has

$$\begin{aligned} \Phi_l(k) &= \sum_{i=0}^{k-1} \left(\Phi_l(k-i) - \Phi_l(k-(i+1))\right) \\ &= \sum_{i=1}^{k} P_l(i). \end{aligned}$$

By Faulhaber's formula, the function $\Phi_l(k)$ is polynomial of degree l, and the proof is complete. □

The next corollaries constitute key steps in our polynomiality proofs. Recall that the notation $\langle P \rangle_i$ denotes the the coefficient of codegree i of a Laurent polynomial $P(q)$.

Corollary 3.6. *Let $i, k \geqslant 0$ and $a_1, \cdots, a_k > i$ be integers. Then one has*

$$\left\langle \prod_{j=1}^{k} [a_j]^2 \right\rangle_i = \Phi_i(k).$$

In particular, the function $(k, a_1, \cdots, a_k) \mapsto \left\langle \prod_{j=1}^{k} [a_j]^2 \right\rangle_i$ only depends on k on the set $\{a_1 > i, \cdots, a_k > i\}$, and is polynomial of degree i.

Proof. Since $\langle [a]^2 \rangle_i = i + 1$ if $a > i$, one has

$$\begin{aligned}\left\langle \prod_{j=1}^{k} [a_j]^2 \right\rangle_i &= \sum_{\substack{i_1+i_2+\cdots+i_k=i \\ i_1,\ldots,i_k \geqslant 0}} \prod_{j=1}^{k} \langle [a_j]^2 \rangle_{i_j} \\ &= \sum_{\substack{i_1+i_2+\cdots+i_k=i \\ i_1,\ldots,i_k \geqslant 0}} \prod_{j=1}^{k} (i_j + 1) \\ &= \sum_{\substack{i_1+i_2+\cdots+i_k=i+k \\ i_1,\ldots,i_k \geqslant 1}} \prod_{j=1}^{k} i_j \\ &= \Phi_i(k),\end{aligned}$$

as announced. □

Corollary 3.7. *Let $P(q)$ be a Laurent polynomial, and $i \geqslant 0$ an integer. Then the function $(k, a_1, \cdots, a_k) \mapsto \left\langle P(q) \times \prod_{j=1}^{k} [a_j]^2 \right\rangle_i$ only depends on k on the set $\{a_1 > i, \cdots, a_k > i\}$, and is polynomial of degree i.*

Proof. One has

$$\left\langle P(q) \times \prod_{j=1}^{k} [a_j]^2 \right\rangle_i = \sum_{\substack{i_1+i_2=i \\ i_1,i_2 \geqslant 0}} \langle P(q) \rangle_{i_1} \times \left\langle \prod_{j=1}^{k} [a_j]^2 \right\rangle_{i_2}.$$

The statement now follows from Corollary 3.6. □

4. The genus 0 case

4.1. Proof of Theorem 1.5

The main step is Lemma 4.1 below. It can be summarized as follows: for (a, b, n, s) satisfying the condition from Theorem 1.5, all floor diagrams of codegree at most i can easily be described. Then Theorem 1.5 simply follows from an explicit computation of the multiplicity and the number of markings of each such floor diagram.

Given $i \in \mathbb{Z}_{\geqslant 0}$, and $(u, \widetilde{u}) \in \mathbb{Z}^i_{\geqslant 0} \times \mathbb{Z}^i_{\geqslant 0}$, we define

$$\operatorname{codeg}(u, \widetilde{u}) = \sum_{j=1}^{i} j\ (u_j + \widetilde{u}_j),$$

and we consider the finite set

$$C_i = \left\{ (u, \widetilde{u}) \in \mathbb{Z}^i_{\geqslant 0} \times \mathbb{Z}^i_{\geqslant 0} \mid \operatorname{codeg}(u, \widetilde{u}) \leqslant i \right\}.$$

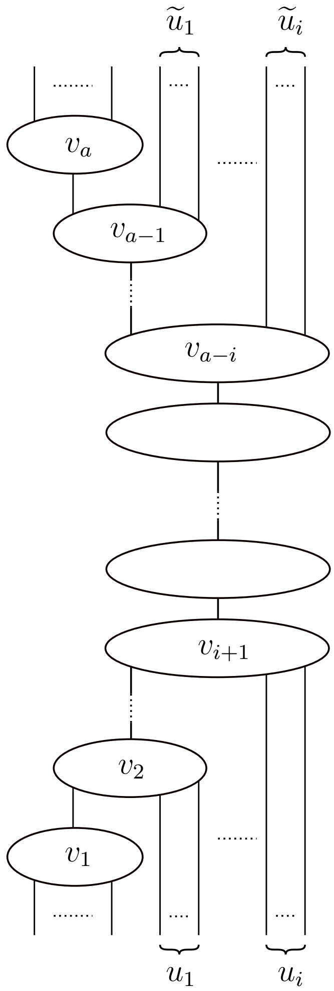


Figure 4.1: The floor diagram $\mathcal{D}_{a,b,n,u,\widetilde{u}}$.

For $(u,\widetilde{u}) \in C_i$, and integers $b, n \geqslant 0$, and $a > i$, we denote by $\mathcal{D}_{a,b,n,u,\widetilde{u}}$ the floor diagram of genus 0 and Newton polygon $\Delta_{a,b,n}$ depicted in Figure 4.1 (we do not specify the weight on elevators in $E^0(\mathcal{D}_{a,b,n,u,\widetilde{u}})$ there since they can be recovered from a, b, n, u, and $\widetilde{u}$). In particular the partial ordering $\preccurlyeq$ on $\mathcal{D}_{a,b,n,u,\widetilde{u}}$ induces a total ordering on its floors

$$v_1 \prec \cdots \prec v_a.$$

Note that $\widetilde{u}_k = 0$ (resp. $u_k = 0$) for $k > i - j$ as soon as $u_j \neq 0$ (resp. $\widetilde{u}_j \neq 0$).

Lemma 4.1. *Let $i, n \in \mathbb{Z}_{\geqslant 0}$, and let $\mathcal{D}$ be a floor diagram of genus 0 with Newton polygon $\Delta_{a,b,n}$ with a, b, and i satisfying*

$$\begin{cases} b > i \\ a > i \end{cases}.$$

Then one has

$$\operatorname{codeg}(\mathcal{D}) \leqslant i \iff \exists (u,\widetilde{u}) \in C_i,\ \mathcal{D} = \mathcal{D}_{a,b,n,u,\widetilde{u}}.$$

Furthermore in this case, any elevator $e \in E^0(\mathcal{D})$ satisfies $\omega(e) > i - \operatorname{codeg}(\mathcal{D})$.

Proof. Given $(u,\widetilde{u})\in C_i$, one has $\operatorname{codeg}(\mathcal{D}_{a,b,n,u,\widetilde{u}})=\sum_{j=1}^{i} j\ (u_j+\widetilde{u}_j)$ by a finite succession of operations $A^{\pm}$ and applications of Lemma 3.2.

Let $\mathcal{D}$ be of codegree at most i, and suppose that the order $\preccurlyeq$ is not total on the set of floors of $\mathcal{D}$. Since $\mathcal{D}$ is a tree, this is equivalent to say that there exist at least two minimal or two maximal floors for $\preccurlyeq$. Denote by k_t and k_b the number of maximal and minimal floors of $\mathcal{D}$, respectively.

By Lemma 3.3 applied to the orthogonal symmetric of the polygon $\Delta_{a,b,n}$ with respect to the x-axis, one has

$$\operatorname{codeg}(\mathcal{D})\geqslant (k_t-1)\ \left(b+n\frac{k_t}{2}\right).$$

Hence $k_t\geqslant 2$ implies that

$$\operatorname{codeg}(\mathcal{D})\geqslant b+n>i,$$

contrary to our assumption.

Analogously, by Lemma 3.3, one has that

$$\operatorname{codeg}(\mathcal{D})\geqslant (k_b-1)\ \left(\left(a-\frac{k_b}{2}\right)n+b\right).$$

Since $k_b\leqslant a-1$, one deduces that $a-\frac{k_b}{2}\geqslant 1$. Hence $k_b\geqslant 2$ implies that

$$\operatorname{codeg}(\mathcal{D})\geqslant b+n>i,$$

contrary to our assumption. Hence we proved that the order $\preccurlyeq$ is total on the set of floors of $\mathcal{D}$.

Denoting by u_j (resp. $\widetilde{u}_j$) the number of elevators in $E^{-\infty}(\mathcal{D})$ (resp. $E^{+\infty}(\mathcal{D})$) adjacent to the floor v_{j+1} (resp. v_{a-j}), we then have $\mathcal{D}=\mathcal{D}_{a,b,n,u,\widetilde{u}}$. Since

$$\operatorname{codeg}(\mathcal{D})=\sum_{j=1}^{a-1} j(u_j+\widetilde{u}_j),$$

we deduce that $(u,\widetilde{u})\in C_i$.

To end the proof of the lemma, just note that the elevator in $E^0(\mathcal{D})$ with the lowest weight is either one of the elevators adjacent to the floors v_k and v_{k+1}, with $1\leqslant k\leqslant i$, or the highest one for $\preccurlyeq$. The former has weight at least

$$(a-k)n+b-\sum_{j=k}^{i} u_j\geqslant b-\operatorname{codeg}(\mathcal{D})>i-\operatorname{codeg}(\mathcal{D}),$$

while the latter has weight at least $n+b-\sum_{j=1}^{i}\widetilde{u}_j>i-\operatorname{codeg}(\mathcal{D})$. □

Let us now count the number of markings of the floor diagram $\mathcal{D}_{a,b,n,u,\widetilde{u}}$. Given $(u,\widetilde{u})\in C_i$, we define the functions

$$\widetilde{\nu}_u(a,b,n,s)=$$

$$\sum_{s_0+s_1+\cdots+s_i=s}\frac{s!}{s_0!s_1!\cdots s_i!}\prod_{j=1}^{i}\binom{an+b+2j-2s_0-2s_1-\cdots-2s_j-u_{j+1}-\cdots-u_i}{u_j-2s_j}$$

and

$$\nu_{u,\widetilde{u}}(a,b,n,s) = \widetilde{\nu}_u(a,b,n,s) \times \widetilde{\nu}_{\widetilde{u}}(0,b,0,0).$$

Lemma 4.2. *If $(u,\widetilde{u}) \in C_i$ and (a,b,n,s) is an element of the subset of $\mathbb{Z}^4_{\geqslant 0}$ defined by*

$$\begin{cases} b \geqslant i \\ an+b \geqslant i+2s \end{cases},$$

then $\nu_{u,\widetilde{u}}(a,b,n,s)$ is the number of markings of the floor diagram $\mathcal{D}_{a,b,n,u,\widetilde{u}}$ that are compatible with the pairing $\{\{1,2\},\{3,4\},\cdots,\{2s-1,2s\}\}$. Furthermore the function $(a,b,n,s) \mapsto \nu_{u,\widetilde{u}}(a,b,n,s)$ is polynomial on this set, and has degree at most $\sum_{j=1}^{i}(u_j+\widetilde{u}_j)$ in each variable. If $(u,\widetilde{u}) = ((i,0,\cdots,0),0)$, then the degree in each variable is exactly i.

