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UNIVERSITY OF CALIFORNIA SAN DIEGO

Dark Matter and Neutrinos in the Lives and Deaths of Supermassive Stars

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Physics

by

Kyle Scott Kehrer

Committee in charge:

Professor George Fuller, Chair
Professor Alexey Arefiev
Professor Adam Burgasser
Professor Raphael Flauger

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The Dissertation of Kyle Scott Kehrer is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2026

DEDICATION

I dedicate this dissertation to (in no particular order): Tony, Robin, Tobias, Bernard, Alex B., Mayia, Sanchit, Varun, Shruti, Yiheng, Thomas, Harry, Izzy, Sophie, Kate, Mia, Will, Emma, Prakamya, Lindsey, Ilyse, Hannah, Casey, Alex W.-K., Beck, Ben, Emily, Julian, Charlie, Eliza, Ollie, Sarah, Julius, Samuel, Vincent, Anaya, Jason, Hena, Lisette, Farouk, Yangxi, Russell, Chesson, Matt, Jake, Aashay, Aniket, Daria, Jarred, Fergus, Julie, and any other graduate students/postdocs here at UCSD who have had a positive impact on my life.

I dedicate this dissertation to Elise, Orion, and Dylan, my best friends from undergrad.

I dedicate this dissertation to Scott and Lori, my parents who manage to support me immensely despite still being in Florida.

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VITA

- 2020 Bachelor of Science, Florida State University
- 2020–2026 Teaching Assistant, Department of Physics
University of California San Diego
- 2021–2026 Graduate Student Researcher, Department of Physics
University of California San Diego
- 2025 Master of Science, University of California San Diego
- 2026 Doctor of Philosophy, University of California San Diego

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ABSTRACT OF THE DISSERTATION

Dark Matter and Neutrinos in the Lives and Deaths of Supermassive Stars

by

Kyle Scott Kehrer

Doctor of Philosophy in Physics

University of California San Diego, 2026

Professor George Fuller, Chair

Supermassive stars offer an intriguing solution to a problem compounded by recent observations that there exists objects in the very early universe which are much too massive than would be expected from standard formation channels. This dissertation explores supermassive stars and their basic properties primarily analytically, thereby discovering a means of stabilizing them with dark matter and possible signatures of their demises induced from the complicated interplay of neutrinos with these stars during collapse.

Introduction

0.1 Dark Matter

Under the mainstream view of modern cosmology, known as Λ CDM, the universe is spatially flat¹ and the contents of the universe are distributed homogeneously and isotropically. While this is not true on small length scales (i.e. planets, galaxies, and galaxy clusters are clearly neither homogeneous nor isotropic), on the scales of hundred of megaparsecs (~ 100 's of millions of lightyears), this is basically accurate. The condition of spatial flatness is predicated on comparing the observed total energy density within the universe to the so-called “critical” energy density, given by solving the Friedmann Equations for a homogeneous and isotropic universe:

$$\rho_{\text{crit.}, \Lambda\text{CDM}} = \frac{3H_0^2}{8\pi G} \approx 5 \times 10^{-42} \text{ MeV fm}^{-3}, \quad (1)$$

where $H_0 \approx 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [1] is the “Hubble Constant,” parameterizing the rate of expansion of our universe today, and $G \approx 6.71 \times 10^{-45} \text{ MeV}^{-2}$ is Newton’s Gravitational Constant. To put this critical density in perspective, this is equivalent to a mass density of about five hydrogen atoms per cubic meter.

Observations tell us that the average total energy density on the largest of cosmological scales agrees with this critical energy density within error bounds [1], with the

¹This is only true for a fixed time-slice; the components of the Riemann curvature tensor in Λ CDM do not vanish so it is not truly a flat manifold. On a given time-slice, the geometry is (at least within error bounds) completely Euclidean.

caveat that the vast majority of it is composed of “stuff” we do not understand at all. Per the Planck Collaboration [1, 2], roughly 70% of this critical energy density is derived from something known as “Dark Energy,” which manifests as constant energy density throughout the entire universe with no known source. Einstein’s Theory of General Relativity suggest that such a constant energy density, i.e. one whose value is unaffected by cosmological expansion, serves to accelerate the expansion already observed in distant galaxies. This dissertation will not touch on dark energy any further, but the reader should be aware that there exists many proposed mechanisms for its origin and properties available elsewhere.

The leftover $\sim 30\%$ of this critical energy density is, for lack of a better word, matter, though the vast majority of this fraction is completely foreign to any usual notions of the term. Less than a quarter of this “matter” refers to things like protons, neutrons, electrons, and anything made of those constituent particles, whereas the remaining matter’s presence is only inferred through careful indirect and independent observations. This “Dark Matter” (DM) is so named for its lack of any directly detectable electromagnetic, weak, or strong interaction signatures; it has so far completely eluded direct observation and appears to not interact with standard model (SM) matter in any way, except through gravity.

The first hints of its presence came from applying the virial theorem to velocity dispersion observations of the Coma cluster of galaxies by the astronomer Fritz Zwicky in 1933 [3]. Based on the observed “luminous” matter (i.e. SM matter) content alone, the galaxies were moving far too quickly to remain bound in a cluster of that size and would have flown apart from one another long ago. This indicated that there exists a great deal of non-luminous matter acting gravitationally on the cluster, thereby permitting the large velocities observed. This paper popularized the term “dark matter” in the field of astronomy, though Zwicky assumed no notion of peculiar properties and instead reasoned (in a later paper) that this missing mass was situated in matter too difficult to see, such as “[...] cool and cold stars, macroscopic and microscopic bodies, and gases.” [4].

Along the same vein in the late 1960’s, though this time with more advanced

instruments focused on the Andromeda galaxy, careful measurements made by astronomers Vera Rubin and Kent Ford revealed later that stars in Andromeda’s periphery were orbiting its center much too quickly than what would be expected if all of its mass were composed of its detectable stars and interstellar media. This same pattern was borne out in the rotation curves of many more galaxies beyond Andromeda [3], all suggesting that there is a substantial portion of basically every galaxy that is completely invisible yet still produces a strong gravitational effect.

While both of these observations could be reasonably well explained by positing Newton’s 2nd Law is altered for small accelerations (i.e. those that would be expected far from a galaxy cluster/galactic center), an observation of two colliding galaxy clusters in 2006 by Douglas Clowe and collaborators would place scientific consensus firmly afield of this idea. Known collectively as the “bullet cluster,” this observation showed that there was a strong distinction between the intergalactic gas of these colliding clusters (where the lion’s share of the SM mass is) and the clusters’ actual gravitational mass. Using a combination of X-ray observations and microlensing data from these colliding clusters, the researchers determined that—during the collision where the galaxies within the clusters passed by each other unimpeded—the intergalactic gas was stripped off each cluster and remained in the center of the collision, causing it to glow bright in the X-ray band from turbulent shocks. The microlensing data suggested, however, that the gravitating mass within the clusters had also passed through each other unimpeded as opposed to remaining with the intergalactic gas at the collision region [5, 3], which would be inexplicable by modifications to gravity alone. A figure from the 2006 paper showing this fact clearly is presented in Fig. 1.

These observations, as well as many others, point to an unsettling fact about our universe: we have no idea what the vast majority of the “stuff” comprising the universe is. Setting aside dark energy, of that which could be reasonably described as matter, we do not have a remotely firm handle on about 80% of it aside from a few basic properties. We

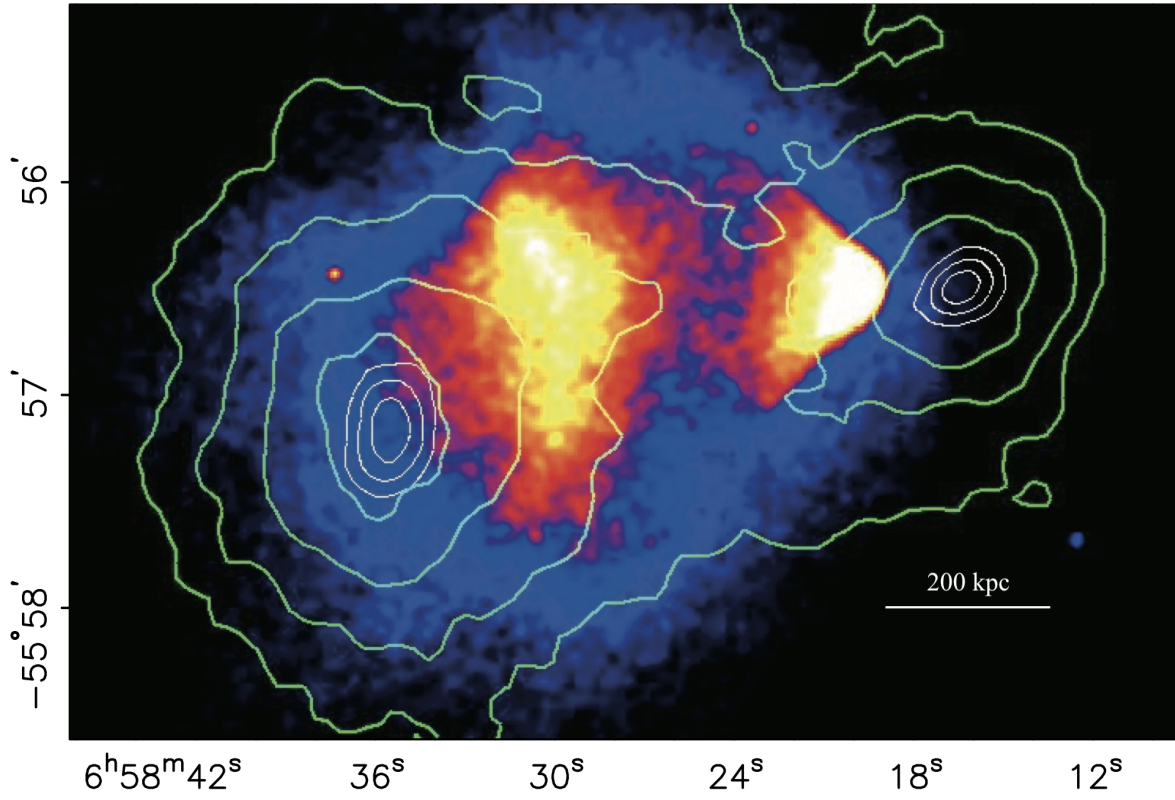


Figure 1: X-ray heatmap (colored regions, no provided scale) from Chandra X-ray Observatory data and weak gravitational lensing contours (green curves, no provided scale) of the bullet cluster. The X-ray data traces the standard model mass in the cluster, and is centered at the collision area between the clusters. The green contours trace the gravitational mass of the clusters and are separated from the collision. Adapted from Ref. [5], Figure 1.

can surmise that this “dark matter” is “cold,” i.e. non-relativistic in its dynamics (a core assumption of the successful Λ CDM model), that its interaction with normal matter is likely only gravitational, and that it may interact with itself though very weakly (or to be more precise, observations similar to and including the bullet cluster place upper rather poor upper limits on its self-interaction cross-section to be $\sigma/m_{\text{DM}} < 1 \text{ cm}^2 \text{ g}^{-1}$ [6]).

This dearth of specifics leaves much room for an entire cottage industry theorizing DM particle physics. The parameter space is wide open for physics at scales smaller than galactic halos, such as: small scale structure, particle mass, DM-SM interactions, DM-DM self-interactions, the presence of a “dark sector” (i.e. are there many types of DM particles with their own internal “dark” interactions?), and many more. This dissertation will take advantage of the wide parameter space available to us for local scale distributions and particle physics, and use that leeway to examine a some theoretical scenarios where dark matter’s presence and impact may be important.

0.2 Supermassive Stars

0.2.1 Background and Motivation

Recent observations from the James Webb Space Telescope (JWST) and the Chandra X-ray observatory at high cosmological redshifts ($z \sim 10$) have revealed scores of supermassive black holes (SMBHs) with masses in excess of $\sim 10^7 M_{\odot}$. While the existence of SMBHs at or above that mass on its own is not noteworthy—the SMBH at the center of the Andromeda galaxy is about $10^8 M_{\odot}$ [7] and is dwarfed by that contained in the galaxy Messier 87—the fact they exist at $z \sim 10$ raises some eyebrows. Being that massive only about 400 million years after the Big Bang, standard formation pathways (already an active area of research for SMBHs in general) seem to not be sufficient to explain their existence.

The largest black holes we know nature produces are born at $\sim 10\text{--}100 M_{\odot}$ and are

the result of core-collapse supernovae or pair-instability collapse of high mass stars [8]. From there, these black holes can grow by accreting matter into them and merging with other black holes, but the timescales for these processes are fixed by some fundamental physics. Accretion cannot rise above a rate known as the Eddington limit, where the intense heat of infalling gas and dust near the event horizon produces enough radiation pressure to drive away any more material from being absorbed by the black hole.² The luminosity at which this is achieved is [9]

$$L_{\text{Edd.}} = \frac{4\pi GMm_p}{\sigma_T} \approx 1.26 \times 10^{28} \left(\frac{M}{M_\odot} \right) \text{ erg s}^{-1}, \quad (2)$$

where M is the black hole mass, $m_p \approx 938 \text{ MeV}$ is the proton mass, and $\sigma_T \approx 6.65 \times 10^{-25} \text{ cm}^2$ is the Thomson cross section for light scattering off electrons. While it is known that this limit can be exceeded for brief periods of time, Ref. [10] found that for a particular high-redshift SMBH observed, this accretion rate would need to be roughly twice the limit to explain its large mass at such a short time following the Big Bang.

The other pathway to large masses from binary SMBH mergers is also fraught. As two gravitating bodies orbit one another, gravitational wave (GW) radiation will carry energy from the system and their distance between one another will decrease. The timescale for GW radiation to bring both bodies to merge in a circular orbit is [11]

$$\tau_{\text{merge}} \sim \frac{1}{G^3} \frac{a^4}{M^2 \mu} \sim 10^{16} \left(\frac{a}{1 \text{ pc}} \right)^4 \left(\frac{10^8 M_\odot}{M} \right)^3 \text{ years}, \quad (3)$$

where a is the initial separation distance, M is the total mass of the binary system, and $\mu = M_1 M_2 / (M_1 + M_2)$ is the reduced mass. Considering that the universe has only been around for $\sim 10^{10}$ years, this poses a clear problem to explaining SMBHs at redshift $z \sim 10$ when the universe was less than 10^9 years old. While it is true that an orbit

²Observers use this Eddington limit to work backward from an observed luminosity to determine the minimum possible mass of an accreting black hole.

with eccentricity $e > 0$ reduces this merger timescale because of larger accelerations, GW radiation also serves to circularize the orbit after several cycles, making the above estimate fit-for-purpose in most cases [11].

These physical limitations indicate that, perhaps, our assumptions about the initial state of these black holes may be wrong. If instead the black hole began its life at a much larger mass than $\sim 10\text{--}100 M_\odot$, both of these problems could be avoided to explain something as massive as $10^7 M_\odot$ existing so soon after the Big Bang. These objects, self-gravitating gas clouds in excess of $\sim 10^4 M_\odot$, could have formed in the early universe, and (as will be explained later) would eventually suffer an instability upon reaching a critical central density and collapse directly into a black hole of similar mass, leaving behind a high-mass seed that could continue accreting at the Eddington limit as investigated by Ref. [10]. This formation pathway shifts the issue from the well-established physics of accretion and mergers to one of forming these supermassive objects in the first place, where large uncertainties in the physics of star formation (especially in the early universe) abound.

0.2.2 Equations of Structure and Polytropes

Owing to their gargantuan mass, these supermassive stars (SMSs) are well described in rather simple ways, specifically that they are fully ionized, radiation dominated in their pressure support, and fully convective. The key parameter governing them is their ratio of gas pressure to total pressure $\beta = P_{\text{gas}}/(P_{\text{gas}} + P_{\text{rad.}})$, where the total pressure is a combination of Maxwell-Boltzmann gas pressure and relativistic radiation pressure. Individually, these contributions are (using units where $c = \hbar = k_B = 1$, *a.k.a.* “natural

units”)

$$P_{\text{gas}} = \frac{\rho}{\mu m_{\text{b}}} T, \quad (4)$$

$$P_{\text{rad.}} = \frac{a}{3} T^4, \quad (5)$$

where ρ is the mass density, μ is the mean atomic weight per particle in the gas, $m_{\text{b}} \approx 939 \text{ MeV}$ is the atomic mass unit, T is the temperature, and $a = \pi^2/15$ is the radiation constant in natural units³. For fully ionized hydrogen gas, where each proton is paired with an electron, $\mu = 0.5$, and for fully ionized “primordial” gas ($\sim 75\%$ hydrogen and $\sim 25\%$ helium by mass), $\mu \approx 0.59$.

If β is constant throughout the star, then it is related to the total mass M via [12]

$$\frac{1 - \beta}{\mu^4 \beta^4} \approx 0.0031 \left(\frac{M}{M_{\odot}} \right)^2. \quad (6)$$

In the radiation dominated regime where $\beta \ll 1$, this reduces to

$$\beta \approx \frac{4.3}{\mu} \left(\frac{M_{\odot}}{M} \right)^{1/2} \approx 7.23 \times 10^{-3} \left(\frac{0.59}{\mu} \right) \left(\frac{10^6 M_{\odot}}{M} \right)^{1/2}. \quad (7)$$

We can follow Ref. [12] to determine the relationship between the total pressure $P = P_{\text{gas}} + P_{\text{rad.}}$ and the density ρ . From the definition of β , we have

$$P_{\text{gas}} = \frac{\rho}{\mu m_{\text{b}}} T = \beta P, \quad (8)$$

$$P_{\text{rad.}} = \frac{a}{3} T^4 = (1 - \beta) P. \quad (9)$$

³The entirety of this dissertation will use “natural units” in equations unless otherwise stated.

We find then that T as a function of β and ρ is

$$T = \left(\frac{3}{a\mu m_b} \frac{1-\beta}{\beta} \right)^{1/3} \rho^{1/3}, \quad (10)$$

which we can plug into Eq. (8) to find P as a function of β and ρ :

$$P = \left[\frac{3}{a} \frac{1}{m_b^4} \frac{1-\beta}{\mu^4 \beta^4} \right]^{1/3} \rho^{4/3}. \quad (11)$$

In principle, the ratio of gas pressure to total pressure β need not be constant throughout the star, and in such cases more information is needed to constrain the above equation of state. However, under the assumption that it is constant, which as said before is expected for an object that is fully convective, we can write the pressure as

$$P = K\rho^\gamma, \quad (12)$$

a so-called “polytropic” equation of state (EOS) where K is the polytropic constant and $\gamma = 4/3$ is the polytropic exponent.

For equations of state of the above kind, the equation governing the resulting density profile under hydrostatic equilibrium can be derived. First, the EOS can be slightly reparameterized as

$$P = K\rho^{1+1/n}, \quad (13)$$

where n is the “polytropic index” defined as $\gamma = 1 + 1/n$. In this definition, $n = 3$ corresponds to $\gamma = 4/3$. Under hydrostatic equilibrium, a fluid element in a star of mass $dm = \rho dV$ feels two forces that must cancel out: a gravitational force pointing inward:

$$dF_g = -\frac{GM_{\text{enc.}}(r)\rho}{r^2}dV, \quad (14)$$

where $M_{\text{enc.}}(r)$ is the mass enclosed within a radius r , and an outward pointing force arising from a pressure gradient acting on an area dA oriented orthogonal to the radial direction:

$$dF_P = (P(r) - P(r + dr)) dA = -\frac{dP}{dr} dV, \quad (15)$$

with the negative sign being required as pressure decreases for radii further from the center. Adding these two together and requiring that the net force on a fluid element vanishes, we obtain the equation for hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM_{\text{enc.}}\rho}{r^2}. \quad (16)$$

This will be used in tandem with the mass continuity equation

$$\frac{dM_{\text{enc.}}}{dr} = 4\pi r^2 \rho \quad (17)$$

to derive the equation for the structure of the polytrope.

We will follow Ref. [11] for this derivation. The two first derivative equations above can be combined into a single second derivative equation to yield

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G\rho, \quad (18)$$

which resembles Poisson's equation in spherical symmetry. We will now posit that the density follows a functional form

$$\rho = \rho_c \theta^n, \quad (19)$$

where ρ_c is the central density and $\theta \in [0, 1]$ is a real-valued unitless function. Therefore, the pressure is now

$$P = K\rho_c^{1+1/n}\theta^{n+1}, \quad (20)$$

so its radial gradient is

$$\frac{dP}{dr} = (n+1)K\rho_c^{1+1/n}\theta^n \frac{d\theta}{dr}. \quad (21)$$

Plugging these in to Eq. (18), we find

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 (n+1) K \rho_c^{1/n} \frac{d\theta}{dr} \right) = -4\pi G \rho_c \theta^n, \quad (22)$$

which can be simplified to

$$\frac{(n+1)K}{4\pi G} \rho_c^{1/n-1} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) = -\theta^n. \quad (23)$$

As θ is unitless, the prefactor on the left must have units of length-squared, so it can be removed with the reparameterization

$$r = \left[\frac{(n+1)K}{4\pi G} \right]^{1/2} \rho_c^{\frac{1-n}{2n}} \xi \equiv \alpha_{\text{LE}} \xi \quad (24)$$

to yield the Lane-Emden equation of order n

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n. \quad (25)$$

The boundary conditions at the core $r = \xi = 0$ are

$$\theta(0) = 1, \quad (26a)$$

$$\theta'(0) = 0, \quad (26b)$$

where the former is required to match the density at the core ρ_c and the latter to force the density to vary smoothly at the core. While analytical solutions exist for some choices of n , it is often more prudent to solve this equation numerically. Another restriction on the solution is that θ remain non-negative, so the first root of each solution is denoted

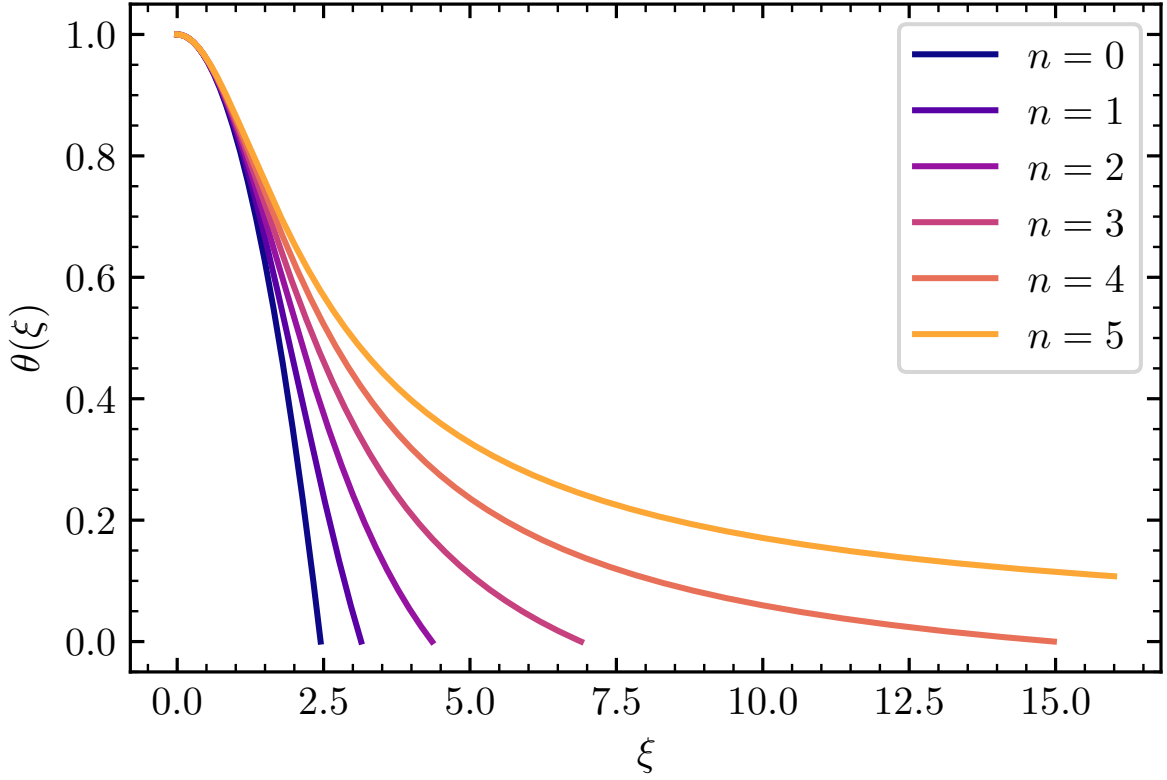


Figure 2: Solutions to the Lane-Emden equation (Eq. (25)) subject to the boundary conditions Eqs. (26). For $n \in [0, 5)$, θ has a root $\xi_n < \infty$.

as $\theta(\xi_n) = 0$ upon which integration ceases. A plot of solutions for various choices of n is given in Fig. 2. Also relevant are the numerical values for ξ_n and $\theta'(\xi_n)$, which are presented in Table 1.

With the solution for θ determined for any n , radius R of the configuration is

$$R = \alpha_{\text{LE}} \xi_n = \left[\frac{(n+1)K}{4\pi G} \right]^{1/2} \rho_c^{\frac{1-n}{2n}} \xi_n, \quad (27)$$

and M is determined from using the relation $r = \alpha_{\text{LE}}\xi$ and Eq. (25):

$$\begin{aligned}
M &= \int_0^R dr 4\pi r^2 \rho(r), \\
&= 4\pi \alpha_{\text{LE}}^3 \rho_c \int_0^{\xi_n} d\xi \xi^2 \theta^n(\xi), \\
&= -4\pi \alpha_{\text{LE}}^3 \rho_c \int_0^{\xi_n} d\xi \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right), \\
&= -4\pi \alpha_{\text{LE}}^3 \rho_c \xi^2 \frac{d\theta}{d\xi} \Big|_0^{\xi_n}, \\
\Rightarrow M &= 4\pi \left[\frac{(n+1)K}{4\pi G} \right]^{3/2} \rho_c^{\frac{3-n}{2n}} \xi_n^2 |\theta'(\xi_n)|,
\end{aligned} \tag{28}$$

where we have used the fact that $\theta'(\xi_n) < 0$ for $n \in [0, 5)$. It is clear that $n = 3$ ($\gamma = 4/3$) is a special polytropic index as it is the only one for which the mass M is independent of the central density. Indeed, it describes the structure of an object for which it costs no energy to expand or contract because the central density is a free parameter for a given mass. These objects are metastable configurations and will be important for the specific study of supermassive stars.

A useful relation between M , R , and ρ_c follows from the above calculations:

$$\frac{M}{R^3} = 4\pi \frac{|\theta'(\xi_n)|}{\xi_n} \rho_c, \tag{29}$$

which reduces to $\rho_c = M/(\frac{4}{3}\pi R^3)$ for $n = 0$, i.e. a constant density sphere. Another useful relation is the ratio of the central density to the average density $\rho_c/\rho_{\text{avg.}}$:

$$\frac{\rho_c}{\rho_{\text{avg.}}} = \frac{\rho_c}{M/(\frac{4}{3}\pi R^3)} = \frac{\xi_n}{3|\theta'(\xi_n)|}, \tag{30}$$

whose values are also presented in Table 1. We will turn focus back to supermassive stars, which take $n = 3$.

Table 1: Tabulated values for the first root of the solution to Eq. (25) ($\theta(\xi_n) = 0$), the absolute value of the derivative of the solution at that root $|\theta'(\xi_n)|$, and the ratio of the central density to the average density $\rho_c/\rho_{\text{avg.}}$ for $n = 0, 1, 2, 3, 4$, and 5.

n	ξ_n	$ \theta'(\xi_n) $	$\rho_c/\rho_{\text{avg.}}$
0	2.4495	0.8165	1.0000
1	3.1416	0.3183	3.2899
2	4.3529	0.1273	11.403
3	6.8969	0.0424	54.183
4	14.972	0.0080	622.41
5	∞	0	∞

0.2.3 Supermassive Stars in Detail

As mentioned before, SMSs being fully convective permits us to use the polytropic equation of state with $\gamma = 4/3$ ($n = 3$) because the ratio of gas pressure to total pressure is constant within the SMS. Being radiation dominated ($\beta \ll 1$), we can approximate the total entropy-per-baryon s to be primarily derived from the radiation field:

$$s \approx s_\gamma = \frac{4a}{3} \frac{T^3}{n_b} = \frac{4am_b}{3} \frac{T^3}{\rho}. \quad (31)$$

With this, the temperature can be written as a function of s and ρ to find that

$$s \approx \frac{4}{4.3} \left(\frac{M}{M_\odot} \right)^{1/2} - \frac{4}{\mu} \quad (32)$$

using the definition of β and its approximate relation to the total mass via Eq. (7). Since $P_{\text{rad.}} = (1 - \beta)P = (1 - \beta)K\rho^{4/3}$, a more useful equation for K can be found in terms of s :

$$K(s) = \frac{1}{1 - \beta} \frac{a}{3} \left(\frac{3}{4am_b} s \right)^{4/3} \approx \frac{0.2843}{1 - \beta} \left(\frac{s}{1000} \right)^{4/3} \text{ MeV}^{-4/3}. \quad (33)$$

Since these SMSs are radiation dominated, their pressure is primarily derived from relativistic particles (specifically: photons). Per a phrase attributed to Nobel laureate

William Fowler, objects supported by relativistic particles are said to be “trembling on the verge of instability,” a fact borne out in several astrophysical scenarios. White dwarfs, for example, contain roughly a solar mass of material in about the same volume as Earth and are supported against collapse—not by any nuclear burning—but by the quantum mechanical effect known as electron degeneracy pressure. While they are stable objects on their own, if they happen to be in a binary system with another star, their deep gravitational well can strip off gas from the other star’s outer layers and grow in mass. Any increase in mass needs to be offset by an increase in pressure to avoid collapse, so the white dwarf reaches a new equilibrium by contracting in size. The electrons in the white dwarf eventually become so dense that the only energy levels available to them are at high momentum states on the top of the Fermi sea as quantum mechanical effects become dominant, meaning that many of the electrons are moving close to the speed of light. When the vast majority of pressure support becomes derived from these ultra-relativistic electrons, the white dwarf becomes unstable to collapse and promptly detonates from the sudden nuclear reactions that take place, causing a Type IA Supernova. A similar process, though this time gravity overcoming neutron neutron degeneracy pressure, results in the collapse of a proto-neutron-star to a black hole during core-collapse supernovae [11].

Besides the quantum mechanical degeneracy pressure, the commonality between these two instabilities is that they are caused by general relativity (GR). In a purely Newtonian theory of gravity, neither the white dwarf nor the neutron star should collapse and should instead approach meta-stable states. Being neither stable nor unstable to collapse, it should cost no energy to compress or expand their size as gravitational forces and pressure forces are always matched in lock-step. Including 1st order effects from GR however, the picture changes. Since it is a non-linear theory, any deepening of the potential well resulting from compressive perturbations feeds back on itself, causing the well to deepen even further. For configurations where most of their pressure support is derived from relativistic particles, this effect is the most pronounced and can induce collapse even

at modest densities.

The moment at which a configuration transitions from marginally stable to unstable is quantified by Chandrasekhar’s Adiabatic Index

$$\Gamma_1 \equiv \left(\frac{d \log P}{d \log \rho} \right) \Big|_s \quad (34)$$

a unitless parameter that represents—since the derivative is taken at constant entropy-per-baryon s —the pressure change for a given change in density with no heat flow. Without including the effects of GR, configurations are stable if $\langle \Gamma_1 \rangle_P > 4/3$, where the angled brackets with a P subscript indicates the pressure-averaged Γ_1 over the entire object. Once GR effects are considered, this condition for stability changes to [11, 13]

$$\langle \Gamma_1 \rangle_P > \frac{4}{3} + \mathcal{O}\left(\frac{2GM}{R}\right), \quad (35)$$

i.e. basically the same as the Newtonian condition but with a correction of order-of-magnitude the ratio of the Schwarzschild radius of an object to its actual radius. Once the configuration crosses the above threshold, it becomes unstable to radial perturbations and will collapse. This instability is known as the Feynman-Chandrasekhar Instability [13, 14]. For extremely dense objects, like white dwarfs and neutron stars, this correction can be substantial as $2GM \sim R$. However, as we will discuss later, this represents a correction of no more than one part in ten-thousand for SMSs. Indeed, this fact means that the structure of SMSs is completely determined by Newtonian gravitation balancing pressure gradients, but their stability is instead completely determined by GR.

While the above approach to determine stability is rather general, we can use an alternative approach that captures the spirit of the Feynman-Chandrasekhar instability and gives the same result for SMSs. We will use an energy extremization approach to determine where an inflection point in the total energy as a function of the central density

occurs, following closely the method of Ref. [15]. To 1st order in GR effects, the total energy E of a configuration is captured by three terms:

$$E = E_{\text{int.}} + E_{\text{grav.}} + \Delta E_{\text{GR}}, \quad (36)$$

representing the internal gas energy (both Maxwell-Boltzmann and radiation contributions), the Newtonian gravitational binding energy, and the 1st order effect from GR, respectively. The terms are calculated from the following integrals [11]:

$$E_{\text{int.}} = \int u \, dM_{\text{enc.}} = n \int \frac{P}{\rho} dM_{\text{enc.}}, \quad (37a)$$

$$E_{\text{grav.}} = - \int \frac{GM_{\text{enc.}}}{r} dM_{\text{enc.}}, \quad (37b)$$

$$\Delta E_{\text{GR}} = -2 \frac{5 + 2n - n^2}{n(5 - n)} \int u \frac{GM_{\text{enc.}}}{r} dM_{\text{enc.}} - \frac{3n - 1}{2(5 - n)} \int \left(\frac{GM_{\text{enc.}}}{r} \right)^2 dM_{\text{enc.}}, \quad (37c)$$

where $u = nP/\rho$ is the internal energy per mass for a polytropic gas of index n . All integrals are taken over the entire configuration. The final complicated term basically serves as an additional binding energy but depends on the central density differently as compared to the Newtonian binding energy. For a polytrope of mass M , polytropic constant K , central density ρ_c , and index n , these integrals can be evaluated to yield the total energy as a function of ρ_c :

$$E(\rho_c) = k_1 MK \rho_c^{1/n} - k_2 GM^{5/3} \rho_c^{1/3} - k_4 G^2 M^{7/3} \rho_c^{2/3}, \quad (38)$$

where $k_1 \approx 1.7558$, $k_2 \approx 0.6390$, and $k_4 \approx 0.9183$ are numerical constants derived from integrating functions involving various powers of $\theta(\xi)$ for $n = 3$. The numerical constant k_3 is skipped as, in this formalism, it is usually reserved for including the effects of rotation. Here we will assume the SMS is static, so that term's contribution has been ignored. Note that the final term in this energy is proportional to G^2 , owing to the non-linearity of

GR. At this 1st order expansion in general relativistic effects, the final term represents a “gravitational self-coupling” of the SMS’s own gravitational potential well.

