

# UC Berkeley

## UC Berkeley Previously Published Works

### Title

Stationary response of a class of nonlinear stochastic systems undergoing markovian jumps

### Permalink

<https://escholarship.org/uc/item/5hn6m7bv>

### Journal

Journal of Applied Mechanics, Transactions ASME, 82(5)

### ISSN

0021-8936

### Authors

Huan, RH  
Zhu, WQ  
Ma, F  
[et al.](#)

### Publication Date

2015-05-01

### DOI

10.1115/1.4029954

Peer reviewed

**Rong-Hua Huan**

Department of Mechanics,  
State Key Laboratory of Fluid Power  
Transmission and Control,  
Zhejiang University,  
Hangzhou 310027, China

**Wei-qiu Zhu<sup>1</sup>**

Department of Mechanics,  
State Key Laboratory of Fluid Power  
Transmission and Control,  
Zhejiang University,  
Hangzhou 310027, China  
e-mail: wqzhu@zju.edu.cn

**Fai Ma**

Department of Mechanical Engineering,  
University of California,  
Berkeley, CA 94720

**Zu-guang Ying**

Department of Mechanics,  
State Key Laboratory of Fluid Power  
Transmission and Control,  
Zhejiang University,  
Hangzhou 310027, China

# Stationary Response of a Class of Nonlinear Stochastic Systems Undergoing Markovian Jumps

*Systems whose specifications change abruptly and statistically, referred to as Markovian-jump systems, are considered in this paper. An approximate method is presented to assess the stationary response of multidegree, nonlinear, Markovian-jump, quasi-nonintegrable Hamiltonian systems subjected to stochastic excitation. Using stochastic averaging, the quasi-nonintegrable Hamiltonian equations are first reduced to a one-dimensional Itô equation governing the energy envelope. The associated Fokker–Planck–Kolmogorov equation is then set up, from which approximate stationary probabilities of the original system are obtained for different jump rules. The validity of this technique is demonstrated by using a nonlinear two-degree oscillator that is stochastically driven and capable of Markovian jumps. [DOI: 10.1115/1.4029954]*

**Keywords:** Markovian jumps quasi-nonintegrable Hamiltonian system, stochastic excitation, stochastic averaging

## 1 Introduction

Markovian-jump systems represent a class of stochastic and hybrid systems whose operational rules can change in accordance with a Markov process. Complex dynamical systems are often Markovian-jump systems because abrupt changes in their configurations may occur due to component or interconnection failure or sudden environmental disturbances. Indeed, important examples of such systems include industrial plants and economic systems. Markovian-jump systems were first introduced by Krasovskii and Lidskii [1–3] in 1961, and they have since constituted an area of continuing research. In the past few decades, issues concerning stability, optimal control, filtering, and robustness have been examined in the literature. However, most of the published results are only applicable to linear Markovian-jump systems. The reader is referred to Mariton [4], Kushner [5], Luo [6], Boukas and Liu [7,8], Farias et al. [9], Souza and Fragoso [10], Kats and Martynyuk [11], Sworder [12], Ghosh et al. [13], Fragoso and Hemerly [14], Ji and Chizeck [15], Natache and Vilma [16], and the references therein. Far less is known about nonlinear Markovian-jump systems, particularly for multi-degree-of-freedom (MDOF) systems. Development of methodology for the analysis of MDOF nonlinear Markovian-jump systems is thus much deserving.

The purpose of this paper is to present an approximate method for evaluating the stationary response of MDOF, nonlinear, Markovian-jump, quasi-nonintegrable Hamiltonian systems subjected to stochastic excitation. The organization of this paper is as follows. In Sec. 2, the equations of Markovian-jump, quasi-nonintegrable, Hamiltonian systems are examined. Stochastic averaging [17–19] is applied to these systems in Sec. 3, which permits the reduction of the Hamiltonian equations to a one-dimensional Itô equation governing the approximate energy envelope of the original system. The Fokker–Planck–Kolmogorov equation associated with

the Itô equation of energy envelope is then set up in Sec. 4. By solving the stationary Fokker–Planck–Kolmogorov equation, stationary probabilities for assessing the long-term behavior of the original system are obtained. In Sec. 5, validity and accuracy of the method thus developed are demonstrated by using a two-degree-of-freedom nonlinear oscillator driven by Gaussian white noise, wherein comparison with direct system simulation is made and detailed calculations are provided. A summary of findings is given in Sec. 6. Throughout the paper, an effort is made to clarify the theoretical development in practical terms.

## 2 Problem Statement

The equations of motion of an  $n$ -degree-of-freedom dynamical system are composed of  $n$  second-order equations in the generalized displacements. These second-order equations can always be recast as  $2n$  first-order equations, usually in the state space or in the Hamiltonian phase space. Consider an  $n$ -degree-of-freedom, stochastically driven, nonlinear Hamiltonian system with Markovian jumps governed by

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad (1)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} - \varepsilon c_{ij}(\mathbf{q}, \mathbf{p}, s(t)) \frac{\partial H}{\partial p_j} + \varepsilon^{1/2} f_{ik}(\mathbf{q}, s(t)) W_k(t) \quad (2)$$

where  $i, j = 1, 2, \dots, n$ ;  $k = 1, 2, \dots, m$ ;  $q_i, p_i$  are, respectively, the generalized displacements and momenta;  $\mathbf{q} = (q_1, \dots, q_n)$  and  $\mathbf{p} = (p_1, \dots, p_n)$ . In accordance with the summation convention, the repeated indices  $j$  and  $k$  in Eq. (2) are summed over their respective ranges. In the above equations,  $H = H(\mathbf{q}, \mathbf{p})$  is the Hamiltonian,  $s(t)$  is a continuous-time Markov jump process,  $\varepsilon$  is a small parameter,  $\varepsilon c_{ij}(\mathbf{q}, \mathbf{p}, s(t))$  denote the jump coefficients of quasi-linear damping, and  $\varepsilon^{1/2} f_{ik}(\mathbf{q}, s(t))$  denote the jump amplitudes of excitations. The stochastic excitations  $W_k(t)$  are independent zero-mean Gaussian white

<sup>1</sup>Corresponding author.

