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Operator algebras in Solovay's model

by

Andre Val Kornell

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

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of the

University of California, Berkeley

Committee in charge:

Professor Marc A. Rieffel, Chair
Professor Leo A. Harrington
Professor Raphael Bousso

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Abstract

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The ultraweak topology on bounded operators on a Hilbert space is given by functionals of the form $x \mapsto \sum_{n=0}^\infty \langle \eta_n | x \xi_n \rangle$ for $\sum_{n=0}^\infty \|\eta_n\|^2 < \infty$ and $\sum_{n=0}^\infty \|\xi_n\|^2 < \infty$. By analogy with the ultraweak topology, we define the continuum-weak topology to be given by functionals of the form $x \mapsto \int_0^\infty \langle \eta_t | x \xi_t \rangle \, dt$ for $\int_0^\infty \|\eta\|^2 \, dt < \infty$ and $\int_0^\infty \|\xi_t\|^2 \, dt < \infty$. In order to make sense of the integral $\int_0^\infty \langle \eta_t | x \xi_t \rangle \, dt$ for arbitrary bounded operators x, we work in a model of set theory where every set of real numbers is Lebesgue measurable: Solovay's model.

Solovay's construction produces transitive models of set theory that satisfy a number of axioms that are convenient for analysis. In any such model, every set of real numbers satisfies Lebesgue measurability, the Baire property, and the perfect set property. By Vitali's theorem, such a model cannot satisfy the full axiom of choice, but it does satisfy the axiom of dependent choices, which allows us to make choices during a countable recursive construction. If we specify that the input model for Solovay's construction satisfies the axiom of constructibility, then the output model also satisfies the axiom of choice almost everywhere, which allows us to make choices for any family of sets indexed by the real numbers, at almost all indices.

Many, but not all, familiar theorems continue to hold in our Solovay model $\mathfrak{N}$. In general, the proof of such a theorem relies only on the axiom of dependent choices, and not on the full axiom of choice. However, it is impractically time consuming to scrutinize the proof of every needed result down to first principles, so we develop an alternative approach. The Solovay model $\mathfrak{N}$ is obtained as an inner model of a forcing extension $\mathfrak{M}[G]$, and it is closed under countable unions, so many properties are absolute for $\mathfrak{N}$ and $\mathfrak{M}[G]$, that is, their truth value is the same in both. We show how this observation can be leveraged to establish sophisticated results in the Solovay model $\mathfrak{N}$.

We develop the properties of the continuum-weak topology by analogy with those of the ultraweak topology. We then define a V*-algebra to be a $*$-algebra of operators that is closed in the continuum-weak topology, by analogy with the definition of von Neumann algebras. For each self-adjoint operator x in some V*-algebra, and each bounded complex-valued function f on the spectrum of x, we can define the operator $f(x)$, which is also in

that V*-algebra. The role of V*-algebras in noncommutative mathematics is not discussed in this dissertation; see [15].

Every C*-algebra A has an enveloping V*-algebra $V^*(A)$ which is universal among the V*-algebras generated by the representations of A. If A is nonseparable, then it may not have any nontrivial representations, so that $V^*(A) \cong 0$. However if A is separable, then it is isomorphic to a C*-subalgebra of $V^*(A)$. If A is a commutative separable C*-algebra, then it is isomorphic to the C*-algebra of all continuous complex-valued functions on some compact metrizable space, and $V^*(A)$ is isomorphic to the von Neumann algebra of all bounded complex-valued functions on that compact metrizable space. More generally, if A is a separable C*-algebra of type I, then its enveloping V^*-algebra is an ℓ^∞-direct sum of type I factors, one for each irreducible representation of A.

Contents

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This research owes a lot to Andrew Marks, John Steel, and Hugh Woodin. They have patiently answered my many, often misguided questions about set theory. In doing so, they introduced me to beautiful mathematics that I might have otherwise never seen. This dissertation incorporates some of their suggestions and advice. I also thank Leo Harrington for his help with appendix B.

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Chapter 1

Working in Solovay's model

In this chapter, we explain how to verify that familiar theorems hold in the Solovay model by appealing to absoluteness, and we briefly survey functional analysis in this new setting. Our primary reference for Solovay's construction is his original paper [27], from which we have taken the notation $\mathfrak{M}$ and $\mathfrak{N}_1$. Jech's *Set Theory* [11] is our reference for set theory in general. Schechter's *Handbook of Analysis and its Foundations* is an excellent reference for the role of the axiom of choice in analysis. Some notation and terminology is reviewed in appendix A.

1.1 The role of set theory

It is sometimes convenient to assume that every subset of $\mathbb{R}$ is Lebesgue measurable. For example, it is difficult to make sense of definition 2.1.2 without this assumption. However, Vitali's theorem implies that this assumption is inconsistent with the standard development of mathematics. Therefore, before proceeding with this assumption, we argue that many familiar mathematical results are compatible with it. Thus, we are led to examine the foundations of mathematics. We work with Zermelo-Fraenkel set theory because it is the established foundational system for mathematics, and because its extensions and fragments are the objects of modern research into consistency.

