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ABSTRACT

Parity tests and ambiguities are discussed for fermion interactions. These include decays into spin-1/2 and spin-3/2 fermions, as well as fermion production from a polarized target. Complete tests for the several-step decay of a high-spin formation resonance are presented.

Parity of Fermions: Tests and Ambiguities*

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This letter presents, through the use of invariance arguments, simple discussions of parity tests and of ambiguities in the following processes: the strong decay of a fermion F_J into a fermion $F_{1/2}$ plus a boson B_0; the strong decay of an F_J into an $F_{3/2}$ plus a B_0; and the production of an $F_{1/2}$ plus a B_0 from a polarized target.[1,2] Decay of a "formation" resonance into an $F_{3/2}$ is treated extensively.

Decay into $F_{1/2}$. -- No parity information can be obtained from the decay angular distribution of a spin-J fermion (F_J) that yields a spin-1/2 fermion ($F_{1/2}$) plus a spinless boson (B_0). A decay matrix (M_+) describing decay of one parity must be multiplied by a pseudoscalar $\bar{\sigma}\cdot\hat{p}$ to obtain the decay matrix (M_-) required for the opposite parity. (The operator $\bar{\sigma}$ is associated with the spin of the final $F_{1/2}$, and $\hat{p}$ is a unit vector along the direction of decay momentum in F_J's rest frame.) Thus

$$M_- \equiv \bar{\sigma}\cdot\hat{p}\; M_+. \tag{1}$$

The initial state is describable by a density matrix ρ_i, so normalized that $\mathrm{Tr}\,\rho_i = 1$. The angular distributions for the two parities are

$$I_+ = \mathrm{Tr}(M_+ \rho_i M_+^\dagger)$$
$$I_- = \mathrm{Tr}[(\bar{\sigma}\cdot\hat{p}\; M_+)\,\rho_i\,(M_+^\dagger\,\bar{\sigma}\cdot\hat{p})] = \mathrm{Tr}[(\bar{\sigma}\cdot\hat{p})^2 M_+\rho_i M_+^\dagger]. \tag{2}$$

Since $(\bar{\sigma}\cdot\hat{p})^2 = I$, the M_+ and M_- transformations are here indistinguishable.[3]

The polarization of the outgoing $F_{1/2}$ is found by evaluating

$$I\,\bar{P}_+ = \mathrm{Tr}(\bar{\sigma}\rho_{f+}) = \mathrm{Tr}[\bar{\sigma}(M_+\rho_i M_+^\dagger)], \tag{3}$$

or, for the opposite parity,

$$I\,\bar{P}_- = \mathrm{Tr}[\bar{\sigma}(\bar{\sigma}\cdot\hat{p}\ \ M_+\rho_i M_+^\dagger\ \ \bar{\sigma}\cdot\hat{p})]\ . \tag{4}$$

By definition, $M_+\rho_i M_+^\dagger$ or ρ_{f+} must equal $\frac{1}{2}I(1+\bar{P}_+\cdot\bar{\sigma})$;[4] thus

$$\bar{P}_- = \bar{\sigma}\cdot\hat{p}\ (\bar{P}_+\cdot\bar{\sigma})\,\bar{\sigma}\cdot\hat{p}. \tag{5}$$

But $i\ \bar{\sigma}\cdot\hat{p}$ is the same as the rotation operator $R(\pi) = \exp(i\ \bar{\sigma}\cdot\hat{p}\ \pi/2)$; hence Eq. (5) may be written

$$\bar{P}_- = R(\pi)\,[\bar{P}_+\cdot\bar{\sigma}]\ \ R^{-1}(\pi). \tag{6}$$

The $F_{1/2}$ vector polarizations for the two decay parities thus differ by a rotation of 180 deg about $\hat{p}$.[5]

<u>Decay into $F_{3/2}$</u>. --The angular distribution for decay of an F_J into $F_{3/2}$ is <u>not</u> parity-ambiguous in the same sense as that for decay into $F_{1/2}$. However, a parity determination from the angular distribution alone is sometimes impossible.

Two orbital angular momenta are possible for each parity in the strong decay into an $F_{3/2}$: $\ell_+ = J - 3/2$ and $\ell'_+ = J + 1/2$, or $\ell_- = J - 1/2$ and $\ell'_- = J + 3/2$.[6] If the transition matrices are separated into lower and higher ℓ-wave contributions, $\mathcal{M}^{\ell}$ and $\mathcal{M}^{\ell'}$, they are related by[7]

$$\mathcal{M}_-^{\ell} + \mathcal{M}_-^{\ell'} = e\,T_{10}\mathcal{M}_+^{\ell} + f\,T_{30}T_{20}^{-1}\,\mathcal{M}_+^{\ell'} \tag{7}$$

(The T_{L0} are spin-3/2 operators expressed in the helicity system, with $T_{10} \propto S_z = \bar{S}\cdot\hat{p}$. The e and f are complex numbers. Cf. Eqs. (3) and (5) of Ref. 7.) Neither of the "parity operators" T_{10} or $T_{30}T_{20}^{-1}$ is unitary, as is $\bar{\sigma}\cdot\hat{p}$ for spin 1/2:

$$T_{10} \propto \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} ; \qquad T_{30}T_{20}^{-1} \propto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} . \quad (8)$$

Thus, in general, the angular distribution

$$I = \mathrm{Tr}\,[\,(\mathcal{M}^{\ell} + \mathcal{M}^{\ell'})\,\rho_i\,(\mathcal{M}^{\ell} + \mathcal{M}^{\ell'})^{\dagger}]/\mathrm{Tr}\,\rho_i \quad (9)$$

differs for even and odd parities.[8]

Although the angular distribution does not involve a Minami-type ambiguity, it does not yield enough information to determine the F_J parity (as well as two partial amplitudes) if J is $\leq 5/2$.

Neither of the (non-unitary) parity operators can be equivalent to a rotation operator that acts on $F_{3/2}$ polarization.[9]

Parity Tests for Formation Resonances. --Decays of fermions into an $F_{3/2}$ have recently been analyzed in "formation" experiments.[10] The two tests utilized may be considerably extended.

The process to be discussed is

$$F_J \xrightarrow[(S)]{(1)} F_{3/2} \xrightarrow[(S)]{(2)} F_{1/2} \xrightarrow[(W)]{(3)} f_{1/2}. \quad (10)$$

A spinless boson is understood to accompany each final fermion. The numbers indicate the step of decay; the letters the strength of decay. The decay of a final-state resonance F_J in this sequence has been treated theoretically, with and without the use of T_{LM} spin operators.[7, 11]

A brief discussion of the T_{LM} tensors will be helpful. These are of great utility for spin-state description, as they make possible the formulation of a complete set of independent spin-parity tests. Each $\langle T_{LM}\rangle$ characterizing a particle's state combines with a $Y_{LM}(\theta, \phi)$ or a $\mathcal{D}^{L}_{MM'}(\phi, \theta, 0)$ in its decay distribution.[12] A system of spin J requires $(2J+1)^2$ parameters for the description of its spin state. For an $F_{3/2}$, the normalization and vector-polarization terms $(\langle T_{00}\rangle = \langle I\rangle,\ \langle T_{10}\rangle \propto \langle S_z\rangle$, etc.) plus twelve additional quantities--such as $\langle T_{20}\rangle \propto \langle 3\,S_z^2 - S^2\rangle$, $\langle T_{21}\rangle \propto \langle S_z(S_x+iS_y)\cdots\rangle$, $\langle T_{22}\rangle \propto \langle (S_x+iS_y)^2\rangle$, and $\langle T_{30}\rangle \propto \langle S_z^3 \cdots\rangle$ --are required. The $\langle T_{2M}\rangle$, which are second-rank tensor polarizations, correspond to alignment of spin. They are quantities similar to moments of inertia or to the nuclear electric quadrupole moment.