Manuscript received January 28, 2015; final manuscript received February 28, 2015; published online March 31, 2015. Editor: Yonggang Huang.

noise processes with correlation  $E[W_k(t)W_l(t+\tau)] = 2D_{kl}\delta(\tau)$ . Note that the system is parametrically driven if  $f_{ik}$  are dependent on  $\mathbf{q}$ .

The Markov process  $s(t)$  points to the model or form in which the system operates [20], and it takes discrete values from a finite set  $S = \{1, 2, \dots, l\}$  with transition probability

$$P\{s(t+\Delta t) = j | s(t) = i\} = \begin{cases} \lambda_{ij}\Delta t + o(\Delta t), & i \neq j \\ 1 + \lambda_{ii}\Delta t + o(\Delta t) & i = j \end{cases} \quad (3)$$

The conditional probability  $P(j, t + \Delta t | i, t) = P\{s(t + \Delta t) = j | s(t) = i\}$  denotes the probability that the system takes the form  $j$  at time  $t + \Delta t$  given that it has the form  $i$  at time  $t$ . Transition rate between form  $i$  and form  $j$  is given by  $\lambda_{ij}$  where  $\lambda_{ij} > 0$  for  $i \neq j$ , and

$$\lambda_{ii} = - \sum_{\substack{j=1 \\ j \neq i}}^l \lambda_{ij} \quad (4)$$

A system governed by Eqs. (1) and (2) is referred to as a quasi-Hamiltonian system. Due to a lack of general techniques to solve nonlinear equations, the primary concern herein is an assessment of the long-term stochastic response of the system.

Theoretically speaking, Eqs. (1) and (2) are equivalent to the following Itô stochastic differential equations:

$$\begin{aligned} dq_i &= \frac{\partial H}{\partial p_i} dt & (5) \\ dp_i &= - \left[ \frac{\partial H}{\partial q_i} + \varepsilon c_{ij}(\mathbf{q}, \mathbf{p}, s(t)) \frac{\partial H}{\partial p_j} - \Delta W \right] dt \\ &+ \varepsilon^{1/2} \sigma_{ik}(\mathbf{q}, s(t)) dB_k(t) & (6) \end{aligned}$$

where  $B_k(t)$  are unit Wiener processes such that  $\sigma_{ik}(\mathbf{q}, s(t)) dB_k(t) = f_{ik}(\mathbf{q}, s(t)) W_k(t) dt$ .  $\Delta W = D_{kl} f_{il} (\partial f_{ik} / \partial p_j)$  is the Wong–Zakai correction term, which is equal to 0 since  $f_{ik}$  are assumed to be independent of  $\mathbf{p}$  in this paper. When  $\varepsilon = 0$ , the resulting Hamiltonian system may be integrable, nonintegrable, or partially integrable [21]. Siegel and Moser [22] showed that, in certain classes of Hamiltonian systems, the nonintegrable ones form a dense set. In view of this, it is assumed that the Hamiltonian system governed by Eqs. (1) and (2) is nonintegrable. That means there is only one independent integral of motion, i.e., the Hamiltonian  $H$ . In general, the system kinetic energy is a quadratic form of the generalized velocities. In this case, the Hamiltonian  $H$  is equal to the energy envelope or total energy of the system.

### 3 Stochastic Averaging of Energy Envelope

Let the Markov jump process be arbitrarily fixed at  $s(t) = u$  where  $1 \leq u \leq l$ . When the system operates in the form  $u$  and jumps are not permitted, denote by  $c_{ij}^{(u)} = c_{ij}(\mathbf{q}, \mathbf{p}, s(t) = u)$  and  $\sigma_{ik}^{(u)} = \sigma_{ik}(\mathbf{q}, s(t) = u)$ . Since  $H = H(\mathbf{q}, \mathbf{p})$ , a stochastic differential for  $H$  can be derived from Eqs. (5) and (6) using the Itô differential rule [23] so that

$$dH = \varepsilon \left[ -c_{ij}^{(u)} \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} + \frac{1}{2} \sigma_{ik}^{(u)} \sigma_{jk}^{(u)} \frac{\partial^2 H}{\partial p_i \partial p_j} \right] dt + \varepsilon^{1/2} \frac{\partial H}{\partial p_i} \sigma_{ik}^{(u)} dB_k(t) \quad (7)$$

Since  $\varepsilon$  is a small parameter, the above relation indicates that  $H$  is a slowly varying process while the generalized displacements  $q_1, q_2, \dots, q_n$  and generalized momenta  $p_1, p_2, \dots, p_n$  are usually rapidly varying processes with respect to time. By a theorem of Khasminskii [24],  $H$  converges to a one-dimensional diffusion process  $E$  as  $\varepsilon \rightarrow 0$ . The Itô equation for this diffusion process is obtained

by time averaging of Eq. (7). The effect of stochastic averaging is to average out the rapidly varying processes so as to yield an equation for the slowly varying process  $H$ , which is essential for describing the long-term behavior of the system.

