Title
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INTUITIONISM AND NUCLEI

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ABSTRACT. Topological spaces that represent locales arising in Beth semantics for intuitionistic logic are examined. The focus is on topological indistinguishablility of points and separation axioms in those spaces, and a few different special classes of Beth frames are considered.

My unexplained notation is that of Bezhanishvili and Holliday [2]. Bezhanishvili and Holliday constructed for a poset X a topological space, which we call $P(X)$, such that each point in $P(X)$ is a path in X, and the algebra $\Omega(P(X))$ of open sets of $P(X)$ is isomorphic to $\mathrm{Up}(X)_{j_\mathrm{b}}$ [2, Theorem 4.18]. One can consider a similar construction for *maximal chains* instead of paths. More specifically, for a poset X, we define $\tilde{P}(X)$ to be the topological space of maximal chains in X whose open sets are exactly the subsets of the form $[U] := \{\alpha \mid \alpha \cap U \neq \{\}\}$ for $U \in \mathrm{Up}(X)_{j_\mathrm{b}}$.

Proposition 1. $\Omega(\tilde{P}(X)) \cong \mathrm{Up}(X)_{j_\mathrm{b}}$ for every poset X.

Proof. The proof of [2, Theorem 4.18] can be adapted for maximal chains. □

Proposition 2. Let $f : P(X) \to \tilde{P}(X)$ be a function that assigns to each path α the maximal path containing α in X. Then f is a quotient map between topological spaces.

Proof. Left as an exercise to the reader. □

Trees. In what follows, we assume that X is a tree. (Our definition of trees is that used in combinatorial set theory.)

Proposition 3. $\tilde{P}(X)$ is homeomorphic to the Kolmogorov quotient of $P(X)$.

Proof. It suffices to show that paths $\alpha, \beta \in P(X)$ are topologically indistinguishable if and only if they are contained in the same maximal chain—i.e., ${\downarrow}\alpha = {\downarrow}\beta$. Suppose that ${\downarrow}\alpha = {\downarrow}\beta$. Let $U \in \mathrm{Up}(X)_{j_\mathrm{b}}$ be arbitrary, and assume that $\alpha \in [U]$ or, equivalently, that α intersects U nontrivially. Let $x \in \alpha \cap U$. Since $\alpha \subseteq {\downarrow}\alpha = {\downarrow}\beta$, there is $y \geq x$ such that $y \in \beta$. Since U is an upset, we also have $y \in U$. Hence β and U intersect nontrivially, and $\beta \in [U]$. Combined with a symmetric argument, this shows that α and β are topologically indistinguishable. To show the converse, suppose that α and β are topologically indistinguishable. we show that ${\downarrow}\alpha \subseteq {\downarrow}\beta$ (the other inclusion can be proved in an entirely symmetric manner). Let $y \in {\downarrow}\alpha$ be arbitrary. Then there exists $x \in \alpha$ with $x \geq y$. Note that $j_\mathrm{b}({\uparrow}y) \supseteq {\uparrow}y \ni x \in \alpha$. Since $j_\mathrm{b}({\uparrow}y) \in \mathrm{Up}(X)_{j_\mathrm{b}}$, and $\alpha \in [j_\mathrm{b}({\uparrow}y)]$, β also intersects $j_\mathrm{b}({\uparrow}y)$ nontrivially by assumption. Let $z \in j_\mathrm{b}({\uparrow}y) \cap \beta$. Then β is a path containing z, which is in turn in $j_\mathrm{b}({\uparrow}y)$. This implies that ${\uparrow}y$ and β intersect nontrivially—i.e., $y \in {\downarrow}\beta$. □

Given the proposition above, we study $\tilde{P}(X)$ instead of $P(X)$ for convenience's sake.

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There is another natural topology on the set of maximal chains in a tree X. This topological space, which we call $\hat{P}(X)$, has a basis $\{[\uparrow x] \mid x \in X\}$ (note that $[\uparrow x]$ is the set of maximal chains containing x). This is called the *branch space* of X by some (e.g., [7]). If X is a subtree of $\omega^{<\omega}$, then $\hat{P}(X) \cap \omega^\omega$ is a subspace of *the Baire space* ω^ω (see, e.g., [3]), which carries the product topology. This is a totally disconnected Polish space of size continuum. More generally, if $X \subseteq \kappa^{<\kappa}$ for a regular cardinal κ with $|\kappa^{<\kappa}| = \kappa$, then $\hat{P}(X) \cap \kappa^\kappa$ is a subspace of a *generalized Baire space*, whose topology is called the *bounded topology* (see [4]).

Proposition 4. $\tilde{P}(X)$ and $\hat{P}(X)$ are homeomorphic.

