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ON STABILITY OF STATIONARY POINTS OF TRANSFORMATION GROUPS

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INTRODUCTION

Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 (= continuously differentiable) vector field on Euclidean n -space, and consider the associated differential equation

$$\frac{dx}{dt} = F(x). \quad (1)$$

For each $y \in \mathbb{R}^n$ let $t \mapsto \phi(t, y)$ be the solution to (1) which passes through y when $t = 0$, so that

$$\phi(0, y) = y, \quad \left. \frac{\partial \phi}{\partial t} \right|_{t=0} = F.$$

Assume for simplicity that solutions are defined for all t . Thus we have a C^1 map $\phi: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. For each $t \in \mathbb{R}$ we obtain a C^1 map

$$\begin{aligned} \phi_t: \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ y &\mapsto \phi(t, y). \end{aligned}$$

Uniqueness of solutions implies that

$$\phi_{s+t} = \phi_s \circ \phi_t,$$

the composition of ϕ_s and ϕ_t , and clearly ϕ_0 is the identity map of $\mathbb{R}^n$. In this way we obtain a homomorphism $t \mapsto \phi_t$ from the additive group of real numbers to the group of homeomorphisms of $\mathbb{R}^n$. Since in fact each ϕ_t is C^1 with a C^1 inverse ϕ_{-t} , we actually have a homomorphism into the group of C^1 diffeomorphisms,

$$\begin{aligned} \alpha: \mathbb{R} &\longrightarrow \text{Diff } \mathbb{R}^n, \\ \alpha(t) &= \phi_t. \end{aligned}$$

We call α the *flow* defined by (1).

Many questions in dynamical systems can be profitably studied in terms of flows. Some flows have a *section*, that is, a hypersurface $S \subset \mathbb{R}^n$ transverse to the vector field, which orbits repeatedly intersect. In this way one is led to consider the "first return" map $f: S \rightarrow S$. Such an f generates (under suitable hypotheses) a homomorphism from the group $\mathbb{Z}$ of integers into $\text{Diff } S$.

A homomorphism of a group G into the group of homeomorphisms of a space X is called an *action* of G on X . In the examples above we have actions of $\mathbb{R}$ and

of $\mathbb{Z}$. Actions of other groups also occur. For example, let $F, E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be two vector fields of the type considered above, such that their Lie bracket $[E, F] = 0$. Then their corresponding flows commute and we obtain an action of $\mathbb{R} \times \mathbb{R}$ on $\mathbb{R}^n$. The presence of a group G of symmetries of a dynamical system often leads to an action of $G \times \mathbb{R}$.

Two central themes in dynamical systems are *recurrence* and *stability*. The book *Topological Dynamics*² of Gottschalk and Hedlund was the first systematic investigation of recurrence in the context of group actions. A great deal of work in this field is currently being done, much of it in ergodic theory and foliation theory. Questions of stability, however, have barely been touched for groups other than $\mathbb{Z}$ or $\mathbb{R}$.^{3,4}

The simplest stability problem is that of stability of a stationary point under perturbation of a group action. The rest of this article discusses sufficient conditions for such stability.

THE PROBLEM

Consider a topological group G acting differentiably on a manifold M by a continuous homomorphism $\alpha: G \rightarrow \text{Diff } M$. Let $p \in M$ be a *stationary point* of the action, that is,

$$\alpha(g)p = p \quad \text{for all } g \in G.$$

Call p *stable* if nearby actions have nearby stationary points (precise definitions are given below). The question is: *under what conditions is a stationary point stable under perturbations of the action?*

For simplicity I consider only the case $M = \mathbb{R}^n$, and G is always assumed to be generated by some compact subset $K \subset G$. For example, G might be a finitely generated discrete group, or a connected Lie group.

Now some precise definitions.

The topology on $\text{Diff } \mathbb{R}^n$ is that of uniform convergence of diffeomorphisms and their first derivatives on compact sets. There is a complete metric for this topology.

