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Publication Date
1978

To be Submitted for Publication

LBL-7292 c. 2
Preprint

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January 1978

Prepared for the U. S. Department of Energy
under Contract W-7405-ENG-48
and the National Science Foundation
Grant MPS 72-04967A02

DISCLAIMER

The pendent liquid drop: asymptotic properties

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ABSTRACT

We obtain estimates for the asymptotic properties of a symmetric pendent liquid drop as the vertex height u_0 approaches negative infinity. The estimates make use of the Delaunay comparison surfaces we have employed previously, but in a manner more precise, so as to yield an improvement of an order of magnitude. For large $|u_0|$ the vertical section of a drop is shown to have near the vertex the general form of a succession of circular arcs joined near the axis by small arcs of large curvature. The section contracts at first toward a certain hyperbola, the circular arcs gradually changing shape but remaining, until a certain fixed height (asymptotically as $u_0 \to -\infty$), within a narrow band surrounding the hyperbola. The continuation of the section eventually projects simply on $u = 0$, separates from the hyperbola, and continues in an oscillatory manner to infinity.

We prove a preliminary (weak) form of our conjecture, that as $u_0 \to -\infty$ the section converges uniformly (as a point set) to a solution with simple projection (for all $r > 0$) on $u = 0$ and an isolated singularity at $r = 0$, whose existence we have proved previously.

This paper is an addendum to our earlier work [1]; its purpose is to provide new information on the asymptotic behavior of axially symmetric "pendent drop" solutions of the capillary equation, as the vertex height u_o becaomes large in magnitude. We assume here familiarity with the format of the problem, as presented in [1]; we use the notation and refer to the results of that paper, without further explanation.

Our specific concern here is the problem discussed in §VI of [1], namely the asymptotic convergence of the pendent drop solution to the singular solution $U(r)$. We do not yet settle completely our conjecture on this matter, but we do provide some new information in that direction.

The crucial new step in the present discussion consists in a more precise use of the Delaunay comparison surfaces as a device to control the behavior of the solutions of

$$(r \sin \psi)_r = - ru . \tag{5}$$

In [1] bounds on these surfaces were used for estimation of integrals of the right side of (5); we now propose to introduce the Delaunay profiles themselves into these integrals. It turns out the results can be expressed succinctly in terms of elliptic integrals, leading to an improvement in an order of magnitude of the estimates of §III. We are led to recurrence relations (107, 136, 137) for the displacement of successive "vertical points" from the hyperbola $ru = -1$, which show that, initially, the solution curve becomes closer to the hyperbola with each successive loop. Integration of these relations shows that the solution curve at first contracts toward the hyperbola, at least until a height of order $|u_o|^{7/9}$, after which it remains confined within a strip whose width has order $|u_o|^{-1}$, until a height of order $|u_o|^{(2\alpha+1)/9}$, for any $\alpha > \frac{23}{9}$. Thus, the solution curve converges asymptotically to the hyperbola, uniformly in $|u| > |u_o|^{(2\alpha+1)/9}$.

For all sufficiently large $|u|$, we show the solution curve is confined to a strip about $ru = -1$, whose width has order $|u|^{-9/7}$, uniformly in u_o as $|u_o| \to \infty$.

In this paper the symbols A, B are used to denote quantities independent of the other terms within a relation, but whose values may however change within a context. Thus, from $y < A\sqrt{1+x^2}$ we may conclude $y < A|x|$ for large $|x|$. The symbol $\sim$ is used to indicate a relationship in which terms of (relatively) small magnitude are neglected.

We start with general estimates on Delaunay arcs $v(r)$, which are solutions of

$$(40) \qquad (r \sin \psi)_r = 2rH, \qquad H > 0,$$

vertical at (r_a, v_a), (r_b, v_b), $r_a < r_b$ (see figure 1). We note

$$(41) \qquad H = \frac{1}{r_a + r_b}$$

and an inflection appears at

$$(42) \qquad r_i = \sqrt{r_a r_b}.$$

We distinguish two cases:

<u>Case a)</u> $\psi \leq \pi/2$: Solving for $r(\psi)$, we find

$$(43) \qquad r = \frac{\sin \psi \pm \sqrt{k^2 - \cos^2\psi}}{2H}, \qquad k = \frac{r_b - r_a}{r_b + r_a},$$

where the upper (lower) sign is to be chosen, according as $r > (<) r_i$.

Setting $\cos \psi = k \sin \varphi$, and using $u'(r) = \tan \psi$, we integrate (43) to obtain[1)]

$$v_b - v_a = \frac{1}{H}\int_0^{\pi/2} \sqrt{1 - k^2\sin^2\varphi}\, d\varphi = \frac{1}{H} E(k)\,, \tag{44}$$

where $E(k)$ is the complete elliptic integral of the second kind, of modulus k.

If $v_i < v < v_b$, then

$$v - v_a = \frac{1}{H} E(k) - \frac{1}{2H}[k \sin\varphi + E(\varphi,k)] \tag{45}$$

where

$$E(\varphi,k) = \int_0^{\varphi} \sqrt{1 - k^2\sin^2\tau}\, d\tau$$

is the incomplete elliptic integral of the second kind.

If $v_a < v < v_i$, then

$$v - v_a = \frac{1}{2H}[-k\sin\varphi + E(\varphi,k)]\,. \tag{46}$$

At the inflection (r_i, v_i)

$$v_i = \frac{1}{2H}(-k + E(k)) + v_a\,. \tag{47}$$

<u>Case b)</u> $\psi \geq \pi/2$: The discussion is unchanged, except in this case $-\pi/2 \leq \varphi \leq 0$. We find now

$$v_a - v_b = \frac{1}{H} E(k)\,. \tag{48}$$

If $v_b \leq v \leq v_i$, then

$$v - v_b = -\frac{1}{2H}[k \sin\varphi + E(\varphi,k)]\,. \tag{49}$$

If $v_i \le v \le v_a$, then

(50) $$v - v_b = \frac{1}{H} E(k) - \frac{1}{2H} [k \sin \varphi - E(\varphi,k)].$$

We have, in this case,

(51) $$v_i = \frac{1}{2H} (k + E(k)) + v_b .$$

We shall need to evaluate integrals of Delaunay arcs, of the form

(52) $$I_{ab} = - \int_{r_a}^{r_b} \rho v(\rho) d\rho$$

$$= - \frac{\rho^2 v}{2}\Big|_{r_a}^{r_b} + \frac{1}{2} \int_{r_a}^{r_b} \rho^2 \frac{dv}{d\rho} d\rho$$

$$= - \frac{\rho^2 v}{2}\Big|_{r_a}^{r_b} + \frac{1}{2} \Big(\int_{o}^{\pi/2} + \int_{\pi/2}^{o} \Big) \rho^2 \frac{dv}{d\varphi} d\varphi$$

for the case $\psi \le \pi/2$; the last two integrals refer to the portions of the curve preceding and following the inflection. For $\psi \ge \pi/2$, $\varphi \le 0$ and the limits in the last two integrals become $-\pi/2$.

<u>Case a)</u> $\psi \le \pi/2$: Taking $r(\varphi)$, $v(\varphi)$ from (43, 45, 46), and setting $\Delta(\varphi,k) = \sqrt{1 - k^2\sin^2\varphi}$, we find, according as $r \gtrless r_i$,

(53) $$r^2 \frac{dv}{d\varphi} = \frac{1}{8H^3} \{\mp(1-k^2)\Delta \mp 4k^2\Delta \cos^2\varphi - 3k\Delta^2 \cos \varphi - k^3 \cos^3\varphi\}.$$

After taking account of some cancellation, we obtain

(54) $$I_{ab} = - \frac{1}{2} [r_b^2 v_b - r_a^2 v_a] + \frac{1}{8H^3} \int_o^{\pi/2} [(1-k^2)\Delta + 4k^2\Delta \cos^2\varphi] d\varphi$$

$$= - \frac{1}{2} [r_b^2 v_b - r_a^2 v_a] + \frac{1}{24H^3} S(k),$$

with

(55) $$S(k) = 8E(k) - (1-k^2)(E(k) + 4K(k))$$

where

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\varphi}}\, d\varphi$$

is the complete elliptic integral of the first kind, of modulus k.