The role of set theory as a foundational system for mathematics may be explained with a simile. Computers handle finite mathematical structures by storing their data as strings of bits. Conceptually, a finite graph is not literally a string of bits, but an ideal computer can search through finite graphs by searching through strings of the appropriate kind. Similarly, the set $\{\{\}, \{\{\}\}\}$ is not literally the number 2, but it is a set that is commonly used to represent the number 2. Thus, a set should be thought of as a block of information that can be used to represent a mathematical structure.

Set theory typically considers only the hereditary sets; a set is said to be hereditary iff all of its elements are sets, all of the elements of its elements are sets, etc. The class $\mathbf{V}$ of all hereditary sets is assumed to satisfy the Zermelo-Fraenkel axioms of set theory, denoted

ZF. These axioms formalize our intuition of sets as classes of limited size. The class **V** is also assumed to satisfy the axiom of choice, denoted **AC**. The theory **ZF** + **AC**, often abbreviated **ZFC**, is the usual foundation for mathematics.

In summary: The class **V** *of hereditary sets satisfies the theory* **ZFC**, *Zermelo-Fraenkel set theory with the axiom of choice.*

1.2 The interpretation of mathematics in set theory

Whatever one's views on the foundation of mathematics, within standard mathematical discourse we assume an external mathematical reality, which consists of mathematical objects which may or may not have various mathematical properties, independent of our ability to determine whether or not a particular mathematical object has a particular mathematical property. For example, it is a simple consequence of classical logic that either there exists an uncountable subset of $\mathbb{R}$ not equinumerous to $\mathbb{R}$ or there does not, despite the fact that this proposition, the continuum hypothesis, cannot be decided from the usual mathematical assumptions.

Thus, we imagine a multitude of mathematical objects, which form our universe of discourse, and each mathematical property yields a function from our universe to the set $\{\top, \bot\}$ of truth-values, true $\top$ and false $\bot$. We also often consider properties that apply jointly to a finite sequence of objects $\mathcal{O}_1, \ldots, \mathcal{O}_n$. For example, group isomorphism is a property of two groups. Each such property yields a function that assigns a truth-value to each n-tuple of objects. The propositions are mathematical properties that do not refer to any variable objects; these are true or false of the universe as whole. For example, the continuum hypothesis is a proposition.

We use the word class to refer to the totality of objects satisfying a particular mathematical property (of a single variable object). For us, classes are not mathematical objects, but are simply properties that are identified if they hold of exactly the same objects.

In summary: The universe of mathematical discourse consists of mathematical objects. Each property holds or fails for all values of variable objects $\mathcal{O}_1, \ldots, \mathcal{O}_n$ *that it references, where* n *depends on the property. A proposition is a property that references no variable objects; it simply holds or fails.*

The above picture is quite inconvenient for mathematical logic because of the diversity of properties that appear in mathematical research. The foundation of mathematics in set theory reduces the study of general mathematical properties to the study of the first-order properties of the hereditary sets, which can be described succinctly and precisely. A first-order property is one that can be expressed by a formula in the language of set theory, i. e., by a grammatically correct string using parantheses, variables, the membership symbol $\in$, and the logical symbols $=$, $\neg$, $\wedge$, $\vee$, $\forall$, and $\exists$.

If hereditary sets are the possible strings of bits in our imaginary computer, then the language of set theory is our programming language. The interpretation of mathematics in set theory is essentially a function that assigns formulas of the language of set theory to mathematical properties, in a way that preserves logical structure and truth. In particular, each property of n variable objects is interpreted as an n-ary formula of the language of set theory, that is, a formula of the language of set theory that has n free variables. The preservation of truth refers to the the requirement that if $\mathbf{p}$ is a proposition that is interpreted as the 0-ary formula π, then $\mathbf{p}$ is true iff π is true in the class $\mathbf{V}$ of all hereditary sets.

The preservation of logical structure refers to the requirement that if a property $\mathbf{p}$ is interpreted as the formula π, then the property "it is not the case that $\mathbf{p}$" is interpreted as the formula $\neg\pi$, the property "there is an object $\mathcal{O}_m$ with property $\mathbf{p}$" is interpreted as the formula $\exists x_m \colon \pi$, and so on. This requirement implies that any mathematical proof can be formalized, i. e, transformed into a sequence of 0-ary set-theoretic formulas that can be mechanically verified to be a proof, provided that the interpretations of our mathematical assumptions themselves have formal proofs from our set-theoretic axioms, **ZFC**.