Time averaging of Eq. (7) can be conducted by traditional methods [17–19] because the system only takes the form  $u$ . Upon stochastic averaging, the limiting process  $E$  satisfies of the equation [18]

$$dE = m^{(u)}(E)dt + \bar{\sigma}^{(u)}(E)dB(t) \quad (8)$$

where  $B(t)$  is unit Wiener process and the drift coefficient  $m^{(u)}(E)$  and diffusion coefficient  $\bar{\sigma}^{(u)}(E)$  are given by

$$m^{(u)}(E) = \frac{1}{T(E)} \int_{\Omega} \left[ -c_{ij}^{(u)} \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} + \frac{1}{2} \sigma_{ik}^{(u)} \sigma_{jk}^{(u)} \frac{\partial^2 H}{\partial p_i \partial p_j} \right] \left( \frac{\partial H}{\partial p_1} \right)^{-1} d\mathbf{z} \quad (9)$$

$$\left\{ \bar{\sigma}^{(u)}(E) \right\}^2 = \frac{1}{T(E)} \int_{\Omega} \left[ \sigma_{ik}^{(u)} \sigma_{jk}^{(u)} \frac{\partial^2 H}{\partial p_i \partial p_j} \right] \left( \frac{\partial H}{\partial p_1} \right)^{-1} d\mathbf{z} \quad (10)$$

where  $\mathbf{z} = (q_1, \dots, q_n, p_2, \dots, p_n)$  is of order  $2n - 1$ , the region of integration is  $\Omega = \{\mathbf{z} : H(q_1, \dots, q_n, 0, p_2, \dots, p_n) \leq E\}$ , and the parameter

$$T(E) = \int_{\Omega} \left( \frac{\partial H}{\partial p_1} \right)^{-1} d\mathbf{z} \quad (11)$$

First, it is intuitive to replace  $E$  by  $H$  in Eq. (8) even though  $E$  is only an approximation and is not equal to  $H$ . Second, Eq. (8) is only valid when  $s(t) = u$ . As Markovian jumps are allowed so that  $s(t)$  takes values from  $S = \{1, 2, \dots, l\}$ , Eq. (8) can be extended so that

$$dH = m(H, s(t))dt + \bar{\sigma}(H, s(t))dB(t) \quad (12)$$

where the drift and diffusion coefficients  $m(H, s(t)), \bar{\sigma}(H, s(t))$  change as  $s(t)$  jumps, so that  $m(H, s(t) = u) = m^{(u)}(H)$  and  $\bar{\sigma}(H, s(t) = u) = \bar{\sigma}^{(u)}(H)$ . In this interpretation, the Markovian-jump system governed by Eqs. (1) and (2) possesses an energy envelope given approximately by the solution  $H$  of Eq. (12). In Sec. 3, probability density of this energy envelope is discussed.

### 4 Diffusion Approach to Energy Envelope

Let  $p(H, j, t + \Delta t | H_0, i, t)$  be the transition probability that the Hamiltonian takes the value  $H$  at time  $t + \Delta t$  when the Markov process  $s(t + \Delta t) = j$ , given that the Hamiltonian is  $H_0$  at  $t$  and  $s(t) = i$ . Recall from Eq. (3) that  $P(j, t + \Delta t | i, t)$  denotes the probability that the system takes the form  $j$  at time  $t + \Delta t$  given that it has the form  $i$  at  $t$ . In order that  $s(t + \Delta t) = j$ , the system either remains in the form  $j$  or it jumps from  $i$  to  $j$  in the interval  $[t, t + \Delta t]$ . Hence the following equation is obtained [25,26]:

$$\begin{aligned} p(H, j, t + \Delta t) &= \int_0^{\infty} p(H_0, j, t) p(H, j, t + \Delta t | H_0, j, t) \\ &P(j, t + \Delta t | j, t) dH_0 \\ &+ \sum_{\substack{i=1 \\ i \neq j}}^l \int_0^{\infty} p(H_0, i, t) p(H, j, t + \Delta t | H_0, i, t) \\ &P(j, t + \Delta t | i, t) dH_0 \end{aligned} \quad (13)$$

Without jumps, the transition probability  $p(H, j, t + \Delta t | H_0, j, t)$  satisfies the Fokker–Planck–Kolmogorov equation

$$\begin{aligned} & \frac{\partial}{\partial t} p(H, j, t + \Delta t | H_0, j, t) \\ &= -\frac{\partial}{\partial H} [m(H, j) p(H, j, t + \Delta t | H_0, j, t)] \\ & \quad + \frac{1}{2} \frac{\partial^2}{\partial H^2} [\bar{\sigma}^2(H, j) p(H, j, t + \Delta t | H_0, j, t)] \end{aligned} \quad (14)$$

In a small time  $\Delta t$ , Eq. (14) can be rewritten as

$$\begin{aligned} & p(H, j, t + \Delta t | H_0, j, t) \\ &= \delta(H - H_0) - \Delta t \frac{\partial}{\partial H} [m(H, j) \delta(H - H_0)] \\ & \quad + \frac{1}{2} \Delta t \frac{\partial^2}{\partial H^2} [\bar{\sigma}^2(H, j) \delta(H - H_0)] + o(\Delta t) \end{aligned} \quad (15)$$

where  $\delta(H - H_0) = p(H, j, t | H_0, j, t)$ . Substitute Eq. (15) into Eq. (13) and simplify it with Eq. (3) to obtain

$$\begin{aligned} p(H, j, t + \Delta t) &= p(H, j, t) - \Delta t \frac{\partial}{\partial H} [m(H, j) p(H, j, t)] \\ & \quad + \frac{1}{2} \Delta t \frac{\partial^2}{\partial H^2} [\bar{\sigma}^2(H, j) p(H, j, t)] \\ & \quad - \Delta t \sum_{\substack{i=1 \\ i \neq j}}^l [\lambda_{ji} p(H, j, t) \\ & \quad - \lambda_{ij} \int_0^\infty p(H_0, i, t) p(H, j, t + \Delta t | H_0, i, t) dH_0] + o(\Delta t) \end{aligned} \quad (16)$$

Divide the above equation by  $\Delta t$  and let  $\Delta t \rightarrow 0$ , the following generalized Fokker-Planck-Kolmogorov equation arises:

$$\begin{aligned} \frac{\partial}{\partial t} p(H, j, t) &= -\frac{\partial}{\partial H} [m(H, j) p(H, j, t)] + \frac{1}{2} \frac{\partial^2}{\partial H^2} [\bar{\sigma}^2(H, j) p(H, j, t)] \\ & \quad - \sum_{\substack{i=1 \\ i \neq j}}^l \left[ \lambda_{ji} p(H, j, t) - \lambda_{ij} \int_0^\infty p(H_0, i, t) \right. \\ & \quad \left. \times p(H, j, t | H_0, i, t) dH_0 \right] \end{aligned} \quad (17)$$

