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Abstract

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In this dissertation, we study complete Calabi-Yau manifolds asymptotic to *Calabi model space*, which appears in singularity formation and limit of Kähler-Einstein metrics. The first example goes back to Tian-Yau [33], where they constructed Kähler Ricci-flat metrics ω_{TY} in the zero class which is exponentially close to the Calabi model space. The tool was systematically generalized by Hein [17] to a powerful package to find Calabi-Yau metrics. In particular, it implies the existence of Calabi-Yau metrics in the compact-supported Kähler classes, which are also exponentially close to the Calabi model space.

Building on previous fundamental work, we further generalized this existence result to non-compact-supported Kähler classes on a type of quasi-projective manifold. Unlike earlier results, these new metrics are not exponentially close, but only polynomially close with a fixed rate. We also proved a uniqueness theorem, which, together with the existence result, gives a relative complete understanding of Calabi-Yau metrics asymptotic to the Calabi model space on this type of quasi-projective manifolds.

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Chapter 1

Introduction

In Chapters 3, 4 of this thesis, we study a special type of complete non-compact Calabi-Yau metrics.

1.1 Overview

A fundamental problem in complex geometry is to determine when a complex n-dimensional manifold M can admit canonical metrics, such as a *Calabi-Yau metric*, defined as a Kähler metric with zero Ricci curvature, or equivalently, with reduced holonomy in $SU(n)$. The study of Calabi-Yau metrics is a fruitful area of differential geometry, closely related to both algebraic geometry and theoretical physics.

The name commemorates Yau's celebrated resolution of Calabi conjecture [34]. This states that given a compact complex manifold M with the topological condition $c_1(M) = 0$, any Kähler class admits a unique Kähler Ricci-flat metric. Although Ricci-flat metrics are generally difficult to find, under the Kähler condition, the Ricci curvature simplifies significantly, reducing the problem to finding a solution to the Monge-Ampère equation. The proof relies on the continuity method, which requires uniform a priori estimates developed by Aubin and Yau. A natural question is whether Yau's theorem can be generalized to non-compact manifolds.

Question 1.1.1. *When does a non-compact complex manifold X admit a complete Kähler Ricci-flat metric? If so, is the metric unique in any sense?*

In the non-compact setting, one cannot hope that $c_1(X) = 0$ is a sufficient condition for the existence of complete Calabi-Yau metrics. There are more obstructions, for example, strictly pseudoconvex bounded domains in $\mathbb{C}^n$ can not admit complete Calabi-Yau metrics since they admit negative Kähler Einstein metrics. On the other hand, traditional methods like the continuity method encounter difficulties such as the unbounded nature of the Kähler and Ricci potentials. So typically, we will impose some constrain of the asymptotic behavior

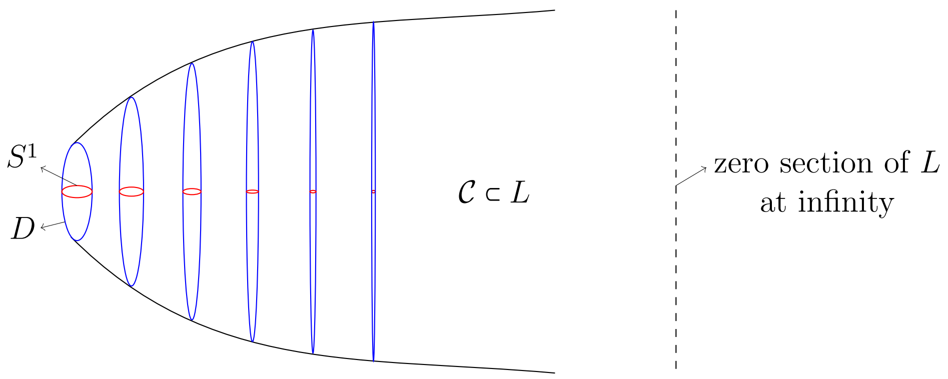

Figure 1.1: The calabi model space $\mathcal{C} \subset L$

at infinity, which would allow us to do estimates in a suitable weighted space. For example, complete Calabi-Yau manifolds with conical and cylindrical behavior were classified by Conlon-Hein [11], Sun-Zhang [27], and Haskins-Hein-Nordström [15].

This thesis focuses on the complete Calabi-Yau manifolds asymptotic to *Calabi model space*, which appears in singularity formation and limit of Kähler-Einstein metrics. The first example goes back to Tian-Yau [33], where they constructed Kähler Ricci-flat metrics ω_{TY} in the zero class which is exponentially close to the Calabi model space. The tool was systematically generalized by Hein [17] to a powerful package to find Calabi-Yau metrics. In particular, it implies the existence of Calabi-Yau metrics in the compact-supported Kähler classes, which are also exponentially close to the Calabi model space. Building on previous fundamental work, we further generalized this existence result to not compactly-supported Kähler classes on a type of quasi-projective manifold. Unlike earlier results, these new metrics are not exponentially close, but only polynomially close with a fixed rate. we also provide a uniqueness theorem, which, together with the existence result, gives a relatively complete understanding of Calabi-Yau metrics asymptotic to the Calabi model space on this type of quasi-projective manifolds.

1.2 Existence

Let me first introduce the Calabi model space.

1.2.0.1 Calabi model space

The ambient manifold is the total space of the disc bundle $\mathcal{C}$ inside an ample line bundle L over a compact Calabi-Yau manifold D. Under the induced holomorphic structure, it has a Calabi-Yau metric $\omega_{\mathcal{C}}$ with S^1 symmetry given by Calabi ansatz. The precise definition of Calabi model space will be given in Section 2.2.2.

The Calabi model space $(\mathcal{C}, \omega_{\mathcal{C}})$ has the following geometric features:

- It behaves nicely like a doubly warped product $\mathbb{R}_+ \times S^1 \times D$ locally at infinity as in Figure 1.1, with the S^1 direction collapsing and the D direction growing sublinearly.
- The growth rate of $\operatorname{Vol} B(x, r)$ is between 1 and 2, which gives an interesting example of non-integer volume growth rate of Calabi-Yau manifold. This behavior makes a crucial difference in the analysis from the higher-order growth case like asymptotically conical case.

1.2.1 Calabi-Yau manifold of Calabi type

We now turn our attention to complete manifolds that are asymptotically close to the model space at infinity. A complete Kähler manifold (X, g) is *of Calabi type* if there exists a diffeomorphism Φ identifying X and $\mathcal{C}$ outside some compact subsets such that the metrics are close at infinity.

A natural candidate for such an X is the complement of a smooth divisor D in a projective manifold M. In this case, a diffeomorphism identifies the punctured neighborhood of a divisor D with the punctured disc bundle $\mathcal{C}$ inside the normal bundle of D. Even though we do not have a holomorphic tubular neighborhood theorem for complex manifold, we can still identify the holomorphic structures on $M \smallsetminus D$ and $\mathcal{C}$ with exponentially small error. If we choose D to be a smooth anticanonical divisor inside a projective manifold M with ample normal bundle, then the total space of its disc normal bundle $\mathcal{C}$ admits the Calabi model metric $\omega_{\mathcal{C}}$.

In 1990, Tian and Yau [33] constructed complete Calabi-Yau metrics ω_{TY} on $M \smallsetminus D$ on these quasi-projective manifolds, called Tian-Yau metric, extending the techniques introduced in [34] to the non-compact case. For any compact supported class in $M \smallsetminus D$, Hein-Sun-Viaclovsky-Zhang [14] find Calabi-Yau metric exponentially close to the Calabi model space. We call the complete Calabi-Yau manifold with this property *asymptotically Calabi*, which is defined as follows.

Definition 1.2.1. *Let X be an n-dimensional complete Kähler manifold with, complex structure I, 2-form ω and $(n, 0)$-form Ω. We say (X, I, ω, Ω) is*

1. weak asymptotically Calabi with rate κ *if there exists $\underline{\delta} > 0$, $\kappa > 0$, a Calabi model space $(\mathcal{C}, I_{\mathcal{C}}, \omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$ with function z on $\mathcal{C}$ defined in Section 2.2.2, and a diffeomorphism $\Phi : \mathcal{C} \smallsetminus \mathcal{K} \to X \smallsetminus K$, where $K \subset X$ and $\mathcal{K} \subset \mathcal{C}$ are compact, such that the following hold uniformly as $z \to +\infty$:*

$$\left|\nabla^k_{\omega_{\mathcal{C}}} \left(\Phi^* I - I_{\mathcal{C}}\right)\right|_{\omega_{\mathcal{C}}} + \left|\nabla^k_{\omega_{\mathcal{C}}} \left(\Phi^* \Omega - \Omega_{\mathcal{C}}\right)\right|_{\omega_{\mathcal{C}}} = O(e^{-\underline{\delta} z^{n/2}}),$$

$$\left|\nabla^k_{\omega_{\mathcal{C}}} \left(\Phi^* \omega - \omega_{\mathcal{C}}\right)\right|_{\omega_{\mathcal{C}}} = O(z^{-\kappa}) \text{ for all } k \in \mathbb{N}_0.$$

2. asymptotically Calabi *if it is weak asymptotically Calabi and*

$$\left|\nabla^k_{\omega_{\mathcal{C}}}\left(\Phi^*\omega - \omega_{\mathcal{C}}\right)\right|_{\omega_{\mathcal{C}}} = O(e^{-\underline{\delta} z^{n/2}}) \text{ for all } k \in \mathbb{N}_0.$$

These manifolds are well understood by Hein-Sun-Viaclovsky-Zhang [14], in which they showed that any asymptotically Calabi manifold that is Calabi-Yau can be compactified complex analytically to a weak Fano manifold, i.e. a smooth projective manifold with nef and big anti-canonical bundle. In this thesis, we are going to show the existence and uniqueness of weak asymptotically Calabi Calabi-Yau metrics that are not asymptotically Calabi. Our setting is as follows:

Definition 1.2.2. *Let M be a compact Kähler manifold with complex dimension $n \geqslant 3$, $D \in |-K_M|$ be a smooth divisor with ample normal bundle and $X = M \setminus D$. We denote $H^2_+(X)$ as the subset of $H^2(X)$ which consists of classes $\mathfrak{k}$ such that $\mathfrak{k}^p$ is positively paired with any compact analytic subset Y of X of pure complex dimension p.*

Remark 1.2.3. *One needs to be slightly careful when talking about paring on noncompact manifolds since we cannot always have integration by parts. The good thing is that in our definition we only care about the compact Y, so $H^2_+(X)$ is well-defined. Assume now we have a Kähler metric ω on X, ω has to be in $H^2_+(X)$. So we can view $H^2_+(X)$ as the Kähler cone of X.*

Building on the work of Tian-Yau [33] and Hein [17], Hein-Sun-Viaclovsky-Zhang [14] showed the existence of Calabi-Yau metrics in all compact-supported Kähler classes on $M \setminus D$ exponentially close to the Calabi model space, generalizing earlier Tian-Yau metric presented in [33].

In [5], I further generalized this existence result to all Kähler classes on $M \setminus D$:

Theorem 1.2.4. *For any class $\mathfrak{k}$ in $H^2_+(X)$, there exists a Calabi-Yau metric ω in the class $\mathfrak{k}$ which is weak asymptotically Calabi with rate 1.*

The proof consists of two main parts. The first step is to find an initial metric close to our model metric in a given cohomology class. A key observation is that all the Kähler classes on X arise from the restriction of a smooth closed form β on M by a vanishing theorem. With this good representative β we can formulate an initial metric ω_0 to start working with. The second part is to modify this almost Calabi-Yau metric to a genuine Calabi-Yau metric by the generalized continuity method of Yau's solution to the Monge-Ampère equation to the complete non-compact case developed in Tian-Yau [33] and Hein [17].

However, we cannot directly apply Tian-Yau-Hein's package here because the Ricci potential of this initial metric will only decay at the rate $r^{-\frac{2}{n+1}}$, whereas Tian-Yau-Hein package requires decay faster than r^{-2}. We need to modify our background metric by solving the linearized operator of Monge-Ampère equation in an iterative way to improve the decay, which motivates the following study of Laplacian operator:

Lemma 1.2.5. *For any smooth function v on $\mathcal{C}$ in a suitable weighted norm space with $|v| = O(r^{\delta})$ with δ not in the indicial roots of $\lambda_{\omega_{\mathcal{C}}}$, there exists a smooth solution u on $\mathcal{C}$ such that $\Delta_{\omega_{\mathcal{C}}} u = v$ with $|u| = O(r^{\delta+2})$ for any $\epsilon > 0$, and with weighted higher regularity estimates.*

This result is of interest from an analytic perspective. Similar elliptic estimate also appears in the work of Xuwen Zhu studying Fredholm theory on ALH* space via microlocal analysis. Furthermore, the slower-than-quadratic volume growth of the Calabi model space imposes an additional integral requirement on the Ricci potential of our initial metric, a condition that plays a crucial role in Tian-Yau-Hein package. Geometrically, this requirement helps determine the choice of the Hermitian metric on the normal bundle.

Once the initial metric is modified to satisfy both the decay and integral conditions, we can proceed by applying Tian-Yau-Hein package to solve the Monge-Ampère equation. Through our iterative process, we demonstrate that the solution decays at any specified polynomial rate. As a result, the Calabi-Yau metric ω admits an expansion at infinity, a phenomenon that also appears in asymptotically conical and cylindrical cases.

1.3 Uniqueness

Having established the existence of these metrics, I next turn to the question of their uniqueness.

Theorem 1.3.1. *If we have another Calabi-Yau metric $\tilde{\omega}$ in the same class $\mathfrak{k}$ satisfying*

$$|\tilde{\omega} - \omega|_{\omega} \leqslant C \cdot r^{-\epsilon}, \text{ for some positive constant } C, \epsilon,$$

where r is a distance function with respect to ω, then $\tilde{\omega} = \omega$.

This result shows that all Calabi-Yau metrics on $M \setminus D$ of Calabi type are parameterized by the Kähler classes of $M \setminus D$. They fall into two categories: exponentially close to $(\mathcal{C}, \omega_{\mathcal{C}})$ for compactly-supported Kähler classes, and polynomially close with a fixed rate for the rest.

Similar uniqueness results are obtained by Conlon-Hein [11] in which they considered the asymptotically conical Calabi-Yau manifolds. Analytically, the proof of Theorem 1.3.1 requires two new ingredients: the $\sqrt{-1}\partial\bar{\partial}$ lemma with the L^2 estimate on manifold X, and a Liouville type theorem on model space $\mathcal{C}$.

1.4 Compactification and Classification

The goal is to classify all complete Calabi-Yau manifolds of Calabi type. Since the topological closure of X at infinity is the zero section of L, which is isomorphic to the compact Calabi-Yau manifold D, it is natural to expect that we can compactify the manifold analytically. Hein-Sun-Viaclovsky-Zhang [14] showed that any Calabi-Yau manifold that is exponentially close to a Calabi model space can be compactified analytically to a weak Fano manifold. For the polynomially close case, we propose the following conjecture:

Conjecture 1.4.1. *Any Calabi-Yau manifold* (X, ω) *which is polynomially close to a Calabi model space can be compactified complex analytically to a weak Fano manifold.*

We should be able to prove this roughly as follows: Repeating the argument in Hein-Sun-Viaclovsky-Zhang [14] via slight modification, we can also construct holomorphic functions on X at infinity using the holomorphic sections of the ample line bundle L over D via Hörmander's L^2 estimates. The next step would be showing that under the metric given by these coordinates, the map into the projective space is an embedding outside a compact subset and showing that under the compactification $\overline{X}'$ given by the image of this map, the holomorphic $(n, 0)$ form has a simple pole along D'. Then we are able to show that the compactification we get is Kähler and hence projective by considering the behavior of the class at the end. By our uniqueness Theorem 1.3.1, this metric ω in fact must come from our existence result in Theorem 1.2.4.

This question fits naturally into the broader framework of complex analytic compactifications of complete Calabi-Yau manifolds raised by Yau, which is known to be true in the case of asymptotically conical end by Conlon-Hein [12] and Chi Li [21] and asymptotically cylindrical end by Haskins-Hein-Norström [15]. Moreover, Sun-Zhang [29] confirmed that for 2-dimensional hyperkähler manifolds with bounded L^2 norm of the curvature can also by compactified analytically, once again utilizing a detailed understanding of the geometry at infinity.

1.5 Outline of the proof

To show the existence, we use the method in Tian-Yau [33], which was subsequently generalized and refined by Hein [17]. This Tian-Yau-Hein's package facilitates the production of complete Calabi-Yau metrics provided a suitable model metric at infinity is known. However, we cannot directly apply it here because the 2-form on X coming from the restriction of 2-forms on M will only decay at the rate $r^{-\frac{2}{n+1}}$, while Tian-Yau-Hein's package requires the Ricci potential decays faster than r^{-2}. We need to modify our background metric by solving the linearized operator of Monge-Ampère equation to improve the decay.

In Section 2.2.2, we provide an introductory overview of the Calabi ansatz, denoted by $\omega_{\mathcal{C}}$, within an open neighborhood $\mathcal{C}$ of the divisor D in its normal bundle, excluding the zero section. Our exposition primarily adheres to the notation and discussion presented by Hein, Sun, Viaclovsky, and Zhang [19]. Furthermore, we briefly review the analytical framework developed by Tian, Yau, and Hein [17], list some geometric properties of the Calabi model space that facilitates the later application of uniform elliptic estimates.

In section 3.2, we have the solution of the Poisson equation with appropriate weighted estimates. The method of variable separation, as detailed by Sun and Zhang [28], allows us to simplify the Poisson equation in the model space into a particular form of ordinary differential equation. Leveraging the solutions' estimates for these ordinary differential equations, as presented in Appendix 3.3, we construct an inversion of the Laplacian operator within suitably weighted spaces. This enables us to initiate the iterative processes detailed in Sections 3.5 and 3.6.

In Sections 3.5 and 3.6, we construct a good background metric ω within the cohomology class $\mathfrak{k}$. We solve the Poisson equation on the model space iteratively to enhance the decay rate of the Ricci potential of ω, as detailed in Section 3.5. In Section 3.6, we refine the Kähler potential by incorporating the harmonic moment map z alongside other pluri-subharmonic functions. This adjustment ensures not only the positivity of ω but also its compliance with the integral condition outlined in Tian-Yau-Hein's package [17]. The iterative method adopted here is inspired by Conlon-Hein [11]. The specific technique of modifying the potential via the harmonic moment map z is equivalent to choosing appropriate scaling of the metric h_D on the normal bundle in Hein-Sun-Viaclovsky-Zhang [14].

In Section 3.7, we deform our good background metric ω to a genuine Calabi-Yau metric on X by Tian-Yau-Hein's package. We also show that the perturbed metric is weak asymptotically Calabi with rate 1. This requires a slight generalization of Hein's decay result in [17].

In Section 3.8, we discuss the uniqueness with restricted asymptotics of the Calabi-Yau metric in the fixed class. We first prove the $i\partial\bar{\partial}$-lemma on X based on the global Hörmander L^2 estimate. Then by our solution of the Poisson equation and the behavior of the harmonic function on the model space we can deduce the global C^0 estimate to do integration by parts.

In Section 3.9, we present some examples of Calabi-Yau manifolds which are weak asymptotically Calabi but not asymptotically Calabi under a fixed diffeomorphism. We also make some conjectures about the stronger uniqueness theorem.

In Chapter 4, we discuss some improved estimates used in the compactification and classification of weak asymptotically Calabi manifold or under even weaker condition.

1.5.1 Other complete non-compact Calabi-Yau manifolds

The topic of complete Calabi-Yau is very rich. There are many progress in the exploration of new complete Calabi-Yau manifolds, extending the seminal work of Tian and Yau to cases where D is singular. Collins-Li [7] constructed new complete Calabi-Yau metrics when D consists of two proportional transversely intersecting smooth divisors. The metric in their construction is $\sqrt{-1}\partial\bar{\partial}$ exact and polynomially closed to the generalized Calabi ansatz. Later Collins-Tong-Yau [8] solved a certain free boundary Monge-Ampère equation crucial to the inductive strategy proposed in [7] to deal with the general case when D is simple normal crossing.

There are also many interesting works on other asymptotic behaviors based on Tian-Yau-Hein's package. For example, the non-flat Calabi-Yau metric on $\mathbb{C}^n$ constructed by Li [22], Székelyhidi [30] and Conlon-Rochon [10] with maximal volume growth, and Min [24] with volume growth $2n-1$ when n is even. Those works constructed Calabi-Yau with singular tangent cone at infinity.

Recently, Apostolov-Cifarelli [1] constructed new complete Calabi-Yau metrics on $\mathbb{C}^n$ with volume growth $2n-1$. Their new method using toric geometry and Hamiltonian 2-

forms is significantly different from Tian-Yau-Hein package and produces exotic complete Calabi-Yau metrics with interesting behavior at infinity.

Chapter 2

Preliminaries

2.1 Kähler-Einstein metric

2.1.1 Kähler structure

Definition 2.1.1. *A* Kähler structure *on a smooth manifold X is a pair (g, J) consisting of a Riemannian metric g and an almost-complex structure $J\colon TX \to TX$ satisfying*

$$g(J\cdot, J\cdot) = g(\cdot, \cdot), \qquad \nabla J = 0,$$

where ∇ is the Levi-Civita connection of g.

Extending J to the cotangent bundle by $(J\theta)(v) := \theta(Jv)$ yields the eigenspace decomposition

$$TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X, \qquad T^*X \otimes \mathbb{C} = T^{*1,0}X \oplus T^{*0,1}X.$$

Consequently,

$$\Lambda^k(T^*X \otimes \mathbb{C}) = \bigoplus_{p+q=k} \Lambda^{p,q}T^*X := \bigoplus_{p+q=k} \Lambda^p T^{*1,0}X \otimes \Lambda^q T^{*0,1}X.$$

Denote by $\mathcal{A}^k(X)$, $\mathcal{A}^k_{\mathbb{C}}(X)$, and $\mathcal{A}^{p,q}(X)$ the spaces of smooth k-forms, complex k-forms, and (p,q)-forms, respectively. The exterior derivative splits as $d = \partial + \bar\partial$; for $f \in C^\infty(X)$,

$$2i\partial\bar\partial f = dJdf.$$

Kähler form. Define the real 2-form

$$\omega(v, w) \; = \; g\big(Jv, \, w\big), \qquad v, w \in TX.$$

Because of the Kähler conditions, ω is closed and of type $(1,1)$. Extend g and ω complex-bilinearly.

Choose holomorphic coordinates $(z^1,\dots,z^n)$ with $z^j = x^j + \sqrt{-1}y^j$. Then

$$J\partial_{x^j} = \partial_{y^j}, \qquad J\partial_{y^j} = -\partial_{x^j},$$

and set

$$\partial_{z^j} = \tfrac{1}{2}\big(\partial_{x^j} - \sqrt{-1}\partial_{y^j}\big), \quad \partial_{\bar z^j} = \tfrac{1}{2}\big(\partial_{x^j} + \sqrt{-1}\partial_{y^j}\big),$$

with dual coframe $dz^j = dx^j + \sqrt{-1}dy^j$ and $d\bar z^j = dx^j - \sqrt{-1}dy^j$.

In this frame,

$$g = \sum_{j,k} g_{j\bar k}\big(dz^j \otimes d\bar z^k + d\bar z^k \otimes dz^j\big), \qquad \omega = \sqrt{-1}\sum_{j,k} g_{j\bar k}\, dz^j \wedge d\bar z^k,$$

where $\big(g_{j\bar k}\big)$ is a positive–definite Hermitian matrix at each point.