Due to the diversity of properties that appear in mathematical research, it is difficult to expound an interpretation of mathematics in set theory here. The interpretations that appear in the literature differ, for example, in their construction of the real numbers, i. e., in their interpretation of the property of being a real number. These differences are not important for us, but for concreteness, we mention the interpretation given in Bourbaki's classic texts [4]. The one unacceptable feature of this interpretation is that it does not respect equality, that is, there exists a hereditary set that codes both a set and a natural number in this interpretation, even though there is no object that is both a natural number and a set. This aspect of Bourbaki's interpretation can be easily corrected, essentially by encoding each object as a pair, with the first element indicating its type, and the second element being its code in Bourbaki's interpretation. This is the interpretation that we accept as our standard interpretation.

In summary: We fix a standard interpretation of mathematics in set theory. The standard interpretation sends each property referring to n variable objects to a set-theoretic formula with n free variables, in a way that preserves logical structure. Each proposition is true iff the corresponding 0*-ary set-theoretic formula is true of the hereditary sets.*

One advantage of Bourbaki's interpretation is that all hereditary sets code themselves, i. e., any property of hereditary sets that is expressible by a formula of the language of set theory, is interpreted as that formula. This is not the case for the standard interpretation. Indeed, suppose that the standard interpretation of the property $\mathbf{p}$ of hereditary sets expressed by the formula $\neg x_0 \in x_0$ is $\neg x_0 \in x_0$. Then, the standard interpretation of the proposition "all objects have property $\mathbf{p}$" is $\forall x_0 \colon \neg x_0 \in x_0$, which is true in the class of hereditary sets, so all objects have property $\mathbf{p}$. But, $\mathbf{p}$ is a property of hereditary sets, so we conclude that all objects are hereditary sets, contrary to our conception of mathematical reality. Nevertheless, the standard interpretation of any proposition expressed by a set-theoretic formula is provably equivalent to that formula, from the axioms of Zermelo-Fraenkel set theory.

Mathematical properties are not mathematical objects, but set-theoretic formulas are. We use boldface for the former and lightface for the latter. For example, **AC** denotes the axiom of choice as a proposition about hereditary sets, while AC denotes the set-theoretic formula expressing this proposition, or equivalently, the standard interpretation of this proposition. Tarski's undefinability theorem implies that the truth of a set-theoretic formula in the class $\mathbf{V}$ of all hereditary sets is not a mathematical property.

1.3 Models of set theory

A model of set theory is a kind of simple directed graph; it is a set $\mathfrak{M}$ equipped with a binary relation $\in_{\mathfrak{M}}$ that satisfies the axioms of Zermelo-Fraenkel set theory, in the same way that a group is a set equipped with a binary operation that satisfies the group axioms. The existence of a model of set theory obviously implies the consistency of ZF, so by Gödel's incompleteness theorem, it is impossible to prove that any model of set theory exists. Furthermore, by Gödel's theory of constructibility, the consistency of ZF implies the consistency of ZFC, so the existence of a model of set theory cannot be proved from ZFC, either. However, if we assume the consistency of ZF in addition to the Zermelo-Fraenkel axioms themselves, then Gödel's completeness theorem does yield a model of set theory. Furthermore, the Löwenheim-Skolem theorem then yields a model of set theory that is countable in the usual sense: it is a directed graph with countably many vertices.

The right notion of subobject for models of set theory is that of transitive submodels. If $\mathfrak{M}$ is a model of set theory, then $\mathfrak{N} \subseteq \mathfrak{M}$ is a transitive submodel if it is itself a model of set theory when equipped with the membership relation inherited from $\mathfrak{M}$, and if it is "closed under elements", i. e., $x \in_{\mathfrak{M}} y \in \mathfrak{N}$ implies that $x \in \mathfrak{N}$. This latter condition is visually suggestive of transitivity in the usual sense, which explains the terminology. This condition is important because it ensures that y represents the same set in $\mathfrak{N}$ as it does in $\mathfrak{M}$; after all, a set is determined by its elements. Each model of set theory $\mathfrak{M}$ has a minimum transitive submodel [5] [24] [25] [26], which may be degenerate in the sense that it fails to have some of the ordinals in $\mathfrak{M}$.

If $\mathfrak{M}$ is a model of set theory, then an inner submodel of $\mathfrak{M}$ is a transitive submodel that contains all the ordinals of $\mathfrak{M}$. Each model $\mathfrak{M}$ has a minimum inner submodel $\mathfrak{L}^{\mathfrak{M}}$, which consists of all of the constructible elements of $\mathfrak{M}$. This minimum inner submodel satisfies the axiom of constructibility, and therefore also the axiom of choice and the continuum hypothesis. Thus, if there is a model of ZF, then there is a model of ZF + AC + CH; so the consistency of the former theory implies the consistency of the latter.