Proof. Recall that $u_j \neq 0$ (resp. $\widetilde{u}_j \neq 0$) implies that $\widetilde{u}_k = 0$ (resp. $u_k = 0$) for $k > i-j$. Next, if $an+b \geqslant i+2s$, then any marking m of $\mathcal{D}_{a,b,n,u,\widetilde{u}}$ satisfies $m(j) \in E^{-\infty}(\mathcal{D}_{a,b,n,u,\widetilde{u}})$ if $j \leqslant 2s$. From these two observations, it is straightforward to compute the number of markings of $\mathcal{D}_{a,b,n,u,\widetilde{u}}$ compatible with $\{\{1,2\},\cdots,\{2s-1,2s\}\}$. This proves the first assertion of the lemma.

To prove the second assertion, notice that the number of possible values of $s_1,\cdots,s_i$ giving rise to a non-zero summand of $\widetilde{\nu}_u(a,b,n,s)$ is finite and only depends on the vector u. Hence this assertion follows from the fact that, for such a fixed choice of $s_1,\cdots,s_i$, the function

$$(a,b,n,s) \longmapsto \frac{s!}{s_0!s_1!\cdots s_i!}\prod_{j=1}^{i}\binom{an+b+2j-2s_0-2s_1-\cdots-2s_j-u_{j+1}-\cdots-u_i}{u_j-2s_j}$$

is polynomial as soon as $an+b \geqslant i+2s$, of degree at most $\sum_{j=1}^{i}(u_j-2s_j)$ in the variables a, b, and n, and of degree at most $\sum_{j=1}^{i}(u_j-s_j)$ in the variable s. The third assertion also follows from this computation. □

Theorem 4.3. *For any $i \in \mathbb{Z}_{\geqslant 0}$, and any (a,b,n,s) in the set $\mathcal{U}_i \subset \mathbb{Z}^4_{\geqslant 0}$ defined by*

$$\begin{cases} an+b \geqslant i+2s \\ b > i \\ a > i \end{cases},$$

one has

$$\left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_i = \sum_{(u,\widetilde{u})\in C_i} \nu_{u,\widetilde{u}}(a,b,n,s) \times \Phi_{i-\mathrm{codeg}(u,\widetilde{u})}(a-1).$$

In particular, the function

$$\begin{array}{ccc} \mathcal{U}_i & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\ (a,b,n,s) & \longmapsto & \left\langle G_{\Delta_{a,b,n}}(0;s)\right\rangle_i \end{array}$$

is polynomial of degree i in each variable.

Proof. Let $(a,b,n,s) \in \mathcal{U}_i$. Since $an + b \geqslant i + 2s$, any marking m of $\mathcal{D}_{a,b,n,u,\widetilde{u}}$ satisfies $m(j) \in E^{-\infty}(\mathcal{D}_{a,b,n,u,\widetilde{u}})$ if $j \leqslant 2s$. In particular, one has

$$\mu_{\{\{1,2\},\cdots,\{2s-1,2s\}\}}(\mathcal{D}_{a,b,n,u,\widetilde{u}}, m) = \mu(\mathcal{D}_{a,b,n,u,\widetilde{u}})$$

for any marking m of $\mathcal{D}_{a,b,n,u,\widetilde{u}}$ compatible with the pairing $\{\{1,2\},\{3,4\},\cdots,\{2s-1,2s\}\}$. Lemma 4.1 and Corollary 3.6 give

$$\langle \mu(\mathcal{D}_{a,b,n,u,\widetilde{u}}) \rangle_{i-\operatorname{codeg}(u,\widetilde{u})} = \Phi_{i-\operatorname{codeg}(u,\widetilde{u})}(a-1).$$

By Lemma 4.2, one has then

$$\begin{aligned}
\left\langle G_{\Delta_{a,b,n}}(0;s) \right\rangle_i &= \left\langle \sum_{(u,\widetilde{u}) \in C_i} \nu_{u,\widetilde{u}}(a,b,n,s) \times \mu(\mathcal{D}_{a,b,n,u,\widetilde{u}}) \right\rangle_i \\
&= \sum_{(u,\widetilde{u}) \in C_i} \nu_{u,\widetilde{u}}(a,b,n,s) \times \langle \mu(\mathcal{D}_{a,b,n,u,\widetilde{u}}) \rangle_{i-\operatorname{codeg}(u,\widetilde{u})} \\
&= \sum_{(u,\widetilde{u}) \in C_i} \nu_{u,\widetilde{u}}(a,b,n,s) \times \Phi_{i-\operatorname{codeg}(u,\widetilde{u})}(a-1).
\end{aligned}$$

Hence Corollary 3.6 and Lemma 4.2 imply that the function $(a,b,n,s) \in \mathcal{U}_i \mapsto \left\langle G_{\Delta_{a,b,n}}(0;s) \right\rangle_i$ is polynomial. Furthermore, its degree in b,n and s is i, since it is the maximal degree of a function $\nu_{u,\widetilde{u}}$. The degree in the variable a of $\nu_{u,\widetilde{u}}(a,b,n,s) \times \Phi_{i-\operatorname{codeg}(u,\widetilde{u})}(a-1)$ is at most

$$i - \operatorname{codeg}(u,\widetilde{u}) + \sum_{j=1}^{i} (u_j + \widetilde{u}_j) = i - \sum_{j=2}^{i} (j-1)\,(u_j + \widetilde{u}_j).$$

Hence this degree is at most i, with equality if $u = \widetilde{u} = 0$. □

4.2. $b = 0$ and n fixed

Here we explain how to modify the proof of Theorem 4.3 in the case when one wants to fix $b = 0$ and $n \geqslant 1$. This covers in particular the case of $X_{\Delta_d} = \mathbb{C}P^2$. The difference with Section 4.1 is that now a floor diagram $\mathcal{D}$ contributing to $\left\langle G_{\Delta_{a,0,n}}(0;s) \right\rangle_i$ may have several maximal floors for the order $\preccurlyeq$. Nevertheless for fixed n and i, we show that the set of possible configurations of these multiple maximal floors is finite and does not depend on a. In order to do so, we introduce the notion of *capping tree*.

Definition 4.4. A *capping tree* with Newton polygon $\Delta_{a,n}$ is a couple $\mathcal{T} = (\Gamma, \omega)$ such that

1. Γ is a connected weighted oriented tree with a floors and with no sources nor sinks;
2. Γ has a unique minimal floor v_1, and $\Gamma \setminus \{v_1\}$ is not connected;
3. for every floor $v \in V(\Gamma) \setminus \{v_1\}$, one has $\operatorname{div}(v) = n$.

Figure 4.2: Two examples of capping trees of codegree 2.

The codegree of a capping tree $\mathcal{T}$ with Newton polygon $\Delta_{a,n}$ is defined as

$$\operatorname{codeg}(\mathcal{T}) = \frac{(a-1)(na-2)}{2} - \sum_{e\in E(\mathcal{T})} (\omega(e)-1)$$

Example 4.5. Examples of capping trees of codegree 2 and with Newton polygon $\Delta_{4,1}$ and $\Delta_{3,2}$ are depicted in Figure 4.2. We use the same convention to depict capping trees as to depict floor diagrams.

Lemma 4.6. *A capping tree with Newton polygon $\Delta_{a,n}$ has codegree at least $n(a-2)$.*

Proof. Let $\mathcal{T}$ be such a capping tree, and denote by $\omega_1, \cdots, \omega_k$ the weight of the elevators of $\mathcal{T}$ adjacent to v_1, and by $a_1, \cdots, a_k$ the number of floors of the corresponding connected component of $\mathcal{T} \setminus \{v_1\}$. By Definition 4.4, one has $\omega_j = na_j$. By a finite succession of operations A^+ and applications of Lemma 3.2, we reduce the proof successively to the case when

1. $\preccurlyeq$ induces a total order on each connected component of $\mathcal{T} \setminus \{v_1\}$;
2. $k = 2$.

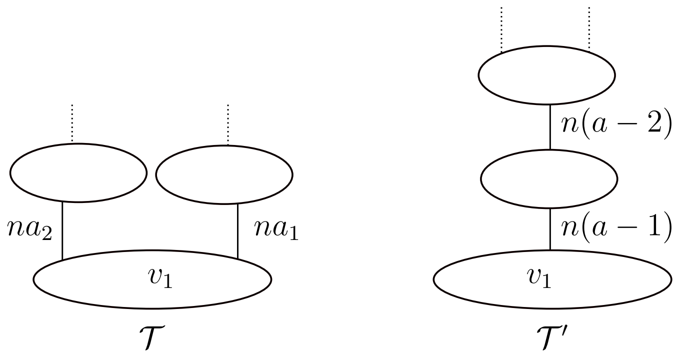


Figure 4.3: Bounding $\operatorname{codeg}(\mathcal{T})$ from below.

By two additional operations A^+, we construct a capping tree $\mathcal{T}'$ such that (see Figure 4.3)

$$\operatorname{codeg}(\mathcal{T}) \geqslant \operatorname{codeg}(\mathcal{T}') + n(a_1 + a_2 - 1) = \operatorname{codeg}(\mathcal{T}') + n(a-2).$$

This proves the lemma since $\operatorname{codeg}(\mathcal{T}') \geqslant 0$. $\square$

Proof of Theorem 1.6. Let $\mathcal{D}$ be a floor diagram of genus 0, with Newton polygon $\Delta_{a,0,n}$, and of codegree at most i. Suppose that $\mathcal{D}$ has $k_b \geqslant 2$ minimal floors for $\preccurlyeq$. Then exactly as in the proof of Lemma 4.1, we have that

$$\operatorname{codeg}(\mathcal{D}) \geqslant n(k_b - 1)\left(a - \frac{k_b}{2}\right) \geqslant n(a-1) \geqslant n(i+1) > i.$$

This contradicts our assumptions, and $k_b = 1$.