2.1.2 Ricci Curvature on Kähler Manifolds

Let (X,ω) be an n-dimensional Kähler manifold with local holomorphic coordinates $(z^1,\dots,z^n)$ and metric coefficients $g_{j\bar k} := g\big(\partial_{z^j},\partial_{\bar z^k}\big)$. Because $\nabla J = 0$, the Levi-Civita connection coincides with the Chern (holomorphic) connection, and the only non-vanishing Christoffel symbols are

$$\Gamma^m_{jk} = g^{m\bar l}\,\partial_j g_{k\bar l}, \qquad \Gamma^{\bar m}_{\bar j\,\bar k} = g^{\bar m l}\,\partial_{\bar j} g_{l\bar k},$$

where $g^{m\bar l}$ is the inverse Hermitian matrix.

The curvature of the Chern connection has components

$$R_{j\bar k\,l\bar m} \ = \ -\partial_{\bar k}\partial_j g_{l\bar m} + g^{p\bar q}\,\partial_j g_{l\bar q}\,\partial_{\bar k} g_{p\bar m},$$

and satisfies the Kähler symmetries $R_{j\bar k\,l\bar m} = R_{l\bar k\,j\bar m} = R_{j\bar m\,l\bar k}$.

Contracting once with the metric gives the Ricci tensor

$$\mathrm{Ric}_{j\bar k} = g^{l\bar m} R_{j\bar k\,l\bar m}\ ,$$

and the associated $(1,1)$-form

$$\mathrm{Ric} \ = \ \sqrt{-1}\,R_{j\bar k}\,dz^j \wedge d\bar z^k.$$

A direct computation shows that Ricci form in this case can be written as

$$\mathrm{Ric} \ = \ -\sqrt{-1}\,\partial\bar\partial \log\det g_{j\bar k},$$

so that Ric is closed. Moreover, its cohomological class in $H^{1,1}(X,\mathbb{R})$ is independent of g and is a multiple of the first Chern class of X.

The volume form of the Kähler metric is

$$dV_g = \frac{\omega^n}{n!} = \det g_{j\bar k}\,(\sqrt{-1})^{2n}\,dz^1 \wedge d\bar z^1 \wedge \cdots \wedge dz^n \wedge d\bar z^n.$$

Hence the Ricci curvature can be rewritten as

$$\mathrm{Ric} \ = \ -\sqrt{-1}\,\partial\bar\partial \log\omega^n,$$

and we denote it as $\mathrm{Ric}\,\omega$ or $\mathrm{Ric}\,g$.

Definition 2.1.2. *Let X be a Kähler manifold. A Kähler metric ω on X is called a* Kähler–Einstein metric *if its Ricci form satisfies the Einstein condition*

$$\operatorname{Ric} g = \lambda g \text{ for some constant } \lambda \in \mathbb{R}. \tag{2.1.1}$$

When the ambient manifold X has complex dimension n, equation 2.1.1 involves $\frac{2n(2n+1)}{2}$ variables and imposes $\frac{2n(2n+1)}{2}$ condition. However, with the Kähler condition, this equation of tensor can be reduced to an equation of a single real-valued function.

2.2 Monge-Ampère equation

2.2.1 On compact Kähler manifold

In this section, we outline the general strategy for finding Kähler–Einstein metrics, following the pioneering works of Calabi, Aubin, Yau and others.

Given a reference Kähler metric ω_0 in $\lambda c_1(X)$, we seek a Kähler metric ω cohomologous to ω_0 of the form

$$\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where φ is a real-valued smooth function on X. The Ricci form of ω satisfies

$$\operatorname{Ric}(\omega) = \operatorname{Ric}(\omega_0) - \sqrt{-1}\partial\bar{\partial}\log\frac{\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right)}{\det\left(g_{i\bar{j}}\right)},$$

where $g_{i\bar{j}}$ are the components of ω_0.

Thus, the Kähler–Einstein equation reduces to solving the following complex Monge–Ampère equation:

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{f-\lambda\varphi}\omega_0^n, \tag{2.2.1}$$

where the smooth function f is determined (up to a constant) by the relation

$$\sqrt{-1}\partial\bar{\partial}f = \operatorname{Ric}(\omega_0) - \lambda\omega_0.$$

The solvability of the Monge-Ampère equation depends on the value of λ, which represents the sign of $c_1(X)$:

when $\lambda < 0$, the independent work of Aubin [31] and Yau [34] guarantees the existence and uniqueness of a solution upto a constant, thus producing a unique Kähler-Einstein metric.

when $\lambda = 0$, Yau's famous solution to the Calabi conjecture [34] provides a solution of 2.2.1 and hence gives Kähler Ricci flat metric in any Kähler class.

when $\lambda > 0$, the existence is more subtle and related to the K-stability in algebraic geometry. There have been many works in this direction and we refer interested readers to the survey of Donaldson [13] and only mention here the break through work of Chen-Donaldson-Sun [2, 3, 4], which solves the Yau-Tian-Donaldson conjecture for Fano manifolds, proving that a Fano manifold admits a Kähler-Einstein metric if and only if it is K-polystable.

2.2.2 On complete Kähler manifold

While the existence results for compact Kähler-Einstein manifolds are well-studied, the explicit behavior of their metrics remains largely unknown. However, we do have a lot of explicit examples of the noncompact complete Calabi-Yau manifolds constructed by many mathematicians which appears in the singular formation and limits of Kähler-Einstein manifolds. One typical example is Tian-Yau metric that is asymptotic to Calabi model space.

2.2.2.1 Calabi model space

Let us give a brief introduction of Calabi ansatz and some notations we will use later. The notations in this section mainly follow Hein-Sun-Viaclovsky-Zhang [19, Section 3].

Let L be an ample line bundle over a compact Calabi-Yau manifold D with a nowhere-vanishing holomorphic $(n-1,0)$-form Ω_D and a Calabi-Yau metric $\omega_D \in c_1(L)$ with

$$\omega_D^{n-1} = i^{(n-1)^2}\Omega_D \wedge \bar{\Omega}_D.$$

We choose an h_D on L such that the curvature form is ω_D. This choice is unique up to rescaling the metric h by multiplying a constant e^{-A}

For any point $\xi \in L$, let

$$t = -\log|\xi|_{h_D}^2$$

and

$$z = t^{\frac{1}{n}}$$

be functions on the complement of the zero section in the total space $L \setminus D$. Let

$$\mathcal{C} = \{0 < |\xi|_{h_D} < 1\} \subseteq L$$

be the disc bundle over D with complex structure $I_{\mathcal{C}}$ restricted from L. On $\mathcal{C}$, we have a Calabi-Yau metric given by the Calabi ansatz:

$$\begin{aligned}\omega_{\mathcal{C}} &= \frac{n}{n+1}\sqrt{-1}\partial\bar{\partial}(t^{\frac{n+1}{n}})\\ &= z\sqrt{-1}\partial\bar{\partial}t + \frac{1}{nz^{n-1}}i\partial t \wedge \bar{\partial}t.\end{aligned}$$

Let $\Omega_{\mathcal{C}}$ denotes the unique holomorphic $(n,0)$-form on L such that

$$Z \lrcorner \Omega_{\mathcal{C}} = \pi^*\Omega_D,$$

where Z denotes the holomorphic vector field generated by the scalar multiplication along the fiber direction and $\pi : L \to D$ is the projection map.

Definition 2.2.1. *The data $(\mathcal{C}, I_{\mathcal{C}}, \omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$ is called a Calabi model space.*

2.2.2.2 Tian-Yau metric

We want to construct a complete Kähler metric on the complement of a divisor D in a projective manifold M, which is asymptotic to the Calabi model space. More precisely, we denote the complement of D in M by $X = M \setminus D$. The metric we want to construct is a Kähler metric ω on X such that:

$$\text{there exists a diffeomorphism } \phi : \mathcal{C} \to X \text{ such that } \phi^* \omega = \omega_{\mathcal{C}} + o(1) \text{ as } r \to \infty.$$

With this diffeomorphism, the topology at infinity of X is the same as that of $\mathcal{C}$, hence the base of line bundle L should be D and hence we require D to be Calabi-Yau and has an ample normal bundle. By adjunction we need D to be a smooth anticanonical divisor in M. Since D is anticanonical, we can find a holomorphic section S of $-K_M$ such that S^{-1} can be seen as an $(n,0)$-form Ω_X on X with a simple pole and residue Ω_D along D.

One natural way to construct such a diffeomorphism is to use the exponential map. The exponential map $\exp : T_{x_0}X \to X$ is a local diffeomorphism around x_0, and we can use it to construct a diffeomorphism Φ from $\mathcal{C}$ to X outside some compact set, mapping the fiber of $N_D = T_{x_0}M / T_{x_0}D$ to the fiber of $(T_{x_0}D)^{\perp}$ over D, then composed with exp. This gives us a local diffeomorphism Φ between the tubular neighborhood of D in M and the disc bundle $\mathcal{C}$ over D. Under this diffeomorphism and the condition that D is anticanonical, we have the following $(1,1)$-form on X that is exponentially close to the Calabi model space as proved by Tian-Yau [33]:

Proposition 2.2.2. *We choose a metric h_M of $-K_M$ such that $h_M|_D = h_D$. Then we can construct the following $(1,1)$-form on X:*

$$\omega_X = \frac{n}{n+1} \sqrt{-1} \partial \bar{\partial} \left(-\log |S|^2_{h_M} \right)^{\frac{n+1}{n}}.$$

It is asymptotically Calabi in the following way: The complex structure I_X and I_C, metric $\omega_{\mathcal{C}}$ and ω_X and canonical form Ω_X and $\Omega_{\mathcal{C}}$ are exponentially closed. To be more precise, there exists a compact set K in X, a compact set $\mathcal{K}$ in $\mathcal{C}$, and a diffeomorphism induced by exponential map $\Phi : \mathcal{C} \setminus \mathcal{K} \to X \setminus K$ such that for all $k \geqslant 0, \epsilon > 0$:

$$\left| \nabla^k_{g_{\mathcal{C}}} \left(\Phi^* I_X - I_{\mathcal{C}} \right) \right|_{g_{\mathcal{C}}} + \left| \nabla^k_{g_{\mathcal{C}}} \left(\Phi^* \Omega_X - \Omega_{\mathcal{C}} \right) \right|_{g_{\mathcal{C}}} + \left| \nabla^k_{g_{\mathcal{C}}} \left(\Phi^* \omega_X - \omega_{\mathcal{C}} \right) \right|_{g_{\mathcal{C}}} = O(e^{-(\frac{1}{2} - \epsilon) z^n}).$$

This metric is constructed by Tian and Yau [33] with the following setting:

Theorem 2.2.3. *Let $X = M \setminus D$, where M is a projective manifold and $D \subset M$ is an ample smooth anticanonical divisor. Then there is a complete Kähler Ricci-flat metric that is asymptotic to Calabi model space.*

2.2.2.3 Tian-Yau-Hein's package

More generally, Tian-Yau [33] proved a perturbation result via the continuity method for Monge-Ampère equation in noncompact case, with some assumptions on boundedness of curvature, growth of the volume and decay of the Ricci potential that we will define in this section.

It is systematically summarized and generalized later by Hein [17] and becomes a very powerful tool in constructing noncompact Calabi-Yau metric. Roughly speaking, the main idea is to find a Kähler metric ω on X such that the Ricci form $\mathrm{Ric}(\omega)$ decays sufficiently fast then we can apply the continuity method to solve the Monge-Ampère equation if we have uniform weighted Sobolev inequalities and uniform Hölder estimates for the Kähler metric ω.

Before we state the main theorem of Tian-Yau-Hein's package, we need to introduce some requirements of the base Riemannian manifold (N, g) in Tian-Yau-Hein's package. We begin with the definition of $\mathrm{SOB}(\nu)$ property:

Definition 2.2.4 (SOB)**.** *Let (N, g) be a complete noncompact Riemannian manifold with real dimension at least 3. We say (N, g) satisfies $\mathrm{SOB}(\nu)$ condition for some $\nu > 0$ if and only if there exists a point $p \in N$ and a positive constant $C > 1$ such that*

1. *Volume growth is at most ν, i.e. $\mathrm{Vol}_g(B(p, R)) \leqslant CR^\nu$ for all $R > C$.*
2. $\mathrm{Ric}(x) \geqslant -C d_g(x, p)^{-2}$, $\forall x \in N$.
3. $\mathrm{Vol}_g(B(x, (1 - \frac{1}{C}) d_g(x, p))) \geqslant \frac{1}{C} d_g(x, p)^\nu$.
4. *For any $D > C$, any two points $x, y \in N$ with $d_g(p, x) = d_g(p, y) = D$ can be joined by a curve of length at most $C \cdot D$, lying in the annulus*

$$A(p, \tfrac{1}{C}D, CD) := \{x \in N \mid \tfrac{1}{C}D < d_g(p, x) < CD\}.$$

Remark 2.2.5. *In the original definition of $\mathrm{SOB}(\nu)$ property in [16], we need that the annulus $A(p, s, t)$ is connected for any $t > s > 0$. To apply Theorem 2.2.7 it suffices to check the Relative Connected Annulus (RCA) property 4. instead of the connectivity of the annulus.*

We continue with the definition of $\mathrm{HMG}(\lambda, k, \alpha)$ property:

Definition 2.2.6 (HMG)**.** *We say that (N^n, g) is $\mathrm{HMG}(\lambda, k, \alpha)$, for some $\lambda \in [0, 1]$, $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$, if there exist $C \geqslant 1$ such that*

1. *There exists a covering for every $x \in I$ with $r(x) \geqslant C$ there exists a local holomorphic diffeomorphism Φ_x from the unit ball $B \subset \mathbb{R}^n$ into N such that $\Phi_x(0) = x$ and $\Phi_x(B) \supset B\left(x, \frac{1}{C} r(x)^\lambda\right)$,*
2. *$h := r(x)^{-2\lambda} \Phi_x^* g$ satisfies $\mathrm{Inj}(h) \geqslant \frac{1}{C}$, $\frac{1}{C} g_{euc} \leqslant h \leqslant C g_{euc}$, and $\|h - g_{euc}\|_{C^{k,\alpha}(B, g_{euc})} \leqslant C$.*

Locally, we can always define $C^{k,\alpha}$ norm of a function f on a ball $B_g(x, r)$ inside a manifold (N, g) with respect to the metric g as follows:

$$\|f\|_{C^{k,\alpha}(B_g(x,r))} = \sum_{j=1}^{k-1} \left\|\nabla_g^j f\right\|_{C^0(B_g(x,r))} + \left\|\nabla_g^k f\right\|_{C^{0,\alpha}(B_g(x,r))}.$$

With this quasi-atlas, we can define the global Hölder norm $C^{l,\gamma}(N)$ as:

$$\|f\|_{C^{l,\gamma}(N)} := \sup\left\{\|u\circ\Phi_x\|_{C^{l,\gamma}(B)} : x\in N\right\}.$$

Now we are ready to present the following result taken from Hein's thesis [17] which is a powerful tool to give the existence of Calabi-Yau metric on complete noncompact Kähler manifold.

Theorem 2.2.7 (Tian-Yau-Hein's Package)**.** *Let* (X^n,ω) *be a complete noncompact Kähler manifold, which satisfies the condition* $\mathrm{SOB}(\nu)$ *and* $\mathrm{HMG}(0,3,\alpha)$ *for some* $\nu>0$, $0<\alpha<1$. *Let* r *be the distance function to a fixed point* $p\in X$ *with respect to the metric* ω. *Let* $f\in C^{2,\alpha}(X)$ *satisfy*

$$|f|\leqslant Cr^{-\mu}\ \text{on } \{r>1\} \text{ for some } \mu>2$$

and

$$\int_X\left(e^f-1\right)\omega^n=0.$$

Then there exist $\bar\alpha\in(0,\alpha]$ *and* $u\in C^{4,\bar\alpha}(X)$ *such that*

$$\left(\omega+i\partial\bar\partial u\right)^n=e^f\omega^n.$$

Moreover $\int_X|\nabla u|^2\omega^n<\infty$. *If in addition* $f\in C^{k,\bar\alpha}_{\mathrm{loc}}(X)$ *for some* $k\geqslant 3$, *then all such solutions* u *belong to* $C^{k+2,\bar\alpha}_{\mathrm{loc}}(X)$.

There are several key ingredients that are different from the compact setting when we apply the continuity method in [34] to solve the Monge-Ampère equation which motivates the definition of the above properties. One of the key ingredients is the Sobolev property which is used to control the volume growth of the manifold and to ensure that the Ricci curvature decays sufficiently fast at infinity. The second key ingredient is the local chart with weighted Hölder continuity which is used to control the local geometry of the manifold and to ensure that the curvature is bounded in a suitable sense. Both of the conditions are motivated to provide the uniform elliptic estimates which are used to control the regularity of the solutions to the Monge-Ampère equation.

2.2.2.4 Poincaré and Sobolev inequalities

The Poincaré inequality is a fundamental result in analysis that provides a bound on the average value of a function over a domain in terms of its gradient. In the context of Riemannian manifolds, it can be used to control the behavior of functions on the manifold in terms of their geometry. The Sobolev property on noncompact manifold is a generalization of the classical Sobolev inequality and only depends on the volume and Ricci curvature at infinity.

We can see the motivation of this definition from the Poincaré inequality locally on a fixed ball, with the coefficient constant depending only volume and Ricci curvature on this ball. We will use it later in proving the decay of the final solution of our perturbed Monge-Ampère equation.

Theorem 2.2.8. *[23, Theorem 1.1] Let* (N,g) *be a complete connected Riemannian manifold of real dimension* n*. Suppose the Ricci curvature on* N *is bounded below by* $\operatorname{Ric} \geq -a^2 g$ *for some* $a \geq 0$*. Then there exists a constant* c_n *depending only on* n*, and for every pair* (p,q) *satisfying* $1 \leq p < n$ *and* $p \leq q \leq \frac{pn}{n-p}$*, there exists a constant* $C_{p,q}$ *such that for all* $x \in N$*, all* $r > 0$*, and all functions* $f \in \mathcal{C}^\infty(B(x,r))$*, we have:*

$$\left(\int_{B(x,r)} |u - \bar{u}|^q \, dv\right)^{1/q} \leq C_{p,q} e^{c_n(1+ar)} r \, |B(x,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(x,r)} |\nabla u|^p \, dv\right)^{1/p},$$

where $\bar{u}$ *denotes the average of* u *over the ball* $B(x,r)$*.*

We also have Poincaré-Sobolev inequality on the annulus.

Theorem 2.2.9. *[17, Proposition 3.4] Let* (N,g) *be complete,* $x \in N$*, and*

$$A(\kappa) := A\left(x, r_1 - \kappa s, r_2 + \kappa s\right)$$

if $r_2 > r_1 > \kappa s$*. Suppose that* $A := A(0)$ *is connected,*

$$\operatorname{Ric} \geq -Cs^{-2} \text{ on } A(6), \qquad \sup_{y \in A(3)} |B(y,s)| \leq C \inf_{y \in A} |B(y,s)|,$$

and

$$\Lambda \geq \max\left\{, s^{-1} \operatorname{diam}(A), \frac{|A(1)|}{\inf_{y \in A} |B(y,s)|}\right\}.$$

Then, for all $u \in C^\infty(A)$*, denoting* $u_A := \oint_A u$*,*

$$\int_A |u - u_A|^2 \leq C \Lambda^2 s^2 \int_{A(6)} |\nabla u|^2.$$

In the continuity method, we also need the complete manifold having good volume growth and curvature condition at infinity to have a uniform weighted Poincaré-Sobolev inequality.

Theorem 2.2.10. *[18, Theorem 1.2] Suppose that* (N,g) *satisfies* $\operatorname{SOB}(\beta)$ *for some* $\beta \in \mathbb{R}^+$*.*

1. *For all* $\varepsilon > 0$ *there exists a positive step function* $\psi_\varepsilon \sim (1+r)^{-\max\{\beta,2\}-\varepsilon}$ *on* N *such that for all* $\alpha \in \left[1, \frac{n}{n-2}\right]$ *and all* $u \in C_0^\infty(N)$*,*

   $$\left(\int_N |u - u_\varepsilon|^{2\alpha} (1+r)^{\alpha(\min\{\beta-2,0\}-\varepsilon)-\beta} d\mathrm{vol}\right)^{\frac{1}{\alpha}} \leq C(\varepsilon) \int_N |\nabla u|^2 d\mathrm{vol}$$

   where u_ε *denotes the average of* u *with respect to the finite measure* $\psi_\varepsilon d\mathrm{vol}$*.*

2. *If* $\beta > 2$*, then, for all* $\alpha \in \left[1, \frac{n}{n-2}\right]$ *and all* $u \in C_0^\infty(N)$*,*

   $$\left(\int_N |u|^{2\alpha} (1+r)^{\alpha(\beta-2)-\beta} d\mathrm{vol}\right)^{\frac{1}{\alpha}} \leq C \int_N |\nabla u|^2 d\mathrm{vol}$$

2.2.2.5 Uniform Schauder estimates

The HMG properties are mainly used to define the weighted global $C^{k,\alpha}$-norm and guarantee that we can do elliptic estimates for a second order elliptic differential operator L if L has coefficients in $C^{l,\gamma}(M)$ with $l \leq k-1$.