In summary: There is a class of simple directed graphs called models of set theory. A transitive submodel $\mathfrak{N}$ *of a model of set theory* $\mathfrak{M}$ *is a subset* $\mathfrak{N} \subseteq \mathfrak{M}$ *that is a model of set theory with the inherited simple directed graph structure, such that all elements with an arrow to some element in the subset* $\mathfrak{N}$ *are themselves in the subset* $\mathfrak{N}$*. An inner submodel is a kind of transitive model.*

We can evaluate each 0-ary set-theoretic formula in a given model $\mathfrak{M}$ of set theory in the obvious way; this is Tarski's definition of truth. For example, the formula $\forall x_1 \colon \neg x_1 \in x_1$ is true in every model of set theory $\mathfrak{M}$, because the binary relation $\in_{\mathfrak{M}}$ is necessarily irreflexive. The set-theoretic formulas that are evaluated as true are closed under logical consequence, so in particular, they are logically consistent. The interpretation of mathematics in $\mathfrak{M}$ refers to the composition of this evaluation with the standard interpretation of mathematics in set theory. Thus, we say that a proposition is true in $\mathfrak{M}$ iff the corresponding 0-ary set-theoretic formula is true for $\mathfrak{M}$. Of course, the propositions true in $\mathfrak{M}$ are also closed under logical consequence, and are therefore consistent with each other. Note that the standard interpretation of any proposition about hereditary sets expressed by a 0-ary set theoretic formula is provably equivalent to that formula, so such a formula is true in a model via the standard interpretation iff it is true in that model in the obvious sense.

The truth or falsehood of a set-theoretic formula with one or more free variables obviously depends not only on the given model of set theory, but also on the values of those variables. For example, in any transitive model, the formula $\forall x_2\colon \neg x_2 \in x_1$ is true only of the empty set. The standard interpretation of a property $\mathbf{p}$ of n object variables is an n-ary set-theoretic formula π which is true or false for any n-tuple of elements of a transitive model $\mathfrak{M}$. If π is true of $x_1, \ldots, x_n$, we say that in $\mathfrak{M}$ these hereditary sets code objects with property $\mathbf{p}$, or, following a standard abuse of terminology, that in $\mathfrak{M}$ these hereditary sets *are* objects with property $\mathbf{p}$. Thus, the phrase "a transitive model in $\mathfrak{M}$" is ambiguous; it may describe an element of $\mathfrak{M}$ that is a transitive model, or an element of $\mathfrak{M}$ that satisfies the standard interpretation of the property "is a transitive model" in $\mathfrak{M}$. In fact, the hereditary sets in $\mathfrak{M}$ in the latter sense form a model of set theory that is canonically isomorphic to $\mathfrak{M}$. If x and y are elements of $\mathfrak{M}$, we say that x codes y iff x is mapped to y by this canonical isomorphism.

In summary: A proposition is true in a model of set theory $\mathfrak{M}$ iff its standard interpretation is true in $\mathfrak{M}$, in the obvious sense. A property of n variable objects is true in a model of set theory $\mathfrak{M}$ for the elements $x_1, \ldots x_n$ of $\mathfrak{M}$ iff its standard interpretation is true for $x_1, \ldots x_n$ in $\mathfrak{M}$, in the obvious sense. In this case, we say that $x_1, \ldots, x_n$ code (or are) objects with the given property in $\mathfrak{M}$.

The class $\mathbf{V}$ of all hereditary sets, equipped with its usual membership relation, is assumed to satisfy the axioms of Zermelo-Fraenkel set theory, but it is not a model of set theory because it is not a set. We may extend the definition of model of set theory to include structures that are large in this sense, just as we sometimes extend the definition of fields to include the field of surreal numbers, which is a proper class. That the axioms of Zermelo-Fraenkel set theory are true in $\mathbf{V}$ implies their consistency in the sense that we cannot obtain a contradiction by reasoning from them, but the axioms themselves do not imply the consistency of ZFC, the set of Zermelo-Fraenkel axioms as mathematical formulas. Though we may speak of the truth or falsehood of a 0-ary formula, as it applies to the class $\mathbf{V}$ of all hereditary sets, it follows by Tarski's undefinability theorem that these are not a mathematical properties. By contrast, the truth of a 0-ary set-theoretic formula in a model of set theory is a mathematical property.

