Lawrence Berkeley National Laboratory

Recent Work

Title

THE LIE-TRANSFORMED VLASOV ACTION PRINCIPLE: RELATIVISTICALLY COVARIANT WAVE PROPAGATION AND SELF-CONSISTENT PONDEROMOTIVE EFFECTS

Permalink

<https://escholarship.org/uc/item/5hv0m7qv>

Authors

Kaufman, A.N.

Holm, D.D.

Publication Date

1984-08-01

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

RECEIVED
LAWRENCE
BERKELEY LABORATORY

NOV 20 1984

LIBRARY AND
DOCUMENTS SECTION

Submitted to Physics Letters A

THE LIE-TRANSFORMED VLASOV ACTION PRINCIPLE:
RELATIVISTICALLY COVARIANT WAVE PROPAGATION AND
SELF-CONSISTENT PONDEROMOTIVE EFFECTS

A.N. Kaufman and D.D. Holm

August 1984

For Reference

Not to be taken from this room

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

**The Lie-Transformed Vlasov Action Principle:
Relativistically Covariant Wave Propagation and
Self-Consistent Ponderomotive Effects***

**Allan N. Kaufman
Physics Dept. and Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720**

and

**Darryl D. Holm
Center for Nonlinear Studies and Theoretical Division
Los Alamos National Laboratory
Los Alamos, NM 87545**

August 1984

* Research was performed at the Aspen Center for Physics Plasma Workshop, June 4-7, 1984, and was supported by the Office of Energy Research, U.S. Dept. of Energy under Contract Number DE-AC03-76SF00098.

Abstract

A covariant action principle is formulated for the Vlasov-Maxwell system. Upon subjecting the particle phase space and invariant Hamiltonian to a Lie transform for interaction with an eikonal wave, we obtain a new action principle for the invariant oscillation-center distribution and the self-consistent wave propagation, with a common kernel for the invariant ponderomotive Hamiltonian and the linear susceptibility.

Some years ago, Dewar [1] showed how the concepts of oscillation center and ponderomotive force followed naturally from a canonical transformation eliminating the linear nonresonant interaction of a particle with a given wave. Since then, there has been an evident need to introduce self-consistency, so that the wave propagation is described at the same conceptual level. Previous Hamiltonian treatments, such as those of Dewar [1], of Johnston [2], of Cary [3], and of Dubin, Krommes, Oberman, and Lee [4], have been constructive, requiring deep insight, and sometimes leading to ambiguity. Recently we have discovered a deductive approach [5,6], by means of applying the standard Lie transform to the phase space and Hamiltonian of the Vlasov action principle of Lewis and Symon [7]. Our first results dealt with the problem of Coulomb interaction, either at the gyrokinetic level [5] (as posed by Dubin et al [4]), or for eikonal waves [6]. This work explicated the intimate relation [8,9,3] between the linear susceptibility and the ponderomotive Hamiltonian. We have also developed an action principle for investigating the ponderomotive stabilization of a mirror-confined plasma [10,11].

Motivated by the upsurge of interest in relativistic plasmas, we have formulated a covariant treatment, which we present here, for the self-consistent nonresonant interaction of an electromagnetic wave (in eikonal form) and an unmagnetized distribution of oscillation centers. In subsequent papers, we plan to discuss resonance [12], magnetized plasma [13], and wave-wave interactions.

Our first step is to self-consistently derive (a) the Ignatiev equation [14]:

$$\{f(z), H(z)\} = 0 \quad (1)$$

for the invariant Vlasov distribution $f(z)$ in terms of the invariant particle Hamiltonian $H(z)$, and (b) the Maxwell equation $F^{\mu\nu}_{, \nu} = 4\pi j^\mu$. Here z is a point in eight-dimensional phase space (r, p) , with $r^\mu = (\vec{r}, t)$, $p_\mu = (\vec{p}, -h)$; $H = (p - eA(r))^2/2m$ yields the correct covariant Hamiltonian equation: $dr^\mu/d\tau = \partial H/\partial p_\mu$, $dp_\mu/d\tau = -\partial H/\partial r^\mu$, implying $du_\mu/d\tau = (e/m)F_{\mu\nu} u^\nu$; the Poisson Bracket is the canonical covariant expression $\{a, b\} = (\partial a/\partial r^\mu)(\partial b/\partial p_\mu) - (\partial a/\partial p_\mu)(\partial b/\partial r^\mu)$; and the four-current density is $j^\mu(x; f) = \int d^8 z f(z) j^\mu(x; z)$, while $j^\mu(x; z) = eu^\mu \delta^4(x - r)$ is the contribution of a particle at z . (We omit the obvious species labels and sums.)