As determined in Ref. [16], Γ_1 in a supermassive star can be calculated to be approximately

$$\Gamma_1 \approx \frac{4}{3} + \frac{\beta}{6} \quad (39)$$

for $\beta \ll 1$. Since this follows from a derivative, it is a local condition at a given point in the configuration, but we will assume this applies throughout the entire SMS with $\beta = \text{constant}$. The associated polytropic index n is therefore

$$n = \frac{1}{\Gamma_1 - 1} = \frac{3}{1 + \beta/2}, \quad (40)$$

which we will use in Eq. (38). Since this index is not exactly 3, $K(s)$ needs to be adjusted to ensure correct units. We will call this adjusted polytropic constant $K'(s)$ and will define it as

$$K'(s) = K(s)\rho^{-\beta/6}, \quad (41)$$

which for $\beta \ll 1$ is a slowly varying enough function of ρ to not offend the assumption of constancy for K in the polytropic EOS. For the rest of this analysis, we will drop the primed notation in K , but it should be in the back of the reader’s mind that it is not necessarily the same as that in Eq. (33).

To make determining the inflection point in that equation easier, we will introduce some reparameterizations to remove the units and simplify some exponents. The

substitutions are as follows:

$$\tilde{E} = \frac{G^{3/2}}{K^{n/2}} E, \quad (42a)$$

$$\tilde{M} = \frac{G^{3/2}}{K^{n/2}} M, \quad (42b)$$

$$x = GM^{2/3} \rho_c^{1/3}, \quad (42c)$$

yielding the dimensionless total energy

$$\tilde{E} = k_1 \tilde{M}^{1-2/n} x^{3/n} - k_2 \tilde{M} x - k_4 \tilde{M} x^2. \quad (43)$$

We will now use $n = 3/(1 + \beta/2)$ and take two derivatives with respect to x , setting both to zero at $x = x_{\text{crit.}}$, we have the system:

$$\left. \frac{\partial \tilde{E}}{\partial x} \right|_{x_{\text{crit.}}} = 0 = \left(1 + \frac{\beta}{2}\right) k_1 \tilde{M}^{1/3-\beta/3} x_{\text{crit.}}^{\beta/2} - k_2 \tilde{M} - 2k_4 \tilde{M} x_{\text{crit.}}, \quad (44a)$$

$$\left. \frac{\partial^2 \tilde{E}}{\partial x^2} \right|_{x_{\text{crit.}}} = 0 = \frac{\beta}{2} \left(1 + \frac{\beta}{2}\right) k_1 \tilde{M}^{1/3-\beta/3} x_{\text{crit.}}^{\beta/2-1} - 2k_4 \tilde{M}. \quad (44b)$$

The first equation gives $\tilde{M}$ in terms of x :

$$\tilde{M} = \left(\frac{k_1}{k_2 + 2k_4 x_{\text{crit.}}} \right)^{3/2} + \mathcal{O}(\beta). \quad (45)$$

If we “turn off” the GR correction by setting $k_4 = 0$, we recover the Newtonian $\tilde{M} = (k_1/k_2)^{3/2} \approx 4.555$ to 0th order in β , a constant with respect to the central density x .

We can divide each equation by $\tilde{M}$ and solve for the resulting term with several annoying exponents $(1 + \beta/2)k_1 \tilde{M}^{-2/3-\beta/3} x_{\text{crit.}}^{\beta/2}$ to arrive at the equation

$$\frac{k_2 + 4k_4 x_{\text{crit.}}}{x_{\text{crit.}}} - 4 \frac{k_4}{\beta} = 0, \quad (46)$$

and finally solving for $x_{\text{crit.}}$ we arrive at

$$x_{\text{crit.}} = \frac{k_2}{4k_4} \frac{\beta}{1 - \beta} = \frac{k_2}{4k_4} \beta + \mathcal{O}(\beta^2). \quad (47)$$

Using the scaling relationship for x and ρ_c (Eq. (42c)) and β as a function of M (Eq. (6)), the critical central density that creates an inflection point in the total energy, where the configuration transitions from stable to unstable, is

$$\rho_{\text{c, crit.}} = \left(\frac{k_2}{4k_4} \right)^3 \frac{\beta^3}{G^3 M^2} \approx 3.98 \left(\frac{0.59}{\mu} \right)^3 \left(\frac{10^5 M_\odot}{M} \right)^{7/2} \text{ g cm}^{-3}. \quad (48)$$

The reader should note the steep dependence on mass M and that a $\sim 10^5 M_\odot$ SMS encounters GR instability at a central density comparable to that of water on Earth's surface.

Once the SMS is formed from an initially diffuse state (the particulars of that formation remain to be investigated, perhaps in a different dissertation), the star will contract on the Kelvin-Helmholtz timescale set by its mass. As we will show, the usual approach for determining this timescale is insufficient for these radiation dominated objects, and as a result the timescale will be comparatively short.

We will begin by trying to find a more convenient way of expressing the total energy in Eq. (38). Ignoring for now the final term in that equation, we can use the internal energy per mass for a polytrope $u = nP/\rho$ and the equation for hydrostatic equilibrium to derive the total energy in terms of $E_{\text{grav.}}$, the Newtonian gravitational binding energy. From the integral for $E_{\text{int.}}$:

$$E_{\text{int.}} = n \int \frac{P}{\rho} dM_{\text{enc.}} = n \int P dV, \quad (49)$$

where $dV = 4\pi r^2 dr$ is the differential volume of a shell at radius r . We can integrate both sides of the equation for hydrostatic equilibrium (Eq. (16)) over the radius of the star

multiplied by the volume enclosed $V = \int dV$:

$$\int_0^R \frac{dP}{dr} V dr = \int_0^R V dP = - \int_0^R \frac{GM_{\text{enc.}} \rho}{r^2} V dr. \quad (50)$$

Integrating by parts and using $V = \frac{4}{3}\pi r^3$, we find

$$\cancel{VP}\Big|_0^R - \int P dV = -\frac{1}{3} \int \frac{GM_{\text{enc.}}}{r} dM_{\text{enc.}}, \quad (51)$$

which means that

$$E_{\text{int.}} = -\frac{n}{3} E_{\text{grav.}}, \quad (52)$$

so the total Newtonian energy is

$$E_{\text{Newt.}} = E_{\text{int.}} + E_{\text{grav.}} = \left(1 - \frac{n}{3}\right) E_{\text{grav.}}. \quad (53)$$

For stars supported primarily by non-relativistic particles instead, the polytropic index is $n \approx 3/2$ ($\gamma \approx 5/3$), so we recover the standard viral relation $2E_{\text{int.}} + E_{\text{grav.}} = 0$, meaning that we can write the Kelvin-Helmholtz contraction time as

$$\tau_{\text{KH}} \sim \frac{|E_{\text{Newt.}}|}{L} \sim \frac{|E_{\text{grav.}}|}{2L} \sim 1.6 \times 10^7 \left(\frac{M}{M_{\odot}}\right)^2 \left(\frac{R_{\odot}}{R}\right) \left(\frac{L_{\odot}}{L}\right) \text{years}. \quad (54)$$

This would be the timescale for the Sun (or any other star on the main sequence) to contract if all nuclear reactions stopped in the core, for example.

For stars supported primarily by relativistic particles on the other hand, where $n \approx 3$, the Newtonian total energy is instead basically zero, so the only contribution to the Kelvin-Helmholtz contraction time scale is the final GR correction term in the energy functional. The SMS will contract from an initially diffuse state ($\rho_c \sim 0$) with almost zero total energy through photons leaving the surface at the Eddington rate (Eq. (2))

until it reaches the critical central density for collapse. The Kelvin-Helmholtz timescale is therefore

$$\tau_{\text{KH}} \stackrel{n=3}{\sim} \frac{|E_{\text{GR}}|}{L_{\text{Edd.}}} \sim \frac{k_4 G^2 M^{7/3} \rho_{\text{c, crit.}}^{2/3}}{L_{\text{Edd.}}} \approx 3,000 \left(\frac{0.59}{\mu} \right)^2 \left(\frac{10^5 M_{\odot}}{M} \right) \text{ years}, \quad (55)$$

a substantially shorter time than what would be expected for an $n \approx 3/2$ configuration.

Once the SMS reaches this GR instability, it will collapse on the free-fall timescale $\tau_{\text{ff}} \sim (G\rho_{\text{c, crit.}})^{-1/2}$. This is an even shorter timescale:

$$\tau_{\text{ff}} \sim 10^{-5} \left(\frac{\mu}{0.59} \right)^{3/2} \left(\frac{M}{10^5 M_{\odot}} \right)^{7/4} \text{ years}, \quad (56)$$

closer to a duration of thousands of seconds rather than thousands of years. When the SMS mass is $\sim 10^8 M_{\odot}$, these two timescales coincide, so there is essentially no point beyond this mass where the SMS is in quasi-static hydrodynamical equilibrium. Therefore, SMSs in excess of $10^8 M_{\odot}$ will not be considered throughout this dissertation.

0.3 Collapse of Supermassive Stars

Upon reaching the critical central density in Eq. (48), an SMS will contract rapidly under free-fall. Before GR considerations become important, the collapse can be modeled analytically in Newtonian gravity following the prescription in Ref. [17], which derived this result from mass continuity, Euler's fluid equation in vortex-free flow, Poisson's equation for the gravitational potential, and $n = 3$. As the SMS contracts, the central density becomes a function of time $\rho_{\text{c}}(t)$, hence the Lane-Emden length scale α_{LE} is also a function of time. Specifically, the equation of motion for α_{LE} was found to be

$$\frac{\dot{\alpha}_{\text{LE}}^2}{2} = \frac{4\lambda}{3} \left(\frac{K^3(s)}{\pi G} \right)^{1/2} \frac{1}{\alpha_{\text{LE}}} + C, \quad (57)$$

where λ is a constant to be discussed shortly and C is determined from the condition that

the SMS is initially at rest $\dot{\alpha}_{\text{LE}}(t = 0) = 0$.

Under these considerations, the structure at any given time for the collapsing SMS is determined by a modified Lane-Emden equation:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^3 + \lambda \quad (58)$$

with the usual boundary conditions $\theta(0) = 1$ and $\theta'(0) = 0$. The case of $\lambda = 0$ is the hydrostatic case, whereas $\lambda > 0$ means that the SMS is in some manner of free-fall. There is a maximum value for lambda that gives a physically reasonable solution, determined numerically to be $\lambda_{\text{max}} \approx 0.0065438$, on account that the solution to θ becomes tangent to the ξ axis when $\theta(\xi) = 0$. A plot of solutions to the above equation with different choices of λ is presented in Figure 3.

For now, we will take $\lambda = \lambda_{\text{max}}$ unless otherwise noted. To continue with our analysis, we can use the condition $\dot{\alpha}_{\text{LE}}(t = 0) = 0$ to determine the constant of integration C . We will take the initial state to be an SMS with initial entropy-per-baryon s_0 determined by its mass (Eq. (32)) having reached its critical central density for the onset of GR instability (Eq. (48)), meaning that

$$\alpha_{\text{LE}, 0} = \left(\frac{K(s_0)}{\pi G} \right)^{1/2} \rho_{\text{c, crit.}}^{-1/3}. \quad (59)$$

The stipulation of an initially stationary SMS at this length scale means that C is found to be

$$C = -\frac{4\lambda}{3} \left(\frac{K(s_0)}{\pi G} \right)^{1/2} \frac{K(s_0)}{\alpha_{\text{LE}, 0}} = -\frac{4\lambda}{3} K(s_0) \rho_{\text{c, crit.}}^{1/3}, \quad (60)$$

so the equation of motion for α_{LE} as the SMS collapses is

$$\dot{\alpha}_{\text{LE}} = - \left[\frac{8\lambda}{3} K(s) \right]^{1/2} \sqrt{\rho_{\text{c}}^{1/3} - \frac{K(s_0)}{K(s)} \rho_{\text{c, crit.}}^{1/3}}, \quad (61)$$

where the negative solution has been taken because the SMS is collapsing instead of

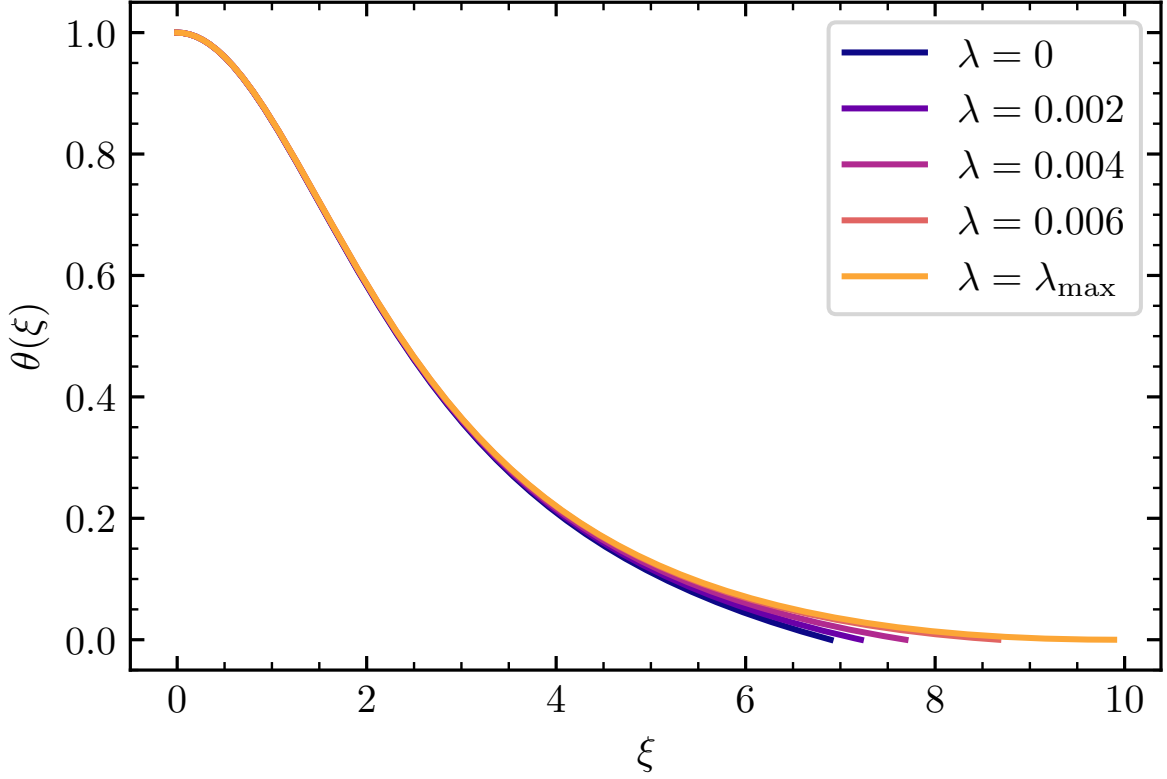


Figure 3: Solutions to the modified Lane-Emden equation (Eq. (58)) for various choices of λ , corresponding to various degrees of free-fall for a collapsing $n = 3$ polytrope. The maximum permitted value of λ is achieved at $\lambda_{\text{max}} \approx 0.0065438$, where $\theta(\xi)$ becomes tangent to the ξ axis where it also vanishes. The interior region of the solutions remain basically unchanged, however the overall size of the configuration does grow for a given α_{LE} . This effect, however, is tempered by the actual density following the cube of these functions, so a very small portion of the SMS is altered.

expanding.

Similar to the static case, we can determine the total radius and mass of the configuration:

$$R = \alpha_{\text{LE}} \xi_{3,\lambda} = \left(\frac{K(s)}{\pi G} \right)^{1/2} \rho_c^{-1/3} \xi_{3,\lambda}, \quad (62)$$

$$\begin{aligned} M &= \int_0^R dr \, 4\pi r^2 \rho, \\ &= 4\pi \alpha_{\text{LE}}^3 \rho_c \int_0^{\xi_{3,\lambda}} d\xi \, \xi^2 \theta^3(\xi), \\ &= 4\pi \left(\frac{K(s)}{\pi G} \right)^{3/2} \int_0^{\xi_{3,\lambda}} d\xi \left[\xi^2 \lambda - \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) \right], \\ M &= \frac{4\pi}{3} \left(\frac{K(s)}{\pi G} \right) \xi_{3,\lambda}^3 \lambda, \end{aligned} \quad (63)$$

where $\xi_{3,\lambda} \approx 9.8894$ is the dimensionless radius when $\lambda = \lambda_{\text{max}}$ and the final total derivative term inside the integral vanishes because $\theta'(\xi_{3,\lambda}) = 0$. The new mass-radius-density relation is therefore

$$\rho_c = \frac{3}{4\pi\lambda} \frac{M}{R^3} = \frac{3}{4\pi\lambda \xi_{3,\lambda}^3} \frac{M}{\alpha_{\text{LE}}^3}, \quad (64)$$

so the time evolution of the central density is

$$\begin{aligned} \dot{\rho}_c &= -\frac{9}{4\pi\lambda \xi_{3,\lambda}^3} \frac{M}{\alpha_{\text{LE}}^4} \dot{\alpha}_{\text{LE}}, \\ &= \frac{9}{4\pi\lambda \xi_{3,\lambda}^3} \left(\frac{\pi G}{K(s)} \right)^2 M \rho_c^{4/3} \left[\frac{8\lambda}{3} K(s) \right]^{1/2} \sqrt{\rho_c^{1/3} - \frac{K(s_0)}{K(s)} \rho_{\text{c,crit.}}^{1/3}}, \\ \dot{\rho}_c &= \frac{9\pi}{4\lambda \xi_{3,\lambda}^3} \sqrt{\frac{8\lambda}{3}} \frac{G^2 M}{K^{3/2}(s)} \rho_c^{4/3} \sqrt{\rho_c^{1/3} - \left(\frac{s_0}{s} \right)^{4/3} \rho_{\text{c,crit.}}^{1/3}}. \end{aligned} \quad (65)$$

Because of the initial condition that $\dot{\alpha}_{\text{LE}}(0) = 0$ at a finite initial density, solving this differential equation numerically will require setting $\rho_{\text{c},0}$ to be slightly larger than $\rho_{\text{c,crit.}}$.

If the entropy-per-baryon s of the SMS is not expected to evolve throughout the collapse, then the above differential equation is simple enough to solve numerically. During

collapse, however, we expect processes that could change s to be present, specifically neutrino processes. While neutrinos in the context of SMSs will be discussed in more detail in the next introduction section, for the purposes of this chapter they will be treated as particles that just sap energy away from the SMS.

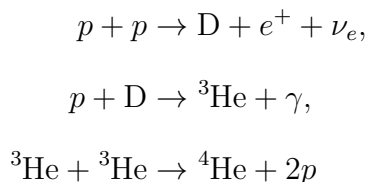
0.3.1 Entropy Evolution from Neutrinos

As the SMS collapses, some of the in-fall kinetic energy will be converted into neutrinos that free-stream out of the SMS without giving their energy back to the configuration. In these high entropy-per-baryon SMSs, there are plentiful relativistic electron (e^-) positron (e^+) pairs in electromagnetic equilibrium with the radiation field (i.e. $e^+ + e^- \rightleftharpoons 2\gamma$). Sometimes, when these pairs annihilate, neutrinos are produced instead of photons ($e^+ + e^- \rightarrow \nu + \bar{\nu}$) which escape the SMS and are therefore not in equilibrium. For a given temperature T , the emissivity (luminosity per volume) of neutrino radiation from these relativistic e^+e^- -pair annihilations is [12, 18]

$$\varepsilon_\nu \approx \frac{128}{\pi^5} G_F^2 T^9 \approx 4.31 \times 10^{25} \left(\frac{T}{1 \text{ MeV}} \right)^9 \text{ erg s}^{-1} \text{ cm}^{-3}, \quad (66)$$

where $G_F \approx 1.167 \times 10^{-11} \text{ MeV}^{-2}$ is the Fermi constant which sets the interaction strength of weak interactions. For reference, G_F is about 8 orders-of-magnitude smaller than the Thomson cross section for photon-electron scattering.

Another process that could change the entropy-per-baryon is nuclear fusion processes. If the SMS is of primordial composition, the dominant nuclear process is the PPI chain:



where D denotes deuterium and ν_e is an electron flavor neutrino. The emissivity for this entire process is [12]

$$\varepsilon_{\text{PPI}} = c_1 X_{\text{H}}^2 \rho_{\text{cgs}}^2 T_6^{-2/3} e^{-c_2 T_6^{-1/3}} \left(1 + c_3 T_6^{1/3} + c_4 T_6^{2/3} + c_5 T_6 \right) \text{erg s}^{-1} \text{cm}^{-3}, \quad (67)$$

where $c_1 = 2.319 \times 10^6$, $c_2 = 33.81$, $c_3 = 0.0123$, $c_4 = 0.0109$, and $c_5 = 0.00095$ are numerical constants, $X_H \approx 0.75$ is the hydrogen mass fraction, ρ_{cgs} is the density expressed in units of grams-per-cubic-centimeter, and $T_6 = T/(10^6 \text{ K})$ is the temperature in units of 10^6 Kelvins. Due in part to the exponential dampening factor and the weak dependence on temperature, however, this emissivity is subdominant to that of neutrinos at basically any conditions involved in SMS collapse (where the maximum temperature is ~ 1 MeV), so it will not be considered going forward.

From the 2nd Law of Thermodynamics, any net heat flow in a system Q is accompanied by a change in total entropy $S = N_{\text{b}} s$ per the relation

$$\delta Q = T dS. \quad (68)$$

We will take $\delta Q = L_{\text{tot.}} dt$, i.e. the net luminosity from processes that add entropy and those that take it away. For now we will only consider the latter from neutrinos, but this leaves the door open for other processes to consider. The total luminosity in this case is therefore

$$\begin{aligned} L_{\text{tot.}} &= - \int dV \varepsilon_{\nu}, \\ &= - \frac{511}{\pi^4} G_{\text{F}}^2 \int_0^R dr r^2 T^9(r), \\ &= - \frac{511}{\pi^4} G_{\text{F}}^2 \alpha_{\text{LE}}^3 T_{\text{c}}^9 \int_0^{\xi_{3,\lambda}} d\xi \xi^2 \theta^9(\xi), \\ L_{\text{tot.}}(s, \rho_{\text{c}}) &= - \frac{511}{\pi^4} G_{\text{F}}^2 \left(\frac{K(s)}{\pi G} \right)^{3/2} \left(\frac{3}{a \mu m_{\text{b}}} \frac{1 - \beta}{\beta} \right)^3 \rho_{\text{c}}^2 I_9, \end{aligned} \quad (69)$$

where the relation between temperature and density is given in Eq. (10) and the shorthand $I_k = \int_0^{\xi_{3,\lambda}} d\xi \xi^2 \theta^k(\xi)$ were used. Here, $I_9 \approx 0.2897$. The time derivative of the entropy-per-baryon s is therefore

$$\frac{ds}{dt} = -\frac{L_{\text{tot.}}(s, \rho_c)}{N_b T} \quad (70)$$

for some choice of T . Clearly the temperature varies throughout the SMS, so some averaging process must be used. Any choice will be in agreement within a prefactor of order 1–10, so to hedge our bets the most we'll take the maximum value $T = T_c$.

We now have a set of coupled differential equations:

$$\frac{d\rho_c}{dt} = \frac{9\pi}{4\lambda\xi_{3,\lambda}^3} \sqrt{\frac{8\lambda}{3}} \frac{G^2 M}{K^{3/2}(s)} \rho_c^{4/3} \sqrt{\rho_c^{1/3} - \left(\frac{s_0}{s}\right)^{4/3} \rho_{c,\text{crit.}}^{1/3}}, \quad (71a)$$

$$\frac{ds}{dt} = -\frac{1}{N_b} \frac{511I_9}{\pi^4} G_F^2 \left(\frac{K(s)}{\pi G}\right)^{3/2} \left(\frac{3}{a\mu m_b} \frac{1-\beta}{\beta}\right)^{8/3} \rho_c^{5/3}, \quad (71b)$$

which dictate the collapse dynamics. Without any further context, these equations imply that the central density will increase without bound and the entropy-per-baryon will decrease without bound, which is surely unphysical. To make our toy model more physically relevant, there should be a final state the collapsing SMS reaches beyond which more sophisticated modeling is required. Since GR was not considered in this analysis, it would be prudent to establish when GR effects are not ignorable.

A gravitational trapped surface is formed in a collapsing body wherever $r = 2GM_{\text{enc.}}(r)$, i.e. wherever a portion of a configuration collapses within its own Schwarzschild radius. For collapsing bodies that are not isotropic (i.e. spherically symmetric but whose density varies throughout, such as polytropes with $n > 0$), this trapped surface will appear somewhere within the configuration as opposed to enclosing the entirety of the matter all

at once. For a freely-falling $n = 3$ polytrope, the enclosed mass function is

$$\begin{aligned}
M_{\text{enc.}}(r) &= 4\pi \int_0^r dr' (r')^2 \rho(r'), \\
&= 4\pi \alpha_{\text{LE}}^3 \rho_c \int_0^\xi d\xi' (\xi')^2 \theta^3(\xi'), \\
&= 4\pi \left(\frac{K(s)}{\pi G} \right)^{3/2} \int_0^\xi d\xi' \left[(\xi')^2 \lambda - \frac{d}{d\xi'} \left((\xi')^2 \frac{d\theta}{d\xi'} \right) \right], \\
M_{\text{enc.}}(r) &= \frac{4\pi}{3} \left(\frac{K(s)}{\pi G} \right)^{3/2} \xi^2 [\xi \lambda - 3\theta'(\xi)],
\end{aligned} \tag{72}$$

where $r = \alpha_{\text{LE}} \xi$. The condition for a trapped surface, now in terms of the scaled radius ξ , is the transcendental equation for $\xi_\bullet$, the location of the trapped surface:

$$\xi_\bullet = \frac{8}{3} K(s) \rho_c^{1/3} \xi_\bullet^2 [\xi_\bullet - 3\theta'(\xi_\bullet)]. \tag{73}$$

For small $K(s) \rho_c^{1/3}$, the L.H.S. of the above equation is always greater than the R.H.S., but there is a particular ρ_c for a given mass (i.e. a given s and β) for which the R.H.S. becomes tangent to the line $y = \xi$ at some point $\xi_\bullet$. It turns out that, no matter the mass, the ξ that satisfies the above equation at the correct ρ_c is always $\xi_\bullet \approx 2.66$. In Chapter 2 the static polytrope enclosed mass function was used at time of publication, however numerically this result and the following results are within $\sim \text{few}\%$ agreement. The central densities at which the equation is satisfied for some SMS masses are given in Table 2. The central density at which a trapped surface is formed $\rho_{c,\bullet}$ is well fit by a power law, found to be specifically

$$\rho_{c,\bullet} \approx 1.44 \times 10^7 \left(\frac{M}{10^5 M_\odot} \right)^{-2.03} \text{ g cm}^{-3}, \tag{74}$$

though this fit is somewhat deficient for higher masses.

This is clearly a crude estimate. GR effects become significant for $r \sim 2GM_{\text{enc.}}(r)$,

Table 2: Tabulated central densities at which a gravitational trapped surface forms somewhere within the collapsing SMS for several masses.

Mass [$M_\odot$]	$\rho_{c,\bullet}$ [g cm^{-3}]
10^4	1.55×10^9
10^5	1.47×10^7
10^6	1.45×10^5
10^7	1.44×10^3
10^8	1.44×10^1

not just when the equality is satisfied. GR would serve to collapse the inner layers of the SMS faster than what would be expected from this analysis because of the high densities. These densities presented should therefore be a strict upper limit on the central densities at which a trapped surface forms. Nonetheless, for the purposes of the following analyses, it will be sufficient as a guide to the effects presented, specifically as a final ρ_c to terminate calculations.

The purpose of the preceding calculations is to eventually determine if the entire collapse of the SMS is homologous—which is to ask—does the entire SMS collapse together as one unit, or does a smaller portion within the SMS instead collapse to form a black hole? In Ref. [17] as well, it was found that the entire $n = 3$ polytrope does indeed collapse all as one unit if the pressure loss from any entropy decrease at a given density was no more than 3% of the initial value, i.e. that $K(s_f)/K(s_0) \lesssim 3\%$. If the pressure loss is greater than this, the homologously collapsing inner portion of the SMS has a mass

$$M^{\text{HC}} \approx \left[\frac{K(s_f)}{K(s_0)} \right]^{3/2} M_{\text{init.}} = \left(\frac{s_f}{s_0} \right)^2 M_{\text{init.}}, \quad (75)$$

where M^{HC} is the mass of the collapsing homologous core, $M_{\text{init.}}$ is the initial mass of the configuration, and $s_0 = s(M_{\text{init.}})$ is the initial entropy-per-baryon. In the original paper’s context of collapsing Chandrasekhar-mass stellar cores preceding supernovae, the entropy loss was quite drastic as it was derived from neutrinos produced from electron capture

on nuclei, but the final result presented above will work well for our purposes. In those situations, about 10% of the *rest mass* of the core is radiated away as neutrinos.

Solving the coupled set of differential equations (Eqs. (71)), s_f can be determined for a range of $M_{\text{init},s}$ between 10^4 – $10^8 M_\odot$ by integrating those equations until $\rho_c = \rho_{c,\bullet}$. The resulting homologous core mass is presented in Figure 4. The effect is much more drastic at “low” masses (i.e. $\sim 10^4$ – $10^5 M_\odot$ where the final densities—and therefore temperatures—are the highest. The strong T^9 dependence on neutrino emissivity means that these smaller mass SMSs lose a lot more of their entropy relative to their initial entropy from neutrino emission.

This toy model is missing a rather important contribution: neutrino scattering and absorption. In the vast majority of astrophysical scenarios, the tiny cross-sections for both mean that neutrinos are basically unaffected by the surrounding matter (as will be discussed in the next Introduction section and Chapter 2. However, as can be seen from Table 2, these are not normal astrophysical scenarios by any means; the densities and hence temperatures involved are quite extreme, and as such the high energy neutrinos being created within high density environments can interact with the surrounding matter significantly. If we ignore energy deposition from elastic scatterings, the total neutrino luminosity (Eq. (69)) should therefore be calculated taking into account the total attenuation from absorption and scattering between the center and the surface, i.e.

$$L_{\text{tot.}} \rightarrow L_{\text{tot.}}^{\text{surf.}} = \int_0^R dr 4\pi r^2 \varepsilon_\nu e^{-\tau(r)}, \quad (76)$$

where the optical depth along the travel path τ is calculated via

$$\tau(r) = \int_0^r dr' n_t(r') \sigma(r'), \quad (77)$$

with n_t is the number density of targets that can absorb and/or scatter neutrinos and σ

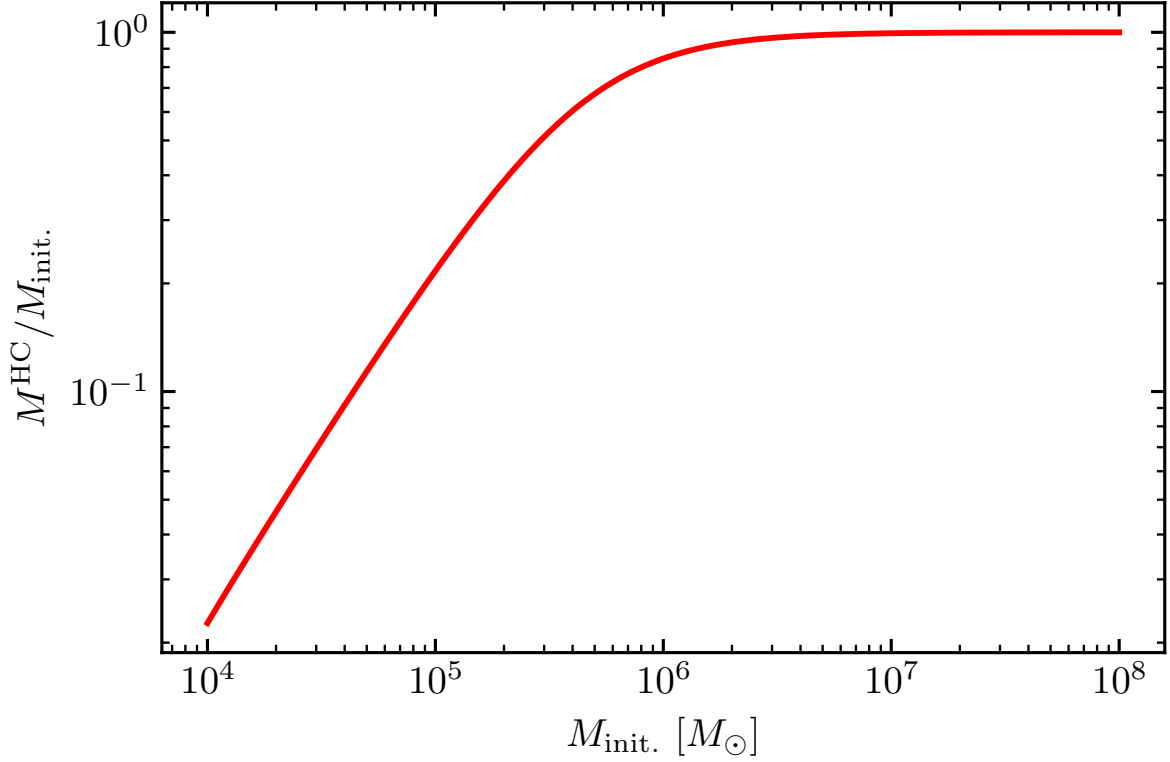


Figure 4: Plot of the ratio of the final homologous core mass to the initial SMS mass $M^{\text{HC}}/M_{\text{init.}}$ against the initial SMS mass, per Eq. (75), following the result of integrating the coupled differential equations Eqs. (71) until $\rho_c = \rho_{c,\bullet}$. The minimum value on the y -axis ($\sim 2\%$) occurs for the lowest initial mass. This is due to lower mass SMSs having higher densities, and hence higher temperatures. The results in the range $M_{\text{init.}} \sim 10^4\text{--}10^5 M_{\odot}$ are rather dubious as the densities reached are quite high, so more processes than what are considered in the toy model are likely to dominate the evolution. The more modest homologous core mass ratios above $\sim 10^5 M_{\odot}$ are more physically reasonable.

is the cross section for absorption. As mentioned previously, more about neutrinos and the relevant cross sections will be discussed later, however since both the number density of targets and the cross section increases with increasing densities, this will tamp down the strong result seen in Figure 4 in the lower mass range. In those cases, τ will be ~ 1 or even $\gg 1$ in the regions near the core at times in collapse where $\rho_c \sim \rho_{c,\bullet}$.

