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Entropy and a convergence theorem for Gauss curvature flow in high dimension

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Abstract. We prove uniform regularity estimates for the normalized Gauss curvature flow in higher dimensions. The convergence of solutions in C^∞-topology to a smooth strictly convex soliton as t goes to infinity is obtained as a consequence of these estimates together with an earlier result of Andrews. The estimates are established via the study of an entropy functional for convex bodies.

Keywords. Gauss curvature flow, entropy, support functions, regularity, convergence

1. Introduction

The Gauss curvature flow was introduced by Firey [14] to model the changing shape of a tumbling stone subjected to collisions from all directions with uniform frequency. Suppose that $\{M_t\} \subset \mathbb{R}^{n+1}$ is a family of smooth compact strictly convex hypersurfaces with $t \in [0, T)$. Denote by $X(x,t)$ and $K(x,t)$ the position vector and the Gauss curvature of M_t. The family $\{M_t\}$ is a solution of the Gauss curvature flow if $X(x,t)$ satisfies the equation

$$\frac{\partial X(x,t)}{\partial t} = -K(x,t)\nu(x,t), \tag{1.1}$$

where $\nu(x,t)$ is the unit exterior normal of the hypersurface M_t.

Assuming the existence, uniqueness and regularity of the solution, Firey proved that if the initial convex surface ($M_0 \subset \mathbb{R}^3$) is symmetric with respect to the origin (abbreviated as *centrally symmetric*), then the flow (1.1) contracts the initial surface to a point in finite time and the evolving surface becomes spherical in shape in the process. The last statement can be rephrased as saying that the normalized flow (with the enclosed volume preserved) converges to a round sphere. Firey conjectured that the result holds in general. After this initial work, the existence and uniqueness of the Gauss curvature flow in any $\mathbb{R}^{n+1}$ was established by Chou [19]. In the same paper it was also proved that the Gauss curvature flow contracts the initial convex hypersurface to a point in finite time.

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More than a decade later, in a breakthrough work [5], Andrews proved that the normalized flow in $\mathbb{R}^3$ converges to a round sphere, that is, evolving surfaces become spherical in the process, hence proving the conjecture of Firey. The proof of Andrews [5] relies on a pinching estimate, whose proof seems to work only in dimension 2. There is an extensive literature devoted to the study of the Gauss curvature flow. Chow [10] established an important differential Harnack inequality (also known as the LYH type estimate) and an entropy monotonicity; Hamilton [15] obtained upper bounds of the support function and the Gauss curvature of the normalized flow; and Daskalopoulos and Hamilton [11] studied the Gauss curvature flow with flat sides. The interested reader may consult [9, 3, 4, 6, 12] for further references on the flow by Gauss curvature and its powers.

In this paper, we establish uniform regularity of the solution of the normalized Gauss curvature flow. By Chou's work, the convex hypersurfaces M_t (and the enclosed convex body Ω_t) shrink to a point along the Gauss curvature flow at a finite time T. If we choose this limiting point as the origin and normalize M_t so that the enclosed volume (the Lebesgue measure $|\Omega_t|$) is the volume of the unit ball, the normalized Gauss flow satisfies the equation

$$\frac{\partial X(x,t)}{\partial t} = (-K(x,t) + u(x,t))\nu(x,t), \tag{1.2}$$

where $u(x,t) = \langle X(x,t), \nu(x,t)\rangle$ is the supporting function.

The following is the main result of this paper.

Theorem 1.1. *Suppose that M_0 is a compact strictly convex hypersurface in $\mathbb{R}^{n+1}$ such that the volume of the enclosed convex body is that of the unit ball $B_1(0) \subset \mathbb{R}^{n+1}$. Assume that the origin is the contracting point of the un-normalized flow (1.1). Let $\{\Omega_t\}$ be the convex bodies enclosed by $\{M_t\}$, the solution to the normalized flow (1.2) with the above normalization. Then there exists a positive constant $\Lambda \geq 1$ depending only on n and M_0 such that*

$$B_{1/\Lambda}(0) \subset \Omega_t \subset B_\Lambda(0), \qquad \forall 0 \leq t < \infty. \tag{1.3}$$

Moreover, for any integer $k \geq 1$, there is a constant $C(n,k,M_0)$, depending on n, k and the initial hypersurface M_0, such that

$$\|M_t\|_{C^k} \leq C(n,k,M_0). \tag{1.4}$$

Finally, the flow (1.2) converges in C^∞-topology to a smooth strictly convex soliton M_∞ satisfying

$$K(x) = u(x) \quad \text{with } K(x) \geq 1/\Lambda. \tag{1.5}$$

As mentioned before, Hamilton [15] obtained upper bounds of the diameter and the Gauss curvature for the normalized flow. In view of the Blaschke selection theorem and a general C^∞-convergence result of Andrews [4] which assumes the regularity of the limiting soliton, the contribution of this paper consists mainly in uniform C^2-estimates for the normalized Gauss curvature flow. The C^2-estimate relies on a C^0-estimate on the support function $u(x,t)$ (particularly a uniform lower bound, which is new and essential) and a uniform lower estimate on the Gauss curvature. To prove that the support function $u(x,t)$

of a solution to (1.2) is uniformly bounded from below by a positive constant, we need to study an entropy functional $\mathcal{E}(\Omega_t)$ for the enclosed convex body Ω_t, which is different from Chow's [10].

Let Ω be a bounded closed convex body such that $0 \in \Omega \subset \mathbb{R}^{n+1}$ and $M := \partial\Omega$. For any $z_0 \in \Omega$, one can define the *support function with respect to* z_0 as

$$u_{z_0}(x) := \max_{z\in\Omega} \langle x, z - z_0\rangle.$$

Define an *entropy functional* $\mathcal{E}(\Omega)$ by

$$\mathcal{E}(\Omega) := \sup_{z_0\in \mathrm{Int}(\Omega)} \mathcal{E}(\Omega, z_0) \quad \text{with} \quad \mathcal{E}(\Omega, z_0) := \frac{1}{\omega_n}\int_{\mathbb{S}^n} \log u_{z_0}(x)\, d\theta(x).$$

Here ω_n is the area of $\mathbb{S}^n$, and $d\theta$ is the induced surface measure. (Later we shall show that given a non-degenerate, that is, full-dimensional, convex body, the entropy can in fact be attained by a positive support function.) It is easy to see that $\mathcal{E}(\Omega)$ is finite. The quantity $\mathcal{E}(\Omega, z_0)$ was introduced by Firey [14] for centrally symmetric convex bodies. The functional $\mathcal{E}(\Omega)$ was first considered by Andrews [4]. Our contribution here consists in deriving a lower estimate of the entropy via the Blaschke–Santaló inequality, which is a new way of using this functional in deriving the lower estimates of the support function and the Gauss curvature along the flow.

Since non-negativity is a defining property of the entropy concept in physics [13], the following result, as well as later monotonicity of $\mathcal{E}(\Omega)$ under the Gauss curvature flow, partially justifies the use of the terminology.

Proposition 1.1. *Let Ω be a bounded convex body in $\mathbb{R}^{n+1}$ with $V(\Omega) = V(B(1))$ (here $V(\Omega)$ denotes the volume of Ω). Let $z_s \in \Omega$ be the Santaló point of Ω. Let u_s be the support function with respect to z_s. Then*

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n} \log u_s \geq 0, \tag{1.6}$$

and equality holds if and only if Ω is a round ball centered at z_s. In particular $\mathcal{E}(\Omega) \geq 0$, and the inequality is strict unless Ω is a round ball centered at z_s. Moreover, for a general convex body Ω without the volume normalization, we have

$$\mathcal{E}(\Omega) \geq \frac{\log V(\Omega) - \log V(B(1))}{n+1}. \tag{1.7}$$

Before the proof, we recall the definition of the Santaló point of Ω. Given Ω and any $z_0 \in \mathrm{Int}(\Omega)$ define $\Omega^*_{z_0}$, the polar dual of Ω with respect to z_0, to be the set $\{y + z_0 \mid \max_{z\in\Omega}\langle y, z - z_0\rangle \leq 1\}$. The *Santaló point* is the unique point z_s such that $\Omega^*_{z_s}$ has the minimum volume among all polar duals with $z_0 \in \Omega$ (in fact it suffices to consider $z_0 \in \mathrm{Int}(\Omega)$, the interior of Ω). When z_s is the Santaló point we also denote $\Omega^*_{z_s}$ by Ω^*_s, and let Ω_s be the translation of Ω by $-z_s$.

Proof of Proposition 1.1. Let Ω_s^* be the polar dual of Ω with respect to z_s, the Santaló point. Its volume can be computed [17] as

$$V(\Omega_s^*) = \frac{1}{n+1}\int_{\mathbb{S}^n} \frac{1}{u_s^{n+1}}\, d\theta.$$

Then by Jensen's inequality ,

$$V(\Omega_s^*) \geq \frac{\omega_n}{n+1}\exp\left(\frac{1}{\omega_n}\int_{\mathbb{S}^n} \log\left(\frac{1}{u_s^{n+1}}\right) d\theta\right) = V(B(1))\exp\left(-\frac{n+1}{\omega_n}\int_{\mathbb{S}^n}\log u_s\right).$$

Since $V(\Omega) = V(B(1))$, together with the Blaschke–Santaló inequality [18]

$$V(\Omega)V(\Omega_s^*) \leq V(B(1))^2 \tag{1.8}$$

we obtain

$$V(B(1))\exp\left(-\frac{n+1}{\omega_n}\int_{\mathbb{S}^n}\log u_s\right) \leq V(\Omega_s^*) \leq \frac{V(B(1))^2}{V(\Omega)}.$$

This implies (1.6). The estimate (1.7) follows similarly. If equality holds, Jensen's inequality in the first step of the proof is an equality. Since e^x is strictly convex, $1/u_s^{n+1}$ and hence u_s must be a constant. The latter constant must be 1 as $V(\Omega_s^*) = V(B(1))$. Hence Ω is a ball centered at z_s.

From the above it is easy to see $\mathcal{E}(\Omega) \geq \mathcal{E}(\Omega, z_s) \geq 0$. Furthermore $\mathcal{E}(\Omega) = 0$ implies that Ω is a ball centered at z_s. □

A refined estimate on the monotonicity of entropy along the flow, as well as estimates of geometric quantities in terms of entropy (as well as the volume of the enclosed body), play a basic role in the proof of the main theorem. The strategy is the following. We first bound rough geometric quantities such as the outer and inner radius in terms of entropy, which in turn can be estimated via refined monotonicity. Then a compactness estimate and a stability estimate, as well as a refined monotonicity estimate, give the desired lower bound of the support function. The lower estimate on the support function seems more subtle and useful than the upper estimate (obtained by Hamilton). By combining the lower estimate on the support function and Chow's Harnack estimate, an iteration argument gives a uniform lower estimate on the Gauss curvature. Once the uniform lower estimate on the Gauss curvature is established, the full regularity follows from previous work of [6], [16] on fully nonlinear parabolic equations.

As a by-product of the monotonicity of the entropy functional we deduce that for any Ω with normalized volume, if z_∞ is the shrinking limit of the Gauss curvature flow then $\int_{\mathbb{S}^n}\log u_{z_\infty} \geq 0$. Hence Firey's entropy with respect to the shrinking limit is nonnegative for any convex body. This yields a geometric property of the shrinking limit. The above mentioned upper bounds of Hamilton on the diameter and the Gauss curvature can also be derived from the uniform lower bound on $u(x,t)$ proved here.