Recall that the $C^{k,\alpha}$ norm of a function f on a ball $B_g(x,r)$ inside a manifold (N,g) is

$$\|f\|_{C^{k,\alpha}(B_g(x,r))} = \sum_{j=1}^{k-1} \left\|\nabla_g^j f\right\|_{C^0(B_g(x,r))} + \left\|\nabla_g^k f\right\|_{C^{0,\alpha}(B_g(x,r))}.$$

If we scale the metric by a constant $\frac{1}{z_0}$, i.e. $\tilde{g} = \frac{1}{z_0} g$, we then have

$$\begin{aligned}
&\left|\nabla_{\tilde{g}}^j f\right|_{\tilde{g}} = z_0^{\frac{j}{2}} \cdot |\nabla_g^j f|_g, \\
&\|f\|_{W^{k,p}(B_{\tilde{g}}(x,r))} = \sum_{j=0}^{k} z_0^{\frac{j}{2}} \left\|\nabla_g^j f\right\|_{L^p(B_g(x,\sqrt{z_0}r))}, \\
&\|f\|_{C^{k,\alpha}(B_{\tilde{g}}(x,r))} = \sum_{j=0}^{k-1} z_0^{\frac{j}{2}} \left\|\nabla_g^j f\right\|_{C^0(B_g(x,\sqrt{z_0}r))} + z_0^{\frac{k+\alpha}{2}} \left\|\nabla_g^k f\right\|_{C^{0,\alpha}(B_g(x,\sqrt{z_0}r))}.
\end{aligned}$$

Now we can prove the uniform Schauder estimate here for future reference:

Proposition 2.2.11. *Let (N,g) be a manifold satisfying* $\mathrm{HMG}(\lambda,k,\alpha)$ *property. Let u and v be smooth functions on M such that $\Delta_g u = v$. Then there exist some constants C_k such that*

$$r(x)^{k\lambda} \left\|\nabla^k u\right\|_{C^0(B_g(x,r(x)^\lambda))} \leqslant C_k \left(\|u\|_{C^0(B_g(x,r(x)^\lambda))} + \sum_{i=0}^{k-1} r(x)^{(i+2)\lambda} \left\|\nabla^i v\right\|_{C^0(B_g(x,r(x)^\lambda))} \right).$$

Proof. Fix any x in N. Let $h = r(x)^{-2\lambda}\Phi^* g$ be the rescaled pull back metric on the unit ball $B(0,1)$ in $\mathbb{R}^n$, let $\tilde{u} = \Phi_x^* u$, $\tilde{v} = r(x)^{2\lambda}\Phi_x^* v$. Since we use the pull back metric, the Laplacian is preserved. We have $\Delta_h \tilde{u} = \tilde{v}$. Write this elliptic equation under the Euclidean coordinate, with $\|\nabla^k h\|_{g_{\mathrm{euc}}} \leqslant C(k)$ for any integer $k \geqslant 0$, we have $\tilde{u}$ satisfies the following elliptic equation

$$\frac{1}{\sqrt{\det h}} \frac{\partial}{\partial x^i} \left(h^{ij} \sqrt{\det h} \frac{\partial}{\partial x^j} \tilde{u} \right) = \tilde{v}. \tag{2.2.2}$$

By the standard elliptic estimates on the Euclidean space and passing to the original metric g we get the required estimate. To be more precise, by $W^{2,q}$ estimates we know that there exists a uniform constant C_q only depends on q such that for any $1 < q < +\infty$

$$\|\tilde{u}\|_{W^{2,q}(B(0,\frac{1}{2}))} \leqslant C_q \cdot \left(\|\tilde{v}\|_{C^0(B(0,1))} + \|\tilde{u}\|_{C^0(B(0,1))} \right).$$

Consequently by Sobolev embedding, we know

$$\|\tilde{u}\|_{C^{1,1-\frac{2n}{q}}(B(0,\frac{1}{2}))} \leqslant C_q \|\tilde{u}\|_{W^{2,q}(B(0,\frac{1}{2}))} \leqslant C_q \left(\|\tilde{v}\|_{C^0(B(0,1))} + \|\tilde{u}\|_{C^0(B(0,1))} \right).$$

Taking derivative of (2.2.2) under the Euclidean coordinate, we have:

$$\begin{aligned}\|\tilde{u}\|_{W^{3,q}(B(0,\frac{1}{2}))} &\leqslant C_q \cdot \left(\|\tilde{v}\|_{W^{1,q}(B(0,\frac{2}{3}))} + \|\tilde{u}\|_{W^{2,q}(B(0,\frac{2}{3}))}\right)\\ &\leqslant C_q \cdot \left(\|\tilde{v}\|_{C^1(B(0,1))} + \|\tilde{u}\|_{C^0(B(0,1))}\right).\end{aligned}$$

And by bootstrapping and Sobolev embedding

$$\begin{aligned}\|\tilde{u}\|_{C^{k+1,1-\frac{2n}{q}}(B(0,\frac{1}{2}))} &\leqslant C_{q,k}\,\|\tilde{u}\|_{W^{k+2,q}(B(0,\frac{1}{2}))}\\ &\leqslant C_{q,k}\left(\|\tilde{v}\|_{W^{k,q}(B(0,\frac{2}{3}))} + \|\tilde{u}\|_{W^{k+1,q}(B(0,\frac{2}{3}))}\right)\\ &\leqslant C_{q,k}\left(\|\tilde{v}\|_{C^k(B(0,1))} + \|\tilde{u}\|_{C^0(B(0,1))}\right).\end{aligned}$$

Passing to the original metric, we know that for any $k \geqslant 1$

$$r(x)^{k\lambda}\left\|\nabla^k u\right\|_{C^0(B_g(x,r(x)^\lambda))} \leqslant C_k\left(\|u\|_{C^0(B_g(x,r(x)^\lambda))} + \sum_{i=0}^{k-1} r(x)^{(i+2)\lambda}\left\|\nabla^i v\right\|_{C^0(B_g(x,r(x)^\lambda))}\right).$$

□

Chapter 3

Asymptotic Calabi metric on weak log Fano pairs

3.1 Geometric properties of Calabi model space

Let us first list some geometry properties of Calabi model space directly coming from the formula of $\omega_{\mathcal{C}}$. Globally, it is complete when $|\xi|_{h_D} \to 0$ and incomplete when $|\xi|_{h_D} \to 1$. Locally, it behaves nicely like a doubly warped product $\mathbb{R}_+ \times S^1 \times D$ as in Figure 1.1.

Lemma 3.1.1. *We have the following quantitative estimates:*

- *Any distance function on* $\mathcal{C}$ *will be comparable to* $z^{\frac{n+1}{2}}$.
- *It has volume growth of order* $\frac{2n}{n+1}$.
- *The* S^1 *direction collapsing at rate* $z^{-\frac{n-1}{2}}$ *and the* D *direction growing at rate* $z^{\frac{1}{2}}$.
- *Its sectional curvature decays at the rate* $r^{-\frac{2}{n+1}}$.

Then we show that our model space $(\mathcal{C}, \omega_{\mathcal{C}})$ satisfies those properties. One may refer to [17] for details and examples.

Proposition 3.1.2. $(\mathcal{C}, \omega_{\mathcal{C}})$ *has* $\mathrm{SOB}(\frac{2n}{n+1})$ *property.*

Proof. (2) follows from the Ricci-flat property of $\mathcal{C}$. For (1), recall that $\omega_{\mathcal{C}} = z\sqrt{-1}\partial\bar{\partial}t + \frac{1}{nz^{n-1}} i\partial t \wedge \bar{\partial}t$ and the volume form $\omega_{\mathcal{C}}^n = (\sqrt{-1}\partial\bar{\partial}t)^{n-1} \wedge i\partial t \wedge \bar{\partial}t$. The distance function $r_{\omega_{\mathcal{C}}}$ to some fixed point p in $\mathcal{C}$ is comparable to $z^{\frac{n+1}{2}}$. Consequently, one can see that $\tilde{A}(z_1, z_2) = \{x \in \mathcal{C} \mid z_1 < z(x) < z_2\}$ is comparable to the annulus $A(p, R_1, R_2)$ in the sense that: for any $z_2 > z_1 > C$ and $R_2 > R_1 > C$, we have

$$\tilde{A}(z_1, z_2) \subseteq A\left(p, \tfrac{1}{C} z_1^{\frac{n+1}{2}}, C z_2^{\frac{n+1}{2}}\right),$$

$$A(p, R_1, R_2) \subseteq \tilde{A}\left(\left(\tfrac{1}{C} R_1\right)^{\frac{2}{n+1}}, (C R_2)^{\frac{2}{n+1}}\right).$$

Similarly, for any $R \leqslant C$, we have:

$$B(p,R) \subseteq \mathcal{K} \cup \tilde{A}\left(0,(CR)^{\frac{2}{n+1}}\right).$$

Moreover, we can see from the ansatz that the diameter of $\{x \in \mathcal{C} \,|\, z = z_0\}$ is comparable to $\sqrt{z_0}$, which shows that

$$\tilde{A}\left(z(x) - \tfrac{1}{C}\left(z(x) - z(p)\right), z(x) + \tfrac{1}{C}\left(z(x) - z(p)\right)\right) \subset B\left(x, \left(1 - \tfrac{1}{C}\right) d_g(x,p)\right).$$

With all these equivalences, we know that

$$\mathrm{Vol}_g(B(p,R)) \leqslant C + \int_{\left\{-(CR)^{\frac{2n}{n+1}} \leqslant t \leqslant (CR)^{\frac{2n}{n+1}}\right\}} (\sqrt{-1}\partial\bar{\partial}t)^{n-1} \wedge dt \wedge d^c t \leqslant CR^{\frac{2n}{n+1}}.$$

Similarly for (3) we have

$$\mathrm{Vol}_g(B\left(x, \left(1 - \tfrac{1}{C}\right) d_g(x,p)\right)) \geqslant C \cdot z(x)^n \geqslant C \cdot d_g(x,p)^{\frac{2n}{n+1}}.$$

To show (4), for D large enough and any two points x and y with $d_g(x,p) = d_g(y,p) = D$, we have $z(x), z(y) \in \left(\frac{1}{C} \cdot D^{\frac{2}{n+1}}, C \cdot D^{\frac{2}{n+1}}\right)$. Then by the formula of $\omega_{\mathcal{C}}$ we can join x and y by a curve of length at most $C \cdot \left(D + D^{\frac{1}{n+1}}\right)$ lying in $A\left(\frac{1}{C} \cdot D^{\frac{2}{n+1}}, C \cdot D^{\frac{2}{n+1}}\right)$ and consequently in the annulus $A(p, \frac{1}{C}D, CD)$.

Hence we have the $\mathrm{SOB}(\frac{2n}{n+1})$ condition on $(\mathcal{C}, \omega_{\mathcal{C}})$. □

Next we show that the model space $\mathcal{C}$ satisfies the $\mathrm{HMG}(0,k,\alpha)$ property. Tian-Yau [33, Proposition 1.2.] provides a simple criterion for a complete Kähler manifold (N,ω) to be $\mathrm{HMG}(0,k,\alpha)$. We refer to Hein's thesis [17, Lemma 4.7.] for a slightly generalized statement and sketch of the proof.

Lemma 3.1.3. *A complete Kähler manifold with* $|\mathrm{Rm}| + \sum_{i=1}^{k} r^{i\lambda} |\nabla^i \mathrm{Scal}| \leqslant C r^{-2\lambda}$ *for some* $k \in \mathbb{N}_0$ *and* $\lambda \in [0,1]$ *is* $\mathrm{HMG}(\lambda, k+1, \alpha)$ *for every* $\alpha \in (0,1)$.

Proposition 3.1.4. $(\mathcal{C}, \omega_{\mathcal{C}})$ *has* $\mathrm{HMG}(\frac{1}{n+1}, k, \alpha)$ *property, for any* $k \in \mathbb{N}_0$ *and* $\alpha \in (0,1)$.

Proof. The proof goes almost verbatim with the proof of Lemma 3.1.3. The proof of Lemma 3.1.3 only used the completeness to guarantee that the injectivity radius of local universal cover around x_0 has uniform lower bound independent of x_0. Since $\mathcal{C}$ is the disc bundle over D, after we do rescaling by $\tilde{\omega}_{\mathcal{C}} = z(x_0)^{-1}\omega_{\mathcal{C}}$, the S^1 action gives the only collapsing direction which disappears after passing to the local universal cover. So the local universal cover has uniform curvature bound and is volume non-collapsing, which leads to uniform injectivity radius lower bound. □

Remark 3.1.5. *We can not directly use Theorem 2.2.7 because in our setting, the decay rate of the class* $\mathfrak{k}$ *is only* $r^{-\frac{2}{n+1}}$ *which is slower than* r^{-2}. *So we need to modify the representative* β *by* $\sqrt{-1}\partial\bar{\partial}u$ *for some function* u *on* X, *which comes from the suitable solution of Poisson equation on model space* $\mathcal{C}$, *which we will discuss in section 3.2.*

3.2 Solving Poisson Equation on the Model Space

In this section, following the approach of Sun-Zhang in [28], we will use separation of variables to solve $\Delta_{\omega_{\mathcal{C}}} u = v$ for u, v functions on $\mathcal{C}$ and give some uniform estimate of our solution. We use the same notation introduced in section 2.2.2.

We first notice that $\mathcal{C}$ is diffeomorphic to $Y \times \mathbb{R}$ where the level set $Y = \{\xi \in \mathcal{C} \mid -\log|\xi|^2_{h_D} = z_0^n\}$ for a fixed $z_0 > 0$ and Y is equipped with an S^1 bundle structure over D and a metric h_Y induced by $\omega_{\mathcal{C}}$. Let $0 = \Lambda_0 < \Lambda_1 < \cdots$ be the spectrum of the Laplacian $-\Delta_{(Y,h_Y)}$ on Y with respect to the metric h_Y. Let $\{\psi_k\}_{k=0}^{\infty}$ be the corresponding eigenfunctions with $\|\psi_k\|_{L^2(Y)} = 1$. They showed that $\Lambda_k = z_0^{-1}\lambda_k + n z_0^{n-1} j_k^2$ for some $\lambda_k \geqslant 0$ and $j_k \in \mathbb{N}$. Moreover, $\{\psi_k\}_{k=0}^{\infty}$ form an orthonormal basis of $L^2(Y)$ and each ψ_k is homogeneous of degree j_k under the S^1 action. The product structure allows us to do Fourier expansion on $\mathcal{C}$. In particular, for any smooth function v on $\mathcal{C}$, if we take $P_k(v)(z) = \int_Y v(z,y)\psi_k(y)$, we can write

$$v(z,y) = \sum_{k=0}^{\infty} P_k(v)(z) \cdot \psi_k(y), \tag{3.2.1}$$

which is convergent in L^2 sense. In fact, we will prove later that the convergence is in C^k if v has proper higher regularity estimate. For the separated function $u(z)\psi(y)$ we have

$$\Delta_{\omega_{\mathcal{C}}} u(z)\psi(y) = \frac{1}{nz^{n-1}}(u''(z) - (\lambda + \frac{j^2 n}{4} \cdot z^n) n z^{n-2} u(z))\psi(y).$$

Moreover, [28] proved the following:

Proposition 3.2.1. *Let $(\mathcal{C}, g_{\mathcal{C}})$ be the Calabi model space, and let u solve the Poisson equation $\Delta_{\mathcal{C}} u = v$ for some $v \in C^{K_0}(\mathcal{C})$ and $K_0 \in \mathbb{N}$ sufficiently large. Let u, v have "fiber-wise" expansions as in (3.2.1). Then for every $k \in \mathbb{N}$, the coefficient functions $u_k(z)$ and $v_k(z)$ satisfy*

$$u_k''(z) - \left(n\lambda_k + \frac{j_k^2 n^2}{4} \cdot z^n\right) z^{n-2} u_k(z) = n z^{n-1} \cdot v_k(z), \quad z \geqslant 1. \tag{3.2.2}$$

Specifically, they found a solution of $\Delta_{\omega_{\mathcal{C}}} u = v$ by solving ODE (3.2.2). With the estimate of the solution of this equation, they showed that the L^2 formal solution given by $\sum u_k \psi_k$ is actually a regular solution to the Poisson equation. Similarly but directly via careful estimates, we can construct the solution of Poisson equation with respect to $\omega_{\mathcal{C}}$ with some weighted regularity and finer polynomial growth order.

3.3 Estimate of the solution of ODE

In this section, we will look closely to the solution of the following ordinary differential equation:

$$u'' - (\frac{j^2 n^2}{4} + n\lambda) z^{n-2} u = n z^{n-1} v,$$

where $\lambda > 0$, $n \geqslant 3$ and $n, j \in \mathbb{N}$.
By the transformation in [28], we have two cases: zero node case when $j = 0$ and non-zero node case when $j > 0$. We will give a brief summary of the estimate of fundamental solutions and have a estimate of u with polynomial rate which slightly generalizes the results in [28].

3.3.1 fundamental solution of zero mode

In this section we focus on the zero mode: the equation

$$u'' - n\lambda z^{n-2}u = nz^{n-1}v. \tag{3.3.1}$$

By [28] we have the decaying solution $\mathcal{D}(z)$ and growth solution $\mathcal{G}(z)$ of the homogeneous equation $u''(z) = nz^{n-2}\lambda u(z)$ given by

$$\mathcal{D}(z) = \sqrt{z}K_{\frac{1}{n}}\left(2\sqrt{\tfrac{\lambda}{n}}\cdot z^{\frac{n}{2}}\right), \tag{3.3.2}$$

$$\mathcal{G}(z) = \sqrt{z}I_{\frac{1}{n}}\left(2\sqrt{\tfrac{\lambda}{n}}\cdot z^{\frac{n}{2}}\right) \tag{3.3.3}$$

where K and I have the following expression: for $\nu \in \mathbb{R}$

$$K_\nu(y) = \int_0^\infty e^{-y\cosh t}\cosh(\nu t)dt,$$
$$I_\nu(y) = \frac{1}{\pi}\int_0^\pi e^{y\cos\theta}\cos(\nu\theta)d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-y\cosh t-\nu t}dt$$

Lemma 3.3.1. *[28][Proposition 3.3.] We have the following uniform estimate:*

1. *For all $\nu \in \mathbb{R}$, there is a constant $C(\nu) > 1$ such that*

$$C^{-1}(\nu)\cdot\tfrac{e^{-y}}{\sqrt{y}} \leqslant K_\nu(y) \leqslant C(\nu)\cdot\tfrac{e^{-y}}{\sqrt{y}}, \quad y \geqslant 1;$$
$$I_\nu(y) \leqslant \begin{cases} C(\nu)\cdot\frac{e^{y}}{\sqrt{y}}, & y \geqslant 1, \\ C(\nu)\cdot y^\nu, & 0 < y \leqslant 1. \end{cases}$$

2. *For all $\nu > -1$, we have*

$$I_\nu(y) \geqslant \begin{cases} C(\nu)^{-1}\cdot\frac{e^{y}}{\sqrt{y}}, & y \geqslant 1 \\ C(\nu)^{-1}\cdot y^\nu, & 0 < y \leqslant 1 \end{cases}$$

Corollary 3.3.2. *[28] For $z > \sqrt[n]{\frac{n}{4\lambda_1}}$, there exists a constant C which only depends on n such that*

$$\frac{1}{C}\cdot\frac{e^{-\frac{2}{\sqrt{n}}\lambda^{\frac{1}{2}}\cdot z^{\frac{n}{2}}}}{\lambda^{\frac{1}{4}}\cdot z^{\frac{n-2}{4}}} < \mathcal{D}(z) < C\cdot\frac{e^{-\frac{2}{\sqrt{n}}\lambda^{\frac{1}{2}}\cdot z^{\frac{n}{2}}}}{\lambda^{\frac{1}{4}}\cdot z^{\frac{n-2}{4}}}$$
$$\frac{1}{C}\cdot\frac{e^{\frac{2}{\sqrt{n}}\lambda^{\frac{1}{2}}\cdot z^{\frac{n}{2}}}}{\lambda^{\frac{1}{4}}\cdot z^{\frac{n-2}{4}}} < \mathcal{G}(z) < C\cdot\frac{e^{\frac{2}{\sqrt{n}}\lambda^{\frac{1}{2}}\cdot z^{\frac{n}{2}}}}{\lambda^{\frac{1}{4}}\cdot z^{\frac{n-2}{4}}}.$$

With those estimates, we can give a C^0 bound of $u(z)$. By computation in [28] we know that the Wronskian

$$\mathcal{W}(\mathcal{G},\mathcal{D}) = \mathcal{G}(z)\mathcal{D}'(z) - \mathcal{G}'(z)\mathcal{D}(z) = -\frac{n}{2}.$$

Hence we have a solution of 3.3.1 as follows:

$$u(z) = -2\left(\mathcal{D}(z)\int_1^z \mathcal{G}(s)s^{n-1}v(s)ds + \mathcal{G}(z)\int_z^\infty \mathcal{D}(s)s^{n-1}v(s)ds\right) \tag{3.3.4}$$

We firstly introduce an estimate of the solution of this ordinary differential equation:

Proposition 3.3.3. *Recall that λ_1 is the first nonzero positive eigenvalue of $-\Delta_Y$. Let v be a function such that $|v(z)| \leqslant C_0 z^\delta$ for $z > 1$. For any λ such that $\lambda > \lambda_1 > 0$, we can find solution of equation $u''(z) = nz^{n-2}\lambda u(z) + nz^{n-1}v$ such that*

$$|u(z)| \leqslant C \cdot C_0 z^{\delta+1},$$

$$|u'(z)| \leqslant C \cdot C_0 z^{\delta+\frac{n}{2}},$$

$$|u''(z)| \leqslant C \cdot C_0 z^{\delta+n-1}$$

on $z > C$ for some constant $C > 1$ only depend on n, λ_1 and δ .

Proof. Now we can estimate $\|u\|_{L^\infty((\max\{1,(\frac{n}{4\lambda_1})^{\frac{1}{n}}\},\infty))}$. Let $\mu = \frac{2}{\sqrt{n}}\lambda^{\frac{1}{2}}$. By integration by parts the first term $\mathcal{D}(z)\int_1^z \mathcal{G}(s)s^{n-1}v(s)ds$ in 3.3.4 is bounded by a constant $C(n)$ times the following term:

$$\begin{aligned}
&\frac{1}{\lambda^{\frac{1}{2}}\cdot z^{\frac{n-2}{4}}}\int_1^z e^{\mu\left(s^{\frac{n}{2}}-z^{\frac{n}{2}}\right)}s^{\delta+\frac{3n-2}{4}}ds\\
\leqslant\ &\frac{1}{\sqrt{n}\lambda}z^{\delta+1} - \frac{\delta+\frac{n+2}{4}}{n\lambda^{\frac{3}{2}}}z^{\delta+1-\frac{n}{2}} + \frac{(\delta+\frac{n+2}{4})(\delta+\frac{-n+2}{4})}{n\sqrt{n}\lambda^2}z^{\delta+1-n}\\
&\qquad - \frac{(\delta+\frac{n+2}{4})(\delta+\frac{-n+2}{4})(\delta+\frac{-3n+2}{4})}{n^2\lambda^{\frac{5}{2}}}z^{\delta+1-\frac{3n}{2}}\\
&\qquad + \frac{\max\{0,(\delta+\frac{n+2}{4})(\delta+\frac{-n+2}{4})(\delta+\frac{-3n+2}{4})(\delta+\frac{-5n+2}{4})\}}{n^2\lambda^{\frac{5}{2}}}z^{\delta+1-\frac{3n}{2}}\\
&\qquad - \left(\frac{1}{\sqrt{n}\lambda} - \frac{(\delta+\frac{n+2}{4})}{n\lambda^{\frac{3}{2}}} + \frac{(\delta+\frac{n+2}{4})(\delta+\frac{-n+2}{4})}{n\sqrt{n}\lambda^2} + C(n,\delta)\frac{1}{\lambda^{\frac{5}{2}}}\right)e^{\mu}\frac{e^{-\mu z^{\frac{n}{2}}}}{z^{\frac{n-2}{4}}}.
\end{aligned}$$

Here we also use the following observation: Since $\mu \geqslant \frac{2}{\sqrt{n}}\lambda_1^{\frac{1}{2}}$, we know that the maximum of $s^{\delta+\frac{-5n-2}{4}}e^{\mu\left(s^{\frac{n}{2}}-z^{\frac{n}{2}}\right)}$ on the interval $[1,z]$ is at z when z is larger than a uniform constant which is independent with respect to λ but only on n and λ_1. So we have

$$\int_1^z s^{\delta+\frac{-5n-2}{4}}e^{\mu\left(s^{\frac{n}{2}}-z^{\frac{n}{2}}\right)}ds \leqslant (z-1)z^{\delta+\frac{-5n-2}{4}} < z^{\delta+\frac{-5n+2}{4}}.$$

For the second term $\mathcal{G}(z)\int_z^\infty \mathcal{D}(s)s^{n-1}v(s)ds$, we have similar estimate:

$$\begin{aligned}
&\frac{1}{\lambda^{\frac{1}{2}}\cdot z^{\frac{n-2}{4}}}\int_z^\infty e^{\mu\left(z^{\frac{n}{2}}-s^{\frac{n}{2}}\right)}s^{\delta+\frac{3n-2}{4}}ds\\
\leqslant{}&\frac{1}{\sqrt{n}\lambda}z^{\delta+1}+\frac{\delta+\frac{n+2}{4}}{n\lambda^{\frac{3}{2}}}z^{\delta+1-\frac{n}{2}}+\frac{(\delta+\frac{n+2}{4})(\delta+\frac{-n+2}{4})}{n\sqrt{n}\lambda^2}z^{\delta+1-n}\\
&+\frac{2\left|(\delta+\frac{n+2}{4})(\delta+\frac{-n+2}{4})(\delta+\frac{-3n+2}{4})\right|}{n^2\lambda^{\frac{5}{2}}}z^{\delta+1-\frac{3n}{2}}.
\end{aligned}$$

So we have the uniform estimate for u that for any $z > C(n,\delta,\lambda_1)$,

$$|u(z)| \leqslant C(n)\cdot C_0\frac{z^{\delta+1}}{\lambda}.$$

For the derivative u' we can do the same computation as in [28] to estimate $\mathcal{D}'(z)$ and $\mathcal{G}'(z)$. In fact, we have the following estimate:

Lemma 3.3.4.

$$\frac{1}{C}\cdot\frac{e^y}{\sqrt{y}} < I'_{\frac{1}{n}}(y) < C\cdot\frac{e^y}{\sqrt{y}},\quad \frac{1}{C}\cdot\frac{e^{-y}}{\sqrt{y}} < -K'_{\frac{1}{n}}(y) < C\cdot\frac{e^{-y}}{\sqrt{y}}$$

for some fixed constant C and any $y > 1$.