0.3.2 Gravitational Wave Signal

Detailed analysis of SMS collapse may be needed to determine if they are detectable by future gravitational wave (GW) observatories. In Ref. [19], the authors examined the GW wave signal from the neutrino burst generated from collapse at the moment just before a trapped surface is formed. If the burst is at least modestly aspherical, there will be a non-vanishing time-varying quadrupole moment in the stress-energy tensor for the neutrinos, which GR predicts will therefore produce gravitational radiation. This is analogous to electrodynamics where a non-vanishing time-dependent dipole moment in electric charge produces electromagnetic radiation. The neutrino burst asphericity in the authors’ model was described by a parameter α , governing the overall spatial distribution of the neutrinos via

$$F(\Omega) = \frac{1 + \alpha \cos^2 \theta}{4\pi(1 + \alpha/3)}, \quad (78)$$

which they chose to be axially symmetric for simplicity in later calculations. The neutrino emission may not necessarily be axially symmetric—indeed it may not follow the above shape at all—though it must not be spherically symmetric in order to emit gravitational radiation. As the gravitational wave passes through an Earth-based detector arm, it will change the length by a fractional amount h known as the “gravitational strain.” This was estimated to be

$$h \approx 6.5 \times 10^{-20} \alpha \varpi \left(\frac{M^{\text{HC}}}{10^5 M_\odot} \right) \left(\frac{10 \text{ Gpc}}{d} \right), \quad (79)$$

where ϖ is the mass-energy fraction of the SMS radiated away as neutrinos, and d is

the proper distance between the SMS collapse event and the detector. In the paper, the authors choose $\alpha = 2\%$ resulting from modest rotation that does not alter the collapse significantly.

We can improve our toy model by trying to determine this parameter ϖ . To do this, we let M be a quantity to be solved for in the coupled set Eqs. (71) by adding its evolution equation. Neutrinos will carry away mass-energy from the SMS exactly in the amount $L_{\text{tot.}}$ in Eq. (69), i.e.

$$\frac{dM}{dt} = -L_{\text{tot.}}. \quad (80)$$

The coupled set then reads

$$\frac{d\rho_c}{dt} = \frac{9\pi}{4\lambda\xi_{3,\lambda}^3} \sqrt{\frac{8\lambda}{3}} \frac{G^2 M}{K^{3/2}(s)} \rho_c^{4/3} \sqrt{\rho_c^{1/3} - \left(\frac{s_0}{s}\right)^{4/3} \rho_{c,\text{crit.}}^{1/3}}, \quad (81a)$$

$$\frac{ds}{dt} = -\frac{1}{N_b} \frac{511I_9}{\pi^4} G_F^2 \left(\frac{K(s)}{\pi G}\right)^{3/2} \left(\frac{3}{a\mu m_b} \frac{1-\beta}{\beta}\right)^{8/3} \rho_c^{5/3}, \quad (81b)$$

$$\frac{dM}{dt} = -\frac{511I_9}{\pi^4} G_F^2 \left(\frac{K(s)}{\pi G}\right)^{3/2} \left(\frac{3}{a\mu m_b} \frac{1-\beta}{\beta}\right)^3 \rho_c^2, \quad (81c)$$

where β and s_0 are now functions of M described by Eqs. (6) and (32), respectively. Upon integrating this set, terminating now once $\rho_c = \rho_{c,\bullet}(M)$ (i.e. now it is a function of the current mass), we determine s_f and M_f . The fraction of mass-energy radiated away of neutrinos is therefore $\varpi = 1 - M_f/M_{\text{init.}}$. Because the mass is changing throughout the collapse, this also slightly alters the evolution of entropy-per-baryon s and hence the size of the homologous core mass. Plots for both ϖ and the HC mass as a function of the initial SMS mass are given in Figure 5. With the final mass determined, the gravitational wave strain h can be calculated per Eq. (79). Fixing $d = 10$ Gpc (corresponding to a cosmological redshift of $z \approx 10$ [20]) and $\alpha = 2\%$, the GW strain from a collapsing SMS is presented in Figure 6.

With a peak strain calculated to be $\sim 10^{-21}$, this would put the signal within the

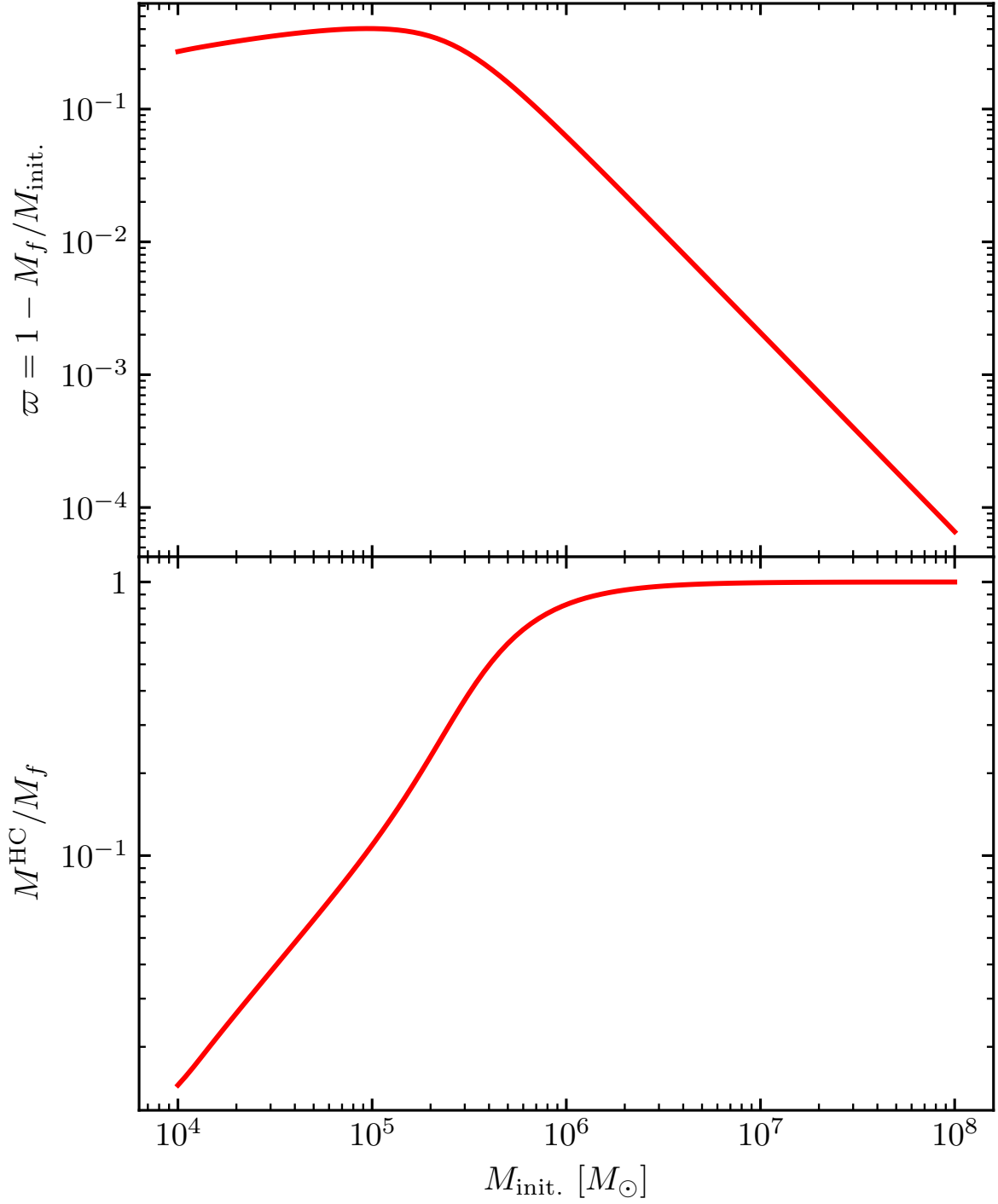


Figure 5: Plots for the fraction of mass-energy of a collapsing SMS radiated as neutrinos ϖ (top) and the ratio of the final homologous core mass to the final mass (bottom), both as a function of the initial SMS mass $M_{\text{init.}}$. The reader should note the local maximum in the top plot at $M_{\text{init.}} \sim 10^5 M_{\odot}$ for which $\varpi \sim 30\%$, and that it also corresponds to a subtle knee in the bottom curve.

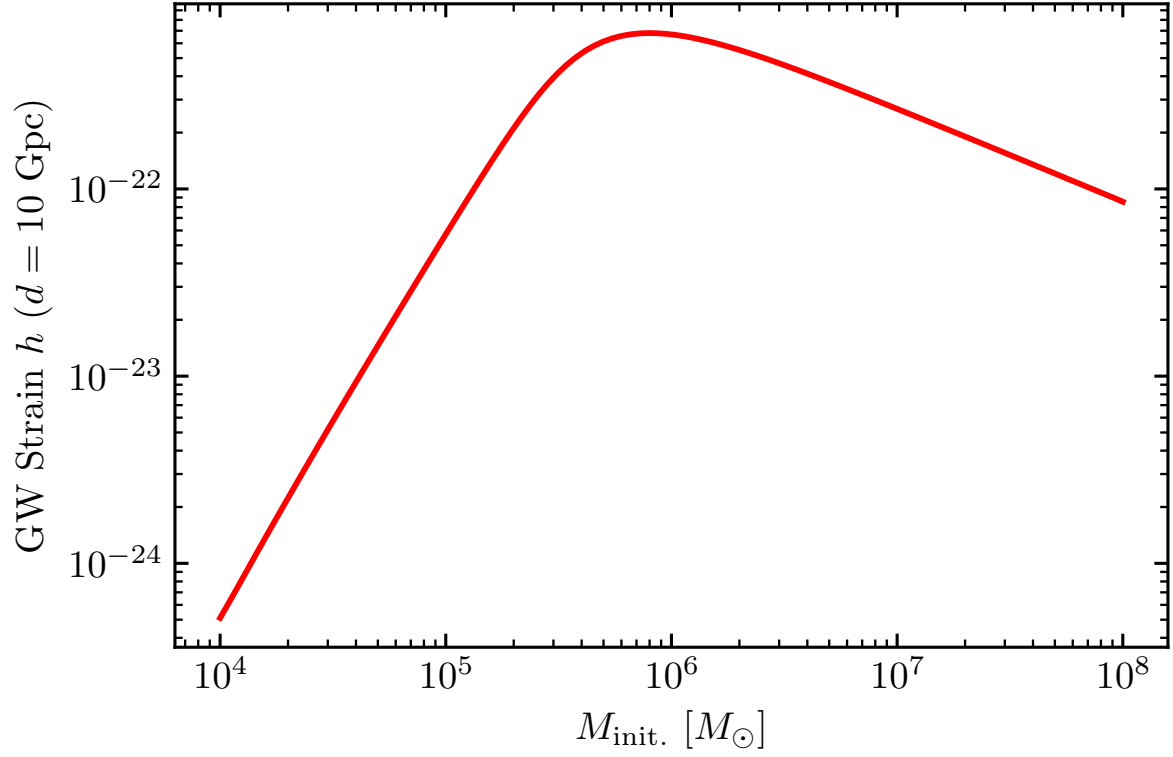


Figure 6: Gravitational wave strain caused by an anisotropic neutrino burst with asphericity $\alpha = 2\%$ at a distance of 10 Gpc ($z \approx 10$) measured by a detector on Earth, as a function of initial SMS mass. There is a peak strain of almost 10^{-21} at $M_{\text{init}} \sim 10^6 M_{\odot}$.

strain sensitivity of LIGO [21]. However, LIGO is optimized for oscillating signals with frequencies in the ~ 100 Hz band, whereas these SMS collapse signals would have their peak power spectrum in frequencies near $1/\tau_{\text{ff}\bullet} \sim (G\rho_{\text{c}\bullet})^{1/2} \sim 0.01\text{--}1$ Hz, depending on the mass. This instead slots nicely in the anticipated sensitivity range for The DECihertz Interferometer Gravitational wave Observatory (DECIGO), a Japanese space-based detector projected to begin operation in the 2030s [22].

This calculation was performed again under the assumption that neutrinos are always free-streaming through the SMS instead of scattering off surrounding material or being absorbed. A more detailed calculation using $L_{\text{tot.}}^{\text{surf.}}$ that takes into account the absorption of neutrinos will be presented in the following section. Including this contribution should serve to push the peak strain in Figure 6 to higher masses and to a lower amount, though the overall shape should not change much.

0.4 Neutrinos in More Detail

As touched upon briefly in the previous section, neutrinos are fundamental spin- $\frac{1}{2}$ particles in the Standard Model of particle physics. Being leptons, they inhabit three “generations” of particles denoted by their “flavor” of electron (e), muon (μ), and tau (τ), though unlike the electrically charged leptons of the same name, the only charge they only possess is weak charge. Therefore, they only interact with themselves and other matter via the weak interaction. This interaction is maximally chirality violating, that is, only so-called “left-handed” neutrinos and “right-handed” antineutrinos participate in any interactions, in contrast to the other three fundamental forces which are chirality blind. Here, chirality describes how each particle transforms under the relativistic Lorentz transformation, but for the purposes of this section it will be unimportant.

Neutrinos are nearly (but not entirely) massless, with the “sum of neutrino masses” Σm_ν constrained to no more than ~ 1 eV [1]. At least two of the three neutrino flavors must have mass because of a peculiar property observed by them: their observed flavor

appears to oscillate as a given neutrino traverses the vacuum. If neutrinos were massless, this flavor oscillation would not be observed since their states could not evolve in time as they cross large distances. This is fundamentally due to a mismatch between their flavor states and their mass states, as will be described below.

0.4.1 Vacuum Flavor Oscillations

Neutrinos are often borne of interactions within nuclei, and as such must be created in a particular flavor state to locally preserve lepton number ℓ . For example, a neutron unbound from a nucleus is unstable, and will decay to the slightly less massive proton with a half-life of about 10 minutes [1]. In order to conserve charge, an electron must also be produced, but as the reactant side did not contain a lepton, which has a lepton number $\ell = +1$, an anti-lepton of the same flavor must also be created on the products side with a lepton number $\ell = -1$. The reaction equation is therefore

$$n \rightarrow p + e^- + \bar{\nu}_e, \quad (82)$$

where $\bar{\nu}_e$ denotes an electron antineutrino. Now, both the total charge and lepton number are conserved, i.e. $\Delta q = 0$ and $\Delta \ell = 0$.

A similar process plays out in many other nuclear reactions, and as such almost any time a neutrino and/or antineutrino are created, they are created in what is known as a “weak eigenstate” that preserves lepton number. The weak eigenstates for neutrinos are $|\nu_e\rangle$, $|\nu_\mu\rangle$, and $|\nu_\tau\rangle$ ($|\bar{\nu}_e\rangle$, $|\bar{\nu}_\mu\rangle$, and $|\bar{\nu}_\tau\rangle$ for antineutrinos), and are created in accordance with whichever charged lepton is also a product of these nuclear reactions. However, these are not necessarily coincident with the *mass* eigenstates $|m_1\rangle$, $|m_2\rangle$, and $|m_3\rangle$ which determine the time evolution of the flavor states as governed by the Schrödinger Equation.

Let us consider a two-flavor toy model where there are two weak eigenstates $|\nu_e\rangle$ and $|\nu_x\rangle$ and two mass eigenstates $|m_1\rangle$ and $|m_2\rangle$ with $m_2 > m_1$. These two sets of states

are related to each other by a unitary rotation transformation $\hat{U}(\theta)$ via

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_x\rangle \end{pmatrix} = \hat{U}(\theta) \begin{pmatrix} |m_1\rangle \\ |m_2\rangle \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} |m_1\rangle \\ |m_2\rangle \end{pmatrix}, \quad (83)$$

where $\theta \neq 0$ is the so-called “mixing angle.” Consider a reaction where the initial quantum state of the neutrino is a ν_e to preserve lepton number, i.e. $|\psi(t=0)\rangle = |\nu_e\rangle$. Per the Schrödinger equation

$$i \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle, \quad (84)$$

where $\hat{H}$ is the Hamiltonian operator describing the total energy, the time-evolved solution to the neutrino’s quantum state (or any quantum state) is

$$|\psi(t)\rangle = \exp(-i\hat{H}t) |\psi(t=0)\rangle \quad (85)$$

where $\exp(\cdot)$ indicates a matrix exponential. For the Hamiltonian, the eigenstates are the mass eigenstates $|m_k\rangle$, where the eigenvalue is determined from

$$\hat{H} |m_k\rangle = E_k |m_k\rangle = \sqrt{p_k^2 + m_k^2} |m_k\rangle, \quad (86)$$

where p_k is the neutrino momentum. When created in nuclear processes, such as nuclear decay, these neutrinos have energies typical of the nuclear shell spacing $E \sim 1$ MeV. As the sum of neutrino masses is no more than ~ 1 eV, neutrinos are almost always

ultra-relativistic (i.e. $E \sim p_k \gg m_k$), so their energy can be well approximated as

$$\begin{aligned}
E_k &= \sqrt{p_k^2 + m_k^2}, \\
&= p_k \sqrt{1 + \frac{m_k^2}{p_k^2}}, \\
&\approx p_k + \frac{m_k^2}{2p_k}, \\
E_k &\approx E_\nu + \frac{m_k^2}{2E_\nu},
\end{aligned} \tag{87}$$

where the energy E_ν is specifically the energy as measured on Earth by a detector. Writing the initial state in terms of the mass eigenstates $|\psi(t=0)\rangle = |\nu_e\rangle = \cos\theta |m_1\rangle - \sin\theta |m_2\rangle$, the solution to the Schrödinger Equation is therefore

$$\begin{aligned}
|\psi(t)\rangle &= \cos\theta e^{-iE_1 t} |m_1\rangle - \sin\theta e^{-iE_2 t} |m_2\rangle, \\
&= \cos\theta e^{-i(E_\nu + m_1^2/(2E_\nu))t} |m_1\rangle - \sin\theta e^{-i(E_\nu + m_2^2/(2E_\nu))t} |m_2\rangle.
\end{aligned} \tag{88}$$

The probability that $|\psi(t)\rangle$ is measured as an electron neutrino is then given by

$$\begin{aligned}
P_e &= |\langle \nu_e | \psi(t) \rangle|^2 = \left| \cos^2\theta e^{-i(E_\nu + m_1^2/(2E_\nu))t} - \sin^2\theta e^{-i(E_\nu + m_2^2/(2E_\nu))t} \right|^2, \\
&= 1 - \sin^2 2\theta \sin^2 \left(\frac{m_2^2 - m_1^2}{4E} t \right) \\
P_e &= 1 - \sin^2 2\theta \sin^2 \left(\frac{\Delta m_{21}^2}{4E_\nu} t \right),
\end{aligned} \tag{89}$$

where $\Delta m_{21}^2 = m_2^2 - m_1^2$ is the mass-squared difference between the mass eigenstates. This probability has a minimum of $1 - \sin^2 2\theta$ and averages over many wavelengths $\lambda = 8\pi E / \Delta m_{21}^2$ to $\langle P_e \rangle = 1 - \frac{1}{2} \sin^2 2\theta$.

The case of three flavor oscillation, i.e. what is observed in nature, follows readily from the two flavor example, but the extra complication is not particularly illustrative. For three flavors, there are three mixing angles: θ_{12} , θ_{13} , and θ_{23} , and three mass-squared

differences: Δm_{21}^2 , Δm_{31}^2 , and Δm_{32}^2 . The values for each of these are measured to be (adapted from Refs. [1, 23])

$$\sin^2(2\theta_{12}) = 0.845 \pm 0.018,$$

$$\sin^2(2\theta_{13}) = 0.086 \pm 0.002,$$

$$\sin^2(2\theta_{23}) = 0.56 \pm 0.04,$$

$$\Delta m_{21}^2 = (7.41 \pm 0.02) \times 10^{-5} \text{ eV}^2,$$

$$|\Delta m_{31}^2| \approx |\Delta m_{32}^2| = (2.44 \pm 0.03) \times 10^{-3} \text{ eV}^2,$$

where the sign of the final two mass-squared differences is unknown. Since the individual masses are not known, this sign ambiguity means that neutrinos fall into one of two “mass hierarchies,” known as “normal” and “inverted.” In the normal hierarchy, mass states m_1 and m_2 are near each other in mass with m_3 residing far above both, whereas in the inverted hierarchy, m_3 is the lightest mass state with m_1 and m_2 near each other but greater in mass. For both cases, if the lightest neutrino is massless, then the largest mass any neutrino could be is ~ 50 meV. A pictorial representation of these hierarchies is given in Figure 7.

These measurements also show that the mixing between all three flavors is not insignificant. Indeed, their large mixing angles led quite quickly to the discovery of flavor oscillations through the Homestake Experiment in the late 1960s, where less than the expected conversion of chlorine from certain Solar neutrinos to a radioactive argon isotope was observed. This became known as the “solar neutrino problem,” however the issue was resolved by proposing that during transit from the Sun to the chlorine tank on Earth, the probability that initially ν_e neutrinos would remain in the same flavor state became only about $\frac{1}{3}$ when averaged over a few orbits of Earth around the Sun, with the remaining $\frac{2}{3}$ having been essentially transmuted into μ and τ flavors that have no interaction with the

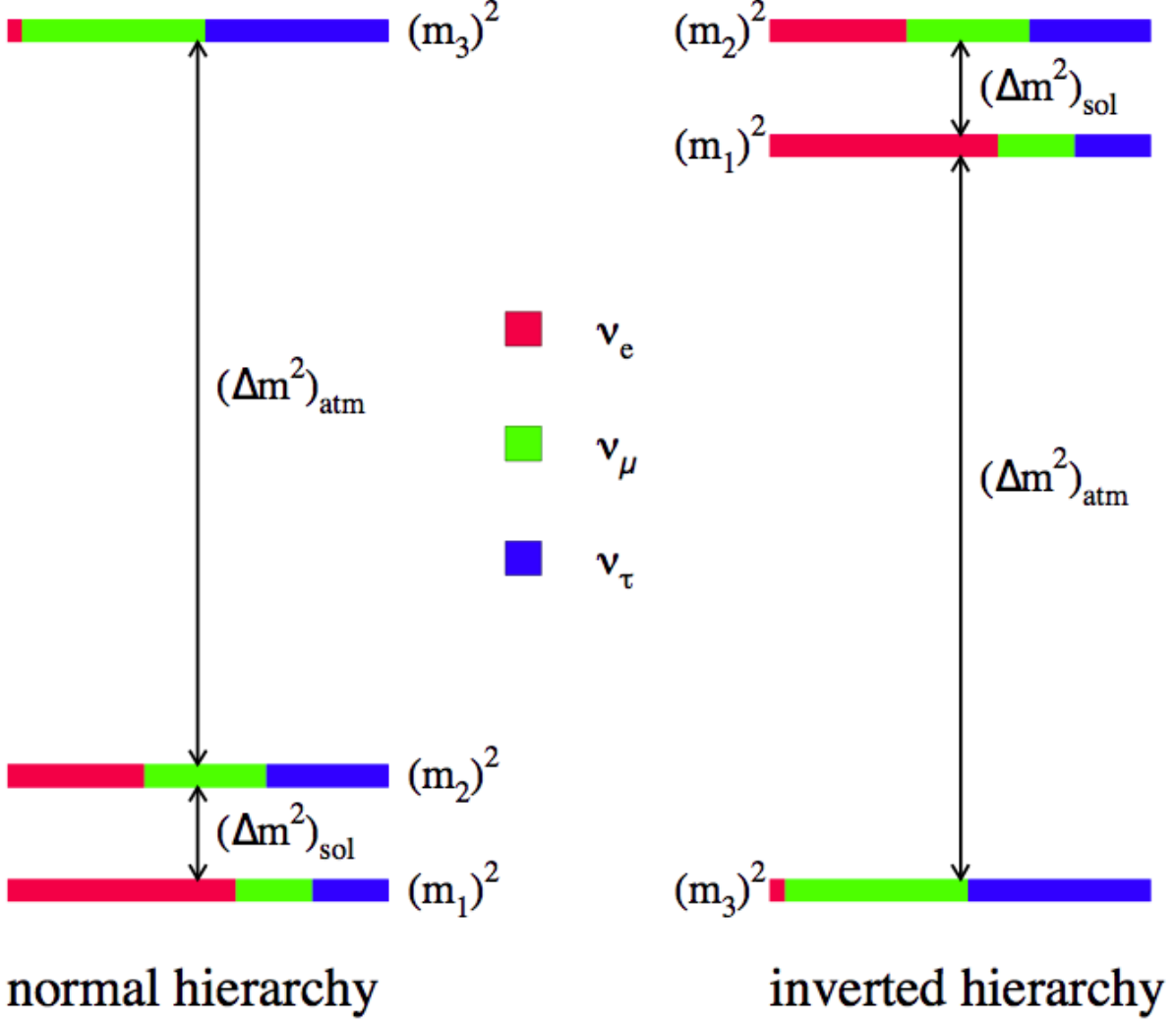


Figure 7: The two possible neutrino mass hierarchies, with colors corresponding to the proportion each flavor state contributes to the mass states' linear combinations. Here, Δm_{sol}^2 corresponds to the previously mentioned Δm_{21}^2 and is determined through solar neutrino observations, and Δm_{atm}^2 corresponds to the previously mentioned $\Delta m_{32}^2 \approx \Delta m_{31}^2$ and is determined from neutrinos produced by cosmic rays. Note that the $|m_3\rangle$ state has almost no $|\nu_e\rangle$ contribution and the $|\nu_\mu\rangle$ and $|\nu_\tau\rangle$ states are almost maximally mixed. Figure from Ref. [24], Figure 1.

chlorine.

0.4.2 In-Media Flavor Oscillations

As stated earlier, neutrinos only interact with themselves and the rest of the standard model through the weak interaction. This interaction is mediated by two force-carrying bosons: the electrically neutral Z^0 boson and the electrically charged $W^\pm$ boson. Interactions mediated by the former are known as “neutral current” (NC), and those mediated by the latter two are known as “charged current” (CC). Both bosons are spin-1 and have masses $m_Z \approx 91.2$ GeV and $m_W \approx 80.4$ GeV, respectively [1].

Consider neutrinos of different flavors encountering material with some total electron number density $n_e = n_{e^-} - n_{e^+}$, i.e. the net number density of electrons minus positrons. All flavors will “feel” the same NC scattering reaction mediated by the Z^0 boson:

$$\bar{\nu}^{(-)} + e^- \xrightarrow{Z^0} \bar{\nu}^{(-)} + e^-, \quad (90)$$

where $\bar{\nu}^{(-)}$ denotes (anti-)neutrinos of any flavor. For just the electron neutrinos, however, they will “feel” an additional scattering reaction mediated by the $W^\pm$ boson:

$$\nu_e + e^- \xrightarrow{W^\pm} e^- + \nu_e. \quad (91)$$

The interaction Hamiltonian for this CC process is [25]

$$H_{CC} = \frac{G_F}{\sqrt{2}} [\bar{\psi}_{\nu_e} P_L \psi_{\nu_e}] [\bar{\psi}_e P_L \psi_e], \quad (92)$$

where ψ_{ν_e} and ψ_e are the electron neutrino and electron spinor fields (barred indicate electron anti-neutrino and positron spinor fields), respectively, and P_L is the left-handed projection operator required for weak interactions. The mathematics involved in spinor fields will not be required to understand the essence of the final result. From this

Hamiltonian, one can derive a potential V_{CC} only “felt” by the electron flavor, given by [25]

$$V_{\text{CC}} = \sqrt{2}G_{\text{F}}n_e. \quad (93)$$

Let us return to our two-flavor toy model, where the x flavor is a stand-in for the μ and τ neutrino flavors. Recall that, in vacuum, the Hamiltonian in the mass basis reads

$$\hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \quad (94)$$

where $E_k = \sqrt{p_k^2 + m_k^2} \approx E_\nu + m_k^2/(2E_\nu)$ as argued in the previous section and $m_2 > m_1$. The Hamiltonian for the CC potential is (in the flavor basis) just $\hat{H}_{\text{CC}} = V_{\text{CC}} |\nu_e\rangle \langle \nu_e|$ because it is only felt by the electron flavor. For later convenience, we can subtract off the constant operator $\hat{E}_\nu = E_\nu \mathbb{1}$ from the total Hamiltonian, which is justified as changes by a constant offset in the energy have no impact on the underlying physics.

With all of these considerations, the total Hamiltonian is therefore

$$\hat{H} = \underbrace{\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}}_{\text{mass basis}} - \underbrace{\begin{pmatrix} E_\nu & 0 \\ 0 & E_\nu \end{pmatrix}}_{\text{mass basis}} + \hat{U}^\dagger \underbrace{\begin{pmatrix} V_{\text{CC}} & 0 \\ 0 & 0 \end{pmatrix}}_{\text{flavor basis}} \hat{U}, \quad (95)$$

where $\hat{U}$ is the rotation matrix in Eq. (83) and $\hat{U}^\dagger$ is the conjugate-transpose of $\hat{U}$. Using the computer algebra software Mathematica [26], the eigenvalues for this Hamiltonian, which in its eigenbasis is represented as

$$\hat{H} = \frac{1}{2E_\nu} \begin{pmatrix} m_{1,\text{M}}^2 & 0 \\ 0 & m_{2,\text{M}}^2 \end{pmatrix}, \quad (96)$$

where $m_{1,M}$ and $m_{2,M}$ are the effective masses in-media, are found to be

$$m_{1M}^2 = \frac{m_1^2 + m_2^2 + 2\sqrt{2}G_F E_\nu n_e}{2} - \frac{1}{2}\sqrt{\left(\Delta m^2 \cos 2\theta - 2\sqrt{2}G_F E_\nu n_e\right)^2 + (\Delta m^2 \sin 2\theta)^2}, \quad (97)$$

$$m_{2M}^2 = \frac{m_1^2 + m_2^2 + 2\sqrt{2}G_F E_\nu n_e}{2} + \frac{1}{2}\sqrt{\left(\Delta m^2 \cos 2\theta - 2\sqrt{2}G_F E_\nu n_e\right)^2 + (\Delta m^2 \sin 2\theta)^2}, \quad (98)$$

which both reduce to $m_{1M}^2 \rightarrow m_1^2$ and $m_{2M}^2 \rightarrow m_2^2$ when $n_e \rightarrow 0$. This change in mass is analogous to photons in dense plasma acquiring an effective mass from electromagnetic interactions with surrounding charged particles. The in-media squared mass difference is therefore

$$\Delta m_M^2 = m_{2M}^2 - m_{1M}^2 = \sqrt{\left(\Delta m^2 \cos 2\theta - 2\sqrt{2}G_F E_\nu n_e\right)^2 + (\Delta m^2 \sin 2\theta)^2}. \quad (99)$$

Clearly, this has a minimum value when $2\sqrt{2}G_F E_\nu n_e = \Delta m^2 \cos 2\theta$, specifically it achieves $\Delta m_{M,\min.}^2 = \Delta m^2 \sin 2\theta$. This matter effect also changes the mixing angle in Eq. (83) to a *matter* mixing angle θ_M , defined via [25]

$$\cos 2\theta_M = \frac{\Delta m^2 \cos 2\theta - 2\sqrt{2}G_F E_\nu n_e}{\sqrt{\left(\Delta m^2 \cos 2\theta - 2\sqrt{2}G_F E_\nu n_e\right)^2 + (\Delta m^2 \sin 2\theta)^2}}, \quad (100)$$

which depends on the local electron number density n_e . For $G_F E_\nu n_e \gg \Delta m^2$ (i.e. in high density environments, such as stellar cores), the above ratio reduces to $\cos 2\theta_M \sim -1$, so $\theta_M \sim \pi/2$. This means that, in high density environments, $|\nu_e\rangle \sim |m_{2M}\rangle$, i.e. the electron flavor state is primarily in the heavier in-media mass eigenstate as opposed to the lighter state, and *vice versa* for $|\nu_x\rangle$. In vacuum ($n_e \sim 0$), $\cos 2\theta_M \sim \cos 2\theta$, which is the normal vacuum mixing. Using $\theta = \theta_{13}$, this means the $|\nu_e\rangle$ is primarily in the *lighter* vacuum mass eigenstate $|m_1\rangle$, and *vice versa* for $|\nu_x\rangle$. The interesting case appears

when $2\sqrt{2}G_F E_\nu n_e = \Delta m^2 \cos 2\theta$, where the above ratio vanishes. Therefore, $\theta_M = \pi/4$ so $\sin \theta_M = \cos \theta_M = 1/\sqrt{2}$, meaning that the unitary transformation $\hat{U}(\pi/4)$ maximally (evenly) mixes the two mass states in the superposition.

For a given neutrino energy, this establishes a resonant electron number density $n_e^{\text{res.}}$ for which the two neutrino states are most indistinguishable:

$$n_e^{\text{res.}} = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}G_F E_\nu}. \quad (101)$$

A schematic of all of the above considerations is given in Figure 8, which plots the two in-media mass eigenstates against the local electron number density.