As a probability density,  $p(H, j, t)$  is finite and, as  $t \rightarrow \infty$ ,  $\partial p(H, j, t) / \partial t$  must vanish. It will be assumed that jumps are independent. In this case [25,26],  $p(H, j, t | H_0, i, t) = \delta(H - H_0)$  and Eq. (17) simplifies to

$$\begin{aligned} \frac{\partial}{\partial t} p(H, j, t) &= -\frac{\partial}{\partial H} [m(H, j) p(H, j, t)] + \frac{1}{2} \frac{\partial^2}{\partial H^2} [\bar{\sigma}^2(H, j) p(H, j, t)] \\ & \quad - \sum_{\substack{i=1 \\ i \neq j}}^l [\lambda_{ji} p(H, j, t) - \lambda_{ij} p(H, i, t)] \end{aligned} \quad (18)$$

The above partial differential equation does not admit an easy solution, analytically or numerically. However, as  $t \rightarrow \infty$ , stationary solution can be readily determined, at least numerically. The stationary probability density  $p(H, j)$  satisfies the ordinary differential equation

$$\begin{aligned} & -\frac{d}{dH} [m(H, j) p(H, j)] + \frac{1}{2} \frac{d^2}{dH^2} [\bar{\sigma}^2(H, j) p(H, j)] \\ & \quad - \sum_{\substack{i=1 \\ i \neq j}}^l [\lambda_{ji} p(H, j) - \lambda_{ij} p(H, i)] = 0 \end{aligned} \quad (19)$$

The stationary probability density of the approximate energy envelope  $H$  can then be expressed as

$$p(H) = c \sum_{j=1}^l p(H, j), \quad c^{-1} = \sum_{j=1}^l \int_0^\infty p(H, j) dH \quad (20)$$

Note that  $c$  is just a normalization constant. In Eq. (20),  $H$  is a positive real number. However, the joint stationary probability  $p(\mathbf{q}, \mathbf{p})$  can be calculated from  $p(H)$  if  $H$  is equated to its functional form  $H = H(\mathbf{q}, \mathbf{p})$  in terms of  $\mathbf{q}$  and  $\mathbf{p}$  [18]. Symbolically,

$$p(\mathbf{q}, \mathbf{p}) = \frac{p(H)}{T(H)} \Big|_{H=H(\mathbf{q}, \mathbf{p})} \quad (21)$$

Subsequently, marginal probability densities of the generalized displacements  $q_i$  and generalized momenta  $p_i$  can be obtained by integration.

It should be pointed out that rigorous analysis of the errors committed by stochastic averaging has not been reported in the open literature. As such, analytical quantification of the errors of the method thus presented cannot be made. While Eqs. (19) and (20) provide approximate formulas for the energy envelope of the stationary response, their accuracy can only be examined by comparison with direct simulations.

## 5 Illustrative Example

To demonstrate the validity and perhaps accuracy of the method presented in this paper, consider a two-degree-of-freedom nonlinear oscillator that is capable of independent Markovian jumps and governed by

$$\ddot{x}_1 + \beta \omega_1^2 x_1 (\omega_1^2 x_1^2 + \omega_2^2 x_2^2)^2 = -c_1(s(t)) \dot{x}_1 + f_1(s(t)) W_1(t) \quad (22)$$

$$\ddot{x}_2 + \beta \omega_2^2 x_2 (\omega_1^2 x_1^2 + \omega_2^2 x_2^2)^2 = -c_2(s(t)) \dot{x}_2 + f_2(s(t)) W_2(t) \quad (23)$$

where  $c_k(s(t))$  are jump coefficients of linear damping,  $f_k(s(t))$  are jump amplitudes of external excitations  $W_k(t)$ , which are independent zero-mean Gaussian white noise processes with intensities  $2D_k$ . The continuous-time Markov process  $s(t)$  takes discrete values from a finite set  $S = \{1, 2, \dots, l\}$  with transition probability given in Eq. (3).

Let  $q_i = x_i$  and  $p_i = \dot{x}_i$ . As a Hamiltonian system, Eqs. (22) and (23) can be expressed as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ -\beta \omega_1^2 q_1 (\omega_1^2 q_1^2 + \omega_2^2 q_2^2)^2 - c_1(s(t)) p_1 + f_1(s(t)) W_1(t) \\ -\beta \omega_2^2 q_2 (\omega_1^2 q_1^2 + \omega_2^2 q_2^2)^2 - c_2(s(t)) p_2 + f_2(s(t)) W_2(t) \end{bmatrix} \quad (24)$$

The system kinetic energy is  $T_2 = (x_1^2 + x_2^2)/2$  and the potential is  $U = \beta(\omega_1^2 q_1^2 + \omega_2^2 q_2^2)^3/6$ . Thus, the Hamiltonian is

$$H = T_2 + U = \frac{1}{2} (p_1^2 + p_2^2) + \frac{\beta}{6} (\omega_1^2 q_1^2 + \omega_2^2 q_2^2)^3 \quad (25)$$

Upon stochastic averaging, Eq. (12) for the Hamiltonian system (24) is obtained, for which  $\Omega = \{(q_1, q_2, p_2) : p_2^2/2 + \beta(\omega_1^2 q_1^2 + \omega_2^2 q_2^2)^3/6 \leq H\}$ , and

$$\begin{aligned} & m(H, s(t) = u) \\ &= \frac{1}{T(H)} \int_{\Omega} \left[ -c_1^{(u)} p_1^2 - c_2^{(u)} p_2^2 + 2\{f_1^{(u)}\}^2 D_1 + 2\{f_2^{(u)}\}^2 D_2 \right] \\ & \quad \times \frac{1}{p_1} dq_1 dq_2 dp_2 \end{aligned} \quad (26)$$

$$\{\bar{\sigma}^{(u)}(H, s(t) = u)\}^2 = \frac{1}{T(H)} \int_{\Omega} \left[ 2\{f_1^{(u)}\}^2 D_1 p_1^2 + 2\{f_2^{(u)}\}^2 D_2 p_2^2 \right] \frac{1}{p_1} dq_1 dq_2 dp_2 \quad (27)$$

$$T(H) = \int_{\Omega} \frac{1}{p_1} dq_1 dq_2 dp_2 \quad (28)$$

Evaluate the above integrals by using the change of variables

$$q_1 = \frac{r}{\omega_1} \cos \theta, \quad q_2 = \frac{r}{\omega_2} \sin \theta \quad (29)$$