A transitive model is, intuitively, a model of set theory that is a transitive submodel of $\mathbf{V}$. Thus, a transitive model is a set $\mathfrak{M}$ of hereditary sets that is a model of set theory if it is equipped with the usual membership relation, inherited from $\mathbf{V}$, and if it "closed under membership" in the sense that $x \in y \in \mathfrak{M}$ implies $x \in \mathfrak{M}$. The existence of a transitive model cannot be proved from the consistency of ZFC, because there is a model of set theory in which the former proposition is false, but the latter proposition is true. The consistency of ZFC is the proposition that there does not exist a finite combinatorial object of a particular kind, namely a proof of a contradiction using axioms of ZFC, and any such proposition is true in any transitive model iff it is actually true, i. e., there is no mathematical object of this kind in the external mathematical reality that we presuppose. In particular, the consistency of ZFC is true in the minimum model of set theory. However, the existence of a transitive model of set theory is false in the minimum model, essentially because the minimum model is the minimum transitive model, so it cannot contain any transitive models as elements. However, if there exists a (strongly) inaccessible cardinal κ, in the sense that κ is uncountable, that a set with fewer than κ elements also has fewer than κ subsets, and that the union of fewer than κ sets, each having fewer than κ elements, itself has fewer than κ elements, then V_κ, the set of all hereditary sets of rank less than κ, is a transitive model of ZFC. The application of the downward Löwenheim-Skolem theorem, followed by Mostowski collapse, then yields a countable transitive model.

In summary: A transitive model is a hereditary set $\mathfrak{M}$ such that every element of an element of $\mathfrak{M}$ is itself an element of $\mathfrak{M}$, and such that $\mathfrak{M}$ becomes a model of set theory if it is equipped with the usual membership relation.

Forcing is a fundamental set theoretic technique, whose simplest formulation is in terms of countable transitive models. If $\mathfrak{M}$ is a countable transitive model, and B codes a complete Boolean algebra in $\mathfrak{M}$, then B codes a countable Boolean algebra in $\mathbf{V}$. The Rasiowa-Sikorski lemma implies that B admits a Boolean algebra homomorphism to the two-element Boolean algebra $\{\bot, \top\}$ that is $\mathfrak{M}$-normal, i. e., it respects the join of any subset of B that is itself in $\mathfrak{M}$. The set of elements mapped to $\top$ by such a Boolean algebra homomorphism is called an $\mathfrak{M}$-generic ultrafilter on B. Usually, forcing over B is thought of as adding an $\mathfrak{M}$-generic ultrafilter on B to the transitive model $\mathfrak{M}$, but it can be equivalently thought of as adding an $\mathfrak{M}$-normal Boolean algebra homomorphism $B \to \{\bot, \top\}$ to the transitive model $\mathfrak{M}$. If G is an $\mathfrak{M}$-generic ultrafilter on B, then $\mathfrak{M}[G]$ is the minimum transitive model that includes $\mathfrak{M}$ and contains G, and careful selection of the complete Boolean algebra B determines many of the properties of $\mathfrak{M}[G]$.

In summary: If $\mathfrak{M}$ is transitive model, and G is a hereditary set not in $\mathfrak{M}$, then $\mathfrak{M}[G]$ denotes the smallest transitive model that contains G and has $\mathfrak{M}$ as a transitive submodel.

For example, if $\mathfrak{M}$ is the minimum transitive model of set theory, λ is Lebesgue measure inside $\mathfrak{M}$, and B is the complete Boolean algebra of projections of $L^\infty(\mathbb{R}, \lambda)$ inside $\mathfrak{M}$, then each $\mathfrak{M}$-generic ultrafilter G on B corresponds to a so-called "random real number", a real number not in $\mathfrak{M}$. A set is constructible in $\mathfrak{M}[G]$ iff it is in the smallest inner model of $\mathfrak{M}[G]$, so random real numbers are not constructible. A variant of this choice of complete Boolean algebra adds $\aleph_2$ random real numbers, so that the continuum hypothesis fails in $\mathfrak{M}[G]$.

1.4 Solovay's Model

Solovay's construction begins with a countable transitive model $\mathfrak{M}$ of set theory that satisfies axiom of choice and the existence of an inaccessible cardinal. The properties guaranteed by Solovay's construction are not quite sufficient for us, so we ask that our countable transitive model additionally satisfy the axiom of constructibility. This approach was suggested to me by **John Steel**.

The existence of such countable transitive models can be established from the existence of two inaccessible cardinals $\kappa_0 < \kappa_1$. In this case, κ_0 is also an inaccessible cardinal in V_{κ_1}, a transitive model satisfying the axiom of choice. Appealing to the downward Löwenheim-Skolem theorem, and then to Mostowski collapse, we obtain a countable transitive model $\tilde{\mathfrak{M}}$ that satisfies both the existence of an inaccessible cardinal and the axiom of choice. Thus, $\mathfrak{M}$ contains an ordinal κ that is an inaccessible cardinal *in* $\tilde{\mathfrak{M}}$; of course κ is actually countable, i. e., countable in $\mathbf{V}$. The countable transitive model $\tilde{\mathfrak{M}}$ has a least inner model $\mathfrak{L}^{\tilde{\mathfrak{M}}}$, which we will call simply $\mathfrak{M}$, and which satisfies the axiom of constructibility. The axiom of constructibility implies both the axiom of choice, and the continuum hypothesis. The ordinal κ is also an inaccessible cardinal in $\mathfrak{M}$.