There remains an interesting question whether or not the sphere is the only compact soliton with positive Gauss curvature. Here we prove that the unit sphere is stable among the admissible variations. We also show that for the solitons with the normalized enclosed volume, there exists a sharp lower estimate on the volume of the dual body, which implies Firey's uniqueness among solitons with *central symmetry*. In the joint paper [7] the method of this paper is generalized to flows by powers of Gauss curvature.

2. Basic properties of entropy

We start with a geometric interpretation of the entropy functional, which also implies the non-negativity of the entropy. For any $z_0 \in \Omega$, by the definition, the dual body $\Omega^*_{z_0}$ is defined by the equation

$$\Omega^*_{z_0} - z_0 = \{w \mid \langle w, z - z_0\rangle \le 1, \forall z \in \Omega\}.$$

Writing w in polar coordinates we obtain

$$\Omega^*_{z_0} - z_0 = \{(r, x) \mid r u_{z_0}(x) \le 1\}. \tag{2.1}$$

Here $u_{z_0}(x)$ is the support function of Ω with respect to z_0. This in particular implies that

$$V(\Omega^*_{z_0}) = \int_0^{1/u_{z_0}(x)} \int_{\mathbb{S}^n} r^n \, d\theta \, dr = \frac{1}{n+1} \int_{\mathbb{S}^n} \frac{1}{u_{z_0}^{n+1}} \, d\theta.$$

If we normalize the volume of Ω to be that of the unit ball, the Blaschke–Santaló inequality implies that there exists $z_0 \in \Omega$ such that $|\Omega^*_{z_0}| \le V(B(1))$. If Ω is not affine equivalent to the unit ball, such z_0's form an open subset. Now observe the following geometric interpretation of the quantity $\int_{\mathbb{S}^n} \log u_{z_0}$.

Proposition 2.1. *Let* $\Omega^0_{z_0} = \Omega^*_{z_0} - z_0$. *Then*

$$\int_{\mathbb{S}^n} \log u_{z_0}(x) \, d\theta(x) = \left(\int_{B(1)\setminus\Omega^0_{z_0}} - \int_{\Omega^0_{z_0}\setminus B(1)} \right) \frac{1}{|w|^{n+1}} \, dw \ge V(B(1)) - V(\Omega^0_{z_0}).$$

Thus $\int_{\mathbb{S}^n} \log u_{z_0}$ *is a weighted (and signed) volume of* $\Omega^0_{z_0} \triangle B(1)$. *In particular, for any* z_0 *with* $|\Omega^*_{z_0}| \le |B(1)|$, *we have* $\int_{\mathbb{S}^n} \log u_{z_0}(x)\, d\theta(x) \ge 0$. *Moreover, if* $z_0 \in \mathrm{Int}(\Omega)$ *is such that* $\mathcal{E}(\Omega) = (1/\omega_n) \int_{\mathbb{S}^n} \log u_{z_0}$, *then*

$$\int_{\Omega^0_{z_0}} \frac{w}{|w|^{n+1}} \, dw = 0.$$

Thus z_0 *is the center of mass of* $\Omega^*_{z_0}$ *with respect to the weighted measure* $dw/|w|^{n+1}$.

Proof. The argument is similar to the above calculation of the dual body volume:

$$\begin{aligned}
\int_{\mathbb{S}^n} \log u_{z_0}(x) \, d\theta(x) &= -\int_{\mathbb{S}^n} \int_1^{1/u_{z_0}(x)} \frac{1}{r} \, dr \, d\theta(x) \\
&= \left(\int_{\{u_{z_0}(x) \ge 1\} \subset \mathbb{S}^n} \int_{1/u_{z_0}(x)}^1 - \int_{\{u_{z_0}(x) < 1\} \subset \mathbb{S}^n} \int_1^{1/u_{z_0}(x)} \right) \frac{1}{|w|^{n+1}} \, dw \\
&= \left(\int_{B(1)\setminus\Omega^0_{z_0}} - \int_{\Omega^0_{z_0}\setminus B(1)} \right) \frac{1}{|w|^{n+1}} \, dw.
\end{aligned}$$

This proves the first identity. The inequality holds since $1/|w|^{n+1} \geq 1$ on $B(1) \setminus \Omega^0_{z_0}$ and $1/|w|^{n+1} \leq 1$ on $\Omega^0_{z_0} \setminus B(1)$.

The last claim can be proved by a similar calculation. □

The following lemma asserts that there exists a unique point $z_e \in \Omega$ such that the entropy $\mathcal{E}(\Omega)$ is attained. It will be called the *entropy point*.

Lemma 2.1. *Given a closed convex body* Ω, *there exists a unique* $z_e \in \Omega$ *such that* $\mathcal{E}(\Omega) = (1/\omega_n)\int_{\mathbb{S}^n} \log u_{z_e}$.

Proof. The quantity $(1/\omega_n)\int_{\mathbb{S}^n} \log u_{z_0}$ is a function of $-z_0 = (t_1, \ldots, t_{n+1})$, say

$$F(t) = \frac{1}{\omega_n}\int_{\mathbb{S}^n} \log\Big(u(x) + \sum_{i=1}^{n+1} t_i x_i\Big)\, d\theta(x).$$

It is easy to see that the convexity of Ω implies that $u_{z_0} \geq 0$ for any $z_0 \in \Omega$, and $F(t)$ is a strictly concave function of t. Let $\{p_n\}$ be a sequence such that $(1/\omega_n)\int_{\mathbb{S}^n} \log u_{p_n} \nearrow \mathcal{E}(\Omega)$ as $n \to \infty$. Without loss of generality we may assume that $p_n \to p$. Then by Fatou's lemma, noting that $\log u_z(x) \leq \log \operatorname{diam}(\Omega)$ for any z and $\log u_{p_n}(x) \to \log u_p(x)$, we have

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n} -\log u_p \leq \frac{1}{\omega_n} \liminf_{n\to\infty} \int -\log u_{p_n} = -\mathcal{E}(\Omega).$$

On the other hand, by the definition $(1/\omega_n)\int_{\mathbb{S}^n} \log u_p \leq \mathcal{E}(\Omega)$. Hence $(1/\omega_n)\int_{\mathbb{S}^n} \log u_p = \mathcal{E}(\Omega)$. The uniqueness follows from the strict concavity of $F(t)$ (as a function of $t \in \mathbb{R}^{n+1}$) and the convexity of Ω. □

We also denote u_{z_e} by u_e. The next lemma strengthens the above result by asserting that in fact $z_e \in \operatorname{Int}(\Omega)$.

Lemma 2.2. *If* Ω *is a bounded convex domain with* $\operatorname{Int}(\Omega) \neq \emptyset$, *then* $\mathcal{E}(\Omega)$ *is attained by a unique support function* $u_e > 0$ *such that*

$$\int_{\mathbb{S}^n} \frac{x_j}{u_e(x)}\, d\theta(x) = 0. \tag{2.2}$$

Moreover for any other support function $u \neq u_e$, $\mathcal{E}(\Omega) > (1/\omega_n)\int_{\mathbb{S}^n} \log u$.

Proof. The main claim here is that $u_e > 0$ everywhere. Assuming this, (2.2) follows easily by the first variation. Namely, we express any support function as

$$u(x) = u_e(x) + \sum_{j=1}^{n+1} t_j x_j.$$

By the maximum property of u_e, the first variation yields

$$\int_{\mathbb{S}^n} \frac{x_j}{u_e(x)}\, d\theta(x) = 0.$$

Suppose $u_e(x_0) = 0$ for some $x_0 \in \mathbb{S}^n$. Then by the convexity of Ω it is easy to see that z_e must be on the boundary of Ω. We may assume $z_e = 0$, the origin. Now we *claim* that there is a support hyperplane of Ω at the origin with outer normal η such that the line segment

$$L = \{-t\eta \mid 0 < t < t_0\} \text{ is inside } \Omega, \text{ for some small } t_0. \tag{2.3}$$

We now prove this claim.[1] First recall that for any $p \in \Omega$, the tangent cone $T_p^{\mathcal{C}}\Omega$ is defined as $\{\xi \mid \langle \xi, p - z_1\rangle \geq 0 \text{ for any } z_1 \text{ with } \mathrm{dist}(z_1, \Omega) = |z_1 - p|\}$. The (outward) normal cone $\mathcal{N}_p(\Omega)$ is then defined as $\{\eta \mid \langle \eta, \xi\rangle \leq 0 \text{ for all } \xi \in T_p^{\mathcal{C}}\Omega\}$. Now it is rather elementary to see that for any support hyperplane H at p, which can be expressed as the zero set of $f(z) = \langle \eta, z - p\rangle$ with $f(z) \leq 0$ for all $z \in \Omega$, we have $\eta \in \mathcal{N}_p(\Omega)$. Thus the outer normal of any support hyperplane must lie inside the normal cone. To prove the claim it suffices to show that $-\mathcal{N}_p(\Omega)$ intersects $\mathrm{Int}(\Omega)$, due to the convexity of Ω. If $-\mathcal{N}_p(\Omega) \cap \mathrm{Int}(\Omega) = \emptyset$, by the separation theorem [18, Theorem 1.3.8] there exists a hyperplane H through the origin which separates $\mathrm{Int}(\Omega)$ and $-\mathcal{N}_p(\Omega)$. This hyperplane must be a support hyperplane. But its outer normal η (with respect to Ω) lies inside $\mathcal{N}_p(\Omega)$. Hence $-\eta \in -\mathcal{N}_p(\Omega)$. This is a contradiction since $-\mathcal{N}_p(\Omega)$ is on the other (outward) side of H than Ω. The claim (2.3) also follows from [8, Theorem 1.12].

We may, without loss of generality, assume that $\eta = (0, \ldots, 0, 1)$, so the north pole has the property that the associated line segment L defined in (2.3) lies inside $\mathrm{Int}(\Omega)$. Hence Ω is contained in the half-space $x_{n+1} \leq 0$ and touches the hyperplane at the origin. For any $x = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{S}^n$ with $x_{n+1} \geq 0$, let $N(x) = (x_1, \ldots, x_n, -x_{n+1})$ be its symmetric image with respect to $x_{n+1} = 0$. By definition, $u_e(x) = \sup_{z\in\Omega}\langle z, x\rangle$. Since Ω is closed, for each $x \in \mathbb{S}^n$ there is $z(x) \in \Omega$ such that $u_e(x) = \langle z(x), x\rangle$. Hence

$$u_e(N(x)) \geq \langle z(x), N(x)\rangle \geq \langle z(x), x\rangle = u_e(x), \qquad \forall x \in \mathbb{S}^n \text{ with } x_{n+1} \geq 0.$$

Here $\langle z(x), \eta\rangle \leq 0$ is used. Since $z_e = 0$ and $u_e(\eta) = 0$ and obviously $u_e(N(\eta)) > 0$, the above inequality is strict for some $x \in \mathbb{S}^n$ forming a set of positive measure. Consider the new support function $u_s(x) = u_e(x) + s x_{n+1}$. As the line segment L defined in (2.3) lies in the interior of Ω, $u_s(x) > 0$ for all $x \in \mathbb{S}^n$ and $0 < s < t_0$. On the other hand,

$$\begin{aligned} \frac{d}{ds}\left(\int_{\mathbb{S}^n} \log u_s\right)\Big|_{s=0} &= \int_{\mathbb{S}^n} \frac{x_{n+1}}{u_e(x)} = \int_{\{x_{n+1}>0\}} \frac{x_{n+1}}{u_e(x)} + \int_{\{x_{n+1}<0\}} \frac{x_{n+1}}{u_e(x)} \\ &= \int_{\{x_{n+1}>0\}} \left(\frac{x_{n+1}}{u_e(x)} - \frac{x_{n+1}}{u_e(N(x))}\right) > 0, \end{aligned}$$

which contradicts the definition of u_e. Therefore, $u_e(x) > 0$ for all $x \in \mathbb{S}^n$.