Case b) $\psi \geq \pi/2$: In this case, we find by an analogous discussion

(56) $$I_{ab} = -\frac{1}{2}[r_b^2 v_b - r_a^2 v_a] - \frac{1}{24H^3} S(k).$$

We indicate in a particular configuration how the above expressions can be used to estimate the solution $u(r)$ of (5). We consider an arc $u(r)$ that is vertical at (r_α, u_α) and at (r_β, u_β), $u_\alpha < u_\beta$ (figure 2). We compare this arc with a Delaunay arc $v(r)$, with curvature $H = -\frac{1}{2}u_\alpha$, and vertical at $(r_a, v_a) = (r_\alpha, u_\alpha)$. The second vertical then appears at (r_b, v_b), determined by

(57) $$-\frac{1}{u_\alpha} = \frac{r_a + r_b}{2}$$

and by

(58) $$v_b - u_\alpha = -\frac{2}{u_\alpha} E(k)$$

with

(59) $$k = \frac{r_b - r_a}{r_b + r_a} = 1 + r_\alpha u_\alpha.$$

The general comparison principle II i applies in the interval $r_\alpha \leq r \leq r_b$, and yields $u'(r) < v'(r)$, $u(r) < v(r)$ in this interval. A consequence is that

(60) $$r_b < r_\beta, \qquad u^* = u(r_b) < v_b .$$

We extend $v(r)$ to the interval (r_α, r_β) by defining $v(r) \equiv u_\beta$, $r > r_b$. From the equation (5) we now find

(61) $$r_\beta - r_\alpha = -\int_{r_\alpha}^{r_\beta} \rho u \, d\rho > -\int_{r_\alpha}^{r_b} \rho v \, d\rho - \int_{r_b}^{r_\beta} \rho u_\beta \, d\rho$$
$$> -\frac{1}{2}[r_b^2 v_b - r_\alpha^2 u_\alpha] + \frac{1}{24H^3} S(k) - \frac{u_\beta}{2}(r_\beta^2 - r_b^2)$$

which we rewrite in the form

(62) $$\frac{1}{2} v_b r_\beta^2 + r_\beta > \frac{1}{2} u_\alpha r_\alpha^2 + r_\alpha + \frac{1}{24H^3} S(k) + \frac{1}{2}(r_b^2 - r_\beta^2)\varepsilon_\beta$$

with $\varepsilon_\beta = u_\beta - v_b$.

We can obtain a similar estimate in the reverse direction by introducing a Delaunay surface $\hat{v}(r)$, vertical at (r_α, u_α) and at $(\hat{r}_b, \hat{v}_b)$, and with mean curvature $\hat{H} = -\frac{1}{2} u_\beta$. The comparison principle now yields

(63) $$r_b < r_\beta < \hat{r}_b; \quad \hat{v}(r) < u(r), \quad r_\alpha \leq r \leq r_\beta; \quad \hat{v}(r_b) < u^* < v_b$$

(see figure 2). Integrating (5) we obtain

(64) $$r_\beta - r_\alpha = -\int_{r_\alpha}^{r_\beta} \rho u \, d\rho < -\int_{r_\alpha}^{r_\beta} \rho \hat{v} \, d\rho = -\int_{r_\alpha}^{\hat{r}_b} \rho \hat{v} \, d\rho + \int_{r_\beta}^{\hat{r}_b} \rho \hat{v} \, d\rho$$
$$< -\frac{1}{2}[\hat{r}_b^2 \hat{v}_b - r_\alpha^2 u_\alpha] + \frac{1}{24\hat{H}^3} S(\hat{k}) + \frac{1}{2} \hat{v}_b(\hat{r}_b^2 - r_\beta^2)$$

by (56, 63). We rewrite (64) in the form

(65) $$(\frac{1}{2} r_\beta^2 \hat{v}_b + r_\beta) < (\frac{1}{2} r_\alpha^2 u_\alpha + r_\alpha) + \frac{1}{24\hat{H}^3} S(\hat{k}) .$$

In order to extract useful information from (62, 65), we need conditions under which a second vertical u will appear, and an estimate for ε_β and the consequent estimates on $\hat{H}$, $\hat{k}$; we proceed to obtain them.

We consider, for the case $\psi \leq \pi/2$, the generic configuration indicated in figure 2. Setting, as before, $u^* = u(r_b)$, $\psi^* = \psi(r_b)$, we find from (5)

(66) $$r_b \sin \psi^* - r_\alpha = - \int_{r_\alpha}^{r_b} \rho u \, d\rho > - \frac{v_b}{2} (r_b^2 - r_\alpha^2) .$$

For the upper Delaunay surface $v(r)$ we have from (40, 41, 57)

(67) $$r_b - r_\alpha = - \frac{u_\alpha}{2} (r_b^2 - r_\alpha^2) .$$

Combining these relations, we obtain

(68) $$r_b(1 - \sin \psi^*) < (v_b - u_\alpha) \frac{r_b + r_\alpha}{2} (r_b - r_\alpha)$$

so that, by (43, 44, 57)

(69) $$1 - \sin \psi^* < 4E(k) \frac{k}{1+k} \frac{1}{u_\alpha^2}$$

from which

(70) $$\cos \psi^* < \frac{2 \sqrt{2} \sqrt{E(k)}}{-u_\alpha} \sqrt{\frac{k}{1+k}} .$$

We now observe $(\sin \psi)_r = - (\cos \psi)_u$ and write (5) in the form

(71) $$\frac{\sin \psi}{r} - (\cos \psi)_u = -u .$$

For all $r > r_b$ for which the solution can be continued in the form $u = u(r)$, we conclude

$$(72) \qquad -(\cos\psi)_u > -u - \frac{1}{r_b} = -u + \frac{u_\alpha}{1+k} .$$

Integrating in u between the values u^* and u, observing $\cos\psi > 0$ and using (70), we find

$$(73) \qquad \frac{1}{2}(\delta u)^2 + (u^* - \frac{u_\alpha}{1+k})\delta u - \frac{2\sqrt{2}\sqrt{E(k)}}{u_\alpha}\sqrt{\frac{k}{1+k}} > 0 ,$$

where we have set $\delta u = u - u^*$, on the arc considered. We have also

$$u^* < v_b = u_\alpha - \frac{2E(k)}{u_\alpha}$$

by (58, 60), and thus

$$(74) \qquad \frac{1}{2}(\delta u)^2 + (\frac{k}{1+k}u_\alpha - \frac{2E(k)}{u_\alpha})\delta u - \frac{2\sqrt{2}\sqrt{E(k)}}{u_\alpha}\sqrt{\frac{k}{1+k}} > 0$$

on any continuation of the solution arc to values $u \geq u^*$. We conclude *a second vertical must appear, in every situation for which*

$$(75) \qquad ku_\alpha^2 \gg 1 .$$

Under this condition we obtain from (74) the simple expression for $\delta^* u = \max_{\psi \leq \pi/2} (u - u^*) = u_\beta - u^*$,

$$(76) \qquad \delta^* u \lesssim \frac{2\sqrt{2}\sqrt{E(k)}\sqrt{k(1+k)}}{ku_\alpha^2 - 2E(k)(1+k)} < \frac{A}{\sqrt{k}\,u_\alpha^2} < \frac{\varepsilon}{|u_\alpha|}$$

which limits the height change between the successive verticals. Here $\varepsilon > 0$ is arbitrarily small, for large ku_α^2.

We note the condition (76) ensures that the second vertical (r_β, u_β) lies to the right of the hyperbola $ru = -1$, that is, $r_\beta u_\beta < -1$. We show that under this condition, *the hyperbola is crossed exactly once between* (r_α, u_α) *and* (r_β, u_β). To see this, we first observe that the comparison function $v(r)$ has exactly one inflection, which must appear in the initial interval determined by $rv > -1$. Also the vertical distance from (r_b, v_b) to the hyperbola $ru = -1$ is

$$d_b = -\frac{1}{r_b} - v_b = -\frac{k}{1+k} u_\alpha + \frac{2E(k)}{u_\alpha}$$

which is positive if $ku_\alpha^2 \gg 1$. Thus, $r_b v_b < -1$, and it follows that $v(r)$ meets the hyperbola exactly once. Since by II i, $u'(r) < v'(r)$, $u(r)$ meets the hyperbola exactly once in the interval $[r_\alpha, r_b]$. We now observe

$$(77) \qquad d_b - \delta^* u > -\frac{k}{1+k} u_\alpha + \frac{2E(k)}{u_\alpha} - \frac{A}{\sqrt{k}\, u_\alpha^2}$$

by (76). The condition $ku_\alpha^2 > B$ implies

$$(78) \qquad d_b - \delta^* u > \frac{1}{|u_\alpha|} \left(\frac{B}{1+k} - 2E(k) - \frac{A}{\sqrt{B}}\right)$$

which is positive for large B. Thus, $u(r)$ cannot cross the hyperbola in the interval $[r_b, r_\beta]$, which completes the proof.