Proof. For $x \in X$ let $b(x)$ stand for the least $y \le x$ such that $[\uparrow y] = [\uparrow x]$. (For instance, if $X \subseteq \omega^{<\omega}$, then $b(x)$ is the longest initial segment of x that is an endpoint, $\langle\rangle$, or an immediate successor of a node with two or more immediate successors.) By definition, $[\uparrow x] = [\uparrow b(x)]$. Moreover, $\uparrow b(x)$ is fixed for every $x \in X$. To see this, let $z \notin \uparrow b(x)$. Assume that z and $b(x)$ are comparable. Since $z \notin \uparrow b(x)$, we have $z < b(x)$. By the minimality of $b(x)$, we have $[\uparrow b(x)] \subsetneq [\uparrow z]$. Then a maximal chain in $[\uparrow z] \setminus [\uparrow b(x)]$ contains z and does not intersect $\uparrow b(x)$. Next, assume that z and x are incomparable. Then no maximal chain containing z intersects $\uparrow b(x)$ nontrivially; indeed, if w is both in a maximal chain containing z and in $\uparrow b(x)$, $\downarrow w$ contains two incomparable elements z and $b(x)$.

The topology of $\tilde{P}(X)$ is at least as fine as that of $\hat{P}(X)$; indeed, it suffices to see that $[\uparrow x]$ is open in $\hat{P}(X)$ for all $x \in X$, which follows from the observations in the paragraph above.

To show that the topology of $\hat{P}(X)$ is at least as fine as that of $\tilde{P}(X)$, let $[U]$ be an arbitrary open set in $\tilde{P}(X)$ with $U \in \mathrm{Up}(X)_{j_\mathrm{b}}$. We have

$$U = j_\mathrm{b}(U) = j_\mathrm{b} \bigcup_{x \in U} \uparrow x = j_\mathrm{b} \bigcup_{x \in U} \uparrow b(x).$$

By the same argument as in the proof of Proposition 1, we have

$$[U] = [j_\mathrm{b} \bigcup_{x \in U} b(\uparrow x)] = \bigcup_{x \in U} [\uparrow b(x)].$$

We conclude that U is the union of some basic open sets in $\hat{P}(X)$. □

Proposition 5. Suppose that $X \subseteq \omega^{<\omega}$. Define a tree $X' \subseteq \omega^{<\omega}$ with no maximal nodes by

$$X' = \{x \mid x \in X, x \text{ is not maximal}\} \cup \{x * 0^n \mid x \in X, x \text{ is maximal}, n \in \omega\},$$

where $*$ stands for concatenation of strings. Then $\tilde{P}(X)$ and $\tilde{P}(X')$ are homeomorphic, and $\mathrm{Up}(X)_{j_\mathrm{b}} \cong \mathrm{Up}(X')_{j_\mathrm{b}}$.

Since $\tilde{P}(X')$ has no maximal nodes, $\tilde{P}(X') \subseteq \omega^\omega$. This shows that, with a locale arising from countable trees of height $\le \omega$ in Beth semantics, we can associate a subspace of the Baire space that represents that locale as the algebra of open sets in it.

Proof. It suffices to work with $\hat{P}(X)$ and $\hat{P}(X')$. There is a bijection $f : \hat{P}(X) \to \hat{P}(X')$ such that $f(\alpha) = \alpha * 0^\omega$ for α finite, and $f(\alpha) = \alpha$ otherwise. Suppose that $\alpha \in \hat{P}(X)$ is finite. Then α is an isolated point in $\hat{P}(X)$; indeed, $[\uparrow_X \max\alpha]_{\hat{P}(X)}$ is open in $\hat{P}(X)$, and $[\uparrow_X \max\alpha]_{\hat{P}(X)} = \{\alpha\}$. $f(\alpha) = \alpha * 0^\omega$ is also isolated in $\hat{P}(X) \subseteq \omega^\omega$, as witnessed by the open set $[\uparrow_{X'} \max\alpha]_{\hat{P}(X')} = \{\alpha * 0^\omega\}$ in $\hat{P}(X')$. The restriction $f|_{\hat{P}(X) \cap \omega^\omega}$ is clearly a homeomorphism. We conclude that f is a homeomorphism. □

Even if X is not a subset of $\kappa^{<\kappa}$, $\tilde{P}(X)$ still has a nice property of Polish spaces:

Proposition 6. $\tilde{P}(X)$ is Hausdorff.