The last claim follows from the strict concavity of $F(t)$ defined in the proof of Lemma 2.1. □

In the rest of this section we derive some geometric estimates in terms of entropy. Let $\rho_+(\Omega)$ $[\rho_-(\Omega)]$ be the outer [inner] radius of a convex body Ω. By definition, the *outer*

[1] We would like thank Gaoyong Zhang to communicating us the proof of claim (2.3).

radius is the radius of the smallest ball which contains Ω, and the *inner radius* is the radius of the biggest ball which is enclosed by Ω. There is also a *width function* $w(x)$ which is defined as $u_{z_0}(x) + u_{z_0}(-x)$, where u_{z_0} is the support function with respect to z_0. It is clear that $w(x)$ is independent of the choice of z_0. Let w_+ and w_- denote the maximum and minimum of $w(x)$. The following estimates have been known [2]:

$$\rho_+ \leq \frac{w_+}{\sqrt{2}}, \qquad \rho_- \geq \frac{w_-}{n+2}. \tag{2.4}$$

Below we prove a result relating these geometric quantities to entropy.

Corollary 2.1. *For a convex body* Ω,

$$\max\{w_+, \rho_+(\Omega)\} \leq C_n e^{\mathcal{E}(\Omega)}, \tag{2.5}$$

where C_n *is a dimensional constant. There is also a lower estimate:*

$$\min\{\rho_-(\Omega), w_-\} \geq C'_n V(\Omega) e^{-n\mathcal{E}(\Omega)}, \tag{2.6}$$

where C'_n *is another dimensional constant.*

Proof. The upper estimate can be reduced to the corresponding upper estimate of w_+ in view of (2.4). Assume that $w(x_0) = w_+$. Without loss of generality we may assume that $u_{z_0}(x_0) \geq u_{z_0}(-x_0)$, $z_0 = 0$. Hence $w_+ \leq 2u_0(x_0)$. Assume that $u_0(x_0) = \langle z_1, x_0\rangle$ for $z_1 \in \partial\Omega$. Applying a rotation we may also assume that $z_1 = (0, \ldots, 0, a\rangle$, with $a = |z_1|$. Then $w_+ \leq 2a$. By convexity, the line segment tz_1 (with $0 \leq t \leq 1$) lies inside Ω. It is also clear that the support function for this segment with respect to $z_1/2$ is $\frac{1}{2}|\langle z_1, x\rangle|$. Hence it is bounded from above by $u_{z_1/2}(x)$. Therefore

$$\begin{aligned} \omega_n \log a - \omega_n \log 2 + \int_{\mathbb{S}^n} \log |x_{n+1}|\, d\theta(x) &= \int_{\mathbb{S}^n} \log \tfrac{1}{2}|\langle z_1, x\rangle|\, d\theta(x) \\ &\leq \int_{\mathbb{S}^n} \log u_{z_1/2}\, d\theta(x) \leq \omega_n \mathcal{E}(\Omega). \end{aligned}$$

Notice that the integral on the left hand side depends only on n. This gives an upper bound of a, hence an estimate for w_+. A lower bound on ρ_- can be derived from this and the observation that Ω can be enclosed in a cylinder with base a ball of radius ρ_+, and of height $2w_-$. Hence

$$n\omega_{n-1}\rho_+^n \cdot 2w_- \geq V(\Omega).$$

The lower bound of ρ_- follows from the estimate of ρ_- in terms of w_-. □

3. Gauss curvature flow and entropies

First we recall the relation between the embedding $X : M \to \mathbb{R}^{n+1}$ of M, a closed convex hypersurface in $\mathbb{R}^{n+1}$, and the related support function $u : \mathbb{S}^n \to \mathbb{R}$ of the enclosed convex body Ω (here we assume that $0 \in \Omega$ and $u(x) = u_0(x)$):

$$u(x) = \langle z, X(\nu^{-1}(z))\rangle$$

where $\nu : M \to \mathbb{S}^n$ is the Gauss map. For convenience we also denote $X(\nu^{-1}(x))$ by $X(x)$ (so $X(x)$, for $x \in \mathbb{S}^n$, denotes the embedding reparametrized via the Gauss map). The following equations are well-known [2]:

$$X(x) = u(x) \cdot x + \bar{\nabla} u, \tag{3.1}$$

$$(W^{-1})^i_j = \bar{g}^{ik}(\bar{\nabla}_k \bar{\nabla}_j u + u\bar{g}_{kj}). \tag{3.2}$$

Here $W = d\nu$ is the Weingarten map, $\bar{\nabla}$ is the covariant derivative of $\mathbb{S}^n$ with respect to the standard induced metric $\bar{g}$ as the boundary of the unit ball in $\mathbb{R}^{n+1}$. It is clear from (3.1) that changing the reference point z_0 in the support function amounts to translating by $-z_0$ the embedding $X(x)$, and (3.2) implies that the Weingarten map W is independent of the choice of z_0. Let $K(x) = \det(W)$ be the Gauss curvature.

First we derive the following estimate on Chow's entropy [10] in terms of the entropy defined in the last section.

Proposition 3.1. *Let Ω be a convex body with smooth boundary $M = \partial\Omega$ and volume $V(\Omega) = V(B(1))$. Let K be the Gauss curvature of M. Then*

$$\mathcal{E}_C(\Omega) := \frac{1}{\omega_n}\int_M K \log K \, d\sigma \geq \mathcal{E}(\Omega) \geq 0. \tag{3.3}$$

Here $d\sigma$ is the induced surface measure on M. Moreover $\mathcal{E}_C(\Omega) = \mathcal{E}(\Omega)$ if and only if $K = u_e$, and $\mathcal{E}_C(\Omega) = 0$ if and only if $\Omega = B(1)$, the unit ball. For general Ω,

$$\mathcal{E}_C(\Omega) \geq \mathcal{E}(\Omega) - \log\left(\frac{V(\Omega)}{V(B(1))}\right).$$

Proof. First observe that $\int_M K \log K \, d\sigma = \int_{\mathbb{S}^n} \log K \, d\theta$. On the other hand, recall

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{u}{K}\, d\theta = \frac{1}{\omega_n}\int_M \langle X, \nu\rangle \, d\sigma = \frac{n+1}{\omega_n} V(\Omega).$$

Hence the estimate via Jensen's inequality gives, in the case $V(\Omega) = V(B(1))$,

$$1 = \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{u}{K}\, d\theta = \frac{1}{\omega_n}\int \exp\left(\log\left(\frac{u}{K}\right)\right) d\theta \geq \exp\left(\frac{1}{\omega_n}\int_{\mathbb{S}^n} \log\left(\frac{u}{K}\right) d\theta\right).$$

This implies that

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n} \log K \, d\theta \geq \frac{1}{\omega_n}\int_{\mathbb{S}^n} \log u \, d\theta.$$

Since this estimate holds for support functions with respect to any $z_0 \in \Omega$, we have the claimed estimate. The equality case follows from Proposition 1.1. $\square$

Remark 3.1. The non-negativity of $\mathcal{E}_C$ also follows from the affine isoperimetric inequality [18]. An alternative argument below, using $x - 1 - \log x \geq 0$, proves a similar result with a weaker estimate

$$\frac{1}{\omega_n}((n+1)V(\Omega) - \omega_n) = \frac{1}{\omega_n}\int_{\mathbb{S}^n}\left(\frac{u}{K} - 1\right) d\theta(x) \geq \frac{1}{\omega_n}\int_{\mathbb{S}^n} \log\frac{u}{K}\, d\theta(x).$$

Hence $\mathcal{E}_C(\Omega) - \mathcal{E}(\Omega) \geq -V(\Omega)/V(B(1)) + 1$.

Corollary 3.2. *Let Ω and M be as in Proposition 3.1. Let $\sigma_k(W) = \sum_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k}$ be the k-th elementary symmetric function of (strictly speaking, eigenvalues $\{\lambda_i\}$ of) the Weingarten map. Then*

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n}\frac{k!(n-k)!}{n!}\sigma_k(W)\,d\theta \geq 1, \qquad \frac{1}{\omega_n}\int_{\mathbb{S}^n}\frac{k!(n-k)!}{n!}K\sigma_k(W)\,d\theta \geq 1. \tag{3.4}$$

Equality holds in any of these inequalities if and only if $\Omega = B(1)$.

In terms of the support function, the Gauss curvature flow (1.1) can be expressed as

$$\frac{\partial u(x,t)}{\partial t} = -\frac{1}{\det(\bar{g}^{ik}(\bar{\nabla}_k\bar{\nabla}_j u + u\bar{g}_{kj}))}. \tag{3.5}$$

Since the convexity of M_t is preserved along the flow (1.1), the equation (3.5) in terms of the support function u always makes sense. In [19] the existence of (1.1) has been proved and it was also shown that the flow will contract a convex hypersurface to a limiting point z_∞. The main concern here is to understand what is the limiting shape of the evolving hypersurfaces M_t. To understand the asymptotic behavior of the flow we also consider the normalized flow:

$$\frac{\partial u(x,t)}{\partial t} = u(x,t) - \frac{1}{\det(\bar{g}^{ik}(\bar{\nabla}_k\bar{\nabla}_j u + u\bar{g}_{kj}))}, \tag{3.6}$$

which preserves the enclosed volume $V(\Omega_t)$, provided that initially $V(\Omega_0) = V(B(1))$. By scaling (multiplying the support function u by a factor e^τ) and reparametrization ($\tau = -\frac{1}{n+1}\log\left(\frac{T-t}{T}\right)$, with T being the terminating time, which equals $\frac{1}{n+1}$ under the above normalization, and relabeling τ as t afterwards), the support function with respect to z_∞ yields a long time *positive* solution to (3.6). Hence the study of the limiting shape is equivalent to finding the asymptotic of (3.6). When Ω is centrally symmetric it was shown by Firey that the solution of (3.6) converges to a round sphere. In dimension $n = 2$, Andrews [5] proved the same result for any convex surfaces in $\mathbb{R}^3$.

In the following we show that the entropy $\mathcal{E}(\Omega)$ is intimately related to the normalized Gauss curvature flow (3.6). First note that the equilibrium for (3.6) satisfies the equation

$$u(x,t)\cdot\det(\bar{g}^{ik}(\bar{\nabla}_k\bar{\nabla}_j u + u\bar{g}_{kj})) = 1. \tag{3.7}$$

Such a solution is also called a *shrinking soliton* of the Gauss curvature flow.