For the "formation resonance" produced from a $B_0 + F_{1/2}$ system, angular-momentum conservation in production permits only even-L, M = 0 $\langle T_{LM}\rangle$ if the incident-beam direction is the z axis.[13] (Only the $m_J = \pm\frac{1}{2}$ spin states are occupied.)

The derivations of Ref. 7 may be readily extended to treat the formation resonance. The initial $\langle T_{LM}\rangle \equiv t_{LM}$ and the helicity amplitudes A_λ [contained in $\mathcal{M}$, Eq. (7)] are used to form the density matrix for the outgoing spin-3/2 particle:

$$[\rho_{(3/2)}]_{\lambda\lambda'} = A_\lambda A^*_{\lambda'} \sum_{L_e}^{2J-1} n^{(2\lambda)}_{L,\lambda-\lambda'}\, t_{L0}\, \mathcal{D}^L_{0,\lambda-\lambda'}(0,\theta,0) \qquad (11)$$

where L_e is even. The $n^{(2\lambda)}_{L,\lambda-\lambda'}$ quantities each contain a Clebsch-Gordan coefficient; they may be expressed in terms of $n^{(1)}_{L0}$ by use of recursion relations.[7]

For initial spin J = 5/2 ,

$$A_+ = \left(\frac{1}{20}\right)^{1/2} \begin{bmatrix} 2a+\sqrt{6}c & 0 & 0 & 0 \\ 0 & \sqrt{6}a-2c & 0 & 0 \\ 0 & 0 & \sqrt{6}a-2c & 0 \\ 0 & 0 & 0 & 2a+\sqrt{6}c \end{bmatrix};$$

$$A_- = \left(\frac{1}{28}\right)^{1/2} \begin{bmatrix} 2\sqrt{3}b+\sqrt{2}d & 0 & 0 & 0 \\ 0 & \sqrt{2}b-2\sqrt{3}d & 0 & 0 \\ 0 & 0 & -\sqrt{2}b+2\sqrt{3}d & 0 \\ 0 & 0 & 0 & -2\sqrt{3}b-\sqrt{2}d \end{bmatrix}; \quad (12)$$

here a, b, c, and d designate the p- through g-wave amplitudes. For a formation resonance of spin 5/2,

$$t_{00} = 1.000;\ t_{20} = -0.478;$$
$$t_{40} = 0.309;\ \text{all other}\ t_{LM} = 0. \quad (13)$$

[14]

The angular distribution for decay (1) is $[\mathrm{Tr}\,\rho_{(J)}$ being 1][15]

$$I(\theta) = \mathrm{Tr}\,\rho_{(3/2)} = \sum_{L_e}^{4} C_L\, t_{L0}\, Y_{L0}(\theta), \quad (14)$$

where each C_L is a function of $|a|^2$, $|c|^2$, and 2 Re a^*c or $|b|^2$, $|d|^2$, and 2 Re b^*d. With the three C_L from $I(\theta)$ data of a J = 5/2 formation resonance, amplitude solutions can be found for either parity.

If some estimate of $|c|$ (or $|d|$) relative to $|a|$ (or $|b|$) can be made, however, a parity determination may be possible. Equation (16) of Ref. 7 with c, d = 0 and J = 5/2 yields the production distributions presented by Minami:[16, 17]

$$I_+(\theta) = (1/2)[1 + 0.800\ P_2(\cos\theta)] \qquad (15)$$

$$I_-(\theta) = (1/2)[1 + 0.409\ P_2(\cos\theta) - 0.976\ P_4(\cos\theta)]. \qquad (16)$$

Decay (2) can be analyzed for F_J parity information. The distribution of $\hat{F}_{1/2}$ (in $F_{3/2}$'s rest frame) $\cdot$ $\hat{F}_{3/2}$ (in the resonance rest frame) will have the form[18]

$$\mathcal{L}(\theta,\psi) \propto I(\theta)[1 - \langle T_{20}\rangle(\theta)\sqrt{5}\ P_2(\cos\psi)] \qquad (17)$$

with $\cos\psi = \hat{F}_{1/2}\cdot\hat{F}_{3/2}$. If the θ of decay (1) and the higher ℓ' wave are ignored [Eqs. (22) and (23) of Ref. 7]:

$$\mathcal{L}'_+(\psi) \propto \{1 + [(2J-3)/4J]\ P_2(\cos\psi)\}$$

$$\mathcal{L}'_-(\psi) \propto \{1 - [(2J+5)/(4J+4)]\ P_2(\cos\psi)\}; \qquad (18)$$

for J = 5/2 these equations are:[19]

$$\mathcal{L}'_+(\psi) \propto [1 + 0.200\ P_2] \text{ and } \mathcal{L}'_-(\psi) \propto [1 - 0.714\ P_2]. \qquad (19)$$

Transformations of $\langle T_{2m}\rangle$ along $\hat{F}_{3/2}$ to $\langle T_{20}\rangle$ along other axes give different P_2 coefficients.[20] With the incident beam as polar axis, these are 0.800 and -0.114 for even and odd parity, respectively.[19] With the production normal as polar axis, these coefficients become -0.700 and 0.786 for even and odd parity.[21] (Some caution should be exercised in interpreting average

$F_{3/2}$ alignment if the formation resonance has any background.) A complete analysis is of course unaffected by the choice of coordinates.

Complete Parity Tests for F_J (formation) $\rightarrow F_{3/2}$. --The above tests [Eqs. (14) and (19)] treat only two "profiles" of a probability distribution. A complete analysis of the distribution involves the full examination of decay (2) for each θ interval in decay (1).