An *action* α of G on $\mathbb{R}^n$ is a continuous homomorphism $\alpha: G \rightarrow \text{Diff } \mathbb{R}^n$. The set of all such actions is denoted $A(G, \mathbb{R}^n)$. A *neighborhood* $\mathfrak{N} \subset A(G, \mathbb{R}^n)$ of α is a set of the form $\mathfrak{N} = \mathfrak{N}(\alpha, s, \delta)$ where $s > 0$, $\delta > 0$, and $\beta \in \mathfrak{N}$ if and only if the following condition holds: for all $g \in K$ and all $x \in \mathbb{R}^n$ with $|x| \leq s$,

$$|\alpha(g)x - \beta(g)x| < \delta$$

and

$$||D\alpha(g)(x) - D\beta(g)(x)|| < \epsilon.$$

Here $D\alpha(g)(x)$ is the linear operator on $\mathbb{R}^n$ which is the derivative of $\alpha(g)$ at x , and $|| \cdot ||$ denotes the usual operator norm.

We can now define what it means for a stationary point $p \in \mathbb{R}^n$ to be *stable* for α : for every $\epsilon > 0$ there is a neighborhood $\mathfrak{N}$ of α such that if $\beta \in \mathfrak{N}$ then β has a stationary point q with $|p - q| < \epsilon$.

EXAMPLES AND THEOREMS

Example 1. Let $G = \mathbb{Z}$. Let $\alpha: G \rightarrow \text{Diff } \mathbb{R}^n$ be generated by the diffeomorphism $\alpha(1) = f$. The stationary points of α are the fixed points of f . It is well known that a fixed point p is stable if 1 is not an eigenvalue of $Df(p)$. This condition is equivalent to the graph of f being transverse to the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ at (p, p) . It is by no means *necessary* for stability. For example $0 \in \mathbb{R}$ is a stable fixed point of $f(x) = x + x^3$.

It is important to notice a certain uniqueness property of the stationary point p in Example 1: There is a neighborhood $U \subset \mathbb{R}^n$ of p such that p is the only stationary point of α in U . The same is true for perturbations of α : it follows easily from the inverse function theorem that neighborhoods U of p and $\mathfrak{N}$ of α can be chosen so that every $\beta \in \mathfrak{N}$ has a unique stationary point in U . A stable stationary point with this property is called *uniquely stable*.

Example 2. Let $\phi_t: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a flow defined by a C^1 vector field F , so that

$$F(x) = \left. \frac{d}{dt} \right|_{t=0} \phi_t(x).$$

Then $p \in \mathbb{R}^n$ is stationary for the action $\phi: \mathbb{R} \rightarrow \text{Diff } \mathbb{R}^n$ if and only if $F(p) = 0$. Suppose that p is stationary and $DF(p)$ is invertible. By the inverse function theorem any C^1 vectorfield sufficiently near F must have a zero near p . Thus p is stable for perturbations of the vector field.

The topology on the space of actions is not defined in terms of the vector field F , but directly in terms of the flow. In fact it is not true that every C^1 flow has a C^1 vector field, although it does have a continuous vector field by a result of Chernoff and Marsden.¹ In any case we can easily replace the condition that $DF(p)$ is nonsingular by a condition expressed purely in terms of the flow. It is equivalent to saying: for some $t \in \mathbb{R}$, $D\phi_t(p)$ does not have 1 for an eigenvalue. The equivalence follows easily from the well-known formula

$$D\phi_t(p) = \exp [tDF(p)].$$

The equivalent condition makes sense for actions of groups other than $\mathbb{R}$, and leads to wide generalization of Example 2.

THEOREM 1. Let $\alpha: G \rightarrow \text{Diff } \mathbb{R}^n$ be an action of an Abelian group G . A stationary point $p \in \mathbb{R}^n$ is stable provided there exists $g_0 \in G$ with $I - D\alpha(g_0)_p$ invertible.

