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TRACE PRESERVING MAPPINGS OF MATRIX ALGEBRAS

By Bernard Russo

1. Introduction. In his survey article on linear transformations of matrix algebras, M. Marcus [4; 838–839] states that not much can be said about a linear transformation T on the full $n \times n$ matrix algebra $M_n(F)$ over a field F if it is assumed only that $E_1(T(A)) = E_1(A)$, for all A in $M_n(F)$, where $E_1(A)$ denotes the sum of all "principal 1-square sub-determinants" of A. If F is the complex field, $E_1(A)$ is the trace of A, denoted by tr (A) in the sequel.

In this paper we show that if it is assumed only that tr $(|T(A)|) =$ tr $(|A|)$ for all A in $M_n(C)$, $C =$ the complex field, and that $T(I) = I =$ the identity matrix, then the transformation T is described as follows: there is a unitary matrix U in $M_n(C)$ such that either $T(A) = UAU^*$ for all A in $M_n(C)$; or $T(A) = UA^tU^*$ for all A in $M_n(C)$, where A^t is the transpose of A, A^* denotes the conjugate transpose of A, and $|A|$ denotes the positive root of A^*A.

Roughly speaking then, by introducing some analysis into an algebraic problem we obtain a complete solution. More precisely, the set $M_n(C)$, together with the function $A \to$ tr $(|A|)$, is a Banach space which is denoted by c_1 in [2], and the transformation T is a linear isometry of this Banach space onto itself.

The author has studied isometries of L^p-spaces, $1 \leq p < \infty$, associated with more general operator algebras, and the Theorem of this paper is contained in [5; Theorem 1]. Nevertheless the proof in the present paper is interesting for two reasons; namely, the extreme points of a certain convex set are determined, and only elementary facts about matrices are used, together with a result of Marcus on matrix algebras [3].

2. Preliminaries. We consider the algebra $M_n = M_n(C)$ of all $n \times n$ complex matrices as the algebra of all linear transformations on an n-dimensional complex inner product space H_n. By a projection we mean an element P in M_n such that $P^* = P^2 = P$. We denote the idensity operator on H_n by I or I_n.

If B is a real or complex Banach space then the unit sphere S_B of B is the set $\{x \in B : ||x|| \leq 1\}$, and is a convex set, i.e. if $x, y \in S_B$, then $\lambda x + (1 - \lambda)y \in S_B$ where $0 \leq \lambda \leq 1$ is arbitrary. A vector x in S_B is an extreme point of S_B if whenever $x = \lambda y + (1 - \lambda)z$ with $0 \leq \lambda \leq 1$ and y, z in S_B, then $y = z$. Equivalently, x is an extreme point of S_B if and only if whenever $||x + y|| \leq 1$ and $||x - y|| \leq 1$ for some y in B, then $y = 0$. It follows that extreme joints of S_B are mapped into extreme points by linear isometries of B onto B.

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The linear space M_n is a Banach space in the norm $||A||_1 = \varphi(|A|)$, where φ is the normalized trace, i.e. $\varphi(A) = (1/n) \operatorname{tr}(A) = (1/n) \sum_{i=1}^n a_{ii}$, $A = (a_{ij})$. We remark that $\varphi(A) = (1/n) \sum_{i=1}^n (A\xi_i, \xi_i)$, where $\xi_1, \cdots, \xi_n$ is any orthonormal basis for H_n and that $|\varphi(A)| \leq ||A||_1$ (cf. [2]).

3. The extreme points. In this section and the next, X will denote the Banach space M_n with the norm $||\cdot||_1$, and S will denote the unit sphere of X.

LEMMA 3.1. *Let X_h denote the real Banach space consisting of all hermitian elements of X and let S_h be the unit sphere of X_h. An operator A in S_h is an extreme point of S_h if and only if $A = \pm nP$, where P is a one dimensional projection.*

Proof. Let A be an extreme point of S_h. Since A is hermitian, $A = \sum_{i=1}^n \lambda_i P_i$ where $P_1, \cdots, P_n$ are one-dimensional pairwise orthogonal projections, and $\lambda_1, \cdots, \lambda_n$ are real numbers [1; 156]. The extremity of A implies that $||A||_1 = 1$. Furthermore, $A = \sum_{i=1}^n |\lambda_i|(\pm P_i)$, $|A| = \sum_{i=1}^n |\lambda_i| P_i$, and $1 = ||A||_1 = \varphi(|A|) = (1/n) \sum_{i=1}^n |\lambda_i|$. Since A is extreme, $A = \pm nP_i$, $i = 1, 2, \cdots, n$.