Proof. Notice that

$$I'_{\frac{1}{n}}(y)=\frac{1}{2\pi}\int_0^\pi e^{y\cos\theta}\left(\cos\tfrac{(n+1)\theta}{n}+\cos\tfrac{(n-1)\theta}{n}\right)d\theta+\frac{\sin\frac{\pi}{n}}{2\pi}\int_0^\infty e^{-y\cosh t}\left(e^{-\frac{(n+1)t}{n}}+e^{\frac{(n-1)t}{n}}\right)dt.$$

As in the proof of [28] Prop. 3.3, we know that for any $\nu\in\mathbb{R}$

$$\int_0^\infty e^{-y\cosh t-\nu t}dt \leqslant e^{-y}\int_0^\infty e^{-\frac{yt^2}{2}-\nu t}dt\leqslant C(\nu)\cdot\frac{e^{-y}}{\sqrt{y}},$$

$$\left|\int_0^\pi e^{y\cos\theta}\cos(\nu\theta)d\theta\right|\leqslant e^y\int_0^{\frac{\pi}{3}}e^{-\frac{y\cdot\theta^2}{4}}d\theta+\frac{2e^{\frac{y}{2}}}{3}\leqslant\frac{2e^y}{\sqrt{\pi}\cdot\sqrt{y}}+\frac{2e^{\frac{y}{2}}}{3}\leqslant\frac{10e^y}{\sqrt{y}}.$$

For $\nu > -1$ and $\nu\neq 0$, let $\eta_\nu=\min\left(\pi,\frac{\pi}{3|\nu|}\right)$,

$$\begin{aligned}
&\int_0^\pi e^{y\cos\theta}\cos(\nu\theta)d\theta\\
={}&\int_0^{\eta_\nu}e^{y\cos\theta}\cos(\nu\theta)d\theta+\int_{\eta_\nu}^\pi e^{y\cos\theta}\cos(\nu\theta)d\theta\\
\geqslant{}&\frac{1}{2}e^y\int_0^{\eta_\nu}e^{-\frac{\theta^2}{2}y}d\theta-\left|\int_{\eta_\nu}^\pi e^{y\cos\theta}\cos(\nu\theta)d\theta\right|\\
\geqslant{}&C(\nu)\frac{e^y}{\sqrt{y}}-\int_{\eta_\nu}^\pi e^{y\cos\theta}d\theta\\
\geqslant{}&C(\nu)\frac{e^y}{\sqrt{y}}-(\pi-\eta_\nu)\,e^{\cos(\eta_\nu)y}\geqslant C(\nu)\frac{e^y}{\sqrt{y}}.
\end{aligned}$$

So we get that $\frac{1}{C}\cdot\frac{e^y}{\sqrt{y}} < I'_{\frac{1}{n}}(y) < C\cdot\frac{e^y}{\sqrt{y}}$.

On the other hand,

$$\begin{aligned}
K'_{\frac{1}{n}}(y) &= -\frac{1}{2}\int_0^\infty e^{-y\cosh t}\left(\cosh\frac{(n+1)t}{n}+\cosh\frac{(n-1)t}{n}\right)dt\\
&= -\frac{1}{2}(K_{\frac{n+1}{n}}(y)+K_{\frac{n-1}{n}}(y)).
\end{aligned}$$

By the estimate of K_ν, we know that $\frac{1}{C}\cdot\frac{e^{-y}}{\sqrt{y}} < -K'_{\frac{1}{n}}(y) < C\cdot\frac{e^{-y}}{\sqrt{y}}$. □

Corollary 3.3.5. *For $z > \sqrt[n]{\frac{n}{4\lambda_1}}$, there exists a constant C which only depends on n such that*

$$\begin{aligned}
\frac{1}{C}\cdot\lambda^{\frac{1}{4}}z^{\frac{n-2}{4}}e^{-\frac{2}{\sqrt{n}}\lambda^{\frac{1}{2}}\cdot z^{\frac{n}{2}}} &< -\mathcal{D}'(z) < C\cdot\lambda^{\frac{1}{4}}z^{\frac{n-2}{4}}e^{-\frac{2}{\sqrt{n}}\lambda^{\frac{1}{2}}\cdot z^{\frac{n}{2}}},\\
\frac{1}{C}\cdot\lambda^{\frac{1}{4}}z^{\frac{n-2}{4}}e^{\frac{2}{\sqrt{n}}\lambda^{\frac{1}{2}}\cdot z^{\frac{n}{2}}} &< \quad\mathcal{G}'(z) < C\cdot\lambda^{\frac{1}{4}}z^{\frac{n-2}{4}}e^{\frac{2}{\sqrt{n}}\lambda^{\frac{1}{2}}\cdot z^{\frac{n}{2}}}.
\end{aligned}$$

Proof. This can be seen directly from computing $\mathcal{D}'$ and $\mathcal{G}'$ with the substitution in (3.3.2).

$$\begin{aligned}
-\mathcal{D}'(z) &= -\frac{1}{2\sqrt{z}}K_{\frac{1}{n}}\left(2\sqrt{\tfrac{\lambda}{n}}\cdot z^{\frac{n}{2}}\right)-\sqrt{n\lambda}z^{\frac{n-1}{2}}K'_{\frac{1}{n}}\left(2\sqrt{\tfrac{\lambda}{n}}\cdot z^{\frac{n}{2}}\right),\\
\mathcal{G}'(z) &= \frac{1}{2\sqrt{z}}I_{\frac{1}{n}}\left(2\sqrt{\tfrac{\lambda}{n}}\cdot z^{\frac{n}{2}}\right)+\sqrt{n\lambda}z^{\frac{n-1}{2}}I'_{\frac{1}{n}}\left(2\sqrt{\tfrac{\lambda}{n}}\cdot z^{\frac{n}{2}}\right).
\end{aligned}$$

By Lemma 3.3.4 we get the estimate. □

Consequently, by integration by parts as before we have the C^1 estimate of u:

$$\begin{aligned}
&|u'(z)|\\
=&\left|2\left(\mathcal{D}'(z)\int_1^z\mathcal{G}(s)s^{n-1}v(s)ds+\mathcal{G}'(z)\int_z^\infty\mathcal{D}(s)s^{n-1}v(s)ds\right)\right|\\
\leqslant&\, C(n)\cdot C_0\frac{z^{\delta+\frac{n}{2}}}{\lambda^{\frac{1}{2}}}
\end{aligned}$$

and

$$u''(z)=n\lambda z^{n-2}u+nz^{n-1}v\leqslant C(n)\cdot C_0z^{\delta+n-1}$$

for $z > C(n,\delta,\lambda_1)$.

In the end, we get if $|v(z)|\leqslant C_0z^\delta$ on $z > C(n,\delta,M)$, then for any $z > C(n,\delta,M)$

$$|u(z)|\leqslant C(n)\cdot C_0\frac{z^{\delta+1}}{\lambda},$$

$$|u'(z)|\leqslant C(n)\cdot C_0\frac{z^{\delta+\frac{n}{2}}}{\lambda^{\frac{1}{2}}},\quad |u''(z)|\leqslant C(n)\cdot C_0z^{\delta+n-1}.$$

□

3.3.2 fundamental solution of non-zero mode

In this section we focus on the non-zero mode: the equation

$$u'' - (\frac{j^2n^2}{4} + n\lambda)z^{n-2}u = nz^{n-1}v. \tag{3.3.5}$$

By [28] we have the decay solution $\mathcal{D}(z)$ and growth solution $\mathcal{G}(z)$ of the homogeneous equation $u''(z) = (\frac{j^2n^2}{4} + n\lambda)z^{n-2}u$ given by

$$\mathcal{D}(z) = e^{\frac{jz^n}{2}} \cdot \Psi^\flat\left(\beta, \alpha, -jz^n\right), \tag{3.3.6}$$

$$\mathcal{G}(z) = e^{\frac{jz^n}{2}} \cdot \Phi^\sharp\left(\beta, \alpha, -jz^n\right), \tag{3.3.7}$$

where $\alpha = 1 - \frac{1}{n}$, $\beta = \frac{n-1}{2n} - \frac{\lambda}{nj} \leqslant 0$, $\Phi^\sharp\left(\beta, \alpha, -jz^n\right)$ and $\Psi^\flat\left(\beta, \alpha, -jz^n\right)$ have the following expression:

$$\Psi^\flat(\beta, \alpha, y) = \frac{e^y}{\Gamma(\alpha-\beta)}\int_0^\infty e^{ys}s^{\alpha-\beta-1}(1+s)^{\beta-1}ds, \tag{3.3.8}$$

$$\Phi^\sharp(\beta, \alpha, y) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \cdot e^y(-y)^{\beta-\alpha} \cdot \int_0^\infty e^{\frac{s}{y}} \cdot s^{\frac{\alpha-1}{2}-\beta} \cdot I_{\alpha-1}(2\sqrt{s})ds. \tag{3.3.9}$$

Let $Q = \alpha - \beta - 1$, $\gamma_n = \frac{1}{2} + \frac{1}{n}$. Denote

$$F(t) = yt + Q\log\frac{t}{t+1}.$$

Then F is strictly concave in $\mathbb{R}$ if $Q > 0$. Let t_0 be the only critical point of F. We have

$$t_0 = \frac{1}{2}\left(-1 + \sqrt{1 + \frac{4Q}{-y}}\right).$$

Denote

$$G(u) = -u^2 + 2(-y)^{\frac{1}{2}} \cdot u + (2Q + \gamma_n)\log u.$$

Then G is strictly concave in $\mathbb{R}_+$. Let u_0 be the only critical point of G. Then

$$u_0 = \frac{(-y)^{\frac{1}{2}}}{2} \cdot \left(1 + \sqrt{1 + \frac{4Q}{-y} + \frac{2\gamma_n}{-y}}\right).$$

In [28] by Laplace method, we can show the following estimate:

Lemma 3.3.6. *[28] There is a constant C which only depends on n such that when $Q \geqslant 1$,*

$$C_n^{-1} \cdot Q^{-\frac{1}{4}-\frac{1}{2n}} \cdot \frac{(-y)^{-1} \cdot e^{y+F(t_0)}}{\Gamma(Q+1)} \leqslant \Psi^\flat(\beta, \alpha, y) \leqslant C_n \cdot Q^{\frac{1}{4}} \cdot \frac{e^{y+F(t_0)}}{\Gamma(Q+1)},$$

$$C_n^{-1} \cdot Q^{-\frac{1}{4}} \cdot \frac{(-y)^{\frac{2-n}{4n}} \cdot e^{y+G(u_0)}}{\Gamma(Q+1)} \leqslant \Phi^\sharp(\beta, \alpha, y) \leqslant C_n \cdot \frac{(-y)^{\frac{2-n}{4n}} \cdot e^{y+G(u_0)}}{\Gamma(Q+1)};$$

when $Q \leqslant 1$,

$$
\begin{aligned}
C_n^{-1} \cdot e^y \cdot (-y)^{\beta-\alpha} \leqslant \Psi^\flat(\beta,\alpha,y) \leqslant e^y \cdot (-y)^{\beta-\alpha}, \\
C_n^{-1} \cdot (-y)^{-\beta} \leqslant \Phi^\sharp(\beta,\alpha,y) \leqslant C_n \cdot (-y)^{-\beta}
\end{aligned}
$$

for any $y \leqslant -1$.

Corollary 3.3.7. *[28] There is a constant* C *which only depends on* n *such that when* $Q \geqslant 1$,

$$
\begin{aligned}
C^{-1} \cdot \frac{Q^{-\frac{1}{4}-\frac{1}{2n}}}{\Gamma(Q+1)} \cdot e^{-\frac{jz^n}{2}+F(t_0(z))} \cdot (jz^n)^{-1} \leqslant \mathcal{D}(z) \leqslant C \cdot \frac{Q^{\frac{1}{4}}}{\Gamma(Q+1)} \cdot e^{-\frac{jz^n}{2}+F(t_0(z))}, \\
C^{-1} \cdot Q^{-\frac{1}{4}} \cdot \frac{(jz^n)^{\frac{2-n}{4n}}}{\Gamma(Q+1)} \cdot e^{-\frac{jz^n}{2}+G(u_0(z))} \leqslant \mathcal{G}(z) \leqslant C \cdot \frac{(jz^n)^{\frac{2-n}{4n}}}{\Gamma(Q+1)} \cdot e^{-\frac{jz^n}{2}+G(u_0(z))};
\end{aligned}
$$

when $Q \leqslant 1$,

$$
\begin{aligned}
C^{-1} \cdot e^{-\frac{jz^n}{2}} \cdot (jz^n)^{\beta-\alpha} \leqslant \mathcal{D}(z) \leqslant C \cdot e^{-\frac{jz^n}{2}} \cdot (jz^n)^{\beta-\alpha}, \\
C^{-1} \cdot e^{\frac{jz^n}{2}} \cdot (jz^n)^{-\beta} \leqslant \mathcal{G}(z) \leqslant C \cdot e^{\frac{jz^n}{2}} \cdot (jz^n)^{-\beta}
\end{aligned}
$$

for any $z > 1$.

We also need the following lemma

Lemma 3.3.8. *For any* $z \geqslant 1$, $e^{F(t_0(z))+G(u_0(z))} \leqslant C \cdot j^{\frac{n+2}{4n}} z^{\frac{n+2}{4}} e^{jz^n} e^{-Q} Q^{Q+\frac{n+2}{4n}}$.

Proof. By similar straight forward computation as in [28]. ☐

With those estimates, we can give a C^0 bound of $u(z)$. By [28] we know that the Wronskian

$$
\mathcal{W}(\mathcal{G},\mathcal{D}) = \mathcal{G}(z)\mathcal{D}'(z) - \mathcal{G}'(z)\mathcal{D}(z) = \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta)} j^{\frac{1}{n}}.
$$

Hence we have a solution of (3.3.5) as follows:

$$
u(z) = \frac{\Gamma(\alpha-\beta)n}{\Gamma(\alpha-1)j^{\frac{1}{n}}} \left(\mathcal{D}(z) \int_1^z \mathcal{G}(s)s^{n-1}v(s)ds + \mathcal{G}(z) \int_z^\infty \mathcal{D}(s)s^{n-1}v(s)ds \right).
$$

Then we can have the following estimate of our solution:

Proposition 3.3.9. *Recall that* λ_1 *is the first nonzero positive eigenvalue of* $-\Delta_Y$. *Let* v *be a smooth function such that* $|v(z)| \leqslant C_0 z^\delta$ *for* $z > 1$. *For any* λ *such that* $\lambda > \lambda_1 > 0$, *we can find solution of equation*

$$
u'' - \left(\frac{j^2 n^2}{4} + n\lambda\right) z^{n-2} u = n z^{n-1} v
$$

such that

$$|u(z)| \leqslant C(n) \cdot C_0 \cdot z^{\delta+1} \cdot \left(\frac{j^2n^2}{4} \cdot z^n + n\lambda\right)^{-1}, \text{ and } |u''(z)| \leqslant C \cdot C_0 z^{\delta+n-1}$$

on $z > C$ for some constant $C > 1$ only depends on n, λ_1 and a.

Proof. Similar as the zero-mode case, we estimate

$$\mathcal{D}(z) \int_1^z \mathcal{G}(s)s^{n-1}v(s)ds + \mathcal{G}(z) \int_z^\infty \mathcal{D}(s)s^{n-1}v(s)ds.$$

By integration by parts we have

$$\begin{aligned}
&\mathcal{D}(z) \int_1^z \mathcal{G}(s)s^{\delta+n-1}ds + \mathcal{G}(z) \int_z^\infty \mathcal{D}(s)s^{\delta+n-1}ds \\
= &-\mathcal{W}(\mathcal{G},\mathcal{D}) \sum_{k=0}^{N-1} P(TP)^k(z^{\delta+n-1}) + \mathcal{D}(z) \int_1^z \mathcal{G}(s)(TP)^N(s^{\delta+n-1})ds \\
&+ \mathcal{G}(z) \int_z^\infty \mathcal{D}(s)(TP)^N(s^{\delta+n-1})ds,
\end{aligned}$$

where $P, T : C^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}_+)$ are given by

$$P(f) = \frac{f}{z^{n-2}\left(\frac{j^2n^2}{4}z^n + n\lambda\right)}, \quad T(f) = f''.$$

By straight forward computation and induction we can see that

$$(TP)^k(z^{\delta+n-1}) \leqslant C(k,n,\delta)\frac{z^{\delta+n-1-2nk}}{j^{2k}}.$$

So by taking N large enough we have $(TP)^N(z^{\delta+n-1}) \leqslant C(n,\delta,j)z^{-M}$ for some $M > 2$ which will be chosen later.

We first consider the case that $Q \geqslant 1$. By Lemma 3.3.8 the first term becomes

$$\begin{aligned}
&\mathcal{D}(z) \int_1^z \mathcal{G}(s)(TP)^N(s^{\delta+n-1})ds \\
\leqslant\, & C \cdot e^{-\frac{jz^n}{2}+F(t_0(z))} \int_1^z (js^n)^{\frac{2-n}{4n}}\, s^{-M} \cdot e^{-\frac{js^n}{2}+G(u_0(s))}ds \\
\leqslant\, & C \cdot e^{-jz^n+G(u_0(z))+F(t_0(z))} \cdot j^{\frac{2-n}{4n}} \cdot z^{\frac{2-n}{4}-M+1} \\
\leqslant\, & C(n,Q) \cdot j^{\frac{1}{n}} \cdot z^{2-M}.
\end{aligned}$$

For the second term, we have similar estimate:

$$\begin{aligned}
&\mathcal{G}(z) \int_z^\infty \mathcal{D}(s)(TP)^N(s^{\delta+n-1})ds \\
\leqslant\, & C \cdot (jz^n)^{\frac{2-n}{4n}} \cdot e^{-\frac{jz^n}{2}+G(t_0(z))} \int_z^\infty s^{-M}e^{-\frac{js^n}{2}+F(u_0(s))}ds \\
\leqslant\, & C \cdot e^{-jz^n+G(u_0(z))+F(t_0(z))} \cdot j^{\frac{2-n}{4n}} \cdot z^{\frac{2-n}{4}-M+1} \\
\leqslant\, & C(n,Q) \cdot j^{\frac{1}{n}} \cdot z^{2-M}.
\end{aligned}$$

Then we consider the case where $Q \leqslant 1$, the first term becomes

$$\begin{aligned}
&\mathcal{D}(z)\int_1^z \mathcal{G}(s)(TP)^N(s^{\delta+n-1})ds\\
&\leqslant C\cdot e^{-\frac{jz^n}{2}}\cdot(jz^n)^{\beta-\alpha}\int_1^z s^{-M}e^{\frac{js^n}{2}}\cdot(js^n)^{-\beta}\,ds\\
&\leqslant C\cdot j^{-\alpha}z^{1-n\alpha-M},
\end{aligned}$$

Also for the second term, we have

$$\begin{aligned}
&\mathcal{G}(z)\int_z^\infty \mathcal{D}(s)(TP)^N(s^{\delta+n-1})ds\\
&\leqslant C\cdot e^{\frac{jz^n}{2}}\cdot(jz^n)^{-\beta}\int_z^\infty s^{-M}e^{-\frac{js^n}{2}}\cdot(js^n)^{\beta-\alpha}\,ds\\
&\leqslant C\cdot j^{-\alpha}z^{1-n\alpha-M}.
\end{aligned}$$

So we have the uniform estimate for u that for any $z > C(n,\delta)$,

$$|u(z)| \leqslant C(n)\cdot C_0\cdot z^{\delta+1}\cdot(\frac{j^2n^2}{4}\cdot z^n + n\lambda)^{-1}.$$

In the end, we get if $|v(z)| \leqslant C_0 z^\delta$ on $z > C(n,\delta,M)$, then for any $z > C(n,\delta,M)$

$$\begin{aligned}
&|u(z)| \leqslant C(n)\cdot C_0\cdot z^{\delta+1}\cdot(\frac{j^2n^2}{4}\cdot z^n + n\lambda)^{-1},\\
&|u'(z)| \leqslant C(n)\cdot C_0\cdot z^{\delta+n},\\
&|u''(z)| \leqslant C(n)\cdot C_0\cdot z^{\delta+n-1}.
\end{aligned}$$

□

Remark 3.3.10. *Even though the separation of variable method is very explicit, we can only get a bound of u with respect to the polynomial growth order of v rather than the function v itself. This is mainly because the behavior of operator T and P is not clear for general function.*

3.4 Solution of Poisson equation

Now we are ready to present the estimate of the solution of Poisson equation:

Proposition 3.4.1. *Assume that v is a function on $\mathcal{C}$ such that for any $k \in \mathbb{N}$ there exist constants C_k and δ such that $|z^{\frac{k}{2}}\nabla^k v|_{\omega_{\mathcal{C}}} \leqslant C_k z^\delta$ on $z > C_k$. Then there exist constants C'_k and a function $u : \mathcal{C} \to \mathbb{R}$ such that $\Delta_{\omega_{\mathcal{C}}} u = v$ and*

$$|u| \leqslant C'_0 z^{\delta+n+1+\epsilon}, \quad |\nabla u|_{\omega_{\mathcal{C}}} \leqslant C'_1 z^{\delta+\frac{n+1}{2}+\epsilon}, \quad \left|\nabla^k u\right|_{\omega_{\mathcal{C}}} \leqslant C'_k z^{\delta-\frac{k-2}{2}+\epsilon}$$

on $z > C'_k$, for any $\epsilon > 0$ and any integer $k \geqslant 2$.

Proof. Let

$$u(z,y)=u_0(z)\psi_0+\sum_{j=1}^{\infty}u_j(z)\psi_j(y)$$

be the formal solution, where u_0 is constructed as follows and u_j is constructed in Section 3.3.

Step 1: We first show that the formal solution converges in C^0 sense with the polynomial order depending on the polynomial order of v.

For u_0, we have

$$\begin{aligned}u_0''(z)&=nz^{n-1}P_0v(z),\\ u_0'(z)&=\int_{C_2}^{z}ns^{n-1}P_0v(s)ds,\\ u_0(z)&=\int_{C_1}^{z}\left(\int_{C_2}^{t}ns^{n-1}P_0v(s)ds\right)dt.\end{aligned}$$

We choose C_1 and C_2 here to be 1 or $+\infty$ depending on the order of P_0v to make $u_0(z)$ and $u_0'(z)$ finite with proper order. Also, the integration here is the only place that we will lose the rate z^{ϵ}.