The following [from Eq. (19), Ref. 7] give the expected θ-dependence of the $F_{3/2}$'s (real) second-rank tensor polarizations.[22] [The first- and third-rank polarizations are not observable in decay (2).]

$$I\langle T_{20}\rangle = Tr[\rho_{(3/2)}T_{20}] = 2\pi(1/5)^{1/2} \sum_{L_e}^{2J-1} [2A_3^2\, n_{L0}^{(3)} - 2A_1^2\, n_{L0}^{(1)}]t_{L0}\, Y_{L0}(\theta)$$

$$I\langle T_{21}\rangle = 2\pi(2/5)^{1/2} \sum_{L_e}^{2J-1} (-A_1 A_3^* - A_{-3}A_{-1}^*)\, n_{L1}^{(3)} t_{L0}\, D_{01}^L(0, \theta, 0)$$

$$I\langle T_{22}\rangle = 2\pi(2/5)^{1/2} \sum_{L_e}^{2J-1} (A_{-1}A_3^* + A_{-3}A_1^*)\, n_{L2}^{(3)}\, t_{L0}\, D_{02}^L(0, \theta, 0) \tag{20}$$

$$I\langle T_{\ell,-m}\rangle = (-)^m\, I\langle T_{\ell,m}\rangle^* = (-)^m I\langle T_{\ell,m}\rangle \quad .$$

For J = 5/2, the first of these becomes

$$I\langle T_{20}\rangle =$$

$$(2\pi/\sqrt{5}) \sum_{L_e}^{4} \begin{Bmatrix} [(2a^2+3c^2+2\sqrt{6}\ \mathrm{Re}\ a^*c)(1-L[L+1]/8)-(3a^2+2c^2-2\sqrt{6}\ \mathrm{Re}\ a^*c)](1/5) \\ [6b^2+d^2+2\sqrt{6}\ \mathrm{Re}\ b^*d)(1-L[L+1]/8) - (b^2+6d^2-2\sqrt{6}\ \mathrm{Re}\ b^*d)](1/7) \end{Bmatrix}$$

$$\times n_{L0}^{(1)} t_{L0} Y_{L0}(\theta) . \quad (21)$$

In these equations, amplitudes have been abbreviated (A_3 instead of $A_{3/2}$, and A^2 instead of $|A|^2$); and $D^L_{0M'}$ has replaced $[(2L+1)/4\pi]^{1/2}\mathscr{D}^L_{0M'}$.

The analysis of the above tensor polarizations may be made by comparing the data with[7]

$$\mathscr{Q}(\theta;\psi,\zeta) = (1/4\pi)\, I(\theta)\, \{1 - \langle T_{20}\rangle(\theta)\sqrt{5}\,(3\cos^2\psi-1)/2$$

$$+2(15/2)^{1/2}\ \mathrm{Re}\,\langle T_{21}\rangle(\theta)\cos\zeta\sin\psi\cos\psi - (15/2)^{1/2}$$

$$-(15/2)^{1/2}\ \mathrm{Re}\langle T_{22}\rangle(\theta)\cos 2\zeta\,\sin^2\psi\} . \quad (22)$$

Histograms of $I(\theta)$ and $I\langle T_{2m}\rangle(\theta)$ may be compared with the following expressions:

$$I(\theta) = \sum_{L_e} \sigma_L Y_{L0}(\theta)\,(4\pi)^{1/2}$$

$$I\langle T_{20}\rangle(\theta) = \sum_{L_e} \tau_L Y_{L0}(\theta)\,(4\pi)^{1/2}$$

$$I\langle T_{21}\rangle(\theta) = \sum_{L_e} \mu'_L\ D^L_{01}(0,\theta,0) = \sum_{L_e} \mu_L Y_{L1}(\theta,0)\,(4\pi)^{1/2} \quad (23)$$

$$I\langle T_{22}\rangle(\theta) = \sum_{L_e} \nu'_L\ D^L_{02}(0,\theta,0) = \sum_{L_e} \nu_L\ Y_{L2}(\theta,0)\,(4\pi)^{1/2}.$$

The coefficients σ_L, τ_L, μ_L, and ν_L depend on spin, parity, and amplitudes. They are given in Table I for J = 5/2 decay with the higher ℓ' amplitude neglected. Figure 1 displays $I\langle T_{21}\rangle(\theta)$ and $I\langle T_{22}\rangle(\theta)$.

After analyzing the data for the $I\langle T_{2m}\rangle(\theta)$, one may evaluate parity (and spin) by taking a ratio of certain moments.[23] The following is valid with any amount of higher ℓ' wave:

$$I\langle T_{22}\rangle \text{ moment}/I\langle T_{21}\rangle \text{ moment} = \langle\langle T_{22}\rangle D^{L*}_{02}\rangle / \langle\langle T_{21}\rangle D^{L*}_{01}\rangle$$

$$= \nu_L/(-\mu_L) = \Gamma(J+\tfrac{1}{2})/[(L+2)(L-1)]^{\frac{1}{2}}, \qquad (24)$$

where Γ = +1 or -1 for "even" ($3/2^-$, $5/2^+$, etc.) or "odd" parity, respectively. [Eq. (24) is similar to Eq. (31) of Ref. 7.] If J = 5/2, two independent tests are possible (for L = 2 and L = 4).

Parity tests may be possible in decay (3) of the formation-resonance decay scheme. The odd-ℓ polarizations resulting from the formation-resonance decay, $F_J \to F_{3/2}$, are

$$I\langle T_{10}\rangle = I\langle T_{30}\rangle = I\langle T_{33}\rangle = 0$$

$$I\langle T_{11}\rangle = -2\pi(2/15)^{1/2} \sum_{L_e}^{2J-1} (A_1A_3^* - A_{-3}A_{-1}^*)\sqrt{3}\, n^{(3)}_{L1}\, t_{L0}\, D^L_{01}(0,\theta,0)$$

$$I\langle T_{31}\rangle = 4\pi(1/35)^{1/2} \sum_{L_e}^{2J-1} (-A_1A_3^* + A_{-3}A_{-1}^*)\, n^{(3)}_{L1}\, t_{L0}\, D^L_{01}(0,\theta,0) \qquad (25)$$

$$I\langle T_{32}\rangle = 2\pi(2/7)^{1/2} \sum_{L_e}^{2J-1} (A_{-1}A_3^* - A_{-3}A_1^*)\, n^{(3)}_{L2}\, t_{L0}\, D^L_{02}(0,\theta,0).$$

These reduce to expressions proportional to Im a^*c or Im b^*d. A ratio of an $I\langle T_{32}\rangle$ moment (for L = 2, 4···) to either an $I\langle T_{11}\rangle$ or an $I\langle T_{31}\rangle$ moment may yield parity (and spin) information.

The $I\langle T_{\ell m}\rangle$ of Eq. (25) may be analyzed by determining the polarization of $F_{1/2}$ from the angular distribution of its weak decay. See Eq. (27) of Ref. 7 (or Addendum to UCRL-16857.)[24]

In conclusion, the following can be said about $F_J \rightarrow F_{3/2}$ decay:

1) A "formation" resonance generally yields considerably less spin-parity information than a "final-state" resonance.

2) Parity cannot be tested in (formation) decay (1) if the higher ℓ wave is taken into account and if $J \leq 5/2$.

3) Parity analysis does not require initial-state vector polarization; F_J alignment yields an excellent test in the strong decay (2) (even with higher ℓ wave).

4) Spin-parity information may be obtained from the weak decay (3), especially for the final-state resonance.

5) If complete angular dependences of decay are investigated, the spin-parity conclusions cannot be affected by the choice of coordinate system.

The above descriptions are complete and are relativistic. For a more extensive discussion, see Ref. 24.

<u>$F_{1/2}$ production from a polarized target.</u> --Invariance arguments may be used to determine parity effects in the distribution and polarization of an $F_{1/2}$ from a polarized $F'_{1/2}$ in the process[25]

$$B_0 + F'_{1/2}\ (\text{polarized}) \rightarrow B'_0 + F_{1/2}. \tag{26}$$

A simple treatment may be made in analogy to the above discussion of the decay $F_J \rightarrow F_{1/2}$.