The result (76) permits us to estimate the error terms in (62, 65). We find, using (58, 60, 76),

$$(79) \qquad 0 < \hat{H} - H = \frac{1}{2}(u_\beta - u_\alpha) = \frac{1}{2}(u^* - u_\alpha) + \frac{1}{2}\delta^* u < \frac{1}{2}(v_\beta - u_\alpha) + \frac{1}{2}\delta^* u$$

$$= -\frac{E(k)}{u_\alpha} + \frac{1}{2}\delta^* u < -\frac{E(k)}{u_\alpha} + \frac{A}{\sqrt{k}\, u_\alpha^2}$$

for large ku^2. Similarly, by (57, 58, 60, 76),

$$(80)\qquad 0 < r_\beta - r_b < \hat{r}_b - r_b = 2\,\frac{u_\beta - u_\alpha}{u_\beta u_\alpha}$$

$$= -\frac{4E(k)}{u_\alpha^3} + \frac{\delta^* u}{u_\alpha^2} + O\left(\frac{1}{|u_\alpha|^5}\right)$$

uniformly in k. It follows that

$$(81)\qquad 0 < r_\beta^2 - r_b^2 < \frac{A}{|u_\alpha|^4}$$

again uniformly in k.

We have $k = 1 + r_\alpha u_\alpha$, $\hat{k} = 1 + r_\alpha u_\beta$, so that by (59, 76)

$$(82)\qquad 0 < \hat{k} - k = r_\alpha(u_\beta - u_\alpha) < r_\alpha\left(\delta^* u - \frac{2E(k)}{u_\alpha}\right)$$

$$< (1-k)\left(\frac{2E(k)}{u_\alpha^2} + \frac{\delta^* u}{|u_\alpha|}\right)$$

$$< \frac{1-k}{u_\alpha^2}\,(2E(k) + \varepsilon)$$

for large ku_α^2. A formal calculation, using the asymptotic estimates for E and K for $k \sim 1$ (cf. [2], Chapter V), now yields

$$(83)\qquad \begin{aligned} E(\hat{k}) - E(k) &= O\left(\frac{1}{u_\alpha^2}\right) \\ S(\hat{k}) - S(k) &= O\left(\frac{1}{u_\alpha^2}\right) \end{aligned}$$

uniformly in k. The singularity of K near $k = 1$ is here canceled by the factor $(1-k)$ in (82).

We note next

$$(84)\qquad -(v_b - \hat{v}(r_b)) < \varepsilon_\beta < \delta^* u + u^* - v_b < \delta^* u < \frac{A}{\sqrt{k}\, u_\alpha^2}\ .$$

We estimate the left side of (84) using the explicit representation (45) for the surface $\hat{v}(r)$. This representation will apply, as $\hat{v}_i < \hat{v}(r_b)$ for $\hat{k}u_\alpha^2 \gg 1$. In the present case we find

$$(85) \quad v_b - \hat{v}(r_b) = -\frac{2E(k)}{u_\alpha} + \frac{2E(\hat{k})}{u_\alpha - (2E(\hat{k})/u_\beta)} + \frac{1}{u_\alpha - (2E(\hat{k})/u_\beta)}\left(\hat{k}\sin\hat{\varphi}_b + \int_0^{\hat{\varphi}_b} \sqrt{1-\hat{k}^2\sin^2\varphi}\, d\varphi\right).$$

From the definition of $v, \hat{v}$ we find

$$(86) \quad r_b - r_\alpha = -u_\alpha \int_{r_\alpha}^{r_b} \rho\, d\rho$$

$$(87) \quad r_b \sin\hat{\psi}(r_b) - r_\alpha = -u_\beta \int_{r_\alpha}^{r_b} \rho\, d\rho .$$

Thus

$$(88) \quad 1 - \sin\hat{\psi}(r_b) = (u_\beta - u_\alpha)\frac{r_b^2 - r_\alpha^2}{2r_b}$$

and from $1 - \sin^2\hat{\psi} = \hat{k}^2\sin^2\hat{\varphi}$ there follows, using (59, 67, 79)

$$(89) \quad \hat{k}^2\sin^2\hat{\varphi} < \frac{A}{u_\alpha^2}\frac{k}{1+k} .$$

We place this result in (85) and use (79) to obtain

$$(90) \quad 0 < v_b - \hat{v}(r_b) < \frac{A}{\sqrt{k}\, u_\alpha^2}$$

uniformly in k.

We are now in position to put (62, 65) into more effective forms. We write first, from (62), with $H = -\frac{1}{2}u_\alpha$,

$$(91) \quad \frac{1}{2}u_\beta\left(r_\beta + \frac{1}{u_\beta}\right)^2 - \frac{1}{2u_\beta} > \frac{1}{2}u_\alpha\left(r_\alpha + \frac{1}{u_\alpha}\right)^2 - \frac{1}{2u_\alpha} - \frac{1}{3u_\alpha^3}S(k) + \frac{1}{2}r_b^2(u_\beta - v_b)$$

from which

$$(92)\quad \frac{1}{2}u_\alpha\{(r_\beta+\frac{1}{u_\beta})^2-(r_\alpha+\frac{1}{u_\alpha})^2\} > \frac{1}{2}(\frac{1}{u_\beta}-\frac{1}{u_\alpha})-\frac{1}{3u_\alpha^3}S(k)$$

$$+\frac{1}{2}\frac{u_\alpha-u_\beta}{u_\beta^2}(r_\beta u_\beta+1)^2+\frac{1}{2}r_b^2(u_\beta-v_b).$$

We have

$$(93)\quad u_\beta-u_\alpha = u_\beta-v_b+v_b-u_\alpha$$

$$= \delta^*u+u^*-v_b+v_b-u_\alpha$$

$$= -\frac{2E(k)}{u_\alpha}+\delta^*u+u^*-v_b$$

which implies, by (76, 90)

$$-\frac{A}{\sqrt{k}\,u_\alpha^2}-\frac{2E(k)}{u_\alpha} < u_\beta-u_\alpha < -\frac{2E(k)}{u_\alpha}+\frac{A}{\sqrt{k}\,u_\alpha^2}.$$

The same calculation yields

$$(95)\quad |u_\beta-v_b| < \frac{A}{\sqrt{k}\,u_\alpha^2}.$$

We have also, by (80, 76)

$$(96)\quad 0 < r_\beta-r_b < \hat{r}_b-r_b = -\frac{4E(k)}{u_\alpha^3}+\frac{A}{\sqrt{k}\,u_\alpha^4}$$

with $|A|$ bounded uniformly in k, u_α for large $|u_\alpha|$, from which we derive

$$(97)\qquad 1+r_\beta u_\beta = 1+r_b v_b + (r_\beta u_\beta - r_b v_b)$$

$$= -k - \frac{2E(k)}{u_\alpha} r_b + r_\beta(u_\beta - v_b) + v_b(r_\beta - r_b)$$

so that the above estimates yield

$$(98)\qquad |(1+r_\beta u_\beta) + k| < \frac{A}{u_\alpha^2}$$

uniformly in k.

Returning to (92), we may now write

$$(99)\qquad \frac{1}{2} u_\alpha\{(r_\alpha + \frac{1}{u_\alpha}) + (r_\beta + \frac{1}{u_\beta})\}\{(r_\beta - r_\alpha) + \frac{1}{u_\beta} - \frac{1}{u_\alpha}\} > \frac{(1+k^2)E(k)}{u_\alpha^3} - \frac{S(k)}{3u_\alpha^3} - \frac{A}{\sqrt{k}\, u_\alpha^4}$$

$$> 2k^2 \frac{q(k)}{u_\alpha^3} - \frac{A}{\sqrt{k}\, u_\alpha^4}$$

with

$$(100)\qquad q(k) = E(k) - \frac{2}{3}\frac{(1+k^2)E(k) - (1-k^2)K(k)}{k^2} .$$

The expression

$$(101)\qquad 3q(k) = -2\int_0^{\pi/2} \frac{1-\sin^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\, d\varphi + \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\, d\varphi$$

shows that $q(k)$ decreases monotonically from $q(0) = 0$ to $q(1) = -1/3$.