For u_j, we first notice that the projection $P_j(v)$ is well-defined and has the following estimate:

$$\begin{aligned}|v_j(z)|=|P_j(v)(z)|&=\left|\int_Y\frac{\Delta_{h_Y}^{K_0}v(z,y)\psi_j}{(\Lambda_j)^{K_0}}\mathrm{dVol_Y}\right| &(3.4.1)\\ &\leqslant\frac{\|v(z,\cdot)\|_{C^{2K_0}(Y,h_Y)}}{(\Lambda_j)^{K_0}} &(3.4.2)\\ &\leqslant C_{K_0}\frac{z^{\delta}}{\Lambda_j^{K_0}} &(3.4.3)\end{aligned}$$

for any K_0 and $z>C$. Consequently, the solution u_j constructed in (3.3.3) and (3.3.9) for the equation (3.2.2) has the following C^0 bound for $z>C_{n,\delta,\lambda_1}$:

$$|u_j(z)|\leqslant\frac{1}{(\Lambda_j)^{K_0}}\frac{C_{n,\delta,\lambda_1,K_0}z^{\delta+1}}{n\lambda_1}. \tag{3.4.4}$$

Here the constant C is uniform for j and z and only depends on K_0 and n. We also have the uniform estimate of ψ_j as the eigenfunctions of $-\Delta_{\omega_Y}$ on Y by its eigenvalues showed in Sun-Zhang [28, Lemma 5.1.]:

$$\|\psi_j\|_{C^k(Y)}\leqslant C_{k,Y}\cdot(\Lambda_j)^{\frac{n+k}{2}}. \tag{3.4.5}$$

Combine (3.4.4) and (3.4.5), we get

$$\begin{aligned}|\sum_{j=1}^{\infty}u_j\psi_j|&\leqslant\sum_{j=1}^{\infty}|u_j(z)|\cdot|\psi_j(y)|\\ &\leqslant C_{n,\delta,\lambda_1,K_0,Y}\sum_{j=1}^{\infty}\frac{z^{\delta+1}}{(\Lambda_j)^{K_0-\frac{n}{2}}}.\end{aligned}$$

Weyl's law gives the bound of Λ_j with $C_Y^{-1} j^{\frac{2}{2n-1}} \leq |\Lambda_j| \leqslant C_Y j^{\frac{2}{2n-1}}$ for some C_Y only depend on (Y, h_Y). Take $K_0 = 2n$ we can conclude that the summation converges.

Step 2: We prove that u is smooth. We mostly follow the proof in Sun-Zhang [28, Proposition 6.2.]. Let

$$U_N = \sum_{j=0}^{N} u_j \psi_j, \quad V_N = \sum_{j=0}^{N} v_j \psi_j.$$

Then we have $\Delta_{\omega_{\mathcal{C}}} U_N = V_N$.
We first show that V_N has C^k bound independent of N. In fact, we have the higher regularity estimate for v_j as in (3.4.1):

$$\begin{aligned} |\nabla^k v_j(z)| &= \left|\nabla^k P_j(v)(z)\right| \\ &= \left|\nabla^k \int_Y (\Lambda_j)^{-K_0} \Delta_{h_Y}^{K_0} v(z,y)\, \psi_j \,\mathrm{dVol_Y}\right| \\ &\leqslant C_{z,Y,K_0,k} \Lambda_j^{-K_0}. \end{aligned}$$

Then given by the estimate of ψ_j (3.4.5), for any integer j, we know that

$$\sum_{j=1}^{N} \left|\nabla^k (v_j \psi_j)\right| \leqslant C_{z,Y,K_0,k} \sum_{j=1}^{N} \Lambda_j^{\frac{n+k}{2} - K_0}.$$

So again by Weyl's law V_N converges to v as a C^k function. Now we can prove that U_N also have the uniform C^k bound with respect to N via local elliptic estimates.

For any fixed point $x \in \mathcal{C}$, we consider the ball $B_{\omega_{\mathcal{C}}}(x, 1)$. Then for any $p > 0$, there exists a constant $C_{p,x}$ such that

$$\|U_N\|_{W^{2,p}(B_{\omega_{\mathcal{C}}}(x,\frac{1}{2}))} \leqslant C_{p,x} \cdot \left(\|V_N\|_{C^0(B_{\omega_{\mathcal{C}}}(x,1))} + \|U_N\|_{C^0(B_{\omega_{\mathcal{C}}}(x,1))}\right) \leqslant C_{p,x}.$$

By bootstrapping and Sobolev embedding, for any $k \geqslant 0$ and $p > 0$, there exists a constant $C_{p,x}$ such that

$$\begin{aligned} &\|U_N\|_{C^{k+1,1-\frac{2n}{p}}(B_{\omega_{\mathcal{C}}}(x,\frac{1}{2}))} \\ \leqslant\, & C_{p,x} \|U_N\|_{W^{k+2,p}(B_{\omega_{\mathcal{C}}}(x,\frac{1}{2}))} \\ \leqslant\, & C_{p,x} \left(\|V_N\|_{W^{k,p}(B_{\omega_{\mathcal{C}}}(x,1))} + \|U_N\|_{W^{k,p}(B_{\omega_{\mathcal{C}}}(x,1))}\right) \\ \leqslant\, & C_{p,k,x}. \end{aligned}$$

Consequently we have U_N converges to u in C^k. Since x is arbitrary, we know that u is smooth on $\mathcal{C}$.

Step 3. We can now give global bound on the C^1, C^2 and higher regularity of u. We treat u_0 and $u - u_0$ separately.

For u_0, we have explicit estimate by computation of the Christoffel symbol under the following holomorphic coordinate: Let $\pi : \mathcal{C} \to D$ be the projection map. For any point $\xi \in \mathcal{C}$ we take the local holomorphic coordinate $\underline{z} = (z_1, z_2, \ldots, z_{n-1})$ on $B \subset D$ around the point $\pi(\xi)$. Take ξ_0 be a local holomorphic section of N_D such that $|\xi_0|_{h_D} = e^{-\varphi}$, where $\sqrt{-1}\partial\bar{\partial}\varphi = \omega_D$ with $\varphi(0) = 0$, $\nabla\varphi(0) = 0$. Then $\xi = \xi_0 \cdot w$ where w is the fiber coordinate. Recall that $z = \left(-\log|\xi|^2_{h_D}\right)^{\frac{1}{n}}$, $w = e^{i\theta - t + \varphi/2}$ and

$$\omega_{\mathcal{C}} = z\pi^*\omega_D + \frac{1}{nz^{n-1}} \cdot \sqrt{-1} \cdot \left(\frac{dw}{w} - \partial\varphi\right) \wedge \left(\frac{d\bar{w}}{\bar{w}} - \bar{\partial}\varphi\right).$$

Under this coordinate, direct computation gives us that

$$\Delta_{\omega_{\mathcal{C}}} t^{\alpha} = C_{n,\alpha} t^{\alpha - \frac{n+1}{2n}},$$

and

$$\left|\nabla t^{\alpha}\right|_{\omega_{\mathcal{C}}} \leqslant C t^{\alpha - \frac{n+1}{2n}}$$

Apply Proposition 2.2.11 with t^{α} and use iteration, we have the following:

$$\begin{aligned}
&\left|\nabla^k z_i\right|_{\omega_{\mathcal{C}}} \leqslant C_k z^{-\frac{k}{2}}, \text{ for any } k \geqslant 1,\\
&|dt|_{\omega_{\mathcal{C}}} \leqslant C z^{\frac{n-1}{2}},\\
&\left|\nabla^k t\right|_{\omega_{\mathcal{C}}} \leqslant C_k z^{-\frac{k}{2}},\\
&\left|\nabla^k t^a\right|_{\omega_{\mathcal{C}}} \leqslant z^{n(a-1)-\frac{k}{2}}, \text{ for any } k \geqslant 2.
\end{aligned} \tag{3.4.6}$$

Now we can estimate the higher derivative of u_0:

$$\begin{aligned}
\nabla^k u_0 &= \sum_{j=1}^{k} u_0^{(j)}(z) \sum_{\substack{i_1+\cdots+i_j=k\\ i_1>0,\cdots,i_j>0}} C_{i_1,\cdots,i_j} \nabla^{i_1} z \otimes \nabla^{i_2} z \otimes \cdots \otimes \nabla^{i_j} z,\\
&= u_0'(z)\nabla^k z + \sum_{j=2}^{k} u_0^{(j)}(z) \sum_{\substack{i_1+\cdots+i_j=k\\ i_1>0,\cdots,i_j>0}} C_{i_1,\cdots,i_j} \nabla^{i_1} z \otimes \nabla^{i_2} z \otimes \cdots \otimes \nabla^{i_j} z,\\
&= u_0'(z)\nabla^k z + \sum_{j=2}^{k} \left(nz^{n-1}P_0 v(z)\right)^{(j-2)} \sum_{\substack{i_1+\cdots+i_j=k\\ i_1>0,\cdots,i_j>0}} C_{i_1,\cdots,i_j} \nabla^{i_1} z \otimes \nabla^{i_2} z \otimes \cdots \otimes \nabla^{i_j} z,
\end{aligned}$$

where $f^{(k)}$ refers to the higher derivative of the function f. Since

$$\left(nz^{n-1}P_0 v(z)\right)^{(j-2)} = \sum_{l=0}^{j-2} C_{l,j,n} z^{n-j+l+1} P_0 v^{(l)}$$

and

$$\left|P_0 v^{(l)}(z)\right| \leqslant C_Y \left|\nabla^l v\right|_{\omega_{\mathcal{C}}} \cdot |dz|_{\omega_{\mathcal{C}}}^{-l} \leqslant C_Y z^{\delta+\frac{(n-2)l}{2}},$$

we know that

$$\begin{aligned}\left|\nabla^k u_0\right|_{\omega_{\mathcal{C}}} &\leqslant C_{n,k,Y} z^{\delta+1-\frac{k}{2}+\epsilon} + C_{n,k,Y} \sum_{j=2}^{k} \sum_{l=0}^{j-2} z^{n-j+1+\delta+\frac{nl}{2}} \\ &\qquad \cdot \sum_{\substack{i_1+\cdots+i_j=k \\ i_1>0,\cdots,i_j>0}} \left|\nabla^{i_1} z\right|_{\omega_{\mathcal{C}}} \cdot \left|\nabla^{i_2} z\right|_{\omega_{\mathcal{C}}} \cdot \cdots \cdot \left|\nabla^{i_j} z\right|_{\omega_{\mathcal{C}}} \\ &\leqslant C_{n,k,Y} z^{\delta+1-\frac{k}{2}+\epsilon} + C_{n,k,Y} z^{1+\delta} \\ &\qquad \cdot \sum_{j=2}^{k} \sum_{\substack{i_1+\cdots+i_j=k \\ i_1>0,\cdots,i_j>0}} \left|z^{\frac{n-2}{2}} \nabla^{i_1} z\right|_{\omega_{\mathcal{C}}} \cdot \left|z^{\frac{n-2}{2}} \nabla^{i_2} z\right|_{\omega_{\mathcal{C}}} \cdot \cdots \cdot \left|z^{\frac{n-2}{2}} \nabla^{i_j} z\right|_{\omega_{\mathcal{C}}}.\end{aligned}$$

By $|\nabla z|_{\omega_{\mathcal{C}}} \leqslant z^{-\frac{n-1}{2}}$ and $|\nabla^i z|_{\omega_{\mathcal{C}}} \leqslant z^{-\frac{i}{2}-n+1}$ for $i \geqslant 2$, we have

$$\sum_{\substack{i_1+\cdots+i_j=k \\ i_1>0,\cdots,i_j>0}} \left|z^{\frac{n-2}{2}} \nabla^{i_1} z\right| \cdot \left|z^{\frac{n-2}{2}} \nabla^{i_2} z\right| \cdot \cdots \cdot \left|z^{\frac{n-2}{2}} \nabla^{i_j} z\right| \leqslant C_k z^{-\frac{k}{2}}.$$

Consequently, we have

$$|\nabla u_0|_{\omega_{\mathcal{C}}} \leqslant C_{n,Y} z^{\delta+\frac{n+1}{2}+\epsilon},$$

and for $k \geqslant 2$

$$\left|\nabla^k u_0\right|_{\omega_{\mathcal{C}}} \leqslant C_{n,k,Y} z^{\delta+1-\frac{k}{2}+\epsilon}.$$

For $u-u_0$ we use our C_0 bound of $u-u_0$ and do elliptic estimate around a point $x \in \{z = z_0\}$ by Proposition 2.2.11. We have the uniform estimate of $u - u_0$ on $\{z \geqslant C'\}$:

$$\begin{aligned}&z^{\frac{k}{2}} \left\|\nabla^k (u-u_0)\right\|_{C^0(B_g(x,\sqrt{z}))} \\ &\leqslant C_k \left(\|u-u_0\|_{C^0(B_g(x,\sqrt{z}))} + \sum_{i=0}^{k-1} z^{\frac{i+2}{2}} \left\|\nabla^i v - P_0(v)\right\|_{C^0(B_g(x,\sqrt{z}))} \right),\end{aligned}$$

which yields

$$\left|z^{\frac{k}{2}} \nabla^k (u-u_0)\right|_{\omega_{\mathcal{C}}} \leqslant C_k' z^{\delta+1}.$$

Together with the estimate of u_0, we have

$$\begin{aligned}&|u| \leqslant C_0' z^{\delta+n+1+\epsilon}, \\ &|\nabla u|_{\omega_{\mathcal{C}}} \leqslant C_1' z^{\delta+\frac{n+1}{2}+\epsilon}, \\ &\left|\nabla^k u\right|_{\omega_{\mathcal{C}}} \leqslant C_k' z^{\delta-\frac{k-2}{2}+\epsilon} \text{ for } k \geqslant 2.\end{aligned}$$

□

Remark 3.4.2. *We see from the proof that the main term in C^0 and C^1 estimate of the solution u is the fiber direction u_0. However, for the higher estimate C^k where $k \geqslant 3$, they will give the same order contribution.*

3.5 Improve the Decay of the Ricci potential

Now we look back at the quasi-projective manifold X and the class $\mathfrak{k} \in H^2(X)$. In this section we are going to find a good representative form β inside the class $\mathfrak{k} \in H^2(X)$. We look at its behavior on the model space and then find some function U by finite step iteration such that

$$\frac{(\Phi^*\beta + \sqrt{-1}\partial\bar{\partial}U)^n}{\omega_{\mathcal{C}}^n} - 1$$

has faster decay rate.

3.5.1 A good representative

Lemma 3.5.1. *For any $\mathfrak{k} \in H^2(X)$, there exists a closed $(1,1)$-form β on M such that $[\beta|_X] = \mathfrak{k}$.*

Proof. By discussion in Section 2 of [14] we know M is weak Fano, and consequently simply connected by [32]. Consider the exact sequence $0 \to \mathcal{O}_M(-D) \to \mathcal{O}_M \to \mathcal{O}_D \to 0$, we have long exact sequence

$$\cdots \to H^1(M, \mathcal{O}_M) \to H^1(D, \mathcal{O}_D) \to H^2(M, \mathcal{O}_M(-D)) \to \cdots$$

On the other hand, by Serre duality we have

$$H^2(M, \mathcal{O}_M(-D)) \simeq H^2(M, K_M) \simeq H^{n,n-2}(M, -K_M).$$

Apply Kawamata–Viehweg vanishing theorem to the nef and big line bundle $-K_M$ so we know that $H^{n,p}(M, -K_M) = H^{0,p}(M) = 0$ for $p \geqslant 1$. So when $n \geqslant 3$, we have $H^1(D, \mathbb{C}) = 0$. The long exact sequence given by the excision theorem and Thom-Gysin sequence

$$H^0(D) \to H^2(M) \to H^2(X) \to H^1(D) \to \cdots$$

yields that the restriction map $j^* : H^2(M) \to H^2(X)$ induced by $j : X \to M$ is surjective with $\dim \mathrm{Ker} j^* = 1$, generated by $c_1(-K_M)$. Hence there exists a closed $(1,1)$-form β on M such that $[\beta|_X] = \mathfrak{k}$. $\square$

Remark 3.5.2. *Here we use the fact that $\dim_{\mathbb{C}} X = n \geqslant 3$ to apply Kawamata–Viehweg vanishing theorem. When $n = 2$ we have $H^2(M, K_M) \simeq H^{0,0}(M) \simeq \mathbb{C}$, so $H^1(D, \mathbb{C})$ may not vanish.*

The global $(1,1)$ form β on M satisfies the following property on the end:

Proposition 3.5.3. *Let $\Phi : \mathcal{C} \setminus \mathcal{K} \to X \setminus K$ be the fixed diffeomorphism and let $p : \mathcal{C} \to D$ be the projection map. Then $|\Phi^*\beta - p^*(\beta|_D)|_{\omega_\mathcal{C}} = O(e^{-(\frac{1}{2}-\epsilon)z^n})$.*

Proof. Let p be a fixed point in D. Let $(w, \underline{z}) = (w, z_1, \cdots, z_n)$ be local holomorphic coordinates around this point such that D is given by $\{w = 0\}$. Then $(w, \underline{z})$ can also be seen as a group of local holomorphic coordinates around p in $\mathcal{C}$ where w represent the fiber direction. We can express β locally around p on M as

$$\beta = \sum_{i,j=1}^{n-1} f_{i\bar{j}} dz_i \wedge d\bar{z}_j + \sum_{i=1}^{n-1} f_{i\bar{w}} dz_i \wedge d\bar{w} + \sum_{i=1}^{n-1} f_{w\bar{i}} dw \wedge d\bar{z}_i + f_{w\bar{w}} dw \wedge d\bar{w},$$

$$\begin{aligned}\Phi^*\beta = &\sum_{i,j=1}^{n-1} f_{i\bar{j}}(\Phi) dz_i(\Phi) \wedge \Phi^*(J_X) dz_j(\Phi) + \sum_{i=1}^{n-1} f_{i\bar{w}}(\Phi) dz_i(\Phi) \wedge \Phi^*(J_X) dw(\Phi) \\ &+ \sum_{i=1}^{n-1} f_{w\bar{i}}(\Phi) dw(\Phi) \wedge \Phi^*(J_X) dz_i(\Phi) + f_{w\bar{w}}(\Phi) dw(\Phi) \wedge \Phi^*(J_X) dw(\Phi).\end{aligned}$$

Notice that we have the estimate of

$$\left|\nabla^k_{g_\mathcal{C}} w\right| = O(e^{-(\frac{1}{2}-\epsilon)z^n}), \quad \left|\nabla^k_{g_\mathcal{C}} z_i\right| = O(1).$$

Since Φ is the exponential map, we know that

$$\Phi|_D = \mathrm{Id},$$

and the complex structure $\Phi^* J_X - J_C$ is exponentially decay as in Proposition 2.2.2, we know that on $\mathcal{C}$ we have

$$(\Phi^*\beta)|_D = \sum_{i,j=1}^{n-1} f_{i\bar{j}}(0, \underline{z}) dz_i \wedge J_\mathcal{C} dz_j + O(e^{-(\frac{1}{2}-\epsilon)z^n}).$$

On the other hand, we know that

$$p^*(\beta|_D)|_D = \sum_{i,j=1}^{n-1} f_{i\bar{j}}(0, z_i) dz_i \wedge d\bar{z}_j.$$

So $\Phi^*\beta - p^*(\beta|_D)$ extends to a smooth form on N_D vanishing on the zero section D, which yields $|\Phi^*\beta - p^*(\beta|_D)|_{\omega_\mathcal{C}} = O(e^{-(\frac{1}{2}-\epsilon)z^n})$. □

3.5.2 Iteration process

With this exponential closeness, we can view $\Phi^*\beta$ as a $(1,1)$-form on $\mathcal{C}$ with only horizontal direction component. This will greatly simplify our computation below.

Definition 3.5.4. *Let η be a $(1,1)$-form on $\mathcal{C}$. We define $F(\eta) := 1 - \frac{(\omega_\mathcal{C}+\eta)^n}{\omega_\mathcal{C}^n}$, called $\omega_\mathcal{C}$-*potential.

Definition 3.5.5. *With the same β as before, we define by iteration*

$$\begin{aligned} F_0 &:= F(p^*(\beta|_D)), \\ F_j &:= F(p^*(\beta|_D) + \sqrt{-1}\partial\bar{\partial}U_j) \\ &= 1 - \frac{(\omega_{\mathcal{C}} + p^*(\beta|_D) + \sqrt{-1}\partial\bar{\partial}U_j)^n}{\omega_{\mathcal{C}}^n} \end{aligned}$$

where

$$U_0 = 0, \quad U_j = U_{j-1} + u_j, \quad \Delta_{\omega_{\mathcal{C}}} u_j = F_{j-1}.$$

Here u_j is the solution constructed in Proposition 3.4.1. We will prove in Proposition 3.5.6 that the derivative of F_j's satisfy the decay condition required for v in Proposition 3.4.1 so this iteration process works.

Proposition 3.5.6. *With U_j's and u_j's defined in 3.5.5, we have $F_n = F(p^*(\beta|_D)+\sqrt{-1}\partial\bar{\partial}U_n)$ decays faster than z^{-n}. More precisely, we have higher order estimate, for any positive integer j and k*

$$|z^{\frac{k}{2}}\nabla^k F_j|_{\omega_{\mathcal{C}}} \leqslant C_{k,j} z^{-j-1+\epsilon}. \tag{3.5.1}$$

Proof. We prove (3.5.1) by induction.

For F_0, we can see this estimate directly follows from computation:

$$\begin{aligned} F_0 &= 1 - \frac{(\omega_{\mathcal{C}} + p^*(\beta|_D))^n}{\omega_{\mathcal{C}}^n} \\ &= \sum_{j=1}^{n} \frac{n-j}{nz^j} \cdot p^*(\frac{\beta|_D^j \wedge \omega_D^{n-j-1}}{\omega_D^{n-1}}). \end{aligned}$$

By (3.4.6) we know that $|z^{\frac{k}{2}}\nabla^k F_0|_{\omega_{\mathcal{C}}} \leqslant C(k)z^{-1}$, for any positive integer k. So when $j = 0$ (3.5.1) holds.

Assume (3.5.1) holds for $i \leqslant j$, i.e. $|z^{\frac{k}{2}}\nabla^k F_i|_{\omega_{\mathcal{C}}} \leqslant C(k,i)z^{-i-1}$, for any $k \geqslant 0$. By straightforward computation,

$$\begin{aligned} F_{j+1} = &- \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{n\sqrt{-1}\partial\bar{\partial}u_{j+1} \wedge (p^*(\beta|_D) + \sqrt{-1}\partial\bar{\partial}U_j)^k \wedge \omega_{\mathcal{C}}^{n-k-1}}{\omega_{\mathcal{C}}^n} \\ &- \sum_{k=2}^{n} \binom{n}{k} \frac{\left(\sqrt{-1}\partial\bar{\partial}u_{j+1}\right)^k \wedge \left(\omega_{\mathcal{C}} + p^*(\beta|_D) + \sqrt{-1}\partial\bar{\partial}U_j\right)^{n-k}}{\omega_{\mathcal{C}}^n}. \end{aligned}$$

Actually, the function in each term is of the from

$$\frac{C\sqrt{-1}\partial\bar{\partial}u_{j+1} \wedge \bigwedge_{q\in Q}\sqrt{-1}\partial\bar{\partial}u_q \wedge p^*(\beta|_D)^m \wedge \omega_{\mathcal{C}}^{n-m-|Q|-1}}{\omega_{\mathcal{C}}^n}, \tag{3.5.2}$$

where Q is a set with repeated elements from $\{1, 2, \cdots, j+1\}$, m is a non-negative integer and positive when $j+1 \notin Q$. By Proposition 3.4.1 we know that $\left|z^{\frac{k}{2}}\nabla^k u_{j+1}\right|_{\omega_{\mathcal{C}}} \leqslant C_{k,j} z^{-j+\epsilon}$. It is easier to deduce the bound for (3.5.2) by passing to the rescaled metric $\tilde{\omega}_{\mathcal{C}}$ on $B_{\tilde{\omega}_{\mathcal{C}}}(x,1)$, where $z(x) = z_0$, $\tilde{\omega}_{\mathcal{C}} = \frac{\omega_{\mathcal{C}}}{z_0}$. We have the uniform weighted bound for each term in the wedge product

$$\begin{aligned}
&\left\|\nabla^k\sqrt{-1}\partial\bar{\partial}u_q\right\|_{C^0(B_{\tilde{\omega}_{\mathcal{C}}}(x,1))} \leqslant C_{k,q} z_0^{-q+1+\epsilon},\\
&\left\|\nabla^k\sqrt{-1}\partial\bar{\partial}u_{j+1}\right\|_{C^0(B_{\tilde{\omega}_{\mathcal{C}}}(x,1))} \leqslant C_{k,j} z_0^{-j+\epsilon},\\
&\left\|\nabla^k p^*(\beta|_D)\right\|_{C^0(B_{\tilde{\omega}_{\mathcal{C}}}(x,1))} \leqslant C_k.
\end{aligned}$$

If we consider the scaled metric $\tilde{\omega}_{\mathcal{C}}$ our function (3.5.2) becomes

$$\frac{C\sqrt{-1}\partial\bar{\partial}u_{j+1} \wedge \bigwedge_{q\in Q}\sqrt{-1}\partial\bar{\partial}u_q \wedge p^*(\beta|_D)^m \wedge \tilde{\omega}_{\mathcal{C}}^{n-m-|Q|-1}}{z_0^{m+|Q|+1}\tilde{\omega}_{\mathcal{C}}^n}.$$

So we have

$$\left\|\nabla^k\frac{C\sqrt{-1}\partial\bar{\partial}u_{j+1} \wedge \bigwedge_{q\in Q}\sqrt{-1}\partial\bar{\partial}u_q \wedge p^*(\beta|_D)^m \wedge \tilde{\omega}_{\mathcal{C}}^{n-m-|Q|-1}}{z_0^{m+|Q|+1}\tilde{\omega}_{\mathcal{C}}^n}\right\|_{C^0(B_{\tilde{\omega}_{\mathcal{C}}}(x,1))} \leqslant C_{k,j,Q,m} z_0^{-j-2+\epsilon}.$$

Consequently,

$$\left\|z_0^{\frac{k}{2}}\nabla^k F_{j+1}\right\|_{C^0(B_{\omega_{\mathcal{C}}}(x,\sqrt{z_0}))} = \left\|\nabla^k F_{j+1}\right\|_{C^0(B_{\tilde{\omega}_{\mathcal{C}}}(x,1))} \leqslant C_{k,j} z_0^{-j-2+\epsilon}.$$

So we finish the proof of (3.5.1). Specially,

$$\left|z^{\frac{k}{2}}\nabla^k F_n\right|_{\omega_{\mathcal{C}}} \leqslant C_k z^{-n-1+\epsilon}.$$

□

3.6 The Integral Condition

For the convenience of statement, let us introduce the following notation.