The transition matrix for the process of Eq. (26) is

$$M_{\pm} = g \pm h\ \vec{\sigma} \cdot \hat{n}, \tag{27}$$

(where $\hat{n}$ is the normal to the production plane and g and h are complex amplitudes) if the intrinsic parity $P(F_{1/2})$ is <u>even</u> relative to $P(B_0) \times P(F'_{1/2}) \times P(B'_0)$. If the parity $P(F_{1/2})$ is relatively odd, then a "parity operator" $\bar{\sigma} \cdot \hat{k}$ changes M_+ to a pseudoscalar form:

$$M_- = (g + h\ \bar{\sigma} \cdot \hat{n})(\bar{\sigma} \cdot \hat{k}). \tag{28}$$

The vector $\hat{k}$ may be any combination of initial and final momenta in the c.m. frame.

The angular distribution of the outgoing $F_{1/2}$ is, with P_t defined as target polarization and $\cos\phi \equiv \hat{n} \cdot \bar{P}_t / P_t$,

$$\begin{aligned} I_+(\phi) &= \mathrm{Tr}\left[M_+\ \tfrac{1}{2}\ (1 + \bar{P}_t \cdot \bar{\sigma}) M_+^\dagger\right] \\ &= |g|^2 + |h|^2 + 2\ \mathrm{Re}\ g^* h\ P_t \cos\phi. \end{aligned} \tag{29}$$

In a separate experiment that produces $F_{1/2}$ from an <u>unpolarized</u> target, the cross section I_0 and polarization $I_0 P_{F0}$ are found. Thus Eq. (29) may be rewritten:

$$I_+(\phi) = I_0(1 + P_{F0}\, P_t \cos\phi). \tag{30}$$

If the relative $F_{1/2}$ parity is <u>odd</u> rather than even, the angular distribution becomes

$$I_-(\phi) = \mathrm{Tr}(M_+\ \bar{\sigma} \cdot \hat{k}\ \ \rho_i\ \bar{\sigma} \cdot \hat{k}\ M_+^\dagger); \tag{31}$$

but as discussed above, [Eqs. (5, 6)], $i\ \bar{\sigma} \cdot \hat{k} = R_k(\pi)$ and thus

$$I_-(\phi) = \mathrm{Tr}\{M_+ [R(\pi)\ \rho_i\ R^{-1}(\pi)] M_+^\dagger\}. \tag{32}$$

This means that the $\bar{P}_t$ in the initial density matrix will appear to be rotated (directed along -z instead of +z). The differential cross section becomes

$$I_-(\phi) = I_0(1 - P_{F0}\ P_t \cos\phi). \tag{33}$$

[We check that P_{F0} has not changed: $IP_{F0-} = \mathrm{Tr}(\bar{\sigma}\cdot\hat{n} M_+ \bar{\sigma}\cdot\hat{k} \frac{1}{2} \bar{\sigma}\cdot\hat{k} M_+^\dagger) = 2\,\mathrm{Re}\, g^* h.$] Evidently the relative parity of $F_{1/2}$ will be manifested in the sign of the $\cos\phi$ term.[26]

The polarization of the outgoing $F_{1/2}$ from a polarized target depends on its relative parity. If events are selected so that the scattering normal is parallel to $\bar{P}_t$, then for even parity

$$I\,\bar{P}_F\cdot\hat{P}_t \equiv I\,\hat{P}_F\cdot\hat{z} = \mathrm{Tr}[\sigma_z M_+ \frac{1}{2}(1 + P_t\sigma_z) M_+^\dagger]$$

$$= I_0(P_{F0} + P_t); \tag{34}$$

for odd parity,

$$I\,\bar{P}_F\cdot\hat{P}_t = \mathrm{Tr}[\sigma_z M_+ \bar{\sigma}\cdot\hat{k}\ \frac{1}{2}(1 + P_t\sigma_z)\,\bar{\sigma}\cdot\hat{k}\ M_+^\dagger]$$

$$= I_0(P_{F0} - P_t). \tag{35}$$

Again the parity operator is equivalent to a rotation of the initial density matrix; and this rotation causes a sign change in P_t. Thus Eqs. (34) and (35) yield a further test for the $F_{1/2}$ parity.

Acknowledgments. --The author acknowledges inspiration derived from the original work of Byers and Fenster,[13] as well as encouragement from Prof. Charles Zemach, and Dr. Henry Stapp.

FOOTNOTES AND REFERENCES

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1. Examples of such treatment are to be found in H. A. Bethe and F. de Hoffman, Mesons and Fields, Vol. II (Row, Peterson and Co., Evanston, Illinois, June, 1965) p. 75 (the Dyson and Nambu proof of the Minami ambiguity); L. Wolfenstein and J. Ashkin, Phys. Rev. 85, 947 (1952); and R. K. Adair, Rev. Mod. Phys. 33, 406 (1951).

2. Here F_J designates a fermion of spin J; and $B_{J'}$, a boson of spin J'.

3. In π-N scattering, this deficiency of parity information in the angular distribution has been known as the "Minami ambiguity."

4. The density matrix equals $(2J+1)^{-1} \sum_{\mu} \langle S_{\mu} \rangle^{*} S_{\mu}$ where the S_{μ} are a complete set of spin operators (Wolfenstein and Ashkin, Ref. 1).

5. Cf. the special cases calculated by Adair, Ref. 1 (J = 1/2), and by J. B. S. (J = 3/2, 5/2), J. B. Shafer, J. J. Murray, and D. O. Huwe, Phys. Rev. Letters 10, 179 (1963), and a discussion of C. Zemach, Phys. Rev. 140, B109 (1965).

6. The + or - subscript designates the $J^P = 1/2^+, 3/2^-, 5/2^+ \ldots$ sequence or the $1/2^-, 3/2^+ \ldots$ sequence, respectively (P being the F_J-$F_{3/2}$ relative parity). Angular-momentum conservation permits only the higher ℓ' waves for $J^P = 1/2^+$ and $1/2^-$.

7. J. Button-Shafer, Phys. Rev. 139, B607 (1965).

8. There is one special case when these are indistinguishable: when the spin J and the partial amplitudes are such that $|e\mathcal{M}_{\pm}^{\ell}|^2 = |f\mathcal{M}_{\pm}^{\ell'} T_{20}^{-1}|^2 (3/7)$, the $\mathcal{M}_-$ terms give incoherent contributions proportional to the identity and thus similar to $\mathcal{M}_+$ contributions. (Interference terms from $\mathcal{M}_-$ and $\mathcal{M}_+$ are always similar.)

9. Any rotation operator is unitary; the parity operators here are not. For the special case of footnote 8, a unitary combination of parity operators exists, but is not equivalent to any $R_p(\phi)$.

10. A "formation" resonance is an s-channel resonance involving all particles produced.

11. S. M. Berman and M. Jacob, Spin and Parity Analysis in Two-Step Decay Processes, unpublished report SLAC-43 (1965); and C. Zemach, Ref. 5.