We now write

$$(102)\qquad \begin{aligned} r_\beta - r_\alpha &> r_b - r_\alpha = -\frac{2k}{u_\alpha} \\ r_\beta - r_\alpha &< \hat{r}_b - r_\alpha = -\frac{2\hat{k}}{u_\beta} < -\frac{2k}{u_\alpha} + \frac{A}{\sqrt{k}\, u_\alpha^2} \end{aligned}$$

and, as in (96)

(103) $$\frac{1}{u_\beta} - \frac{1}{u_\alpha} = -\frac{u_\beta - u_\alpha}{u_\alpha u_\beta} = \frac{2E(k)}{u_\alpha^3} + \frac{A}{\sqrt{k}\, u_\alpha^4} .$$

We put these estimates into (99) to obtain

(104) $$(1+\varepsilon)\{(r_\alpha + \frac{1}{u_\alpha}) + (r_\beta + \frac{1}{u_\beta})\} < -\frac{2k\,q(k)}{u_\alpha^3} + \frac{A}{\sqrt{k}\, u_\alpha^4}$$

where $|\varepsilon|$ is small and $|A|$ is bounded, depending only on $ku_\alpha^2 \gg 1$.

Repeating the entire procedure starting with (65), we are led to the reverse inequality, with k replaced by $\hat{k}$ on the right. Applying (82) we obtain (104) with the inequality reversed, and thus

(105) $$(r_\alpha + \frac{1}{u_\alpha}) + (r_\beta + \frac{1}{u_\beta}) = -\frac{2k\,q(k)}{u_\alpha^3}(1+\varepsilon) + \frac{A}{\sqrt{k}\, u_\alpha^4} .$$

We place this estimate back into (99) (and the corresponding expression with $\hat{k}$) to find, using (102) and (103),

(106) $$|\varepsilon\{(r_\alpha + \frac{1}{u_\alpha}) + (r_\beta + \frac{1}{u_\beta})\}| < \frac{A}{\sqrt{k}\, u_\alpha^4} ,$$

the ε being the one that appears in (104). We are led to the basic relation for an outgoing arc (on which $0 \le \psi \le \pi/2$)

(107) $$(r_\alpha + \frac{1}{u_\alpha}) + (r_\beta + \frac{1}{u_\beta}) = -\frac{2k\,q(k)}{u_\alpha^3} + \frac{A}{\sqrt{k}\, u_\alpha^4}$$

with bounded $|A|$, depending only on $ku_\alpha^2 \gg 1$.

The case of a returning arc $(\psi \ge \pi/2)$ does not yield immediately to the same discussion, and it is necessary to distinguish the case $k \sim 1$. We note (fig. 3) that the comparison Delaunay surface $v(r)$ of curvature $H = -\frac{1}{2} u_\beta$ now lies *below* $u(r)$, and now provides an upper bound rather than a lower bound for $r_\beta - r_\alpha$. To obtain a lower bound, we observe that since $1 \leq E(k) \leq \pi/2$ and $E'(k) < 0$, there is (for large $|u_\beta|$) a unique positive solution $\hat{\tau}$ of

$$\hat{\tau}^2 + u_\beta \hat{\tau} + 2E(-1 - u_\beta r_\beta - \hat{\tau} r_\beta) = 0\,, \tag{108}$$

that is, there exists a unique Delaunay comparison surface $\hat{v}(r)$ through (r_β, u_β) with mean curvature $H = -\frac{1}{2}\hat{v}_a$, so that

$$\hat{\tau} = \hat{v}_a - u_\beta = -\frac{2E(\hat{k})}{\hat{v}_a} \tag{109}$$

with

$$\hat{k} = \frac{r_\beta - \hat{r}_a}{r_\beta + \hat{r}_a} = 1 + \hat{r}_a \hat{v}_a = -1 - r_\beta \hat{v}_a\,. \tag{110}$$

The solution curve $u(r)$ satisfies $u'(r) > \hat{v}'(r)$, $u(r) < \hat{v}(r)$, and hence $u(r)$ can be continued from r_β through decreasing r at least to the value $\hat{r}_a$. Letting $v(r)$ denote now the Delaunay surface through (r_β, u_β) with mean curvature $H = -\frac{1}{2}u_\beta$, we find $u'(r) < v'(r)$, $u(r) > v(r)$; it follows $u(r)$ cannot be continued to the minimum value r_a of definition of $v(r)$.

The relations analogous to (62, 65) become

$$\frac{1}{2} r_\beta^2 u_\beta + r_\beta < \frac{1}{2} r_\alpha^2 v_a + r_\alpha + \frac{1}{3u_\beta^3} S(k) \tag{111}$$

$$\frac{1}{2} r_\beta^2 u_\beta + r_\beta > \frac{1}{2} \hat{r}_a^2 \hat{v}_a + r_\alpha + \frac{1}{3\hat{v}_a^3} S(\hat{k}) - \frac{1}{2} u_\alpha(\hat{r}_a^2 - r_\alpha^2)\,. \tag{112}$$

As before, we may rewrite these relations:

$$\frac{1}{2} u_\beta\left[\left(r_\beta + \frac{1}{u_\beta}\right)^2 - \left(r_\alpha + \frac{1}{u_\alpha}\right)^2\right] < \frac{u_\alpha - u_\beta}{2}\left\{\frac{1}{u_\alpha u_\beta} + \frac{(1 + r_\alpha u_\alpha)^2}{u_\alpha^2}\right\} \tag{113}$$

$$+ \frac{1}{3u_\beta^2} S(k) + \frac{1}{2} r_\alpha^2 (v_a - u_\alpha)$$

$$(114)\quad \frac{1}{2}u_\beta[(r_\beta+\frac{1}{u_\beta})^2-(r_\alpha+\frac{1}{u_\alpha})^2] > \frac{u_\alpha-u_\beta}{2}\{\frac{1}{u_\alpha u_\beta}+\frac{(1+r_\alpha u_\alpha)^2}{u_\alpha^2}\} + \frac{1}{3\hat{v}_a^3}S(\hat{k})+\frac{1}{2}\hat{r}_a^2(\hat{v}_a-u_\alpha).$$

The further estimates must proceed differently, at least in the range $k \sim 1$.

From the defining relation (108) for $\hat{\tau}$ and the analogous one for $\tau = v_a - u_\beta$, follow $\tau, \hat{\tau} < A|u_\beta|^{-1}$ for large $|u_\beta|$. From

$$(115)\qquad \hat{r}_a - r_a = \frac{-4E(\hat{k})}{u_\beta(u_\beta+\hat{\tau})^2}$$

thus follows

$$(116)\qquad 0 < \hat{r}_a - r_a = A|u_\beta|^{-3}$$

with bounded A.

Let $u^* = u(\hat{r}_a)$. For given $\lambda > 0$, consider a rectangle R of width $A|u_\beta|^{-3}$ and height $\lambda A|u_\beta|^{-3}$ as in figure 4. Since $u(r)$ cannot be extended to $r = r_a$, there must be at least one point (r_p, u_p) in R at which $|\tan\psi| > \lambda$, i.e., at which

$$(117)\qquad |\cos\psi| < \frac{1}{\sqrt{1+\lambda^2}}, \qquad \sin\psi > \frac{\lambda}{\sqrt{1+\lambda^2}}.$$

From (71), which holds also on a returning arc, we find, for all $r \leq \hat{r}_a$,

$$(118)\qquad (\cos\psi)_u > u + \frac{\sin\psi}{\hat{r}_a}$$

and hence, at the given point,

$$(119)\quad (\cos\psi)_u > u + \frac{\lambda}{\hat{r}_a\sqrt{1+\lambda^2}} = u - \frac{1}{1-\hat{k}}\frac{\lambda}{\sqrt{1+\lambda^2}}(u_\beta - \frac{2E(\hat{k})}{\hat{v}_a}).$$

We note $E(\hat{k}) < \pi/2$; for any given k_o, $0 < k_o < 1$, we choose λ so that

(120) $$\frac{1}{1-k_o} \frac{\lambda}{\sqrt{1+\lambda^2}} > 1 .$$

For all sufficiently large $|u_\beta|$, the right side of (119) will then be positive for all $u \geq u_\beta$, for any $\hat{k}$ in $k_o < \hat{k} < 1$. Thus, $\cos\psi$ is increasing (from a negative value) at $r = r_p$, and we conclude that (120), and hence also (119), continue to hold for all $r < r_p$ to which $u(r)$ can be continued. Integrating (119), we find that a vertical must appear within a height change

(121) $$\delta^* u < A \frac{1-\hat{k}}{|u_\beta|} , \qquad \delta^* u = u_\alpha - u^* ,$$

uniformly in $k_o < \hat{k} < 1$.