In summary: If there exist two strongly inaccessible cardinals, then there exists a countable transitive model $\mathfrak{M}$ that satisfies the axiom of constructibility, the axiom of choice, the continuum hypothesis, and the existence of an inaccessible cardinal.

We now apply Solovay's construction to $\mathfrak{M}$, which may be sketched as follows: There is a complete Boolean algebra in $\mathfrak{M}$, such that the inaccessible cardinal κ becomes the least uncountable ordinal in the corresponding forcing extension $\mathfrak{M}[G]$. This forcing is termed the Lévy collapse of κ to $\aleph_1$. One of the consequences of this collapse is that almost every real number in $\mathfrak{M}[G]$ is random over $\mathfrak{M}[s]$ for every real number s in $\mathfrak{M}[G]$.

Solovay shows that all sets of real numbers definable in $\mathfrak{M}[G]$ by a set-theoretic formula using parameters in $\mathfrak{M}$ and $\mathbb{R}$ satisfy Lebesgue measurability, the Baire property and the perfect set property. A set X of real numbers satisfies Lebesgue measurability iff there is a G_δ set that differs from X on a null set, that is, on a subset that can be covered by a countable family of intervals, the sum of whose lengths is arbitrarily small. A set X of real numbers satisfies the Baire property iff there exists an open set that differs from X on a meager set, that is, a subset that can be covered by a countable family of closed nowhere dense sets. A set X satisfies the perfect set property if it is either countable (possibly finite) or contains a perfect set, that is, a nonempty subset that is closed and has no isolated points. Solavay also shows that if $X \subseteq \mathbb{R}^2$ is definable in $\mathfrak{M}[G]$ using parameters in $\mathfrak{M}$ and $\mathbb{R}$, and for all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $(x, y) \in X$, then there exists a Borel function $h \colon \mathbb{R} \to \mathbb{R}$ such that $(x, h(x)) \in X$ for almost all $x \in \mathbb{R}$. This is a weak form of the axiom of uniformization, itself a weak form of the axiom of choice.

We will work in the inner model $\mathfrak{N}_1$ of $\mathfrak{M}[G]$, which consists of all sets hereditarily definable using sequences of ordinals as parameters, i. e., all sets that are definable using sequences of ordinals as parameters, all of whose elements are also definable using sequences of ordinals as parameters, and so on. Solovay shows that every sequence of ordinals is definable from parameters in $\mathfrak{M}$ and $\mathbb{R}$, so in particular any set of real numbers in $\mathfrak{N}_1$ satisfies Lebesgue measurability, the Baire property, and the perfect set property in $\mathfrak{M}[G]$. It is then straightforward to show these same properties in $\mathfrak{N}_1$, appealing to their absoluteness, which is described below.

Theorem 1.4.1 (Solovay [27, lemma III.2.6, theorem 1]). *The inner submodel $\mathfrak{N}_1$ of $\mathfrak{M}[G]$ is closed under countable unions. Also, the transitive model $\mathfrak{N}_1$ satisfies:*

1. **DC***, the axiom of dependent choices (there is an infinite walk through any directed graph starting from any vertex),*
2. **LM***, the Lebesgue measurability axiom (every set of real numbers is Lebesgue measurable),*
3. **BP***, the Baire property axiom (every set of real numbers has the Baire property), and*
4. **PSP***, the perfect set property axiom (every set of real numbers is either countable or contains a perfect subset).*

Solovay establishes the four above principles in $\mathfrak{N}_1$ without assuming that $\mathfrak{M}$ satisfies the axiom of constructibility. This assumption implies that the Solovay model $\mathfrak{N}_1$ also satisfies the following choice principle, which is established in appendix B:

5. $\mathbf{AC_{ae}}$*, the axiom of choices almost everywhere (for every family of nonempty sets indexed by $\mathbb{R}$ there is a function on $\mathbb{R}$ whose value at almost every real is in the corresponding set).*

The principles **DC**, **LM**, **BP**, and **PSP** are known to imply the following:

6. *All ultrafilters on $\mathbb{N}$ are principal. All ultrafilters are countably additive.*
7. *Every linear function between Fréchet spaces is continuous* [8]*. Every linear function between Banach spaces is bounded.*
8. *For every Banach space, its algebraic dual is equal to its continuous dual. For all σ-finite measure spaces S, $L^\infty(S)^* = L^1(S)$* [30]*, and in particular $\ell^\infty(\mathbb{N})^* = \ell^1(\mathbb{N})$* [22].