Conversely, let P be a one-dimensional projection and suppose that $nP = \frac{1}{2}(A + B)$ where $A, B \in S_h$. The proof will be complete if it is shown that $A = B$. Indeed, this will imply that nP is an extreme point of S_h and since the map $A \to -A$ is an isometry, it will follow that $-nP$ is also extreme. Now $\varphi(A)$ and $\varphi(B)$ are complex numbers of absolute value at most one, and $\varphi(A) + \varphi(B) = \varphi(A + B) = \varphi(2nP) = 2$. Hence $\varphi(A) = \varphi(B) = 1$. Also $1 = |\varphi(A)| \leq ||A||_1 \leq 1$, so that $||A||_1 = 1$ and similarly $||B||_1 = 1$. It follows that $|A| - A$ and $|B| - B$ are positive operators of trace zero, so that A and B are positive operators. Thus $2nP = A + B \geq A$ and therefore A is a scalar multiple of P. The same is true of B and it follows easily that $A = nP = B$.

LEMMA 3.2. *For each A in M_n, $n\,||A||_1 \geq ||A||$, where $||A||$ denotes the norm of the linear transformation A.*

Proof. If A is hermitian, say $A = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$, then

$$||A|| = \max_{1 \leq i \leq n} |\lambda_i| \leq \sum_{i=1}^n |\lambda_i| = n\left((1/n) \sum_{i=1}^n |\lambda_i|\right) = n\,||A||_1 .$$

For a general A, let $A = U\,|A|$ be the polar decomposition so that U is unitary [1; 169]. Then $||A|| = ||U\,|A|\,|| = ||\,|A|\,|| \leq n\,||\,|A|\,||_1 = n\,||A||_1$.

PROPOSITION 3.1. *An operator A in S is an extreme point of S if and only if $A = nUP$, where U is a unitary matrix and P is a one-dimensional projection.*

Proof. Let A be an extreme point of S. Letting $A = U\,|A|$ be the polar decomposition of A it follows (since $B \to U^{-1}B$ is an isometry of X) that $|A|$ is an extreme point of S. A fortiori, $|A|$ is an extreme point of S_h. By Lemma 3.1 $|A| = \pm nP$, so that $A = U\,|A| = n(\pm U)P$.

For the converse it is sufficient, as in the proof of Lemma 3.1 to prove that nP is an extreme point of S for P a one dimensional projection. Accordingly suppose $nP = \frac{1}{2}(A + B)$ with $A, B \in S$. Then $2n = ||2nP|| = ||A + B|| \leq ||A|| + ||B|| \leq n\,||A||_1 + n\,||B||_1 \leq 2n$, by Lemma 3.2. Hence $||A|| = ||B|| = n$. Trivially $PAP = nP = PBP$. Let $\xi_1, \cdots, \xi_n$ be an orthonormal basis for H_n chosen so that ξ_1 belongs to the range of P. Then $1 = \varphi(nP) = \varphi(PAP) = (1/n)\sum_{i=1}^{n}(PAP\xi_i, \xi_i) = (PAP\xi_1, \xi_1)/n = (A\xi_1, \xi_1)/n$, so that $(A\xi_1, \xi_1) = n$. Since $||A|| \leq n$, we have equality in the Schwarz inequality so $A\xi_1 = n\xi_1$, and similarly $B\xi_1 = n\xi_1$. Since $||\,|A|\,|| = n$, n is an eigenvalue of $|A|$. Since $\varphi(|A|) = 1$, the multiplicity of n is one and all other eigenvalues of $|A|$ are zero. Let $\eta_1, \cdots, \eta_n$ be an orthonormal basis for H_n consisting of eigenvectors for $|A|$ chosen so that $|A|\,\eta_1 = n\eta_1$ and $|A|\,\eta_i = 0$, $i = 2, \cdots, n$, [1; 156]. Writing $\xi_1 = \sum_{i=1}^{n}(\xi_1, \eta_i)\eta_i$, we have $n\xi_1 = A\xi_1 = \sum_{i=1}^{n}(\xi_1, \eta_i)A\eta_i = \sum_{i=1}^{n}(\xi_1, \eta_i)U\,|A|\,\eta_i = (\xi_1, \eta_1)nU\eta_1$. Again we have equality in the Schwarz inequality so that ξ_1 is a scalar multiple of η_1; in particular ξ_1 is orthogonal to $\eta_2, \cdots, \eta_n$. It is now easy to check that A and nP agree on the orthonormal basis $\xi_1, \eta_2, \cdots, \eta_n$. The proof is complete.