For given $k < 1$ and large $|u|$ we may improve this result by estimating $\cos\bar{\psi}^*$ explicitly. We have

(122) $$r_\beta - \hat{r}_a \sin\psi^* = -\int_{\hat{r}_a}^{r_\beta} \rho u \, du < -\frac{u_\beta}{2}(r_\beta^2 - \hat{r}_a^2)$$

(123) $$r_\beta - \hat{r}_a = -\frac{\hat{v}_a}{2}(r_\beta^2 - \hat{r}_a^2)$$

from which

(124) $$\hat{r}_a(1-\sin\psi^*) < (\hat{v}_a - u_\beta)\frac{r_\beta + \hat{r}_a}{2}(r_\beta - \hat{r}_a)$$

so that

(125) $$1 - \sin\psi^* < \frac{4E(\hat{k})}{\hat{v}_a^2} \frac{\hat{k}}{1-\hat{k}}$$

and hence

(126) $$\cos\psi^* < \frac{2\sqrt{2}\sqrt{E(\hat{k})}}{-\hat{v}_a} \sqrt{\frac{\hat{k}}{1-\hat{k}}} .$$

We note (126) is similar to (70), however the term $(1+k)$ of (70) is replaced here by $1-\hat{k}$. This is the reason the range $k \sim 1$ requires special consideration on a returning arc.

Repeating now the reasoning that led to (121), with (117) replaced by (126), leads to

$$\delta^{*}u < \frac{A}{\sqrt{k}\, u_\beta^2} \tag{127}$$

for all $\hat{k} \leq k_o < 1$. This estimate holds for all sufficiently large $|u_\beta|$.

A returning arc has in all cases exactly one inflection between the vertical points (III iii). It is obvious a returning arc meets the hyperbola $ru = -1$ in exactly one point.

We proceed to obtain further estimates for $k \sim 1$, analogous to (79-90).

We have

$$k = \frac{r_\beta - r_a}{r_\beta + r_a}, \qquad \hat{k} = \frac{r_\beta - \hat{r}_a}{r_\beta + \hat{r}_a}, \tag{128}$$

thus

$$0 < k - \hat{k} = \frac{1}{2} r_\beta u_\beta \hat{v}_a (\hat{r}_a - r_a) < A\, u_\beta^{-2} \tag{129}$$

by (115). We note the factor $(1-k)$ of (82) no longer appears.

The estimate for $\hat{v}_a - v(\hat{r}_a)$ is complicated by the strong dependence on k of the position of the inflection on $v(r)$. We avoid this difficulty by noting that the hemispherical surface $w(r)$ of constant mean curvature r_β^{-1}, that passes through (r_β, u_β), has larger mean curvature than does $v(r)$, hence $v(r) - w(r) > 0$. It follows that

$$(130)\qquad \hat{v}_a - v(\hat{r}_a) < (\hat{v}_a - w(0)) + (w(0) - w(\hat{r}_a))$$

$$= (\hat{v}_a - u_\beta - r_\beta) + (r_\beta - \sqrt{r_\beta^2 - \hat{r}_a^2})$$

$$= -\frac{2E(\hat{k})}{\hat{v}_a} - r_\beta \sqrt{1 - (\hat{r}_a^2/r_\beta^2)}\,.$$

Formal estimation gives

$$(131)\qquad E(k) \sim 1 - \frac{1-k^2}{4}\left(\log \frac{1-k^2}{16} + 1\right)$$

for k near 1. Further,

$$(132)\qquad \frac{1}{\hat{v}_a} \sim \frac{1}{u_\beta} + \frac{2E(\hat{k})}{u_\beta^3}\,,$$

thus

$$(133)\qquad \hat{v}_a - v(\hat{r}_a) \mathrel{\tilde{<}} -\frac{2E(\hat{k}) + r_\beta u_\beta}{u_\beta} + \frac{1}{2}\left(\frac{1-\hat{k}}{1+\hat{k}}\right)^2 r_\beta$$

$$< \frac{A(1-\hat{k})|\log(1-\hat{k})|}{u_\beta}$$

with bounded $|A|$, uniformly in k for large $|u_\beta|$.

A repetition of previous procedures, using (128-133) in place of (79-90), leads after some calculation to

$$(134)\qquad \frac{2\hat{k}\, q(\hat{k})}{u_\beta^3} + \frac{A(1-\hat{k})\log(1-\hat{k})}{u_\beta^3} < \left(r_\alpha + \frac{1}{u_\alpha}\right) + \left(r_\beta + \frac{1}{u_\beta}\right)$$

$$< \frac{2k\, q(k)}{u_\beta^3} + \frac{B(1-\hat{k})\log(1-\hat{k})}{u_\beta^3}$$

with $|A|$ and $|B|$ bounded uniformly in k for large $|u_\beta|$. A formal, if tedious, calculation, based on asymptotic estimates for E and K for $k \sim 1$, yields

(135) $$|q(k) - q(\hat{k})| < \frac{A}{u_\beta^2} .$$

We place this estimate and (129) into (134) to obtain the basic estimate, for a returning arc with $k \sim 1$,

(136) $$(r_\alpha + \frac{1}{u_\alpha}) + (r_\beta + \frac{1}{u_\beta}) = \frac{2k\, q(k)}{u_\beta^3} + A\, \frac{(1-\hat{k})\log(1-\hat{k})}{u_\beta^3} .$$

If, for some fixed k_o, there holds $0 < k \leq k_o < 1$, $ku_\beta^2 \gg 1$, then the same procedure, using (127) in place of (121), yields for large $|u_\beta|$

(137) $$(r_\alpha + \frac{1}{u_\alpha}) + (r_\beta + \frac{1}{u_\beta}) = \frac{2k\, q(k)}{u_\beta^3} + \frac{A}{\sqrt{k}\, u_\beta^4}$$

with $|A| < A_o(k_o)$.

We summarize the information obtained thus far.

VI i: *A solution vertical at* (r_α, u_α)*, such that* $r_\alpha u_\alpha > -1$ *and* (75) *holds with* $k = 1 + r_\alpha u_\alpha$*, will again become vertical at* (r_β, u_β)*, with* $r_\beta u_\beta < -1$*. Between the two verticals there holds* $0 < \psi < \pi/2$*. The height change is estimated by*

(138) $$u_\beta = u_\alpha - \frac{2E(k)}{u_\alpha} + \varepsilon$$

with

(139) $$|\varepsilon| < \frac{A}{\sqrt{k}\, u_\alpha^2} .$$

The solution arc meets the hyperbola $ru = -1$ *in exactly one point. The change in horizontal distance to the hyperbola at the two vertical points is controlled by* (107).

VI ii: *Let* $k = -1 - r_\beta u_\beta$, *let*

(140) $$\hat{k} = -1 - r_\beta \hat{v}_a = -1 - r_\beta\left(u_\beta - \frac{2E(\hat{k})}{u_\beta + \hat{\tau}}\right) \qquad \text{(cf. (109))}.$$

A solution vertical at (r_β, u_β), *such that* $r_\beta u_\beta < -1$ *and* $ku_\beta^2 \gg 1$, *will again become vertical at* (r_α, u_α) *with* $r_\alpha u_\alpha > -1$. *Between the two verticals there holds* $\frac{\pi}{2} < \psi < \pi$. *There is exactly one inflection and one intersection with* $ru = -1$. *The height change is estimated by*

(141) $$u_\alpha = u_\beta - \frac{2E(\hat{k})}{u_\beta} + \varepsilon$$

with

(142) $$|\varepsilon| < A\,\frac{1 - \hat{k}}{|u_\beta|}$$

and $A < A_o(k_o) < \infty$ *in any range* $0 < k_o \leq k < 1$. *The change in horizontal distance to the hyperbola* $ru = -1$ *is controlled by* (136).

In any range $0 < k \leq k_o < 1$, *if* $ku_\beta^2 \gg 1$, *then the solution will again become vertical at* (r_α, u_α), $r_\alpha u_\alpha > -1$; *the height change is again estimated by* (141), *but with*

(143) $$|\varepsilon| < \frac{A}{\sqrt{k}\, u_\beta^2}$$

in place of (142). *The change in horizontal distance to the hyperbola* $ru = -1$ *is controlled by* (137).