To see 6, observe that any nonpricipal ultrafilter on $\mathbb{N}$ yields a subset $U \subseteq \mathbb{Z}_2^{\mathbb{N}}$, such that the membership of any infinite sequence $s \in \mathbb{Z}_2^{\mathbb{N}}$ in U does not depend on any finite initial segment of s, and furthermore $\{U, U+1\}$ is a partition of $\mathbb{Z}_2^{\mathbb{N}}$. Suppose that U is measurable with respect to the coin-flip measure. The symmetry between 0 and 1 in the definition of this measure then implies that the measure of U is $\frac{1}{2}$; this contradicts Kolmogorov's zero-one law, which implies that the measure of U must be 0 or 1. Thus, U is not measurable, which contradicts $\mathbf{DC} + \mathbf{LM}$; see 1.7.5. We conclude that every ultrafilter on $\mathbb{N}$ is principal. Now, suppose that on some set X there exists an ultrafilter $\mathscr{U}$ that is not countably additive. It is straightforward to partition X into a countably many subsets that are not in the ultrafilter. The pushforward of $\mathscr{U}$ to the set of partition blocks is then a nonprincipal ultrafilter on a countable set, which contradicts our previous finding that there are no nonprincipal ultrafilters on $\mathbb{N}$.

That the inner submodel $\mathfrak{N}_1$ of $\mathfrak{M}[G]$ is closed under countable unions means that $\bigcup_{x\in X} x$ is an element of $\mathfrak{N}_1$ whenever $X \subseteq \mathfrak{N}_1$, $X \in \mathfrak{M}[G]$, and X is countable in $\mathfrak{M}[G]$. This last condition is a bit awkward to define within the framework of our approach; it does not mean that X codes a countable hereditary set in $\mathfrak{M}[G]$, but rather that the hereditary set coding X in $\mathfrak{M}[G]$ codes a countable set in $\mathfrak{M}[G]$. It is equivalent to saying that there is an enumeration (x_n) of X such that the set $\{\{0, x_0\}, \{1, x_1,\}, \{2, x_2\} \ldots\}$ is in $\mathfrak{M}[G]$, where 0 denotes the natural number 0 in $\mathfrak{M}[G]$, i. e., the unique element of $M[G]$ that satisfies the standard interpretation of the property "is zero", and likewise for the other natural numbers.

The fact that the inner submodel $\mathfrak{N}_1$ of $\mathfrak{M}[G]$ is closed under countable unions implies that every countable sequence in $\mathfrak{M}[G]$ of objects in $\mathfrak{N}_1$ is in $\mathfrak{N}_1$. This has the consequence that every Cauchy sequence in $\mathfrak{M}[G]$ from a metric space in $\mathfrak{N}_1$ is in $\mathfrak{N}_1$, so in particular every real number in $\mathfrak{M}[G]$ is in $\mathfrak{N}_1$, and any metric space in $\mathfrak{N}_1$ that is complete in $\mathfrak{N}_1$ is also a complete metric space in $\mathfrak{M}[G]$. The absoluteness of many items in section 1.6 relies crucially on these facts.

1.5 Absoluteness

Some familiar results fail in the Solovay model $\mathfrak{N}_1$, but many continue to hold. How can we verify that a familiar theorem continues to hold in $\mathfrak{N}_1$?

The brute force approach is to carefully check the proof to see whether it uses only a fragment of choice provable from $\mathbf{ZF} + \mathbf{DC} + \mathbf{AC_{ae}}$. It is necessary to check the proof down to the foundations, i. e., to check also the proofs of logically preceding results. It would take a good deal of time and care to scrutinize the material of the standard textbooks in some subject area. Yet, after this work is complete, we will have established only the most basic results of a single branch of mathematics. The task of scrutinizing a significant portion of published mathematical research in this way is effectively insurmountable.

In summary: If the usual proof of a theorem uses only the choice principles $\mathbf{DC}$ *and* $\mathbf{AC_{ae}}$*, then it is true in the Solovay model* $\mathfrak{N}_1$*.*

The better approach is to appeal to the absoluteness of many mathematical properties. This approach demands the we scrutinize the *statement* rather than the proof of a familiar mathematical theorem. The statement of Fermat's last theorem is very simple, and we can easily verify that it holds in the Solovay model $\mathfrak{N}_1$ without examining its proof at all.

Absoluteness may be defined for any pair of models, but we will appeal to absoluteness only for the pair $(\mathfrak{N}_1, \mathfrak{M}[G])$, as they are defined in the section above. A proposition is absolute for this pair in case it is true in $\mathfrak{N}_1$ iff it is true in $\mathfrak{M}[G]$. In general, a property of n variable objects is absolute for this pair in case for all $x_1, \ldots, x_n$ from $\mathfrak{N}_1$, the given property holds of $x_1, \ldots, x_n$ in $\mathfrak{N}_1$ iff it holds of $x_1, \ldots, x_n$ in $\mathfrak{M}[G]$. We are concerned only with the Solovay model $\mathfrak{N}_1$, so we use the unqualified term "absolute" to mean absolute for the pair $(\mathfrak{N}_1, \mathfrak{M}[G])$.