It can be checked that

$$m(H, s(t) = u) = -\frac{3}{4} [c_1^{(u)} + c_2^{(u)}] H + \{f_1^{(u)}\}^2 D_1 + \{f_2^{(u)}\}^2 D_2 \quad (30)$$

$$\{\bar{\sigma}^{(u)}(H, s(t) = u)\}^2 = \frac{3}{2} \left[ \{f_1^{(u)}\}^2 D_1 + \{f_2^{(u)}\}^2 D_2 \right] H \quad (31)$$

$$T(H) = \frac{2\pi^2}{\omega_1 \omega_2} \left( \frac{6H}{\beta} \right)^{1/3} \quad (32)$$

When the jumps are independent, the conditional probability density  $p(H, u, t)$  satisfies Eq. (18), while the stationary probability  $p(H, u)$  satisfies Eq. (19). In this example, Eq. (19) is solved numerically. The stationary probability  $p(H)$  of the energy envelope is then obtained by using Eq. (20). Numerical results are given explicitly for a two-form jump process and a three-form jump process.

**5.1 Two-Form System.** In this case,  $l = 2$  and  $S = \{1, 2\}$ . In Eqs. (22) and (23), put  $\omega_1 = 1$ ,  $\omega_2 = 2.2$ ,  $\beta = 1$ ,  $D_1 = 0.04$ ,  $D_2 = 0.04$ , and

$$c_1^{(1)} = 0.01, \quad c_2^{(1)} = 0.03, \quad c_1^{(2)} = 0.03, \quad c_2^{(2)} = 0.05 \quad (33)$$

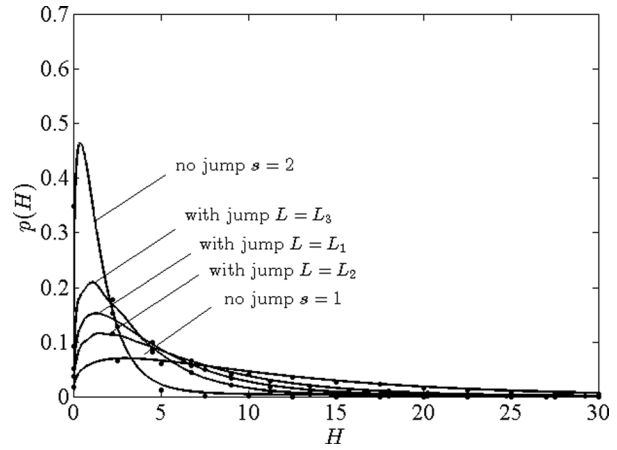
$$h_1^{(1)} = 2, \quad h_2^{(1)} = 2, \quad h_1^{(2)} = 1, \quad h_2^{(2)} = 1 \quad (34)$$

The above specifications indicate that additional viscous dampers are deployed and the random excitations also decrease in amplitudes if the system takes the form  $s(t) = 2$ . Prescribe the transition rate  $\lambda_{ij}$  between the form  $i$  and the form  $j$  by a transition matrix  $L = [\lambda_{ij}]$ . Three special cases are considered with

$$L_1 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}, \quad L_3 = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \quad (35)$$

Observe that  $\lambda_{12} = \lambda_{21}$  if  $L = L_1$  so that the system jumps between the two forms with equal probabilities. On the other hand,  $\lambda_{12} < \lambda_{21}$  if  $L = L_2$  so that the system is more likely to take the form  $s(t) = 1$ . Finally, the system is more likely to take the form  $s(t) = 2$  if  $L = L_3$ .

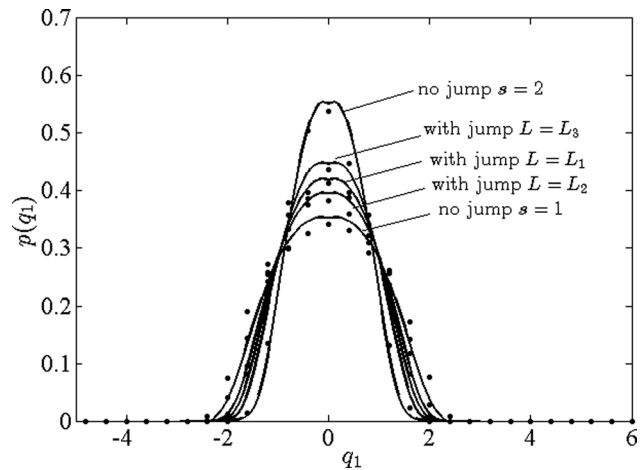
In Fig. 1, the stationary probability  $p(H)$  of the energy envelope  $H$  is shown when the transition rates are specified by the three transition matrices in Eq. (35). Also plotted are the probability densities when Markov jump process is fixed at either  $s(t) = 1$  or  $s(t) = 2$ . When the system operates in the form  $s(t) = 2$ , it has larger damping coefficients  $c_1^{(2)}$ ,  $c_2^{(2)}$  and smaller amplitudes  $h_1^{(2)}$ ,  $h_2^{(2)}$  of stochastic excitations than when it operates in the form  $s(t) = 1$ . Thus the system energy dissipates faster when  $s(t) = 2$  and, as a consequence,  $p(H)$  decreases faster as  $H$  increases. For large values of  $H$ ,  $p(H)$  is the smallest when  $s(t) = 2$ , and it increases as the system cycles through  $L = L_3$ ,  $L = L_1$ ,  $L = L_2$ , and  $s(t) = 1$ . In Fig. 1, the lines are obtained by numerical



**Fig. 1** Stationary probability densities  $p(H)$  of the energy envelope  $H$  of two-form system (24) for  $L = L_1$ ,  $L = L_2$ ,  $L = L_3$  in Eq. (35), and when the system is fixed at  $s(t) = 1$  and  $s(t) = 2$ . The lines are obtained by numerical solution of Eqs. (19) and (20), while the dots • are obtained by direct simulation of system (24).

solution of Eqs. (19) and (20), while the dots are obtained by direct simulation of system (24). Observe that the dots match closely with the corresponding lines, demonstrating the validity and accuracy of Eq. (19), derived by analysis to assess the stationary response.