Proof. Let $\alpha, \beta \in \tilde{P}(X)$, and assume that $\alpha \neq \beta$. Since α and β are maximal chains, neither of the two chains is contained in the other; one can take $x \in \alpha \setminus \beta$ and $y \in \beta \setminus \alpha$. Note that x and y are incomparable by maximality of α and β. Then $j_{\mathrm{b}}(\uparrow x)$ and $j_{\mathrm{b}}(\uparrow y)$ are open sets, and they are clearly neighborhoods of α and β, respectively. Suppose by way of contradiction that $j_{\mathrm{b}}(\uparrow x) \cap j_{\mathrm{b}}(\uparrow y) \neq \{\}$; take $z \in j_{\mathrm{b}}(\uparrow x) \cap j_{\mathrm{b}}(\uparrow y)$. Let γ be a maximal chain containing z. Then γ nontrivially intersects both $\uparrow x$ and $\uparrow y$. This is a contradiction; for, if $z \in \gamma \cap \uparrow x$, and $w \in \gamma \cap \uparrow y$, then $\downarrow \max\{z, w\}$ contains two incomparable elements—namely, x and y. □

Posets where every principal upset is fixed. As suggested by [2, Example 4.15], posets every principal upset of which is fixed often give rise to locales that are easy to study.

Proposition 7. Suppose that every principal upset in X is fixed. Let P be either $P(X)$ or $\tilde{P}(X)$. $\alpha, \beta \in P$ are topologically indistinguishable if and only if one of the two points is dense in the other—i.e.,

$$\forall x \in \alpha\, \exists y \geq x\; y \in \beta, \qquad \text{and} \qquad \forall y \in \alpha\, \exists x \geq y\; z \in \alpha. \tag{1}$$

Proof. Suppose that α and β are topologically indistinguishable. Let $x \in \alpha$ be arbitrary. By hypothesis, $\uparrow x$ is fixed. Since $\uparrow x$ and α intersect nontrivially, so do $\uparrow x$ and β by the topological indistinguishablity of α and β. This means $\exists y \geq x\; y \in \beta$. Likewise, we have $\forall y \in \alpha\, \exists x \geq y\; z \in \alpha$. Conversely, assume (1). Suppose that $U \in \mathrm{Up}(X)_{j_{\mathrm{b}}}$ intersects α nontrivially; let $x \in \alpha \cap U$. By hypothesis, $\exists y \geq x\; y \in \beta$. Since U is an upset, $y \in U$. We have shown that U intersects β nontrivially. The other direction can be shown in the same way. □

We exhibit a class of posets the satisfies the assumption of the preceding proposition. (For set-theoretic ideas that appear later in this subsection, see [8].) A poset X is *separative* if for every $x, y \in X$ we have

$$x \geq y \iff \forall x' \geq x\, \exists z \geq x'\, z \geq y.$$

For a boolean algebra of B, let B^+ be the subposet of B consisting of every element of B but $\max B$. This is called a *topless boolean algebra*. Topless boolean algebras are separative. In fact, they are special separative posets: every separative posets densely embeds into some topless boolean algebra.

Proposition 8. Let X be a separative poset. Then every principal upset in X is fixed.

Proof. Let B be a boolean algebra such that X is a dense subset of B^+. Let $x \in X$ be arbitrary. Let $y \notin \uparrow_X x$. It suffices to show that there is a path in X containing y that does not intersect $\uparrow_X x$ nontrivially. Since $y \ngeq x$, we have $z := \neg x \vee y \neq 1$ and $z \in B^+$, where the operations are those of B. Since X is dense in B^+, there exists $z' \in X$ such that $z' \geq z$. Take a path α in X starting at y and containing z', such that an end segment of α is in $\uparrow_X z'$. Assume for contradiction that α and $\uparrow_X x$ intersect nontrivially; let $w \in \alpha \cap \uparrow_X x$. Suppose that $w \leq z'$. This implies $x \leq z' \geq \neg x$, and it contradicts $z' \in X \subseteq B^+$. Suppose, on the other hand, that $w \geq z'$. This implies $x \leq w \geq \neg x$, which contradicts $w \in \alpha \subseteq X \subseteq B^+$. We conclude that α does not intersect $\uparrow_X x$ nontrivially and that $\uparrow_X x$ is fixed. □

Our next goal is to prove that Beth semantics restricted to separative trees is as general as that restricted to trees.

Lemma 9. Let X be a tree, and let b be as in the proof of Proposition 4. Let X' be the image of X under b with the induced order. Then X' is (isomorphic to) the separative quotient of X.