As mentioned before, when neutrinos are created in nuclear reaction processes, they are created in a particular flavor eigenstate to locally conserve lepton number. Consider an initially electron flavor neutrino created in a dense environment where $n_e \gg n_e^{\text{res.}}$, such as the core of a star. This ν_e will be primarily in the heavier in-media mass eigenstate m_{2M} because $\theta \approx \pi/2$. As the ν_e travels from the core to the surface, its in-media mass will change depending on the local electron number density n_e , until it finally exits the star. Assuming the change in density it experienced was “gentle” (to be elaborated upon later), this means that it will stay on the top track of Figure 8 and exit the star in the heavier mass eigenstate *in vacuum*. However, this mass state is primarily composed of the ν_x flavor in vacuum, not ν_e , so this neutrino will have been essentially transmuted into a ν_x from a ν_e purely from sustained weak interactions with the electrons in the star. The probability that an initially electron flavor neutrino created in a dense environment remains a ν_e in a diffuse environment is given by [25]

$$P_{e \rightarrow e} = \frac{1}{2} \left(1 + \cos 2\theta_M^{(i)} \cos 2\theta \right), \quad (102)$$

where the (i) indicates the initial in-media mixing angle. If the origination region for the ν_e

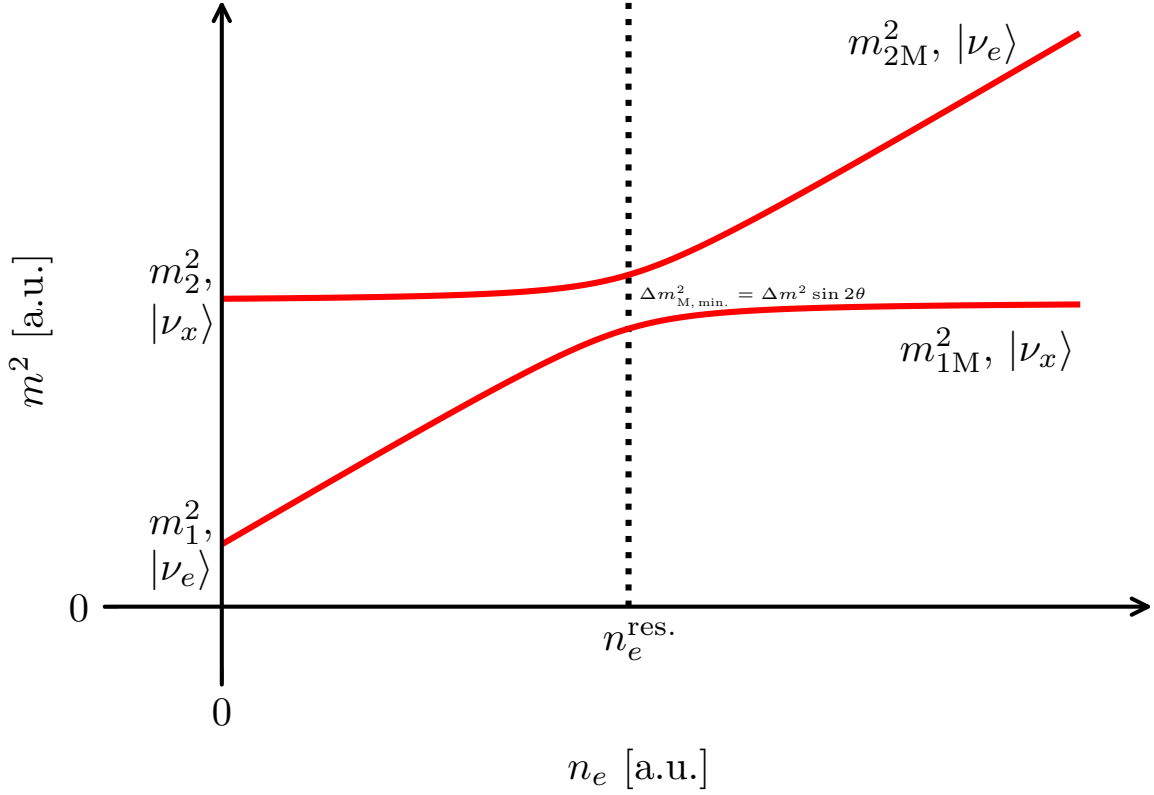


Figure 8: Squares of both mass eigenstates in media m_{1M}^2 (bottom) and m_{2M}^2 (top) (Eqs. (97) and (98), respectively) as a function of local electron number density n_e , both presented in arbitrary units on a linear scale. In vacuum ($n_e = 0$), the masses reduce to their vacuum values m_1^2 and m_2^2 , which (for $\theta = \theta_{13} \approx 0.15$) means that $|\nu_e\rangle$ is primarily in the lighter $|m_1\rangle$ mass eigenstate and $|\nu_x\rangle$ is primarily in the heavier $|m_2\rangle$ mass eigenstate. On the right side of the graph (very dense environments), the mixing angle is $\theta_M \sim \pi/2$, so the correlations between the flavor and mass eigenstates is flipped compared to vacuum. Between these two extremes, there exists an electron number density such that $\theta_M = \pi/4$, meaning that both flavor eigenstates are an equal linear combination of the mass eigenstates, which occurs at $n_e^{\text{res.}}$. This also corresponds to a minimal squared-mass difference in media, given by $\Delta m_{M, \min.}^2 = \Delta m^2 \sin 2\theta$.

is sufficiently dense, then $\cos 2\theta_{\text{M}}^{(\text{i})} \approx -1$, so this probability will be $P_{e \rightarrow e} \approx \frac{1}{2}(1 - \cos 2\theta)$, which for $\theta = \theta_{13} \approx 0.15$ means a probability of about 2% that it will remain a ν_e upon leaving the star.

The above analysis is only true if the change in density over the neutrino’s path is “gentle,” i.e. that the length scale of density changes is large relative to the flavor oscillation wavelength. To quantify this relationship, we introduce an “adiabaticity parameter” $\gamma_{\text{res.}}$ defined via [27]

$$\gamma_{\text{res.}} = \frac{\Delta m^2 \sin^2 2\theta}{2E_\nu \cos 2\theta} \left| \frac{1}{n_e} \frac{dn_e}{dr} \right|_{\text{res.}}^{-1}, \quad (103)$$

where the derivative is taken at $n_e^{\text{res.}}$. This parameter indicates “gentleness” (adiabaticity) if $\gamma_{\text{res.}} \gg 1$. This condition is analogous to the so-called geometric optics limit for light traveling through a lens—if the size of the refractor is much larger than the wavelength of light, then the light can be treated as a straight beam being deflected as prescribed by Snell’s Law instead of invoking complicated wave mechanics to determine the light path.

For the vast majority of astrophysical scenarios, $\gamma_{\text{res.}} \gg 1$ if a resonance occurs because the density-change length scale (the derivative portion of the above equation) is of order-of-magnitude the size of the object. However, in extreme environments, like the moments following core-collapse, the density-change length scale may be quite small across the edge of the proto-neutron star, so the adiabaticity condition may not be satisfied. In either case, the probability that the neutrino will “jump” to the other track and therefore not experience this matter-induced flavor conversion is [27]

$$P_{\text{jump}} \sim \exp\left(-\frac{\pi}{2}\gamma_{\text{res.}}\right). \quad (104)$$

It is a valid question to ask if the two-flavor toy model we discussed for vacuum and matter flavor oscillations is even physically illuminating given that neutrinos actually have three flavors. By a quirk of nature, it turns out to be quite accurate. Specifically, it

turns out that the μ and τ flavors are almost evenly mixed as $\sin \theta_{23} \approx 1/\sqrt{2}$. This means that the new x flavor can be written as an even and equal linear combination of those two flavors via [28]:

$$|\nu_x\rangle = \frac{1}{\sqrt{2}}(|\nu_\mu\rangle + |\nu_\tau\rangle), \quad (105)$$

and introduce a new flavor x' that is an odd and equal linear combination:

$$|\nu_{x'}\rangle = \frac{1}{\sqrt{2}}(|\nu_\mu\rangle - |\nu_\tau\rangle). \quad (106)$$

Now we have three flavors (e , x , and x') as observed in nature, however the final flavor x' is orthogonal to both e and x and therefore does not contribute to any e and x flavor evolution, nor does it evolve itself. As a linear combination in vacuum, this condenses to

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_x\rangle \\ |\nu_{x'}\rangle \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |m_1\rangle \\ |m_2\rangle \\ |m_3\rangle \end{pmatrix}. \quad (107)$$

Of course, more careful considerations can be included since θ_{23} is not exactly $\pi/4$, however for the broad brushes of what is expected in nature, this two-flavor toy model therefore suffices.

0.4.3 Scatterings and Absorptions

In most astrophysical scenarios, neutrinos will either scatter off or absorb onto baryons (protons and neutrons), composite nuclei, electrons, or themselves. Elastic scattering off of electrons is the simplest interaction, and all possible incident neutrinos have the same general form for the cross-section σ , being [25]

$$\sigma \propto 2G_F^2 m_e E_\nu \quad (108)$$

if the electrons are non-relativistic and neutrino mass is neglected. The proportionality constant depends on the particular neutrino involved in the interaction.

Due to both NC and CC channels, the reaction $\nu_e + e^- \rightarrow \nu_e + e^-$ has the largest cross section, specifically

$$\sigma_{\nu_e + e^-} = \frac{G_F^2 m_e E_\nu}{2\pi} \left[(1 + 2 \sin^2 \vartheta_W)^2 + \frac{4}{3} \sin^4 \vartheta_W \right] \approx 9.49 \times 10^{-45} \left(\frac{E_\nu}{1 \text{ MeV}} \right) \text{ cm}^2, \quad (109)$$

where $\sin^2 \vartheta_W \approx 0.23$ [1] is the sine-squared of the Weinberg mixing angle. The next largest is $\bar{\nu}_e + e^- \rightarrow \bar{\nu}_e + e^-$, which has the form

$$\sigma_{\bar{\nu}_e + e^-} = \frac{G_F^2 m_e E_\nu}{2\pi} \left[\frac{1}{3} (1 + 2 \sin^2 \vartheta_W)^2 + 4 \sin^4 \vartheta_W \right] \approx 3.97 \times 10^{-45} \left(\frac{E_\nu}{1 \text{ MeV}} \right) \text{ cm}^2, \quad (110)$$

which is slightly smaller despite also having both NC and CC channels because of the particle physics particulars involved with antiparticles incident on particles. The final two interactions $\nu_x + e^- \rightarrow \nu_x + e^-$ and $\bar{\nu}_x + e^- \rightarrow \bar{\nu}_x + e^-$, where $x = \mu, \tau$ are similar to each other in magnitude and take the forms

$$\sigma_{\nu_x + e^-} = \frac{G_F^2 m_e E_\nu}{2\pi} \left[(1 - 2 \sin^2 \vartheta_W)^2 + \frac{4}{3} \sin^4 \vartheta_W \right] \approx 1.56 \times 10^{-45} \left(\frac{E_\nu}{1 \text{ MeV}} \right) \text{ cm}^2, \quad (111)$$

$$\sigma_{\bar{\nu}_x + e^-} = \frac{G_F^2 m_e E_\nu}{2\pi} \left[\frac{1}{3} (1 - 2 \sin^2 \vartheta_W)^2 + 4 \sin^4 \vartheta_W \right] \approx 1.33 \times 10^{-45} \left(\frac{E_\nu}{1 \text{ MeV}} \right) \text{ cm}^2. \quad (112)$$

For reference, the Thomson cross section σ_T , which governs *photon* scattering off of electrons, is energy independent and is about 20 orders of magnitude larger than any of these.

Neutrinos can also scatter elastically off of free baryons through NC channels. Due to being NC, the cross-sections are independent on the incident neutrino flavor and parity and only depends on whether the baryon is a proton p or neutron n . Each cross section

is [25]

$$\begin{aligned}\sigma_{\nu+p}^{\text{NC}} &= \frac{G_{\text{F}}^2 E_{\nu}^2}{4\pi} \left[(1 - 4 \sin^4 \vartheta_{\text{W}})^2 + 3g_A^2 \right] \\ &\approx 2.14 \times 10^{-44} \left(\frac{E_{\nu}}{1 \text{ MeV}} \right)^2 \text{ cm}^2,\end{aligned}\tag{113}$$

$$\begin{aligned}\sigma_{\nu+n}^{\text{NC}} &= \frac{G_{\text{F}}^2 E_{\nu}^2}{4\pi} [1 + 3g_A^2] \\ &\approx 2.56 \times 10^{-44} \left(\frac{E_{\nu}}{1 \text{ MeV}} \right)^2 \text{ cm}^2,\end{aligned}\tag{114}$$

where ν refers to neutrinos of any flavor and parity (both neutrinos and antineutrinos) and $g_A \approx 1.3$ is the so-called “axial coupling constant.” Note that these cross sections depend quadratically on the incident neutrino energy instead of linearly as with electrons.

Neutrinos may also scatter elastically off of composite nuclei with atomic mass A composed of Z protons and N neutrons. This scattering has the form [29]

$$\sigma_{\nu+A}^{\text{NC}} = \frac{G_{\text{F}}^2}{4\pi} [(4 \sin^2 \vartheta_{\text{W}} - 1)Z + N]^2 E_{\nu}^2,\tag{115}$$

however since $\sin^2 \vartheta_{\text{W}} \approx 0.25$, the proton number contribution can be ignored. Therefore, the cross section scales quadratically with both the neutrino energy and the number of neutrons⁴:

$$\sigma_{\nu+A}^{\text{NC}} = \frac{G_{\text{F}}^2}{4\pi} N^2 E_{\nu}^2 \approx 4.22 \times 10^{-45} N^2 \left(\frac{E_{\nu}}{1 \text{ MeV}} \right)^2 \text{ cm}^2.\tag{116}$$

We will not be considering so-called “deep NC inelastic scatterings” where the final state of the nucleus may be excited, split into several components, or otherwise structurally impacted by the incident neutrino which would require $E_{\nu} \sim$ the nuclear binding energy.

Neutrinos may also scatter off nucleons inelastically through the CC channel, which converts the neutrino to its associated charged lepton in exchange for transmuting the

⁴This differs slightly from Eq. (2.2) which appears in a paper sent for review for publication in Physical Review D, and will be rectified upon responses from the reviewers. The main results of that paper are unaffected.

nucleon to its isospin mirror. For certain processes, this requires a “threshold energy” for the neutrino that it must have to complete the reaction as the total mass of the final state may be larger than the initial state. For the purposes of this dissertation, we will be considering the reactions $\nu_e + n \rightarrow p + e^-$ and $\bar{\nu}_e + p \rightarrow n + e^+$ and ignoring the reactions with other leptons in the final state as their high mass requires very energetic neutrinos. These reactions have the form

$$\begin{aligned}\sigma_{\bar{\nu}_e+p}^{\text{CC}} &= \frac{G_F^2 |V_{ud}|^2}{\pi} (1 + 3g_A^2) (E_\nu - Q)^2, \\ &\approx 9.70 \times 10^{-44} \left(\frac{E_\nu - Q}{1 \text{ MeV}} \right)^2 \text{ cm}^2,\end{aligned}\tag{117}$$

$$\begin{aligned}\sigma_{\nu_e+n}^{\text{CC}} &= \frac{G_F^2 |V_{ud}|^2}{\pi} (1 + 3g_A^2) (E_\nu + Q)^2, \\ &\approx 9.70 \times 10^{-44} \left(\frac{E_\nu + Q}{1 \text{ MeV}} \right)^2 \text{ cm}^2,\end{aligned}\tag{118}$$

where $|V_{ud}|^2 \approx 0.948$ is the squared magnitude of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element corresponding to up-down quark flavor mixing [1], and $Q = m_n - m_p \approx 1.29 \text{ MeV}$ is the mass difference between a neutron and a proton. Note that the first reaction only applies for $E_\nu > Q$, however the second can happen for any $E_\nu > 0$ because there is no kinematics blocking the reaction.

0.4.4 SMS Collapse Including Neutrino Interactions

We turn our attention back to SMSs, specifically to determine the neutrino luminosity at the surface of the SMS (Eq. (76)) that includes the effects of scatterings and absorptions. An SMS of primordial chemical composition is composed of $\sim 75\%$ by mass protons and $\sim 25\%$ by mass helium nuclei (a.k.a. α particles) with $Z = N = 2$. Denoting the hydrogen mass fraction as $X \approx 0.75$ and the helium mass fraction $Y = 1 - X$, the

number density of electrons can be determined from overall charge neutrality:

$$n_e = n_p + 2n_\alpha, \quad (119)$$

which is to say (assuming the gas is fully ionized) each species p and α contribute 1 and 2 electrons, respectively. In terms of the total mass density ρ , the number densities for each species is

$$n_p = X \frac{\rho}{m_b}, \quad (120)$$

$$n_\alpha = \frac{1-X}{4} \frac{\rho}{m_b}, \quad (121)$$

where $m_b \approx 939$ MeV is the atomic mass unit. Therefore,

$$n_e = \frac{1+X}{2} \frac{\rho}{m_b}. \quad (122)$$

The total (inverse) mean free path is then a sum-product of the cross-sections discussed in the previous section with the relevant number densities of targets:

$$\begin{aligned} (\sigma n_t)_{\text{tot.}} &= (\sigma_{\nu+p}^{\text{NC}} + \tfrac{1}{2}\sigma_{\bar{\nu}e+p}\Theta[E_\nu - Q])n_p + \sigma_{\nu+A}^{\text{NC}}n_\alpha + \tfrac{1}{2}(\sigma_{\nu_e+e^-} + \sigma_{\bar{\nu}_e+e^-})n_e, \\ &= \left[X(\sigma_{\nu+p}^{\text{NC}} + \tfrac{1}{2}\sigma_{\bar{\nu}e+p}\Theta[E_\nu - Q]) + \frac{1-X}{4}\sigma_{\nu+A}^{\text{NC}} + \frac{1+X}{4}(\sigma_{\nu_e+e^-} + \sigma_{\bar{\nu}_e+e^-}) \right] \frac{\rho}{m_b}, \\ &= \sigma_{\text{tot.}}(E_\nu) \frac{\rho}{m_b} \end{aligned} \quad (123)$$

where we assume all of the neutrinos are an even split of ν_e s and $\bar{\nu}_e$ s (hence the odd factors of $1/2$ appearing next to some cross sections) and the Heaviside function $\Theta[E_\nu - Q]$ ensures that only neutrinos with $E_\nu > Q$ contribute to that cross section. In reality, neutrinos in collapsing SMSs are produced in $\nu\bar{\nu}$ pairs of all flavors (though not equally and in fact primarily in the electron flavor, see e.g. Ref. [30]) and are produced in a spectrum of

energies (again see Ref. [30]), for the purposes of this calculation we will take the E_ν that appears in these cross sections to be the average neutrino energy given the spectrum in Ref. [30], which is approximately $\langle E_\nu \rangle \approx 5.8T$, i.e. almost six times the local temperature. Due to the strong dependency on temperature on the neutrino emissivity (Eq. (66)), we will also consider only the central temperature in the optical depth, i.e. $T = T_c$ from Eq. (10) by setting $\rho = \rho_c$. Therefore,

$$\begin{aligned}\tau(r) &= \sigma_{\text{tot.}}(5.8T_c) \frac{1}{m_b} \int_0^r dr' \rho(r') \\ &= \sigma_{\text{tot.}}(5.8T_c) \frac{\rho_c \alpha_{\text{LE}}}{m_b} \int_0^{\xi=r/\alpha_{\text{LE}}} d\xi' \theta^3(\xi'), \\ \tau(r) &= \sigma_{\text{tot.}}(5.8T_c) \left(\frac{K(s)}{\pi G} \right)^{1/2} \frac{\rho_c^{2/3}}{m_b} \int_0^\xi d\xi' \theta^3(\xi'),\end{aligned}\tag{124}$$

where θ is the solution to the freely falling Lane-Emden Equation (Eq. (58)). Therefore, the total neutrino luminosity at the surface taking into account scatterings and absorptions is (rather clumsily)

$$\begin{aligned}L_{\text{tot.}}^{\text{surf.}} &= \int_0^R dr 4\pi r^2 \varepsilon_\nu \exp \left[-\sigma_{\text{tot.}}(5.8T_c) \left(\frac{K(s)}{\pi G} \right)^{1/2} \frac{\rho_c^{2/3}}{m_b} \int_0^\xi d\xi' \theta^3(\xi') \right], \\ &= \frac{512}{\pi^4} G_F^2 \alpha_{\text{LE}}^3 T_c^9 \int_0^{\xi_{3,\lambda}} d\xi \xi^2 \theta^9(\xi) \exp \left[-\sigma_{\text{tot.}}(5.8T_c) \left(\frac{K(s)}{\pi G} \right)^{1/2} \frac{\rho_c^{2/3}}{m_b} \int_0^\xi d\xi' \theta^3(\xi') \right], \\ L_{\text{tot.}}^{\text{surf.}} &= \frac{512}{\pi^4} G_F^2 \left(\frac{K(s)}{\pi G} \right)^{3/2} \left(\frac{3}{a\mu m_b} \frac{1-\beta}{\beta} \right)^3 \rho_c^2 \\ &\quad \times \int_0^{\xi_{3,\lambda}} d\xi \xi^2 \theta^9(\xi) \exp \left[-\sigma_{\text{tot.}}(5.8T_c) \left(\frac{K(s)}{\pi G} \right)^{1/2} \frac{\rho_c^{2/3}}{m_b} \int_0^\xi d\xi' \theta^3(\xi') \right].\end{aligned}\tag{125}$$

The above outer integral reduces to $I_9 = \int_0^{\xi_{3,\lambda}} d\xi \xi^2 \theta^9(\xi)$ for $\sigma_{\text{tot.}} \rightarrow 0$, i.e. in the limit of low energy neutrinos. Subsequently, the differential equations governing the collapse of

the SMS now read

$$\frac{d\rho_c}{dt} = \frac{9\pi}{4\lambda\xi_{3,\lambda}^3} \sqrt{\frac{8\lambda}{3}} \frac{G^2 M}{K^{3/2}(s)} \rho_c^{4/3} \sqrt{\rho_c^{1/3} - \left(\frac{s_0}{s}\right)^{4/3} \rho_{c,\text{crit.}}^{1/3}}, \quad (126a)$$

$$\frac{ds}{dt} = -\frac{L_{\text{tot.}}^{\text{surf.}}(s, \rho_c)}{N_b T_c}, \quad (126b)$$

$$\frac{dM}{dt} = -L_{\text{tot.}}^{\text{surf.}}(s, \rho_c). \quad (126c)$$

Following the same process as in Section 0.4, the gravitational strain at a distance of $d = 10$ Gpc as a function of the initial SMS mass is calculated and presented in Figure 9.

This calculation shows that, indeed, the peak gravitational strain is slightly decreased and shifted to a slightly higher mass. An interesting note is that, for smaller masses (i.e. higher neutrino energies), the strain is increased relative to the case of not considering neutrino interactions within the SMS. This is due to the larger homologous core mass predicted as the scatterings and absorptions of neutrinos serve to “trap” at least some of them within the SMS during collapse and hence prevent them from carrying away as much entropy. This is also despite the associated decrease in the mass-energy fraction radiated away by neutrinos ϖ .

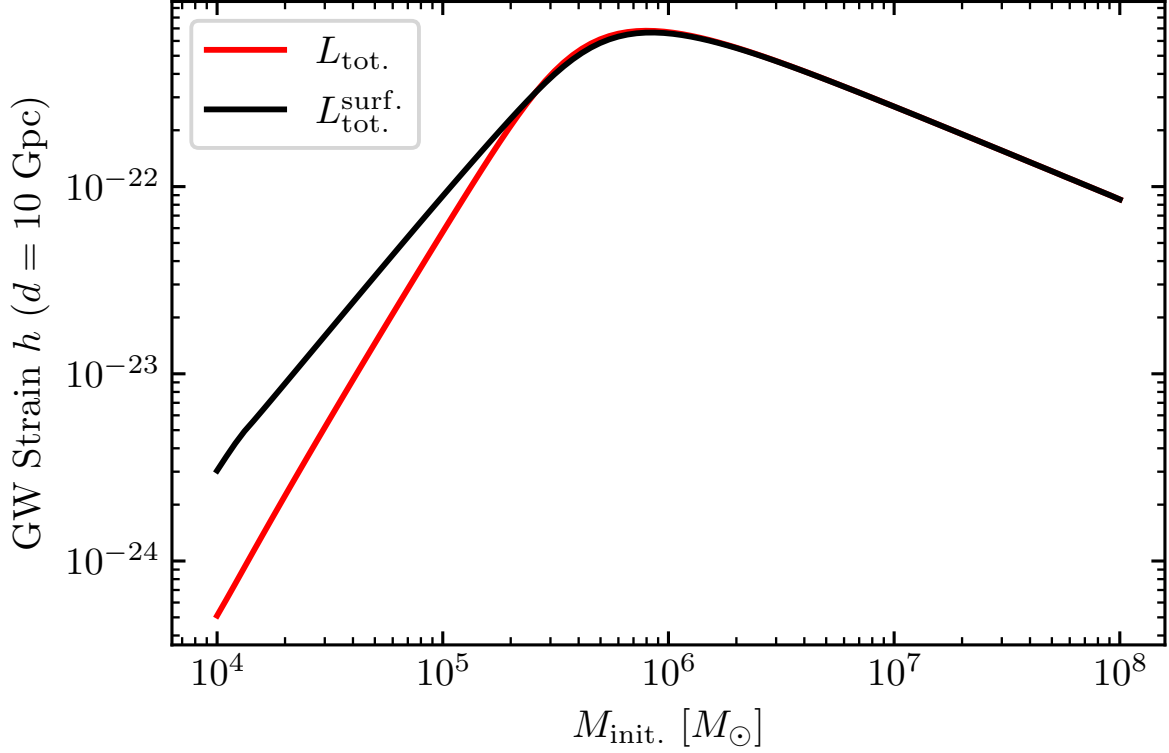


Figure 9: Plots of both the gravitational strain when not including the effects of neutrino scatterings/absorptions (red curve) and when including those effects (black curve). The latter's peak is slightly reduced and slightly shifted to a higher mass, though the strain is increased at lower masses by at most almost a factor of 10 and is in good agreement at higher masses. For the lower mass range, the increase is due to an increase in the homologous core mass as fewer neutrinos are able to escape the SMS to carry away entropy due to considering scattering and absorption. In the higher mass range, the neutrinos produced in a given SMS up to and including the black hole formation density are at too low of an energy to appreciably interact with the surrounding matter, hence the agreement with the model that does not consider these interactions.

Chapter 1

Dark Matter and General Relativistic Instability in Supermassive Stars

We calculate the extent to which collisionless dark matter impacts the stability of supermassive stars ($M \gtrsim 10^4 M_\odot$). We find that, depending on the star's mass, a dark matter content in excess of $\sim 1\%$ by mass throughout the entire star can raise the critical central density for the onset of general relativistic instability, in some cases by orders of magnitude. We consider implications of this effect for the onset of nuclear burning and significant neutrino energy losses.

1.1 Introduction

In this dissertation chapter, we examine how dark matter (DM) could impact the onset of general relativistic instability in massive self-gravitating configurations, i.e., supermassive stars (SMSs), as well as their nuclear and neutrino evolution. SMSs are supported principally by radiation pressure, making them vulnerable to general relativistic instability. If these extreme objects had existed in the early universe, and then collapsed to black holes, they could provide massive seeds that facilitate early production of supermassive black holes (SMBHs).

Indeed, SMBHs seem to be extant at an embarrassingly early epoch in the history of the universe. For example, recent high redshift surveys from the James Webb Space Telescope (JWST) [31] and even the Chandra X-Ray Observatory [10] suggest the existence of compact objects with masses $\gtrsim 10^7 M_\odot$ at redshift $z \sim 10$, when the universe is a scant ~ 400 Myr in age. These observations are consistent with lower redshift observations (e.g. [32, 33, 34, 35, 36, 37]) that also suggest a significant inventory of SMBHs early on. Moreover, these black holes may be growing too quickly to be consistent with building up the observationally-inferred SMBH masses given a $\sim 1\text{--}10 M_\odot$ progenitor black hole, as expected in the standard pictures of stellar evolution and mass accretion.

By contrast, at the end of their lives, SMSs would likely collapse directly to black holes, providing large SMBH “seeds” that would be able to grow to the SMBH sizes observed in the early universe. Such scenarios are favored by some modelers since they do not invoke super-Eddington accretion rates to explain the high masses observed [38, 39]. Ref. [10] reports having observational evidence that supports this argument. The collapse of the SMS itself is potentially observable with future gravitational wave observatories like DECIGO [22]. This gravitational wave signal stems, in part, from an assumed (slightly) non-spherically symmetric neutrino burst generated from the collapsing homologous core [19].

For the purposes of this chapter, an “SMS” is any hydrostatic and self-gravitating gas cloud with a mass in excess of $\sim 10^4 M_\odot$. These are fully convective, high entropy systems whose pressure support is almost entirely derived from radiation (i.e. relativistic particles). They will eventually suffer general relativistic instability [40, 41, 42, 43]. Such configurations, those that in general rely on pressure from relativistic particles, are “trembling on the verge of instability¹,” and are most widely realized in compact scenarios like white dwarfs and neutron stars. In Newtonian gravitation, radiation dominated systems are nearly neutrally buoyant under radial perturbations. If the pressure is entirely

¹Phrase attributed to W.A. Fowler.

derived from radiation, the outward pressure forces always perfectly match the inward gravitational forces as the object is expanded and contracted, meaning such perturbations cost no energy to perform, i.e., a metastable configuration.

The situation changes, however, when first order general relativistic (GR) corrections are included in the analysis. Being non-linear, GR predicts that both stress-energy (primarily baryon rest mass, in this case) *and spacetime curvature itself* cause local spacetime curvature. This makes the metastable system under Newtonian gravity actually marginally unstable under GR. This is the Feynman-Chandrasekhar post-Newtonian instability [13, 14].

Specifically, upon the star’s core exceeding a critical central density, it will become unstable to collapse and unable to establish hydrostatic equilibrium. It will most likely collapse directly into a black hole, but may explode completely if energy derived from nuclear reactions (or some other source) can be added quickly enough to make up for the in-fall kinetic energy of collapse and neutrino energy losses [44]. If the energy source in this scenario indeed stems from nuclear reactions, then an explosion is only possible if the star has sufficient metallicity to burn Hydrogen with the more efficient Carbon-Nitrogen-Oxygen (CNO) cycle instead of through the Weak Interaction limited proton-proton (PPI) chain [43].

While the formation of such SMSs seems dubious at first, especially when considering cloud fragmentation, Ref. [45] argues that tantalizing evidence of such objects may have been observed by JWST. This, however, begs the question as to how they would have been observed in the first place since they are so close to instability. Ref. [46] puts forward the idea that these SMSs could be supported through particle DM self-annihilation into standard model particles which would keep them “puffy” (i.e. lower the central density) and prevent GR instability, at least until they exhaust their DM reserves.

Alternatively, a SMS also could be stabilized by DM without assuming such a self-annihilation cross section [47, 48]. If it is assumed that the DM particles are collisionless

and only interact with the Standard Model component of the star through gravitation, then it is possible to derive how much its presence alone raises the critical central density beyond which the star suffers the Feynman-Chandrasekhar instability.

Non-intuitively, adding more matter to the meta-stable star, specifically a non-interacting non-relativistic fluid, actually tends to stabilize the configuration as opposed to tipping it closer to collapse. The physical mechanism of this increased stability arises from the DM acting as a source of non-relativistic stress.

Absent a significant DM content, heavier ($\gtrsim 10^5 M_\odot$) SMSs will undergo the Feynman-Chandrasekhar instability prior to, or just after Hydrogen burning ignition, meaning they effectively have no main sequence phase (see e.g. Ref. [43]). An interesting question is then whether a sufficient DM content in these objects could delay the onset of instability, allowing a hydrostatic burning phase, and thereby changing the downstream nuclear and entropy evolution of these stars.

It will be shown in what follows that the amount of DM needed to substantially change where this instability occurs is rather extreme. For example, the DM density at the galactic center or in DM halos considered in standard cosmology are small compared to the those considered here. However, the early universe and some proposed DM candidates, e.g. self interacting models, may provide the necessary DM densities in the compact configurations considered below. It has been argued that the DM density can be greatly increased via adiabatic contraction, a purely gravitational process. For example, this has been proposed in works on “Dark Stars,” e.g. in Ref. [49]. We return to these issues in the conclusion of this chapter.

In Section 1.2, we establish the overall framework to analyze stability of SMSs by considering the case without any DM. We determine the critical central density from extremizing the total energy of the configuration with a 1st Order GR correction term to derive the standard result.

In Section 1.3, we examine two kinematic limits of the DM based on its specific

kinetic energy relative to the gravitational potential at the SMS’s surface, determine the total energy added by this DM, and derive how its presence impacts the critical central density as a function of the DM mass fraction. We then present our results for several SMS masses to examine how the effect also depends on the total mass.

In Section 1.4, we approximate how the total fraction of DM changes with radial perturbations of the SMS, providing insight into the physical mechanism for the enhanced stability discussed in the previous section.

In Section 1.5, we determine the extent to which the equilibrium SMS density distribution is impacted by explicitly considering the gravitational influence of the DM. This analysis is performed for several SMS masses and DM mass fractions.

Finally, in Section 1.6, we investigate nuclear burning rates, neutrino emission rates, and their respective luminosities as a fraction of the stars’ Eddington luminosities. This allows us to ascertain how “important” nuclear burning is to an SMS’s evolution and when neutrino emission overpowers it. We also determine by how much these luminosities change with the addition of DM, and what that may mean for the downstream stability and entropy evolution.

1.2 Stability of Supermassive Stars

The analysis in this section will closely follow Ref. [15]’s analytical treatment, specifically their “Approach II” in Section 2.5, to derive the onset of stability for SMSs. The SMSs of concern in this chapter are taken to be spherically symmetric, with primordial metallicity, fully ionized and fully convective. They are high entropy gas clouds supported against gravity almost entirely by radiation pressure $P_{\text{rad.}}$, with a marginal but important contribution from classical Boltzmann gas pressure P_{gas} . We define the parameter β as the

ratio of gas pressure to total pressure (the final approximation following from Ref. [16])

$$\beta = \frac{P_{\text{gas}}}{P_{\text{gas}} + P_{\text{rad.}}} \approx \frac{4.3}{\mu} \left(\frac{M}{M_{\odot}} \right)^{-1/2}, \quad (1.1)$$

where $\mu = 0.59$ is the mean molecular weight (in atomic mass units) per particle for a primordial composition gas.

Given our approximations, the equation of state (EOS) for this configuration can be modeled accurately as a polytrope:

$$P(s, \rho) = K(s)\rho^{\Gamma} = K(s)\rho^{1+1/n}, \quad (1.2)$$

where P is the total pressure, s is the entropy per baryon, ρ is the mass density, and Γ is the so-called polytropic exponent, with the associated quantity n being the polytropic index. For a radiation dominated system, we can safely assume that the radiation field dominates the entropy per baryon $s \approx s_{\text{rad.}} = \frac{4}{3}m_b a T^3 / \rho$, where T is the plasma temperature, and $m_b \approx 939$ MeV is the average free baryon mass. Here, in natural units, $a = \pi^2/15$ is the radiation constant. Since s is constant throughout the fully convective SMS, we can use Eq. (1.1) and $P_{\text{gas}} = \rho T / (\mu m_b)$ and $P_{\text{rad.}} = a T^4 / 3$ to derive

$$s \approx \frac{4}{4.3} \left(\frac{M}{M_{\odot}} \right)^{1/2} - \frac{4}{\mu}, \quad (1.3)$$

where here we express s in units of Boltzmann's constant k_B per baryon. Note that for the SMSs considered here the entropy per baryon is large, e.g., $s \sim 1000$ for $M \sim 10^6 M_{\odot}$. This large entropy implies a large number of photons per baryon, resulting in implications for the $e^{\pm}$ content of the star during its eventual collapse which, in turn, has implications for neutrino emissivity [18, 50, 30].

Using $P(s, \rho) = P_{\text{gas}} + P_{\text{rad}}$ and $n = 3$, $K(s)$ is found to be

$$K(s) = \frac{1}{1-\beta} \frac{a}{3} \left(\frac{3}{4a} \frac{s}{m_b} \right)^{4/3} \approx \frac{0.2843}{1-\beta} \left(\frac{s}{1000} \right)^{4/3} \text{ MeV}^{-4/3}. \quad (1.4)$$

For the stars discussed in this chapter, their EOS and structure are well approximated by an $n = 3$ ($\Gamma = 4/3$) polytropic model.