In Fig. 2, the stationary probability  $p(q_1)$  of the displacement  $q_1$  is shown with the same transition rates as in Fig. 1, while  $p(q_2)$  is given in Fig. 3. As pointed out previously, the system has larger damping coefficients and smaller amplitudes of stochastic excitations if  $s(t) = 2$ . When the system spends more time in the form  $s(t) = 2$ , there is a higher probability that  $q_1$ ,  $q_2$  will be located near their equilibrium in the long term. Hence, as shown in Fig. 2,  $p(q_1)$  has the largest mode and the smallest dispersion around  $q_1 = 0$  if  $s(t) = 2$ . The mode around  $q_1 = 0$  decreases and the dispersion increases as the system cycles through  $L = L_3$ ,  $L = L_1$ ,  $L = L_2$ , and  $s(t) = 1$ . The same observation can be made about  $p(q_2)$  in Fig. 3. Again, the dots obtained by direct simulation of Eq. (24) closely match the data obtained by using Eqs. (19) and (20). Finally, a sample time history of the two-form Markov jump process  $s(t)$  with independent jumps is shown in Fig. 4.



**Fig. 2** Stationary probability densities  $p(q_1)$  of the displacement  $q_1$  of two-form system (24) with the same transition rates as in Fig. 1

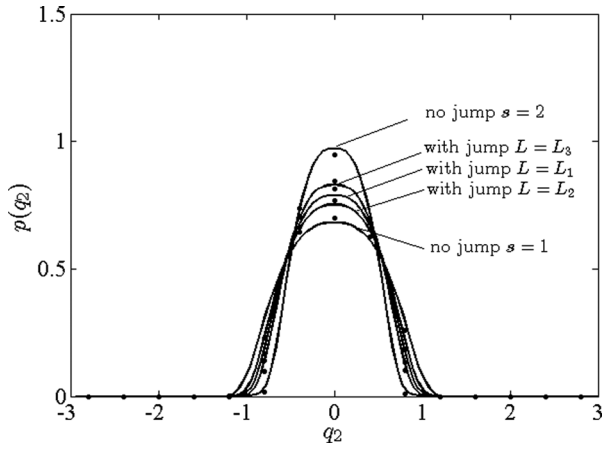


Fig. 3 Stationary probability densities  $p(q_2)$  of the displacement  $q_2$  of two-form system (24) with the same transition rates as in Fig. 1

**5.2 Three-Form System.** In this case,  $l = 3$  and  $S = \{1, 2, 3\}$ . The Markov process may be regarded as a direct generalization of the two-form jump process considered earlier, so that the previous parameters remain unchanged, with the additional specifications that

$$c_1^{(3)} = 0.05, \quad c_2^{(3)} = 0.07, \quad h_1^{(3)} = 1, \quad h_2^{(3)} = 1 \quad (36)$$

Thus the system has the largest damping coefficients and among the smallest amplitudes of stochastic excitations if  $s(t) = 3$ . Four special cases are considered with

$$L_1 = \begin{bmatrix} -3 & 1.5 & 1.5 \\ 1.5 & -3 & 1.5 \\ 1.5 & 1.5 & -3 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -2 & 1 & 1 \\ 1.5 & -3 & 1.5 \\ 1.5 & 1.5 & -3 \end{bmatrix} \quad (37)$$

$$L_3 = \begin{bmatrix} -3 & 1.5 & 1.5 \\ 1 & -2 & 1 \\ 1.5 & 1.5 & -3 \end{bmatrix}, \quad L_4 = \begin{bmatrix} -3 & 1.5 & 1.5 \\ 1.5 & -3 & 1.5 \\ 1 & 1 & -2 \end{bmatrix} \quad (38)$$

Observe that all three forms are equally likely when  $L = L_1$ . If  $L = L_2$ , the form  $s(t) = 1$  is favored. Likewise, the system favors the form  $s(t) = 2$  if  $L = L_3$ , while it favors the form  $s(t) = 3$  if  $L = L_4$ . The stationary probability  $p(H)$  of the energy envelope  $H$  is evaluated and plotted in Fig. 5 for different cases. Stationary

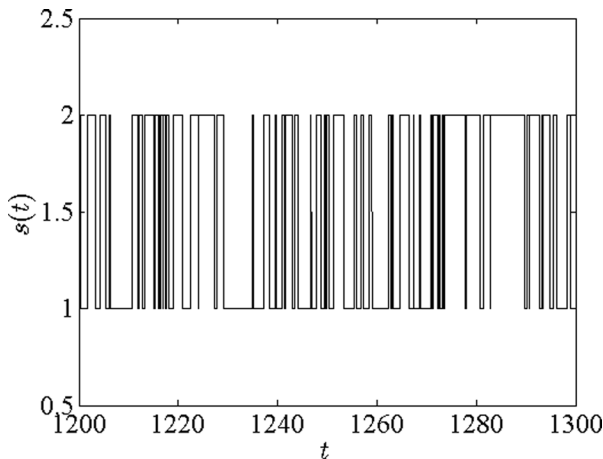


Fig. 4 Sample time history of two-form Markov jump process  $s(t)$  with independent jumps