Consider the family of particle world lines in phase space, each parameterized by its proper time τ . With a smooth but arbitrary assignment of $\tau=0$ on each line, we denote the seven-dimensional hypersurface $\tau=0$ as the "initial-condition surface." Introducing seven arbitrary coordinates η on that surface, we let $g(\eta)d^7\eta$ represent the particle density at $\tau=0$. The particle orbits $z(\tau, \eta)$ are thus an eight-component field on the eight-dimensional space (τ, η) .

We now introduce the action functional:

$$S[z(\tau, \eta), A(x)] = \int d^7\eta g(\eta) \int d\tau [p_\mu(\tau, \eta) dr^\mu(\tau, \eta)/d\tau - H(p, r; A)] \\ - \int d^4x F_{\mu\nu} F^{\mu\nu}/16\pi, \quad (2)$$

the sum of the particle action (in phase-space form) and the Maxwell action. Requiring that S be stationary with respect to $z(\tau, \eta)$ yields the Hamiltonian equations stated above. The invariant Vlasov distribution is defined as

$$f(\bar{z}) = \int d^7\eta g(\eta) \int d\tau \delta^8(\bar{z} - z(\tau, \eta)). \quad (3)$$

To obtain (1), we introduce the intermediate distribution $\bar{g}(\bar{z};\tau) = \int d^7\eta g(\eta) \delta^8(\bar{z} - z(\tau, \eta))$, and take its τ -derivative, using the Hamiltonian equations. We find $\partial \bar{g} / \partial \tau = -\{\bar{g}, H\}$. Integration over τ then yields (1), since g vanishes for finite z and infinite τ .

Variation of S with respect to $A_\mu(x)$ yields the Maxwell equation, with $j^\mu(x) = \int d^7\eta g(\eta) \int d\tau (-)\delta H(z(\eta, \tau)) / \delta A_\mu(x)$. Use of (3) and H then produces $j^\mu(x) = \int d^8z f(z) e u^\mu \delta^4(x - r)$, as desired. Thus the action principle yields the self-consistent Maxwell and Ignatiev equations.

We now restrict the Maxwell potential to represent an eikonal wave:

$A_\mu(x) = \tilde{A}_\mu(x) \exp i\theta(x)/\epsilon + \text{c.c.}$, where $\tilde{A}$ and θ are the slowly varying amplitude and phase, while the eikonal infinitesimal ϵ will be omitted from subsequent formulas. The Maxwell field is thus $F_{\mu\nu}(x) = i \tilde{F}_{\mu\nu}(x) \exp i\theta(x) + \text{c.c.}$, with $\tilde{F}_{\mu\nu} = k_\mu \tilde{A}_\nu - k_\nu \tilde{A}_\mu$, and $k_\mu(x) = \partial\theta(x)/\partial x^\mu = (\vec{k}, -\omega)$. The Maxwell action in (2) is thus $-\int d^4x \tilde{F}^*_{\mu\nu} \tilde{F}^{\mu\nu} / 8\pi$.

The Lie generator $w(z)$ is determined [15] by $\{w, H^{(0)}\} = -H^{(1)}$, where $H^{(0)} = p^2/2m$ is the zero-order Hamiltonian, and $H^{(1)} = \tilde{H} \exp i\theta(r) + \text{c.c.}$, with $\tilde{H} = -(e/m)p \cdot \tilde{A}(r)$. We find the solution $w(z) = -ie(p \cdot \tilde{A}(r)/p \cdot k(r)) \exp i\theta(r) + \text{c.c.}$, and proceed to the new Hamiltonian $K = [\exp i\{w, \cdot\}]H = p^2/2m + \Psi(z)$ to second order, with $\Psi = \frac{1}{2}\{w, H^{(1)}\} + H^{(2)}$, where $H^{(2)} = (e^2/m)|\tilde{A}(r)|^2$ and oscillatory terms are omitted (being eliminated by a second Lie transform). Evaluation of the relativistically invariant ponderomotive Hamiltonian $\Psi(z;A)$ yields