We now consider the first variation of $\mathcal{E}(\Omega)$ under the constraint $V(\Omega) = V(B(1))$. For a fixed Ω, by Lemma 2.2, there exists a unique $z_e \in \mathrm{Int}(\Omega)$ such that $\mathcal{E}(\Omega) = (1/\omega_n)\int_{\mathbb{S}^n}\log u_e(x)\,d\theta(x)$. Moreover u_e satisfies

$$\int_{\mathbb{S}^n}\frac{x_j}{u_e}\,d\theta(x) = 0, \qquad \forall j = 1,\ldots,n+1. \tag{3.8}$$

Let Ω_η be a family of convex bodies such that $\Omega_0 = \Omega$. In terms of support functions, we have a family of functions $v_\eta \in C^2(\mathbb{S}^n)$ such that

$$A_\eta = ((v_\eta)\delta_{ij} + (v_\eta)_{ij}) > 0.$$

We assume in addition that v_η satisfies (3.8). Hence $\mathcal{E}(\Omega_\eta) = (1/\omega_n)\int_{\mathbb{S}^n} \log v_\eta \, d\theta$. Write $v_\eta(x) = u_e(x) + \rho(\eta, x)$, where $\rho(0, x) = 0$ for all $x \in \mathbb{S}^n$. Below we abbreviate v_η by v, and u_e by u. As before, the constraint $V(\Omega_\eta) = V(B(1))$ implies

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n} v \det(A_v) = 1. \tag{3.9}$$

Recall that we also have

$$\int_{\mathbb{S}^n} \frac{x_j}{v} = 0, \qquad \forall j = 1, \dots, n+1,\ \forall \eta, \tag{3.10}$$

$$\mathcal{E}(\Omega_\eta) = \frac{1}{\omega_n}\int_{\mathbb{S}^n} \log v. \tag{3.11}$$

Proposition 3.2. *If u, the unique support function which achieves the entropy, is a critical point of $\mathcal{E}(\Omega)$, viewed as a functional of Ω, under the constraint that $V(\Omega) = V(B(1))$, it must be a solution to (3.7), that is, a shrinking soliton. Thus a critical point to $\mathcal{E}(\Omega)$ must be a shrinking soliton to the Gauss curvature flow. Moreover, the converse is also true.*

Proof. Differentiate (3.9) and (3.11) in η and then set $\eta = 0$. By the Lagrangian multiplier method, for any critical point u there exists a $\lambda \in \mathbb{R}$ such that (in view of (3.9))

$$\int_{\mathbb{S}^n} \rho' \det(A_u) = \lambda \int_{\mathbb{S}^n} \frac{\rho'}{u}, \quad \forall \rho', \quad \text{with} \quad \int_{\mathbb{S}^n} \frac{\rho' x_j}{u^2} = 0,\ \forall j = 1, \dots, n+1. \tag{3.12}$$

Here we have used the fact that $\frac{\partial \det(A_u)}{\partial A_{ij}} \bar{\nabla}_i \bar{\nabla}_j$ is self-adjoint. Let $\mathcal{N}_u = \operatorname{span}\{x_j/u \mid j = 1, \dots, n+1\}$. Note that

$$\int_{\mathbb{S}^n} \det(A_u) x_j = \int_{\partial\Omega} \langle \nu, e_j \rangle = 0.$$

Since both $u(\det(A_u) - \lambda/u)$ and ρ'/u belong to $\mathcal{N}_u^\perp$, and ρ'/u is arbitrary in $\mathcal{N}_u^\perp$ and $u > 0$, we must have

$$\det(A_u) = \lambda/u. \tag{3.13}$$

As $V(\Omega) = V(B(1))$, we conclude that $\lambda = 1$. To check the converse, from (3.9) we conclude that

$$\int_{\mathbb{S}^n} \rho' \det(A_u) = 0,$$

which readily implies that $\int_{\mathbb{S}^n} \rho'/u = 0$. ◻

The next result gives a lower estimate on the volume of Ω_0^*, the dual of Ω with respect to the origin, when Ω (more precisely u, the support function with respect to the origin) is a soliton of the Gauss curvature flow.

Proposition 3.3. *Assume u is a soliton with associated body Ω (so $u = K$). Then:*

(i) *The origin is the entropy point of Ω and $V(\Omega) = V(B(1))$.*
(ii) *The volume of Ω_0^* satisfies*

$$V(\Omega_0^*) \ge V(B(1)). \tag{3.14}$$

In particular, if the origin is the Santaló point of Ω then $\Omega = B(1)$.

Proof. Observe that for any $1 \leq j \leq n+1$ we have $0 = \int_M \langle \nu(z), e_j \rangle \, d\sigma = \int_{\mathbb{S}^n} (x_j/K) \, d\theta(x)$, which implies that $\int_{\mathbb{S}^n} (x_j/u) \, d\theta(x) = 0$. This shows that the origin is the entropy point. Similarly $V(\Omega) = \frac{1}{n+1} \int_M \langle X, \nu \rangle = \frac{1}{n+1} \int_{\mathbb{S}^n} \frac{u}{K} = V(B(1))$.

Let $X(x) = u(x)x + \bar{\nabla} u(x)$ be the position vector of M_t. Observe that for any support function u of a convex body,

$$\frac{1}{\omega_n} \int_{\mathbb{S}^n} \frac{u}{K|X|^{n+1}} \, d\theta(x) = \frac{1}{\omega_n} \int_{\partial \Omega} \frac{\langle X, \nu \rangle}{|X|^{n+1}} \, d\sigma = \frac{1}{\omega_n} \int_{\partial B(\epsilon)} \frac{1}{\epsilon^n} \, d\sigma = 1.$$

Here we have used the fact that $\operatorname{div}(X/|X|^{n+1}) = 0$. The claimed lower estimate on the dual volume follows, as

$$\frac{V(\Omega_0^*)}{\omega_n} = \frac{1}{n+1} \fint_{\mathbb{S}^n} \frac{1}{u^{n+1}} \geq \frac{1}{n+1} \fint_{\mathbb{S}^n} \frac{u}{K|X|^{n+1}} = \frac{1}{n+1}.$$

The last statement follows, since when the origin is the Santaló point, $V(\Omega_0^*) \leq V(B(1))$ by the Blaschke–Santaló inequality, hence equality holds in the above estimates. In particular, it implies that $|X| = u$ and $\bar{\nabla} u = 0$, so u is a constant. □

Remark 3.3. One can also prove the estimate (3.14) using the isoperimetric inequality $\fint_{\mathbb{S}^n} (1/K) \, d\theta \geq 1$.

For the normalized Gauss curvature flow (3.6), Chow [10] proved that $\mathcal{E}_C(\Omega_t)$ is non-increasing in t. A refinement of the following theorem (Lemma 4.3) is of fundamental importance to the C^0-estimate. The monotonicity of $\mathcal{E}(\Omega_t)$ first appeared in [4, Corollary 9].

Theorem 3.4. *Along the flow* (3.6) *the entropy $\mathcal{E}(\Omega_t)$ is non-increasing. Moreover, for any $t_1 \leq t_0$,*

$$\mathcal{E}(\Omega_{t_0}) - \mathcal{E}(\Omega_{t_1}) \leq \int_{t_1}^{t_0} (\mathcal{E}(\Omega_t) - \mathcal{E}_C(\Omega_t)) \, dt \leq 0. \tag{3.15}$$

Proof. Assume that $\mathcal{E}(\Omega_{t_0}) = (1/\omega_n) \int_{\mathbb{S}^n} \log u_{e(t_0)}$ at some point t_0, where $u_{e(t_0)}$ is the support function with respect to the unique entropy point $z_e(t_0) \in \operatorname{Int}(\Omega)$. Hence for $t < t_0$ but very close to t_0, one still has $u_{e(t)}(x,t) := u(x,t) - \langle \exp(t - t_0) z_e(t_0), x \rangle > 0$. If $u(x,t)$ is a solution to (3.6), so is $u_{e(t)}(x,t)$. Now we calculate

$$\begin{aligned} \frac{d}{dt} \fint_{\mathbb{S}^n} \log u_{e(t)}(x,t) &= \fint_{\mathbb{S}^n} \frac{u_{e(t)} - K}{u_{e(t)}} = 1 - \fint_{\mathbb{S}^n} \frac{K}{u_{e(t)}} \\ &= -\fint_{\mathbb{S}^n} \left(\sqrt{\frac{K}{u_{e(t)}}} - \sqrt{\frac{u_{e(t)}}{K}} \right)^2 \leq 0. \end{aligned}$$

This implies that there exists $\delta > 0$ such that for $t \in (t_0 - \delta, t_0)$,

$$\mathcal{E}(\Omega_t) \geq \fint_{\mathbb{S}^n} \log u_{e(t)}(x,t) \geq \fint_{\mathbb{S}^n} \log u_{e(t_0)}(x,t_0) = \mathcal{E}(\Omega_{t_0}),$$

which proves the first claim. Making use of the above calculation again we arrive at

$$\mathcal{E}(\Omega_{t_0}) - \mathcal{E}(\Omega_{t_1}) \leq \int_{t_1}^{t_0} \fint_{\mathbb{S}^n} \left(1 - \frac{K}{u_{e(t)}}\right) d\theta\, dt.$$

Using $1 - x \leq -\log x$ and some elementary estimates, we establish (3.15) for $t_1 \in (t_0 - \delta, t_0)$. The continuity argument can be applied to conclude the same for all $t_1 \leq t_0$. $\square$

The proof above is a modification of that of Firey [14], in which he introduced the entropy $\mathcal{E}_F(\Omega_t) = \fint_{\mathbb{S}^n} \log u(x,t)$ for the centrally symmetric case and showed that it is non-increasing along the flow. Now we have $\mathcal{E}_C(\Omega_t) \geq \mathcal{E}(\Omega_t) \geq \mathcal{E}_F(\Omega_t)$.

4. C^0-estimates

Let $u(x,t)$ be a long time solution to (3.6). By translation we may assume that $z_\infty = 0$. Combining Corollary 2.1 and Theorem 3.4 we obtain an upper bound of ρ_+, hence an upper bound of $u(x,t)$, and a lower bound on ρ_-. An upper estimate of $u(x,t)$ was first proved by Hamilton [15] using a different argument.

The main result of this section is a uniform lower bound of $u(x,t)$. Since we assume that z_∞, the limit point which lies inside all Ω_t evolving by (3.5), is the origin, we have a solution $u(x,t)$ to (3.6) with $u(x,t) > 0$ for all $(x,t) \in \mathbb{S}^n \times [0,\infty)$. As ρ_- is bounded from below, if one is willing to shift the origin, a lower bound of the support function would follow. The subtle point here is to bound the support function from below without shifting for all t.

We start with a similar lower bound for the support function with respect to the Santaló point, which motivates the C^0-estimates. This is based on the following gradient estimate on the support function u of a convex body:

$$\max_{\mathbb{S}^n} |\bar\nabla u| \leq \max_{\mathbb{S}^n} u. \tag{4.1}$$

This gradient estimate can be proved by the following observation. Due to the positivity of $\bar\nabla_i \bar\nabla_j u + u\delta_{ij}$, one can conclude that $\bar\nabla u = 0$ at the maximum point of $|\bar\nabla u|^2 + u^2$. Hence $\max_{\mathbb{S}^n} |\bar\nabla u| \leq \max_{\mathbb{S}^n} u$. Geometrically this is clear since $X = \bar\nabla u + u\,x$ is the position vector with length square $|X|^2 = |\bar\nabla u|^2 + u^2$, which attains its maximum for some X_0 parallel to x.

Proposition 4.1. *If u_s is the support function with respect to the Santaló point of Ω, then*

$$u_s(x) \geq c(n) V(\Omega) e^{-n\mathcal{E}(\Omega)}, \tag{4.2}$$

where $c(n) > 0$ is a dimensional constant.