12. N. Byers and S. Fenster, Phys. Rev. Letters 11, 52 (1963). The $\langle T_{LM}\rangle$ are referred to by Byers and Fenster as "multipole parameters." All $\langle T_{LM}\rangle$ with $0 \leq L \leq 2J$ and $-L \leq M \leq L$ are allowed.

13. The incident-beam direction is the only possible choice in a "formation" experiment because the decay must be referred to axes from a prior system.

14. These are formed by taking $\mathrm{Tr}[\rho_{(J)} T_{LM}] = \frac{1}{2}[(T_{LM})_{\frac{1}{2},\frac{1}{2}} + (T_{LM})_{-\frac{1}{2},-\frac{1}{2}}]$ where $(T_{LM})_{mm'} \equiv C(JLJ;m'M)$ with $m-m' = M$.

15. Cf. Eq. (16) of Ref. 7. The values of the $n_{L0}^{(1)}$ coefficients required for $J = 5/2$ are $n_{00}^{(1)} = (4\pi)^{-1/2}$, $n_{20}^{(1)} = -1.07(4\pi)^{-1/2}$, and $n_{40}^{(1)} = 0.925(4\pi)^{-1/2}$.

16. S. Minami, Nuovo Cimento 31, 258 (1964).

17. R. W. Birge, R. P. Ely, G. E. Kalmus, A. Kernan, J. Louie, J. S. Sahouria, and W. M. Smart, Proceedings of the Athens Topical Conference on Recently Discovered Resonant Particles, June 1965 (Ohio University Physics Dept., 1966); and R. Armenteros, M. Ferro-Luzzi, D. W. G. Leith, R. Levi-Setti, A. Minten, R. D. Tripp, H. Filthuth, V. Hepp, E. Kluge, H. Schneider, R. Barloutaud, P. Granet, J. Meyer, J.-P. Porte, Phys. Letters 19, 338 (1965).

A brief reanalysis of CERN (Armenteros et al.) data has recently appeared; this takes account of higher ℓ waves for just the two distributions examined by experimenters. [G. F. Wolters and D. J. Holthuizen, Phys. Letters 19, 701 (1966)].

18. The customary Byers-Fenster distribution for decay into $F_{1/2}$ yields the expression in brackets. Here the notation $\langle T_{\ell m} \rangle$ is reserved for $F_{3/2}$ and t_{LM} for F_J.

19. Cf. J. D. Jackson, University of Illinois preprint, Particle and Polarization Angular Distributions for Two- and Three-Body Decays, prepared for Les Houches École d'Été, July-August, 1965. C. Zemach also presents the $\hat{F}_{1/2} \cdot \hat{F}_{3/2}$ distribution (Ref. 5).

20. The T_{LM} transform according to $R\, T_{LM}\, R^{-1} = \sum_{M'} \mathcal{D}^{L}_{M'M}(\alpha, \beta, \gamma) T_{LM'}$, where R is the rotation operator and α, β, and γ are the Euler angles.

21. A simple method is to retain the usual $\hat{z} = \hat{F}_{3/2}$ representation and to calculate the expectation value, $\mathrm{Tr}[\rho_{(3/2)} T_{20}]$, of $T_{20}(\hat{n}) = T_{20}(\hat{y}) =$

$$(1/3\sqrt{5})(3S_y^2 - S^2) = \frac{1}{2\sqrt{5}} \begin{bmatrix} -1 & 0 & -\sqrt{3} & 0 \\ 0 & 1 & 0 & -\sqrt{3} \\ -\sqrt{3} & 0 & 1 & 0 \\ 0 & -\sqrt{3} & 0 & -1 \end{bmatrix} .$$

Alignment along the normal was first calculated by R. Barloutaud and R. D. Tripp and was presented in Armenteros et al., Ref. 17.

22. The $n^{(3)}_{L0}$, $n^{(3)}_{L1}$, and $n^{(3)}_{L2}$ follow from Eqs. (43), (45), and (46) of Ref. 7.

23. The "moment" of a distribution is defined as the coefficient of some orthonormal function.

24. Janice Button-Shafer, Parity of Fermions: Tests and Ambiguities, Lawrence Radiation Laboratory Report Addendum to UCRL-16857, (unpublished).

25. These have been discussed with different language by S. M. Bilenky, Nuovo Cimento 10, 1049 (1958), and A. Bohr, Nucl. Phys. 10, 486 (1959).

26. One could also write $M_- = (\bar{\sigma} \cdot \hat{k})(g + h\bar{\sigma} \cdot \hat{n})$. The fact that $\bar{\sigma} \cdot \hat{k}$ precedes M_+ causes I_- to have the same form as I_+, but "rotates" P_{F0} to $-2 \,\mathrm{Re}\, g^* h/I_0$; actually redefining M_- has changed the sign of h. Equation 33 again is obtained.

Table I. Coefficients for $F_{3/2}$ distributions

$J = 5/2$, lower ℓ wave only [Eq. (23)].

	Even Parity			
	σ_L	τ_L	μ_L	ν_L
L = 0	0.500	-0.0446	0.000	0.000
L = 2	0.179	-0.0574	0.0685	-0.103
L = 4	0.000	-0.0765	0.0700	-0.0496
	Odd Parity			
	σ_L	τ_L	μ_L	ν_L
L = 0	0.500	0.159	0.000	0.000
L = 2	0.0914	0.0081	0.0488	0.0732
L = 4	-0.163	-0.0911	0.0500	0.0352

Fig. 1. Tensor polarization components of $F_{3/2}$ resulting from the decay F_J (formation resonance) $\rightarrow F_{3/2}$. The angle θ is that of the $F_{3/2}$ relative to the incident beam. The labels indicate J^P (parity relative to $F_{3/2}$) of the F_J resonance. The higher ℓ' amplitude is neglected here. The ratio of each $I\langle T_{22}\rangle$ moment to the corresponding $I\langle T_{21}\rangle$ moment yields $(J+\frac{1}{2})(-)^P$. (See Table I for the two coefficients or moments of each J = 5/2 curve. For $J^P = 3/2^-$, $\mu_2 = 0.100$ and $\nu_2 = -0.100$; for $J^P = 3/2^+$, $\mu_2 = 0.060$ and $\nu_2 = 0.060$.)