VII Asymptotic Estimates

The results of VI show that for large $|u|$, the solution curve contracts toward the hyperbola $ru = -1$ between any two successive verticals. The estimates (107, 136, 137) contain quantitative information, which we now proceed to integrate to obtain new global information on the behavior of the solution, when $|u_o|$ is large. We set $r_o = 0$, denote the successive vertical points by (r_j, u_j) and write

(144)
$$c_j = |r_j + \frac{1}{u_j}|, \qquad k_j = -c_j u_j, \qquad \delta c_j = c_{j+1} - c_j,$$
$$\hat{\tau}_j = -\frac{2E(\hat{k}_j)}{u_j + \hat{\tau}_j}, \qquad \hat{k}_j = |1 + r_j(u_j + \hat{\tau}_j)| .$$

Using (144), (107, 136) now take the form, for $k, \hat{k} \sim 1$,

(145)
$$\delta c_j = -\frac{2k_j q(k_j)}{u_j^3} + A_j \frac{1}{\sqrt{k_j}\, u_j^4}, \qquad j \text{ even},$$

(146)
$$\delta c_j = -\frac{2k_j q(k_j)}{u_j^3} + A_j \frac{(1-\hat{k}_j)\log(1-\hat{k}_j)}{u_j^3}, \qquad j \text{ odd},$$

with $|A_j| < A < \infty$, uniformly for all sufficiently large $|u_j|$, in any range $0 < k_o \leq k < 1$.

We are interested in (145, 146) for large $|u|$. We note $-q(k) \leq -q(1) = \frac{1}{3}$, $E(k) \leq \frac{\pi}{2}$, and choose $u_{m_1}^2$ to be the (unique) solution of the equation

(147)
$$-\frac{4\pi}{u^2} \log \frac{4\pi}{u^2} = \frac{1}{6A} .$$

Let

(148)
$$k^{(1)} = \max\{k: \ -2k\,q(k) \leq -2A(1-k + \frac{4\pi}{u_{m_1}^2})\log(1-k + \frac{4\pi}{u_{m_1}^2})\} .$$

Clearly, $0 < k^{(1)} < 1$, and

(149)
$$k^{(1)} > \max\{k: \ -2k\,q(k) \leq -2A(1-\hat{k})\log(1-\hat{k})\} .$$

For all $k_j > k^{(1)}$, there holds $-A(1-\hat{k}_j)\log(1-\hat{k}_j) < -k_j q(k_j)$.

Now choose u_{m_2} so that $(k^{(1)})^3 q^2(k^{(1)})u_{m_2}^2 > A^2$. For values

$$u_j^2 > \max\{u_{m_1}^2, u_{m_2}^2\} \tag{150}$$

we may write, since $k_j = -c_j u_j$,

$$\delta c_j = -P_j c_j^3 \tag{151}$$

with

$$\min_{k \geq k^{(1)}} \frac{|q(k)|}{k^2} < P_j < \max_{k \geq k^{(1)}} \frac{3|q(k)|}{k^2} . \tag{152}$$

Integration of (151), with $c_o = -u_o^{-1}$, yields

$$2NP \sim c_N^{-2} - u_o^2 \tag{153}$$

for some P in the range indicated by (152).

We consider also the relation, which follows from (141-143),

$$\delta u_j = -\Lambda_j u_j^{-1}, \qquad 2 \gtrsim \Lambda_j \gtrsim \pi, \tag{154}$$

and which integrates to

$$u_N^2 \sim u_o^2 - 2\Lambda N, \qquad 2 \gtrsim \Lambda \gtrsim \pi. \tag{155}$$

From (153) and (154) we calculate

$$k_N^2 \sim \frac{pu_N^2}{(1+p)u_0^2 - u_N^2}, \qquad p = \Lambda P^{-1}, \tag{156}$$

and setting $u_N^2 = (1-\eta)u_o^2$,

$$k_N^2 \sim \frac{p(1-\eta)}{p+\eta} . \tag{157}$$

Given $k^{(0)}$, $k^{(1)} < k^{(0)} < 1$, there will be, for all sufficiently large $|u_o|$, a unique smallest $N = N^{(1)}$ for which

(158) $$k^{(1)} < [\frac{p(1-\eta)}{p+\eta}]^{\frac{1}{2}} < k^{(0)},$$

the value of the expression in (158) tends to $k^{(0)}$ with increasing $|u_o|$.

We reformulate our result slightly, and summarize the information thus far obtained. We note that any set of points k_j = const lies on the hyperbola $1 + ru$ = const, and that the singular solution $U(r)$ is asymptotic to the hyperbola $ru = -1$, as $r \to 0$. The following result holds for all $|u_o|$ sufficiently large.

VII i: *Given any* $k^{(0)}$*,* $k^{(1)} < k^{(0)} < 1$*, there exists* $\eta(k^{(0)}) > 0$. *such that the solution curve, starting at* $(0, u_o)$*, "separates" from the axis* $r = 0$ *and from the hyperbola* $ru = -2$*, after an interval* $u_N - u_o \sim \frac{1}{2}\eta|u_o|$*, in the sense that near the height* u_N *all points on the curve lie between the hyperbolas* $ru = -1 \pm k^{(0)}$*. Between* u_o *and* u_N *a number* $N^{(1)} \sim \frac{\eta}{2\Lambda} u_o^2$ *of vertical points appear, and each vertical point is followed by another (on the opposite side of* $ru = -1$*) at a height change* $\delta u_j \sim -\Lambda u_j^{-1}$.

To proceed further, we return to the relations (107, 136, 137); since $(1-k^{(0)}) \neq 0$ ($k^{(0)}$ independent of u_o), we may use (143) to write (145, 146), for $j > N^{(1)}$, in the common form

(159) $$\delta c_j = - \frac{2k_j q(k_j)}{u_j^3} + \frac{A_j}{\sqrt{k_j}\, u_j^4} .$$

We consider an interval in which the last two terms on the right in (122) will be small in relation to the first term. Since

(160) $$\lim_{k \to 0} \frac{kq(k)}{k^3} = -\frac{\pi}{16}$$

the condition takes the form

(161) $$k > A|u|^{-2/7}$$

for suitable A. Integration of (159) and of (154) yields, as above,

(162) $$k_N \sim \left(\frac{p}{1+p}\right)^{\frac{1}{2}} \frac{u_N}{u_o}$$

so that (161) now reads

(163) $$|u_N| > A|u_o|^{7/9}.$$

We can in fact achieve the situation

(164) $$k \sim A|u|^{-2/7}$$ $$|u| \sim A|u_o|^{7/9}$$

for suitably large A (independent of u_o), asymptotically for large $|u_o|$, in a number $N^{(2)} \sim \frac{1}{2\Lambda} u_o^2$ steps. In this configuration, *the solution curve has "contracted" towards the hyperbola* $ru = -1$; *we compute in fact from* (153, 155)

(165) $$\frac{c_{N^{(2)}}}{c_o} \sim \left(\frac{\Lambda}{\Lambda+P}\right)^{\frac{1}{2}}$$

as $|u_o| \to \infty$.

At the level $u_{N^{(2)}}$ the relation (159) no longer ensures a contraction at each step. The conditions for appearance of successive vertical points are, however, still satisfied, and (159) still suffices to bound the change δc_j at each step.