Proof. By the Blaschke–Santaló inequality,

$$V(\Omega_s^*) = \frac{1}{n+1} \int_{\mathbb{S}^n} \frac{1}{u_s^{n+1}} \leq \frac{V(B(1))^2}{V(\Omega)}.$$

Let $m = u_s(x_0)$ be the minimum value of u_s (attained at some x_0). By (4.1), $\max_{\mathbb{S}^n} |\bar{\nabla} u_s| \leq \max_{\mathbb{S}^n} u_s \leq 2\rho_+$. Therefore, in a geodesic ball $\bar{B}_{x_0}(r)$ (inside $\mathbb{S}^n$) with $r = m/\rho_+$, we have $u_s(x) \leq 2m$. In turn,

$$\frac{V(B(1))^2}{V(\Omega)} \geq \frac{1}{n+1} \int_{\mathbb{S}^n} \frac{1}{u_s^{n+1}} \geq \tilde{C}_n m^{-(n+1)} r^n = \tilde{C}_n \frac{\rho_+^{-n}}{m}.$$

The result now follows from Corollary 2.1. □

Now we prove the main result of this section, which is based on establishing a similar result for $u_{e(t)}$ where $e(t)$ is the entropy point of the convex body Ω_t.

Theorem 4.1. *Suppose $u(x,t) > 0$ is the solution of (3.6) with initial data $u(x,0) = u_0(x) > 0$, where $u_0(x)$ is the support function of Ω_0 with $V(\Omega_0) = V(B(1))$ and $\mathcal{E}(\Omega_0) \leq A$. Then there are $\epsilon = \epsilon(n, \mathcal{E}(\Omega_0)) > 0$ and $T_0 = T(\Omega_0)$ such that for $t \geq T_0$,*

$$u(x,t) \geq \epsilon, \qquad \forall t \geq 0,\ \forall x \in \mathbb{S}^n. \tag{4.3}$$

The proof is built upon several lemmas. For each bounded closed convex body Ω, we denote by $e(\Omega)$ the unique entropy point of Ω. For each $p \in \Omega$, let u_p be the support function of Ω with respect to p.

Lemma 4.1. *For each Ω, there is $D > 0$ depending only on n and the diameter of Ω such that for any $p \in \Omega$,*

$$\frac{1}{\omega_n} \int_{\mathbb{S}^n} \log u_p \leq \mathcal{E}(\Omega) - D\, \mathrm{dist}^2(p, e(\Omega)). \tag{4.4}$$

Proof. Since u_p is bounded from above by $2\rho_+$, $1/u_p$ is bounded from below. As in Lemma 2.1, consider $F(t) = (1/\omega_n) \int_{\mathbb{S}^n} \log u_p = (1/\omega_n) \int_{\mathbb{S}^n} \log(u_e + \langle x, e - p \rangle)$ with $t = e - p$. A direct calculation shows that

$$\frac{\partial^2 F(t)}{\partial t_i \partial t_j} = -\int_{\mathbb{S}^n} \frac{x_i x_j}{(u_e + \langle x, t \rangle)^2}\, d\theta(x).$$

By Taylor's theorem, if we write $t = |t| a$ with $a = \frac{e-p}{|e-p|}$, we have

$$F(t) \leq F(0) - C|t|^2 \int_{\mathbb{S}^n} \langle a, x \rangle^2\, d\theta(x).$$

Here C is a constant only depending on the upper bound of ρ_+. Now (4.4) follows from the fact that the integral on the right hand side is a constant depending only on n. □

Note that by Corollary 2.1, there exists an upper bound of ρ_+ depending only on A, the upper bound of the entropy.

For any $A, B > 0$, consider the collection of bounded closed convex sets

$$\Gamma_B^A = \{\Omega \subset \mathbb{R}^{n+1} \mid \Omega \text{ is a closed convex subset, } 0 \in \Omega,\ V(\Omega) \geq B,\ \mathcal{E}(\Omega) \leq A\}. \tag{4.5}$$

Lemma 4.2. *Suppose $\Omega_k \in \Gamma_B^A$ is a sequence of convex bodies with $0 \in \Omega_k$ for all k. Suppose $\lim_{k\to\infty} \Omega_k = \Omega_0$. Then*

$$\lim_{k\to\infty} \mathcal{E}(\Omega_k) = \mathcal{E}(\Omega_0).$$

Moreover, there is $\delta(A, B, n) > 0$, depending only on n, A, B, such that the entropy point e_Ω satisfies

$$\operatorname{dist}(e_\Omega, \partial\Omega) \geq \delta(A, B, n), \qquad \forall \Omega \in \Gamma_B^A. \tag{4.6}$$

Proof. By Lemma 2.1, for all $\Omega \in \Gamma_B^A$, $\rho_+(\Omega) \leq C(n, A)$ for some $C(n, A) > 0$. Since the volume is bounded from below, we also have, for all $\Omega \in \Gamma_B^A$, $\rho_-(\Omega) \geq c(n, A, B) > 0$. By Lemma 2.2, the entropy point e_{Ω_0} is in Ω_0. Therefore, $e_{\Omega_0} \in \Omega_k$ for k large. Again by Lemma 2.2,

$$\mathcal{E}(\Omega_0) = \frac{1}{\omega_n}\int_{\mathbb{S}^n} \log u_{e(\Omega_0)}^{\Omega_0} = \lim_{k\to\infty} \frac{1}{\omega_n}\int_{\mathbb{S}^n} \log u_{e(\Omega_0)}^{\Omega_k} \leq \lim_{k\to\infty} \mathcal{E}(\Omega_k). \tag{4.7}$$

Here $u_p^{\Omega_k}$ is the support function of Ω_k with respect to p.

On the other hand, since $u_p^{\Omega_k} \leq 2\rho_+(\Omega_k) \leq 2C(n, A)$ for each $p \in \Omega_k$, $\log u_p^{\Omega_k}$ is bounded from above. As $\Omega_k \in \Gamma_B^A$, by estimate (1.7), we have

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n} \log\left(\frac{u_{e(\Omega_k)}^{\Omega_k}}{2C(n, A)}\right) \geq \mathcal{E}(\Omega_k) - \log(2C(n, A)) \geq \frac{\log\left(\frac{B}{V(B(1))}\right)}{n+1} - \log(2C(n, A)).$$

That is,

$$\int_{\mathbb{S}^n} \left|\log\left(\frac{u_{e(\Omega_k)}^{\Omega_k}}{2C(n, A)}\right)\right| \leq C, \qquad \forall k. \tag{4.8}$$

Let $p = \lim_{k\to\infty} e(\Omega_k)$. Noticing that $\log\left(\frac{u_{e(\Omega_k)}^{\Omega_k}}{2C(n,A,B)}\right) \leq 0$, by Fatou's Lemma we get

$$\int_{\mathbb{S}^n} \log\left(\frac{u_p^{\Omega_0}}{2C(n, A, B)}\right) \geq \limsup_{k\to\infty} \int_{\mathbb{S}^n} \log\left(\frac{u_{e(\Omega_k)}^{\Omega_k}}{2C(n, A)}\right).$$

This yields

$$\mathcal{E}(\Omega_0) \geq \limsup_{k\to\infty} \mathcal{E}(\Omega_k). \tag{4.9}$$

Combining (4.7) and (4.9) proves the first claim of the lemma.

For the second part, suppose that (4.6) is not true. Then there is a sequence $\{\Omega_k\}$ $(\subset \Gamma_B^A)$ such that

$$\operatorname{dist}(e_{\Omega_k}, \partial\Omega_k) \to 0, \qquad k \to \infty.$$

By the Blaschke selection theorem [18, Theorem 1.8.6], there exists a subsequence of $\{\Omega_k\}$ in Γ_B^A, still denoted by Ω_k, that converges to a convex body Ω_0. Let $p = \lim_{k\to\infty} e(\Omega_k)$. By the assumption $\operatorname{dist}(e_{\Omega_k}, \partial\Omega_k) \to 0$, we have $p \in \partial\Omega_0$. The support function u_p of Ω_0 vanishes at p. By the first part of the lemma, $\mathcal{E}(\Omega_0) = \lim_{k\to\infty} \mathcal{E}(\Omega_k)$. Hence $\Omega_0 \in \Gamma_B^A$. Again, we argue as before using Fatou's Lemma:

$$\mathcal{E}(\Omega_0) = \lim_{k\to\infty} \mathcal{E}(\Omega_k) = \lim_{k\to\infty} \frac{1}{\omega_n} \int_{\mathbb{S}^n} \log u_{e(\Omega_k)}^{\Omega_k} \le \frac{1}{\omega_n} \int_{\mathbb{S}^n} \log u_p.$$

This contradicts Lemma 2.2. $\square$

Now consider the positive solution to (3.6). We first observe an easy consequence of the uniqueness.

Proposition 4.2. *For any given convex body Ω with normalized volume, there is at most one positive solution of* (3.6) *which exists on $\mathbb{S}^n \times [0, \infty)$ such that $u(x, 0)$ is a support function of Ω.*

Proof. Suppose v is another positive solution. Then at $t = 0$, $v(x, 0) = u(x, 0) - \sum_{i=1}^{n+1} a_i x_i$. It is easy to check $\tilde{v}(x, t) = u(x, t) - e^t \sum_{i=1}^{n+1} a_i x_i$ is a solution of the normalized Gauss curvature flow, so that $\tilde{v}_t = -K + \tilde{v}$. Since $A_{\tilde{v}} = A_u$, we have $\Psi(A_u) = \Psi(A_{\tilde{v}})$. Therefore, $\tilde{v} = v$. Hence if $a \neq 0$, v cannot be bounded! Therefore there exists only one positive solution to (3.6) on $\mathbb{S}^n \times [0, \infty)$. $\square$

For each Ω_t corresponding to $u(x, t)$, let $\mathcal{E}(t) := \mathcal{E}(\Omega_t)$. We know that $\mathcal{E}(t) \ge 0$ and $\mathcal{E}(t)$ is decreasing. Let $\mathcal{E}_\infty := \lim_{t\to\infty} \mathcal{E}(t)$.

Lemma 4.3. *Let $u(x, t)$ be the unique positive solution of* (3.6). *Then*

$$\fint_{\mathbb{S}^n} \log u(x, t) \ge \mathcal{E}_\infty + \int_t^\infty \fint_{\mathbb{S}^n} \left(\sqrt{\frac{K}{u}} - \sqrt{\frac{u}{K}} \right)^2, \quad \forall t \ge 0. \tag{4.10}$$

In particular, $\mathcal{E}(t) \ge \mathcal{E}_F(t) \ge \mathcal{E}_\infty$.