Suppose that $\mathcal{D}$ has at least two maximal floors. Denote by v_o the lowest floor of $\mathcal{D}$ having at least two adjacent outgoing elevators. Since $k_b = 1$, the order $\preccurlyeq$ induces a total ordering on floors v of $\mathcal{D}$ such that $v \preccurlyeq v_o$. Let $\mathcal{T}$ be the weighted subtree of $\mathcal{D}$ obtained by removing from $\mathcal{D}$ all elevators and floors strictly below v_o, and denote by a_o the number of floors of $\mathcal{T}$. Suppose that $\mathcal{T}$ is not a capping tree, i.e. $E^{-\infty}(\mathcal{T}) \neq \varnothing$. By a finite succession of A^- operations, we construct a floor diagram $\mathcal{D}'$ with the same floors as $\mathcal{D}$, the same elevators as well, except for elevators in $E^{-\infty}(\mathcal{T})$, which become adjacent to v_o in $\mathcal{D}'$. By Lemma 3.2, we have

$$\operatorname{codeg}(\mathcal{D}) > \operatorname{codeg}(\mathcal{D}').$$

Let $\mathcal{T}'$ be the capping tree obtained by removing from $\mathcal{D}'$ all elevators and floors strictly below v_o. By Lemma 4.6, it has codegree at least $n(a_o - 2)$. Since at least one elevator in $E^{-\infty}(\mathcal{D}')$ is adjacent to v_o, we deduce that

$$\operatorname{codeg}(\mathcal{D}') \geqslant n(a_o - 2) + a - a_o = a + (n-1)a_o - 2n.$$

Since $a_o \geqslant 3$, we obtain

$$\operatorname{codeg}(\mathcal{D}') \geqslant a + n - 3 \geqslant i.$$

As a consequence we get that $\operatorname{codeg}(\mathcal{D}) > i$, contrary to our assumption that $\mathcal{T}$ is not a capping tree.

Hence the floor diagram $\mathcal{D}$ either is $\mathcal{D}_{a,0,n,u,0}$, or looks like the floor diagram $\mathcal{D}_{a,0,n,u,0}$, except that the top part is replaced by a capping tree of codegree at most i. In any case $\mathcal{D}$ looks like the floor diagram depicted in Figure 4.4 where $\mathcal{T}$ is either a single vertex or a capping tree of codegree at most i. Note that the number of edges e of $\mathcal{D}$ with $\omega(e) \leqslant i - \operatorname{codeg}(\mathcal{D})$, as well as the Laurent polynomial

$$P(q) = \prod_{\substack{e \in E^0(\mathcal{D}) \\ \omega(e) \leqslant i - \operatorname{codeg}(\mathcal{D})}} [w(e)]^2$$

do not depend on a. Indeed, let k be such that there exists $l \geqslant k$ with $u_l \neq 0$. Denoting by e the elevator $e \in E^0(\mathcal{D})$ adjacent to the floors v_k and v_{k+1}, we have that

$$\omega(e) = n(a-k) - \sum_{j=k}^{i} u_j > i - k + 1 - \sum_{j=k}^{i} u_j \geqslant i - \sum_{j=k}^{i} j u_j \geqslant i - \operatorname{codeg}(\mathcal{D}).$$

Thus by Corollary 3.7, the coefficient $\langle \mu(\mathcal{D}) \rangle_{i - \operatorname{codeg}(\mathcal{D})}$ is polynomial in a of degree $i - \operatorname{codeg}(\mathcal{D})$. Furthermore since $an \geqslant i + 2s$, any increasing bijection

$$\{\eta(\mathcal{D}) - \operatorname{Card}(V(\mathcal{T}) \cup E(\mathcal{T})) + 1, \cdots, \eta(\mathcal{D})\} \longrightarrow V(\mathcal{T}) \cup E(\mathcal{T})$$

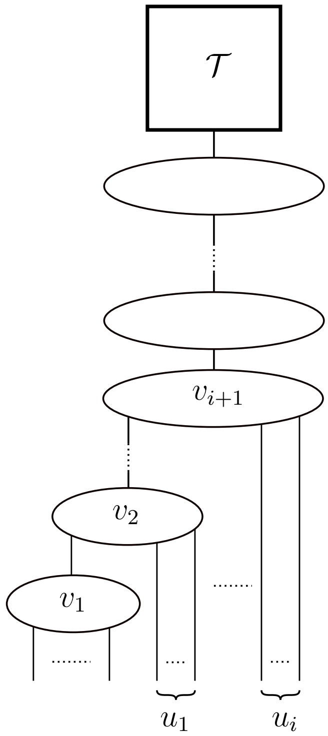


Figure 4.4: $\text{codeg}(\mathcal{T}) + \sum_{j=1}^{i} ju_j \leqslant i$.

extends to exactly $\widetilde{\nu}_u(a, 0, n, s)$ markings of $\mathcal{D}$ compatible with $\{\{1, 2\}, \cdots, \{2s - 1, 2s\}\}$.

Since there exists finitely many such increasing maps, and finitely many capping trees of codegree at most i by Lemma 4.6, the end of the proof is now entirely analogous to the proof of Theorem 4.3. □

4.3. Polynomiality with respect to s

We use a different method to prove polynomiality with respect to s when Δ is fixed, namely we prove that the i*-th discrete derivative* of the map $s \mapsto \langle G_\Delta(0; s)\rangle_i$ is constant. Recall that the n-th discrete derivative of a univariate polynomial $P(X)$ is defined by

$$P^{(n)}(X) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} P(X + l).$$

Lemma 4.7. *One has*

$$(P^{(n)})^{(1)}(X) = P^{(n+1)}(X) \qquad \textit{and} \qquad \deg P^{(n)}(X) = \deg P(X) - n.$$

Furthermore, if the leading coefficient of $P(X)$ *is* a*, then the leading coefficient of* $P^{(n)}(X)$ *is*

$$(-1)^n \, a \deg P(X)(\deg P(X) - 1) \cdots (\deg P(X) - n + 1).$$

Proof. The first assertion is a simple application of Descartes' rule for binomial coefficients:

$$\begin{aligned}(P^{(n)})^{(1)}(X) &= P^{(n)}(X) - P^{(n)}(X+1)\\ &= \sum_{l=0}^{n}(-1)^l\binom{n}{l}P(X+l) - \sum_{l=1}^{n+1}(-1)^{l-1}\binom{n}{l-1}P(X+l)\\ &= \sum_{l=0}^{n+1}(-1)^l\left(\binom{n}{l}+\binom{n}{l-1}\right)P(X+l)\\ &= P^{(n+1)}(X).\end{aligned}$$

Hence the second and third assertions follow by induction starting with the straightforward case $n=1$. □

Proof of Theorem 1.7. Recall that

$$\eta(\Delta) = \mathrm{Card}(\partial\Delta\cap\mathbb{Z}^2)-1,\ \iota(\Delta) = \mathrm{Card}(\Delta\cap\mathbb{Z}^2)-\mathrm{Card}(\partial\Delta\cap\mathbb{Z}^2),\ \text{and } s_{max} = \left\lceil\frac{\eta(\Delta)}{2}\right\rceil.$$

We denote by $a_i(X)$ the polynomial of degree at most s_{max} that interpolates the values

$$\langle G_\Delta(0;0)\rangle_i, \cdots, \langle G_\Delta(0;s_{max})\rangle_i.$$

By Lemma 4.7, the polynomial $a_i^{(i)}(X)$ has degree at most $s_{max}-i$, and we are left to prove that

$$a_i^{(i)}(0) = \cdots = a_i^{(i)}(s_{max}-i) = 2^i.$$

Let $s\in\{0,1,\cdots,s_{max}-i\}$, and S be a pairing of order s of the set $\{2i+1,\cdots,\eta(\Delta)\}$. Given $I\subset\{1,\ldots,i\}$, we denote by S^I the pairing

$$S^I = S\cup\bigcup_{j\in I}\{\{2j-1,2j\}\}.$$

Given $(\mathcal{D},m)$ a marked floor diagram with Newton polygon Δ and of genus 0, we define

$$\kappa(\mathcal{D},m)(q) = \sum_{l=0}^{i}\sum_{\substack{I\subset\{1,\cdots,i\}\\|I|=l}}(-1)^l\mu_{S^I}(\mathcal{D},m)(q).$$

By Theorem 2.13, we have

$$\begin{aligned}\sum_{j=-\iota(\Delta)}^{\iota(\Delta)} a^{(i)}_{\iota(\Delta)-|j|}(s)q^j &= \sum_{l=0}^{i}\sum_{\substack{I\subset\{1,\cdots,i\}\\|I|=l}}(-1)^l\sum_{(\mathcal{D},m)}\mu_{S^I}(\mathcal{D},m)(q)\\ &= \sum_{(\mathcal{D},m)}\kappa(\mathcal{D},m)(q),\end{aligned}$$

where the sum over $(\mathcal{D},m)$ runs over all isomorphism classes of marked floor diagrams with Newton polygon Δ and of genus 0.

Let $(\mathcal{D},m)$ be one of these marked floor diagrams, and denote by i_0 the minimal element of $\{1,\cdots,\eta(\Delta)\}$ such that $m(i_0)\in V(\mathcal{D})$. We also denote by $J\subset\{1,\cdots,2i\}$ the set of elements j such that $m(j)$ is mapped to an elevator in $E^{-\infty}(\mathcal{D})$ adjacent to $m(i_0)$.

Step 1. We claim that if the set $J \cup \{i_0\}$ contains a pair $\{2k-1, 2k\}$ with $k \leqslant i$, then $\kappa(\mathcal{D}, m)(q) = 0$.

Let $I \subset \{1, \cdots, i\} \setminus \{k\}$. It follows from Definition 2.11 that

$$\mu_{S^I}(\mathcal{D}, m)(q) = \mu_{S^{I\cup\{k\}}}(\mathcal{D}, m)(q).$$

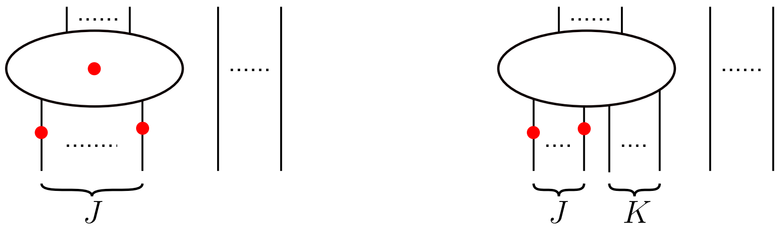
Hence one has

$$\begin{aligned}
\kappa(\mathcal{D}, m)(q) &= \sum_{l=0}^{i} \sum_{\substack{I \subset \{1,\cdots,i\} \\ |I| = l}} (-1)^l \mu_{S^I}(\mathcal{D}, m)(q) \\
&= \sum_{l=0}^{i-1} \sum_{\substack{I \subset \{1,\cdots,i\}\setminus\{k\} \\ |I| = l}} \left((-1)^l \mu_{S^I}(\mathcal{D}, m)(q) + (-1)^{l+1} \mu_{S^{I\cup\{k\}}}(\mathcal{D}, m)(q)\right) \\
&= 0,
\end{aligned}$$

and the claim is proved. We assume from now on that the set $J \cup \{i_0\}$ contains no pair $\{2k-1, 2k\}$ with $k \leqslant i$.