Let α satisfy $\frac{23}{9} < \alpha \leq 3$. In any range $A|u|^{-2/7} \geq k \geq B|u|^{\frac{2\alpha-8}{2\alpha+1}}$, we find

(166) $$\frac{1}{\sqrt{k}\, u_\alpha^4} < A\, c^\alpha$$

and we consider the inequality

(167) $$|\delta c_j| < A\, c^\alpha .$$

We integrate and simplify, noting

(168) $$c_{N(2)}^{1-\alpha} \gg u_{N(2)}^2$$

for large $|u_o|$, to obtain

(169) $$A < \frac{c_N}{c_o} < B$$
$$|u_N| > |u_o|^{\frac{2\alpha+1}{9}}$$
$$k_N > |u_o|^{\frac{2\alpha-8}{9}} \quad ;$$

thus the solution remains in a strip of sensibly constant width about $ru = -1$, until a height

$$|u_{N(3)}| \sim |u_o|^{\frac{2\alpha+1}{9}} .$$

We conclude in particular the existence of a constant A such that *in an interval*

(170) $$|u_o|^{\frac{2\alpha+1}{9}} < |u| < 2|u_o|^{\frac{2\alpha+1}{9}}$$

there holds $k_N < A|u|^{\frac{2\alpha-8}{2\alpha+1}}$. We assert that *for all sufficiently large* $|u|$, *the solution curve lies interior to a strip determined by*

(171) $$k = -\, cu = (A+1)|u|^{-2/7}$$

uniformly in u_o *as* $|u_o| \to \infty$, *for any* A *sufficiently large to justify* (164). This is clearly the case in the interval (170). If the curve $u(r)$ were to extend outside the strip (171) for values of u exceeding $-|u_o|^{(2\alpha+1)/9}$, there must be a first point p on the boundary of the strip. By comparison with Delaunay surfaces through the point p, one then sees (note either the condition (75) or the corresponding condition with u_β is satisfied at p) that a vertical would appear on or outside the strip $k = A|u|^{-2/7}$. Let q_1 be the first such point. The estimate (159), applied now in the direction of increasing $|u|$, shows that a preceding vertical can be found at a point q_o, with horizontal distance to $ru = -1$ exceeding that from q_1. The strip $k = A|u|^{-2/7}$ is however narrower at q_o than at q_1. This contradiction establishes the assertion.

We summarize:

VII ii: *Given* $k^{(0)} > k^{(1)}$, $|u_o|$ *large, there is an* $\eta(k^{(0)})$ *(determined by* (158)*) so that* $k < k^{(0)}$ *for* $|u_N| < |u_{N(1)}| \sim \sqrt{1-\eta}\,|u_o|$. *The curve can be continued through successive verticals to a height* $|u_{N(2)}| \sim A|u_o|^{7/9}$, *for suitably large* A, *at which level it has contracted towards* $ru = -1$ *in a ratio given by* (165). *For any* α, $\frac{23}{9} < \alpha < 3$, *the curve can be continued further through successive verticals till a height* $|u_{N(3)}| \sim |u_o|^{(2\alpha+1)/9}$, *and is confined to a strip of sensibly constant width, as indicated by* (169). *For smaller values of* $|u|$ *(relative to* $|u_o|$*) vertical points presumably cease to appear, however the curve lies within a strip about* $ru = -1$, *of width determined by* $k = A|u|^{-2/7}$, *for sufficiently large* A *(independent of* u_o*). Specifically, there exists* A *such that for any fixed (sufficiently large)* $\hat{u}$, *there holds, for* $(\hat{r},\hat{u})$ *on the solution curve,*

(172) $$\left|\hat{r} - \frac{1}{\hat{u}}\right| < A|\hat{u}|^{-9/7}$$

uniformly in u_o, *as* $|u_o| \to \infty$.

The global asymptotic behavior is sketched in figure 5.

VIII A Compactness Property

Let us consider the family of solution curves, represented in the form $r = f(u;u_o)$, with u_o as parameter, $|u_o| \to \infty$. The result (172) shows that for large $|u|$ the curve is confined to a narrow strip about $ru = -1$, and the method of proof of (172) yields as corollary the existence of a constant A such that on any fixed interval $a \leq u \leq b$,

$$\left|\frac{\partial f}{\partial u}\right| < A < \infty ,$$

for all sufficiently large $|u_o|$.

It follows there is a subsequence of values $u_o \to -\infty$ such that the corresponding functions $f(u;u_o)$ converge, uniformly on compact intervals, for all $|u|$ sufficiently large that (172) applies. The limit curve $\mathscr{C}$: $r = \mathfrak{F}(u)$, when described with arc length as parameter, yields a solution of the parametric system (3) of [1]. There holds

$$(173) \qquad |1 + u\,\mathfrak{F}(u)| < A|u|^{-2/7}$$

for all large $|u|$.

Each of the curves $f(u;u_o)$ can be extended globally without self-intersection as indicated in Theorem 6 of [1]. Applying the general continuous dependence theorem, we find that the limit curve $\mathscr{C}$ has the same property (a reasoning similar to the proof of Theorem 6 excludes self-intersection). The curve $\mathscr{C}$ has the asymptotic property $u\,\mathfrak{F}(u) \sim -1$ for large $|u|$, and the oscillatory behavior indicated in figure 1 of [1] for large r. It seems likely the curve $\mathscr{C}$ is the singular solution $U(r)$, and we conjecture that is the case.

Part of this work was carried out while the latter author was at Universität Bonn. The work was supported in part by the U.S. Department of Energy and by the National Science Foundation.

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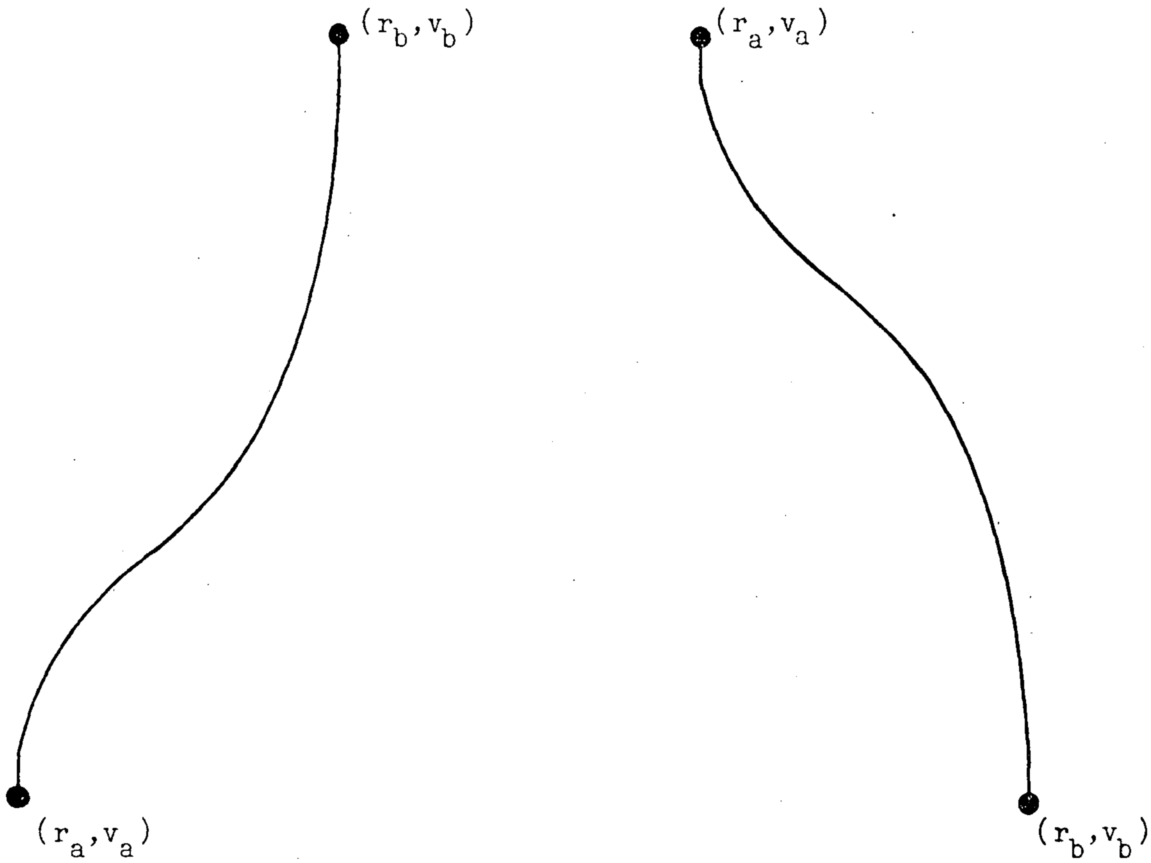
Footnote, p. 2

1) We note for reference the alternative representation $v_b - v_a = r_a K(\tilde{k}) + r_b E(\tilde{k})$, where K and E are complete elliptic integrals of first and second kind, and $\tilde{k} = (r_b^2 - r_a^2)^{\frac{1}{2}}/r_b$. Similarly, (45) takes the form $v - v_a = r_a F(\varphi, \tilde{k}) + r_b[E(\tilde{k}) - E(\Phi, \tilde{k})]$, where F is the incomplete integral of the first kind, and $r(1 - \tilde{k}^2 \sin^2\varphi)^{\frac{1}{2}} = r_a$, $r_b(1 - \tilde{k}^2 \sin^2\Phi)^{\frac{1}{2}} = r$. In this form of the representation there is no need to distinguish the inflection, however the formulae become technically more complicated in other respects.