Proof. For each $T_0 >$ fixed, pick $T > T_0$. Let $a^T = (a_1^T, \ldots, a_{n+1}^T)$ be the entropy point of Ω_T. Set $u^T = u - e^{t-T} \sum_{i=1}^{n+1} a_i^T x_i$. It can be checked that

$$u_t^T = -K + u^T. \tag{4.11}$$

Since both the origin and the entropy point a^T are in $\mathrm{Int}(\Omega_T)$,

$$|a^T| \le 2\rho^+(t) \le C.$$

If T is large enough, $u^T(x, 0) > 0$ for all $x \in \mathbb{S}^n$. We also know that $u^T(x, T) > 0$ for all $x \in \mathbb{S}^n$ since the entropy point is an interior point of Ω_T. If $u^T(x_0, t_0) \le 0$ for some $0 < t_0 < T$ and $x_0 \in \mathbb{S}^n$, then (4.11) implies $u^T(x_0, t) < 0$ for all $t > t_0$, which contradicts $u^T(x, T) > 0$. Hence $u^T(x, t) > 0$ for all $0 \le t \le T$ and $x \in \mathbb{S}^n$. By (4.11), a similar calculation to that in Theorem 3.4 shows

$$\frac{d}{dt}\left(\fint_{\mathbb{S}^n} \log u^T(x, t) \right) = -\fint_{\mathbb{S}^n} \left(\sqrt{\frac{K(x, t)}{u^T(x, t)}} - \sqrt{\frac{u^T(x, t)}{K(x, t)}} \right)^2, \quad \forall\, 0 \le t \le T. \tag{4.12}$$

Hence

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n}\log u^T(x,0)-\mathcal{E}(T)=\frac{1}{\omega_n}\int_{t=0}^{T}\int_{\mathbb{S}^n}\left(\sqrt{\frac{K(x,t)}{u^T(x,t)}}-\sqrt{\frac{u^T(x,t)}{K(x,t)}}\right)^2.$$

Since $T_0<T$,

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n}\log u^T(x,0)-\mathcal{E}(T)\geq\frac{1}{\omega_n}\int_{t=0}^{T_0}\int_{\mathbb{S}^n}\left(\sqrt{\frac{K(x,t)}{u^T(x,t)}}-\sqrt{\frac{u^T(x,t)}{K(x,t)}}\right)^2.$$

Now let $T\to\infty$; as $u^T(x,t)\to u(x,t)$ uniformly for $0\leq t\leq T_0$, $x\in\mathbb{S}^n$, we obtain

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n}\log u(x,0)-\mathcal{E}_\infty\geq\frac{1}{\omega_n}\int_{t=0}^{T_0}\int_{\mathbb{S}^n}\left(\sqrt{\frac{K(x,t)}{u(x,t)}}-\sqrt{\frac{u(x,t)}{K(x,t)}}\right)^2. \tag{4.13}$$

Now (4.10), for $t=0$, follows directly from (4.13) since T_0 is arbitrary. If in the above we replace 0 by any $t\leq T$, we obtain (4.10). □

Lemma 4.3 has the following immediate consequence.

Corollary 4.2.

$$\lim_{t\to\infty}\mathcal{E}_C(\Omega_t)=\lim_{t\to\infty}\mathcal{E}(\Omega_t)=\mathcal{E}_\infty.$$

Proof. Since $\mathcal{E}_C(\Omega_t)\geq\mathcal{E}(\Omega_t)$, we have $\lim_{t\to\infty}\mathcal{E}_C(\Omega_t)\geq\mathcal{E}_\infty$. Assume that the inequality is strict. Then there exists $\delta>0$ such that for sufficiently large t_0 we have $\mathcal{E}_C(\Omega_t)-\mathcal{E}(\Omega_t)\geq\delta$ for $t\geq t_0$. This contradicts (3.15). □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Since $\mathcal{E}(\Omega_t)\to\mathcal{E}_\infty$, by (4.10),

$$\mathcal{E}_\infty\leq\frac{1}{\omega_n}\int_{\mathbb{S}^n}\log u(x,t)\leq\mathcal{E}(\Omega_t).$$

That is,

$$0\leq\mathcal{E}(\Omega_t)-\frac{1}{\omega_n}\int_{\mathbb{S}^n}\log u(x,t)\to 0\quad\text{ as } t\to\infty.$$

As u is the support function of Ω_t with respect to the origin, by Lemma 4.1, $e(\Omega_t)\to 0$ as $t\to\infty$. The claimed lower estimate now follows from (4.6) in Lemma 4.2. □

The proof effectively shows that there exists $C=C(\Omega_0,n)$ such that if $e(t)=e(\Omega_t)$ is the entropy point of Ω_t, then

$$|e(t)|^2\leq C\left(\mathcal{E}(t)-\fint_{\mathbb{S}^n}\log u(x,t)\right). \tag{4.14}$$

Finally the following corollary summarizes Corollary 2.1, Theorem 3.4 and Theorem 4.1.

Corollary 4.3. *Let $u(x,t)$ be as in Theorem 4.1. Then there exists $\Lambda=\Lambda(\Omega_0,n)>0$ such that*

$$1/\Lambda\leq u(x,t)\leq\Lambda. \tag{4.15}$$

5. C^2-estimates and convergence

In this section we derive uniform C^2-estimates from the C^0-estimate (4.15). The first is an upper estimate, which was first proved by Hamilton [15].

Theorem 5.1. *Suppose $u(x,t) \geq a > 0$ is the solution of (3.6) with initial data $u(x,0) = u_0(x)$, where $u_0(x) > 0$ is the support function of Ω_0 with $V(\Omega_0) = V(B(1))$. There exists a constant $C = C(a,n) > 0$ such that*

$$K(x,t) \leq C. \tag{5.1}$$

In the Appendix we include an alternative proof of this result using the lower estimate of $u(x,t)$. For the C^2-estimate the key is the following lower bound on the Gauss curvature $K(x,t)$.

Theorem 5.2. *Suppose $u(x,t) > 0$ is a positive solution of (3.6), obtained from the un-normalized flow (3.5), with initial data $u(x,0) = u_0(x)$, where $u_0(x) > 0$ is the support function of Ω_0 with $V(\Omega_0) = V(B(1))$. Then there exists a constant $\epsilon_1 = \epsilon(n,\Omega_0) > 0$ such that*

$$K(x,t) \geq \epsilon_1. \tag{5.2}$$

Proof. For this estimate, it is more convenient to work with the un-normalized flow (3.5). Let T be the terminating time (which is $\frac{1}{n+1}$ by our normalization). Then the claimed estimate is equivalent to

$$K(x,t)(T-t)^{n/(n+1)} \geq \epsilon_1. \tag{5.3}$$

For the proof we recall Theorem 3.7 of [10] under the Gauss map parametrization:

$$K(x,t)t^{n/(n+1)} \leq K(x,t')t'^{n/(n+1)} \tag{5.4}$$

for any $0 < t \leq t' < T$. Since it is sufficient to prove (5.3) for $t \geq T/2$, the estimate (5.4) implies that

$$K(x,t) \leq 2^{n/(n+1)} K(x,t'). \tag{5.5}$$

The two-sided C^0-estimate (4.15) implies that the un-normalized support function $u(x,t)$ satisfies

$$\frac{1}{\Lambda}(T-t)^{1/(n+1)} \leq u(x,t) \leq \Lambda(T-t)^{1/(n+1)}. \tag{5.6}$$

Let

$$\alpha = \left(\frac{1}{2\Lambda^2}\right)^{n+1}, \qquad h_j = \frac{T}{2}\alpha^j, \qquad t_j = T - h_j \quad \text{for } j = 0, 1, \ldots.$$

Clearly $t_j \to T$ as $j \to \infty$. The estimate (5.6) implies that

$$\begin{aligned} u(x,t_j) - u(x,t_{j+1}) &\geq \frac{1}{\Lambda}h_j^{1/(n+1)} - \Lambda h_{j+1}^{1/(n+1)} \\ &= \frac{1}{\Lambda}\left(\frac{T}{2}\right)^{1/(n+1)}\alpha^{j/(n+1)} - \Lambda\left(\frac{T}{2}\right)^{1/(n+1)}\alpha^{(j+1)/(n+1)} = \frac{1}{2\Lambda}h_j^{1/(n+1)}. \end{aligned} \tag{5.7}$$

The Gauss curvature flow equation implies that for any $t' < T$,

$$u(x, t') = \int_{t'}^{T} K(x, t)\, dt,$$

which in turn yields

$$u(x, t_j) - u(x, t_{j+1}) = \int_{t_j}^{t_{j+1}} K(x, t)\, dt. \tag{5.8}$$

Now we claim that there exists $s_j \in [t_j, t_{j+1}]$ such that

$$K(x, s_j)(T - s_j)^{n/(n+1)} \geq \frac{1}{4(n+1)\Lambda}. \tag{5.9}$$

Indeed, otherwise we would have

$$\begin{aligned}\int_{t_j}^{t_{j+1}} K(x, t)\, dt &\leq \frac{1}{4(n+1)\Lambda} \int_{t_j}^{t_{j+1}} (T - t)^{-n/(n+1)}\, dt \\ &= \frac{1}{4(n+1)\Lambda} \int_{h_{j+1}}^{h_j} \tau^{-n/(n+1)}\, d\tau \leq \frac{1}{4\Lambda} h_j^{1/(n+1)},\end{aligned}$$

contradicting (5.7) and (5.8).

Now the claimed estimate (5.3) can be derived from (5.9) and (5.5). First we claim that

$$K(x, t_{j+1})(T - t_{j+1})^{n/(n+1)} \geq \frac{1}{4(n+1)\Lambda} \left(\frac{\alpha}{2}\right)^{n/(n+1)}. \tag{5.10}$$

This can be proven via the estimates

$$\begin{aligned}K(x, t_{j+1})(T - t_{j+1})^{n/(n+1)} &\geq \frac{1}{2^{n/(n+1)}} K(x, s_j) h_{j+1}^{n/(n+1)} \\ &= \frac{1}{2^{n/(n+1)}} K(x, s_j) \alpha^{n/(n+1)} h_j^{n/(n+1)} \\ &\geq \left(\frac{\alpha}{2}\right)^{n/(n+1)} K(x, s_j)(T - s_j)^{n/(n+1)}\end{aligned}$$

and (5.9). The claimed estimate (5.3) follows by another iteration of the above argument applying (5.10) instead. Namely for $t \in [t_j, t_{j+1}]$, we have

$$\begin{aligned}K(x, t)(T - t)^{n/(n+1)} &\geq \frac{1}{2^{n/(n+1)}} K(x, t_j)(T - t)^{n/(n+1)} \geq \frac{1}{2^{n/(n+1)}} K(x, t_j) h_{j+1}^{n/(n+1)} \\ &\geq \left(\frac{\alpha}{2}\right)^{n/(n+1)} K(x, t_j)(T - t_j)^{n/(n+1)}.\end{aligned}$$

Hence we conclude that for any $t \in [t_1, T]$,

$$K(x, t)(T - t)^{n/(n+1)} \geq \left(\frac{\alpha}{2}\right)^{2n/(n+1)} \frac{1}{4(n+1)\Lambda}.$$

The claimed result follows from the above easily. □

Now the proof of [6, Theorem 10] gives the following estimate. For completeness a proof is included in the Appendix.

Theorem 5.3. *Suppose $u(x,t) > 0$ is the solution of (3.6) with initial data $u(0,x) = u_0(x)$, where $u_0(x) > 0$ is the support function of Ω_0 with $V(\Omega_0) = V(B(1))$. There exists a constant $C > 0$, depending on n, Ω_0, such that*

$$\mathrm{trace}(\bar{\nabla}_i \bar{\nabla}_j u + u\delta_{ij}) \le C. \tag{5.11}$$

Moreover the symmetric tensor A has the lower estimate

$$\bar{\nabla}_i \bar{\nabla}_j u + u\bar{g}_{ij} \ge \frac{1}{C}\bar{g}_{ij}. \tag{5.12}$$

Combining Corollary 2.1, Theorems 3.4, 4.1, 5.1 and 5.3, as well as the gradient estimate (4.1), we conclude that there exists a positive constant C depending only on the initial data such that the unique positive solution to (3.6) satisfies

$$\|u(\cdot,t)\|_{C^2(\mathbb{S}^n)} \le C. \tag{5.13}$$

Since (3.6) is a concave parabolic equation, by Krylov's theorem [16] and the standard theory of parabolic equations, estimates (5.13) and (5.12) imply bounds on all (space and time) derivatives of $u(x,t)$. More precisely, for any $k \ge 3$, there exists $C_k \ge 0$, depending only on the initial value, such that for $t \ge 1$,

$$\|u(\cdot,t)\|_{C^k(\mathbb{S}^n)} \le C_k. \tag{5.14}$$

Now for any $T > 0$ and any sequence $\{t_j\} \to \infty$, consider $u_j(x,t) := u(x, t - t_j)$. We have the following result on sequential convergence.