Since these stars are fully convective, we can take their entropy per baryon to be constant throughout. With constant entropy, the polytropic exponent $\Gamma = 1 + 1/n$ is equivalent to Chandrasekhar's adiabatic index Γ_1 , defined as [51]

$$\Gamma_1 \equiv \left(\frac{\partial \log P}{\partial \log \rho} \right) \Big|_s. \quad (1.5)$$

In the purely Newtonian gravitation case, the frequency of oscillations in response to radial perturbations is proportional to $(\langle \Gamma_1 \rangle_P - 4/3)^{1/2}$, where the pressure-averaged value of Γ_1 throughout the star is $\langle \Gamma_1 \rangle_P$. Therefore, in order for oscillations to have a real frequency and thereby stable against perturbations, $\langle \Gamma_1 \rangle_P$ must be strictly greater than $4/3$. When including 1st Order GR corrections, however, this condition of stability changes to

$$\langle \Gamma_1 \rangle_P > \frac{4}{3} + \mathcal{O} \left(\frac{R_S}{R} \right), \quad (1.6)$$

where R is the radius of the star and $R_S = 2GM$ is the star's Schwarzschild radius. For the types of stars considered in this chapter, this correction is typically no more than one part in ten-thousand.

As the star quasistatically contracts, however, $\langle \Gamma_1 \rangle_P$ asymptotically approaches $4/3$ from above, meaning that the adiabatic index will eventually fall below this threshold and make the whole configuration unstable. This point highlights a striking fact about the nature of these stars: while the internal structure is accurately determined by just Newtonian gravitation, their stability is entirely determined by GR.

To quantify this instability point, we can extremize the total energy of the star. As a function of the central density ρ_c , the total energy E of this star is [11]

$$E(\rho_c) = k_1 MK \rho_c^{1/n} - k_2 GM^{5/3} \rho_c^{1/3} - k_4 G^2 M^{7/3} \rho_c^{2/3}, \quad (1.7)$$

where $k_1 \approx 1.7558$, $k_2 \approx 0.6390$, and $k_4 \approx 0.9183$ are structure coefficients calculated for an $n = 3$ polytrope (see Ref. [11] for derivation). The first two terms of Eq. (1.7) are the standard internal gas energy and Newtonian gravitational binding energy, respectively, and the final term represents the non-linear gravitational self coupling term arising from 1st Order GR corrections. A convenient dimensionless re-scaling follows from performing the transformations

$$\tilde{E} = \frac{G^{3/2}}{K^{\tilde{n}/2}} E, \quad (1.8a)$$

$$\tilde{M} = \frac{G^{3/2}}{K^{\tilde{n}/2}} M, \quad (1.8b)$$

$$x = GM^{2/3} \rho_c^{1/3}, \quad (1.8c)$$

where here $\tilde{n} = 1/(\Gamma_1 - 1)$ and we take the standard adiabatic index for a radiation pressure dominated plasma to be [16]:

$$\Gamma_1 \approx \frac{4}{3} + \frac{\beta}{6}. \quad (1.9)$$

We assume Γ_1 is constant throughout the SMS since it is fully convective. Substituting the above into Eq. (1.7) gives

$$\tilde{E} = k_1 \tilde{M}^{1/3-\beta/3} x^{1+\beta/2} - k_2 \tilde{M} x - k_4 \tilde{M} x^2. \quad (1.10)$$

The extremization follows on finding the first and second derivatives of Eq. (1.10)

with respect to x , setting them both to zero, and solving simultaneously. The first derivative vanishing determines the point of hydrostatic equilibrium. The second derivative also vanishing determines where this hydrostatic configuration changes from stable to unstable. If we ignore the correction from GR and take the $\beta \rightarrow 0$ limit for the first derivative, we find the standard Newtonian rescaled mass for an $n = 3$ polytrope:

$$\tilde{M} = \left(\frac{k_1}{k_2}\right)^{3/2} \approx 4.5547, \quad (1.11)$$

which is independent of x (and hence the central density) as expected for an $n = 3$ polytrope. In other words, a star with Newtonian gravitation and supported entirely by radiation (particles with relativistic kinematics) has zero total energy, and hence there is no energy cost for expanding or contracting this configuration. This self gravitating configuration is therefore very close to instability.

Such metastability means that even very small effects can have a dramatic effect on stability to push a configuration over the edge. As we have argued above, GR gives a negligible effect on the structure of the star, i.e., the run of pressure with density. However, the inclusion of a very simple 1st-Order correction from GR, namely one that captures the non-linearity of gravitation through self-coupling, can make stable configurations in Newtonian gravitation unstable.

Including general relativistic corrections in this way and simultaneously solving the two extremization equations, leads to a critical value of x where a hydrostatic configuration changes from stable to unstable:

$$x_{\text{crit.}} = \frac{k_2}{4k_4}\beta + \mathcal{O}(\beta^2). \quad (1.12)$$

This corresponds to a critical central density of

$$\rho_{\text{c,crit.}}^0 \approx \left(\frac{k_2}{4k_4}\right)^3 \frac{\beta^3}{G^3 M^2} \approx 3.98 \left(\frac{0.59}{\mu}\right)^3 \left(\frac{10^5 M_\odot}{M}\right)^{7/2} \text{ g cm}^{-3} \quad (1.13)$$

where we have used Eq. (1.1) to relate β to the star's composition and mass. The physical interpretation of Eq. (1.13) has two parts. First, note that the rest mass of an SMS stems almost entirely from baryons, whereas the pressure is mostly from radiation. Increasing the mean molecular weight μ by fusing baryons in composite nuclei has the effect of lowering the gas pressure contribution to the total pressure. That, in turn, decreases the critical density for the onset of instability. Secondly, increasing the mass M of the SMS requires a higher pressure for hydrostatic equilibrium, implying a higher temperature and a higher pressure contribution from radiation. Hence, increasing mass also serves to push the critical central density lower.

1.3 Collisionless Dark Matter

We will now consider the effects of including a cloud of collisionless DM that permeates the entire star. Assuming standard spherical accretion, we expect the density profile of this DM cloud to be given by [11]

$$\rho_{\text{DM}}(r) = \rho_{\text{DM}}^\infty \left(1 + \frac{2GM(r)}{v_\infty^2 r}\right)^{1/2} \quad (1.14)$$

where ρ_∞ and v_∞ are the typical DM density and DM particle velocities far from any gravitational sources, respectively, and $M(r)$ is the total mass enclosed in a sphere of radius r .

The final term in the above equation represents the ratio of the gravitational potential of a DM particle to its specific kinetic energy. It permits two natural limits: $\frac{1}{2}v_\infty^2 \gg GM(r)/r$ and $\frac{1}{2}v_\infty^2 \ll GM(r)/r$ which, following Ref. [15]'s convention, we will

call “hot” and “cold” DM, respectively. It should be noted that this nomenclature is not to be confused with the cosmological notion of “Hot Dark Matter” and “Cold Dark Matter,” which refers to the kinematics of DM at its decoupling epoch and how these kinematics affect early structure formation. Here, we will assume that both our “hot” and “cold” limits refer to DM particles with non-relativistic kinematics.

1.3.1 “Hot” Dark Matter

In the “hot” DM limit, Eq. (1.14) reduces to the simple relation

$$\rho_{\text{DM}}(r) = \rho_{\text{DM}}^{\infty} \equiv \rho_{\text{DM}}, \quad (1.15)$$

i.e. a constant density fluid to first order. The energy added to the system by the DM is calculated by first determining the gravitational potential created by the fluid. From Poisson’s Equation, this DM has the harmonic potential

$$\Phi_{\text{HDM}}(r) = \frac{2\pi}{3} G \rho_{\text{DM}} r^2 + C, \quad (1.16)$$

where we have assumed regularity at the origin and C is a constant of integration. To calculate the total gravitational energy contributed by this fluid, we use

$$E_{\text{HDM}} = \int dm \Phi_{\text{HDM}} = \frac{8\pi^2}{3} G \rho_{\text{DM}} \int_0^R \rho(r) r^4 dr + CM. \quad (1.17)$$

We will assume that M , dm , and $\rho(r)$ are dominated by the baryon rest mass. Note that $\rho(r)$ very closely follows the profile of an $n = 3$ polytrope. We will extremize the total energy while keeping the mass M fixed, so the final term in Eq. (1.17) will be ignored. The first term in Eq. (1.17) can be evaluated by employing the well known solutions to

the Lane-Emden Equation of order 3, yielding

$$E_{\text{HDM}} = k_{\text{HDM}} M K \rho_{\text{DM}} \rho_{\text{c}}^{-2/3}, \quad (1.18)$$

where $k_{\text{HDM}} = 3.5845$ is the structure coefficient for the included DM and K is the same from Eq. (1.2). Following the same transformation in Eqs. (1.8a)-(1.8c), the total energy now reads

$$\tilde{E} = k_1 \tilde{M}^{1/3-\beta/3} x^{1+\beta/2} - k_2 \tilde{M} x - k_4 \tilde{M} x^2 + k_{\text{HDM}} \tilde{M}^{1/3} x_{\text{DM}}^3 x^{-2}, \quad (1.19)$$

where $x_{\text{DM}}^3 = G^3 M^2 \rho_{\text{DM}}$. Extremization of this yields the following set of equations to be solved simultaneously after expanding in powers of the ratio of gas pressure to total pressure β :

$$0 = \frac{k_1}{\tilde{M}^{2/3}} \left(1 - 2 \frac{k_{\text{HDM}}}{k_1} \frac{x_{\text{DM}}^3}{x_{\text{crit.}}^3} \right) - k_2 - 2k_4 x_{\text{crit.}} + \mathcal{O}(\beta), \quad (1.20a)$$

$$0 = -2k_4 + \frac{k_1}{2\tilde{M}^{2/3}} \frac{\beta}{x_{\text{crit.}}} \left(1 + 12 \frac{k_{\text{HDM}}}{k_1 \beta} \frac{x_{\text{DM}}^3}{x_{\text{crit.}}^3} \right) + \mathcal{O}(\beta^2). \quad (1.20b)$$

After identifying that the ratio x_{DM}^3/x^3 is related to the DM mass fraction f via

$$\frac{x_{\text{DM}}^3}{x_{\text{crit.}}^3} = \frac{\rho_{\text{DM}}}{\rho_{\text{c, crit.}}} = \frac{\rho_{\text{DM}}}{C_3 \rho_{\text{avg.}}} = \frac{M_{\text{DM}}}{C_3 M} = \frac{f}{C_3}, \quad (1.21)$$

where $C_3 \approx 54.18$ is the core-to-average-density ratio (a.k.a “condensation parameter”) for a $n = 3$ polytrope, we find that

$$\rho_{\text{c, crit.}} \approx \frac{1}{G^3 M^2} \left(\frac{k_2}{4k_4} \beta + \frac{3k_1 k_{\text{HDM}}}{k_2 k_4} \frac{f}{C_3} \right)^3, \quad (1.22)$$

which for typical values of μ , M , and f is

$$\rho_{\text{c, crit.}}^{\text{HDM}} \approx \rho_{\text{c, crit.}}^0 \left[1 + 0.197 \left(\frac{\mu}{0.59} \right) \left(\frac{M}{10^5 M_\odot} \right)^{1/2} \left(\frac{f}{0.01} \right) \right]^3, \quad (1.23)$$

where $\rho_{\text{c, crit.}}^0$ is the critical central density without DM given in Eq. (1.13). Here, we have ignored $\mathcal{O}(\beta^4)$ and $\mathcal{O}(f^4)$ terms. It should be noted that all higher order terms in f have positive coefficients, so Eq. (1.23) give a slight underestimate to the true increase in $\rho_{\text{c, crit.}}$ for larger f . Eq. (1.23) recovers the standard case when $f \rightarrow 0$, as expected.

In Figure 1.1, we show the critical central density $\rho_{\text{c, crit.}}$ at the onset of GR instability as a function of DM mass fraction f in the hot DM case (dashed lines) for various SMS masses between 10^4 and $10^8 M_\odot$. While the relative increase is not too significant for smaller mass SMSs, the effect is quite strong for larger masses (e.g. $\gtrsim 10^5 M_\odot$), increasing $\rho_{\text{c, crit.}}$ by as much as 4 orders of magnitude at $f \sim 20\%$.

1.3.2 “Cold” Dark Matter

The alternative limit of Eq. (1.14) reduces to the profile

$$\rho_{\text{DM}}(r) = \frac{\rho_{\text{DM}}^\infty}{v_\infty} \sqrt{\frac{GM}{r}}, \quad (1.24)$$

where we have also made the same approximation as Ref. [15] that since an $n = 3$ polytrope is so centrally condensed, it is reasonable to treat the SMS as a point source of mass M . Without this assumption, the DM density profile would feature a flattened core as opposed to the above divergent cusp at the origin. Such a flat shape would be closer to the limit described in Section 1.3.1, so the ‘cuspy’ profile offers sufficiently disparate extreme from the “hot” DM case to allow us to explore the full range of DM effects we would expect to see.

The mass fraction of this DM is found by integrating the density throughout the

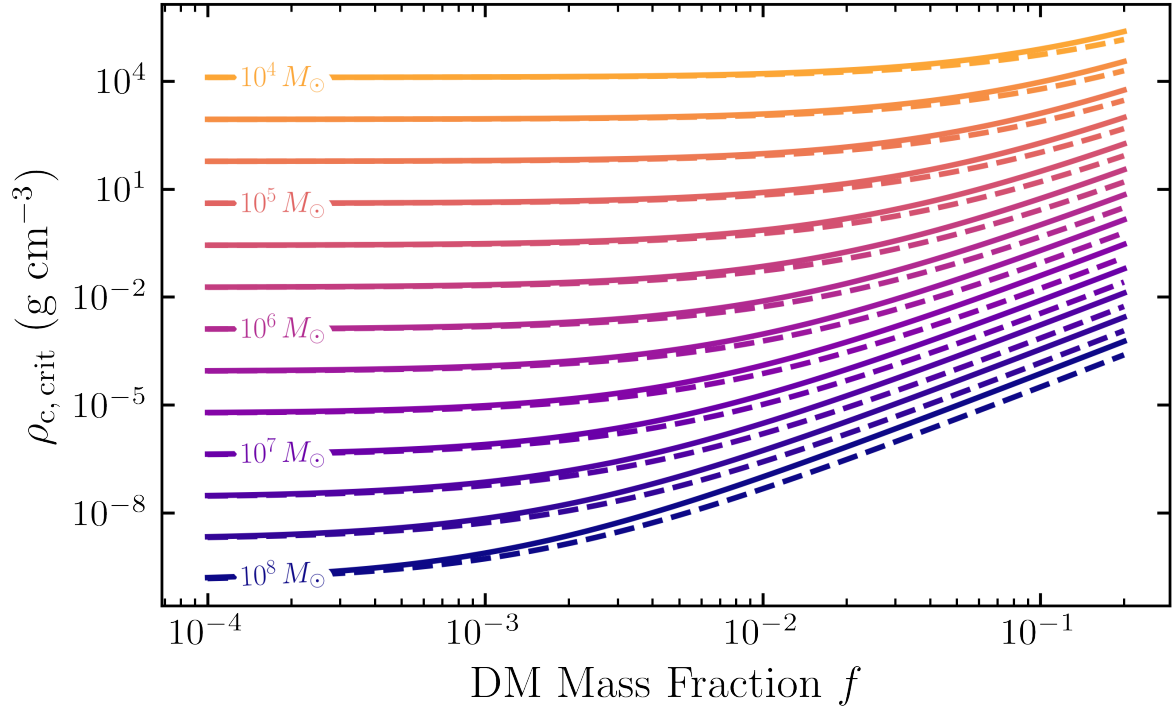


Figure 1.1: Critical central density for the onset of general relativistic instability, $\rho_{c, \text{crit}}$, for SMSs vs. total DM mass fraction f for different SMS masses. Dashed lines are the “hot” DM limit and solid lines are the “cold” DM limit (Eqs. (1.23) and (1.32), respectively).

entire star and dividing by the mass:

$$\begin{aligned}
f = \frac{M_{\text{DM}}}{M} &= 4\pi \sqrt{\frac{2G}{M}} \frac{\rho_{\text{DM}}}{v_{\infty}} \int_0^R dr r^{3/2}, \\
&= \frac{8\pi\sqrt{2}}{5} \sqrt{\frac{GR^5}{M}} \frac{\rho_{\text{DM}}}{v_{\infty}}, \\
f &= C_{\text{CDM}} \frac{x_{\text{DM}}^3}{x^{5/2}},
\end{aligned} \tag{1.25}$$

where $C_{\text{CDM}} \approx 60.0151$ is the “cold” DM equivalent to the condensation parameter in the previous section and is calculated in part from solutions to the Lane-Emden equation of order 3, x is the reparameterized central baryonic density given by Eq. (1.8c), and x_{DM} is defined as

$$x_{\text{DM}}^3 = G^3 M^2 \frac{\rho_{\text{DM}}^{\infty}}{v_{\infty}}. \tag{1.26}$$

This definition is similar to x_{DM} of the previous section with the inclusion of v_{∞} , the typical DM particle velocity far from the SMS. Since the only the ratio $\rho_{\text{DM}}^{\infty}/v_{\infty}$ appears in our calculations, both have been collapsed into the single quantity x_{DM} .

Solving Poisson’s Equation again for the gravitational potential caused by this “cold” DM, we find that

$$\Phi_{\text{CDM}}(r) = \frac{16\pi\sqrt{2}}{15} G^{3/2} M^{1/2} \frac{\rho_{\text{DM}}^{\infty}}{v_{\infty}} r^{3/2}, \tag{1.27}$$

where we again assume regularity at the origin and have ignored the constant of integration for the same reason as the previous section. The total gravitational energy is then calculated to be

$$E_{\text{CDM}} = \int dm \Phi_{\text{CDM}} = k_{\text{CDM}} M K^{3/2} \frac{\rho_{\text{DM}}^{\infty}}{v_{\infty}} \rho_{\text{c}}^{-1/2} \tag{1.28}$$

where $k_{\text{CDM}} \approx 14.066$ is calculated following a similar process to the previous section. The

rescaled energy per Eqs. (1.8a)-(1.8c) and Eq. (1.26) is then

$$\tilde{E}_{\text{CDM}} = k_{\text{CDM}} x_{\text{DM}}^3 x^{-3/2}, \quad (1.29)$$

so the total rescaled energy is

$$\tilde{E} = k_1 \tilde{M}^{1/3-\beta/3} x^{1+\beta/2} - k_2 \tilde{M} x - k_4 \tilde{M} x^2 + k_{\text{CDM}} x_{\text{DM}}^3 x^{-3/2}. \quad (1.30)$$

Following the same process as the “hot” DM case with the mass fraction in Eq. (1.25), we find that

$$\rho_{\text{c, crit.}} \approx \frac{1}{G^3 M^2} \left(\frac{k_2}{4k_4} \beta + \frac{15k_2 k_{\text{CDM}}}{8\tilde{M}^{1/3} k_1 k_4} \frac{f}{C_{\text{CDM}}} \right)^3, \quad (1.31)$$

where $\tilde{M} = 4.5547$ (per Eq. (1.11)) and we have again ignored 4th order terms in β and f . Again, for typical values of μ , M , and f , this reduces to

$$\rho_{\text{c, crit.}}^{\text{CDM}} \approx \rho_{\text{c, crit.}}^0 \left[1 + 0.262 \left(\frac{\mu}{0.59} \right) \left(\frac{M}{10^5 M_{\odot}} \right)^{1/2} \left(\frac{f}{0.01} \right) \right]^3, \quad (1.32)$$

which has a slightly stronger dependence on the DM mass fraction than the “hot” DM case, as evidenced in Fig. 1.1.

1.4 DM Response to SMS Contraction

As the star quasistatically contracts, we would like to quantify how much the mass fraction of DM within the star changes. If the fraction of DM decreases for a hypothetical decrease in radius (enclosing the baryons), the SMS would be enclosing less total mass. That may help stabilize the configuration. Alternatively, or perhaps in addition to this effect, if SMS contraction serves to increase the velocity dispersion of the DM, the effective pressure provided by the dispersion would increase and also stabilize the configuration.

From Eqs. (1.21) and (1.25), the fractional change in DM fraction f is

$$\frac{\delta f}{f} = \frac{\delta \rho_{\text{DM}}}{\rho_{\text{DM}}} - \kappa \frac{\delta \rho_c}{\rho_c}, \quad (1.33)$$

where $\kappa = 1$ or $5/6$ for the “hot” and “cold” limits, respectively.

The largest potential deviation in DM density from its background value would occur where $M(r)/r$ achieves its maximum value, which is at $r = R$, so we will consider that region and small changes in the SMS’s radius $\delta R < 0$ where $\delta R/R \ll 1$. Since $\rho_c \sim M/R^3$,

$$\frac{\delta \rho_c}{\rho_c} = -3 \frac{\delta R}{R} \quad (1.34)$$

for a fixed mass, and from Eq. (1.14) we find that

$$\frac{\delta \rho_{\text{DM}}}{\rho_{\text{DM}}} = -\frac{GM}{v_\infty^2 R} \left(1 + \frac{2GM}{v_\infty^2 R}\right)^{-1} \frac{\delta R}{R}. \quad (1.35)$$

We will now consider both the “hot” and “cold” DM limits to calculate this fractional change.

1.4.1 “Hot” Dark Matter Case

To first order in $GM/(v_\infty^2 R)$, Eq. (1.35) reduces to

$$\frac{\delta \rho_{\text{DM}}}{\rho_{\text{DM}}} \approx -\frac{GM}{v_\infty^2 R} \frac{\delta R}{R}, \quad (1.36)$$

so

$$\frac{\delta f}{f} \approx 3 \left(1 - \frac{GM}{3v_\infty^2 R}\right) \frac{\delta R}{R}. \quad (1.37)$$

Therefore, the total mass fraction of DM would tend to decrease since $\delta R < 0$, unless $GM/(6R) > \frac{1}{2}v_\infty^2$ which is the opposite of the limit we are considering, so this can be

further reduced to

$$\left(\frac{\delta f}{f}\right)_{\text{HDM}} \approx 3 \frac{\delta R}{R}. \quad (1.38)$$

1.4.2 “Cold” Dark Matter Case

If $GM/(v_\infty^2 R) \gg 1$, then Eq. (1.35) becomes simply

$$\frac{\delta \rho_{DM}}{\rho_{DM}} \approx -\frac{1}{2} \frac{\delta R}{R}, \quad (1.39)$$

so

$$\left(\frac{\delta f}{f}\right)_{\text{CDM}} \approx 2 \frac{\delta R}{R}, \quad (1.40)$$

which is again negative for $\delta R < 0$, indicating that the total DM mass fraction also decreases in this limit but by a less severe amount compared to the “hot” DM case.

1.4.3 Physical Origin of Enhanced Stability

The stability of a purely standard model star (i.e., composed of baryons, electrons, photons, etc.) is determined by its pressure-averaged adiabatic index as given in Eq. (1.5). A heuristic physical picture for this effective “spring constant” for the star’s material is as follows: Squeeze the star radially and rapidly (so rapidly that heat does not flow and so entropy remains constant) to higher density and ask how much the pressure increases – i.e., how much does the material “push back” when quickly perturbed to higher density?

Of course, when the pressure support for a self-gravitating configuration stems from particles with relativistic kinematics (e.g., photons in the SMS case), that configuration is close to instability. For example, this case would correspond to a zero total energy configuration in purely Newtonian gravitation, therefore pressure forces will grow in lock-step with the increased gravitational force from compressing the star. This means the configuration is in a meta-stable state as it costs no energy to expand or contract the star,

making it ripe for instability once the non-linearity of GR comes into play.

When collisionless DM is added to this picture there are several effects to consider. We envision that the standard model portion of the star, contained inside a radius R , is embedded in an extended halo of DM. When the star is radially and adiabatically squeezed to higher baryon density there are two questions that arise: (1) What is the effect of reducing R in the limit where the DM distribution in space is fixed so that less DM is enclosed inside the SMS radius?; and (2) If the standard model star is squeezed to higher density, what is the response of the DM spatial distribution and how does this affect stability?

To the first issue, as discussed previously, enclosing less total mass as the SMS is squeezed is a contributing factor in the enhanced stability afforded by DM. Indeed, as evidenced in Eqs. (1.37) and (1.40), the total DM mass fraction decreases in both DM kinematic limits. Curiously, this total mass decrement effect is less pronounced for the “cold” case despite it producing a stronger stabilizing effect (c.f. the coefficients in the correction terms in Eqs. (1.32) and (1.23) and Figure 1.1). This indicates that DM induced stabilization is not only due to enclosing less *total* mass.

As for the second issue, the gravitational response of the DM spatial distribution to a perturbation in the standard model star is second order, so it is not accounted for in our calculations. The “cold” DM case possesses a more centrally condensed DM distribution, which changes the outcome of the extremization calculation as compared to the “hot” case. In the end, adding non-relativistic stress from DM to the supermassive star in general, and doing so in a way that provides a nearly fixed background, is inherently stabilizing.

We note that the increased stability stemming from a fixed DM distribution inside our SMS may have similarity with the enhanced stability found in giant-planet envelope models with a fixed, incompressible core [52]. In both the giant-planet envelope and our SMS cases there is little or no change in the background gravitational potential of the core in the former case, or the DM envelope in the latter.

1.5 Equilibrium Structure of Star with Dark Matter

Let us consider the “hot” dark matter case where the density profile follows a flat distribution. Considering that SMSs are well approximated by the structure of an $n = 3$ polytrope, they are highly centrally condensed relative to their average density. For any non-zero DM mass fraction f , there should be a radius $r_{\text{eq.}} < R$ beyond which the DM density is greater than the standard model matter density. Since the pressure gradient of the SMS depends on the local gravitational field strength, we should expect that around and beyond this radius the SMS structure may be altered by this extra gravitation source.

Specifically, if we assume the DM provides no direct pressure support but does interact with the SMS gravitationally, the pressure gradient at a given radius r is

$$\frac{dP_0}{dr} = -\frac{G(m_0 + m_{\text{DM}})(\rho_0 + \rho_{\text{DM}})}{r^2}, \quad (1.41)$$

where P_0 , m_0 , and ρ_0 are the standard model pressure, mass enclosed, and density, respectively, and m_{DM} and ρ_{DM} are the DM mass enclosed and density, respectively. Both m_0 and m_{DM} have the following continuity conditions:

$$\frac{dm_0}{dr} = 4\pi r^2 \rho_0, \quad (1.42)$$

$$\frac{dm_{\text{DM}}}{dr} = 4\pi r^2 \rho_{\text{DM}}. \quad (1.43)$$

With these three equations, Eq. (1.41) can be manipulated to derive

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho_0 + \rho_{\text{DM}}} \frac{dP_0}{dr} \right) = -4\pi G(\rho_0 + \rho_{\text{DM}}). \quad (1.44)$$

Let us now assume that the standard model density is parameterized by its central density ρ_{0c} and a function of radius θ via $\rho_0 = \rho_{0c}\theta^n$, where n is the polytropic index. Plugging this into Eq. (1.2) yields $P_0 = K\rho_{0c}^{1+1/n}\theta^{n+1}$. Collecting these relations together in Eq. (1.44)

gives

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\theta^n + \frac{f}{C_3}} \alpha_{\text{LE}}^2 \theta^n \frac{d\theta}{dr} \right) = - \left(\theta^n + \frac{f}{C_3} \right), \quad (1.45)$$

where $\alpha_{\text{LE}}^2 = (n+1)K\rho_{0c}^{1/n-1}/(4\pi G)$ is the square of the standard Lane-Emden length scale and f/C_3 is the DM mass fraction scaled by the central condensation as per Eq. (1.21), specifically for a flat DM distribution.

Scaling the radius r via $r = \alpha_{\text{LE}}\xi$, we arrive at a modified Lane-Emden equation which reads

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\frac{\xi^2}{\theta^n + \frac{f}{C_3}} \theta^n \frac{d\theta}{d\xi} \right) = - \left(\theta^n + \frac{f}{C_3} \right), \quad (1.46)$$

which reduces to the standard Lane-Emden equation for $f \rightarrow 0$. We can solve this equation numerically for different values of f , as presented in Figure 1.2 for $n = 3$. The solutions agree with the standard (no DM) case for most of the x-axis range of the figure, but begin to diverge radically at roughly where the standard model and DM densities are comparable.

Using these functions, the altered values of the structure coefficients k_1 , k_2 , k_4 , and k_{HDM} are tabulated as a function of DM mass fraction f in Table 1.1. These were calculated following the prescriptions in Chapter 6 of Ref. [11]. In Figure 1.3, the relative change in the critical central density for several SMS masses and DM mass fractions is visualized. For all SMS masses and DM mass fractions, including the effect of the altered equilibrium distribution of the standard model matter lowers the critical central density slightly from Eq. (1.23).

Clearly, considering the change in the SMS equilibrium density profile only confers significant changes in the critical central density for DM mass fractions f in the larger end of the range we are considering ($\gtrsim 1\%$).

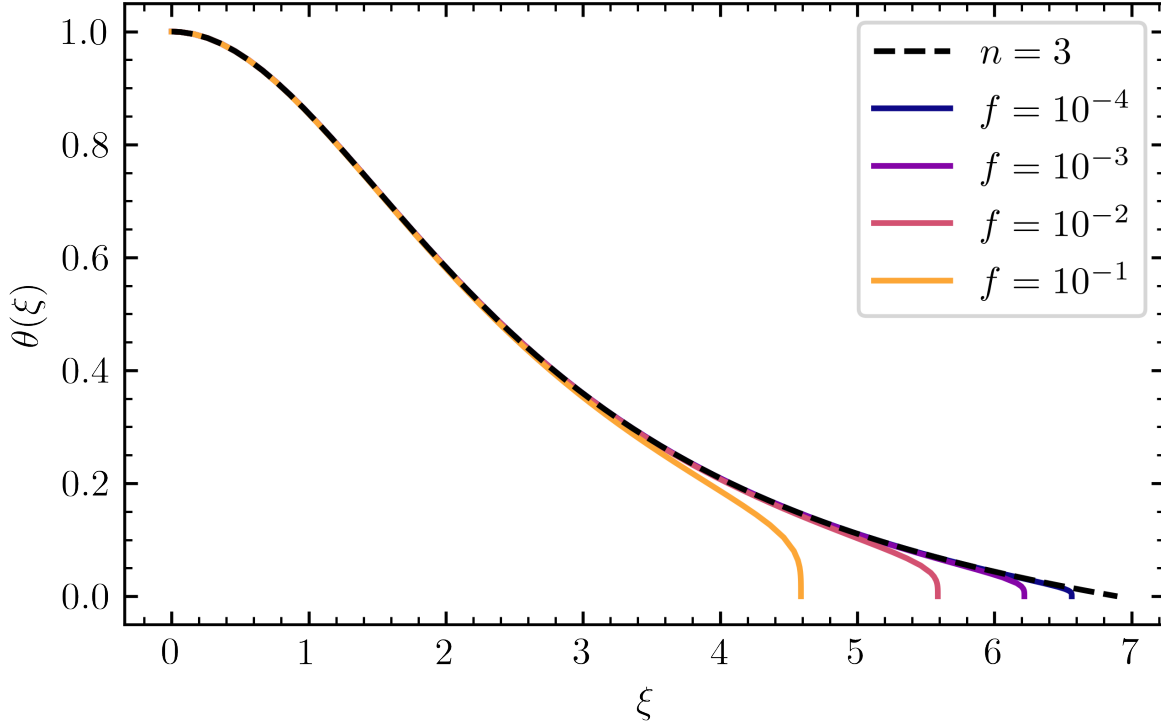


Figure 1.2: Solutions to Eq. (1.46) of the function θ for different DM mass fractions f . The actual physical density is $\rho_0(r) = \rho_{c0}\theta^3(\xi)$, where $\xi = r/\alpha_{\text{LE}}$.

Table 1.1: Tabulated structure coefficients determined upon integrating the solutions to Eq. (1.46). The “ $n = 3$ ” column corresponds to the standard Lane-Emden solution (i.e. $f = 0$), and the following columns are the values calculated for several chosen and increasing DM mass fractions f .

	$n = 3$	$f = 10^{-4}$	$f = 10^{-3}$	$f = 10^{-2}$	$f = 10^{-1}$
k_1	1.75579	1.75595	1.75714	1.76558	1.80913
k_2	0.63900	0.63911	0.63999	0.64650	0.68533
k_4	0.91829	0.91853	0.92044	0.93485	1.02386
k_{HDM}	3.58450	3.58409	3.58044	3.55002	3.37575

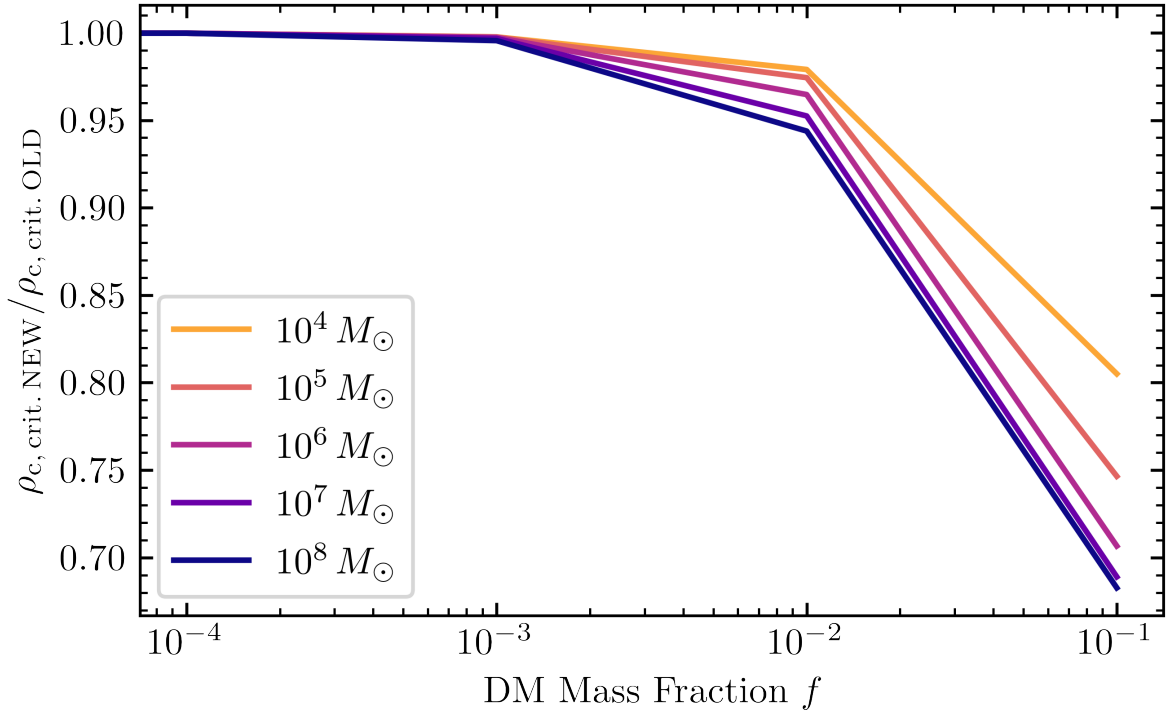


Figure 1.3: As a function of the DM mass fraction f , the relative change in the critical central density is plotted. Here, the quantity $\rho_{c, \text{crit. NEW}} / \rho_{c, \text{crit. OLD}}$ is the ratio of the critical central densities considering the change in the equilibrium density profile of the SMS *vs.* not considering such a change.