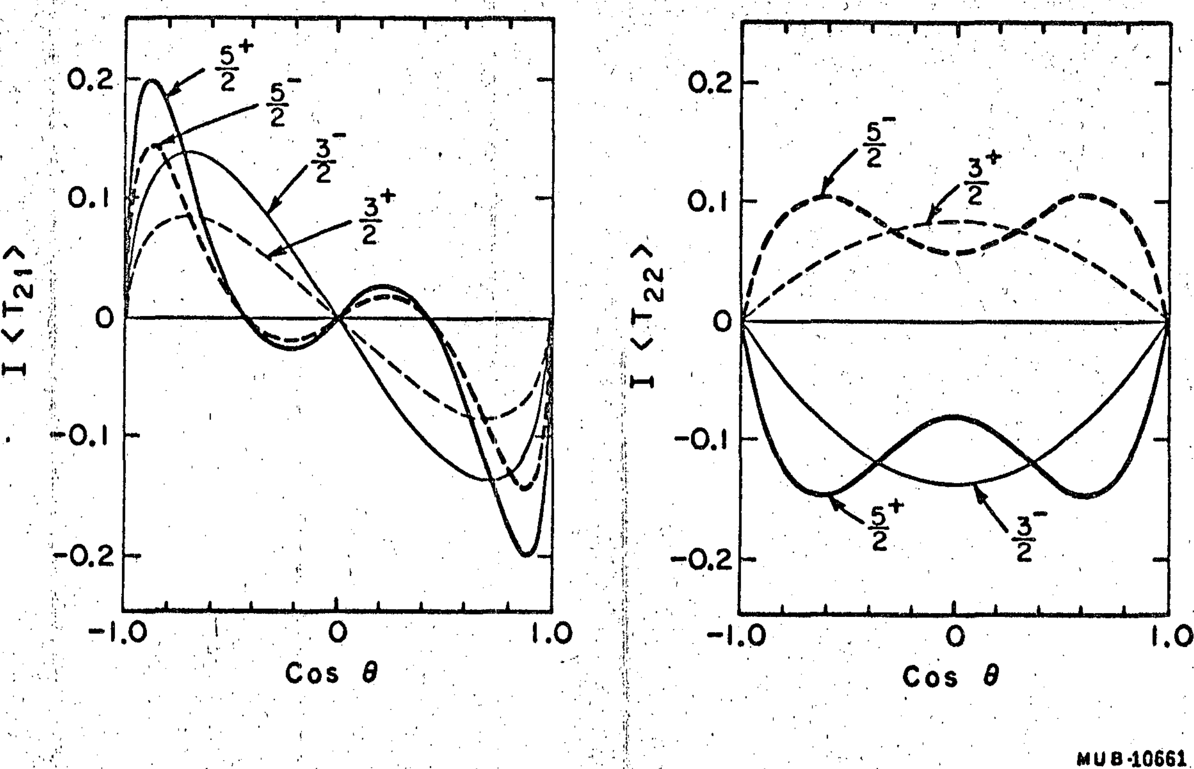
0.2
0.1
0
-0.1
-0.2
-1.0
0
1.0
Cos θ
I ⟨T21⟩
5/2+
5/2-
3/2-
3/2+
I ⟨T22⟩
5/2-
3/2+
5/2+
3/2-
MUB-10661

Fig. 1

ADDENDUM

The following represents additional explanations of material in the text. New equations and new footnotes are labelled by letters.

Page 2:

Two orbital angular momenta are possible for each parity in the strong decay of a fermion (with spin $\geqslant 3/2$) into an $F_{3/2}$ plus B_0: $\ell_+ = J - 3/2$ and $\ell'_+ = J + 1/2$, or $\ell_- = J - 1/2$ and $\ell'_- = J + 3/2$.[6] If only the lower waves, ℓ_+ and ℓ_-, are considered, there is a simple relationship between the transition matrices for the two parities of decay. The "parity operator" in this case is $\bar{S} \cdot \hat{p}$, where S_x, S_y, and S_z are the usual spin-3/2 operators and $\hat{p}$ the direction of decay momentum. With the initial density matrix given by ρ_i (normalized so that Tr $\rho_i = 1$) and the "plus-parity" transition matrix represented by $\mathcal{M}_+$, the angular distribution of the decay $F_J \to F_{3/2}$ (plus parity) is

$$I_+ = \mathrm{Tr}\ (\mathcal{M}_+ \rho_i \mathcal{M}_+^\dagger); \tag{a}$$

if the higher ℓ' waves are neglected, $\mathcal{M}_- \propto (\bar{S} \cdot \hat{p})\mathcal{M}_+$ and the angular distribution for the decay of opposite parity is

$$I_- \propto \mathrm{Tr}\ [(\bar{S} \cdot \hat{p})^2 \mathcal{M}_+^\ell \rho_i \mathcal{M}_+^{\ell\,\dagger}]. \tag{b}$$

Unlike the case for a final $F_{1/2}$, the parity operator related to the $F_{3/2}$ system is not proportional to the identity when squared. (This can be seen by squaring the S_i matrices given in Schiff[m] or the $T_{\ell m}$ matrices given in Ref. 7.) In fact, with $\hat{p}$ considered the z axis,

$$(\bar{S} \cdot \hat{p})^2 \propto \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}^2 . \tag{c}$$

Thus the angular distributions for opposite parities, I_+ and I_-, differ from each other if higher ℓ' waves are neglected.

The inclusion of the higher waves requires closer examination of the transition matrices. With the spherical tensors T_{1M}, T_{2M}, and T_{3M} serving as spin operators in the spin-3/2 space, the transition matrix may be written (in the helicity system) as follows for each parity:[7,n]

$$\mathcal{M} = \mathcal{M}^{\ell} + \mathcal{M}^{\ell'} ;$$

$$\text{with} \quad \mathcal{M}_+^{\ell} = eG, \; \mathcal{M}_+^{\ell'} = f' H T_{20} p_{20} = f H T_{20};$$

$$\text{or} \quad \mathcal{M}_-^{\ell} = g' G T_{10} p_{10} = g G T_{10}, \qquad (7)$$

$$\mathcal{M}_-^{\ell'} = h' H T_{30} p_{30} = h H T_{30}.$$

Here e, f, g, and h are complex numbers, and G and H are (real) diagonal matrices in spin space. The p_{L0} are components of spherical tensors constructed from $\hat{p}$ (defined as $\hat{z}$). The above are the forms demanded by invariance principles.

The $\mathcal{M}_\pm$ for lower ℓ waves have been discussed above, where it was noted that $\mathcal{M}_-^{\ell} \propto \mathcal{M}_+^{\ell} \bar{S} \cdot \hat{p}$ (or $\mathcal{M}_+^{\ell} T_{10}$). The $\mathcal{M}_\pm^{\ell'}$ for the higher waves are also simply related:*

$$\mathcal{M}_-^{\ell'} \propto \mathcal{M}_+^{\ell'} T_{20}^{-1} T_{30} = \mathcal{M}_+^{\ell'} (1/\sqrt{7}) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} . \qquad (8)$$

The contribution to the negative-parity distribution made by the higher wave alone is (ρ_i^R representing ρ_i rotated to helicity axes)

$$I_-^{\ell'} = \mathrm{Tr}\,[(T_{20}^{-1} T_{30})^2 \mathcal{M}_+^{\ell'} \rho_i^R \mathcal{M}_+^{\ell'\dagger}] ; \qquad (9a)$$

*Note in proof: It would be better form to write the $\bar{S} \cdot \hat{p}$ or $T_{20}^{-1} T_{30} (= T_{30} T_{20}^{-1})$ parity operator before the $\mathcal{M}_+$; but since $\mathcal{M}_+$ is diagonal here, the notation used is equivalent.

obviously this is different from

$$I_{+}^{\ell'} = \mathrm{Tr}(\mathcal{M}_{+}^{\ell'} \rho_i^R \mathcal{M}_{+}^{\ell' \dagger}). \tag{9b}$$

(However, the contribution from the interference of lower and higher ℓ waves is similar for the two parity cases.)