Step 2. We first study the case when $2i \leqslant d_b(\Delta)$.

If $i_0 \leqslant 2i$, then $|J| \leqslant i-1$, and no element $k > 2i$ is mapped to an elevator in $E^{-\infty}(\mathcal{D})$ adjacent to $m(i_0)$. The codegree of $(\mathcal{D}, m)$ is then at least $d_b(\Delta) - |J| \geqslant d_b(\Delta) - i + 1$ by Lemma 3.2, see Figure 4.5a). Hence this codegree is at least $i+1$ by assumption, which means that $\kappa(\mathcal{D}, m)(q)$ does not contribute to $a_i^{(i)}(s)$.



a) $i_0 \leqslant 2i$ b) $i_0 > 2i$

Figure 4.5: Illustration of Step 2; red dots represent points in $m(\{1, \cdots, 2i\})$.

Suppose now that $i_0 > 2i$, so in particular $m(\{1, \cdots, 2i\}) \subset E^{-\infty}(\mathcal{D})$. We denote by $K \subset \{2i+1, \cdots, \eta(\Delta)\}$ the set of elements j such that $m(j)$ is mapped to an elevator in $E^{-\infty}(\mathcal{D})$ adjacent to $m(i_0)$. Note that $|K| \leqslant d_b(\Delta) - 2i$. Hence Lemma 3.2 implies that $(\mathcal{D}, m)$ has codegree at least

$$d_b(\Delta) - |J| - |K| \geqslant d_b(\Delta) - i - |K| = i + (d_b(\Delta) - 2i - |K|),$$

see Figure 4.5b). Hence $\kappa(\mathcal{D}, m)(q)$ can contribute to $a_i^{(i)}(s)$ only if $|K| = d_b(\Delta) - 2i$. It follows from Lemma 3.2 again that $\kappa(\mathcal{D}, m)(q)$ contributes to $a_i^{(i)}(s)$ if and only if

- the order $\preccurlyeq$ is total on the set of floors of $\mathcal{D}$;
- elevators in $E^{+\infty}(\mathcal{D})$ are all adjacent to the maximal floor of $\mathcal{D}$;
- $m(\{1,\cdots,2i\}\setminus J)$ consists of elevators in $E^{-\infty}(\mathcal{D})$ adjacent to the second lowest floor of $\mathcal{D}$;
- any elevator in $E^{-\infty}(\mathcal{D})\setminus m(\{1,\cdots,2i\})$ is adjacent to $m(i_0)$;
- The set J contains exactly i elements, and no pair $\{2k-1,2k\}$;
- the function $l: V(\mathcal{D})\to d_l(\Delta)$ is decreasing, and the function $r: V(\mathcal{D})\to d_r(\Delta)$ is increasing.

For such $(\mathcal{D},m)$, we have
$$\kappa(\mathcal{D},m)(q)=\mu_S(\mathcal{D},m)(q),$$
since $\mu_{S^I}(\mathcal{D},m)(q)=0$ if $I\neq\varnothing$. The coefficient of codegree 0 of $\mu_S(\mathcal{D},m)(q)$ is 1 by Definition 2.11. The floor diagram $\mathcal{D}$ has codegree i, and there are exactly 2^i such marked floor diagrams $(\mathcal{D},m)$, one for each possible set J, so we obtain that $a_i^{(i)}(s)=2^i$ as claimed.

Step 3. We assume now that $2i\in\{d_b(\Delta)+1,d_b(\Delta)+2\}$. In this case we necessarily have $i_0\leqslant 2i$. As in Step 2, we have $|J|\leqslant i-1$, and the codegree of $(\mathcal{D},m)$ is at least $d_b(\Delta)-|J|\geqslant d_b(\Delta)-i+1$ by Lemma 3.2. Hence $\kappa(\mathcal{D},m)(q)$ can contribute to $a_i^{(i)}(s)$ only if one of the following sets of conditions is satisfied:

1. $(\mathcal{D},m)$ has codegree i, with $2i=d_b(\Delta)+1$ and $|J|=i-1$;
2. $(\mathcal{D},m)$ has codegree $i-1$, with $2i=d_b(\Delta)+2$ and $|J|=i-1$;
3. $(\mathcal{D},m)$ has codegree i, with $2i=d_b(\Delta)+2$ and $|J|=i-1$;
4. $(\mathcal{D},m)$ has codegree i, with $2i=d_b(\Delta)+2$ and $|J|=i-2$.

We end by studying these cases one by one. Recall that in the last three cases, we make the additional assumption that $\Delta=\Delta_{a,b,n}$. In this case, the conditions $an+b+2=2i$ and $\iota(\Delta)\geqslant i$ ensure that $n\leqslant i-2$.

1. $(\mathcal{D},m)$ has codegree i, with $2i=d_b(\Delta)+1$ and $|J|=i-1$. As in Step 2, the Laurent polynomial $\kappa(\mathcal{D},m)(q)$ contributes to $a_i^{(i)}(s)$ if and only if (see Figure 4.6a):
    - the order $\preccurlyeq$ is total on the set of floors of $\mathcal{D}$;
    - elevators in $E^{+\infty}(\mathcal{D})$ are all adjacent to the maximal floor of $\mathcal{D}$;
    - $m(\{1,\cdots,2i\}\setminus(J\cup\{i_0\}))$ consists of all elevators in $E^{-\infty}(\mathcal{D})$ adjacent to the second lowest floor of $\mathcal{D}$;
    - the function $l: V(\mathcal{D})\to d_l(\Delta)$ is decreasing, and the function $r: V(\mathcal{D})\to d_r(\Delta)$ is increasing.

a) $d_b(\Delta) = 2i - 1$ and $|J| = i - 1$ b) $d_b(\Delta) = 2i - 2$ and $|J| = i - 1$ or $|J| = i - 2$

Figure 4.6: Illustration of Step 3; red dots represent points in $m(\{1, \cdots, 2i\})$.

For such $(\mathcal{D}, m)$, we have

$$\kappa(\mathcal{D}, m)(q) = \mu_S(\mathcal{D}, m)(q),$$

since $\mu_{S^I}(\mathcal{D}, m)(q) = 0$ if $I \neq \varnothing$. The coefficient of codegree 0 of $\mu_S(\mathcal{D}, m)(q)$ is 1, and there are exactly 2^i such marked floor diagrams, one for each possible set $J \cup \{i_0\}$. We obtain again that $a_i^{(i)}(s) = 2^i$.

2. $(\mathcal{D}, m)$ has codegree $i-1$, with $2i = d_b(\Delta)+2$ and $|J| = i-1$. As in Step 2, the Laurent polynomial $\kappa(\mathcal{D}, m)(q)$ contributes to $a_i^{(i)}(s)$ if and only if (see Figures 4.1 and 4.6b):
   - $\mathcal{D} = \mathcal{D}_{a,b,n,(i-1),0}$;
   - $i_0 = 2i-1$, and $m(2i)$ is the elevator of $\mathcal{D}$ adjacent to and oriented away from $m(i_0)$.
   - $m(\{1, \cdots, 2i-2\} \setminus J)$ consists of all elevators in $E^{-\infty}(\mathcal{D})$ adjacent to the second lowest floor of $\mathcal{D}$.

   For such $(\mathcal{D}, m)$, we have

$$\kappa(\mathcal{D}, m)(q) = \mu_S(\mathcal{D}, m)(q) - \mu_{S^{\{i\}}}(\mathcal{D}, m)(q),$$

   since $\mu_{S^I}(\mathcal{D}, m)(q) = 0$ if $I \not\subset \{i\}$. We have $[w]^2(q) - [w](q^2) = 0$ if $w = 1$, and

$$[w]^2(q) - [w](q^2) = 0q^{-w+1} + 2q^{-w+2} + \dots$$

   if $w \geqslant 2$. Since $w = i - 1 - n$ in Figure 4.6b), we have by Definition 2.11 that the coefficient of codegree 1 of $\kappa(\mathcal{D}, m)(q)$ is 0 if $n = i-2$, and is 2 if $n \leqslant i-3$. There are exactly 2^{i-1} such marked floor diagrams, one for each possible set J. So the total contribution of such $(\mathcal{D}, m)$ to $a_i^{(i)}(s)$ is 0 if $n = i-2$ and is $2 \times 2^{i-1} = 2^i$ if $n \leqslant i-3$.

3. $(\mathcal{D}, m)$ has codegree i, with $2i = d_b(\Delta) + 2$ and $|J| = i - 1$. As in the previous cases $\kappa(\mathcal{D}, m)(q)$ can contribute to $a_i^{(i)}(s)$ only if $i_0 = 2i - 1$, and $m(2i)$ and $m(2i-1)$ are

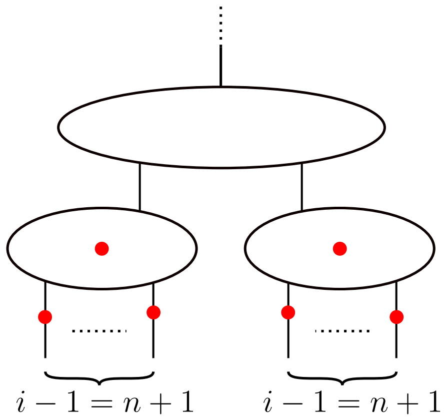


Figure 4.7: Illustration of Step 3; red dots represent points in $m(\{1,\cdots,2i\})$.

not adjacent. This is possible if and only if both $m(2i-1)$ and $m(2i)$ are floors and $n=i-2$, see Figure 4.7. In this case $\kappa(\mathcal{D},m)(q)=\mu_S(\mathcal{D},m)(q)$, and the coefficient of codegree 0 of $\mu_S(\mathcal{D},m)(q)$ is 1. There are exactly 2^i such marked floor diagrams, so the total contribution of such $(\mathcal{D},m)$ to $a_i^{(i)}(s)$ is 2^i.

4. $(\mathcal{D},m)$ has codegree i, with $2i=d_b(\Delta)+2$ and $|J|=i-2$. As in Step 2, the marked floor diagram $\kappa(\mathcal{D},m)(q)$ may contribute to $a_i^{(i)}(s)$ only if (see Figures 4.1 and 4.6b):

   - $\mathcal{D}=\mathcal{D}_{a,b,n,(i),0}$;
   - $i_0=2i-3$ or $i_0=2i-2$;
   - $m(2i-1)$ or $m(2i)$ is the elevator of $\mathcal{D}$ adjacent to and oriented away from $m(i_0)$.
   - $m(\{1,\cdots,2i\}\setminus(J\cup\{i_0\}))$ consists of all elevators adjacent to and oriented toward the second lowest floor of $\mathcal{D}$.