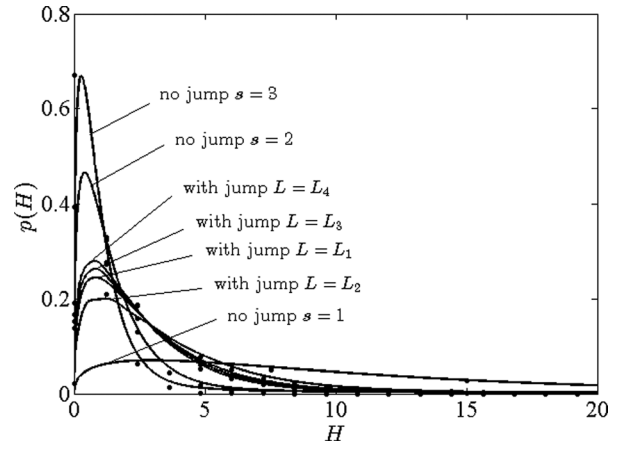


Fig. 5 Stationary probability densities  $p(H)$  of the energy envelope  $H$  of three-form system (24) for  $L = L_1$ ,  $L = L_2$ ,  $L = L_3$ ,  $L = L_4$  in Eqs. (37) and (38), and when the system is fixed at  $s(t) = 1$ ,  $s(t) = 2$  and  $s(t) = 3$ . The lines are obtained by numerical solution of Eqs. (19) and (20), while the dots • are obtained by direct simulation of system (24).

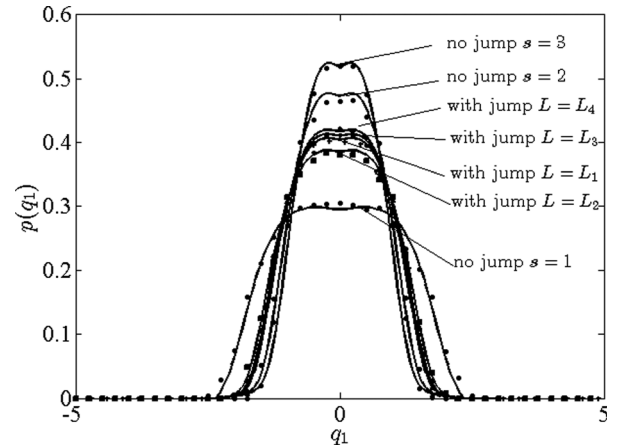


Fig. 6 Stationary probability densities  $p(q_1)$  of the displacement  $q_1$  of three-form system (24) with the same transition rates as in Fig. 5

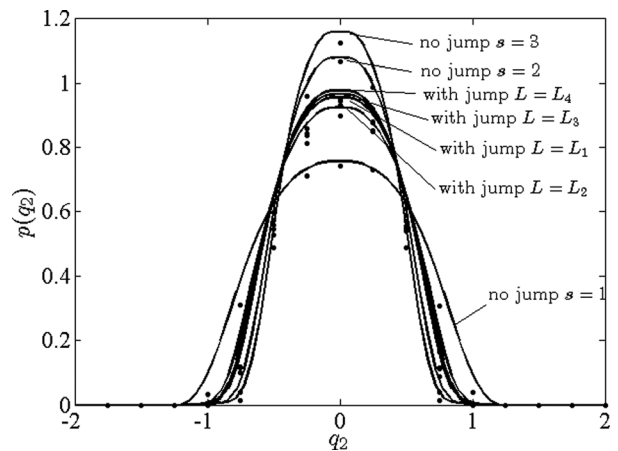
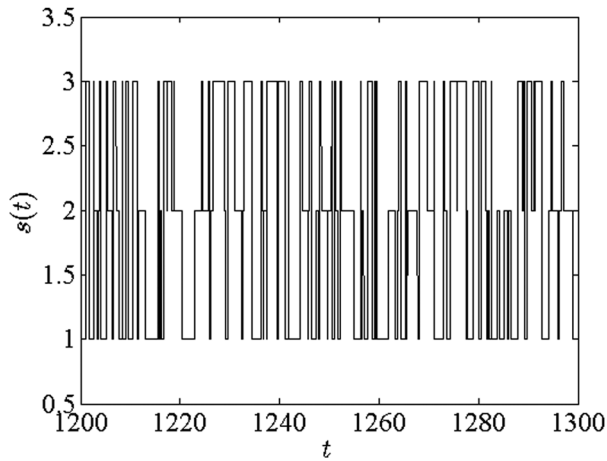


Fig. 7 Stationary probability densities  $p(q_2)$  of the displacement  $q_2$  of three-form system (24) with the same transition rates as in Fig. 5



**Fig. 8 Sample time history of three-form Markov jump process  $s(t)$  with independent jumps**

probability densities  $p(q_1)$ ,  $p(q_2)$  of the displacement  $q_1$ ,  $q_2$  are exhibited in Figs. 6 and 7. The graphs can be interpreted in a fashion similar to the two-form system considered earlier, resulting in analogous observations. Again, the dots obtained by direct simulation of Eq. (24) match closely with the lines obtained by using Eqs. (19) and (20), which are derived by analysis to assess the stationary response. Finally, a sample time history of the three-form Markov jump process  $s(t)$  with independent jumps is shown in Fig. 8.

## 6 Conclusions

Markovian-jump systems are practically significant since they include many industrial plants and economic systems. In this paper, an approximate method has been presented to assess the stationary response of MDOF nonlinear stochastic systems whose equations of motion, when cast in first-order form, are Markovian-jump, quasi-nonintegrable Hamiltonian equations. Important results reported in the paper are summarized in the following statements:

- (1) Using stochastic averaging, the quasi-nonintegrable Hamiltonian equations are reduced to a one-dimensional Itô equation governing the approximate energy envelope.
- (2) A stationary Fokker–Planck–Kolmogorov equation for the stationary probability density  $p(H)$  of the approximate energy envelope  $H$  has been derived. Solution of this equation generates  $p(H)$  and, by integration, the stationary marginal densities of various coordinates.
- (3) The validity of the method presented herein has been demonstrated by using a two-degree-of-freedom nonlinear oscillator that is capable of independent Markovian jumps. By comparison with direct system simulation, it has been observed that the method is fairly accurate in assessing the stationary response.