$$\Psi(z) = (e^2/m) |p \cdot \tilde{F}|^2 / (p \cdot k)^2, \quad (4)$$

where $|p \cdot \tilde{F}|^2 = p_\mu \tilde{F}^{\mu\nu}(r) p^\sigma \tilde{F}_{\sigma\nu}^*(r)$, and the denominator is the familiar resonance $p \cdot k = m\gamma(\vec{k} \cdot \vec{v} - \omega)$. To this order, Ψ has the interesting

gauge-invariant expression $\Psi = m|d\tilde{u}/d\tau|^2/d\Theta(r)/d\tau)^2$, and reduces, in the unperturbed rest frame, to the familiar $e^2|\tilde{E}|^2/m\omega^2$. We express the dependence of $\Psi(z)$ on $\tilde{A}_\mu(x)$ explicitly:

$$\Psi(z;A) = \int d^4x \tilde{A}_\mu^*(x) \Psi_\nu^\mu(x;z) \tilde{A}^\nu(x); \quad (5)$$

by (4), the kernel is (remember that $k = k(x)$)

$$\Psi_\nu^\mu(x;z) = \delta^4(x-r)(e^2/m)(p \cdot k)^{-2} [k^2 p^\mu p_\nu + \delta_\nu^\mu (k \cdot p)^2 - (k \cdot p)(p^\mu k_\nu + k^\mu p_\nu)]. \quad (6)$$

We now use the property of the phase-space Lagrangian action, of invariance under canonical transformations [16]:

$$\int d\tau [p_\mu dr^\mu/d\tau - H(p,r)] = \int d\tau [\bar{p}_\mu d\bar{r}^\mu/d\tau - K(\bar{p},\bar{r})], \quad (7)$$

where the overbar (omitted subsequently) denotes oscillation-center variables.

Thus the action now reads

$$S = \int d^7\eta g(\eta) \int d\tau [p_\mu dr^\mu/d\tau - p^2/2m - \Psi(z;A)] - \int d^4x \tilde{F}_{\mu\nu}^* \tilde{F}^{\mu\nu}/8\pi. \quad (8)$$

Variation with respect to the oscillation-center orbit $z(\tau,\eta)$ yields $dr^\mu/d\tau = \partial K/\partial p_\mu$, $dp_\mu/d\tau = -\partial K/\partial r^\mu$, for the covariant ponderomotive effects.

Introducing the invariant oscillation-center distribution (not to be confused with $F_{\mu\nu}$): $F(z) = \int d^7\eta g(\eta) \int d\tau \delta^8(z - \bar{z}(\tau,\eta))$, we obtain, in analogy to the steps leading to (1), the corresponding Ignatiev equation:

$$\{F(z), p^2/2m + \Psi(z;A)\} = 0. \quad (9)$$

In order to vary S with respect to $\tilde{A}_\mu(x)$ and $\Theta(x)$, we first substitute (5) into (8), obtaining (for the terms bilinear in A)

$$S^{(2)} = \int d^4x \tilde{A}_\mu^*(x) D_\nu^\mu(x;F) \tilde{A}^\nu(x), \quad (10)$$

where the dielectric matrix D_ν^μ is the sum of the vacuum part

$$D_{\text{vac}}^{\mu}{}_{\nu}(k(x)) = (k^{\mu}k_{\nu} - k^2\delta_{\nu}^{\mu})/4\pi \quad (11)$$

and the susceptibility matrix

$$\chi^{\mu}{}_{\nu}(x;F) = - \int d^8z F(z) \Psi^{\mu}{}_{\nu}(x;z). \quad (12)$$

This relation (12) is the "K-x theorem" [8,9,3], which thus is the essential ingredient of the action functional, coupling $F(z)$ to $A_{\mu}(x)$.

It is convenient to express the dielectric matrix in terms of its local eigenvalues and eigenvectors. Since $S^{(2)}$ is a scalar and $\tilde{A}^{\nu}\tilde{A}_{\mu}^*$ is hermitian, $D^{\mu}{}_{\nu}(x)$ is hermitian, with real eigenvalues $D_{\alpha}(x)$ (labeled by α)

and with orthonormal (complex) eigenvectors $\hat{e}_{\alpha}^{\mu}(x)$: $e_{\mu}^{\alpha*} e_{\beta}^{\mu} = \delta_{\beta}^{\alpha}$.