Proposition 5.1. *After passing to a subsequence, on $\mathbb{S}^n \times [-T,T]$, $\{u_j\}$ converges in C^∞-topology to a smooth function u_∞ which is a self-similar solution to (3.7).*

Proof. By the proof of Theorem 4.1 we have, for $t \in [-T,T]$,

$$\lim_{j\to\infty} \frac{1}{\omega_n}\int_{\mathbb{S}^n} \log u_j(x,t)\, d\theta(x) = \mathcal{E}_\infty.$$

Hence $u_\infty(x,t)$ satisfies

$$\frac{1}{\omega_n}\int_{\mathbb{S}^n} \log u_\infty(x,t)\, d\theta(x) = \mathcal{E}_\infty.$$

u_∞ is also a solution to (3.6) and positive by Theorem 4.1. Hence by the proof of Theorem 3.4 we conclude that

$$\frac{u_\infty(x,t)}{K(x,t)} = \frac{K(x,t)}{u_\infty(x,t)},$$

which implies that $(u_\infty)_t(x,t) = 0$. Hence we get the claimed result. □

6. Uniform convergence and the stability of the solitons

Combining [4, Theorem 2] with Proposition 5.1 we obtain the following result.

Theorem 6.1. *The normalized GCF (3.6) converges in C^∞-topology to a smooth soliton u_∞ (M_∞) which satisfies $K(x) > 0$ and the soliton equation*

$$u\det(u\,\mathrm{id} + \bar{\nabla}^2 u) = 1.$$

There remains an interesting question whether the round sphere (ball) is the unique compact soliton. For this we consider the following functional for $u > 0$, with A_u being positive definite:

$$\mathcal{J}_1(u) := \fint_{\mathbb{S}^n} \log u - \frac{1}{n+1}\log\Big(\fint_{\mathbb{S}^n} u\det(A_u)\Big) + \frac{1}{2}\Big(\fint_{\mathbb{S}^n} u\det(A_u) - 1\Big)^2.$$

Here $\fint_{\mathbb{S}^n} = (1/\omega_n)\int_{\mathbb{S}^n}$. If $v = u + \eta\rho$ is a variation, then

$$\frac{d}{d\eta}\mathcal{J}_1(v)\Big|_{\eta=0} = \fint_{\mathbb{S}^n}\frac{\rho}{u} - \frac{\fint_{\mathbb{S}^n}\rho\det(A_u)}{\fint_{\mathbb{S}^n} u\det(A_u)} + (n+1)\Big(\fint_{\mathbb{S}^n} u\det(A_u) - 1\Big)\fint_{\mathbb{S}^n}\rho\det(A_u).$$

Here we have used the fact that

$$\int u\sigma_n^{ij}(A)(A_\rho)_{ij} = \int \rho\sigma_n^{ij}(A)(A_u)_{ij} = n\int\rho\det(A_u),$$

where $\sigma_n^{ij}(A)$ denotes the cofactor of A_{ij} in $\det(A)$, which can also be expressed as KW^{ij} with (W^{ij}) being the Weingarten map. Hence the Euler–Lagrange equation of $\mathcal{J}_1(u)$ is

$$0 = \frac{1}{u} - \frac{\det(A_u)}{\fint_{\mathbb{S}^n} u\det(A_u)} + (n+1)\Big(\fint_{\mathbb{S}^n} u\det(A_u) - 1\Big)\det(A_u). \tag{6.1}$$

Multiplying (6.1) by u and integrating on $\mathbb{S}^n$ we obtain

$$\fint_{\mathbb{S}^n}(u\det(A_u) - 1)\,dx = 0.$$

This together with (6.1) implies $u = 1/\det(A_u)$. Hence we have the following proposition.

Proposition 6.1. *The critical point of the functional $\mathcal{J}_1(u)$ over positive smooth functions u with $A_u > 0$ satisfies the soliton equation $u = K$.*

Similarly we can compute the second variation of the functional $\mathcal{J}_1$:

$$\begin{aligned}\frac{d^2}{d\eta^2}\mathcal{J}_1(v_\eta)\Big|_{\eta=0} &= -\fint_{\mathbb{S}^n}\frac{\rho^2}{u^2} - \frac{\fint \rho\sigma_n^{ij}(\rho_{ij} + \rho\delta_{ij})}{\fint_{\mathbb{S}^n} u\det(A_u)} + (n+1)\Big(\frac{\fint_{\mathbb{S}^n}\rho\det(A_u)}{\fint_{\mathbb{S}^n} u\det(A_u)}\Big)^2\\ &\quad + (n+1)^2\Big(\fint_{\mathbb{S}^n}\rho\det(A_u)\Big)^2.\end{aligned}$$

Hence if $u \equiv 1$, as u is a critical point with $\fint u \det(A_u) = 1$, we deduce that

$$\frac{d^2}{d\eta^2}\mathcal{J}_1(v_\eta)\bigg|_{\eta=0} = -\fint_{\mathbb{S}^n} \rho^2 - \fint_{\mathbb{S}^n} \eta(\bar{\Delta}\rho + n\rho) + (n+1)(n+2)\left(\fint_{\mathbb{S}^n} \rho\right)^2$$
$$= \fint_{\mathbb{S}^n} |\bar{\nabla}\rho|^2 - (n+1)\fint_{\mathbb{S}^n} \rho^2 + (n+1)(n+2)\left(\fint_{\mathbb{S}^n} \rho\right)^2.$$

This computation, together with the detailed knowledge of the spectrum of the Laplace operator of the sphere, proves the following stability result.

Proposition 6.2. *The unit sphere/ball, the soliton with $u \equiv 1$, is stable among the variations $v_\eta = u + \eta\rho$ with $\rho \perp \mathrm{span}\{1, x_1, \ldots, x_{n+1}\}$.*

A similar consideration was applied by Andrews [6] to construct solitons of the flow with speed being the power of the Gauss curvature.

7. Appendix

1. Here we collect some equations related to (3.5) and its normalization. We denote $\bar{g}^{ik}(\bar{\nabla}_k\bar{\nabla}_j u + u\bar{g}_{kj})$ by A (or A_u to make clear the dependence), and $-1/\det(A)$ by Ψ, viewed as a function of the tensor A. It is known that Ψ is $-n$-concave [1]:

$$\ddot{\Psi}(X, X) \le \frac{n+1}{n}\,\frac{\dot{\Psi}(X)^2}{\Psi}. \tag{7.1}$$

When we discuss a solution to (3.6) we always assume that $A > 0$. The elliptic operator $\mathcal{L} := (\dot{\Psi}_A)_{ij}\bar{\nabla}_i\bar{\nabla}_j$, written in terms of a normal coordinate of $\mathbb{S}^n$, appears in the linearization of (3.6):

$$\frac{\partial}{\partial t} - \mathcal{L} - KH - 1.$$

If u_1 and u_2 are two convex (being the support functions of a convex body) solutions to (3.6) with $u_1(x, 0) = u_2(x, 0)$, then $v = u_1 - u_2$ satisfies, under the normal coordinates,

$$\frac{\partial}{\partial t}v = \left(\int_0^1 (\dot{\Psi}(A_s))_{ij}\,ds\right)\bar{\nabla}_i\bar{\nabla}_j v + \left(\int_0^1 \dot{\Psi}(A_s)(\delta_{ij})\,ds\right)v + v$$

with $A_s = \bar{\nabla}_i\bar{\nabla}_j u_s + u_s\delta_{ij}$ and $u_s = su_1 + (1-s)u_2$. Hence $u_1(x, t) \equiv u_2(x, t)$. The following evolution equations are also well-known [5, 1].

Proposition 7.1. *Under the normalized flow (3.6), the following hold:*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)u = (n+1)\Psi + u - u\Psi H, \tag{7.2}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\Psi = -\Psi^2 H - n\Psi, \tag{7.3}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)P = P - \Psi HP + \ddot{\Psi}_A(Q, Q). \tag{7.4}$$

Here H is the mean curvature of $M_t := \partial\Omega_t$, $P = \frac{\partial\Psi}{\partial t}$, the time derivative of the speed, that is, the acceleration, and $Q = A_t$.

Noticing that $-\Psi H = \dot{\Psi}_A(\mathrm{id})$, we see that the above two equations can be written as

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)u = (n+1)\Psi + u + u\dot{\Psi}_A(\mathrm{id}), \tag{7.5}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\Psi = \Psi\dot{\Psi}_A(\mathrm{id}) - n\Psi, \tag{7.6}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)P = P + \dot{\Psi}_A(\mathrm{id})P + \ddot{\Psi}_A(Q, Q). \tag{7.7}$$

From these equations it is easy to see that (3.6) preserves the volume of the enclosed body. Precisely,

$$\begin{aligned}\Sigma(t) := \int_{\mathbb{S}^n} \frac{u}{-\Psi}\, d\theta(x) &= \int_{M_t} \langle X(y,t), \nu(y)\rangle\, d\sigma(y)\\ &= \int_{\Omega_t} \mathrm{div}(X)\, d\mu_y = (n+1)V(\Omega_t).\end{aligned}$$

A direct calculation using (7.2), (7.3) and the divergence structure of the operator $\mathcal{L}/\Psi^2$ yields

$$\Sigma'(t) = (n+1)(\Sigma(t) - \omega_n).$$

Since $\Sigma(0) - \omega_n = 0$, this implies that $\Sigma(t) \equiv \omega_n$ for all t.

The evolution equation for $A_{ij} := u_{ij} + u\delta_{ij}$, the inverse of the Weingarten map W^{-1}, in the normal coordinates is useful. The equation was first proved in [6].

Proposition 7.2. *In the normal coordinates, for a solution to (3.6) the tensor A_{ij} satisfies*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)A_{ij} = -KHA_{ij} + A_{ij} + (n-1)K\bar{g}_{ij} + \ddot{\Psi}_A(\bar{\nabla}_i A, \bar{\nabla}_j A). \tag{7.8}$$

Here $\Psi = -K$ and H is the mean curvature, that is, the sum of the eigenvalues of A^{-1}.