Proof. It suffices to show that

$$x \leq y \to b(x) \leq b(y) \tag{2}$$

and that

$$x \between y \iff b(x) \between b(y), \tag{3}$$

where $w \between z$ for $w, z \in X$ denotes w and z being compatible. This is because the separative quotient of X is determined up to isomorphism as the image of a map that satisfies the properties above. Note that, since X is a tree, $w \between z$ if and only if w and z are comparable. It is clear that b satisfies the first condition by definition. Suppose that $x \between y$. Since $x \geq b(x)$, $y \geq b(x)$, and X is a tree, $x \between y$. Suppose that $b(x) \between b(y)$. The two points $b(x)$ and $b(y)$ are comparable; without loss of generality we may assume that $b(x) \leq b(y)$. Then

$$\uparrow y = \uparrow b(y) \subseteq \uparrow b(x) = \uparrow x.$$

This implies $y \leq x$ and $y \between x$. □

Proposition 10. Suppose that X is a tree and that X' is its separative quotient. Then $\tilde{P}(X) \cong \tilde{P}(X')$, and thus $\mathrm{Up}(X)_{j_b} \cong \mathrm{Up}(X')_{j_b}$.

Proof. By the previous lemma, we may assume that $X' = b(X)$. Note that X' is also the set of fixed-points in X with respect to b.

The map $b : X \to X'$ induces a bijection $b_* : \tilde{P}(X) \to \tilde{P}(X')$, where $b_*(\alpha)$ is the image $b``\alpha$ for $\alpha \in \tilde{P}(X)$. Indeed, suppose that $\alpha \in \tilde{P}(X)$. Since $X$ is a tree, and $b(x) = x$ for all $x \in X'$, the image $b``\alpha$ is equal to $\alpha \cap X'$. Thus, the image $b``\alpha$ is a chain in $X' \subseteq X$, and it is maximal in X' by the maximality of α in X. Hence, b_* is well-defined as a map to $\tilde{P}(X')$. Next, suppose that $\beta \in \tilde{P}(X')$. We show that there is a unique $\alpha \supseteq \beta$ that is a maximal chain in X, and that thus b_* is bijective. Assume otherwise; let $\alpha_0, \alpha_1 \supseteq \beta$ be distinct maximal chains in X. By the maximality of α_0 and α_1, there exist $x_0 \in \alpha_0$ and $x_1 \in \alpha_1$ such that x_0 and x_1 are incomparable. Since x_0 and x_1 are incomparable, so are $b(x_0)$ and $b(x_1)$ by (3). Since $x_0 \geq b(x_0)$, and $x_1 \geq b(x_0)$, for $i < 2$ the two points x_i and $b(x_{1-i})$ are incomparable. By the maximality of β in X', exactly one of $b(x_0), b(x_1) \in X'$ must be in β; we may assume $b(x_0) \in \beta$ without loss of generality. However, a chain α_1 in X cannot contain incomparable points $b(x_0)$ and x_1.

We show that b_* is in fact a homeomorphism. By Proposition 4, it suffices to consider the topology of $\tilde{P}(X)$ and $\tilde{P}(X')$ and their basic open sets. Specifically, since $\uparrow_X x = \uparrow_X b(x)$ for every $x \in X$, it suffices to show for every $x \in X'$

$$b_*``\{\alpha \in \tilde{P}(X) \mid x \in \alpha\} = \{\beta \in \tilde{P}(X') \mid x \in \beta\},$$
$$\{\alpha \in \tilde{P}(X) \mid x \in \alpha\} = b^*``\{\beta \in \tilde{P}(X') \mid x \in \beta\},$$

where $b^* : \tilde{P}(X') \to \tilde{P}(X)$ is the inverse of b_*. These follow because $b_*(\alpha) = \alpha \cap X'$, and $b^*(\beta)$ is the unique maximal chain $\alpha \supseteq \beta$ in X. □

It is easy to see that a tree is separative if and only if every principal upset in it is fixed. Therefore, the last proposition also shows the generality of Beth semantics restricted to trees whose principal upsets are all fixed.

The Beth nucleus and double negation. The results in the previous section suggest that the study of $\mathrm{Up}(X)_{\neg\neg}$ is useful in that of $\mathrm{Up}(X)_{j_b}$ and $\tilde{P}(X)$. In general, $\mathrm{Up}(X)_{\neg\neg} \subseteq \mathrm{Up}(X)_{j_b}$. Moreover, we have the following.

Proposition 11. For any poset X, $\mathrm{Up}(X)_{j_b} = \mathrm{Up}(X)_{\neg\neg}$ if and only if $\mathrm{Up}(X)_{j_b}$ is boolean.

Proof. This is in fact true of any locale A and a dense nucleus $j : A \to A$: $A_j = A_{\neg\neg}$ if and only if A_j is boolean. This fact is well known, but we include the proof for completeness.

It suffices to show the "if" direction. Note that $0_A \in A_j$. Since A_j is boolean, $A_j = \{x \to a \mid x \in A\}$ for some $a \in A$ (see, e.g., [6, III.10.4]). In particular, for some $x \in A$, we have $x \to a = 0_A$. Since $0_A \geq (x \to a) \geq a$, we have $a = 0_A$. As is well known, $A_{\neg\neg} = \{x \to 0_A \mid x \in A\}$. □

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