i) outgoing arc

ii) returning arc

Figure 1. Delaunay arcs

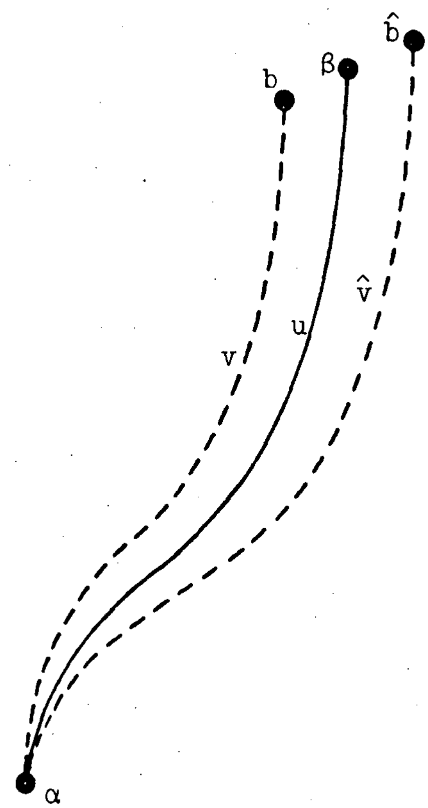


Figure 2. Comparison with Delaunay arcs; outgoing case

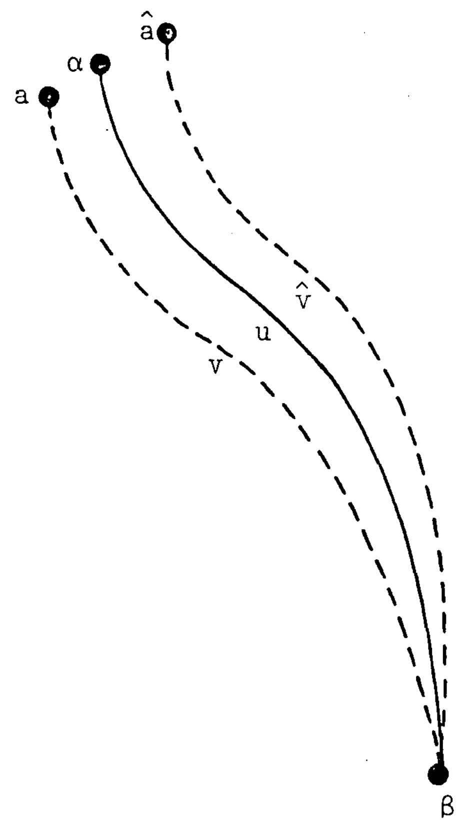


Figure 3. Comparison with Delaunay arcs; returning case

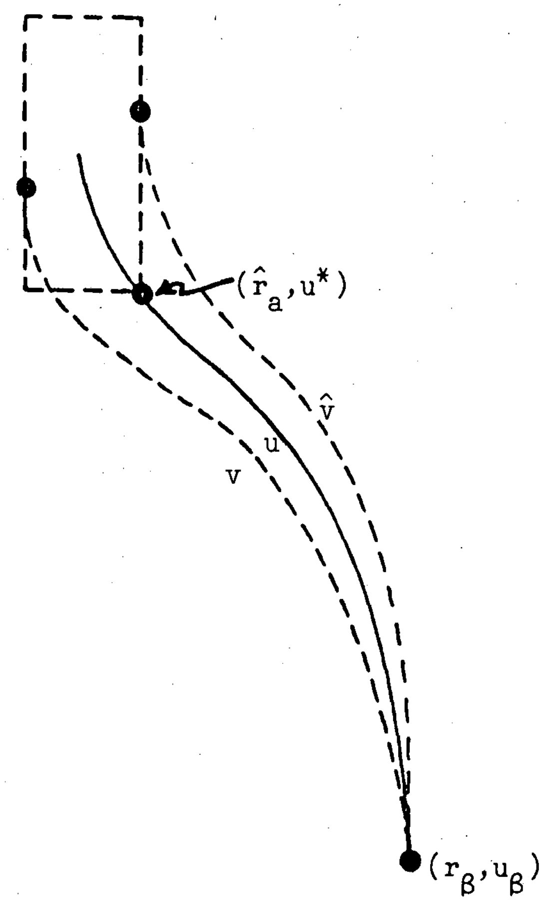


Figure 4. Estimate for u_α when $k \sim 1$

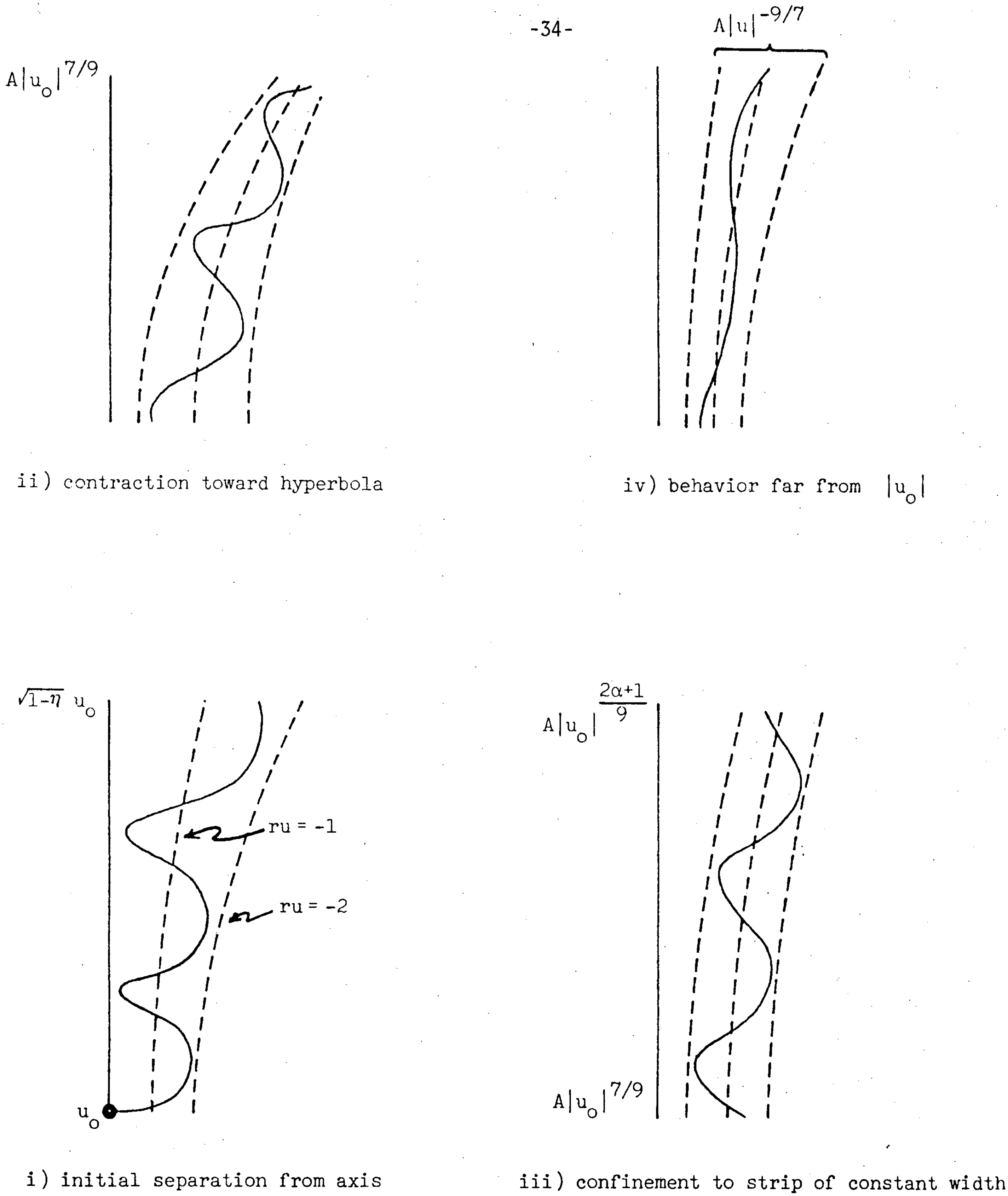


Figure 5. Asymptotic behavior for large $|u_o|$ at four levels (scales differ)

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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