1.6 Nuclear Burning and Neutrino Losses

It should be expected that, at least for some masses in the range $\sim 10^4\text{--}10^8 M_\odot$, nuclear (hydrogen) burning on the PPI chain is taking place at a high enough rate to significantly contribute to the total energy budget of the SMS. This rate depends on both the density and temperature throughout the star, both of which decrease as the mass increases. Specifically, *at the onset of collapse* where the density and temperature are highest, $\rho_c \sim M^{-7/2}$ and $T_c \sim M^{-1}$, the latter following from the definition of an $n = 3$ polytrope. Since including DM can raise the critical central density (and hence the temperature) required before collapse, we can also investigate how the DM mass fraction impacts the energy released by the PPI chain throughout the star. Ref. [53] investigates Helium burning in SMSs, which could be relevant for SMSs light enough to have a stable Hydrogen burning phase. The following calculations in this section are determined for the non-altered SMS density distribution, *contra* the previous section’s treatment, for simplicity.

While significant nuclear burning would potentially lead the star through a stable “main-sequence” phase of evolution, neutrino emission from the high entropy plasma could carry that extra energy generation away from the core, providing no direct pressure support and hence adding no more stability to the configuration. At such high entropies and temperatures, the neutrino luminosity may outshine that of nuclear burning completely, causing the star to contract more quickly and leading it closer to GR instability.

1.6.1 PPI Chain Nuclear Burning

For zero metallicity, protons p can only fuse to Helium nuclei α via the PPI chain $4p \rightarrow \alpha + 2e^+ + 2\nu_e$, where e^+ indicates positrons and ν_e indicates electron neutrinos. In total, each such reaction releases about 26 MeV of energy to the plasma while the neutrinos carry away about 0.5 MeV away from the star as the likelihood they will transfer energy

to the plasma is vanishingly small [12]. The energy generation rate for this reaction is calculated to be (in $\text{erg cm}^{-3} \text{s}^{-1}$)

$$Q_{\text{PPI}} = c_1 \rho^2 X_{\text{H}}^2 T_6^{-2/3} \exp\left(-c_2 T_6^{-1/3}\right) \left(1 + c_3 T_6^{1/3} + c_4 T_6^{2/3} + c_5 T_6\right), \quad (1.47)$$

where ρ is the density in g cm^{-3} , $X_{\text{H}} \approx 0.75$ is the total hydrogen mass fraction, $T_6 = T/(10^6 \text{ K})$ is the temperature in units of 10^6 Kelvins, and $c_1 = 2.319 \times 10^6$, $c_2 = 33.81$, $c_3 = 0.0123$, $c_4 = 0.0109$, $c_5 = 0.00095$ are numerical constants (see Ref. [12] for derivation).

To determine if the PPI chain provides a significant contribution to the energy budget of the star, we will compare the integrated luminosity at the surface, i.e.

$$L_{\text{PPI}} = \int_0^R Q_{\text{PPI}} 4\pi r^2 dr \quad (1.48)$$

to the Eddington luminosity, which is approximately [11]

$$L_{\text{Edd.}} \approx 1.3 \times 10^{38} \left(\frac{M}{M_{\odot}}\right) \text{ erg s}^{-1}, \quad (1.49)$$

which comes from considering at what photon luminosity momentum transfer via photon-electron Thomson scattering balances the gravitational force.

1.6.2 Neutrino Losses

The high entropy per baryon ($s \gtrsim 1000$) required for hydrostatic equilibrium in SMSs implies a significant $e^{\pm}$ pair content in electromagnetic equilibrium $e^+ + e^- \leftrightarrow 2\gamma$. These electrons and positrons can both annihilate and interact with the local radiation

field to produce neutrinos via [18, 50, 54]:

$$e^- + e^+ \rightarrow \nu + \bar{\nu} \quad (\text{pair production})$$

$$\gamma + e^\pm \rightarrow e^\pm + \nu + \bar{\nu} \quad (\text{photo-emission})$$

where γ represents an incident photon and ν ($\bar{\nu}$) refers to (anti-)neutrinos of any flavor. These processes are expected to be the dominant source of neutrino emissivity in the low density and low-to-moderate temperature regimes in SMSs. There is also a process where plasmons ($\tilde{\gamma}$) decay to neutrino pairs ($\tilde{\gamma} \rightarrow \nu + \bar{\nu}$), but this only dominates in high density and temperature regimes.

If the electrons and positrons are non-relativistic and non-degenerate, the energy density loss rates for the two dominant neutrino production processes are calculated to be [12, 18] (in $\text{erg cm}^{-3} \text{s}^{-1}$):

$$Q_{\text{pair}} \approx (4.9 \times 10^{18}) T_9^3 \exp\left(-\frac{11.86}{T_9}\right), \quad (1.50)$$

$$Q_{\text{photo.}} \approx (0.98 \times 10^8) \frac{\rho}{\mu} T_9^8, \quad (1.51)$$

where ρ is the mass density in g cm^{-3} , μ is the mean molecular weight, and $T_9 = T/(10^9 \text{ K})$ is the temperature in units of 10^9 Kelvins. While the pair production mechanism has a much higher prefactor than the photo-emission process, the former suffers an exponential suppression for low temperatures that the latter channel does not, actually making that channel dominant in higher mass SMSs.

Using a similar process as in the previous section, we calculate the combined luminosity from these neutrino emission processes to be

$$L_{\nu, \text{Tot.}} = \int_0^R (Q_{\text{pair}} + Q_{\text{photo.}}) 4\pi r^2 dr. \quad (1.52)$$

While the PPI chain does emit two ν_e per reaction, the energy they carry away is far smaller than the energy released from fusing four protons into one α particle, so we will ignore that contribution from the total neutrino luminosity.

To examine how the PPI chain luminosity compares to the neutrino luminosity as the star contracts, we can simply scale down the central density and let it approach the critical central density from below. Assuming that the run of density and temperature closely follows an $n = 3$ polytrope, this is a valid analysis as such a polytrope is entirely self similar (i.e. the mass is independent of the central density and only depends on s and μ via the polytropic constant of proportionality K).

In Figure 1.4, we present this result in three symmetric log-log plots showing the total luminosity from both sources ($L_{\text{PPI}} - L_{\nu, \text{Tot.}}$) as a fraction of the Eddington luminosity for various SMS masses and DM mass fractions. We subtract the neutrino luminosity since it carries away energy from the star, as opposed to creating energy like the PPI process. This plot should not be treated as an evolutionary plot since the curves shown in the figure are calculated with a fixed mean molecular weight, $\mu = 0.59$, i.e., the value corresponding to a fully ionized primordial gas. Nuclear evolution, in the sense of burning via incorporating baryons into nuclei, should serve to increase the mean molecular weight μ as higher mass nuclei are created.

We would also expect that when nuclear burning is “important” (for the purposes of this chapter chosen to be $\sim 1\%$ of a star’s Eddington luminosity) the central density is held almost constant until its fuel is exhausted. This would prevent an unabated contraction of the star to the critical central density for GR instability, if only for a relatively short duration.

Since adding DM serves to increase the critical central density, more DM can push massive stars to higher central densities without causing them to become unstable. Both the PPI process and the relevant neutrino processes depend sensitively on the central temperature (and hence the central density), so a delicate balancing act ensues between

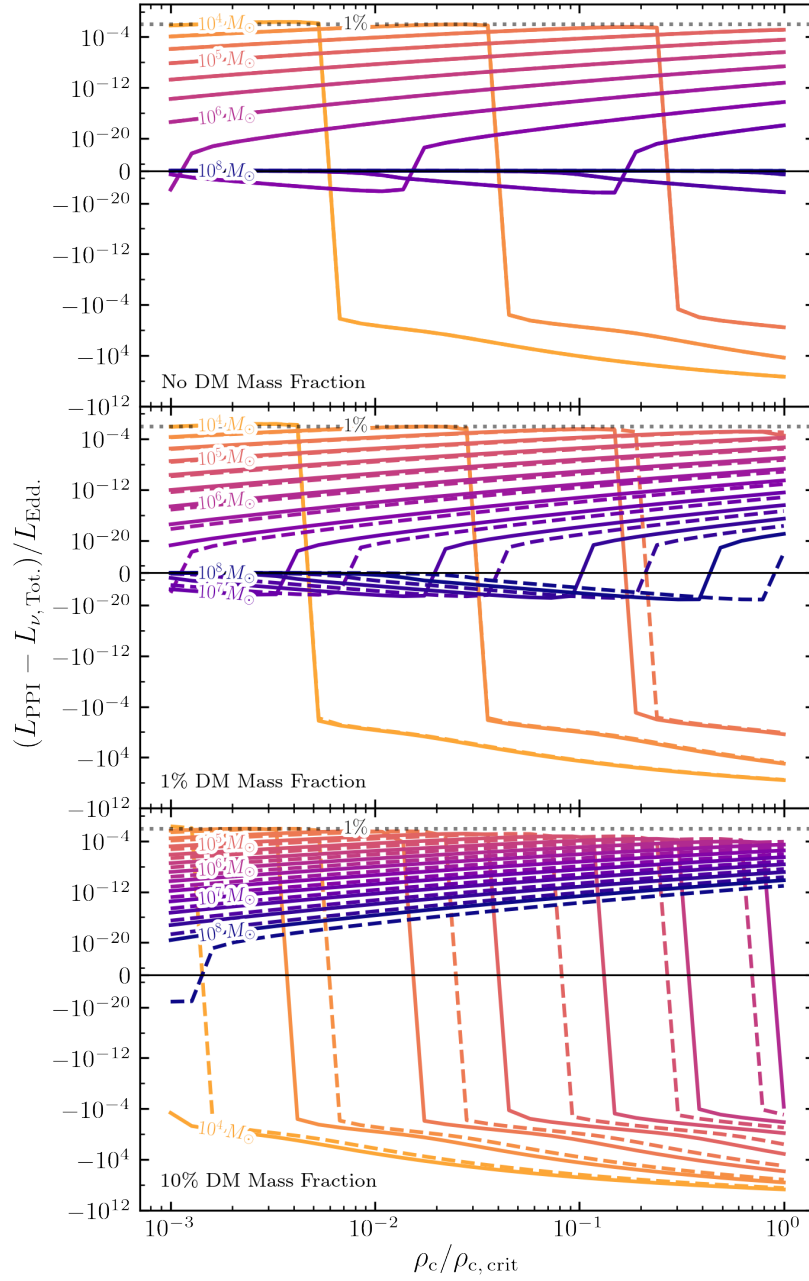


Figure 1.4: Symmetric log-log triptych plot showing total luminosity (burning minus neutrino losses) normalized to each SMSs’ Eddington luminosity versus the SMS’s central density as a fraction of its critical central density for different DM mass fractions. Solid (dashed) curves correspond to the “cold” (“hot”) kinematic limit. The top panel is for a star with no DM, the middle for a star with 1% DM by mass, and the bottom for a star with 10% DM by mass, all for various SMS masses between 10^4 – $10^8 M_\odot$. Indicated in each panel is a dashed gray line at 1% Eddington luminosity to indicate when burning is “important” for the evolution of the star. Note that these should not be taken to be evolutionary plots since the mean molecular weight μ is fixed.

energy generation from proton fusion and energy loss from neutrino emission.

As evidenced from the top plot (i.e. zero DM) in Figure 1.4, specifically for SMSs of masses $\lesssim 10^5 M_\odot$, increasing the central density increases the fraction of PPI energy generation, and hence net energy deposition, but only up to a point. Eventually, the neutrino losses from the increasing temperature win out, and the star is net losing energy instead of generating energy.

Should the quasi-statically contracting SMS encounter this situation during its evolution, the neutrino emission would cause the star to contract more quickly. This is because the neutrinos carry away the plasma's entropy faster than Hydrogen burning can generate it throughout the configuration. Per the solutions to the Lane-Emden equations, an $n = 3$ polytrope's radius scales as $R \sim K^{1/2} \sim s^{2/3}$, and since $\rho_c \sim R^{-3}$, the central density will increase as the radius decreases as neutrinos remove entropy from the SMS. This serves to push the SMS closer to instability.

The situation changes slightly for the $\sim 1\%$ DM mass fraction case (middle panel) and quite dramatically for the $\sim 10\%$ case (bottom panel). At a given central density fraction $\rho_c/\rho_{c, \text{crit}}$, increasing DM mass fraction corresponds to a higher absolute density on each mass curve. If we examine the $10^5 M_\odot$ curve, for example, this mechanism allows for the energy generation fraction from PPI to quite closely approach 1% of the Eddington luminosity as the DM mass fraction increases. This potentially permits such stars to burn stably on the main sequence instead of a direct collapse as expected in the standard case. This would drastically change their evolution as opposed to the case without DM.

1.7 Conclusion

These SMSs are quite interesting case studies, primarily because their existence provides a potential physical explanation for several observational mysteries that have only deepened as more advanced telescopes have come online. Not only has JWST, as an example, seen plenty of high mass quasars at high redshift which puts the standard picture

of solar mass remnant accretion into doubt, it may also have observed SMSs supported in some way by DM prior to collapse in the JADES survey, per Ref. [55].

Considering that these stars are “trembling on the verge of instability” due to the Feynman-Chandrasekhar Instability, it begs the question as to how they would be observed at all prior to collapse. One possible avenue to support these stars is through some form of non-interacting and non-relativistic DM. The enhanced stability would certainly also have implications for the nuclear and neutrino evolution of the stars.

In Section 1.2, we derived the standard result for the critical central density beyond which a hydrostatic SMS is unstable to collapse. This rested upon extremizing the total energy of the star with respect to the central density. We found that, as expected, higher mean molecular weight plasmas and higher mass SMSs are more unstable to collapse. Both of these factors serve to decrease the critical central density.

In Sections 1.3.1 and 1.3.2, we derived analytically how the critical central density changes as a function of the DM mass fraction within the star. We considered two limiting cases of the DM kinematics, the “hot” case and the “cold” case, which compared the typical DM particle’s specific kinetic energy far away from the SMS to the gravitational potential felt at the SMS surface, and found that in both cases the critical central density for the onset of general relativistic instability increased as the DM mass fraction increases. The “cold” case shows a slightly stronger effect, and the relative increase is greater for more massive stars in both kinematic limits.

In Section 1.4 we investigated a possible mechanism for the increased stability. We showed that if the star contracts, the total mass fraction of DM inside the star decreases, so the total mass enclosed by the star also decreases. This added stability is therefore completely agnostic to any particle physics specific to the DM, so long as any consideration of such physics keeps the DM non-relativistic and non-interacting with the baryonic matter. In fact, if the DM possesses some self-annihilation cross-section, like in the model discussed in Ref. [55], this may further bolster the stability by providing an extra source of energy

and entropy.

This brings up a potential limitation of our stability analysis. Our procedure was preformed under the assumption that the mass, total entropy, and angular momentum were fixed. If we include additional physics, such as rotation, or DM self interactions that inject entropy into the star, our analysis may fall short. Specifically, in Ref. [56], it is shown that a sufficient but not necessary condition for the onset of instability is that both the first and second derivatives of the total energy vanish for the case that total energies parameterized by multiple quantities. However, our analysis involves a sequence of configurations with only one parameter, the central density ρ_c , so this condition is sufficient *and* necessary.

In Section 1.5, we have determined the equilibrium stellar density profile for an SMS embedded in a field of DM at a constant density. We note that the profile closely follows that of a standard $n = 3$ polytrope until the two densities become comparable, whereupon they radically diverge to form a slightly smaller total configuration. The standard and new profiles are quite similar for low DM mass fractions, as expected. Using these new profiles, the change in critical central density is determined quantitatively by integrating them to calculate the structure coefficients that directly determine the critical central density. We find that for all SMS masses and DM mass fractions considered, considering the new equilibrium density profile would serve to slightly lower the increased critical central density due to the DM presence alone.

Finally, in Section 1.6, we considered by what degree Hydrogen burning could impact the evolution of these SMSs and how DM influences that burning. We also analyzed how neutrino emission from the high-entropy plasma would take away energy, potentially faster than what is generated from fusion, as the star approaches the critical central density from below. We found that, considering both energy generation from fusion and energy losses from neutrino emission, DM could allow heavier stars than previously predicted to stably burn on the main sequence at some point in their lives, increasing the rough

upper limit of $\sim 30,000 M_{\odot}$ to about $50,000 M_{\odot}$, depending on when one considers nuclear burning to be “important” to the evolution of the star.

More investigation is needed to understand these extreme objects. If they are to explain the high mass SMBHs in the early universe by providing a supermassive seed black hole, there should be enough of them formed to account for all or most of them. This is complicated by a suspect formation history, since if the collapsing gas is cool enough to contain molecular Hydrogen instead of atomic Hydrogen, the enhanced cooling efficiency via rotational and vibrational modes [57] could favor a fragmented formation into many stars instead of one SMS. However, enhanced cooling from early molecular Hydrogen formation via Beyond the Standard Model (BSM) channels (e.g. in Ref. [58]) can lead to very early star formation. Whether SMS formation may accompany this early star formation is an open question. Additionally, if a sufficient population of primordial black holes (PBHs) formed with low enough mass to fully evaporate before $z \sim 10$, the extra heating provided can suppress molecular Hydrogen formation [59], possibly allowing another channel to form SMSs. A similar result can also be reached by considering the decay of relic particles [60].

We would expect that nuclear burning, i.e. fusing bare nucleons into larger composite nuclei, would lower the critical central density for the onset of collapse since the mean molecular weight increases. The gas pressure is provided primarily by non-relativistic baryons, which is proportional to the total number of particles, so nuclear fusion reduces this number and hence lowers the gas pressure if sufficient burning takes place. While sufficient burning would serve to slow the SMS’s contraction by injecting energy in the region surrounding the core, the critical central density is being reduced nonetheless, hastening the approach to instability if the chemical composition of the star is significantly changed.

Including DM further complicates this picture. We saw that sufficient DM will change the evolution of some SMS masses that would normally collapse via GR instability,

making them also vulnerable to burning induced instability. As DM raises the critical central density, it also permits these stars to host greater central temperatures, making neutrino emission more efficient at carrying away entropy. This would cause the SMS to collapse on a different adiabat, specifically one at lower entropy than expected, potentially altering the prospects for post-collapse detection.

If such stars existed and underwent sufficient burning, they could be detected from their collapse directly if they then explode (instead of directly onto a black hole), and indirectly from enriching their surrounding medium with burning products both pre and post-collapse [61, 62]. The enhanced neutrino emission from DM would also alter the gravitational wave signature from an anisotropic neutrino pulse as the homologous core’s entropy would be reduced for a given mass (see e.g. Ref. [19] for details on this neutrino driven gravitational radiation for standard SMSs).

The authors would also caution those who may be unfazed by the supposedly ‘modest’ amount of DM invoked in this study. Consider a $10^6 M_\odot$ SMS with $\sim 1\%$ DM by mass. If the DM is at a constant density within the star, such a DM mass fraction corresponds to a density of $\sim 10^{20} \text{ GeV cm}^{-3}$, which would be an over-dense region of DM about 20 orders of magnitude more dense than the expected background at redshift $z \sim 10$.

If these DM supported SMSs had existed, this huge accumulation and concentration of DM greatly constrains models for DM particles. Specifically, models where these particles cannot alter their phase space density and clump together through self and/or standard model interactions would clearly be disfavored in this regard. An example of a model where particles can alter their phase space density appropriately is discussed in Ref. [63], which in turn also provides a different mechanism for seeding SMBHs. Densities approaching the requisite amount of DM needed to stabilize SMSs through purely gravitation can be reached via adiabatic contraction of a baryonic cloud in a DM halo (e.g. Ref. [49]), but would still need an additional source of heating to substantially change the stability of the SMS (e.g. Ref. [64] for the case of DM self-annihilation).

Moreover, the huge concentration of DM that we discuss here likely also constrains the manner and environments of SMS formation. Whether such environments can be found, and whether a component of the DM can be compatible with the concentrations discussed here, remain intriguing and open questions. In the end, such speculation may be warranted given our relative ignorance of DM and dark sector physics, and given the existence of SMBHs at very high redshift.

Here we have used a very simplistic model, a non-rotating and isolated SMS, to explore the physical effects of a large DM content. Nonetheless, our findings pose several open questions. First, with the inclusion of DM, what is the interplay of nuclear burning, hydrodynamics, energy transport, neutrino emission, etc., in the evolution of the SMS up to instability and the subsequent collapse to a black hole? Second, are there specific observational “handles” on the collapse of a SMS with or without DM? For example, is the gravitational radiation signature [19] different in these two cases, and to what extent would it depend on the amount of DM? Third, in this chapter we have not considered the effects of DM on scenarios where a supermassive core undergoes rapid accretion, as in the *hylotropic* configurations proposed in Ref. [65]. The stabilizing effect of DM on such hylotropic configurations has been recently considered [66]. These scenarios may provide the most plausible route for the production of an SMBH from an SMS-like object.

Finally, perhaps the most significant issue is how an SMS would form with the huge quantities of DM considered in this work and in others that invoke DM effects in the production of SMBHs. However, there is urgency for consideration of these issues because of the existence of SMBHs at very early epochs in the history of the Universe, and because both the sources and properties of DM remain mysterious.

1.8 Acknowledgments

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Chapter 2

Neutrino Flavor Transformation in Collapsing Supermassive Objects

The collapse of supermassive stars (SMSs, $M \gtrsim 10^4 M_\odot$) to black holes is accompanied by a prodigious flux of neutrinos of all flavors. These are produced thermally via $e^\pm$ annihilations, mostly in the core and just before gravitational trapped surface formation. There, the ratio of fluxes for $\nu_e \bar{\nu}_e$ -pairs to $\nu_\mu \bar{\nu}_\mu / \nu_\tau \bar{\nu}_\tau$ -pairs is ~ 5 -to-1. This is because at SMS temperature scales, $\nu_e \bar{\nu}_e$ pairs have both charged and neutral current production channels, whereas $\nu_\mu \bar{\nu}_\mu / \nu_\tau \bar{\nu}_\tau$ -pairs only have neutral current production channels. We point out that the typical energies of these neutrinos, and the run of density in collapsing radiation-dominated supermassive configurations, leads to Mikheyev–Smirnov–Wolfenstein (MSW) resonances inside these objects for the atmospheric neutrino mass splitting scale, $\Delta m_{\text{atm.}}^2 \sim 2.4 \times 10^{-3} \text{ eV}^2$. In the normal neutrino mass hierarchy, adiabatic flavor transformation through the MSW resonances would then swap the fluxes $\nu_e \rightleftharpoons \nu_{\mu,\tau}$, whereas, in the inverted neutrino mass hierarchy, the anti-neutrino fluxes are swapped, $\bar{\nu}_e \rightleftharpoons \bar{\nu}_{\mu,\tau}$. We also examine the prospects for collective neutrino flavor oscillations in these environments. Implications for flavor oscillation’s effects on neutrino energy deposition and neutrino-induced nucleosynthesis in the SMS’s outer layers are examined, as are prospects

for detections of SMS collapses through various means.

2.1 Introduction

In this dissertation chapter, we point out an intriguing connection between neutrino flavor physics and the conditions and neutrino fluxes accompanying the gravitational collapse of supermassive stars (i.e., stars with masses in excess of $\sim 10^4 M_\odot$). The formation and collapse of supermassive stars may figure in the solution of several important cosmological mysteries. Perhaps the most vexing of these is the existence of supermassive black holes (SMBHs) early in the history of the universe [32, 33, 34, 35, 31]. Specifically, Ref. [10] suggests that if these objects had existed no later than about 100 million years after the Big Bang and collapsed directly into black holes at the end of their lives, the high mass “seeds” left behind would be able to accrete at the Eddington rate into such massive quasars. Since black holes created from supernovae are at most $\sim 10 M_\odot$, they would need to accrete at *super*-Eddington rates for hundreds of millions of years to achieve the same mass as these quasars. This fact could make the scenario involving prompt collapse of individual SMSs and standard Eddington accretion rates preferred.

Ignoring for now the issue of how such supermassive stars (SMSs) would have formed into single gravitationally bound objects instead of fragmenting, investigating their properties is still of fruitful academic interest at the very least. Moreover, beyond-standard-model (BSM) physics has been invoked as a way for SMSs to circumvent several formation pathway problems [55, 45, 60, 67, 68, 69]. In any case, hydrostatic supermassive objects share some common features regardless of the particulars of their formation mechanisms.

As SMSs are so massive, the vast majority of their pressure support is derived from relativistic particles (in this case, photons) and will therefore eventually suffer general relativistic (GR) instability [40, 41, 42, 43]. Such configurations can be fully convective and hence have a constant entropy profile throughout the star. In that case, the entropy is determined only by the total mass and overall chemical composition.

In purely Newtonian gravity, these SMSs would be nearly neutrally buoyant, neither stable nor unstable to radial perturbations. Any contraction or expansion of the star would see the gravitational and pressure forces change together in lock-step and therefore cost no energy, leaving the SMS in a metastable state. Being rather large ($R \sim 1$ AU) and diffuse ($\rho_c \sim 10 \text{ g cm}^{-3}$), SMSs' structure can be accurately modeled by just Newtonian theory.

Their stability (or lack thereof), however, is entirely determined by GR. In general, objects whose pressure is derived primarily from relativistic particles are said to be “trembling on the verge of instability,” and the inclusion of 1st order GR corrections to Newtonian theory is sufficient to push them over the edge to instability. The inherent non-linearity of GR predicts that both stress-energy *and local spacetime curvature itself* both generate spacetime curvature, which in this context materializes as a slightly stronger than expected gravitational force. Inward radial perturbations would serve to condense the metastable SMS, increasing the pressure forces throughout the star while also deepening the gravitational potential well. While GR at this scale has nothing to say about this increase in pressure, the slightly stronger gravitational forces would then break this delicate balance and shift the configuration from metastable to unstable, leading to a direct collapse into a black hole. This is the so-called Feynman-Chandrasekhar Instability [13, 14].

The SMS evolutionary timescale from the onset of GR instability to black hole formation is of particular interest to this chapter. During collapse, the central density and temperature will increase dramatically until a gravitational trapped surface is formed. Prior to black hole formation, the high densities and temperatures will convert some of the kinetic energy of the infalling stellar material into neutrinos, sapping energy from the interior as they mostly free stream from the core through the SMS basically unimpeded. The chance scatterings on their way out may physically deposit some of their energy into the SMS's outer envelope, or transmute free nucleons into their isospin mirrors thereby also creating e^- s and e^+ s.

As for neutrino inelastic scattering, Ref. [62] determined that there would be

fairly substantial production of deuterium through this mechanism, potentially above the primordial deuterium abundance created during Big Bang Nucleosynthesis (BBN). This could offer tantalizing observational signatures of SMSs having ever existed during the universe’s early history.

A possible enhancement of the deuterium production would be realized through the Mikheyev-Smirnov-Wolfenstein (MSW) mechanism, assuming that neutrinos exist in the “normal mass hierarchy,” where the lightest mass eigenstate for neutrinos is primarily associated with the e flavor state. Neutrinos already oscillate between the e , μ , and τ flavors in vacuum, however interactions with the electrons in the SMS interior can alter this oscillation. Specifically, ν_e s would “feel” both a charged current and neutral current interaction with the ambient electrons that energetics forbid for the other flavor neutrinos, giving the ν_e s a larger in-medium mass that depends on the local electron number density.

Depending on the neutrino energy, there exists a resonant electron number density at which the flavor oscillation frequency becomes maximal, causing initially e -flavor neutrinos produced in the dense core to convert to ν_μ s and ν_τ s in the diffuse outer layers. As there is no net positron density, $\bar{\nu}_e$ s are unaffected in the normal mass hierarchy. This means that the reaction $\nu_e + n \rightarrow p + e^-$ is now substantially disfavored relative to the reaction $\bar{\nu}_e + p \rightarrow n + e^+$, creating a large population of free neutrons in the SMS. Of course, this only is the case if the MSW resonant density is experienced somewhere within the star, which we will show is the case for a range of SMS progenitor masses.

The first few sections of this chapter will serve as an overview of well established neutrino processes in SMSs, focusing on the formation channels, the cross-sections for interaction on various targets, and the overall structure of SMS. Specifically, Section 2.2 will establish the relative initial amounts of each flavor of neutrino at creation and the total thermally averaged cross section of all neutrinos combined. These considerations will be relevant for determining the interaction rates of each species.

Section 2.3 will discuss the structure of SMSs pre-instability, the point of instability,

and the point at which a gravitational trapped surface is formed. Any analysis in this chapter begins at the point of instability and ends when the subsequent black hole appears, as further numerical study is needed to explore conditions that arise after the formation of an event horizon.

Section 2.4 will consider all kinds of neutrino flavor oscillation. First, we will consider collective effects that would arise from the high neutrino number densities we expect to occur in SMSs. If collective oscillations are important, they will wash out the initial preference given to $\nu_e\bar{\nu}_e$ -pair creation, equalizing all flavors of both neutrinos and antineutrinos. We will show that these non-linear effects are likely unimportant except potentially at the lower end of the SMS mass range. We will then discuss the aforementioned MSW effect, the result of which will suppress the presence of ν_e s relative to $\bar{\nu}_e$ s in the SMS's outer layers. We will show that there is an initial mass range of SMSs where the MSW resonances are present within the SMS, and that where it occurs in the SMS also depends on the mass. We will finally show that this flavor oscillation occurs adiabatically, that is, the flavor oscillation wavelength at resonance is much smaller than the resonance width, so the net effect is a true *en mass* flavor conversion.

Section 2.5 will discuss the process of $\bar{\nu}_e$ capture on free protons to convert them to neutrons in the SMS's outer layers. We expect, assuming this transmuted material does not subsequently fall directly into the newly formed black hole, that these free neutrons will readily combine with free protons to form deuterium and potentially even α particles. We then speculate on whether this could provide an observational signature for detection of these SMSs.

2.2 Neutrino Creation and Interactions

Despite modest temperatures, SMSs possess large entropy-per-baryon ($s/k_B \sim 100\text{--}1000$), meaning there exists a significant abundance of $e^\pm$ -pairs in electromagnetic equilibrium. Neutrinos are created as $\nu\bar{\nu}$ -pairs of all flavors in the centers of collapsing

SMSs, primarily from relativistic $e^\pm$ annihilations [43, 50, 62]. These can be mediated by the neutral Z^0 boson (“neutral current,” NC) and the charged $W^\pm$ boson (“charged current,” CC). NC production channels are flavor agnostic, so the $\nu_e\bar{\nu}_e$, $\nu_\mu\bar{\nu}_\mu$, and $\nu_\tau\bar{\nu}_\tau$ -pairs are produced in equal amounts. CC production channels only produce e flavor pairs to preserve lepton number of the reactants at tree-level, and the energetics involved forbid $\mu^\pm$ or $\tau^\pm$ annihilation to create the other flavors in the CC channel. This production channel asymmetry leads to a flux enhancement of $\nu_e\bar{\nu}_e$ -pairs over $\nu_\mu\bar{\nu}_\mu$ or $\nu_\tau\bar{\nu}_\tau$ -pairs, as calculated in Ref. [30], to be about 5-to-1, i.e., ν_e -pairs constitute about 70% of the total neutrino flux. This will be relevant for CC scattering on nucleons, as discussed later in this section.

As neutrinos travel through an SMS, they may scatter on free nuclei, specifically free protons and composite nuclei of atomic mass A . Neutrinos may also scatter on electrons, but these are greatly subdominant to nucleon interactions at the energy scales considered in this chapter [25]. NC scatterings are elastic and have the form

$$\begin{aligned}\bar{\nu}_\ell^{(-)} + p &\xrightarrow{Z^0} \bar{\nu}_\ell^{(-)} + p, \\ \bar{\nu}_\ell^{(-)} + A &\xrightarrow{Z^0} \bar{\nu}_\ell^{(-)} + A,\end{aligned}$$

where $\bar{\nu}_\ell^{(-)}$ refers to neutrinos and anti-neutrinos of lepton flavor $\ell = (e, \mu, \tau)$ and p denotes protons.

CC scatterings are inelastic and have the form

$$\begin{aligned}\nu_e + n &\xrightarrow{W^\pm} e^- + p, \\ \bar{\nu}_e + p &\xrightarrow{W^\pm} e^+ + n.\end{aligned}$$

Assuming the protons and neutrons are nearly at rest in the COM frame, these CC reactions require $E_\nu > E_\nu^{\text{th}} = \Delta m + m_{e^\pm}$ to occur, where $\Delta m \sim \pm 1.29 \text{ MeV}$ is the nucleon mass difference after the reaction and $m_{e^\pm} \sim 0.511 \text{ MeV}$ is the mass of the resulting

electron or positron (both in units of $c = \hbar = 1$, *a.k.a* “natural” units, which we will use throughout the chapter). The maximum temperatures achieved in SMS collapse (which we will show is $\sim 0.5\text{--}5\text{ MeV}$, in units of $k_B = 1$) therefore forbid the other neutrino flavors from participating in the above CC reactions as they do not have the requisite threshold energy to produce the heavy charged leptons.

The NC scattering on nucleons by neutrinos of any lepton flavor has the cross sections [25, 11]

$$\begin{aligned}\sigma_{\text{NC}}^{\nu p}(E_\nu) &= \frac{G_F^2}{4\pi} [(1 - 4\sin^2\theta_W)^2 + 3g_A^2] E_\nu^2, \\ &\approx 2.2 \times 10^{-44} \left(\frac{E_\nu}{1\text{ MeV}} \right)^2 \text{ cm}^2,\end{aligned}\tag{2.1}$$

$$\begin{aligned}\sigma_{\text{NC}}^{\nu A}(E_\nu) &= \frac{G_F^2}{4\pi} A^2 E_\nu^2, \\ &\approx 6.7 \times 10^{-44} \left(\frac{A}{4} \right)^2 \left(\frac{E_\nu}{1\text{ MeV}} \right)^2 \text{ cm}^2,\end{aligned}\tag{2.2}$$

where $G_F \approx 1.1664 \times 10^{-11} \text{ MeV}^{-2}$ is the Fermi constant, $\sin^2\theta_W \approx 0.23$ is the sine-squared of the Weinberg mixing angle [1], $g_A \approx 1.3$ is the axial-to-vector weak strength ratio for free nucleons, and E_ν is the neutrino energy. Note that these cross sections are energy dependent, in contrast to, say, the energy independent Thomson cross section ($\sigma_T \sim 6.65 \times 10^{-25} \text{ cm}^2$) which governs electromagnetic interactions, and are also ~ 20 orders of magnitude smaller for MeV scale neutrinos.