As before, (7.8) can be written as

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)A_{ij} = -\dot{\Psi}_A(\mathrm{id})A_{ij} + A_{ij} - (n-1)\Psi\bar{g}_{ij} + \ddot{\Psi}_A(\bar{\nabla}_i A, \bar{\nabla}_j A). \tag{7.9}$$

Below we derive the corresponding equation for A_{ij} when u is instead a solution of (3.5) since the corresponding equation readily yields an upper estimate for the Hessian of u, for the un-normalized solution u. By (3.5) we have $\frac{\partial}{\partial t}A_{ij} = \bar{\nabla}_i\bar{\nabla}_j\Psi + \Psi\bar{g}_{ij}$. Now we compute

$$\begin{aligned}\bar{\nabla}_j\Psi &= \dot{\Psi}_A(\bar{\nabla}_j A),\\ \bar{\nabla}_i\bar{\nabla}_j\Psi &= \dot{\Psi}_A(\bar{\nabla}_i\bar{\nabla}_j A) + \ddot{\Psi}_A(\bar{\nabla}_i A, \bar{\nabla}_j A),\\ \bar{\nabla}_i\bar{\nabla}_j A_{kl} &= \bar{\nabla}_i\bar{\nabla}_j\bar{\nabla}_k\bar{\nabla}_l u + \bar{\nabla}_i\bar{\nabla}_j u\bar{g}_{kl}.\end{aligned}$$

The commutator formulae yield

$$\begin{aligned}
\bar\nabla_j\bar\nabla_k\bar\nabla_l u &= \bar\nabla_k\bar\nabla_j\bar\nabla_l u - \bar R_{lpkj}\bar\nabla_p u,\\
\bar\nabla_i\bar\nabla_j\bar\nabla_k\bar\nabla_l u &= \bar\nabla_i(\bar\nabla_k\bar\nabla_l\bar\nabla_j u - \bar R_{lpkj}\bar\nabla_p u)\\
&= \bar\nabla_k\bar\nabla_i\bar\nabla_l\bar\nabla_j u - \bar R_{lpkj}\bar\nabla_i\bar\nabla_p u - \bar R_{jpki}\bar\nabla_p\bar\nabla_l u - \bar R_{lpki}\bar\nabla_j\bar\nabla_p u,\\
\bar\nabla_k\bar\nabla_i\bar\nabla_l\bar\nabla_j u &= \bar\nabla_k\bar\nabla_l\bar\nabla_i\bar\nabla_j u - \bar R_{jpli}\bar\nabla_p\bar\nabla_k u.
\end{aligned}$$

Here $\bar R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ is the curvature tensor of $\mathbb{S}^n$. Putting all together we have

$$\begin{aligned}
\bar\nabla_i\bar\nabla_j\bar\nabla_k\bar\nabla_l u = \bar\nabla_k\bar\nabla_l\bar\nabla_i\bar\nabla_j u - \bar R_{jpli}\bar\nabla_p\bar\nabla_k u - \bar R_{lpkj}\bar\nabla_i\bar\nabla_p u\\
- \bar R_{jpki}\bar\nabla_p\bar\nabla_l u - \bar R_{lpki}\bar\nabla_j\bar\nabla_p u.
\end{aligned}$$

Now using $(\dot\Psi_A)^{kl} = KA^{kl}$, where (A^{ij}) is the inverse of (A_{ij}), we obtain

$$\begin{aligned}
\bar\nabla_i\bar\nabla_j\Psi &= KA^{kl}(\bar\nabla_i\bar\nabla_j\bar\nabla_k\bar\nabla_l u) + KH\bar\nabla_i\bar\nabla_j u + \ddot\Psi_A(\bar\nabla_i A, \bar\nabla_j A)\\
&= KA^{kl}(\bar\nabla_k\bar\nabla_l(A_{ij} - u\bar g_{ij})) - 2KH\bar\nabla_i\bar\nabla_j u + 2KA^{kl}(\bar\nabla_k\bar\nabla_l u)\bar g_{ij}\\
&\quad + KH\bar\nabla_i\bar\nabla_j u + \ddot\Psi_A(\bar\nabla_i A, \bar\nabla_j A)\\
&= KA^{kl}(\bar\nabla_k\bar\nabla_l A_{ij}) - KH\bar\nabla_i\bar\nabla_j u + +KA^{kl}(\bar\nabla_k\bar\nabla_l u)\bar g_{ij} + \ddot\Psi_A(\bar\nabla_i A, \bar\nabla_j A)\\
&= KA^{kl}(\bar\nabla_k\bar\nabla_l A_{ij}) - KHA_{ij} + nK\bar g_{ij} + \ddot\Psi_A(\bar\nabla_i A, \bar\nabla_j A).
\end{aligned}$$

Combining the above we arrive at the following parabolic equation for A_{ij}:

$$\frac{\partial}{\partial t}A_{ij} = KA^{kl}(\bar\nabla_k\bar\nabla_l A_{ij}) - KHA_{ij} + (n-1)K\bar g_{ij} + \ddot\Psi_A(\bar\nabla_i A, \bar\nabla_j A). \tag{7.10}$$

The equation (7.8) follows similarly if u satisfies (3.6) instead. Let $B_{ij} = \bar\nabla_i\bar\nabla_j u$, the Hessian of u. Then if u is a solution to (1.1), B satisfies

$$\frac{\partial}{\partial t}B_{ij} = KA^{kl}(\bar\nabla_k\bar\nabla_l B_{ij}) - KHB_{ij} + 2nK\bar g_{ij} - 2uHK\bar g_{ij} + \ddot\Psi_A(\bar\nabla_i A, \bar\nabla_j A). \tag{7.11}$$

An immediate consequence of the above is an upper bound on B_{ij}. Let

$$B_S(t) = \max_{x\in\mathbb{S}^n}\max_{X\in T_x\mathbb{S}^n,\,|X|=1} X^iX^j\bar\nabla_i\bar\nabla_j u.$$

If $B_S(t_0) = \max_{t\in[0,T)} B_S(t)$, using the concavity of $\ddot\Psi$ we see that at an extremal point (x_0, t_0) where $B_S(t_0)$ is achieved, by the maximum principle,

$$HB_S(t_0) \le 2n - 2uH.$$

Hence, via the Cauchy–Schwarz estimate $H \ge nK^{1/n}$,

$$B_S(t_0) \le 2/K^{1/n}.$$

Using $\inf_{M_t} K \ge \inf_{M_0} K$ we have the uniform upper bound

$$(\bar\nabla_i\bar\nabla_j u)(x,t) \le \frac{2}{\inf_{M_0} K^{1/n}}\bar g_{ij}(x,t) + \max_x(\bar\nabla_i\bar\nabla_j u)(x,0), \tag{7.12}$$

which recovers a key C^2-estimate of [19] in the proof of the existence and convergence to a point for the un-normalized flow.

Making use of the computation above we also have the following evolution equation for $|X|^2 = |\bar\nabla u|^2 + u^2$:

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)|X|^2 = 2|X|^2 - 2(\dot\Psi_A)_{ij}\bar\nabla_i\bar\nabla_k u\bar\nabla_j\bar\nabla_k u + 2(n+1)u\Psi + 2u^2(\dot\Psi_A)(\mathrm{id}). \tag{7.13}$$

2. Here we give an alternative proof of Theorem 5.1. Consider the quantity $Q := K/(2u-a)$. The evolution equations (7.2) and (7.3) yield

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \mathcal{L}\right)Q &= \frac{K^2H - nK}{2u-a} - 2K\frac{-(n+1)K + u + uKH}{(2u-a)^2} + 2\dot\Psi_{ij}\bar\nabla_i Q\bar\nabla_j \log(2u-a)\\ &= \frac{-aK^2H + 2(n+1)K^2 - (2u-a)nK - 2uK}{(2u-a)^2} + 2\dot\Psi_{ij}\bar\nabla_i Q\bar\nabla_j \log(2u-a).\end{aligned}$$

Now apply the maximum principle: if $m(t) = \max_{x\in\mathbb{S}^n} Q(x,t)$ is achieved at (x_0,t), then at that point we have

$$\begin{aligned}0 &\le \frac{-aK^2H + 2(n+1)K^2 - (2u-a)nK - 2uK}{(2u-a)^2}\\ &\le m(t)^2(-aH + 2(n+1)).\end{aligned}$$

Noting that $K \le (H/n)^n$, we then deduce that at (x_0,t),

$$K \le \left(\frac{2(n+1)}{na}\right)^n,$$

which in turn implies that

$$m(t) \le \left(\frac{2(n+1)}{n}\right)^n \frac{1}{a^{n+1}}.$$

The claimed estimate in Theorem 5.1 follows from the above.

We remark that in [15, Corollary, p. 156], Hamilton obtained the above estimate by using the sharp differential estimate of Chow (which is also referred to as a differential Harnack estimate, as well as a Li–Yau–Hamilton type estimate) and the entropy formula of Chow [10]. Hamilton's estimate is built upon a lower estimate of $u(x,t)/K(x,t)$. Our proof of Theorem 5.1 avoids the use of Chow's entropy formula and his differential estimate [10], but is based on the C^0-lower bound. Below we include a slightly stronger result lower estimate of $u(x,t)/K(x,t)$.

Proposition 7.3. *Let u be a solution to the un-normalized flow (3.5) with the reference point being the limit point when $t \to T$. Then*

$$\frac{u(x,t)}{K(x,t)} \ge (n+1)t^{n/(n+1)}(T^{1/(n+1)} - t^{1/(n+1)}). \tag{7.14}$$

Since $T \geq t$, the above (7.14) implies $u(x,t)/K(x,t) \geq (t/T)^{n/(n+1)}(T-t)$, a result of Hamilton [15].

Proof of Proposition 7.3. By the differential estimate of Chow [10, Theorem 3.7], we deduce that, with respect the parametrization via the Gauss map,

$$-\Psi_t - \frac{n}{(n+1)t}\Psi \geq 0.$$

Then a direct calculation shows that $y(t) = u/(-\Psi)$ satisfies the estimate

$$y'(t) \leq -1 + \frac{n}{(n+1)t} y(t).$$

Noticing that $y(t) \to 0$ as $t \to T$, integrating the above from t to T yields

$$-t^{-n/(n+1)} y(t) \leq -(n+1)(T^{1/(n+1)} - t^{1/(n+1)}).$$

Hence we have the claimed estimate. □

Note that for the solution $u(x,t)$ to the normalized flow (3.6), the estimate (7.14) implies

$$\frac{u(x,t)}{K(x,t)} \geq \frac{1}{n+1}(1-e^{-(n-1)t})^{n/(n+1)}, \tag{7.15}$$

which together with Corollary 2.1 and Theorem 3.4 gives another proof of Theorem 5.1.

3. Here we include a proof of Theorem 5.3. We denote by $\sigma_i(A)$ (or simply σ_i) the i-th symmetric function of the symmetric tensor $A_{ij} = \bar\nabla_i\bar\nabla_j u + u\delta_{ij}$. The previous result implies that $\sigma_n \geq 1/C_1$, where C_1 is the positive constant from Theorem 5.1. We recall Newton's inequality (the function $\log(\sigma_k/C_n^k)$, with C_n^k being the binomial coefficient, is a concave function of k):

$$\frac{\sigma_{n-1}}{n} \geq \left(\frac{\sigma_1}{n}\right)^{1/(n-1)} \sigma_n^{(n-2)/(n-1)}. \tag{7.16}$$

The concavity of $\ddot\Psi$ together with (7.8) implies that

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\sigma_1 \leq -\frac{\sigma_1\sigma_{n-1}}{\sigma_n^2} + \sigma_1 + \frac{n(n-1)}{\sigma_n} - \frac{n+1}{n}\frac{|\bar\nabla K|^2}{K}. \tag{7.17}$$

Let $m(t) := \max_{x\in\mathbb{S}^n}\sigma_1(x,t)$. Then at (x_0,t) where $m(t)$ is achieved we have

$$\begin{aligned} 0 &\leq -\frac{\sigma_1\sigma_{n-1}}{\sigma_n^2} + \sigma_1 + \frac{n(n-1)}{\sigma_n} \leq -n^{(n-2)/(n-1)}\frac{\sigma_1^{n/(n-1)}}{\sigma_n^{n/(n-1)}} + \sigma_1 + n(n-1)C_1 \\ &\leq -C_2\sigma_1^{n/(n-1)} + \sigma_1 + C_1'. \end{aligned}$$

Here in the second last inequality we applied (7.16) and the upper estimate of $K(x,t)$, and in the last inequality we applied the lower estimate of $K(x,t)$ established in Theorem 5.1.

The claimed result (5.11) follows from the application of the maximum principle to the above estimate. The estimate (5.12) follows from Theorem 5.1 and (5.11).

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