   For such $(\mathcal{D},m)$, we have

$$\kappa(\mathcal{D},m)(q)=\mu_S(\mathcal{D},m)(q)-\mu_{S^{\{i\}}}(\mathcal{D},m)(q),$$

   since $\mu_{S^I}(\mathcal{D},m)(q)=0$ if $I\not\subset\{i\}$. We have

$$[w]^2(q)-\frac{[w][w+1]}{[2]}(q)=0q^{-w+1}+\ldots,$$

   so by Definition 2.11 the coefficient of codegree 0 of $\mu_S(\mathcal{D},m)(q)$ is 0. Hence the total contribution of such $(\mathcal{D},m)$ to $a_i^{(i)}(s)$ is 0.

Summing up all contributions, we obtain that $a_i^{(i)}(s)=2^i$ as announced. $\square$

5. Higher genus case

The generalization of Theorems 1.5 and 1.6 to higher genus is quite technical and requires some care. Following [FM10] and [AB13], we prove Theorems 1.1, 1.2, and 1.3 by decomposing floor diagrams into elementary building blocks that we call *templates*. Although templates from this paper differ from those from [FM10] and [AB13], we borrow their terminology since we follow the overall strategy exposed in [FM10].

5.1. Templates

Recall that the orientation of an oriented acyclic graph Γ induces a partial ordering $\preccurlyeq$ on Γ. Such an oriented graph Γ is said to be *layered* if $\preccurlyeq$ induces a total order on vertices of Γ. A layered graph Γ is necessarily connected. We say that an edge e of Γ is *separating* if $\Gamma \setminus \{e\}$ is disconnected, and if e is comparable with any element of $\Gamma \setminus \{e\}$. A *short edge* of Γ is an edge connecting two consecutive vertices of Γ, and we denote by $E^c(\Gamma)$ the set of short edges of Γ.

Definition 5.1. A *pre-template* is a couple $\Theta = (\Gamma, \omega)$ such that

1. Γ is a layered acyclic oriented graph with no separating edge;
2. ω is a weight function $E(\Gamma) \setminus E^c(\Gamma) \to \mathbb{Z}_{>0}$;
3. every edge in $E^{\pm\infty}(\Gamma)$ has weight 1.

One says that $\Theta = (\Gamma, \omega)$ is a *template* if it satisfies the additional condition:

(4) $E^{+\infty}(\Gamma) = \varnothing$ or $E^{-\infty}(\Gamma) = \varnothing$.

Similarly to floor diagrams, we will not distinguish between a pre-template Θ and its underlying graph, and the *genus* of Θ is defined to be its first Betti number. A template Θ which is not reduced to a vertex and for which $E^{\pm\infty}(\Theta) = \varnothing$ is called *closed*. Denoting by $v_1 \prec v_2 \prec \cdots \prec v_{l(\Theta)}$ the vertices of Θ, we define $c(e)$ for a non-short edge e by

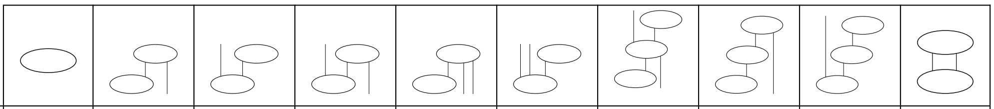
- $c(e) = j - 1$ if $e \in E^{-\infty}(\Theta)$ is adjacent to v_j;
- $c(e) = j$ if $e \in E^{+\infty}(\Theta)$ is adjacent to $v_{l(\Theta)-j}$;
- $c(e) = (k - j - 1)\,\omega(e)$ if $e \in E^0(\Theta) \setminus E^c(\Theta)$ is adjacent to v_j and v_k with $v_j \preccurlyeq v_k$.

Finally, we defined the *codegree* of Θ by

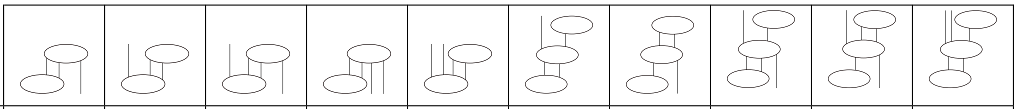
$$\operatorname{codeg}(\Theta) = \sum_{e \in E(\Gamma) \setminus E^c(\Gamma)} c(e).$$

The integer $l(\Theta)$ is called the *length* of Θ.

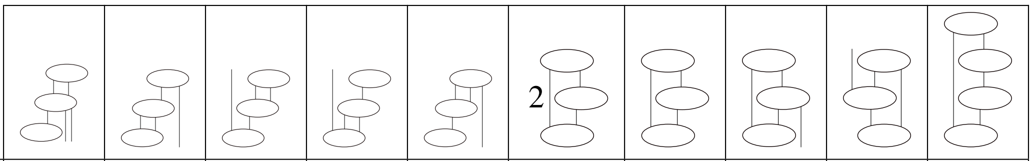
Example 5.2. We depict in Figure 5.1 all templates of genus at most 1 and codegree at most 2. Note that for a fixed g and i, there are finitely many templates of genus g and codegree i.



genus	0	0	0	0	0	0	0	0	0	1
codegree	0	1	1	2	2	2	2	2	2	0
length	1	2	2	2	2	2	3	3	3	2



genus	1	1	1	1	1	1	1	1	1	1
codegree	1	1	2	2	2	1	1	2	2	2
length	2	2	2	2	2	3	3	3	3	3



genus	1	1	1	1	1	1	1	1	1	1
codegree	2	2	2	2	2	2	1	2	2	2
length	3	3	3	3	3	3	3	3	3	4

Figure 5.1: Templates of genus at most 1 and codegree at most 2.

Lemma 5.3. *Any pre-template Θ satisfies*

$$\mathrm{codeg}(\Theta) + g(\Theta) \geqslant l(\Theta) - 1.$$

Proof. The proof goes by induction on $\mathrm{codeg}(\Theta)$. The lemma holds if $\mathrm{codeg}(\Theta) = 0$, since any two consecutive vertices of Θ are connected by at least two edges. If $\mathrm{codeg}(\Theta) > 0$, then an operation $A^{\pm}$ produces a graph Θ' with

$$l(\Theta') = l(\Theta), \qquad g(\Theta') = g(\Theta), \qquad \text{and} \qquad \mathrm{codeg}(\Theta') \leqslant \mathrm{codeg}(\Theta) - 1.$$

There are now two cases: either Θ' is a template, or it contains a separating edge. In the former case, the lemma holds by induction. In the latter case, denote by e the separating edge of Θ', and Θ'_1 and Θ'_2 the two connected components of $\Theta' \setminus \{e\}$. Both Θ'_1 and Θ'_2 are templates, and one has

$$l(\Theta'_1) + l(\Theta'_2) = l(\Theta), \ \mathrm{codeg}(\Theta'_1) + \mathrm{codeg}(\Theta'_2) \leqslant \mathrm{codeg}(\Theta) - 1, \ \text{and } g(\Theta'_1) + g(\Theta'_2) = g(\Theta).$$

Hence the lemma holds by induction again. $\square$

Given a layered floor diagram $\mathcal{D} = (\Gamma, \omega)$, we denote by $E^u(\mathcal{D})$ the union of

- the set of separating edges e of $\mathcal{D}$,
- the set of edges in $E^{-\infty}(\Gamma)$ and $E^{+\infty}(\Gamma)$ adjacent to the minimal and maximal floor of $\mathcal{D}$, respectively,

and we denote by $\mathcal{D}_1, \cdots, \mathcal{D}_l$ the connected components of $\mathcal{D} \setminus E^u(\mathcal{D})$ that are not reduced to a non-extremal vertex. Each $\mathcal{D}_j$ equipped with the the weight function $\omega|_{E(\mathcal{D}_j)\setminus E^c(\mathcal{D}_j)}$ is a pre-template. If $\mathcal{D}_1$ is not a template, then necessarily $E^u(\mathcal{D}) \subset E^{-\infty}(\Gamma) \cup E^{+\infty}(\Gamma)$ and $\mathcal{D}_1 = \mathcal{D} \setminus E^u(\mathcal{D})$.

Definition 5.4. A layered floor diagram $\mathcal{D} = (\Gamma, \omega)$ is said to be *strongly layered* if each $\mathcal{D}_j$ equipped with the the weight function $\omega|_{E(\mathcal{D}_j)\setminus E^c(\mathcal{D}_j)}$ is a template.

Now we explain how to reverse this decomposing process. A collection of templates $\Xi = (\Theta_1, \cdots, \Theta_m)$ is said to be *admissible* if $E^{+\infty}(\Theta_1) = E^{-\infty}(\Theta_m) = \varnothing$, and $\Theta_2, \cdots, \Theta_{m-1}$ are closed. Given $a \in \mathbb{Z}_{>0}$, we denote by $A_a(\Xi)$ the set of sequences of positive integers $\kappa = (k_1 = 1, k_2, \cdots, k_m)$ such that

- $\forall j \in \{1, \cdots, m-1\},\ k_{j+1} \geqslant k_j + l(\Theta_j)$;
- $k_m + l(\Theta_m) = a + 1$.

Given $\kappa \in A_a(\Xi)$ and additional integers $n \geqslant 0$ and $b \geqslant \mathrm{Card}(E^{+\infty}(\Theta_m))$, we denote by $B_{a,b,n}(\Xi, \kappa)$ the set of collections $\Omega = (\omega_1, \cdots, \omega_m)$ where $\omega_j : E(\Theta_j) \to \mathbb{Z}_{>0}$ is a weight function extending $\omega_j : E(\Theta) \setminus E^c(\Theta_j) \to \mathbb{Z}_{>0}$ such that

- $\mathrm{div}(v) = n$ for any non-extremal vertex v of Θ_j;
- $\mathrm{div}(v) = -\left((a - k_j)n + b - \mathrm{Card}(E^{+\infty}(\Theta_j)\right)$ if v is the minimal vertex of Θ_j, when Θ_j is not reduced to v.

Note that by definition Θ_j may be reduced to v only if $j = 1$ or $j = m$. We denote by

$$\omega_{\Xi,\Omega} : \bigsqcup_{j=1}^{m} \Theta_j \longrightarrow \mathbb{Z}_{>0}$$

the weight function whose restriction to Θ_j is ω_j.