The CC capture cross section of $\bar{\nu}_e$ s on protons is [25, 70, 71]:

$$\begin{aligned}\sigma_{\text{CC}}^{\bar{\nu}_e p} &= \frac{G_F^2 |V_{ud}|^2}{\pi} (1 + 3g_A^2) (E_\nu - Q)^2, \\ &\approx 10^{-43} \left(\frac{E_\nu - Q}{1\text{ MeV}} \right)^2 \text{ cm}^2,\end{aligned}\tag{2.3}$$

for $E_\nu > Q$, where $|V_{ud}|^2 \approx 0.948$ is the squared magnitude of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element corresponding to up-down quark flavor mixing [1], and

$Q = m_n - m_p \approx 1.29 \text{ MeV}$ is the mass difference between a neutron and a proton. Here, we assume that the electron's momentum after creation is negligible and that there is no phase-space blocking from ambient positrons resulting from Fermi statistics. We will also not consider the cross section for the isospin mirror reaction $\nu_e + n \rightarrow p + e^-$. Any free neutrons would be quickly incorporated into nuclei via strong and electromagnetic interactions. Moreover, the high entropies that attend the collapse of SMSs inhibit the formation of heavy nuclei.

In the context of collapsing SMSs, Ref. [30] found that the energy spectrum of neutrinos produced by annihilating $e^\pm$ pairs near the growing event horizon is well fit by the normalized Fermi-Dirac distribution

$$f_\nu(E_\nu) = \frac{1}{T_\nu^3 F_2(\eta_\nu)} \frac{E_\nu^2}{e^{E_\nu/T_\nu - \eta_\nu} + 1}, \quad (2.4)$$

where $T_\nu \approx 1.6 T$ is the “neutrino temperature,” about 60% greater than the local plasma temperature T , $\eta_\nu \sim 2$ is the “degeneracy parameter,” and $F_2(\eta_\nu)$ is a numerical factor calculated in general to be

$$F_k(\eta_\nu) = \int_0^\infty \frac{x^k}{e^{x-\eta_\nu} + 1} dx, \quad (2.5)$$

hereafter referred to as Fermi integrals of order k . It is worth noting that these are fitting parameters and should not be misconstrued as the thermodynamic notions of temperature and degeneracy.

The spectrum-averaged NC elastic scattering cross section is then

$$\langle \sigma_{\text{NC}}^i \rangle = \int_0^\infty f_\nu(E_\nu) \sigma_{\text{NC}}^i(E_\nu) dE_\nu, \quad (2.6)$$

where $i = \nu p$ or νA . For SMSs with primordial element abundances, the only other scattering targets besides protons are helium nuclei (α particles) with $A = 4$. As a function

of the hydrogen mass fraction X , the number fractions of each target are

$$\begin{aligned}\mathbf{n}_p(X) &= \frac{4X}{1+3X}, \\ \mathbf{n}_\alpha(X) &= 1 - \mathbf{n}_p,\end{aligned}$$

assuming the SMS is composed of only hydrogen and helium.

The *total* thermally-averaged NC elastic scattering cross section is then

$$\begin{aligned}\langle \sigma_{\text{NC}} \rangle &= \mathbf{n}_p \langle \sigma_{\text{NC}}^{\nu p} \rangle + \mathbf{n}_\alpha \langle \sigma_{\text{NC}}^{\nu \alpha} \rangle, \\ &= \frac{G_{\text{F}}^2}{4\pi} \frac{F_4(\eta_\nu)}{F_2(\eta_\nu)} T_\nu^2 g(X)\end{aligned}\tag{2.7}$$

where $g(X) = \mathbf{n}_p \left([1 - 4 \sin^2 \theta_{\text{W}}]^2 + 3g_A^2 \right) + 16(1 - \mathbf{n}_p)$. For a hydrogen mass fraction of $\sim 75\%$, typical of a primordial gas, $g(X) \approx 5.9$. In standard units, this cross section is therefore

$$\langle \sigma_{\text{NC}} \rangle \approx 4.0 \times 10^{-43} \left(\frac{T_\nu}{1 \text{ MeV}} \right)^2 \text{ cm}^2\tag{2.8}$$

for $\eta_\nu = 2$.

To find the thermally averaged cross section for the CC case, we use [70]

$$\langle \sigma_{\text{CC}}^{\bar{\nu} e p} \rangle = \int_{E_\nu^{\text{th.}}}^{\infty} \sigma_{\text{CC}}^{\bar{\nu} e p} f_\nu(E_\nu) \text{d}E_\nu,\tag{2.9}$$

where $E_\nu^{\text{th.}} = Q + m_e \sim 1.8 \text{ MeV}$ is the threshold energy required for this interaction. This evaluates to

$$\langle \sigma_{\text{CC}}^{\bar{\nu} e p} \rangle \approx 10^{-43} \left(\frac{T_\nu}{1 \text{ MeV}} \right)^2 \mathfrak{F}(T_\nu) \text{ cm}^2,\tag{2.10}$$

where the function $\mathfrak{F}(T_\nu)$ is a linear combination of Fermi integrals of several orders:

$$\mathfrak{F}(T_\nu) = \frac{1}{F_2(\eta_\nu)} \sum_{i=0}^4 \alpha_i(T_\nu) F_i(\eta_\nu - E_\nu^{\text{th.}}/T_\nu)\tag{2.11}$$

with temperature dependent coefficients

$$\begin{aligned}
\alpha_0(T_\nu) &= (Q^2 m_e^2 + 2Q m_e^3 + m_e^4)/T_\nu^4, \\
\alpha_1(T_\nu) &= 2(Q^2 m_e + 3Q m_e^2 + 2m_e^3)/T_\nu^3, \\
\alpha_2(T_\nu) &= (Q^2 + 6Q m_e + 6m_e^2)/T_\nu^2, \\
\alpha_3(T_\nu) &= 2(Q + 2m_e)/T_\nu, \\
\alpha_4(T_\nu) &= 1.
\end{aligned}$$

This function is rather involved but has convenient limiting behaviors. $\mathfrak{G}(T_\nu)$ vanishes precipitously for $T_\nu \ll Q$, illustrative of the fact that neutrinos with energies sufficient to carry out the reaction are exponentially suppressed. For $T_\nu \gg Q$, $\mathfrak{G}(T_\nu)$ approaches a limiting value $F_4(\eta_\nu)/F_2(\eta_\nu)$, which for $\eta_\nu = 2$ is about 16.

The total thermally-averaged cross section from both the NC and CC interactions is therefore (taking into account that $\sim 40\%$ of neutrinos are $\bar{\nu}_e$):

$$\begin{aligned}
\langle \sigma \rangle &= \langle \sigma_{\text{NC}} \rangle + 0.4 \mathfrak{n}_p(X) \langle \sigma_{\text{CC}}^{\bar{\nu}_e p} \rangle, \\
&\approx 4.0 \times 10^{-43} \left(\frac{T_\nu}{1 \text{ MeV}} \right)^2 [1 + 0.1 \mathfrak{G}(T_\nu)] \text{ cm}^2,
\end{aligned} \tag{2.12}$$

for $X = 0.75$, or in terms of the local plasma temperature T :

$$\langle \sigma \rangle \approx 10^{-42} \left(\frac{T}{1 \text{ MeV}} \right)^2 [1 + 0.1 \mathfrak{G}(1.6T)] \text{ cm}^2. \tag{2.13}$$

This total cross section will determine where, if anywhere, neutrinos are effectively “trapped” within the SMS as they stream from the center during collapse. Neutrinos that are not trapped by the time they encounter the MSW flavor resonance will be of particular interest for this chapter.

2.3 SMS Structure, Instability, and Black Hole Formation

The supermassive stars we consider here are radiation-pressure dominated, fully ionized, totally convective, isentropic, and non-rotating hydrostatic configurations of gas with masses in excess of $\sim 10^4 M_\odot$. Here, we consider only non-rotating and non-magnetic configurations in order to focus on the neutrino physics of collapse. For these SMSs, the ratio of gas pressure to total pressure β is constant throughout the star, and is calculated to be [72]

$$\beta \equiv \frac{P_{\text{gas}}}{P_{\text{rad.}} + P_{\text{gas}}} \approx \frac{4.3}{\mu} \left(\frac{M}{M_\odot} \right)^{-1/2}, \quad (2.14)$$

where $\mu \approx 0.59$ is the mean molecular weight for a fully ionized primordial gas, and P_{gas} and $P_{\text{rad.}}$ are the Maxwell-Boltzmann pressure and radiation pressure contributions, respectively. β is much less than unity for the mass range considered in this paper ($10^4 \lesssim M/M_\odot \lesssim 10^8$). The entropy-per-baryon s is similarly calculated as

$$s \approx \frac{4}{4.3} \left(\frac{M}{M_\odot} \right)^{1/2} - \frac{4}{\mu}, \quad (2.15)$$

in units of Boltzmann's Constant k_B , which we will use hereafter.

The above properties combine to make an SMS's equation of state match closely to that of a polytrope, i.e.:

$$P(s, \rho) = K(s) \rho^{1+1/n}, \quad (2.16)$$

where P is the pressure, ρ is the mass density, n is the “polytropic index,” and $K(s)$ is the “polytropic constant” that depends on the entropy-per-baryon. Since s is constant throughout the SMS, $K(s)$ is also a constant. Specifically, we take $n \sim 3$ for SMSs, so

$K(s)$ is calculated to be

$$\begin{aligned} K(s) &= \frac{1}{1-\beta} \frac{a}{3} \left(\frac{3}{4a} \frac{s}{m_b} \right)^{4/3}, \\ &\approx \frac{0.2843}{1-\beta} \left(\frac{s}{1000} \right)^{4/3} \text{ MeV}^{-4/3}, \end{aligned} \quad (2.17)$$

where $a = \pi^2/15$ is the radiation constant and $m_b \approx 939$ MeV is the average baryon mass.

The total energy for such a configuration is given by [15]

$$E(\rho_c) = k_1 M K(s) \rho_c^{1/n} - k_2 G M^{5/3} \rho_c^{1/3} - k_4 G^2 M^{7/3} \rho_c^{2/3}, \quad (2.18)$$

where $k_1 \approx 1.7558$, $k_2 \approx 0.6390$, and $k_4 \approx 0.9183$ are numerical constants calculated from integrating over an $n = 3$ polytrope, ρ_c is the central density, and $G \approx 6.7 \times 10^{-45} \text{ MeV}^{-2}$ is Newton's gravitational constant.

The terms represent, in order, the internal kinetic energy (non-relativistic Maxwell-Boltzmann and relativistic photon contributions), the gravitational binding energy, and first order corrections from GR, all as a function of M and ρ_c . For $n \sim 3$, the virial theorem dictates that the first two terms nearly perfectly cancel, so the total energy is dominated by the GR correction term. Since that energy is proportional to G^2 , it is in effect a gravitational “self-coupling,” hinting at the true non-linear nature of GR.

As these SMSs are radiation dominated, they will also radiate photons from their surface at the Eddington limited rate L_{Edd} . [11], defined to be

$$\begin{aligned} L_{\text{Edd}} &= \frac{4\pi G M m_b}{\sigma_T}, \\ &\approx 1.3 \times 10^{38} \left(\frac{M}{M_\odot} \right) \text{ erg s}^{-1}. \end{aligned} \quad (2.19)$$

This energy loss causes the SMS to quasi-statically contract, raising the central density and temperature over time. As discussed in the Appendix (Sec. A), this will not change

the total entropy of the SMS by much, but it is key to the inevitable instability brought about by GR.

2.3.1 SMS Structure and Instability

With the above equation of state (Eq. (2.16), $n = 3$) and the following rescalings for radius r and density:

$$r = \alpha_{\text{LE}} \xi = \left(\frac{K}{\pi G} \right)^{1/2} \rho_c^{-1/3} \xi, \quad (2.20)$$

$$\rho = \rho_c \varphi^3, \quad (2.21)$$

the equations for hydrostatic equilibrium can be manipulated to yield the Lane-Emden Equation of order 3:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\varphi}{d\xi} \right) + \varphi^3 = 0, \quad (2.22a)$$

$$\varphi(0) = 1, \quad (2.22b)$$

$$\varphi'(0) = 0, \quad (2.22c)$$

the solution of which is used to find the density profile as a function of radius. Denoting the location of the first zero of $\varphi(\xi)$ as $\varphi(\xi_3) = 0$, the total mass M and radius R of the configuration are

$$M = 4\pi \left(\frac{K}{\pi G} \right)^{3/2} \xi_3^2 |\varphi'(\xi_3)|, \quad (2.23)$$

$$R = \left(\frac{K}{\pi G} \right)^{1/2} \rho_c^{-1/3} \xi_3, \quad (2.24)$$

where $\xi_3 \approx 6.9868$ and $|\varphi'(\xi_3)| \approx 0.0424$.

Under a purely Newtonian theory of gravity, these $n = 3$ polytropes are neutrally

stable, i.e. it costs no energy to expand or contract the configuration. This fact is also realized by the hydrostatic mass in Eq. 2.23 having no dependence on the central density, lending them a self-similar structure.

Upon considering first order GR corrections to Newtonian theory, one can derive the critical central density $\rho_{\text{c, crit.}}$ above which the configuration is unstable to collapse [72, 11]:

$$\rho_{\text{c, crit.}} \approx 3.98 \left(\frac{\mu}{0.59} \right)^{-3} \left(\frac{M}{10^5 M_{\odot}} \right)^{-7/2} \text{ g cm}^{-3}. \quad (2.25)$$

Per Ref. [11], the SMS will contract from a diffuse state ($\rho_{\text{c}} \sim 0$) to the critical state on roughly the Kelvin-Helmholtz thermal timescale τ_{KH} . For $n \sim 3$ polytropes radiating at the Eddington luminosity $L_{\text{Edd.}}$, this timescale is calculated using the final term in Eq. (2.18) at the critical state via

$$\tau_{\text{KH}} \sim \frac{k_4 G^2 M^{7/3} \rho_{\text{c, crit.}}^{2/3}}{L_{\text{Edd.}}} \approx 8.9 \times 10^3 \left(\frac{M}{10^5 M_{\odot}} \right)^{-1} \text{ yr}, \quad (2.26)$$

since the internal energy and Newtonian gravitational binding energy cancel each other out.

During this quasi-static contraction phase, the central density will approach $\rho_{\text{c, crit.}}$ from below (Eq. (2.25)), at which point it will become unstable to collapse and begin a rapid contraction phase roughly at the free-fall timescale $\tau_{\text{ff}} \sim (G\rho)^{-1/2} \approx 4.8 \times 10^{-5} (M/10^5 M_{\odot})^{7/4} \text{ yr}$. For $M \sim 10^8 M_{\odot}$, both of the prior timescales are equal, so masses greater than this are not considered as there is basically no distinction between a quasi-static contraction phase and free-fall collapse.

2.3.2 Trapped Surface Formation and Neutrino Trapping

As the SMS collapses after instability sets in, neutrino production in the center will sap entropy from the SMS. Ref. [43] shows that this loss in entropy will cause a

smaller so-called “homologous core” (HC) with mass $M^{\text{HC}} \sim 0.1 M^{\text{init.}}$ to collapse while maintaining the same overall $n = 3$ polytropic structure. The central density will rise and the radius will decrease per Eq. (2.24), while the total HC mass stays the same. This process continues until a trapped surface is formed at a radius $r_\bullet$ such that

$$r_\bullet = 2GM_{\text{enc.}}(r_\bullet) \quad (2.27)$$

where $M_{\text{enc.}}(r)$ is the mass enclosed at radius r with boundary conditions $M_{\text{enc.}}(0) = 0$ and $M_{\text{enc.}}(R) = M$.

The enclosed mass function can be calculated directly to be, using the Lane-Emden rescalings:

$$M_{\text{enc.}}(r) = \int_0^r 4\pi(r')^2 \rho(r') dr' = 4\pi\alpha_{\text{LE}}^3 \rho_c \xi^2 |\varphi'(\xi)|. \quad (2.28)$$

The condition for a trapped surface now simplifies to the transcendental equation

$$\xi_\bullet = 8K(s)\rho_c^{1/3}\xi_\bullet^2 |\varphi'(\xi_\bullet)|, \quad (2.29)$$

where for fixed entropy-per-baryon s , the above equation is just parameterized by ρ_c .

There exists some minimum central density $\rho_\bullet$ such that Eq. (2.29) has one solution for $\xi_\bullet$ for any choice of HC mass. It turns out that, for an $n = 3$ polytrope, $\xi_\bullet$ is independent of the mass and always occurs at $\xi_\bullet \approx 2.632$, a bit under 40% of the total radius. No GR considerations went into this calculation, so it should be emphasized that this “quick-and-dirty” approach is likely an over-estimate of both the central density at trapped surface formation and the subsequent trapped surface radius.

Indeed, Ref. [73] have already performed several full-GR simulations of collapses of initially $n = 3$ polytropes with masses ranging from $5 \times 10^5 M_\odot$ to $10^9 M_\odot$. They found that—regardless of the mass—a gravitational trapped surface forms during collapse that encloses the innermost $\sim 25\%$ of the total SMS mass. Assuming the collapsing SMS

structure is still close to that of an $n = 3$ polytrope (this is surely not the case when GR considerations become important), this means the trapped surface forms at about 20% of the total radius, which would make our naïve calculation an overestimate for the trapped surface size. The authors, however, do not report the central densities at which these trapped surfaces form, so for the purpose of this paper the $\rho_\bullet$ determined in Eq. (2.29) will be sufficient. The densities that satisfy that condition for various SMS homologous core masses have been computed numerically and are presented in Table 2.1.

Despite the small cross-section for NC elastic scattering and CC inelastic scattering (Eq. (2.13)), the high densities and temperatures involved behoove us to consider if the neutrinos become “trapped” within the SMS. We will consider the neutrinos trapped if their mean free path λ_{mfp} is of order the radius R and determine at what ρ_c this is the case. This condition yields the equation:

$$\lambda_{\text{mfp}} \sim R(\rho_{\nu \text{ trap}}) \sim \frac{\mu m_b}{\rho_{\nu \text{ trap}} \langle \sigma(\rho_{\nu \text{ trap}}) \rangle}, \quad (2.30)$$

where $R(\rho_c)$ is given in Eq. (2.24).

The cross section’s dependency on central density is determined from noting that a radiation dominated $n = 3$ polytrope’s central temperature is related to the central density via

$$T_c = \left(\frac{3K}{a} (1 - \beta) \right)^{1/4} \rho_c^{1/3}, \quad (2.31)$$

where $T_\nu \approx 1.6 T_c$ as discussed previously. The neutrino trapping densities $\rho_{\nu \text{ trap}}$ are also calculated numerically and are presented in Table 2.1.

It is clear that for some homologous core mass $\sim 10^5 M_\odot$, the neutrinos are trapped by scatterings at the same time as the formation of a gravitational trapped surface. Therefore, any SMS mass above this will permit neutrinos to escape the star even after the trapped surface is formed.

Table 2.1: Calculated central densities at which a trapped surface is formed ($\rho_\bullet$) and at which the neutrinos’ mean free path becomes less than the radius ($\rho_{\nu\text{ trap}}$). For homologous core masses $\gtrsim 10^5 M_\odot$, the black hole formation density is lower than the neutrino trapping density, meaning that neutrinos are free streaming from the core to the surface before a trapped surface is formed.

$M^{\text{HC}} [M_\odot]$	$\rho_\bullet [\text{g}/\text{cm}^3]$	$\rho_{\nu\text{ trap}} [\text{g}/\text{cm}^3]$
10^4	1.61×10^9	4.89×10^7
10^5	1.52×10^7	1.52×10^7
10^6	1.50×10^5	4.81×10^6
10^7	1.49×10^3	1.52×10^6
10^8	1.49×10^1	4.80×10^5

2.4 Neutrino flavor conversion

Neutrino flavor conversion is a complex quantum mechanical phenomenon that can have important consequences on the macroscopic scale. Dense astrophysical environments—such as core-collapse supernovae and neutron stars mergers—are two important venues where neutrinos are expected to undergo flavor conversion because of extremely high neutrino number densities near the neutrino-baryon decoupling regions. Neutrino forward scattering on background matter offers a simple phenomenology described by the MSW effect; however, neutrino forward scattering on background neutrinos can lead to a plethora of non-linear collective effects where neutrinos of different momenta couple to each other significantly [74]. These non-linear interactions lend a great deal of complexity to the oscillation phenomenology. Here we explore whether neutrino collective effects are important for SMSs.

In order to describe the oscillation phenomenology, one needs to know the energy scales present in the neutrino interaction Hamiltonian, which reads

$$H = H_{\text{vac.}} + H_{\text{matt.}} + H_{\nu\nu}, \quad (2.32)$$

where $H_{\text{vac.}}$ is the vacuum (i.e. free streaming) contribution, $H_{\text{matt.}}$ is the matter interaction

term, which in our case will describe forward scatterings of neutrinos on electrons, and $H_{\nu\nu}$ is the neutrino self-interaction term, describing forward scatterings of neutrinos with other neutrinos. The vacuum term is proportional to the vacuum oscillation frequency

$$\omega = \frac{\Delta m^2}{2E_\nu} \sim 6.3 \left(\frac{1 \text{ MeV}}{E_\nu} \right) \text{ km}^{-1}, \quad (2.33)$$

where $\Delta m^2 = m_2^2 - m_1^2$ is the difference between the squared mass eigenvalues in vacuum, E_ν is the neutrino energy, and we used $\Delta m^2 \sim 10^{-3} \text{ eV}^2$. This is roughly the scale for the wavenumber of flavor oscillations in vacuum. For the following section of this paper, wherein we discuss the matter term in the Hamiltonian in more detail, we will take $\Delta m^2 = \Delta m_{\text{atm.}}^2 \approx 2.4 \times 10^{-3} \text{ eV}^2$, the so-called “atmospheric” mass-squared splitting.

The matter term is only felt by ν_e s, and it interferes with the vacuum contribution to produce the MSW effect. It is proportional to the neutrino-electron interaction strength, with its scale being set by

$$\lambda = \sqrt{2} G_F n_e, \quad (2.34)$$

where $n_e = \rho/\mu m_b$ is the local number density of electrons. In the dense core, just before gravitational trapped surface formation, we expect $\lambda \gg \omega$, whereas at the relatively diffuse SMS outer layers we expect $\lambda \ll \omega$. Somewhere within the collapsing SMS homologous core, we expect $\lambda \sim \omega$, wherein the interesting MSW effect lies. A detailed exploration of this effect is provided in the forthcoming second, third, and fourth sub-sections of this section.

Before discussing the MSW effect, we will discuss collective flavor oscillations borne from the $H_{\nu\nu}$ term in the Hamiltonian and whether we expect these non-linear interactions to be important in collapsing SMS homologous cores.

2.4.1 Collective Flavor Oscillations

The Hamiltonian contribution that describes neutrino forward scattering on other neutrinos is fixed by the neutrino-neutrino interaction strength $\mu_{\nu\nu} = \sqrt{2}G_F n_\nu$, where n_ν is the number density of neutrinos. For a collapsing HC, Ref. [30] calculated the peak luminosity of e^+e^- pair-produced neutrinos to be $L_\nu \sim 10^{56} (M^{\text{HC}}/10^5 M_\odot)^{-3/2} \text{ erg s}^{-1}$.

Take for example a collapsing $10^5 M_\odot$ homologous core that forms a trapped surface at $R_\bullet \sim 2GM^{\text{HC}}$. The neutrinos have average energy $\langle E_\nu \rangle \sim 1 \text{ MeV}$ (since $E_\nu \sim T_\nu \sim T_c$ calculated from using $\rho_c = \rho_\bullet$ for $M^{\text{HC}} = 10^5 M_\odot$ in Eq. (2.31)), so the neutrino number density produced at the gravitational trapped surface $R_\bullet \sim 10^{11} \text{ cm}$ is $n_\nu \sim L_\nu / (4\pi c R_\bullet^2 E_\nu) \sim 10^{28} \text{ cm}^{-3}$. We can now calculate the neutrino-neutrino interaction strength scale $\mu_{\nu\nu}$ to be

$$\mu_{\nu\nu} \simeq \frac{\sqrt{2}G_F L_\nu}{4\pi c R_\bullet^2 E_\nu} \sim 10 \text{ km}^{-1}. \quad (2.35)$$

Thus, one can see that roughly $\mu_{\nu\nu} \sim \omega$, meaning that collective effects, if any, would occur on the same timescale as neutrino vacuum oscillations. SMS masses larger than $10^5 M_\odot$ see a precipitous drop in $\mu_{\nu\nu}$ relative to ω (both L_ν and $1/R_\bullet^2$ decrease quickly with increasing mass), meaning that collective effects are even less prominent.

Moreover, the neutrino emission derived entirely from annihilating thermal e^+e^- pairs, so the spatial distribution of these neutrinos is completely isotropic. Therefore, there are no crossings in the (anti)neutrino angular distributions, suggesting that fast flavor conversion (FFC) mechanisms [75, 76, 77] (see [78, 79] for recent reviews) are most likely not operative [80, 81, 82]. Slow flavor conversion (SFC), on the other hand, can lead to unstable solutions of the flavor evolution even for isotropic systems [74, 83] and non-isotropic systems without angular crossings [84]. If the distributions are perfectly isotropic, the system is described by the “single-angle approximation” which is unstable in the inverted mass ordering, i.e. $\omega \propto \Delta m^2 < 0$, and stable in the normal mass hierarchy $\omega > 0$ [74, 83]. In reality, however, the neutrino distributions are not exactly isotropic, and

even small perturbations in the emission can lead to multi-angle instabilities that break the axial symmetry [85]. The single-energy Hamiltonian describing neutrino-neutrino forward scattering is

$$H_{\nu\nu} = \mu_{\nu\nu} \int dv' (\hat{\rho}' - \alpha \hat{\bar{\rho}}')(1 - vv'), \quad (2.36)$$

where $\alpha = n_{\bar{\nu}_e}/n_{\nu_e}$ is the asymmetry parameter ($\alpha = 1$ in SMSs) and $v = \cos \theta_\nu$ is the cosine of the neutrino emission angle. $\hat{\rho}$ and $\hat{\bar{\rho}}$ are the neutrino and anti-neutrino density matrices, respectively, whose diagonal terms are proportional to the neutrino number densities and the off-diagonal terms encode coherence among flavors.

Lastly, a novel type of instability, the so-called collisional flavor instability (CFI), has been identified in core-collapse supernovae and neutron star mergers [86, 87, 88, 89, 90]. This type of instability, arises due to sufficiently distinct scattering rates on matter of ν_e and $\bar{\nu}_e$ and is most effective when $n_{\nu_e} \approx n_{\bar{\nu}_e}$, both conditions present in SMSs. SMSs can guarantee a large asymmetry between collision rates $\Gamma_{\nu_e} \sim n\sigma_{\text{NC}}$ and $\Gamma_{\bar{\nu}_e} \sim n(\sigma_{\text{NC}} + \sigma_{\text{CC}})$, where n is the number density of targets (roughly the number density of protons). These can differ by as much as $\Gamma_{\bar{\nu}_e}/\Gamma_{\nu_e} \simeq 4$ despite the exponential temperature cutoff dependence for $T \lesssim 1.8 \text{ MeV}$ in σ_{CC} .

The collision rates for neutrinos in SMSs are, however, very small, reaching only $\Gamma_{\nu_e}, \Gamma_{\bar{\nu}_e} \sim 10^{-3} \text{ km}^{-1}$ in the dense core for even the most favorable low mass scenario with $M^{\text{HC}} \sim 10^4 M_\odot$. Higher mass SMS homologous cores are less dense at gravitational trapped surface formation, and therefore less hot, lowering these scattering rates and their asymmetry. In SMSs, we can therefore conclude that collisional rates are so low that CFI is most likely unimportant even where it does arise. We leave a more thorough exploration of flavor instabilities in lower mass SMSs for future work.

All of that said, in this paper we can rather safely say that collective effects are not important. We will assume the normal mass hierarchy, so since $\mu_{\nu\nu} \sim \omega$ and because our neutrino distributions are perfectly isotropic (single-angle), no flavor instabilities will

occur. The scattering rates on matter for both ν_e s and $\bar{\nu}_e$ s are small, so collisional flavor instabilities are likely unimportant. These considerations indicate that there is basically no influence from collective effects in general, so in-medium flavor conversion will proceed exclusively through the MSW effect.

2.4.2 MSW Flavor Conversion

The primary neutrino production channel in collapsing SMSs is $e^+ + e^- \rightarrow \nu + \bar{\nu}$. This produces neutrino/anti-neutrino pairs of any flavor. If the reaction is mediated by a Z^0 boson, then the flavors are produced in equal amounts. If the mediator is a $W^\pm$ boson, however, $\nu_e \bar{\nu}_e$ -pairs will be produced to preserve lepton number at tree-level. As discussed previously, this causes a flavor asymmetry of about 5-to-1 in the relative numbers of $\nu_e \bar{\nu}_e$ -pairs over $\nu_\mu \bar{\nu}_\mu$ and $\nu_\tau \bar{\nu}_\tau$ -pairs.

After being created, neutrinos will oscillate between the e , μ , and τ flavors as they propagate through matter and vacuum. The former case is of particular interest to this paper as there exists a matter density—depending on the neutrino energy—such that this flavor mixing is maximal, *a.k.a.* the MSW effect. For simplicity, we will consider a scenario where there are only two neutrino flavors (weak eigenstates) $|\nu_e\rangle$ and $|\nu_x\rangle$, the latter representing both the μ and τ flavors. Neutrinos also have mass eigenstates $|m_1\rangle$ and $|m_2\rangle$ where $m_1 < m_2$. In this 2×2 case, we take the unitary transformation between flavor states and mass states in vacuum to be:

$$|\nu_e\rangle = \cos \theta |m_1\rangle + \sin \theta |m_2\rangle, \quad (2.37a)$$

$$|\nu_x\rangle = -\sin \theta |m_1\rangle + \cos \theta |m_2\rangle, \quad (2.37b)$$

where $\theta \neq 0$ is the vacuum mixing angle. Here, we will take $\theta = \theta_{13} \approx 0.15$ [1] so that the electron flavor is primarily in the smaller mass eigenstate in vacuum, i.e. the normal mass hierarchy. Neutrinos are always created in flavor eigenstates to preserve lepton number, but

the Schrödinger equation predicts that $|m_1\rangle$ and $|m_2\rangle$ will accumulate different complex phases as they propagate since $m_1 \neq m_2$.

The situation becomes more complicated in the presence of matter. While both flavors can forward scatter on electrons via NC Z^0 exchange:

$$\begin{aligned}\nu_e + e^- &\xrightarrow{Z^0} \nu_e + e^-, \\ \nu_x + e^- &\xrightarrow{Z^0} \nu_x + e^-, \end{aligned}$$

the lower mass of the electron also allows charged current $W^\pm$ -mediated forward scattering between electrons and ν_e only:

$$\nu_e + e^- \xrightarrow{W^\pm} e^- + \nu_e,$$

in effect giving the ν_e s an extra charged current potential A_{CC} . One can derive this extra potential to be [25]

$$A_{CC} = 2\sqrt{2}G_F E_\nu n_e, \tag{2.38}$$

where n_e is the electron number density. This additional potential is completely unfelt by the ν_x s as they propagate.

Similar to how photons traveling in a medium acquire an effective mass from electromagnetic interactions with the ambient electrons, the ν_e s now have an effective mass that increases with increasing A_{CC} . A higher electron number density forces the ν_e s to be mostly in the heavier mass eigenstate $|m_2\rangle$ instead of the lighter mass eigenstate $|m_1\rangle$, meaning that the in-medium mixing angle θ_M now depends on the surrounding matter.

This matter dependent mixing manifests as the superposition

$$|\nu_e\rangle = \cos \theta_M |\nu_1\rangle + \sin \theta_M |\nu_2\rangle, \quad (2.39a)$$

$$|\nu_x\rangle = -\sin \theta_M |\nu_1\rangle + \cos \theta_M |\nu_2\rangle, \quad (2.39b)$$

where θ_M is related to the vacuum mixing angle θ and the CC potential A_{CC} via [25]

$$\cos 2\theta_M = \frac{\Delta m^2 \cos 2\theta - A_{CC}}{\sqrt{(\Delta m^2 \cos 2\theta - A_{CC})^2 + (\Delta m^2 \sin 2\theta)^2}} \quad (2.40)$$

where $\Delta m^2 = m_2^2 - m_1^2$ is the difference of the squares of the mass eigenvalues in vacuum.

As stated previously, we will take $\Delta m^2 = \Delta m_{\text{atm.}}^2 \approx 2.4 \times 10^{-3} \text{ eV}^2$.

2.4.3 MSW Resonance Density

For dense environments where $A_{CC} \gg \Delta m^2 \cos 2\theta$, the fraction in Eq. (2.40) reduces to $\cos 2\theta_M \approx -1$, hence $\theta_M \approx \pi/2$. Therefore, the superposition in Eqs. (2.39a) and (2.39b) is such that $|\nu_e\rangle$ is primarily in the heavy mass state and $|\nu_x\rangle$ is primarily in the lighter state. Outside of the SMS, where $n_e \sim 0$ so $A_{CC} \ll \Delta m^2 \cos 2\theta$, it is instead the case that $\cos 2\theta_M \rightarrow \cos 2\theta$, so the superposition reduces to the standard vacuum case.

Between these two extremes, where $A_{CC} \sim \Delta m^2 \cos 2\theta$, the numerator vanishes so $\cos 2\theta_M \sim 0$, meaning that $\theta_M \sim \pi/4$. This condition corresponds to maximal mixing between the two flavor states which establishes a “resonant density” $n_e^{\text{res.}}$ given by

$$n_e^{\text{res.}} = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}G_F E_\nu} \approx 9.6 \times 10^{27} \cos 2\theta \left(\frac{E_\nu}{1 \text{ MeV}} \right)^{-1} \text{ cm}^{-3}, \quad (2.41)$$

which corresponds to a mass density of

$$\rho_{\text{res.}} = \frac{2}{1+X} m_b n_e^{\text{res.}} \approx 1.9 \times 10^4 \cos 2\theta \left(\frac{E_\nu}{1 \text{ MeV}} \right)^{-1} \frac{\text{g}}{\text{cm}^3}, \quad (2.42)$$

where $X \approx 0.75$ is the mass fraction of hydrogen in a primordial gas cloud.

Averaging over the distribution described in Eq. (2.5) we find that the resonant density in terms of the plasma temperature T is

$$\langle \rho_{\text{res.}} \rangle \approx 4.1 \times 10^3 \cos 2\theta \left(\frac{T}{1 \text{ MeV}} \right)^{-1} \frac{\text{g}}{\text{cm}^3}, \quad (2.43)$$

where we have again used $T_\nu \approx 1.6 T$ as per Ref. [30].