4. The Theorem.

THEOREM. *Let T be a linear transformation of $M_n(C)$ onto $M_n(C)$ such that* $\mathrm{tr}\,(|T(A)|) = \mathrm{tr}\,(|A|)$ *for each A in $M_n(C)$ and $T(I) = I$. Then there is a unitary matrix U in $M_n(C)$ such that either $T(A) = UAU^*$ for all A in $M_n(C)$; or $T(A) = UA^tU^*$ for all A in $M_n(C)$.*

LEMMA 4.1. *Let $K_1, K_2, \cdots, K_r$ be $(n - 1)$-dimensional subspaces of H_n, where $n \geq 3$ and $2 \leq r \leq n-1$. Then the dimension of $K_1 \cap K_2 \cap \cdots \cap K_r$ is not less than $n - r$; in particular $K_1 \cap K_2 \cap \cdots \cap K_r \neq \{0\}$.*

Proof. Induction based on the equation $\dim(K_1 \cap \cdots \cap K_r) = \dim((K_1 \cap \cdots \cap K_{r-1}) \cap K_r) = -\dim((K_1 \cap \cdots \cap K_{r-1}) + K_r) + \dim(K_1 \cap \cdots \cap K_{r-1}) + \dim(K_r)$.

LEMMA 4.2 *If $E_1, \cdots, E_k$, and $E_1 + \cdots + E_k$ are all projections, then $E_iE_j = 0$ if $i \neq j$.*

Proof. An easy induction.

LEMMA 4.3. *Suppose that $I_n = \sum_{i=1}^{n} V_i$, where $V_i^*V_i = Q_i$ is a one dimensional projection, $i = 1, \cdots, n$. Then $V_i = Q_i$ for $i = 1, \cdots, n$; and $Q_iQ_j = 0$ for $i \neq j$.*

Proof. Applying Lemma 4.1 to the ranges of $I - Q_i$ for $i \neq j$, there is for each fixed j, a non-zero vector ξ_j such that $Q_i\xi_j = 0$ for $i \neq j$. Then $V_i\xi_j = 0$ for $i \neq j$ and thus $V_j\xi_j = I\xi_j = \xi_j \neq 0$. Let P_j be the projection on the non-zero subspace $\{\xi : V_j\xi = \xi\}$. Then $0 \neq P_j \leq Q_j$; and since Q_j is one-dimensional, $P_j = Q_j = V_j$. Since j is arbitrary, the proof of the first part of the lemma is complete. The rest follows from Lemma 4.2.

Proof of the theorem. The mapping T is an isometry of the Banach space X onto itself. Let V be a unitary matrix in $M_n(C)$. Then $V = \sum_{i=1}^n \lambda_i P_i$ and $I = \sum_{i=1}^n P_i$, where $P_1, \cdots, P_n$ are pairwise orthogonal one-dimensional projections and $|\lambda_i| = 1, i = 1, \cdots, n$ [1; 161]. Using Proposition 3.1 $V_i \equiv T(P_i) = U_i Q_i$, where Q_i is a one dimensional projection and U_i is a unitary matrix, $i = 1, \cdots, n$. It follows that $T(V) = \sum_{i=1}^n \lambda_i V_i$ and $I = T(I) = \sum_{i=1}^n V_i$ where $V_i^* V_i = Q_i, i = 1, \cdots, n$. Thus by Lemma 4.3 $T(V)$ is a unitary matrix, i.e. T maps the unitary group of $M_n(C)$ into itself. The result now follows from a theorem of Marcus [3; 155].

5. Remarks. 1. We have assumed in the theorem that $T(I) = I$. As shown in [5], this assumption is not essential in general. However, it does seem to be essential here for an elementary proof (cf. Lemma 4.3).

2. It is curious to note that whereas the theorem of Marcus used above is valid in more general operator algebras [6; Corollary 2], it is not used in the proof of [5, Theorem 1].

Addendum. The ideas of this paper have been extended to cover the infinite dimensional case. Details will appear in the Proc. Amer. Math. Soc.

References

1. P. R. Halmos, *Finite Dimensional Vector Spaces*, New York, 1958.
2. C. A. McCarthy, c_p, Israel J. Math., vol. 5(1967), pp. 249–271.
3. M. Marcus, *All linear operators leaving the unitary group invariant*, Duke Math. J., vol. 26 (1959), pp. 155–164.
4. M. Marcus, *Linear operations on matrices*, Amer. Math. Monthly, vol. 69(1962), pp. 837–847.
5. B. Russo, *Isometries of* L^p*-spaces associated with finite von Neumann algebras*, Bull. Amer. Math. Soc., vol. 74(1968), pp. 228–232.
6. B. Russo and H. A. Dye, *A note on unitary operators in* C**-algebras*, Duke Math. J., vol. 33(1966), pp. 413–416.

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