Given three integers $a, b, n \geqslant 0$, an admissible collection of templates $\Xi = (\Theta_1, \cdots, \Theta_m)$, and two elements $\kappa \in A_a(\Xi)$ and $\Omega \in B_{a,b,n}(\Xi, \kappa)$, we construct a strongly layered floor diagram $\mathcal{D}$ with Newton polygon $\Delta_{a,b,n}$ as follows:

1. for each $j \in \{1, \cdots, m-1\}$, connect the maximal vertex of Θ_j to the minimal vertex of Θ_{j+1} by a chain of $k_{j+1} - k_j - l(\Theta_j) + 1$ edges, oriented from Θ_j to Θ_{j+1}; denote by $\widetilde{\Gamma}_{\Xi,\kappa}$ the resulting graph;
2. extend the weight function $\omega_{\Xi,\Omega}$ to $\widetilde{\Gamma}_{\Xi,\kappa}$ such that each non-extremal vertex has divergence n; this extended function is still denoted by $\omega_{\Xi,\Omega}$;

3. add $an+b-\mathrm{Card}(E^{-\infty}(\Theta_1))$ edges to $E^{-\infty}(\widetilde{\Gamma}_{\Xi,\kappa})$, all adjacent to the minimal vertex of $\widetilde{\Gamma}_{\Xi,\kappa}$, and extend $\omega_{\Xi,\Omega}$ by 1 on these additional edges;

4. add $b-\mathrm{Card}(E^{+\infty}(\Theta_m))$ edges to $E^{+\infty}(\widetilde{\Gamma}_{\Xi,\kappa})$, all adjacent to the maximal vertex of $\widetilde{\Gamma}_{\Xi,\kappa}$, and extend $\omega_{\Xi,\Omega}$ by 1 on these additional edges; denote by $\Gamma_{\Xi,\kappa}$ the resulting graph.

The resulting weighted graph $\mathcal{D}_{\Xi,\kappa}=(\Gamma_{\Xi,\kappa},\omega_{\Xi,\Omega})$ is a strongly layered floor diagram with Newton polygon $\Delta_{a,b,n}$ as announced. Note also that

$$g(\mathcal{D}_{\Xi,\kappa})=\sum_{j=1}^{m} g(\Theta_j) \qquad \text{and} \qquad \mathrm{codeg}(\mathcal{D}_{\Xi,\kappa})=\sum_{j=1}^{m}\mathrm{codeg}(\Theta_j).$$

These two quantities are called the genus and the codegree of Ξ, respectively. The next proposition generalizes Lemma 4.1 to higher genera.

Lemma 5.5. *Let $a,b,n,i\in\mathbb{Z}_{\geqslant 0}$ be such that*

$$\begin{cases} b>i \\ a>i+g+1 \end{cases}.$$

Then any floor diagram with Newton polygon $\Delta_{a,b,n}$ and of codegree at most i is strongly layered. In particular, the construction above establishes a bijection between the set of triples (Ξ,κ,Ω), with Ξ admissible of genus g and codegree i, with $\kappa\in A_a(\Xi)$ and $\Omega\in B_{a,b,n}(\Xi,\kappa)$ on one hand, and the set of floor diagram with Newton polygon $\Delta_{a,b,n}$, of genus g and codegree i on the other hand.

Proof. The second assertion follows immediately from the first one. Assume that there exists a non-strongly layered floor diagram $\mathcal{D}$ with Newton polygon $\Delta_{a,b,n}$ and of codegree at most i.

Suppose first that $\mathcal{D}$ is not layered. This means that there exist two floors v_1 and v_2 of $\mathcal{D}$ that are not comparable for $\preccurlyeq$. As in the proof of Lemma 4.1, the floor diagram $\mathcal{D}$ has a unique minimal floor and a unique maximal floor. By finitely many applications of moves $A^{\pm}$ and Lemma 3.2, we reduce to the case where

- $\preccurlyeq$ induces a total order on $V(\mathcal{D})\setminus\{v_1,v_2\}$;
- $\mathcal{D}\setminus\{v_1,v_2\}$ is disconnected;
- elevators in $E^{\pm\infty}(\mathcal{D})$ are adjacent to an extremal floor of $\mathcal{D}$;
- elevators in $E^0(\mathcal{D})$ not adjacent to v_1 nor v_2 are adjacent to two consecutive floors;
- elevators in $E^0(\mathcal{D})$ adjacent to v_1 or v_2 are as depicted in Figure 5.2 (where weights are not mentioned).

Defining

$$w=\sum_{v_0\xrightarrow{e}}\omega(e),$$

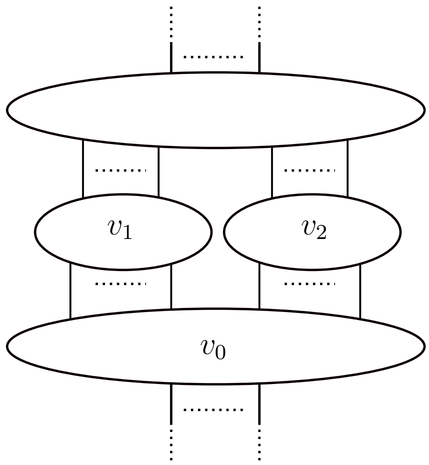


Figure 5.2: A non-layered floor diagram.

we have that

$$w \geqslant b + 3n.$$

Finitely many applications of moves $A^{\pm}$ and Lemma 3.2 also give

$$\operatorname{codeg}(\mathcal{D}) \geqslant w - n \geqslant b + 2n > i,$$

in contradiction to our assumption. Hence $\mathcal{D}$ is layered.

Since $\mathcal{D}$ is not strongly layered, this means by Definition 5.4 that $E^u(\mathcal{D}) \subset E^{-\infty}(\Gamma) \cup E^{+\infty}(\Gamma)$ and $\mathcal{D}_1 = \mathcal{D} \setminus E^u(\mathcal{D})$. According to Lemma 5.3, one has

$$\operatorname{codeg}(\mathcal{D}_1) + g \geqslant a - 1.$$

Since $a > i + g + 1$, we deduce that $\operatorname{codeg}(\mathcal{D}_1) > i$ in contradiction to our assumption. $\square$

5.2. Polynomiality of $(a, b, n) \mapsto \langle G_{\Delta_{a,b,n}}(g) \rangle_i$

Similarly to floor diagrams, we define a *marking of a template* Θ as a bijective map

$$m\colon \{1, 2, \ldots, \operatorname{Card}(V(\Theta) \cup E^0(\Theta))\} \longrightarrow V(\Theta) \cup E^0(\Theta)$$

such that $j \leqslant k$ whenever $m(j) \preccurlyeq m(k)$. All markings of a given template Θ are considered up to automorphisms of oriented partially weighted graph $\varphi : \Theta \longrightarrow \Theta$ such that and $m = \varphi \circ m'$.

Denoting by $v_1 \prec v_2 \prec \cdots \prec v_{l(\Theta)}$ the vertices of Θ, we define γ_j to be the number of edges connecting v_j and v_{j+1}, and

$$\mathcal{A}(\Theta) = \prod_{j=1}^{l(\Theta)-1} \frac{1}{\gamma_j!}.$$

Next, given an admissible collection of templates $\Xi = (\Theta_1, \cdots, \Theta_m)$, we set

$$\mathcal{A}(\Xi) = \prod_{j=1}^{m} \mathcal{A}(\Theta_j).$$

If $\kappa \in A_a(\Xi)$ and $\Omega \in B_{a,b,n}(\Xi,\kappa)$, any collection $M = (M_1,\cdots,M_m)$ of markings of $\Theta_1,\cdots,\Theta_m$ extends uniquely to the graph $\widetilde{\Gamma}_{\Xi,\kappa} \setminus (E^{-\infty}(\widetilde{\Gamma}_{\Xi,\kappa}) \cup E^{+\infty}(\widetilde{\Gamma}_{\Xi,\kappa}))$ constructed out of Ξ, κ, and Ω. The number of ways to extend this marking to a marking of the floor diagram $\mathcal{D}_{\Xi,\kappa}$ depends on neither κ nor Ω, and is denoted by $\nu_{\Xi,M}(a,b,n)$. Analogously to the function $\nu_{u,\widetilde{u}}$ from Section 4.1, the function $\nu_{\Xi,M}$ is polynomial and has degree at most $\mathrm{Card}(E^{-\infty}(\Theta_1)) + \mathrm{Card}(E^{+\infty}(\Theta_m))$ in each of the variables a, b, and n.

Lemma 5.6. *Let $a,b,n,i \in \mathbb{Z}_{\geqslant 0}$ be such that*

$$\begin{cases} b > i \\ a > i+g+1 \end{cases}.$$

Then for any $g \geqslant 0$ one has

$$\langle G_{\Delta_{a,b,n}}(g)\rangle_i = \sum_{\Xi,M} \mathcal{A}(\Xi) \times \nu_{\Xi,M}(a,b,n) \sum_{\kappa\in A_a(\Xi)} \sum_{\Omega\in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega})\rangle_{i-\mathrm{codeg}(\Xi)},$$

where the first sum ranges over all admissible collections of templates $\Xi = (\Theta_1,\cdots,\Theta_m)$ of genus g and codegree at most i, and over all collections of markings M of $\Theta_1,\cdots,\Theta_m$.

Proof. Given a floor diagram $\mathcal{D}$, we denote by $\nu(\mathcal{D})$ its number of markings. By Theorem 2.7, we have

$$\langle G_{\Delta_{a,b,n}}(g)\rangle_i = \sum_{\mathcal{D}} \nu(\mathcal{D})\, \langle \mu(\mathcal{D})\rangle_{i-\mathrm{codeg}(\mathcal{D})},$$

where the sum is taken over all floor diagrams $\mathcal{D}$ of genus g and codegree at most i. Now the result follows from Lemma 5.5. □

Lemma 5.6 provides a decomposition of $\langle G_{\Delta_{a,b,n}}(g)\rangle_i$ into pieces that are combinatorially manageable. We prove the polynomiality of $\sum_{\Omega\in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega})\rangle_{i-\mathrm{codeg}(\Xi)}$ in the next lemma, from which we deduce a proof of Theorem 1.1.