A plot of where this resonance occurs in the homologous core at the black hole forming density $\rho_\bullet$ as a function of homologous core mass is in Figure 2.1. The plasma temperature T was taken to be the central temperature T_c , related to $\rho_\bullet$ by Eq. (2.31). Also plotted in the same figure is the neutrino mean free path at $\rho_\bullet$ for neutrinos emitted from the core. The region following the intersection of these two curves is of interest as those neutrinos would undergo MSW resonance and are likely to escape the homologous core without being trapped by scatterings. A final addition to the figure is the location of the gravitational trapped surface at $\rho_\bullet$ using the naïve prescription detailed in Section 2.3.2. While the precise location of this curve should be calculated with an approach that tackles the collapse with full GR, it should still be a guide to the eye for where effects from the MSW resonances are suppressed as material may be trapped behind an event horizon.

2.4.4 Adiabatic Flavor Conversion

Analogous to the geometric optics limit for light waves and lenses/mirrors, *en masse* flavor conversion between neutrino species in non-uniform media is only observed if the length scale on which the density changes is much larger than the flavor oscillation wavelength. The location where this matters most is at densities around ρ_{res} since the

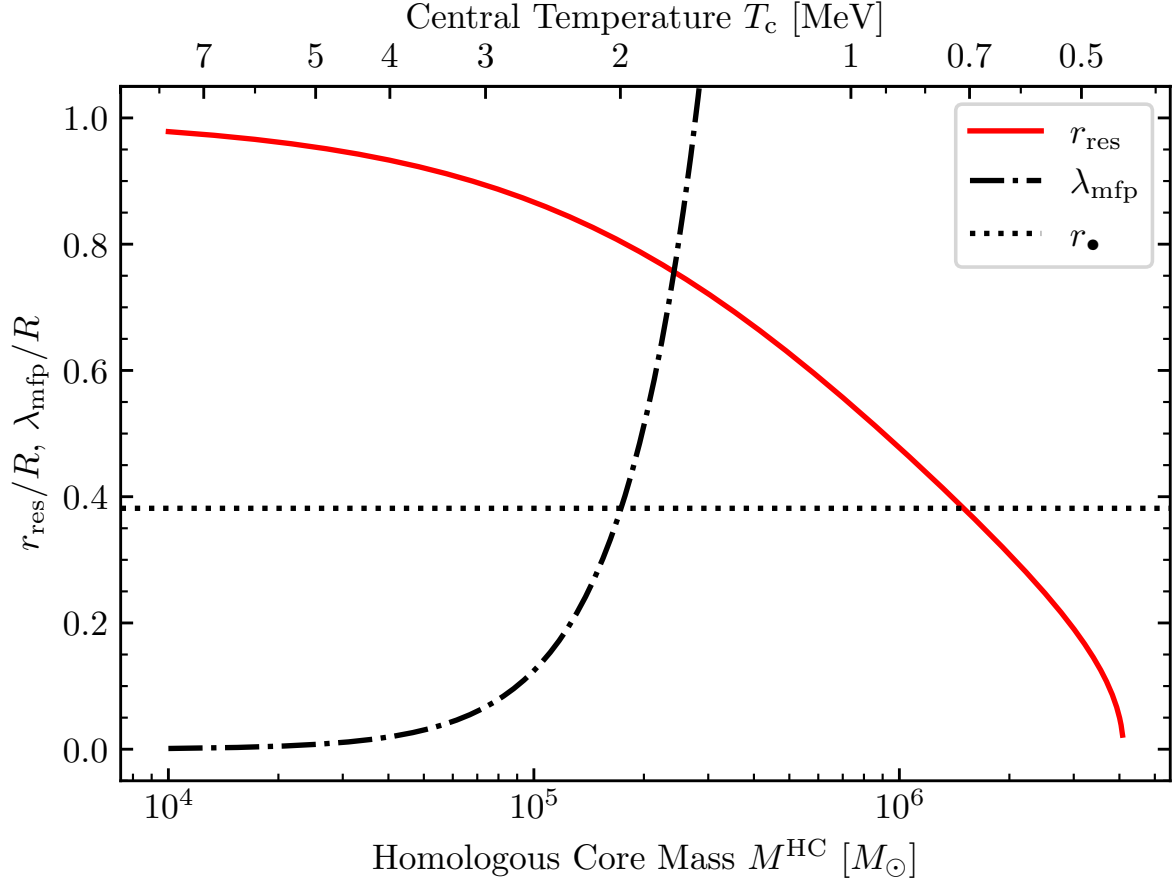


Figure 2.1: The radial location of the resonant density $\langle \rho_{\text{res.}} \rangle$ (solid red line, scaled by the HC radius) at the black hole forming central density $\rho_{\bullet}$ as a function of the HC mass M^{HC} on the bottom x -axis and the central temperature T_c on the top x -axis. The mean free path of neutrinos produced at the center as a ratio of the HC radius is also plotted as the dashed-dotted black line. We have chosen the neutrino mass-splitting and mixing angle to be, respectively, $\Delta m^2 = \Delta m_{\text{atm.}}^2 \approx 2.4 \times 10^{-3} \text{ eV}^2$ and $\theta = \theta_{13} \approx 0.15$. The radial location of the trapped surface (per Eq. (2.29)) as a fraction of the HC radius is plotted as a dotted black line.

flavor mixing is maximal and the in-media mass-squared difference, Δm_{M}^2 , is minimal:

$$\begin{aligned}\Delta m_{\text{M}}^2 &= \sqrt{(\Delta m^2 \cos 2\theta - A_{\text{CC}})^2 + (\Delta m^2 \sin 2\theta)^2}, \\ &\stackrel{\text{res.}}{=} \Delta m^2 \sin 2\theta.\end{aligned}\tag{2.44}$$

Following the prescription in Ref. [27], we can introduce an “adiabaticity parameter” at the resonant density, notated as $\gamma_{\text{res.}}$, given by

$$\gamma_{\text{res}} = \frac{\Delta m^2 \sin^2 2\theta}{2E_\nu \cos 2\theta} \left| \frac{1}{\rho} \frac{d\rho}{dr} \right|_{\text{res.}}^{-1},\tag{2.45}$$

which is the ratio of the resonance width $\delta r = |d \ln(\rho)/dr|^{-1} \tan 2\theta \sim R \tan 2\theta$ to the flavor oscillation wavelength at the resonance density. The probability that a neutrino will “jump” to the other mass state at resonance is $P \sim \exp(-\frac{\pi}{2} \gamma_{\text{res.}})$ [91]. If the neutrino does indeed jump from one mass state to the other at the resonant density, there will be no flavor conversion, i.e. the correlation between the flavor and mass states remains the same between the high density and vacuum environments.

On the other hand, if $\gamma_{\text{res}} \gg 1$, the probability of a mass state jump is negligible and *en masse* flavor transformation will occur. If we calculate the adiabaticity parameter for $M^{\text{HC}} = 10^4 M_\odot$, we find $\gamma_{\text{res}} \sim 500$, so the jump probability is exponentially suppressed. For masses greater than $10^4 M_\odot$, the neutrino energy decreases and the density scale height increases, so γ_{res} increases in turn and the jump probability continues to be exponentially suppressed.

If a ν_e is created in an initially dense environment where $\cos 2\theta_{\text{M}} \sim -1$, passes through the resonance density in Eq. (2.41), then continues out to where $\cos 2\theta_{\text{M}} \sim \cos 2\theta$, the probability that it remains in the ν_e state upon interacting with another particle is

found to be approximately [25]

$$P_{\nu_e \rightarrow \nu_e} \approx \frac{1}{2} \left(1 + \cos 2\theta_M^{(i)} \cos 2\theta \right), \quad (2.46)$$

if $\gamma_{\text{res.}} \gg 1$, where $\theta_M^{(i)}$ is the initial matter mixing angle when the ν_e was created. Clearly, for $\cos 2\theta_M^{(i)} \sim -1$ and $\cos 2\theta \sim 1$, almost all of the ν_e s created will be converted to ν_μ s and ν_τ s.

The same resonant flavor transformation applies to the ν_x s, so we would therefore expect nearly complete swapping of flavor labels for the e and x flavor neutrinos after encountering adiabatic resonance. While it may seem at first that these effects cancel out and there is no net change in the relative abundances of flavors, recall that neutrinos produced via $e^+ + e^- \rightarrow \nu + \bar{\nu}$ have an overabundance of $\nu_e \bar{\nu}_e$ -pairs relative to $\nu_\mu \bar{\nu}_\mu / \nu_\tau \bar{\nu}_\tau$ -pairs by a factor of about 5-to-1 [30]. As discussed in the next subsection, this asymmetry at creation will have drastic downstream effects for the final flavor flux ratios after MSW adiabatic flavor transformation.

We have performed this analysis in the normal mass hierarchy, so $\bar{\nu}_e$ will not experience this maximal mixing resonance with the other anti-neutrino flavors. In the inverted mass hierarchy $m_1 > m_2$, however, we would see maximal flavor mixing between the $|\bar{\nu}_e\rangle$ and $|\bar{\nu}_x\rangle$ states with the $|\nu_e\rangle$ and $|\nu_x\rangle$ states unaffected, the opposite of the case just discussed.

2.4.5 Ratio of Flavor Fluxes

Our simplistic picture of a 2 flavor case can be easily cast into the real 3 flavor case by notable coincidence: the $|\nu_\mu\rangle$ and $|\nu_\tau\rangle$ weak eigenstates are nearly maximally mixed ($\sin \theta_{23} \sim 1/\sqrt{2}$ [1]). We can therefore rewrite the “ x ” flavor as (following Refs. [28, 92])

$$|\nu_x\rangle = \frac{1}{\sqrt{2}}(|\nu_\mu\rangle + |\nu_\tau\rangle), \quad (2.47)$$

and introduce a new flavor x' that is fully decoupled from the e and x flavors:

$$|\nu_{x'}\rangle = \frac{1}{\sqrt{2}}(|\nu_\mu\rangle - |\nu_\tau\rangle). \quad (2.48)$$

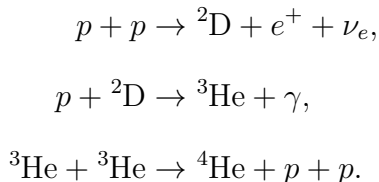
Recall now that Ref. [30] found that $\nu_e\bar{\nu}_e$ pairs have a ~ 4.7 times greater flux than the $\nu_\mu\bar{\nu}_\mu/\nu_\tau\bar{\nu}_\tau$ pairs for $e^\pm$ annihilation-produced neutrinos in the centers of collapsing SMSs. This dense environment gives the neutrinos created in the $|\nu_e\rangle$ state a higher effective mass than those in the $|\nu_x\rangle$ state on account of the extra CC potential $A_{CC} \propto n_e$. These neutrinos stream through the SMS HC almost entirely unimpeded, pass through the MSW resonant flavor mixing density somewhere in the bulk of the HC, and continue to the diffuse outer layers. Since this process is adiabatic for these neutrino energies and SMS density profiles, the net result is an almost complete swapping of flavor labels between e flavor and x flavor neutrinos, the latter representing about one-half of the μ and τ flavor neutrinos. Whether this occurs for neutrinos or anti-neutrinos depends on the mass hierarchy.

In the normal mass hierarchy $m_1 < m_2$, the neutrino flavors are maximally transformed. The new ratio of flavors is closer to $\nu_e/\nu_{\mu,\tau} \rightarrow 1/5$, an inversion of the original ratio at the SMS center. The other half of μ and τ flavor neutrinos, represented by the x' state, are decoupled and hence unaffected. The anti-neutrino flavors are unaffected. This is in contrast to the inverted mass hierarchy $m_1 > m_2$, where the MSW resonance inverts the anti-neutrino flavor flux ratio to $\bar{\nu}_e/\bar{\nu}_{\mu,\tau} \rightarrow 1/5$, leaving the neutrino flavors unaffected.

Per Figure 2.1, we predict a wide range of SMS homologous core masses to exhibit nearly complete flavor transformation of core-produced neutrinos or anti-neutrinos at some point in their collapse.

2.5 Neutrino Capture Induced Burning

For stars burning on the main sequence with low metallicity, the principle p - p reaction chain follows as [12]



The first step of this chain involves converting a proton into a neutron, a weak interaction process, making this the primary bottleneck for this reaction.

This bottleneck can be overcome if there is already a supply of free neutrons in the gas. While, normally, free neutrons decay into protons on a rather short time scale, neutrons can be made via inverse- β decay through the reaction $\bar{\nu}_e + p \rightarrow n + e^+$. Assuming equal amounts of ν_e and $\bar{\nu}_e$, the reverse reaction $\nu_e + n \rightarrow p + e^-$ and neutron decay would basically ensure no such supply can be maintained.

If, however, the neutrinos experience resonant flavor mixing on their way out of the core, the amount of ν_e will be heavily suppressed relative to the amount of $\bar{\nu}_e$. Ref. [62] found that the probability that inverse- β decay occurs at radius r is given approximately by

$$P_{\beta^{-1}} \approx 4.5 \times 10^{-6} f_E f_{\bar{\nu}_e} M_5^{\text{HC}} \left(\frac{\langle E_\nu \rangle}{1 \text{ MeV}} \right) \left(\frac{10^{13} \text{ cm}}{r} \right)^2 \quad (2.49)$$

where $f_E \approx 0.1$ is the amount of gravitational redshift experienced by the neutrinos, $f_{\bar{\nu}_e} \sim 0.4$ is the fraction of all neutrinos which are $\bar{\nu}_e$, and M_5^{HC} is the homologous core mass in units of $10^5 M_\odot$. Most neutrons created in this process will almost immediately find a proton and fuse to ${}^2\text{D}$. If the rate of $n + p \rightarrow {}^2\text{D} + \gamma$ is fast compared to both the material ejection and neutron decay rates *and* slow compared to compared to the $p + {}^2\text{D} \rightarrow {}^3\text{He} + \gamma$ rate, then the probability in Eq. (2.49) is also the fraction of hydrogen

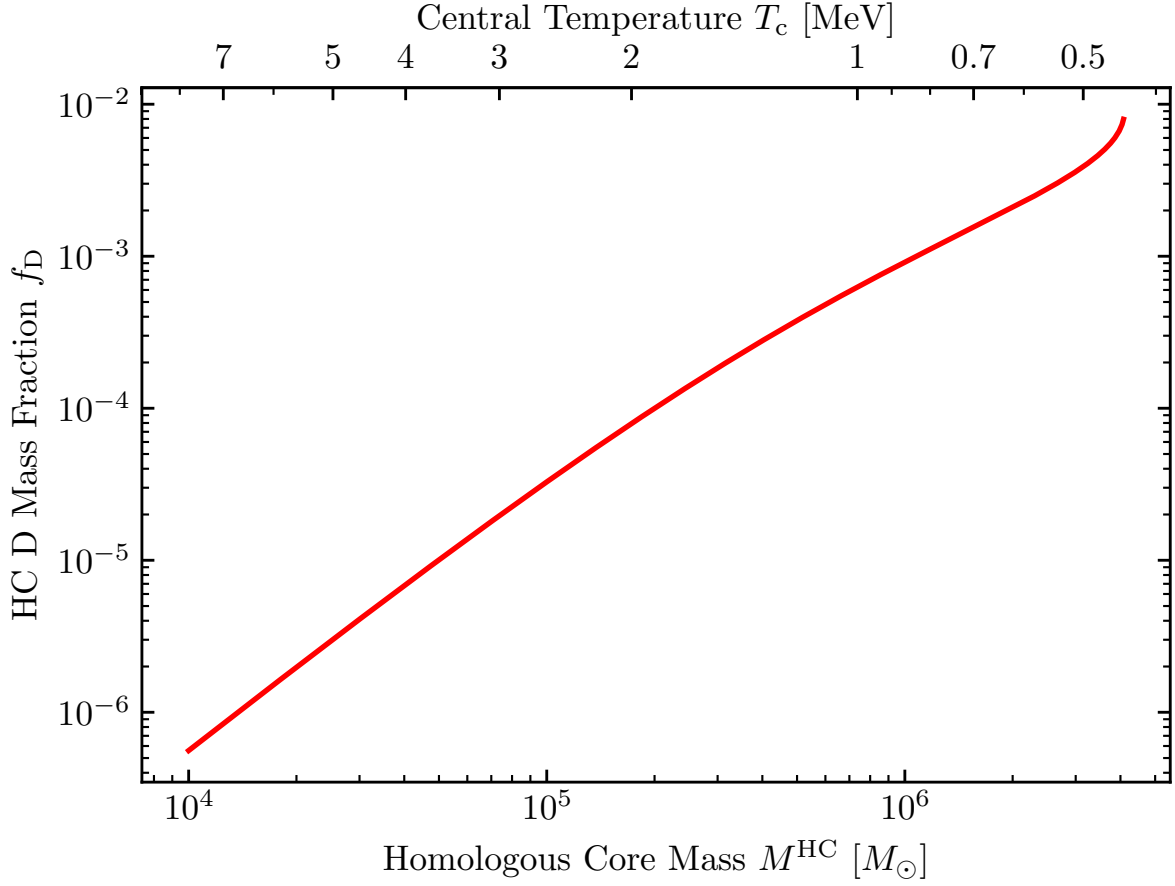


Figure 2.2: The mass fraction of the SMS homologous core that is converted to deuterium in the region after MSW resonance as a function of the homologous core mass. For $M^{\text{HC}} \gtrsim \text{few} \times 10^5 M_{\odot}$, the mass fraction calculation may be dubious (see Figure 2.1) as the MSW resonance density occurs within the naïvely calculated gravitational trapped surface horizon at $\rho_c = \rho_{\bullet}$.

converted to ${}^2\text{D}$ at a given radius r .

Following this assumption, the mass fraction of the SMS homologous core that is converted to deuterium is

$$f_{\text{D}} = \frac{1}{M^{\text{HC}}} \int_{r_{\text{res}}}^R 4\pi r^2 \rho_p(r) P_{\beta-1}(r) dr, \quad (2.50)$$

where $\rho_p = X\rho$ is the mass density of free protons if X is the mass fraction of hydrogen.

With the standard Lane-Emden coordinate transformation, this mass fraction is

$$f_D \approx 3.3 \times 10^{-4} f_E f_{\bar{\nu}_e} \left(\frac{T_c}{1 \text{ MeV}} \right) \frac{X \rho_c (10^{13} \text{ cm})^2 \alpha_{\text{LE}}}{10^5 M_\odot} \int_{\xi_{\text{res}}}^{\xi_3} \varphi^3(\xi) d\xi, \quad (2.51)$$

where $\langle E_\nu \rangle \approx 1.6 \frac{F_3(\eta_\nu)}{F_2(\eta_\nu)} T_c$ is the thermally averaged neutrino energy as a function of the central temperature, r_{res} is the location of the MSW resonance density (see the solid red line in Fig. 2.1), and $\xi_{\text{res}} = r_{\text{res}}/\alpha_{\text{LE}}$. Evaluation of this mass fraction at $\rho_c = \rho_\bullet$ as a function of SMS homologous core mass is presented in Figure 2.2. For a homologous core mass of $M^{\text{HC}} \sim 10^6 M_\odot$, this means that around a few tenths of a percent of the HC is converted to deuterium, with some unspecified amount more transmuted in the SMS envelope that did not collapse homologously. This makes what is presented in Figure 2.2 a conservative underestimate of the amount of deuterium produced because $\bar{\nu}_e$ s will continue reacting on free protons outside of the homologous core. Depending on local temperatures and densities, the deuterium could then readily react with the surrounding matter, potentially up to α particles.

2.6 Conclusion

Supermassive stars may yet be a key to understanding the very early universe and the structure formation therein, providing an explanation—however unlikely—to the apparent overabundance of very massive black holes soon after the Big Bang. In this paper, we have discussed their structure, which is well approximated by an $n = 3$ polytrope, and how general relativistic corrections to Newtonian gravity cause their eventual rapid collapse. During this infalling period on the short time scale of $\tau_{\text{ff}} \sim 4.8 \times 10^{-5} (M/10^5 M_\odot)^{7/4} \text{ yr}$, the central core will rapidly increase in density and temperature, causing copious creation of $\nu\bar{\nu}$ pairs. Charged and neutral current weak interactions, as well as the energetics involved in this environment, favor the creation of $\nu_e \bar{\nu}_e$ pairs over $\nu_\mu \bar{\nu}_\mu$ and $\nu_\tau \bar{\nu}_\tau$ pairs by a factor of ~ 5 -to-1. We showed that the conditions, at least for homologous core

masses of $H^{\text{HC}} \gtrsim 10^4 M_\odot$, likely do not facilitate appreciable collective oscillation effects as the neutrino number densities are too low. We then proceeded to show that, for SMS homologous core masses between $10^4 M_\odot < M^{\text{HC}} \lesssim 10^6 M_\odot$ and at the central density where a gravitational trapped surface is formed, there exists a density away from the core where there are MSW resonances where flavor conversion is adiabatic. This maximal flavor mixing (at least in the normal mass hierarchy) serves to completely swap the flavor labels between ν_e s with ν_μ and ν_τ , leaving the $\bar{\nu}$ s of all flavors unaffected. Therefore, the electron anti-neutrinos will outnumber the electron neutrinos by $n_{\bar{\nu}_e}/n_{\nu_e} \sim 2.5$ in the outer layers of the SMS beyond the MSW resonance location. We then show this enhanced $\nu_e \bar{\nu}_e$ -asymmetry leads to an appreciable creation of free neutrons in the outer layers of the SMS. Clearly, however, questions for future work abound.

While it is clear that the neutron-to-proton ratio would change in the SMS as a result of these MSW resonance neutrinos, what implications would that have? Ref. [62] showed that there are a wide range of conditions where any free neutrons created would readily find a proton to create deuterium, implying that Eq. 2.49 also gives the fraction of hydrogen that that is converted to deuterium. With the weak bottleneck in the p - p chain overcome, it would also be reasonable to assume at least a substantial amount of the deuterium would eventually be incorporated into α particles, releasing ~ 32 MeV of energy into the diffuse outer layers of the SMS per reaction. As SMSs are very well described by $n \sim 3$ polytropes, there is very little gravitational binding energy holding on to the outer layers. Is this energy deposition enough to perhaps eject material from the already loosely held together SMS? Does the SMS reach a neutrino Eddington-like luminosity from elastic scatterings? That is, can outer layers be ejected by substantial neutrino flux alone? Are these questions made moot by the entirety of the SMS falling into its own black hole regardless? As opposed to our analysis, which tried to be as analytic as possible, all of these questions likely need detailed simulations that includes robust neutrino processes, energy transport, and full general relativity to be answered satisfactorily. Nevertheless,

our results suggest that the MSW resonance in SMSs is something worthy of consideration when analyzing these objects, and warrants future study.

Collective flavor oscillation effects were touched upon in this paper, and we determined that their contribution is likely unimportant at least for $M^{\text{HC}} \gtrsim 10^4 M_\odot$. In this mass range, the number density of neutrinos is such that $\sqrt{2}G_F n_\nu \sim \Delta m^2/(2E_\nu) \ll \sqrt{2}G_F n_e$ in the core, i.e. any collective effects occur on the same timescale as vacuum oscillations and are swamped by matter effects. Collective flavor oscillations would serve to equilibrate the initial flavor asymmetry at the core, severely dampening the MSW enhanced neutron production after MSW resonance. Since collective flavor oscillations are a non-linear effect, however, a full integration of the neutrino flavor quantum kinetic equations in these SMSs is warranted instead of just examining relative energy scales.

Furthermore, energy scale comparisons suggest that collective effects would be important for the lighter SMS homologous core masses. As the neutrinos are effectively trapped in the core by scatterings for $M^{\text{HC}} \lesssim \text{few} \times 10^5 M_\odot$ anyway, this regime was not studied in detail in this work, but future work is needed to examine it further. At these higher densities, there may also be substantial ν_e flux from electron capture on protons, thereby altering the neutron-to-proton ratio in the core and creating an asymmetry in ν_e s and $\bar{\nu}_e$ s. The former consideration has implications for energy deposition in the core region during collapse, and the latter has further implications for collective flavor effects as the assumption of $n_{\nu_e}/n_{\bar{\nu}_e} \sim 1$ in the core breaks down.

Finally, being such massive objects in the early universe, it is also worth examining dark matter's effect on this entire process. It has been shown that (see, e.g. [48, 47, 67, 66, 67]) a substantial dark matter mass fraction in SMSs increases their stability, that is, the critical density at which the SMS is unstable to collapse is increased. This means that, before the gravitational trapped surface is formed, the core is denser and hotter than what's considered in this paper, so the neutrino emissivity during collapse is greatly amplified. This means that more entropy is carried away from the core, causing the

homologous core mass to likely be less massive than the figure of $M^{\text{HC}} \sim 0.1M^{\text{init.}}$ from Ref. [30]. Another possibility is that dark matter plays a role via dynamical heating, for example in Supermassive Dark Stars [68].

SMSs are of great interest for study, and may hold the key to understanding why there are so many supermassive black holes in the early universe of such high masses. We have examined, being as analytic as possible, the realization that a confluence of neutrino energies and matter densities in SMSs are such that an MSW resonance occurs for the prodigious amount of neutrinos produced in and leaving the core. This effect greatly alters the initial neutrino/anti-neutrino flavor distribution, leading to a substantial amount of free neutrons in the outer layers. Whether this process may provide an observational handle on these objects remains to be seen, but the subsequent evolution of these SMSs post instability will be impacted by this effect. Future work that studies this evolution in more detail with this MSW flavor oscillation resonance will be more complete, offering greater insight into these objects and the early universe in general.

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This work made use of the following open-and-closed-source software and programming packages: `Jupyter` [93, 94], `numpy` [95], `matplotlib` [96], `python` [97], `scipy` [98, 99], and `Mathematica` [26]. This research has made use of NASA’s ADS. Software citation information aggregated using `The Software Citation Station` [100, 101].

Appendix A

Validity of Complete Homologous Contraction Pre-Instability

For $n \sim 3$ polytropes, quasi-static contraction will be complete and homologous if $K(s)$ decreases by no more than $\sim 2.9\%$ before free-fall begins [17]. For radiation dominated SMSs, this change in $K(s)$ would be due to photons leaving the SMS surface at the Eddington rate $L_{\text{Edd.}} \approx 1.26 \times 10^{38} (M/M_{\odot}) \text{ erg s}^{-1}$ and carrying away entropy to the surrounding vacuum.

The rate at which the central density would increase from this radiation emitted is given by

$$-\frac{dE}{dt} = L_{\text{Edd.}} = \frac{2}{3} k_4 G^2 M^{7/3} \rho_c^{-1/3} \frac{d\rho_c}{dt}, \quad (\text{A.1})$$

where we have ignored the first two terms for the total energy in Eq. (2.18) as per the virial theorem applied to an $n \sim 3$ polytrope.

Since the SMS is not in thermal equilibrium, the luminosity profile as a function of the enclosed mass coordinate $L(m)$ satisfies the differential equation

$$\frac{\partial L}{\partial m} = \epsilon - \frac{T}{m_b} \frac{\partial s}{\partial t}, \quad (\text{A.2})$$

with boundary conditions $L(0) = 0$ and $L(M) = L_{\text{Edd.}}$, where ϵ is the net heat generation/loss, typically from fusion and neutrino emission respectively. Per Chapter 1, these are insubstantial compared to the Eddington luminosity during the quasistatic contraction (pre-GR instability) phase, so we will ignore ϵ . These SMSs are also fully convective, so their entropy profile is constant throughout the structure, allowing us to replace the partial derivative with respect to time with a full derivative.

Integrating across the entire mass of the star, we have

$$\int_0^M \frac{\partial L}{\partial m} dm = L_{\text{Edd.}} = -\frac{1}{m_b} \frac{ds}{dt} \int_0^M T dm. \quad (\text{A.3})$$

Mass continuity allows us to integrate instead over r via $dm = 4\pi r^2 \rho dr$:

$$L_{\text{Edd.}} = -\frac{4\pi}{m_b} \frac{ds}{dt} \int_0^R \rho T r^2 dr. \quad (\text{A.4})$$

The standard Lane-Emden substitutions are as follows:

$$r = \alpha_{\text{LE}} \xi = \left(\frac{K}{\pi G} \right)^{1/2} \rho_c^{-1/3} \xi, \quad (\text{A.5})$$

$$R = \alpha_{\text{LE}} \xi_3, \quad (\text{A.6})$$

$$\rho(r) = \rho_c \varphi^3(\xi), \quad (\text{A.7})$$

$$T(r) = T_c \varphi(\xi) = \left[\frac{3K}{a} (1 - \beta) \right]^{1/4} \rho_c^{1/3} \varphi(\xi), \quad (\text{A.8})$$

leaving us with

$$L_{\text{Edd.}} = -\zeta \frac{(1 - \beta)^{1/4} K^{7/4}}{m_b G^{3/2} \rho_c^{1/3}} \frac{ds}{dt}, \quad (\text{A.9})$$

where ζ is a numerical factor calculated as

$$\zeta = \frac{4}{\sqrt{\pi}} \left(\frac{3}{a} \right)^{1/4} \int_0^{\xi_3} \varphi^4(\xi) \xi^2 d\xi \approx 3.895. \quad (\text{A.10})$$

Equating (A.1) and (A.9), we find that

$$ds = -\frac{2k_4}{3\zeta} \frac{G^{7/2} M^{7/3}}{(1-\beta)^{1/4} K^{7/4}} \frac{m_b}{\rho_c^{2/3}} d\rho_c. \quad (\text{A.11})$$

We can re-write $K = K_0 s^{4/3}/(1-\beta)$ where $K_0 = 0.28429/1000^{4/3} \text{ MeV}^{-4/3}$ to isolate the explicit dependence on s :

$$s^{7/3} ds = -\frac{2k_4}{3\zeta} (1-\beta)^{3/2} \frac{G^{7/2} M^{7/3} m_b}{K_0^{7/4}} \frac{d\rho_c}{\rho_c^{2/3}}, \quad (\text{A.12})$$

and integrating both sides from the initial state ($\rho_c = 0$, $s_0 = s(M)$ (Eq. (2.15)) to the final state ($\rho_c = \rho_{c,\text{crit.}}$, $s = s_f$), we find that

$$s_f = \left[s_0^{10/3} - \frac{20k_4}{3\zeta} (1-\beta)^{3/2} \frac{G^{7/2} M^{7/3} m_b}{K_0^{7/4}} \rho_{c,\text{crit.}}^{1/3} \right]^{3/10}. \quad (\text{A.13})$$

The corresponding fractional change in $K(s)$ is plotted in Figure A.1. From Ref. [17], The mass of the homologous core that will undergo contraction is

$$M^{\text{HC}} \approx \left(\frac{K(s_f)}{K(s_0)} \right)^{3/2} M_{\text{init.}} \quad (\text{A.14})$$

if $\Delta K/K_0$ is more than about 2.9%. Fig. A.1 shows that for most of the SMS mass range between 10^4 – $10^8 M_\odot$, this fractional change is below 2.9%, so the entire SMS will quasistatically contract homologously before GR instability sets in.

Chapter 2, in full, except for the included appendix, has been submitted for review and eventual publication of the material as it may appear in Phys. Rev. D, 2026, Kehrer, Kyle; Fuller, George; Padilla-Gay, Ian; Kishimoto, Chad. The dissertation author was the primary investigator and author of this paper.

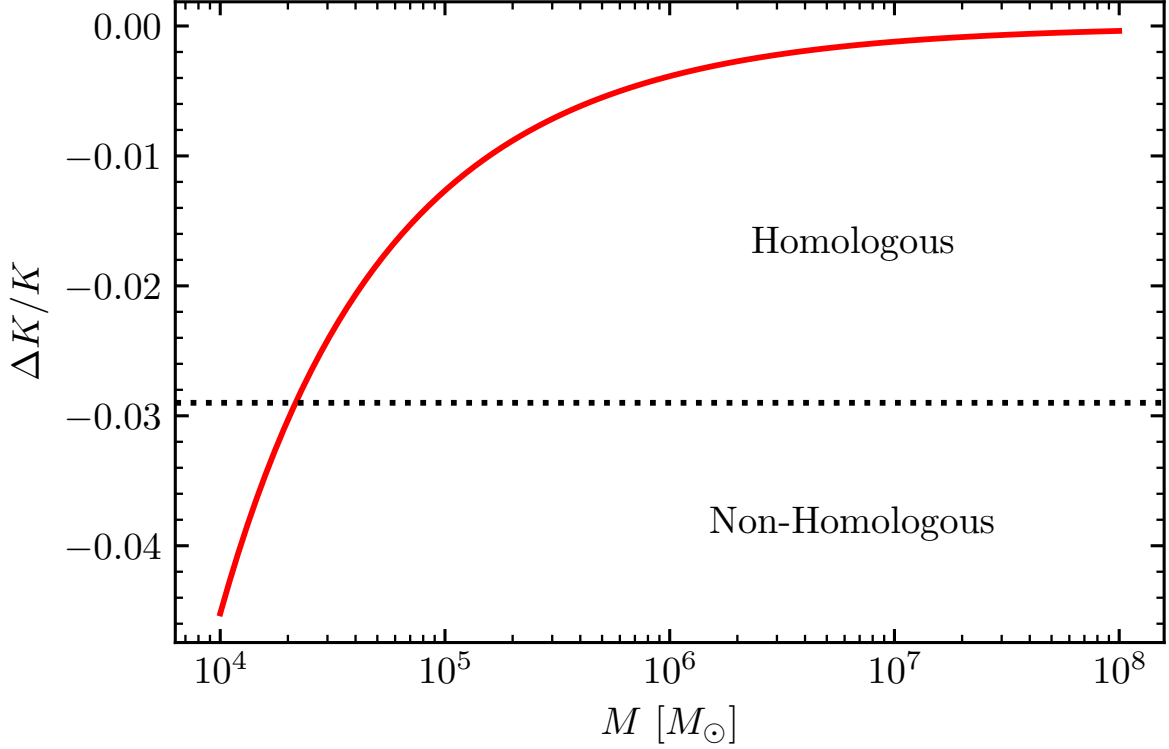


Figure A.1: Fractional change in the polytropic constant $K(s)$ of an SMS from the initially diffuse state ($\rho_c \sim 0$) to the critical state as a function of the SMS mass M . The fractional change in the polytropic constant $\Delta K/K$ is found from determining the change in entropy-per-baryon via Eq. (A.13) and comparing it to the initial entropy, calculated from Eq. (2.15). SMSs with a fractional drop in $K(s)$ less than $\sim 2.9\%$ will quasistatically contract entirely and homologously, whereas those with a greater loss than this threshold will homologously contract with a slightly smaller mass, specifically found via Eq. (A.14). The quasistatic contraction is only marginally non-homologous for SMS masses less than $\sim 2 \times 10^4 M_\odot$.

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