Definition 3.6.1. *For an $(n,0)$ form Ω, we say that a (1,1)-form α is* Ω-compatible *if* $\int_X \Omega \wedge \overline{\Omega} - \alpha^n = 0$.

Remark 3.6.2. *Since $\Omega \wedge \overline{\Omega}$ and α^n are not integrable for most of the time, this integration identity means that the function $f = \frac{\alpha^n}{\Omega\wedge\overline{\Omega}} - 1$ satisfies $\int_X f\Omega \wedge \overline{\Omega} = 0$.*

In this section, we will show that by adding a suitable potential we can make $\beta+\sqrt{-1}\partial\bar{\partial}U$ to be Ω_X-compatible.

Proposition 3.6.3. *There exists a smooth function $\tilde{U}$ on X such that $\beta+\sqrt{-1}\partial\bar{\partial}\tilde{U}$ is an Ω_X-compatible Kähler form. Meanwhile, we have that*

$$\left|\Phi^*\tilde{U}-(z^{n+1}+U_n+\lambda z)\right|\leqslant Ce^{-\delta z^n}$$

when $z>C$ for some constant $C>0$ and $\lambda\in\mathbb{R}$.

Proof. We first show that we can find U such that $(\beta+\sqrt{-1}\partial\bar{\partial}U+\omega_X)^n$ is integrable on the end. Recall that we have $U_j=\sum_{p=1}^{j}u_p$ and u_j are functions on $\mathcal{C}$ such that

$$\Delta_{\omega_{\mathcal{C}}}u_{j+1}=1-\frac{\left(p^*(\beta|_D)+\omega_{\mathcal{C}}+\sqrt{-1}\partial\bar{\partial}U_j\right)^n}{\omega_{\mathcal{C}}^n}=F_j.$$

We know that the following integration is finite since $|F_n|\leqslant C_nz^{-n-1+\epsilon}$:

$$\begin{aligned}&\left|\int_{\mathcal{C}\setminus\mathcal{K}}(\Phi^*\beta+\omega_{\mathcal{C}}+\sqrt{-1}\partial\bar{\partial}U_n)^n-\omega_{\mathcal{C}}^n\right|\\ \leqslant&\int_{\mathcal{C}\setminus\mathcal{K}}|F_n|(\sqrt{-1}\partial\bar{\partial}t)^{n-1}dt\wedge d^ct\\ &\qquad+\left|\int_{\mathcal{C}\setminus\mathcal{K}}(\Phi^*\beta+\omega_{\mathcal{C}}+\sqrt{-1}\partial\bar{\partial}U_n)^n-(p^*(\beta|_D)+\omega_{\mathcal{C}}+\sqrt{-1}\partial\bar{\partial}U_n)^n\right|\\ =&\int_{\{t=T\}}nT^{\frac{-1+\epsilon}{n}}(\sqrt{-1}\partial\bar{\partial}t)^{n-1}\wedge d^ct+C<+\infty.\end{aligned}$$

Since we have the exponentially closed estimate between $X\setminus K$ and $\mathcal{C}\setminus\mathcal{K}$, we will have

$$\begin{aligned}&\int_{X\setminus K}\left(\beta+\sqrt{-1}\partial\bar{\partial}(\Phi^{-1})^*U_n+\omega_X\right)^n-\omega_X^n\\ =&\int_{\mathcal{C}\setminus\mathcal{K}}(\Phi^*\beta+\sqrt{-1}\partial\bar{\partial}U_n+\omega_{\mathcal{C}})^n-\omega_{\mathcal{C}}^n+O(e^{-\delta z^n})<+\infty.\end{aligned}$$

Now we can construct the Kähler potential following the construction in Hein-Sun-Viaclovsky-Zhang [14, Lemma 2.7.]:

The ampleness of N_D implies that X is 1-convex. Hence by Remmert reduction we know that $-K_M$ is semi-ample, we denote its non-ample locus by E. Recall that $[\beta]^p\cdot Y>0$ for any p-dimensional compact subvariety Y in X, by the generalized Demailly-Păun criterion in [9] we know that there exists a smooth function u_0 on X such that $\beta+\sqrt{-1}\partial\bar{\partial}u_0$ is positive on the neighborhood U of E. Let χ_0 be a smooth function on X support on U and $\chi_0=1$ on E. Then

$$\beta+\sqrt{-1}\partial\bar{\partial}\left(\chi_0\cdot u_0\right)\ \text{is positive around } E$$

and

$$\sqrt{-1}\partial\bar{\partial}\left(\chi_0\cdot u_0\right)\ \text{is supported on } U.$$

Let $\mathfrak{t} = -\log|S|^2_{h_M}$, then the curvature form $\sqrt{-1}\partial\bar{\partial}\mathfrak{t} \geqslant 0$ and $\sqrt{-1}\partial\bar{\partial}\mathfrak{t} > 0$ on $X \setminus E$. Let χ_1 be a smooth cutoff function on $[0, +\infty)$ such that $\chi_1 = 1$ on $[0,1]$ and $\chi_1 = 0$ on $[2, +\infty)$. Then $\sqrt{-1}\partial\bar{\partial}(\chi_1(\frac{\mathfrak{t}}{C_2}) \cdot \mathfrak{t})$ is positive on $\{\mathfrak{t} \leqslant C_2\}$ and supported on $\{\mathfrak{t} \leqslant 2C_2\}$.

Let ρ_A be a smooth convex function on $\mathbb{R}$ with $\rho_A(x) = \frac{2A}{3}$ on $(-\infty, \frac{A}{2}]$ and $\rho_A(x) = x$ on $(A, +\infty)$. Then we can obtain that $\sqrt{-1}\partial\bar{\partial}(\rho_{C_3}(\mathfrak{t}^{\frac{n+1}{n}})) \geqslant 0$ on X and $\sqrt{-1}\partial\bar{\partial}(\rho_{C_3}(\frac{n}{n+1}\mathfrak{t}^{\frac{n+1}{n}})) = \sqrt{-1}\partial\bar{\partial}(\frac{n}{n+1}\mathfrak{t}^{\frac{n+1}{n}})$ on $\{\mathfrak{t} \geqslant 2C_3^{\frac{n}{n+1}}\}$.

Let

$$\beta_1 = \beta + \sqrt{-1}\partial\bar{\partial}\left(\chi_0 \cdot u_0 + C_1\chi_1\left(\frac{\mathfrak{t}}{C_2}\right) \cdot \mathfrak{t} + \rho_{C_3}\left(\tfrac{n}{n+1}\mathfrak{t}^{\frac{n+1}{n}}\right)\right).$$

By our choice of u_0, $\beta + \sqrt{-1}\partial\bar{\partial}u_0$ is positive around E. By choosing C_1 and C_2 large we can make β_1 is positive on U. Then choosing C_2 large enough depending on C_1 and C_3 we have that β_1 is positive on $\{C_2 \leqslant \mathfrak{t} \leqslant 2C_2\}$ hence Kähler on X.

Then we can glue our perturbation function U_n via a cut-off function χ_2 supported outside a compact set K' with $\chi_2 = 1$ outside a open neighborhood U of K' and let

$$\beta_2(\lambda) = \beta_1 + \sqrt{-1}\partial\bar{\partial}\left(\chi_2 \cdot \left((\Phi^{-1})^*(U_n + \lambda z)\right)\right).$$

Our goal next is to find suitable λ and K' such that $\beta_2(\lambda)$ is Kähler and Ω_X-compatible.

Let us first show that the Ω_X-compatible condition is a linear equation of λ and only the constant term depends on the choice of χ_2. By our previous estimate of U_n we know as a starting point that $\int_X \beta_2(0)^n - \Omega_X \wedge \bar{\Omega}_X = C$ is finite. The Ω_X-compatible condition becomes

$$\begin{aligned}
0 &= \int_X \beta_2(\lambda)^n - \Omega_X \wedge \bar{\Omega}_X \\
&= \int_X \beta_2(\lambda)^n - \beta_2(0)^n + \int_X \beta_2(0)^n - \Omega_X \wedge \bar{\Omega}_X \\
&= \sum_{k=0}^{n-1}\binom{n}{k}\lim_{\varepsilon\to 0}\int_{\{(\Phi^{-1})^*t \leqslant -\log\varepsilon\}} \lambda\sqrt{-1}\partial\bar{\partial}\left(\chi_2 \cdot (\Phi^{-1})^*z\right) \\
&\qquad\qquad \wedge \beta_2(0)^k \wedge \sqrt{-1}\partial\bar{\partial}\left(\lambda\chi_2 \cdot (\Phi^{-1})^*z\right)^{n-k-1} + C \\
&= \sum_{k=0}^{n-1}\binom{n}{k}\lim_{\varepsilon\to 0}\int_{\{t = -\log\varepsilon\}} \lambda d^c t \wedge \frac{1}{nz^{n-1}}(\omega_{\mathcal{C}} + \beta + \sqrt{-1}\partial\bar{\partial}U_n)^k \wedge (\lambda\sqrt{-1}\partial\bar{\partial}z)^{n-k-1} + C.
\end{aligned}$$

Expanding the terms in the bracket, we notice that only $\omega_{\mathcal{C}}^{n-1}$ remains non-vanishing after we take the limit, so we have the equation

$$0 = \lim_{\varepsilon\to 0}\int_{\{t=-\log\varepsilon\}} \lambda d^c t \wedge (\sqrt{-1}\partial\bar{\partial}t)^{n-1} + C = \lambda \cdot \int_D \omega_D^{n-1} + C.$$

This is a linear equation on λ. On the other hand, we also notice by the previous computation that χ_2 does not affect the integral, so we can choose λ_0 first to satisfy the integral condition and then choose K' large enough such that $\beta_2(\lambda_0)$ is Kähler. So by choosing

$$\tilde{U} = \chi_0 \cdot u_0 + C_1\chi_1\left(\tfrac{\mathfrak{t}}{C_2}\right) \cdot \mathfrak{t} + \rho_{C_3}\left(\tfrac{n}{n+1}\mathfrak{t}^{\frac{n+1}{n}}\right) + \chi_2 \cdot \left(\left(\Phi^{-1}\right)^*(U_n + \lambda z)\right)$$

we finish our proof. □

In order to apply Tian-Yau-Hein's package, we need to repeat the iteration process for one more step such that the $\omega_{\mathcal{C}}$-potential of $\beta + \sqrt{-1}\partial\bar{\partial}U$ decays faster than r^{-2}.

Proposition 3.6.4. *Furthermore, we can construct a Kähler Ω_X-compatible form $\beta+\sqrt{-1}\partial\bar{\partial}U$ on X such that*

$$\left|1 - \frac{(\beta + \sqrt{-1}\partial\bar{\partial}U)^n}{\Omega_X \wedge \bar{\Omega}_X}\right| \leqslant Cr^{-2-\epsilon}, \qquad \left|\omega_{\mathcal{C}} - \Phi^*(\beta + \sqrt{-1}\partial\bar{\partial}U)\right|_{\omega_{\mathcal{C}}} \leqslant Cz^{-1},$$

where r is the distance function to some point $p \in X$ under metric $\beta + \sqrt{-1}\partial\bar{\partial}U$, $C > 0$.

Proof. Let $u_{n+1} = \lambda z$, $U_{n+1} = U_n + u_{n+1}$. Let u_{n+2} be the solution of $\Delta_{\omega_{\mathcal{C}}} u_{n+2} = F_{n+1}$ constructed in Proposition 3.4.1.

$$F_{n+1} = F_n - \frac{n\sqrt{-1}\partial\bar{\partial}\lambda z \wedge \sum_{k=1}^{n-1}\binom{n-1}{k}(p^*(\beta|_D) + \sqrt{-1}\partial\bar{\partial}U_n)^k \wedge \omega_{\mathcal{C}}^{n-k-1}}{\omega_{\mathcal{C}}^n}$$
$$- \frac{\sum_{k=2}^{n}\binom{n}{k}\left(\sqrt{-1}\partial\bar{\partial}\lambda z\right)^k \wedge \left(\omega_{\mathcal{C}} + p^*(\beta|_D) + \sqrt{-1}\partial\bar{\partial}U_n\right)^{n-k}}{\omega_{\mathcal{C}}^n}.$$

By Proposition 3.4.1, we know that

$$|z^{\frac{k}{2}}\nabla^k F_{n+1}(z,\cdot)|_{\omega_{\mathcal{C}}} \leqslant C_K z^{-n-1+\epsilon},$$

which is of the same order of F_n. Then

$$F_{n+2} = -\frac{n\sqrt{-1}\partial\bar{\partial}u_{n+2} \wedge \sum_{k=1}^{n-1}\binom{n-1}{k}(p^*(\beta|_D) + \sqrt{-1}\partial\bar{\partial}U_{n+1})^k \wedge \omega_{\mathcal{C}}^{n-k-1}}{\omega_{\mathcal{C}}^n}$$
$$- \frac{\sum_{k=2}^{n}\binom{n}{k}\left(\sqrt{-1}\partial\bar{\partial}u_{n+2}\right)^k \wedge \left(\omega_{\mathcal{C}} + p^*(\beta|_D) + \sqrt{-1}\partial\bar{\partial}U_{n+1}\right)^{n-k}}{\omega_{\mathcal{C}}^n}.$$

Again by the estimate in Proposition 3.4.1 we have

$$\left|z^{\frac{k}{2}}\nabla^k u_{n+2}\right| \leqslant Cz^{-n+\epsilon}.$$

So $F_{n+2} \leqslant Cz^{-n-2+\epsilon}$.

Let $U = \tilde{U} + \chi_2(\Phi^{-1})^* u_{n+2}$, we can choose K' large enough such that $\beta + \sqrt{-1}\partial\bar{\partial}U$ is Kähler on X. Also we see from the construction of U that

$$\left|\omega_{\mathcal{C}} - \Phi^*(\beta + \sqrt{-1}\partial\bar{\partial}U)\right|_{\omega_{\mathcal{C}}} = |\Phi^*(\beta)|_{\omega_{\mathcal{C}}} + \sum_{k=1}^{n+2}\left|\sqrt{-1}\partial\bar{\partial}u_k\right|_{\omega_{\mathcal{C}}} \leqslant Cz^{-1}. \tag{3.6.1}$$

Fix a point $p \in X$. Let $r(x)$ denote the distance function to p with the metric $\beta + \sqrt{-1}\partial\bar{\partial}U$. With this asymptotic behavior, we know that $r(x)$ is in the same order of the distance function on $\mathcal{C}$. Then outside a compact set on X we have the estimate that

$$\begin{aligned}
&\left|1 - \frac{(\beta + \sqrt{-1}\partial\bar{\partial}U)^n}{\Omega_X \wedge \bar{\Omega}_X}\right| \\
=&(\Phi^{-1})^* \left(\left|1 - \frac{(p^*(\beta|_D) + \omega_{\mathcal{C}} + \sqrt{-1}\partial\bar{\partial}U_{n+2})^n}{\omega_{\mathcal{C}}^n} + O(e^{-\delta z^n})\right|\right) \\
=&(\Phi^{-1})^* \left(|F_{n+2}| + O(e^{-\delta z^n})\right) \\
\leqslant& C(\Phi^{-1})^* \left(z^{-n-2+\epsilon}\right) \\
\leqslant& Cr^{-2-\epsilon}.
\end{aligned}$$

For the Ω_X-compatible condition, we notice that the small term u_{n+2} does not affect the integration of the form $\beta + \sqrt{-1}\partial\bar{\partial}U$:

$$\begin{aligned}
&\int_X (\beta + \sqrt{-1}\partial\bar{\partial}U)^n - \Omega_X \wedge \bar{\Omega}_X \\
=&\int_X (\beta + \sqrt{-1}\partial\bar{\partial}U)^n - (\beta + \sqrt{-1}\partial\bar{\partial}\tilde{U})^n \\
=&\sum_{k=0}^{n-1} \binom{n}{k} \lim_{\varepsilon \to 0} \int_{\{(\Phi^{-1})^* t \leqslant -\log\varepsilon\}} \sqrt{-1}\partial\bar{\partial}(\rho_{A_3}((\Phi^{-1})^* u_{n+2})) \\
&\qquad \wedge \left(\beta + \sqrt{-1}\partial\bar{\partial}\tilde{U}\right)^k \wedge \sqrt{-1}\partial\bar{\partial}\left(\rho_{A_2}((\Phi^{-1})^* u_{n+2})\right)^{n-k-1} \\
=&\sum_{k=0}^{n-1} \binom{n}{k} \lim_{\varepsilon \to 0} \int_{\{t = -\log\varepsilon\}} d^c u_{n+2} \wedge \frac{1}{nz^{n-1}} \left((\beta + \omega_{\mathcal{C}} + \sqrt{-1}\partial\bar{\partial}U_{n+1})^k \wedge (\sqrt{-1}\partial\bar{\partial}u_{n+2})^{n-k-1}\right) \\
=&0.
\end{aligned}$$

So $\beta + \sqrt{-1}\partial\bar{\partial}U$ satisfies both Ω_X-compatible condition and decay condition required in Tian-Yau-Hein's package. □

3.7 Existence and decay of the solution

3.7.1 Existence result

Now we are ready to apply Tian-Yau-Hein's package to deform our metric $\beta + \sqrt{-1}\partial\bar{\partial}U$ to a Calabi-Yau metric.

Theorem 3.7.1. *For any class $\mathfrak{k}$ in $H^2_+(X)$, there exists a Calabi-Yau metric ω in the class $\mathfrak{k}$.*

Proof. Let β be the good representative we chose in U be the potential constructed in section 3.6 with the form β. We know that $\beta + \sqrt{-1}\partial\bar{\partial}U$ is a Kähler metric on X such that

$$F(\beta + \sqrt{-1}\partial\bar{\partial}U) := 1 - \frac{(\beta + \sqrt{-1}\partial\bar{\partial}U)^n}{\Omega_X \wedge \bar{\Omega}_X}$$

decays in order $r^{-2-\epsilon}$ and $\int_X (\beta + \sqrt{-1}\partial\bar{\partial}U)^n$, here r is any distance function under the metric $\beta + \sqrt{-1}\partial\bar{\partial}U$. Let

$$f = \log \frac{\Omega_X \wedge \bar{\Omega}_X}{(\beta + \sqrt{-1}\partial\bar{\partial}U)^n}.$$

We have f satisfies integral condition $\int_X (e^f - 1)(\beta + \sqrt{-1}\partial\bar{\partial}U)^n = 0$ and the decay condition $|f| \leqslant Cr^{-2-\epsilon}$.

On the other hand, we have higher regularity estimate of u_i's:

$$\begin{aligned} &\left| z^{\frac{k}{2}} \nabla^k \left(\Phi^* \left(\beta + \sqrt{-1}\partial\bar{\partial}U \right) - \omega_{\mathcal{C}} \right) \right|_{\omega_{\mathcal{C}}} \\ = &\left| z^{\frac{k}{2}} \nabla^k \Phi^*(\beta) + \sum_{j=1}^{n+2} z^{\frac{k}{2}} \nabla^k \sqrt{-1}\partial\bar{\partial}u_j \right|_{\omega_{\mathcal{C}}} \leqslant Cz^{-1} \end{aligned}$$

for any $z > C$ with some $C > 0$. Then we have higher estimate of metric and scalar curvature. So $(X, \beta + \sqrt{-1}\partial\bar{\partial}U)$ satisfies the SOB$(\frac{2n}{n+1})$ condition by Lemma 3.1.2 and HMG$(\frac{1}{n+1}, k, \alpha)$ by Lemma 3.1.3 for any $k > 0$ and $0 < \alpha < 1$.

So we know that there exists a function ϕ on X such that

$$(\beta + \sqrt{-1}\partial\bar{\partial}U + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^f (\beta + \sqrt{-1}\partial\bar{\partial}U)^n = \Omega_X \wedge \bar{\Omega}_X, \tag{3.7.1}$$

with $\phi \in C^4(X)$. □

3.7.2 Decay of the solution

The iteration process shows that for any $K > 0$, there exists function U_K and constant C_K such that

$$|f_K| := |\log \frac{\Omega_X \wedge \bar{\Omega}_X}{(\beta + \sqrt{-1}\partial\bar{\partial}U_K)^n}| \leqslant C_K r^{-K}.$$

If we choose U such that the $\omega_{\mathcal{C}}$ potential $F(\beta + \sqrt{-1}\partial\bar{\partial}U)$ decays fast enough, we can show that the solution ϕ of (3.7.1) also decays fast to a constant. To do this, we first present the following local Poincaré lemma for SOB(ν) manifold with $\nu \in (0, 2]$:

Lemma 3.7.2. *Assume (X, ω_0) is a complete Kähler manifold satisfying*

SOB(ν) condition with $\nu \in (0, 2]$, *and*

$$r(x)^{\kappa}|B(x,1)| \leqslant C \text{ as } r(x) \to \infty$$

for some fixed $\kappa > 0$*,* $C > 0$*.*
Let $u, f \in C^{\infty}(X)$ *such that*

$$\sup|\nabla^i u| + \sup|\nabla^i f| < \infty \text{ for all } i \in \mathbb{N}_0,$$

and

$$(\omega_0 + i\partial\bar{\partial}u)^n = e^f \omega_0^n.$$

Then for any $\delta > 0$*, there exists* $K_\delta > 0$ *such that if*

$$\int_X |\nabla u|^2 \omega^n < \infty \text{ and } |f| \leqslant C r^{-K_\delta},$$

then

$$\sup_{B(x,1)} |u - u_{B(x,1)}| \leqslant C r(x)^{-\delta}$$

for any $x \in X$*.*

Remark 3.7.3. *The proof is entirely same as the proof in [17, Proposition 4.8(ib)]. The only difference is that we choose* r_i *to be* i*. For reader's convenience we give a sketch of proof here.*

Proof. We prove this via Moser iteration.