Lemma 5.7. *Let $i,g \in \mathbb{Z}_{\geqslant 0}$, and $\Xi = (\Theta_1,\cdots,\Theta_m)$ be an admissible collection of templates of genus g and codegree at most i. Given $(a,b,n) \in \mathbb{Z}_{\geqslant 0}$ such that*

$$\begin{cases} n \geqslant 1 \\ b \geqslant \mathrm{Card}(E^{+\infty}(\Theta_m)) \\ b+n > (g+2)i+g \\ a \geqslant l(\Theta_1)+\cdots+l(\Theta_m) \end{cases},$$

and $\kappa \in A_a(\Xi)$, the sum

$$\sum_{\Omega\in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega})\rangle_{i-\mathrm{codeg}(\Xi)}$$

is polynomial in $a,b,n,k_2,\cdots,k_{m-1}$, of total degree at most $i-\mathrm{codeg}(\Xi)+g$, and of

- *degree at most $i-\mathrm{codeg}(\Xi)+g$ in the variable a;*

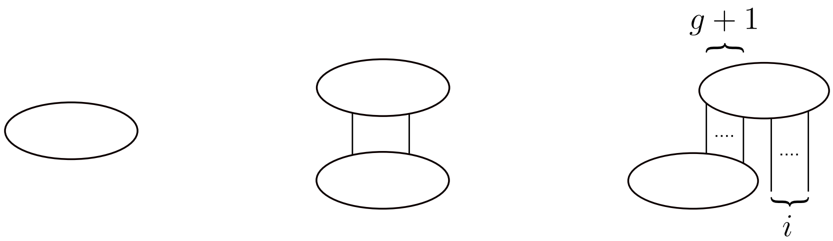
- *degree at most g in the variables b and n;*
- *degree at most $g(\Theta_j)$ in the variable k_j.*

If $\Xi = (\widetilde{\Theta}_1, \widetilde{\Theta}_2, \widetilde{\Theta}_2, \cdots, \widetilde{\Theta}_2, \widetilde{\Theta}_1)$, with $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$ depicted in Figure 5.3, then the sum

$$\sum_{\Omega \in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega}) \rangle_i$$

is polynomial in $a, b, n, k_2, \cdots, k_{g+1}$, of total degree $i + g$, and of

- *degree $i + g$ in the variable a;*
- *degree g in the variables b and n;*
- *degree $g(\widetilde{\Theta}_2) = 1$ in the variable k_j.*



a) $\widetilde{\Theta}_1$ b) $\widetilde{\Theta}_2$ c) $\widetilde{\Theta}_{g,i}$

Figure 5.3

If $\Xi = (\widetilde{\Theta}_{g,i}, \widetilde{\Theta}_1)$, with $\widetilde{\Theta}_{g,i}$ as depicted in Figure 5.3, then the sum

$$\sum_{\Omega \in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega}) \rangle_0$$

is polynomial in a, b, and n of total degree g, and of degree g in each of the variables a, b, and n.

Proof. Let $v_{j,1} \prec \cdots \prec v_{j,l(\Theta_j)}$ be the vertices of Θ_j, and let $e_{j,k,1}, \cdots, e_{j,k,g_{j,k}+1}$ be the edges of Θ_j connecting $v_{j,k}$ and $v_{j,k+1}$. In particular we have

$$\sum_{k=1}^{l(\Theta_j)-1} g_{j,k} \leqslant g(\Theta_j).$$

Given $\Omega \in B_{a,b,n}(\Xi, \kappa)$, we also have

$$\sum_{u=1}^{g_{j,k}+1} \omega_{\Xi,\Omega}(e_{j,k,u}) = (a - k_j - k + 1)n + b - c_{j,k},$$

with $c_{j,k} \in \{0, 1, \cdots, i\}$ that only depends on Θ_j. Hence $B_{a,b,n}(\Xi, \kappa)$ is in bijection with subsets of $\prod_{j,k} \mathbb{Z}_{>0}^{g_{j,k}}$ which correspond to decompositions of each integer

$$\beta_{j,k} = (a - k_j - k + 1)n + b - c_{j,k}$$

in an ordered sum of $g_{j,k} + 1$ positive integers. In particular we have

$$\mathrm{Card}(B_{a,b,n}(\Xi, \kappa)) = \prod_{j,k} \binom{\beta_{j,k} - 1}{g_{j,k}}.$$

Note that since $b + n > (g + 2)i + g \geqslant i + g$ by assumption, and $\beta_{j,k} \geqslant b + n - i$, one has

$$\forall j, k, \qquad \beta_{j,k} - 1 \geqslant g \geqslant g_{j,k}.$$

In particular $\mathrm{Card}(B_{a,b,n}(\Xi, \kappa))$ is polynomial in $a, b, n, k_2, \cdots, k_{m-1}$ of total degree at most g, and of degree at most $g(\Theta_j)$ in the variable k_j. If $\langle \mu(\mathcal{D}_{\Xi,\Omega}) \rangle_{i-\mathrm{codeg}(\Xi)}$ were not depending on Ω, then the lemma would be proved. This is unfortunately not the case, nevertheless there exists a partition of $B_{a,b,n}(\Xi, \kappa)$ for which the independency holds on each subset of this partition.

To show this, let $F = \prod_{j,k} \{0, \cdots, i\}^{g_{j,k}}$ and

$$\begin{array}{rccl} \Upsilon : & B_{a,b,n}(\Xi, \kappa) & \longrightarrow & F \\ & (\omega_1, \cdots, \omega_m) & \longmapsto & \left\{ \begin{array}{ll} f_{j,k,u} = 0 & \text{if } \omega_j(e_{j,k,u}) > i - \mathrm{codeg}(\Xi) \\ f_{j,k,u} = \omega_j(e_{j,k,u}) & \text{if } \omega_j(e_{j,k,u}) \leqslant i - \mathrm{codeg}(\Xi) \end{array} \right. . \end{array}$$

Given $f \in F$, we denote by $\lambda_{j,k}(f)$ the number of non-zero coordinates $f_{j,k,u}$, and we define

$$\lambda(f) = \sum_{j,k} \lambda_{j,k}(f).$$

Since $b + n > (g + 2)i + g \geqslant (g + 2)i$, we have that

$$\beta_{j,k} \geqslant b + n - i > i(g + 1) \geqslant i(g_{j,k} + 1),$$

which in its turn implies that $\lambda_{j,k}(f) \leqslant g_{j,k}$ and $\lambda(f) \leqslant g$ if $\Upsilon^{-1}(f) \neq \varnothing$. As above, we have

$$\mathrm{Card}(\Upsilon^{-1}(f)) = \prod_{j,k} \binom{\beta_{j,k} - \sum_u f_{j,k,u} - 1}{g_{j,k} - \lambda_{j,k}}.$$

Hence if $\Upsilon^{-1}(f) \neq \varnothing$, then for any j and k one has

$$\beta_{j,k} - \sum_u f_{j,k,u} - 1 \geqslant \beta_{j,k} - i\lambda(f) - 1 \geqslant \beta_{j,k} - ig - 1 \geqslant b + n - (g+1)i - 1 \geqslant g + i \geqslant g_{j,k} - \lambda_{j,k}.$$

In particular $\mathrm{Card}(\Upsilon^{-1}(f))$ is polynomial in $a, b, n, k_2, \cdots, k_{m-1}$ of total degree at most $g - \lambda(f)$, and of degree at most $g(\Theta_j) - \lambda_j(f)$ in the variable k_j.

Furthermore, for any $\Omega \in \Upsilon^{-1}(f)$, we have

$$\mu(\mathcal{D}_{\Xi,\Omega}) = P_{\Xi,f}(q) \times \prod_{\omega_j(e_{j,k,u})>i-\mathrm{codeg}(\Xi)} [\omega_j(e_{j,k,u})]^2,$$

where $P_{\Xi,f}(q)$ is a Laurent polynomial that only depends on Ξ and f. In particular it follows from Corollary 3.7 that $\langle \mu(\mathcal{D}_{\Xi,\Omega})\rangle_{i-\mathrm{codeg}(\Xi)}$ is a polynomial $Q_{\Xi,f}(a)$ in a of degree $i-\mathrm{codeg}(\Xi)$, which only depends on Ξ and f. We deduce that

$$\sum_{\Omega\in\Upsilon^{-1}(f)} \langle \mu(\mathcal{D}_{\Xi,\Omega})\rangle_{i-\mathrm{codeg}(\Xi)} = \mathrm{Card}(\Upsilon^{-1}(f)) \times Q_{\Xi,f}(a)$$

is polynomial in $a, b, n, k_2, \cdots, k_{m-1}$, of total degree at most $i - \mathrm{codeg}(\Xi) + g - \lambda(f)$, and of

- degree at most $i - \mathrm{codeg}(\Xi) + g - \lambda(f)$ in the variable a;
- degree at most $g - \lambda(f)$ in the variables b and n.
- degree at most $g(\Theta_j) - \sum_k \lambda_{j,k}(f)$ in the variable k_j.

The first part of the lemma now follows from the equality

$$\sum_{\Omega\in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega})\rangle_{i-\mathrm{codeg}(\Xi)} = \sum_{f\in F} \sum_{\Omega\in\Upsilon^{-1}(f)} \langle \mu(\mathcal{D}_{\Xi,\Omega})\rangle_{i-\mathrm{codeg}(\Xi)}.$$

The second part of the lemma follows from a direct application of the computations above in both specific situations. $\square$

Proof of Theorem 1.1. Recall that $\mathcal{U}_{i,g} \subset \mathbb{Z}^3_{\geqslant 0}$ is the set of triples (a, b, n) satisfying

$$\begin{cases} n \geqslant 1 \\ b > i \\ b + n > (g+2)i + g \\ a \geqslant i + 2g + 2 \end{cases}.$$

Let $\Xi = (\Theta_1, \cdots, \Theta_m)$ be an admissible collection of templates of genus g and codegree at most i. By Lemma 5.3, we have

$$l(\Theta_1)+\cdots+l(\Theta_m) \leqslant i+g+m \leqslant i+2g+2 \leqslant a \qquad \text{and} \qquad b+2n > b+n > (g+2)i+g \geqslant i.$$

Hence the set of such collections of templates is finite, and the assumptions of Lemma 5.6 are satisfied. Since

$$\mathrm{codeg}(\Xi) \geqslant \mathrm{Card}(E^{-\infty}(\Theta_1)) + \mathrm{Card}(E^{+\infty}(\Theta_m)),$$

to prove the polynomiality of the function $(a, b, n) \mapsto \left\langle G_{\Delta_{a,b,n}}(g)\right\rangle_i$ and to get an upper bound on its degree, it is enough to prove that on $\mathcal{U}_{i,g}$, the function

$$\sum_{\kappa\in A_a(\Xi)} \sum_{\Omega\in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega})\rangle_{i-\mathrm{codeg}(\Xi)}$$

is polynomial, of degree at most g in the variables b and, and of degree at most $i+2g-\mathrm{codeg}(\Xi)$ in the variable a.