It follows that the $F_J \to F_{3/2}$ angular distributions of opposite parity, I_+ and I_-, are generally distinguishable.[8, o]

The inclusion of both orbital angular momenta in strong decay introduces a new ambiguity into a parity determination from the angular distribution of $F_J \to F_{3/2}$. The two complex amplitudes represent three independent real parameters that must be extracted from the data. When only the polar angle θ is observable (as in the "decay" of a formation resonance), the spin J of the resonance must be $> 5/2$ if parity discrimination is to be made.

For the decay $F_J \to F_{3/2}$, the question arises as to the effect of the "parity operator(s)" on polarization components. Is it possible that the $\bar{S} \cdot \hat{p}$ or the $T_{20}^{-1} T_{30}$ operator is equivalent to a rotation operator, in analogy to the $\bar{\sigma} \cdot \hat{p}$ of Eq. (5)? The operator for rotation of the spin-3/2 system through angle ϕ about $\hat{p}$ (or $\hat{z}$) is

$$R_p(\phi) = e^{i\bar{S}\cdot \hat{p}\phi} = e^{iS_z\phi} ; \tag{d}$$

and on expansion of the exponential, one obtains higher powers of S_z which do not reduce as in the case of the spin-1/2 system.[9] This rotation operator does not reduce (nor does any other) to the $\bar{S} \cdot \hat{p}$ "parity operator" for the lower ℓ waves or to the $T_{20}^{-1} T_{30}$ parity operator for the higher ℓ' waves.

Page 4:

The $\langle T_{2M}\rangle$ are quantities similar to moments of inertia in mechanics problems. A familiar nuclear-physics analog for $\langle T_{20}\rangle$ is the electric quadrupole moment Q_{20}, defined as $\int r^2\, Y_{20}(\theta)\, \rho(\bar{r})\, d\bar{r}$, with $\rho(\bar{r})$ the nuclear charge density. [The decay distribution $I(\theta)$ yields $\langle T_{20}\rangle \propto \int Y_{20}\,(\theta)\, I(\theta)\, d\Omega$.] The $\langle T_{20}\rangle$ multipole parameter represents the polar spin alignment; $\langle T_{22}\rangle$, the azimuthal alignment; and $\langle T_{21}\rangle$, a combination of polar and azimuthal alignment.

For the "formation resonance" produced from a $B_0 + F_{1/2}$ system, angular-momentum conservation in production permits only even -L, M = 0 $\langle T_{LM}\rangle$ if the incident-beam direction is the z axis.[13] As only the $m_J = +\frac{1}{2}$ and $-\frac{1}{2}$ spin states are occupied, the density matrix describing the resonance has the simple form

$$[\rho]_{\frac{1}{2},\frac{1}{2}} = [\rho]_{-\frac{1}{2},-\frac{1}{2}} = 1/2, \qquad \text{(e)}$$

$$\text{all other } [\rho]_{m,m'} = 0.$$

It follows that the only nonzero polarization parameters describing the resonant state are the "alignment" terms $\langle T_{20}\rangle$, $\langle T_{40}\rangle$, $\langle T_{60}\rangle$, etc.[14]

Note: The $n_{L,\lambda-\lambda'}^{(2\lambda)}$ of Eq. (11) are expressible as functions of $n_{L0}^{(1)} = (-)^{J-1/2}[(2J+1)/4\pi]^{1/2}\, C(JJL;\, 1/2, -1/2)$. See Ref. 7, Appendix II.

Page 5:

The angular distribution for decay (1) is[15]

$$I(\theta) = \mathrm{Tr}\ \rho_{(3/2)} = \sum_{L_{even}}^{4} C_L\ t_{L0}\ Y_{L0}(\theta) \qquad (14)$$

where, for even parity,

$$C_0 = 2\pi(a^2+c^2)\, n_{00}^{(1)}$$

$$C_2 = \frac{\pi}{5}(7\, a^2 + 5.5\, c^2 - 3\sqrt{6}\ \mathrm{Re}\ a^* c)\, n_{20}^{(1)} \qquad (f)$$

$$C_4 = \frac{\pi}{5}(-5\, c^2 - 10\sqrt{6}\ \mathrm{Re}\ a^* c)\, n_{40}^{(1)}$$

and, for odd parity,

$$C_0 = 2\pi(b^2 + d^2)\, n_{00}^{(1)}$$

$$C_2 = \frac{\pi}{7}(5\, b^2 + 12.5\, d^2 - 3\sqrt{6}\ \mathrm{Re}\ b^* d)\, n_{20}^{(1)} \qquad (g)$$

$$C_4 = \frac{\pi}{7}(-16\, b^2 + 9\, d^2 - 10\sqrt{6}\ \mathrm{Re}\ b^* d)\, n_{40}^{(1)}\ .$$

With the three C_L from $I(\theta)$ data, both Eqs. (f) and (g) will generally be soluble. Hence, no determination of parity can be made from just $I(\theta)$ for a $J = 5/2$ formation resonance.

Page 8:

For J = 5/2 Eq. (20) becomes

$$I\langle T_{20}\rangle = (2\pi/\sqrt{5}) \sum_{L_e}^{4} \begin{Bmatrix} [(2a^2+3c^2+2\sqrt{6}\ \mathrm{Re}\ a^*c)[1-L(L+1)/8]-(3a^2+2c^2-2\sqrt{6}\ \mathrm{Re}\ a^*c)](1/5) \\ [(6b^2+d^2+2\sqrt{6}\ \mathrm{Re}\ b^*d)[1-L(L+1)/8]-(b^2+6d^2-2\sqrt{6}\ \mathrm{Re}\ b^*d)](1/7) \end{Bmatrix} \times n_{L0}^{(1)}\, t_{L0}\, Y_{L0}(\theta) \qquad (21)$$

$$I\langle T_{21}\rangle = (\pi/\sqrt{5}) \sum_{L_e}^{4} \begin{Bmatrix} (-\sqrt{6}a^2+\sqrt{6}c^2-\mathrm{Re}\ a^*c)(1/5) \\ (-\sqrt{6}b^2+\sqrt{6}d^2+5\ \mathrm{Re}\ b^*d)(1/7) \end{Bmatrix} [L(L+1)]^{1/2}\, n_{L0}^{(1)}\, t_{L0}\, D_{01}^{L}(0,\theta,0)$$

$$I\langle T_{22}\rangle = (\pi/\sqrt{5}) \sum_{L_e}^{4} \begin{Bmatrix} (-\sqrt{6}a^2+\sqrt{6}c^2-\mathrm{Re}\ a^*c)(1/5) \\ (\sqrt{6}b^2-\sqrt{6}d^2-5\mathrm{Re}\ b^*d)(1/7) \end{Bmatrix} (J+1/2)\,[L(L+1)/(L+2)(L-1)]^{1/2} \times n_{L0}^{(1)}\, t_{L0}\, D_{02}^{L}(0,\theta,0).$$

We note that the above are all real functions. {In these equations, amplitudes have been abbreviated (A_3 instead of $A_{3/2}$, and A^2 instead of $|A|^2$); and $D_{0M'}^{L}$ has replaced $[(2L+1)/4\pi]^{1/2}\mathscr{D}_{0M'}^{L}$.}

Page 9:

After the coefficients of the ψ and ζ functions in Eq. (22) have been determined from the data as functions of θ, "moments" of the $I(\theta)$ and $I\langle T_{2m}\rangle(\theta)$ distributions can be found and checked against values predicted for formation-resonance decay.[p] Alternatively, the data may be directly compared with the predicted function $\mathcal{I}(\theta, \psi, \zeta)$ throughout the θ-ψ-ζ space. (Predictions for $\mathcal{I}$ will of course depend on spin, parity, and a_ℓ assumptions.)