Step 1. We first show that local L^2-norm of ∇u decays. Let $A_i := \{i \le r \le i+1\}$ and ends $E_i := \{r \ge i\}$. since

$$\left|\nabla |u|^{\frac{p}{2}}\right|^2 = \frac{p^2}{4}|u|^{p-1}\left|\nabla|u|\right|^2,$$

Multiplying by $\xi u|u|^{p-2}$ to both sides of Monge-Ampère equation and taking integration by parts we have the following inequality:

$$\int \zeta|\nabla|u|^{\frac{p}{2}}|^2\omega_0^n \leqslant C_n \cdot \frac{p^2}{p-1}\left(\int \zeta u|u|^{p-2}\left(e^f - 1\right)\omega_0^n + \int u|u|^{p-2} d\zeta \wedge d^c u \wedge \omega^{n-1}\right) \tag{3.7.2}$$

For any $\mu \in \mathbb{R}$, $u - \mu$ also solves the Monge-Ampère equation. Choosing $p = 2$ and $\zeta = \chi(\frac{r}{R})\zeta_i$ to be a suitable bump function supporting on an increasing annulus $A(x_0, R, 2R)$, we have the following inequality.

$$\begin{aligned}
&\int_{E_{i+1}} |\nabla u|^2 \le \int \zeta_i |\nabla u|^2 \\
&\le C\int |\nabla\zeta_i||u-\mu||\nabla u| + C\int \zeta_i|u-\mu||f| \\
&\le C\left(r_{i+1} - r_i\right)^{-1}\|u-\mu\|_{L^2(A_i)}\|\nabla u\|_{L^2(A_i)} + C\sum_{j=i}^{\infty}\left(r_{j+1}^{\beta} - r_j^{\beta}\right)\|f\|_{L^{\infty}(A_j)}
\end{aligned}$$

The diameter of the annulus is bounded by $r_{i+1} - r_i + r_{i+1}^{\frac{1}{n+1}}$. Sobolev inequality on the annulus as in 2.2.9 yields that

$$\|u - \mu\|_{L^2(A_i)} \leq C\left(r_{i+1} - r_i + r_{i+1}^{\frac{1}{n+1}}\right)\|\nabla u\|_{L^2(A(r_i-6,r_i+6))}$$

Take $Q_i = \int_{E_i}|\nabla u|^2$, we have

$$\begin{aligned}Q_{i+1} &\leqslant C\cdot\left((1+\frac{r_{i+1}^{\frac{1}{n+1}}}{r_{i+1}-r_i})\left(Q_{i-1}-Q_{i+2}\right)+\sum_{j=i}^{\infty}\left(r_{j+1}^{\beta}-r_j^{\beta}\right)r_j^{-K_\delta}\right)\\&\leqslant C\cdot\left(1+\frac{r_{i+1}^{\frac{1}{n+1}}}{r_{i+1}-r_i}\right)\left(Q_{i-1}-Q_{i+2}\right)+C\cdot r_i^{-K_\delta+3}\end{aligned}$$

Choosing $r_i = i^{\frac{1}{n+1}}$, thus there exists some constant C such that

$$\frac{r_{i+1}^{\frac{1}{n+1}}}{r_{i+1}-r_i} \leqslant C,$$

which implies

$$\|\nabla u\|_{L^2(E_i)} \leq Cr^{-K'_\delta}$$

for some other number K'_δ that could be arbitrary large if K_δ is large enough.

Step 2. We do the iteration via local Sobolev inequality:

$$\left(\fint_{B(x,s)}|v-\bar{v}|^{2\alpha}\right)^{\frac{1}{2\alpha}} \leqslant Cs\cdot\left(\fint_{B(x,s)}|\nabla v|^2\right)^{\frac{1}{2}}$$

$$\left(\int_{B(x,s)}|v-\bar{v}|^{2\alpha}\right)^{\frac{1}{2\alpha}} \leqslant Cs|B(x,s)|^{-\frac{\alpha-1}{2\alpha}}\left(\int_{B(x,s)}|\nabla v|^2\right)^{\frac{1}{2}}$$

By triangular inequality we have

$$\left(\int_{B(x,s)}|v|^{2\alpha}\right)^{\frac{1}{2\alpha}} \leqslant |B(x,s)|^{\frac{1}{2\alpha}}\bar{v} + C|B(x,s)|^{-\frac{\alpha-1}{2\alpha}}\cdot\left(\int_{B(x,s)}|\nabla v|^2\right)^{\frac{1}{2}}$$

Take $v = |u|^{\frac{p}{2}}$:

$$\begin{aligned}&\left(\int_{B(x,s)}|u|^{p\alpha}\right)^{\frac{1}{2\alpha}}\\&\leqslant |B(x,s)|^{\frac{1-2\alpha}{2\alpha}}\int_{B(x,s)}|u|^{\frac{p}{2}} + Cs|B(x,s)|^{\frac{1-\alpha}{2\alpha}}\cdot\left(\int_{B(x,s)}\left|\nabla|u|^{\frac{p}{2}}\right|^2\right)^{\frac{1}{2}}\\&\leqslant C|B(x,s)|^{\frac{1-\alpha}{2\alpha}}\left(\left(\int_{B(x,s)}|u|^p\right)^{\frac{1}{2}}+\left(\int_{B(x,s)}\left|\nabla|u|^{\frac{p}{2}}\right|^2\right)^{\frac{1}{2}}\right)\end{aligned}$$

Take λ_i be a decreasing sequence in $(1,2)$, Choose a cut-off function support ζ_i on $B(x,\lambda_i)$ and is constant 1 on $B(x,\lambda_{i+1})$. By 3.7.2, the integration in the second term is bounded by

$$\left(\int_{B(x,\lambda_{i+1})} \left|\nabla |u|^{\frac{p}{2}}\right|^2\right)^{\frac{1}{2}} \tag{3.7.3}$$

$$\leqslant C\left(\frac{p^2}{p-1}\right)^{\frac{1}{2}}\left(\int_{B(x,\lambda_i)} |u|^{p-1}\left(e^f-1\right)+\int_{A(x,\lambda_{i+1},\lambda_i)} |u|^{p-1}\cdot|\nabla\zeta_i|\cdot|\nabla u|\right)^{\frac{1}{2}} \tag{3.7.4}$$

$$\leqslant C\left(\frac{p^2}{p-1}\right)^{\frac{1}{2}}\left(|B(x,\lambda_i)|^{\frac{1}{p}}\cdot\left(|f|\cdot+\|\nabla u\|_{L^\infty(X)}\cdot\|\nabla\zeta_i\|_{L^\infty(X)}\right)\cdot\left(\int_{B(x,\lambda_i)}|u|^p\right)^{\frac{p-1}{p}}\right)^{\frac{1}{2}}. \tag{3.7.5}$$

Let $\lambda_i = 1+\alpha^{-\frac{i}{2}}$. We know that f decays, $|u|$ is bounded, and we can choose ζ_i such that $|\nabla\zeta_i| < C\,\alpha^{\frac{i}{2}}$. Then substituting (3.7.5) into (3.7.3), we have

$$\begin{aligned}\|u\|_{L^{p\alpha}(B(x,\lambda_{i+1}))} &= \left(\left(\int_{B(x,\lambda_{i+1})}|u|^{p\alpha}\right)^{\frac{1}{2\alpha}}\right)^{\frac{2}{p}}\\ &\leqslant C^{\frac{2}{p}} r^{\frac{\alpha-1}{p(n+1)\alpha}}\left(1+\frac{\alpha^{\frac{i}{2}}}{r^{\frac{1}{p(n+1)}}}\cdot\left(\frac{p^2}{p-1}\right)^{\frac{1}{2}}\right)^{\frac{2}{p}}\max\left\{\|u\|_{L^p(B(x,\lambda_i))},\|u\|^{\frac{p-1}{p}}_{L^p(B(x,\lambda_i))}\right\}\end{aligned}$$

Let $B_i = B(x,\lambda_i)$, $p_i = 2\alpha^i$, and abusing notation we let $Q_i = \|u\|_{L^{p_i}(B_i)}$ again. We can do Moser iteration:

$$Q_{i+1} \leqslant C^{\frac{2}{p_i}} r^{\frac{\alpha-1}{p_i(n+1)\alpha}}\left(1+p_i\right)^{\frac{2}{p_i}}\max\left\{Q_i, Q_i^{\frac{p_i-1}{p_i}}\right\}$$

By our choice of p_i, there exists some constant a such that

$$\|u\|_{L^\infty(B(x,1))} \leqslant C\cdot\|u\|^a_{L^2(B(x,2))}.$$

Since $u-\bar{u}$ also solve the Monge-Ampère equation, where $\bar{u} = \oint_{B(x,1)} u$, we have

$$\|u-\bar{u}\|_{L^\infty(B(x,1))} \leqslant Cr^{-\delta}$$

as long as we choose K_δ large enough. □

Then we can improve the C^0 bound of our solution ϕ to get the optimal close rate of our weak asymptotically Calabi metric:

Theorem 3.7.4. *For any class $\mathfrak{k}$ in $H^2_+(X)$, there is a Calabi-Yau metric ω in $\mathfrak{k}$ which is weak asymptotically Calabi with rate* 1.

Proof. With Lemma 3.7.2, together with [17, Proposition 4.8(ii)], we know that if we choose K large enough, there exists a constant $\bar{\phi}$ and C such that the solution ϕ satisfies that

$$|\phi - \bar{\phi}| \leqslant Cr^{-\delta+\frac{n}{n+1}}, \text{ for any } x \text{ such that } r(x) > C.$$

Then we can replace ϕ by $\phi-\bar{\phi}$ to get a better candidate for the solution of (3.7.1), so ϕ could be chosen to decay at any polynomial rate. Repeat our local rescaling and local Schauder estimate, we know that the Calabi-Yau metric $\beta+\sqrt{-1}\partial\bar{\partial}U_K+\sqrt{-1}\partial\bar{\partial}\phi$ is polynomially closed to the Calabi model space with the leading error term $\beta + \sqrt{-1}\partial\bar{\partial}U_K$.

If $\beta|_D = 0$, the error term is exponentially close to Calabi model space. If $\beta|_D$ is nonzero, the decay rate of β is exactly $r^{-\frac{2}{n+1}}$. If we choose β such that $\beta|_D$ is primitive with respect to ω_D, the decay of $\sqrt{-1}\partial\bar{\partial}U_K$ would be $r^{-\frac{4}{n+1}+\epsilon}$, which is strictly lower order term compared with β. Thus, the Calabi-Yau metric $\beta + \sqrt{-1}\partial\bar{\partial}(U + \phi)$ decays exactly at the rate $r^{-\frac{2}{n+1}}$, which is equivalent to z^{-1}. □

3.8 Uniqueness

In this section, we prove that the Calabi-Yau metric asymptotic to $\omega_{\mathcal{C}}$ in the class $\mathfrak{k}$ is unique.

Theorem 3.8.1. *Let (M, D) be the pair we considered before. If we have another Calabi-Yau metric $\tilde{\omega}$ in the same class $\mathfrak{k}$ satisfying $|\tilde{\omega} - \omega|_{\omega} \leqslant r^{-\kappa}$, when $r \to \infty$, for some distance function r with respect to ω and some $\kappa > 0$, then $\tilde{\omega} = \omega$.*

Remark 3.8.2. *We are also interested in the problem that how different choice of the diffeomorphism Φ will change our Calabi-Yau metric. For example, the scaling in the fiber direction will change the metric by the rate $r^{-\frac{2n}{n+1}}$ and by our uniqueness theorem, we get the same Calabi-Yau metric.*

The proof of the theorem can be sketched as follows. We start with a $\partial\bar{\partial}$-lemma by solving $\bar{\partial}$ equation via the L^2 method. Then we can write $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}l$ with some estimate on l. By pulling back to $\mathcal{C}$, we construct f on the model space to solve the Poisson equation $\Delta_{\omega_{\mathcal{C}}} f = \Delta_{\omega_{\mathcal{C}}} l$. Via the estimate of harmonic function on $\mathcal{C}$ in Sun-Zhang [28], we can use the equation $(\omega + \sqrt{-1}\partial\bar{\partial}l)^n = \omega^n$ and take integration by parts to deduce that $\sqrt{-1}\partial\bar{\partial}l = 0$.

Lemma 3.8.3. *There exists a smooth function l on X such that $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}l$ with $|\Phi^* l| < Ce^{\epsilon t}$ on $\{t \geqslant C\}$ for any $\epsilon > 0$ and some $C > 0$.*

Proof. We prove the lemma by several steps:

Step 1: We show that there exists a smooth 1-form σ on X such that $\tilde{\omega} - \omega = d\sigma$ with $|\sigma|_{\omega} \leqslant Cz^{n+\frac{1}{2}-\kappa}$.

After pulling back to the model space $\mathcal{C}$ we have $\Phi^*(\tilde{\omega} - \omega)$ is a closed 2-form with $|\Phi^*(\tilde{\omega} - \omega)|_{\omega_{\mathcal{C}}} \leqslant Cz^{-\kappa}$. By viewing $\mathcal{C}$ as $Y \times (0, +\infty)$, we can write it as

$$\Phi^*(\tilde{\omega} - \omega) = \eta + dz \wedge \gamma$$

with $\partial_z \lrcorner \eta = 0$, $\partial_z \lrcorner \gamma = 0$. Then the fact that $d(\eta + dz \wedge \gamma) = 0$ implies $d_Y \eta = 0$, $\partial_z \eta = d_Y \gamma$. So we can choose

$$\tilde{\sigma} = \int_1^z \gamma dz$$

such that

$$d\tilde{\sigma} = \eta + dz \wedge \gamma = \Phi^*(\tilde{\omega} - \omega).$$

Since $|\Phi^*(\tilde{\omega} - \omega)|_{\omega_{\mathcal{C}}} \leqslant C z^{-\kappa}$, we have the decay of $dz \wedge \gamma$ which implies that

$$|\gamma|_{\omega_{\mathcal{C}}} \leqslant C z^{\frac{n-1}{2} - \kappa}.$$

Given the formula of $\omega_{\mathcal{C}}$ we can have an estimate of $|\tilde{\sigma}|_{\omega_{\mathcal{C}}}$ at the point $(y, z_0) \in \mathcal{C}$:

$$\begin{aligned} |\tilde{\sigma}(y, z_0)|_{\omega_{\mathcal{C}}(y, z_0)} &\leqslant \int_1^{z_0} |\gamma(y, z)|_{\omega_{\mathcal{C}}(y, z_0)} dz \\ &\leqslant \int_1^{z_0} |\gamma(y, z)|_{\omega_{\mathcal{C}}(y, z)} z_0^{\frac{n-1}{2}} z^{\frac{1}{2}} dz \\ &\leqslant C z_0^{n + \frac{1}{2} - \kappa}. \end{aligned}$$

After extending $(\Phi^{-1})^* \tilde{\sigma}$ as a smooth 1-form on X, we can write $\tilde{\omega} - \omega = d((\Phi^{-1})^* \tilde{\sigma}) + \theta$ for some smooth compact supported closed 2-form θ on X.

Recall that X is 1-convex. Then by the vanishing theorem for 1-convex manifold from Van Coevering [6, Proposition 4.2.], $\theta = \sqrt{-1}\partial\bar{\partial} s = dd^c s$ for some compact supported function s on X. Then

$$\sigma = (\Phi^{-1})^* \tilde{\sigma} + d^c s$$

is the smooth 1-form that we are looking for.

Step 2: Recall that E is the non-ample locus of $-K_M$. The $X \setminus E$ admits a complete Kähler metric by Proposition 4.1 in Ohsawa [26]. So we can use L^2-estimate on $X \setminus E$ to solve the $\bar{\partial}$ equation to construct the potential l such that $\tilde{\omega} - \omega = \sqrt{-1}\partial\bar{\partial} l$.

Let $\tau = \epsilon \cdot \rho_{B_1}(\mathsf{t}) - \delta \cdot \rho_{B_2}(\mathsf{t})^{\frac{1}{n}}$. Choose z_0, B_1 and B_2 large, then choose δ small depending on ϵ, we can guarantee that the $(1,1)$ form

$$\sqrt{-1}\partial\bar{\partial}\tau = \sqrt{-1}\partial\bar{\partial}(\epsilon \cdot \rho_{B_1}(\mathsf{t}) - \delta \cdot \rho_{B_2}(\mathsf{t})^{\frac{1}{n}})$$

is a Kähler form on $X \setminus E$. We have $\sqrt{-1}\partial\bar{\partial}\tau \geqslant C_{\epsilon,\delta} \mathsf{t}^{-1} \omega$ outside a compact set.

If we take the type decomposition of

$$\left(\Phi^{-1}\right)^* \tilde{\sigma} = \left(\left(\Phi^{-1}\right)^* \tilde{\sigma}\right)^{1,0} + \left(\left(\Phi^{-1}\right)^* \tilde{\sigma}\right)^{0,1},$$

we have the estimate of $\left(\left(\Phi^{-1}\right)^* \tilde{\sigma}\right)^{0,1}$ that

$$\left| \left(\left(\Phi^{-1}\right)^* \tilde{\sigma}\right)^{0,1} \right|_{\omega} \leqslant C \mathsf{t}^{1 + \frac{1 - 2\kappa}{2n}}$$

for $\mathfrak{t} > C$ and $(\Phi^{-1})^* \tilde{\sigma}$ supported on $\mathfrak{t} > C$.

So with the same weighted L^2 estimate in Hein-Sun-Viaclovsky-Zhang [14, Proposition 2.2.], we have

$$\int_{X\smallsetminus E} \mathfrak{t}\cdot \left|(\Phi^{-1})^*\tilde{\sigma}^{0,1}\right|_{\omega} e^{-\tau}\omega^n \leqslant C\int_{X\smallsetminus E} \mathfrak{t}\cdot \mathfrak{t}^{\frac{2n+1-2\kappa}{2n}} e^{-\epsilon\mathfrak{t}+\delta\mathfrak{t}^{\frac{1}{n}}}\omega^n < \infty$$

which yields that we have a solution ι such that $\bar{\partial}\iota = \sigma^{0,1}$ with

$$\int_{X\smallsetminus E} |\iota|^2 e^{-\tau}\omega^n \leqslant \int_{X\smallsetminus E} \mathfrak{t}\cdot |(\Phi^{-1})^*\tilde{\sigma}^{0,1}|_{\omega} e^{-\tau}\omega^n.$$

Consequently, we have $\sqrt{-1}\partial\bar{\partial}(2Im\iota) = d(\Phi^{-1})^*\tilde{\sigma}$. Set $l = 2Im\iota + s$ then we have $\tilde{\omega} - \omega = \sqrt{-1}\partial\bar{\partial}l$.

Step 3: We give the C^0 bound and C^k bound for l via elliptic estimates on the scaled metric.

Let x be any point in $X \smallsetminus K$. With the same local elliptic estimate under the scaled metric $\hat{\omega} = \mathfrak{t}(x)^{-\frac{1}{n}}\omega$ as in Proposition 3.4.1, we can give a global C^0 bound of l. We know that l satisfies the elliptic equation $(\omega + \sqrt{-1}\partial\bar{\partial}l)^n = \omega^n$ with

$$\begin{aligned}
&\int_{B_{\hat{\omega}}(x,1)} |l|^2\omega^n \\
&\leqslant e^{\epsilon\cdot C\mathfrak{t}(x)}\int_{X\smallsetminus E} |l|^2 e^{-\epsilon\cdot\rho_{B_1}(\mathfrak{t})+\delta\cdot\rho_{B_2}(\mathfrak{t})^{\frac{1}{n}}}\omega^n \\
&\leqslant C_\epsilon e^{\epsilon\cdot C\mathfrak{t}(x)},
\end{aligned}$$

since we have some uniform constant C such that $\mathfrak{t}(y) \leqslant C\cdot\mathfrak{t}(x)$ for any $y \in B_{\hat{\omega}}(x,1)$ and any $x \in X \smallsetminus K$. By adjusting ϵ small enough we have

$$\|l\|_{L^2(B_{\hat{\omega}}(x,1))} \leqslant C_\epsilon e^{\epsilon\mathfrak{t}(x)}.$$

Now we can do local elliptic estimates on the scaled metric after lifting to the universal cover. Since the S^1 direction on $\mathcal{C}$ collapsing in polynomial order with respect to z, we know that

$$\|l\|_{L^2(\widetilde{B}_{\hat{\omega}}(x,1))} \leqslant C\cdot\mathfrak{t}(x)^{\frac{n-1}{2n}}\cdot\|l\|_{L^2(B_{\hat{\omega}}(x,1))} \leqslant C_\epsilon e^{\epsilon\mathfrak{t}(x)}.$$

We have the global C^0 bound for l:

$$|l|(x) \leqslant \|l\|_{W^{2,2}(\widetilde{B}_{\hat{\omega}}(x,1))} \leqslant C\cdot\|l\|_{L^2(\widetilde{B}_{\hat{\omega}}(x,1))} \leqslant C_\epsilon e^{\epsilon\mathfrak{t}(x)}$$

for any $\epsilon > 0$. □

Remark 3.8.4. *In the proof of the $\sqrt{-1}\partial\bar{\partial}$-lemma 3.8.3 we did not use the polynomial decay of $\tilde{\omega} - \omega$. In fact, we can always find l even when $|\tilde{\omega} - \omega|$ is polynomially growth.*

Furthermore, we can prove that $\tilde{\omega}-\omega$ has weighted higher regularity bound.

Lemma 3.8.5. *There exists a constant $C>0$ such that*

$$\left|z^{\frac{k}{2}}\nabla^k(\tilde{\omega}-\omega)\right|_{\omega}\leqslant Cz^{-\kappa}$$

for any $z>C$.