Let us describe precisely the set $A_a(\Xi)$ when $m \geqslant 3$, which is by definition the subset of $\mathbb{Z}_{>0}^{m-2}$ defined by the system of inequalities

$$\begin{cases} k_2 & \geqslant 1 + l(\Theta_1) \\ k_3 & \geqslant k_2 + l(\Theta_2) \\ \vdots & \\ k_{m-1} & \geqslant k_{m-2} + l(\Theta_{m-2}) \\ a + 1 - l(\Theta_m) & \geqslant k_{m-1} + l(\Theta_{m-1}) \end{cases}.$$

Hence, in order to get a parametric description of $A_a(\Xi)$, we need to estimate $l(\Theta_1)+\cdots+l(\Theta_m)$. By Lemma 5.3, we have

$$\sum_{j=1}^{m} l(\Theta_j) \leqslant g + i + m.$$

Furthermore since $g(\Theta_j) \geqslant 1$ if $j \in \{2, \cdots, m-1\}$, we have $m \leqslant g+2$, and we deduce that

$$\sum_{j=1}^{m} l(\Theta_j) \leqslant i + 2g + 2.$$

In particular, since $a \geqslant i+2g+2$ the set $A_a(\Xi)$ can be described as the set of $(k_2, \cdots, k_{m-1}) \subset \mathbb{Z}_{>0}^{m-2}$ such that

$$\begin{cases} 1 + l(\Theta_1) + \cdots + l(\Theta_{m-3}) + l(\Theta_{m-2}) & \leqslant & k_{m-1} & \leqslant & a + 1 - l(\Theta_m) - l(\Theta_{m-1}) \\ 1 + l(\Theta_1) + \cdots + l(\Theta_{m-3}) & \leqslant & k_{m-2} & \leqslant & k_{m-1} - l(\Theta_{m-2}) \\ & & \vdots & & \\ 1 + l(\Theta_1) & \leqslant & k_2 & \leqslant & k_3 - l(\Theta_2) \end{cases},$$

in other words the sum over $A_a(\Xi)$ can be rewritten as

$$\sum_{\kappa \in A_a(\Xi)} = \sum_{k_{m-1}=1+l(\Theta_1)+\cdots+l(\Theta_{m-2})}^{a+1-l(\Theta_m)-l(\Theta_{m-1})} \; \sum_{k_{m-2}=1+l(\Theta_1)+\cdots+l(\Theta_{m-3})}^{k_{m-1}-l(\Theta_{m-2})} \cdots \sum_{k_2=1+l(\Theta_1)}^{k_3-l(\Theta_2)}.$$

Combining Faulhaber's formula with Lemma 5.7, we obtain that the sum

$$\sum_{k_2=1+l(\Theta_1)}^{k_3-l(\Theta_2)} \; \sum_{\Omega \in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega}) \rangle_{i-\mathrm{codeg}(\Xi)}$$

is polynomial in $a, b, n, k_3, \cdots, k_{m-1}$, of total degree at most $i - \mathrm{codeg}(\Xi) + g + 1$, and of

- degree at most $i - \mathrm{codeg}(\Xi) + g$ in the variable a;
- degree at most g in the variables b and n;

- degree at most $g(\Theta_2) + g(\Theta_3) + 1$ in the variable k_3;
- degree at most $g(\Theta_j)$ in the variable k_j with $j \geqslant 4$.

As in the end of the proof of [FM10, Theorem 5.1], we eventually obtain by induction that

$$\sum_{\kappa \in A_a(\Xi)} \sum_{\Omega \in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega}) \rangle_{i-\mathrm{codeg}(\Xi)}$$

is polynomial of degree at most g in the variables b and n, and of degree at most $i - \mathrm{codeg}(\Xi) + g + m - 2$ in the variable a. Since $m - 2 \leqslant g$, we obtain that the function $(a, b, n) \in \mathcal{U}_{i,g} \mapsto \langle G_{\Delta_{a,b,n}}(g) \rangle_i$ is polynomial, of degree at most $i + g$ in the variables b and n, and of degree at most $i + 2g$ in the variable a. The fact that it is indeed of degree $i + g$ in the variables b and n, and of degree $i + 2g$ in the variable a follows from the second part of Lemma 5.7. □

The proof of Theorem 1.2 is identical to the proof of Theorem 1.1. The only place where the assumption $n > 0$ comes into play is Lemma 5.7, in the estimation of the degrees of

$$\sum_{\Omega \in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega}) \rangle_{i-\mathrm{codeg}(\Xi)}$$

with respect to its different variables, and one sees easily how to adapt Lemma 5.7 when $n = 0$.

Proof of Theorem 1.2. If $n = 0$, then Lemma 5.7 still holds with the following edition: the sum

$$\sum_{\Omega \in B_{a,b,n}(\Xi,\kappa)} \langle \mu(\mathcal{D}_{\Xi,\Omega}) \rangle_{i-\mathrm{codeg}(\Xi)}$$

is polynomial in a and b, of total degree at most $i - \mathrm{codeg}(\Xi) + g$, and of

- degree at most $i - \mathrm{codeg}(\Xi)$ in the variable a;
- degree at most g in the variables b.

Indeed in this case we have

$$\beta_{j,k} = b - c_{j,k},$$

which implies exactly as in the proof of Lemma 5.7 that $\mathrm{Card}(\Upsilon^{-1}(f))$ is polynomial in b of total degree at most $g - \lambda(f)$. Now the remaining of the proof of Lemma 5.7 proves the claim above. The proof of Theorem 1.2 follows eventually from this adapted Lemma 5.7 exactly as Theorem 1.1 follows from Lemma 5.7. □

5.3. $b = 0$ and n fixed

As in the genus 0 case, one easily adapts the proof of Theorem 1.1 in the case when one wants to fix $b = 0$ and $n \geqslant 1$. There is no additional technical difficulty here with respect to Sections 4.2 and 5.2, so we briefly indicate the main steps. Again, the difference with the case $b \neq 0$ is that now a floor diagram $\mathcal{D}$ contributing to $\left\langle G_{\Delta_{a,0,n}}(0)\right\rangle_i$ may not be layered because of some highest vertices.

Definition 5.8. A *capping template* with Newton polygon $\Delta_{a,n}$ is a couple $\mathcal{C} = (\Gamma, \omega)$ such that

1. Γ is a connected weighted oriented acyclic graph with a vertices and with no sources nor sinks;
2. Γ has a unique minimal vertex v_1, and $\Gamma \setminus \{v_1\}$ has at least two minimal vertices;
3. for every vertex $v \in V(\Gamma) \setminus \{v_1\}$, one has $\operatorname{div}(v) = n$.

The codegree of a capping template $\mathcal{C}$ with Newton polygon $\Delta_{a,n}$ is defined as

$$\operatorname{codeg}(\mathcal{C}) = \frac{(a-1)(na-2)}{2} - g(\mathcal{C}) - \sum_{e \in E(\Gamma)} (\omega(e) - 1)$$

The proof of the next lemma is analogous to the proof of Lemma 4.6.

Lemma 5.9. *A capping template with Newton polygon $\Delta_{a,n}$ has codegree at least $n(a-2)$.*

Proof of Theorem 1.6. Let $\mathcal{D}$ be a floor diagram of genus g, Newton polygon $\Delta_{a,0,n}$, and of codegree at most i. As in the proof of Theorem 1.6, we have that $\mathcal{D}$ has a unique minimal floor. Suppose that $\mathcal{D}$ is not layered, and let v_o be the lowest floor of $\mathcal{D}$ such that $\mathcal{D} \setminus \{v_o\}$ is not connected and with a non-layered upper part. Let $\mathcal{C}$ be the weighted subgraph of $\mathcal{D}$ obtained by removing from $\mathcal{D}$ all elevators and floors strictly below v_o. As in the proof of Theorem 1.6, one shows that $\mathcal{C}$ is a capping template. For a fixed i and g, there exist finitely many capping templates of codegree at most i and genus at most g. The end of the proof is now entirely analogous to the end of the proof of Theorem 1.6. □

A. Some identities involving quantum numbers

For the reader's convenience, we collect some easy or well-known properties of quantum integers. Recall that given an integer $n \in \mathbb{Z}$, the quantum integer $[k](q)$ is defined by

$$[k](q) = \frac{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{\frac{k-1}{2}} + q^{\frac{k-3}{2}} + \cdots + q^{-\frac{k-3}{2}} + q^{-\frac{k-1}{2}} \in \mathbb{Z}_{\geqslant 0}[q^{\pm\frac{1}{2}}].$$

Given two elements $f, g \in \mathbb{Z}_{\geqslant 0}[q^{\pm\frac{1}{2}}]$, we write $f \geqslant g$ if $f - g \in \mathbb{Z}_{\geqslant 0}[q^{\pm\frac{1}{2}}]$.

Lemma A.1. *For any $k, l \in \mathbb{Z}_{\geqslant 0}$, one has*

$$[k] \cdot [k+l] = [2k+l-1] + [2k+l-3] + \cdots + [l+3] + [l+1].$$

Proof. This is an easy consequence from the fact that, given $c \in \{1, \cdots k-1\}$, one has

$$(q^{-\frac{k-c}{2}} + q^{\frac{k-c}{2}}) \cdot [k+l](q) = [2k+l-c](q) + [l+c](q). \qquad \square$$

Corollary A.2. *For any positive integers k and l, one has*

$$[k] \cdot [k+l-1] = [k-1] \cdot [k+l] + [k].$$

In particular, one has $[k] \cdot [k+l-1] \geqslant [k-1] \cdot [k+l]$

Proof. It follows from Lemma A.1 that

$$\begin{aligned}[k] \cdot [k+l-1] &= [2k+l-2] + [2k+l-4] + \cdots + [l+2] + [l] \\ &= [k-1] \cdot [(k-1)+l+1] + [l],\end{aligned}$$

and the statement is proved. $\square$

Lemma A.3. *For any positive integer k, one has*

$$\frac{[2k]}{[2]}(q) = [k](q^2).$$

In particular $\frac{[2k]}{[2]} \in \mathbb{Z}_{\geqslant 0}[q^{\pm 1}]$, and one has

$$[2k-1] \geqslant \frac{[2k]}{[2]}.$$

Proof. One has

$$\frac{[2k]}{[2]}(q) = \frac{q^{-2k} - q^{2k}}{q^{-1} - q} = \frac{(q^2)^{-k} - (q^2)^k}{(q^2)^{-\frac{1}{2}} - (q^2)^{\frac{1}{2}}}$$

as announced. $\square$

Corollary A.4. *For any positive integers k and l, one has*

$$[k]^2 \cdot [l]^2 \geqslant \frac{[k] \cdot [l] \cdot [k+l]}{[2]}.$$

Proof. Suppose first that $k+l$ is even. By Lemmas A.3 and A.1, one has

$$\frac{[k+l]}{[2]} \leqslant [k+l-1] \leqslant [k] \cdot [l],$$

and the lemma is proved in this case.

If $k+l$ is odd, we may assume that k is even. Then by Lemmas A.3 and A.1, and Corollary A.2, one has

$$\frac{[k] \cdot [k+l]}{[2]} \leqslant [k-1] \cdot [k+l] \leqslant [k] \cdot [k+l-1] \leqslant [k]^2 \cdot [l],$$

and the lemma is proved in this case as well. $\square$

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