Proof. Put $f = \alpha(g_0) \in \text{Diff } \mathbb{R}^n$. Then p is a uniquely stable fixed point for f , by Example 1 and the remark following it. Therefore there exists a neighborhood U of p , and a neighborhood $\mathfrak{N}$ of α , such that for every $\beta \in \mathfrak{N}$, $\beta(g_0)$ has a unique stationary point $q_\beta \in U$. Moreover $q_\beta \rightarrow p$ as $\beta \rightarrow \alpha$. Let $\mathfrak{N}_1 \subset \mathfrak{N}$ be a smaller neighborhood such that $\beta \in \mathfrak{N}_1$ implies $\beta(g)q_\beta \in U$ for all $g \in K$ where $K \subset G$ is a compact generating set. Such an $\mathfrak{N}_1$ exists because $\beta(g)q_\beta \rightarrow p$ as $\beta \rightarrow \alpha$, uniformly for $g \in K$.

I claim q_β is stationary for β if $\beta \in \mathfrak{N}_1$. It suffices to prove $\beta(g)q_\beta = q_\beta$ for all $g \in K$. Now

$$\begin{aligned}\beta(g)q_\beta &= \beta(g)\beta(g_0)q_\beta \\ &= \beta(g_0)\beta(g)q_\beta.\end{aligned}$$

Therefore $\beta(g)q_\beta$ is a fixed point of $\beta(g_0)$, and it lies in U . By uniqueness of such a fixed point, $\beta(g)q = q_\beta$. Thus q_β is stationary for β because K generates G .

Example 3. Let $\alpha: \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Diff } \mathbb{R}^2$ be the linear action generated by

$$\begin{aligned}\alpha(1, 0) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = f_1, \\ \alpha(0, 1) &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = f_2.\end{aligned}$$

Then the stationary point $0 \in \mathbb{R}^2$ is stable. What must be shown is that if $g_1, g_2 \in \text{Diff } \mathbb{R}^2$ commute and are sufficiently C^1 close to f_1, f_2 , then they have a common fixed point near 0. I do not know how to prove this directly. But the proof is easy if we use the fact that $\mathbb{Z} \oplus \mathbb{Z}$ is an Abelian group. For we apply Theorem 1 to $g_0 = (1, 1)$:

$$\alpha(g_0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus $I - D\alpha(g_0)(0)$ is invertible, so 0 is a stable stationary point.

The method of proof used in Theorem 1 extends to more general situations. All that was really needed was the *normality* of the subgroup generated by g_0 . A more general result is:

THEOREM 2. Let $\alpha: G \rightarrow \text{Diff}(\mathbb{R}^n)$ be an action and $p \in \mathbb{R}^n$ a stationary point. Suppose $H \subset G$ is a subgroup such that p is uniquely stable for the restricted action $\alpha|_H: H \rightarrow \text{Diff}(\mathbb{R}_n)$. If H is a normal subgroup then p is uniquely stable for α .

Proof. Let $U \subset \mathbb{R}^n$ be a neighborhood of p for which there is a neighborhood $\mathfrak{N}_0 \subset A(H, \mathbb{R}^n)$ of $\alpha|_H$ such that if $\gamma \in \mathfrak{N}_0$ then γ has a unique stationary point $q(\gamma) \in U$, and $q(\gamma) \rightarrow p$ has $\gamma \rightarrow \alpha|_H$. Let $\mathfrak{N} \subset A(G, \mathbb{R}^n)$ be a neighborhood of α so small that if $\beta \in \mathfrak{N}$ then $\gamma = \beta|_H \in \mathfrak{N}_0$, and also $\beta(k)q(\beta|_H) \in U$ for all $k \in K$ ($=$ a compact generating set for G). Let $\beta \in \mathfrak{N}$. I claim $q(\beta|_H) = q$ is fixed for all $\beta(k), k \in K$. For if $k \in K$ then

$$\begin{aligned}\beta(k)q &= \beta(k)\beta(H)q \quad (\text{as sets}) \\ &= \beta(H)\beta(k)q \quad (\text{by normality}).\end{aligned}$$

Hence $\beta(k)q$ is a stationary point in U for $\alpha|_H$. By uniqueness $\beta(k)q = q$. Q.E.D.