It should be mentioned that, due to the lack of general techniques to treat nonlinear systems, rigorous analysis of the errors of approximation of the method thus presented cannot be made. Among other things, it is hoped that this paper would point to directions along which further research efforts in Markovian-jump nonlinear systems can be made.

## Acknowledgment

This research was supported in part by the Natural Science Foundation of China through various grants (Grant Nos. 11272279, 11372271, 11432012, 11272201), by the 973 Program under Grant No. 2011CB711105, and by the Natural Science Foundation of Zhejiang Province under Grant No. LY12A02004. Partial support was also provided by the Powley Fund of the University of California at Berkeley. Opinions, findings, and conclusions expressed in this paper are those of the authors and do not necessarily reflect the views of the sponsors.

## References

- [1] Krasovskii, N. N., and Lidskii, E. A., 1961, “Analytical Design of Controllers in Systems With Random Attributes I,” *Autom. Remote Control*, **22**(9), pp. 1021–1025.
- [2] Krasovskii, N. N., and Lidskii, E. A., 1961, “Analytical Design of Controllers in Systems With Random Attributes II,” *Autom. Remote Control*, **22**(10), pp. 1141–1146.
- [3] Krasovskii, N. N., and Lidskii, E. A., 1961, “Analytical Design of Controllers in Systems With Random Attributes III,” *Autom. Remote Control*, **22**(11), pp. 1289–1294.
- [4] Mariton, M., 1988, “Almost Sure and Moment Stability of Jump Linear Systems,” *Syst. Control Lett.*, **11**(5), pp. 393–397.
- [5] Kushner, H., 1967, *Stochastic Stability and Control*, Academic Press, New York.
- [6] Luo, J. W., 2006, “Comparison Principle and Stability of Itô Stochastic Differential Delay Equations With Poisson Jump and Markovian Switching,” *Nonlinear Anal.*, **64**(2), pp. 253–262.
- [7] Boukas, E. K., and Liu, Z. K., 2001, “Robust Stability and Stabilizability of Markov Jump Linear Uncertain Systems With Mode Dependent Time Delays,” *J. Optim. Theory Appl.*, **109**(3), pp. 587–600.
- [8] Boukas, E. K., and Liu, Z. K., 2002, *Deterministic and Stochastic Systems With Time-Delay*, Birkhauser, Boston, MA.
- [9] de Farias, D. P., Geromel, J. C., do Val, J. B. R., and Costa, O. L. V., 2000, “Output Feedback Control of Markov Jump Linear System in Continuous-Time,” *IEEE Trans. Autom. Control*, **45**(5), pp. 944–949.
- [10] de Souza, C. E., and Fragoso, M. D., 1993, “ $H_\infty$  Control for Linear Systems With Markovian Jumping Parameters,” *Control Theory Adv. Technol.*, **9**, pp. 457–466.
- [11] Kats, I. Y., and Martynyuk, A. A., 2002, *Stability and Stabilization on Nonlinear Systems With Random Structure*, Taylor and Francis, London.
- [12] Sworder, D., 1969, “Feedback Control of a Class of Linear Systems With Jump Parameters,” *IEEE Trans. Autom. Control*, **14**(1), pp. 9–14.
- [13] Ghosh, M. K., Arapostathis, A., and Marcus, S. L., 1997, “Ergodic Control of Switching Diffusions,” *SIAM J. Control Optim.*, **35**(6), pp. 152–198.
- [14] Fragoso, M. D., and Hemery, E. M., 1991, “Optimal Control for a Class of Noisy Linear Systems With Markovian Jumping Parameters and Quadratic Cost,” *Int. J. Syst. Sci.*, **22**(12), pp. 2553–2561.
- [15] Ji, Y. D., and Chizeck, H. J., 1992, “Jump Linear Quadratic Gaussian Control in Continuous Time,” *IEEE Trans. Autom. Control*, **37**(12), pp. 1884–1892.
- [16] Natache, S. D. A., and Vilma, A. O., 2004, “State Feedback Fuzzy-Model-Based Control for Markovian Jump Nonlinear Systems,” *SBA Controle Automacao*, **15**(3), pp. 279–290.
- [17] Zhu, W. Q., 2006, “Nonlinear Stochastic Dynamics and Control in Hamiltonian Formulation,” *ASME Appl. Mech. Rev.*, **59**(4), pp. 230–248.
- [18] Zhu, W. Q., and Yang, Y. Q., 1997, “Stochastic Averaging of Quasi-Nonintegrable Hamiltonian Systems,” *ASME J. Appl. Mech.*, **64**(1), pp. 157–164.
- [19] Huan, R. H., Wu, Y. J., and Zhu, W. Q., 2009, “Stochastic Optimal Bounded Control of MDOF Quasi-Nonintegrable Hamiltonian Systems With Actuator Saturation,” *Arch. Appl. Mech.*, **79**(2), pp. 157–168.
- [20] Wonham, W. M., 1970, *Random Differential Equations in Control Theory*, Academic Press, New York.
- [21] Tabor, M., 1989, *Chaos and Integrability in Nonlinear Dynamics*, Wiley, New York.
- [22] Siegel, C. L., and Moser, J. K., 1971, *Lecture on Celestial Mechanics*, Springer-Verlag, New York.
- [23] Itô, K., 1951, “On Stochastic Differential Equations,” *Mem. Am. Math. Soc.*, **4**, pp. 289–302.
- [24] Khasminskii, R. Z., 1968, “On the Averaging Principle for Itô Stochastic Differential Equations,” *Kibernetika*, **3**, pp. 260–279.
- [25] Fang, Y. W., Wu, Y. L., and Wang, H. Q., 2012, *Optimal Control Theory of Stochastic Jump System*, National Defense Industry Press, Beijing.
- [26] Wu, S. T., 2007, *The Theory of Stochastic Jump System and Its Application*, Science Press, Beijing.