Proof. Fix any point x in X with $\mathfrak{t}(x)=z_0^n$. We still work on the scaled metric $\hat{\omega}=z_0^{-1}\omega$ with uniform bounded curvature. The injectivity radius of the universal covering around x is bounded below by a universal constant δ independent of x. Now we are working on the ball $\tilde{B}(\tilde{x},\delta)$ in the universal cover. Since $\tilde{\omega}-\omega$ is d-exact, locally we can take integration of $\tilde{\omega}-\omega$ along the geodesic lines to have 1-form σ on $\tilde{B}(\tilde{x},\delta)$ such that

$$\begin{aligned} d\sigma &= \tilde{\omega}-\omega,\\ \|\sigma\|_{C^0_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))} &\leqslant \|\tilde{\omega}-\omega\|_{C^0_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}\leqslant Cz_0^{1-\kappa}. \end{aligned}$$

Consider the type decomposition of $\sigma=\sigma^{0,1}+\sigma^{1,0}$. The operator

$$N: L^2(\tilde{B}(\tilde{x},\delta),\Omega^{0,1})\to L^2(\tilde{B}(\tilde{x},\delta),\Omega^{0,1})$$

constructed in [20, Theorem 8.9] satisfies that

$$\begin{aligned} &\Delta_{\bar{\partial}}(N\sigma^{0,1})=\sigma^{0,1},\\ &\|N\sigma^{0,1}\|_{L^2_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}\leqslant C\|\sigma^{0,1}\|_{L^2_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))} \end{aligned}$$

and N commutes with ∂ and $\bar{\partial}$. Then

$$\|N\sigma^{0,1}\|_{W^{2,2}_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}\leqslant C\|\sigma^{0,1}\|_{L^2_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}\leqslant C\|\sigma^{0,1}\|_{C^0_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}.$$

Then we know by Sobolev lemma and iteration process that for any $q>1$

$$\|N\sigma^{0,1}\|_{W^{2,q}_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}\leqslant C\|\sigma^{0,1}\|_{L^q_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}\leqslant C\|\sigma^{0,1}\|_{C^0_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}.$$

Take $q>n$, there exists $\alpha>0$ such that

$$\|N\sigma^{0,1}\|_{C^{1,\alpha}_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}\leqslant C\|N\sigma^{0,1}\|_{W^{2,q}_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}\leqslant C\|\sigma^{0,1}\|_{C^0_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}.$$

Let $f=\bar{\partial}^*N\sigma^{0,1}$. We have

$$\begin{aligned} &\|f\|_{C^{0,\alpha}_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))}\leqslant C\|\sigma^{0,1}\|_{C^0_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))},\\ &\bar{\partial}f=\Delta_{\bar{\partial}}(N\sigma^{0,1})=\sigma^{0,1}. \end{aligned}$$

Let $\hat{l} = z_0^{-1} \cdot 2\mathrm{Im} f$. We have $\sqrt{-1}\partial\bar{\partial}\hat{l} = z_0^{-1}(\tilde{\omega} - \omega)$. Hence $(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\hat{l})^n = \hat{\omega}^n$ with

$$\|\hat{l}\|_{C^{0,\alpha}_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))} \leqslant C z_0^{-\kappa}.$$

By Schauder estimates we have higher regularity

$$\|\hat{l}\|_{C^{k,\alpha}_{\hat{\omega}}(\tilde{B}(\tilde{x},\delta))} \leqslant C z_0^{-\kappa}$$

which yields

$$\left| z_0^{\frac{k}{2}} \nabla^k (\tilde{\omega} - \omega) \right|_{\omega} \leqslant C z_0^{-\kappa}$$

for any $z_0 > C$. □

Now we are ready to prove the uniqueness:

Proof of Theorem 3.8.1. Given by previous estimate, we have $\tilde{\omega} - \omega = \sqrt{-1}\partial\bar{\partial} l$ with $|l| \leqslant C_\epsilon e^{\epsilon t}$. If we pull back l to $\mathcal{C}$, by the closeness of complex structure we have

$$|dJ_{\mathcal{C}} dl|_{\omega_{\mathcal{C}}} \leqslant |d(J_{\mathcal{C}} - J_X) dl|_{\omega_{\mathcal{C}}} + |dJ_X dl|_{\omega_{\mathcal{C}}} \leqslant C(e^{-(\frac{1}{2}-\epsilon) z^n} + z^{-\kappa}).$$

The function $F_l = \Delta_{\omega_{\mathcal{C}}} l$ has higher regularity bound on $\mathcal{C}$:

$$\left| z^{\frac{k}{2}} \nabla^k F_l \right|_{\omega_{\mathcal{C}}} \leqslant C z^{-\kappa} \text{ for any } z > C.$$

By Proposition 3.4.1 there exists a smooth function f on $\mathcal{C}$ such that $\Delta_{\omega_{\mathcal{C}}} f = F_l$ with

$$\begin{aligned} &|\sqrt{-1}\partial\bar{\partial} f|_{\omega_{\mathcal{C}}} \leqslant C z^{-\kappa+\epsilon}, \\ &|df|_{\omega_{\mathcal{C}}} \leqslant C z^{\frac{n+1}{2}-\kappa+\epsilon}, \\ &|f| \leqslant C z^{n+1-\kappa+\epsilon} \text{ for any } \epsilon > 0. \end{aligned}$$

Since $\Delta_{\omega_{\mathcal{C}}}(l - f) = 0$ and $|l - f| \leqslant e^{\epsilon t}$ for any $\epsilon > 0$, from the behavior of harmonic function [28, Proposition 5.3.] we know that

$$l = f + \lambda z + g + O(e^{-\delta z})$$

for some $\lambda > 0$ and some harmonic S^1-invariant function g on $\mathcal{C}$ with

$$|g| \leqslant C e^{\delta z^{\frac{n}{2}}}.$$

Since

$$|z^{n-1} g_{tt}(t, q)| \leqslant |\sqrt{-1}\partial\bar{\partial} g|_{\omega_{\mathcal{C}}} \leqslant C z^{-\kappa+\epsilon}$$

holds uniformly for any $q \in D$, integration along the $\mathbb{R}$-fiber direction shows that g and hence l is at most polynomially growth. Again by [28, Proposition 5.3.] we know that

$$l = f + \lambda z + O(e^{-\delta z}).$$

Recall that l satisfies that

$$\Delta_{\omega_{\mathcal{C}}} l = \Delta_{\omega_{\mathcal{C}}} l - \Delta_{\Phi^*\omega} l + \sum_{k=2}^{n} \binom{n}{k} (d\Phi^* J_X dl)^k \wedge \Phi^*\omega^{n-k},$$

by our previous construction we know that $|\Phi^*\omega - \omega_{\mathcal{C}}|_{\omega_{\mathcal{C}}} \leqslant Cz^{-1}$, so

$$\begin{aligned} &|\Delta_{\omega_{\mathcal{C}}} l - \Delta_{\Phi^*\omega} l\,|_{\omega_{\mathcal{C}}} \leqslant Cz^{-1-\kappa} \\ &\Delta_{\omega_{\mathcal{C}}} l \leqslant C\left(z^{-1}|\sqrt{-1}\partial\bar{\partial} l|_{\omega_{\mathcal{C}}} + |\sqrt{-1}\partial\bar{\partial} l|^2_{\omega_{\mathcal{C}}}\right) \\ &\qquad \leqslant Cz^{-\min\{1+\kappa, 2\kappa, 1+n\}}. \end{aligned} \tag{3.8.1}$$

Thus by finite step iteration we can find a better candidate $\tilde{f}$ and another constant $\tilde{\lambda}$ such that

$$\begin{aligned} &l = \tilde{f} + \tilde{\lambda} z + O(e^{-\delta z}), \\ &|\nabla^2 \tilde{f}|_{\omega_{\mathcal{C}}} \leqslant Cz^{-n-1+\epsilon}, \\ &|\nabla \tilde{f}|_{\omega_{\mathcal{C}}} \leqslant Cz^{-\frac{n+1}{2}+\epsilon}, \\ &|\tilde{f}| < Cz^{\epsilon}. \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 &= \int_X \sqrt{-1}\partial\bar{\partial} l \wedge \sum_{k=1}^{n} \binom{n}{k} (\sqrt{-1}\partial\bar{\partial} l)^{k-1} \wedge \omega^{n-k} \\ &= \lim_{\varepsilon \to 0} \int_{\{(\Phi^{-1})^* t \leqslant -\log\varepsilon\}} \sqrt{-1}\partial\bar{\partial} l \wedge \sum_{k=1}^{n} \binom{n}{k} (\sqrt{-1}\partial\bar{\partial} l)^{k-1} \wedge \omega^{n-k} \\ &= \lim_{\varepsilon \to 0} \int_{\{t = -\log\varepsilon\}} \tilde{\lambda} d^c z \wedge \sum_{k=1}^{n} \binom{n}{k} (\sqrt{-1}\partial\bar{\partial} l)^{k-1} \wedge \omega^{n-k} \\ &= \lim_{\varepsilon \to 0} \int_{\{t = -\log\varepsilon\}} \tilde{\lambda} d^c t \wedge (\sqrt{-1}\partial\bar{\partial} t)^{n-1} = \tilde{\lambda} \mathrm{Vol}(D). \end{aligned}$$

Consequently, $\tilde{\lambda} = 0$, $l \leqslant Cz^{\epsilon}$ for any $\epsilon > 0$. From the equation of l we know that

$$\begin{aligned} &\lim_{\varepsilon \to 0} \left| \int_{\{(\Phi^{-1})^* t = -\log\varepsilon\}} l d^c l \wedge \sum_{k=1}^{n} \binom{n}{k} (\sqrt{-1}\partial\bar{\partial} l)^{k-1} \wedge \omega^{n-k} \right| \\ \leqslant &\lim_{\varepsilon \to 0} \int_{\{t = -\log\varepsilon\}} \left| \frac{C}{z^{1-\epsilon}} \cdot d^c t \wedge (\sqrt{-1}\partial\bar{\partial} t)^{n-1} \right| = 0. \end{aligned}$$

Hence by integration by parts and

$$0 = l(\tilde{\omega}^n - \omega^n) = l \cdot \sqrt{-1}\partial\bar{\partial} l \wedge \sum_{k=0}^{n-1} \omega^k \wedge \tilde{\omega}^{n-1-k},$$

we have:

$$\begin{aligned} 0 &= -\int_X l \cdot \sqrt{-1}\partial\bar{\partial} l \wedge \sum_{k=0}^{n-1} \omega^k \wedge \tilde{\omega}^{n-1-k} \\ &= \int_X dl \wedge d^c l \wedge \sum_{k=0}^{n-1} \omega^k \wedge \tilde{\omega}^{n-1-k}. \end{aligned}$$

Since $\sum_{k=0}^{n-1} \omega^k \wedge \tilde{\omega}^{n-1-k}$ is a positive form, we know that $dl = d^c l = 0$. Hence $\omega = \tilde{\omega}$. □

3.9 Examples and further discussion

3.9.1 Example

We present examples that (X, ω) is a Calabi-Yau manifold not asymptotically Calabi but weak asymptotically Calabi under the fixed diffeomorphism Φ. As discussed in the end of the proof of Theorem 1.2.4, we have the following:

Claim. *Let (M, D) be the pair in Definition 1.2.2 with $X = M \setminus D$. Let $H^2_{+,c}(X) = \mathrm{Im}(H^2_c(X) \to H^2(X)) \cap H^2_+(X)$. Fix a diffeomorphism $\Phi : \mathcal{C} \setminus \mathcal{K} \to X \setminus K$. Then for any $\mathfrak{k}$ in $H^2_+(X)$ but not $H^2_{+,c}(X)$ the metric ω we constructed in Theorem 1.2.4 is a Calabi-Yau metric not asymptotically Calabi but weak asymptotically Calabi.*

Example 3.9.1. *Let $M = \mathbb{P}^1 \times \mathbb{P}^2$ with two projection maps $\pi_1 : M \to \mathbb{P}^1$ and $\pi_2 : M \to \mathbb{P}^2$. Then we have $D = -K_M = \pi_1^* O_{\mathbb{P}^1}(2) + \pi_2^* O_{\mathbb{P}^2}(3)$. $\mathrm{Pic}(M)$ is generated by $\pi_1^* O_{\mathbb{P}^1}(1)$ and $\pi_2^* O_{\mathbb{P}^2}(1)$ and the image of $\pi_1^* O_{\mathbb{P}^1}(1) + \pi_2^* O_{\mathbb{P}^2}(1)$ under the map $i^* : H^2(M) \to H^2(D)$ induced by the inclusion map $i : D \to M$ is not parallel to $[\omega_D] = c_1(N_D)$. Choose a primitive representative of this class and apply Theorem 1.2.4 we will find a Calabi-Yau metric not asymptotically Calabi but weak asymptotically Calabi.*

These kind of examples could be found on any Fano manifold M with $\dim_{\mathbb{C}} M \geqslant 3$ and $h_2(M) \geqslant 2$. We can find many examples in Mori-Mukai [25]. Besides, there are also many examples in the weak Fano case but we do not have a simple topological sufficient condition.

3.9.2 Weaker Decay Condition

In our statement of uniqueness Theorem 1.3.1, we need the metric $\tilde{\omega}$ to be polynomially closed to ω. The main difficulty to get rid of this condition lies in how to deduce the decomposition of $l = f + \lambda z + O(e^{-\delta z})$ with $|f| \leqslant z^\epsilon$ for any $\epsilon > 0$, where we cannot do iteration to improve the decay of f as in (3.8.1).

It is natural to ask the following question:

Question 3.9.2. *Can we prove a stronger uniqueness theorem: If we have another Calabi-Yau metric* $\tilde{\omega}$ *such that* $|\tilde{\omega} - \omega|_{\omega} \to 0$ *when* $r \to \infty$ *for some distance function* r *with respect to* ω*, then* $\tilde{\omega} = \omega$*?*

One possible obstruction of this stronger uniqueness theorem is that we cannot rule out the possibility that there is a Calabi-Yau metric ω closed to the Calabi model space in a logarithm rate rather than any polynomial rate. The existence of this type of Calabi-Yau metric is also an interesting question to study.

Chapter 4

Compactification of asymptotically Calabi space

4.1 Overview

Now that we know that for any weak log Fano pair, we have Calabi-Yau metrics of Calabi type, the next goal is to classify all complete Calabi-Yau manifolds of Calabi type. Since the topological closure of X at infinity is the zero section of L, which is isomorphic to the compact Calabi-Yau manifold D, it is natural to expect that we can compactify the manifold analytically. Hein-Sun-Viaclovsky-Zhang [14] showed that any asymptotically Calabi manifold which is Calabi-Yau can be compactified complex analytically to a weak Fano manifold and the Calabi-Yau metric comes from the construction by Tian-Yau-Hein's package. In our setting of weak asymptotically Calabi case, we can repeat the argument in Hein-Sun-Viaclovsky-Zhang [14] with slight modification, we can also construct holomorphic functions on X at infinity using the holomorphic sections of the ample line bundle L over D via Hörmander's L^2 estimates. The next step would be showing that under the metric given by these coordinates, the map into the projective space is an embedding outside a compact subset and showing that under the compactification $\overline{X}'$ given by the image of this map, the holomorphic $(n, 0)$ form has a simple pole along D'. Then we are able to show that the compactification we get is Kähler and hence projective by considering the behavior of the class at the end. By our uniqueness Theorem 3.8.1, this metric ω in fact must come from our existence result in Theorem 1.2.4.

4.2 From holomorphic section to holomorphic function

Definition 4.2.1. *For any holomorphic section* $s \in H^0(D, L^k)$, *we define a holomorphic function* f_s *on* $L \setminus \mathbf{0}_L$ *as*

$$f_s(\xi) = \frac{s(\pi(\xi))}{\xi^{\otimes k}}$$

where $\xi \in L \setminus \mathbf{0}_L$, *and* $\pi : L \to D$ *is the bundle projection.*

Under this definition, f is a holomorphic function on L that is complex homogeneous along fiber with degree $-k$, i.e.

$$f(\lambda \cdot \xi) = \lambda^{-k} f(\xi), \text{ for } \lambda \in \mathbb{C}.$$

Pulling back f to X via the diffeomorphism $\Phi^{-1} : X \setminus K \to \mathcal{C} \setminus \mathcal{K}$, we get an almost holomorphic function on $X \setminus K$. Abusing notation we still denote it by f_s. We can also extend the function to the whole space X by using a cutoff function χ which is equal to 1 on $X \setminus K$ and vanishes on a compact subset of K.

We can say χf_s is almost holomorphic because of the closeness of the complex structure between X and $\mathcal{C}$.

We will use Hörmander's L^2 estimates to construct a holomorphic function $\mathcal{L}(s)$ that is a perturbation of f_s on X outside a compact set. To do this, we need to have C^2 estimate for f_s on $\mathcal{C}$ as in [14]. More precisely, we have the following improved estimates for f.

Lemma 4.2.2. *We have*

$$|\nabla^l f|_{\omega_{\mathcal{C}}} \leq C_l \cdot e^{\frac{k}{2} z^n} \cdot z^{\frac{l}{2}(n-1)}$$

for all $l \geqslant 0$ *and any* $z > z_l$.

Proof. Assume we have local holomorphic charts $(W)_\alpha \simeq (U)_\alpha \times \mathbb{C}$ with $(U)_\alpha \subset \mathbb{C}^{n-1}$ and the coordinates denoted by $(\underline{z}, w) = (z_1, z_2, \cdots, z_{n-1}, w)$. We can write

$$f_s(\underline{z}, w) = f_s(\underline{z}, 1) w^{-k}.$$

For $l = 0$, we have $|f_s| \leqslant C_s |w|^{-k} = C_s e^{\frac{k}{2} z^n}$. Since f_s is holomorphic, we know that f_s is also harmonic under the metric $g_{\mathcal{C}}$.

Given $x \in \mathcal{C}$, let $z_0 = z(x)$. The injectivity radius of $\omega_{\mathcal{C}}$ is of order $z^{-\frac{n-1}{2}}$, so scaling up the metric by $z_0^{\frac{n-1}{2}}$ we have a local ball with definite size c in $\mathbb{C}^n$ with Kähler metric $\tilde{\omega} = z_0^{\frac{n-1}{2}} \cdot \omega_{\mathcal{C}}$ on $\mathcal{C}$. By suitable scaling again we can denote it by $B_{\tilde{\omega}}(x, 1)$. Since the Riemannian curvature of $\omega_{\mathcal{C}}$ is of order z^{-1} we know

$$|\mathrm{Rm}_{\tilde{\omega}}|_{\tilde{\omega}} = O(z_0^{-n})$$

on $B_{\tilde{\omega}}(x,1)$. Together with the Ricci flatness we know that we have uniform Schauder estimate as in Proposition 2.2.11, which yields that

$$z_0^{-l\frac{n-1}{2}} \left\|\nabla^l f_s\right\|_{C^0\left(B_g\left(x, z_0^{-\frac{n-1}{2}}\right)\right)} \leqslant C_l \left\|f_s\right\|_{C^0\left(B_g\left(x, z_0^{-\frac{n-1}{2}}\right)\right)}.$$

Hence,

$$\begin{aligned}\left|\nabla^l f_s\right|(x) &\leqslant C_l \cdot \left\|f_s\right\|_{C^0\left(B_g\left(x, z_0^{-\frac{n-1}{2}}\right)\right)} \\ &\leqslant C_l \cdot \sup_{y \in B_g\left(x, z_0^{-\frac{n-1}{2}}\right)} e^{\frac{k}{2} z(y)^n} \cdot z_0^{l\frac{n-1}{2}} \\ &\leqslant C_l \cdot e^{\frac{k}{2} z(x)^n + C} \cdot z_0^{\frac{l(n-1)}{2}}\end{aligned}$$

The last inequality uses the fact that

$$|dt|_{\omega_{\mathcal{C}}} = O(z^{\frac{n-1}{2}}).$$

□

Proposition 4.2.3. *For any $\delta > 0$, any holomorphic section $s \in H^0(D, L^k)$, we have a holomorphic function $L(s)$ on X such that*

$$\left\|e^{-\frac{kt}{2}} t^{k\delta}(L(s) - f_s)\right\|_{L^2(X)} \leqslant +\infty.$$

Proof. The proof is similar to the one in [14]. We can use the Hörmander's L^2 estimates to get a holomorphic function $L(s)$. We only point out some key differences here.

Notice that for $z > C$ with Lemma 4.2.2, we have

$$|\bar{\partial}_I(\chi f_s)|_g = |\bar{\partial}_I f_s|_g = |\left(\bar{\partial}_I - \bar{\partial}_{\mathcal{C}}\right) f_s|_g \leqslant e^{\frac{k}{2} z^n} \cdot z^{\frac{n-1}{2}} \cdot |\bar{\partial}_I - \bar{\partial}_{\mathcal{C}}|_g$$

Then if we take $\varphi_0 = t - \delta \log t$, then by simple computation we have

$$\sqrt{-1}\partial\bar{\partial}\varphi \geq \frac{1}{z^{n+1}} \omega_{\mathcal{C}}.$$

Then by gluing process we have the references metric $\sqrt{-1}\partial\bar{\partial}\varphi$ on X such that $\sqrt{-1}\partial\bar{\partial}\varphi \geq \Phi\omega$ with $\Phi = z^{-n-1}$ with on $X \backslash E$ where E is the exceptional set of Remmert reduction as in [14].

Then $\bar{\partial}_I(\chi f_s)$ has weighted L^2 norm if the complex structure is close enough to the one in $\mathcal{C}$:

$$\begin{aligned}&\int_{X \backslash E} \frac{1}{\Phi(z)} \left|\bar{\partial}_I(\chi f_s)\right|_g^2 e^{-k\phi} d\,\mathrm{Vol}_g \\ \leqslant\, & C \cdot \int_{X \backslash E} z^{n+1} \cdot e^{kz^n} \cdot z^{n-1} \cdot |\bar{\partial}_I - \bar{\partial}_{\mathcal{C}}|_g^2 \cdot e^{-kz^n} \cdot z^{k\delta} d\,\mathrm{Vol}_g \\ \leqslant\, & C \cdot \int_{X \backslash E} z^{2n+k\delta} \cdot |\bar{\partial}_I - \bar{\partial}_{\mathcal{C}}|_g^2 d\,\mathrm{Vol}_g\end{aligned}$$

Now in order to apply the Hörmander's L^2 estimates for the $\bar{\partial}$-operator on $X\backslash E$, we need the complex structure to be close enough to the one in $\mathcal{C}$, faster than z^{3n}. And when the integral is finite, we get a solution u to the equation

$$\bar{\partial}_I u = \bar{\partial}_I \left(\chi f_s\right)$$

which is perpendicular to the holomorphic functions on X and

$$\begin{aligned} &\int_{X\backslash E} \frac{1}{\Phi(z)} |u|^2 \cdot e^{-kt} \cdot t^{k\delta} d\operatorname{Vol}_g \\ \leqslant &\int_{X\backslash E} \frac{1}{\Phi(z)} \left|\bar{\partial}_I \left(\chi f_s\right)\right|_g^2 e^{-k\phi} d\operatorname{Vol}_g < \infty \end{aligned}$$

Let $L(s)$ be $\chi f_s - u$, then we have

$$\bar{\partial}_I L(s) = 0$$

and

$$\left\| e^{-\frac{kt}{2}} t^{k\delta} (L(s) - f_s) \right\|_{L^2(X)} \leqslant +\infty.$$

□

4.3 polynomially decay complex structure

We would like to make the following conjecture to further generalize this into slower decay assumption.

Definition 4.3.1. *Let X be a complete Kähler manifold with complex dimension n, complex structure I, Kähler form ω and $(n,0)$-form Ω. We say (X, I, ω, Ω) is* polynomial asymptotically Calabi with rate (κ_1, κ_2) *if:*

there exists $\kappa_1, \kappa_2 > 0$, a Calabi model space $(\mathcal{C}, I_{\mathcal{C}}, \omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$, and a diffeomorphism $\Phi : \mathcal{C} \setminus \mathcal{K} \to X \setminus K$, where $K \subset X$ and $\mathcal{K} \subset \mathcal{C}$ are compact, such that the following hold uniformly as $z \to +\infty$:

$$\left|\nabla_{\omega_{\mathcal{C}}}^k \left(\Phi^* I_X - I_{\mathcal{C}}\right)\right|_{\omega_{\mathcal{C}}} + \left|\nabla_{\omega_{\mathcal{C}}}^k \left(\Phi^* \Omega - \Omega_{\mathcal{C}}\right)\right|_{\omega_{\mathcal{C}}} = O\left(z^{-\kappa_1}\right), \quad \left|\nabla_{\omega_{\mathcal{C}}}^k \left(\Phi^* \omega - \omega_{\mathcal{C}}\right)\right|_{\omega_{\mathcal{C}}} = O\left(z^{-\kappa_2}\right)$$

for all $k \in \mathbb{N}_0$.

For the polynomially close case, we propose the following conjecture:

Conjecture 4.3.2. *There are optimal constants λ and μ such that for any $\kappa_1 > \lambda$ and $\kappa_2 > \mu$, any polynomial asymptotically Calabi Calabi-Yau manifold with rate (κ_1, κ_2) can be compactified complex analytically to a weak Fano manifold. Furthermore, the Calabi-Yau metric comes from our generalized Tian-Yau construction in Theorem 1.2.4.*

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