A nilpotent group G may be defined as one having the property that for any subgroup $H_0 \subset G$ there is a finite chain of subgroups $H_0 \subset \dots \subset H_n = G$ where each H_{i-1} is normal in H_i . The group of upper triangular $m \times m$ matrices with values of 1 on the diagonal is nilpotent. One can use Theorem 2 to prove:

THEOREM 3. Let G be a nilpotent group and p a stationary point for an action α of G on $\mathbb{R}^n$. Then p is uniquely stable provided

$$I - D\alpha(g)(p)$$

is invertible for some $g \in G$.

Example 4. Let $G = SL(n)$ be the group of $m \times m$ real matrices of determinant 1, acting as usual on $\mathbb{R}^n$. Then the origin $0 \in \mathbb{R}^n$ is stable if n is even. For let $H = \{\pm I\}$. Then H is Abelian and the matrix $-I$ does not have 1 as an eigenvalue. By Theorem 1, 0 is stable for the action of H . Since H is normal, 0 is stable for $SL(m)$ by Theorem 2. This result is also true for odd m , but needs a different proof; see Example 6.

Example 5. Let S be the group of 2×2 real matrices

$$A_{a,b} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \quad a > 0.$$

S is solvable but not nilpotent. For every real number t an action $\alpha_t: S \rightarrow \text{Diff}(\mathbb{R})$ is defined by

$$\alpha_t(A_{a,b})(x) = ax + tb.$$

The action α_0 has $0 \in \mathbb{R}$ as a stationary point; 0 is *unstable* because α_t has no stationary point for $t \neq 0$, and $\alpha_t \rightarrow \alpha_0$ as $t \rightarrow 0$. This example also shows that Theorem 2 fails for solvable groups.

I mention without proof two other sufficient conditions for stability, Theorems 4 and 5.

THEOREM 4 (Palais⁵). If G is compact every stationary point is stable.

Example 6. The origin in $\mathbb{R}^n$ is stable for the usual action α of $SL(n)$. The case n even was covered in Example 4. Since $SO(n)$ is compact, 0 is stable for it by Palais' Theorem. If β is a small perturbation of α then $\beta|_{SO(n)}$ has a stationary point near 0. The isotropy group of q is a closed subgroup Γ , and $SO(n) \subset \Gamma \subset SL(n)$. The orbit of q under $\beta(SL(n))$ is the image of an immersion of $SL(n)/\Gamma$, so that

$$\dim SL(n) - \dim \Gamma \leq n.$$

It follows that $\Gamma \neq SO(n)$ if $n \geq 3$. But $SO(n)$ is a maximal proper closed subgroup; hence $\Gamma = SL(n)$.

THEOREM 5. Let G be a Lie group; $H \subset G$ a closed subgroup; α an action of G on $\mathbb{R}^n$; $p \in \mathbb{R}^n$ a stationary point of $\alpha|_H$. Let E be the quotient of the Lie algebra of G by that of H , and for $h \in H$ let $A(h): E \rightarrow E$ be the linear operator induced by the adjoint representation of G . Then p is a stable stationary point for α provided there exists $h \in H$ such that $D\alpha(h)(p)$ and $A(h)$ have no eigenvalues in common.

Example 7. Let S be as in Example 5. For every $s \in \mathbb{R}$ define an action

$$\begin{aligned} \gamma_s: S &\rightarrow \text{Diff } \mathbb{R}, \\ \gamma_s(A_{a,b})x &= a^s x. \end{aligned}$$

Then 0 is unstable for $s = 1$ (Example 6) and for $s = 0$ (Exercise!). For $s \neq 0, 1$ it is not hard to use Theorem 3 to prove that 0 is stable: take $H = \{A_{a,b}: b = 0\}$.

Theorem 5 can be used to prove that many representations of semisimple groups have a stable stationary point at the origin.

I end with a conjecture and an unsolved problem.

Conjecture. Every stationary point of every semisimple Lie group is stable.

Problem. Is $0 \in \mathbb{R}^3$ stable for the action of $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ generated by 180° rotations about the three coordinate axes?

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