For J = 5/2, Eq. (25) becomes

$$I\langle T_{11}\rangle = \pi(1/5)^{1/2}\, i\, \mathrm{Im}\left\{{a^*c \atop b^*d}\right\} \sum_{L_e} \sqrt{L(L+1)}\; n^{(1)}_{L0}\, t_{L0}\, D^{L}_{01}(0,\theta,0)$$

$$I\langle T_{31}\rangle = \pi(2/35)^{1/2}\, i\, \mathrm{Im}\left\{{a^*c \atop b^*d}\right\} \sum_{L_e} \sqrt{L(L+1)}\; n^{(1)}_{L0}\, t_{L0}\, D^{L}_{01}(0,\theta,0) \qquad \text{(h)}$$

$$I\langle T_{32}\rangle = \Gamma\; (J+1/2)\, \pi(1/7)^{1/2}\, i\, \mathrm{Im}\left\{{a^*c \atop b^*d}\right\}$$

$$\times \sum_{L_e} [L(L+1)/(L+2)(L-1)]^{1/2}\, n^{(1)}_{L0}\, t_{L0}\, D^{L}_{02}(0,\theta,0),$$

where Γ is the parity parameter defined as above. Evidently a ratio of an $I\langle T_{32}\rangle$ moment (for L = 2 or 4) to either an $I\langle T_{11}\rangle$ moment or an $I\langle T_{31}\rangle$ moment may yield parity (and spin) information.

The $I\langle T_{\ell m}\rangle$ of Eq. (h) may be analyzed by determining the polarization of $F_{1/2}$ from the angular distribution of decay:

$$\mathscr{P}\,\bar{P}\cdot\hat{F}_{1/2} = (4\pi)^{-1/2}\, I\ \{0.896\ \mathrm{Im}\,\langle T_{11}\rangle\ \mathrm{Im}\,Y_{11}(\psi,\zeta)$$

$$-\,2.68\,[\mathrm{Im}\,\langle T_{31}\rangle\ \mathrm{Im}\,Y_{31}(\psi,\zeta) + \mathrm{Im}\,\langle T_{32}\rangle\ \mathrm{Im}\,Y_{32}(\psi,\zeta)]\}$$

$$\mathscr{P}(\bar{P}\cdot\hat{x}' + i\,\bar{P}\cdot\hat{y}') = -\gamma\,(4\pi)^{-1/2}\, I\ \{1.27\, i\, \mathrm{Im}\,\langle T_{11}\rangle\,[D^{1}_{11}(\zeta,\psi,0) \quad \text{(i)}$$

$$+\,D^{1}_{-1,1}(\zeta,\psi,0)] - 1.55\, i\, \mathrm{Im}\,\langle T_{31}\rangle\,[D^{3}_{11}(\zeta,\psi,0) + D^{3}_{-1,1}(\zeta,\psi,0)]$$

$$+\, i\, \mathrm{Im}\,\langle T_{32}\rangle\,[D^{3}_{21}(\zeta,\psi,0) - D^{3}_{-2,1}(\zeta,\psi,0)]\},$$

where γ is +1 or -1 in accordance with the $F_{3/2}$-$F_{1/2}$ relative parity.[q] These $F_{1/2}$ polarization components are readily found, as functions of ψ and ζ; for example,

$$\mathscr{P}\,\bar{P}\cdot\hat{F}_{1/2} = (3/4\pi\alpha N) \sum^{N} (\hat{p}\cdot\hat{F}_{1/2}) \quad \text{(j)}$$

where α is the usual asymmetry parameter in $F_{1/2}$ decay; $\hat{p}$ is the decay momentum in the $F_{1/2}$ rest frame; and the sum is taken over all events with $\hat{F}_{1/2}$ at some particular ψ, ζ orientation. (See Ref. 7 for explanation of axes x' and y'.)

Moments of the I and $I\langle T_{2m}\rangle$ distributions provide information on $|c|^2$ and 2 Re a^*c (or $|d|^2$ and 2 Re b^*d) and thus may indicate whether the higher ℓ' wave contribution is significant in F_J decay. Examination of the Im a^*c or Im b^*d terms of the Eq. (h) distributions may also help to establish the contribution of the higher ℓ' wave. However, even without knowledge of the relative ℓ-wave contributions, the L = 2, 4, ··· (2J-1) moments of the $I\langle T_{2m}\rangle$ and possibly of the $I\langle T_{1m}\rangle$ and $I\langle T_{3m}\rangle$ may give an answer for the parity of the formation resonance.

Page 10:

Relativistic considerations.--The descriptions above are relativistically correct, although they utilize three-vector language. Each density matrix describes a particle in its rest frame; and helicity amplitudes are invariant under transformation to a rest frame. In the application of the formalism, the usual rules must be followed: transformations must be made from the c.m. to each rest frame (in the reaction sequence), and momentum vectors in each frame must be referred to axes prescribed by the "direct Lorentz transformation." (See H. P. Stapp.[r])

FOOTNOTES AND REFERENCES

m. L. I. Schiff, Quantum Mechanics, McGraw-Hill Book Co., 1955 (p. 146).

n. The rotation-function part of the transition matrix {the $\mathcal{D}$ in $\mathcal{M}_{\lambda M} = A_\lambda [(2J+1)/4\pi]^{1/2} \mathcal{D}_{M\lambda}^{J*}(\phi, \theta, 0)$} is here ignored; it is parity-independent. The coefficients e, f, g, and h are proportional to a_ℓ amplitudes. The elements of G and H depend on J.

o. There is one special case when I_+ and I_- are indistinguishable: when the spin J and the partial amplitudes are such that $|g|^2 G^2/3 = |h|^2 H^2/7$, the $\mathcal{M}_-^{\ell}$ and $\mathcal{M}_-^{\ell'}$ give incoherent contributions proportional to the identity (and an interference term proportional to T_{20}).

p. The "moment" of a distribution is defined as the coefficient of some orthonormal function; e.g., $\langle\langle T_{\ell m}\rangle Y_{LM}\rangle$ is the Y_{LM} moment of the $I\langle T_{\ell m}\rangle$ distribution.

q. The value of γ is +1 if the relative parity demands $\ell = J - \frac{1}{2}$ in $F_{3/2}$ decay and is -1 if the parity demands $\ell = J + \frac{1}{2}$.

r. H. P. Stapp, University of California Radiation Laboratory Report No. UCRL-8096, 1957 (unpublished).