Thus we express

$$D^{\mu}{}_{\nu}(x) = \sum_{\alpha} D_{\alpha}(x) e_{\alpha}^{\mu}(x) e_{\nu}^{\alpha*}(x), \quad (13)$$

so that

$$S^{(2)} = \int d^4x D_{\alpha}(x) |A_{\alpha}(x)|^2, \quad (14)$$

where $A_{\alpha}(x) = e_{\nu}^{\alpha*}(x) \tilde{A}^{\nu}(x)$ is the projection of $\tilde{A}$ on the α eigenvector of $D^{\mu}{}_{\nu}(x)$.

Noting that $D_{\alpha}(x) = D_{\alpha}(x, k(x))$ [see (11) and (6)], we proceed to vary $S^{(2)}$, first with respect to $A_{\alpha}(x)$, obtaining the eikonal equation for the phase:

$$D_{\alpha}(x, k = \partial\theta/\partial x) = 0, \quad (15)$$

associated with polarization e_{α} . This yields the covariant ray equations:

$$dx^{\mu}/d\sigma = - \partial D_{\alpha}/\partial k_{\mu}, \quad dk_{\mu}/d\sigma = \partial D_{\alpha}/\partial x^{\mu}. \quad (16)$$

Variation with respect to $\Theta(x)$ (the phase for a given polarization) yields the wave-action conservation law [17]: $\partial J^\mu(x)/\partial x^\mu = 0$, where the action density four-vector is

$$J^\mu(x) = -|A_\alpha(x)|^2 \partial D_\alpha(x, k) / \partial k_\mu, \quad (17)$$

(evaluated at $k(x)$). Since the local eigenvalues $D_\alpha(x)$ are functionals of the oscillation-center distribution $F(z)$, by (12), we have now obtained a closed self-consistent set of coupled equations for $F(z)$ and the wave amplitude and phase.

In a future paper, we shall investigate the conservation laws [18] associated with these action principles.

We have benefitted greatly from discussions with B. M. Boghosian, R. G. Littlejohn, S. M. Omohundro, P. L. Similon, and J. S. Wurtele.

References

- [1] R. L. Dewar, Phys. Fluids 16 (1973) 1102.
- [2] S. Johnston, Phys. Fluids 19 (1976) 93.
- [3] J. R. Cary and A. N. Kaufman, Phys. Fluids 24 (1981) 1238.
- [4] D. H. Dubin, J. A. Krommes, C. Oberman, and W. W. Lee, Phys. Fluids 26 (1983) 3524.
- [5] A. N. Kaufman and B. M. Boghosian, Contemporary Math 28 (1984) 169.
- [6] A. N. Kaufman, XIII Int'l Colloq. on Group Theoretical Methods in Physics, College Park, Md. (1984, in press).
- [7] H. R. Lewis and K. R. Symon, J. Math. Phys. 20 (1979), 413 (Appendix).
- [8] J. R. Cary and A. N. Kaufman, Phys. Rev. Lett. 39 (1977) 402.
- [9] S. Johnston and A. N. Kaufman, Phys. Rev. Lett. 40 (1978) 1266.
- [10] P. L. Similon and A. N. Kaufman, Phys. Rev. Lett. 53 (1984) 1061.
- [11] P. L. Similon, A. N. Kaufman, and D. D. Holm (submitted to Phys. Lett. A, June 1984).
- [12] J. S. Wurtele, S. M. Omohundro, and A. N. Kaufman (in preparation).
- [13] B. M. Boghosian and A. N. Kaufman (in preparation).
- [14] Yu. G. Ignatiev, Zh. Eksp. Teor. Fiz. 81 (1981) 3 [Sov. Phys. JETP 54 (1981) 1].
- [15] J. R. Cary, Physics Reports 79 (1981) 131.
- [16] R. G. Littlejohn, J. Math. Phys. 23 (1982) 742.
- [17] R. L. Dewar, Ap. J. 174 (1972) 301.
- [18] R. L. Dewar, Aust. J. Phys. 30 (1977) 533.

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.

TECHNICAL INFORMATION DEPARTMENT
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720