In summary: A property $\mathbf{p}$ *of* n *variable objects being absolute means that for all* $x_1, \ldots, x_n$ *in* $\mathfrak{N}_1$*, these hereditary sets code objects with property* $\mathbf{p}$ *in* $\mathfrak{N}_1$ *iff they code objects with property* $\mathbf{p}$ *in* $\mathfrak{M}[G]$.

If we imagine each n-ary set-theoretic formula as a program which takes n blocks of data, stored as hereditary sets, as input, and outputs either true or false, then we might imagine the transitive models $\mathfrak{N}_1$ and $\mathfrak{M}[G]$ as computers with different capabilities. The registers of the computer $\mathfrak{M}[G]$ can store more data than the registers of computer $\mathfrak{N}_1$, so a program that asks the computer to search through blocks of data of the appropriate kind searches through more blocks on $\mathfrak{M}[G]$ than on $\mathfrak{N}_1$, and so might yield a different result. Intuitively, a property is absolute for the pair $(\mathfrak{N}_1, \mathfrak{M}[G])$ if the corresponding program produces the same result on either machine, for any input that can be made to both machines.

Suppose that $\mathbf{p}$ is an absolute proposition that is a theorem of ordinary mathematics. It follows that the corresponding 0-ary set-theoretic formula is provable from **ZFC**. The transitive model $\mathfrak{M}[G]$ satisfies the axioms of **ZFC**, so $\mathbf{p}$ is true in $\mathfrak{M}[G]$. Appealing to absoluteness, we find that $\mathbf{p}$ is true in the Solovay model $\mathfrak{N}_1$. Thus, any absolute proposition that is a theorem of ordinary mathematics is true in the Solovay model.

Suppose now that $\mathbf{p}$ is an absolute property of a single variable object, and that it is a theorem of ordinary mathematics that all objects have this property. As above, it follows that in $\mathfrak{M}[G]$ every element codes an object with property $\mathbf{p}$, and since this property is absolute, we conclude that in $\mathfrak{N}_1$ every element codes an object with property $\mathbf{p}$. Thus, if it is a theorem of ordinary mathematics that every object has the absolute property $\mathbf{p}$, then, in $\mathfrak{N}_1$, every object has the property $\mathbf{p}$. This reasoning generalizes easily to properties that refer to more than one variable object.

In summary: If $\mathbf{p}$ *is an absolute property of the variable objects* $\mathcal{O}_1, \ldots, \mathcal{O}_n$*, and it is a theorem of ordinary mathematics that all tuples of objects* $\mathcal{O}_1, \ldots, \mathcal{O}_n$ *have property* $\mathbf{p}$*, then in the Solovay model* $\mathfrak{N}_1$*, all tuples of objects* $\mathcal{O}_1, \ldots, \mathcal{O}_n$ *have property* $\mathbf{p}$.

For example, we will show that the property "if $\mathcal{O}_1$ is a finite abelian group, then $\mathcal{O}_1$ is isomorphic the direct sum of a finite set of cyclic groups" is an absolute property. Since, it is a theorem of ordinary mathematics that this property holds of all objects, we can conclude that, in the Solovay model $\mathfrak{N}_1$, it holds of all objects.

1.6 Establishing absoluteness

The core of this section is a list of absolute properties that the reader can use to verify theorems as they come up. This list is preceded by several clarifications to help orient the reader.

1.6.1. The absoluteness of defined symbols, e. g., of defined operations, is a well established concept in set theory; for example, see [24, section 2]. The presentation here is designed to be applied by mathematicians not working in logic, and it proceeds further into ordinary mathematics than the other presentations of absoluteness known to me. The absoluteness of a formula of the language of set theory is usually a straightforward consequence of the following basic principles, whose meaning we do not explain:

i. Absolute formulas are closed under Boolean combination and bounded quantification.

ii. Being the initial segment $V_{\omega+1}$ of the set theoretic universe is absolute.

iii. Being the set of countable sequences from a given set is absolute.

iv. Being the image of a given set under an absolute functional class is absolute.

These principles are sufficient to obtain the provided list of absolute properties in the standard interpretation. Furthermore, a minority of the properties on this list is sufficient to obtain the rest of the list without reference to the standard interpretation, because complex mathematical objects such as C*-algebras are widely understood to be defined in terms of basic mathematical objects such as sets, functions, and numbers.

1.6.2. For convenience, we rely on one convention of the standard interpretation: we assume that each structure, such as a group, a metric spaces, or an operator algebra, is the tuple of its parts, in the style of Bourbaki. For example a group is a 4-tuple $(G, \cdot, e, ^{-1})$. This frees us from having to state separately that, for example, being the metric of a metric space is an absolute property.

1.6.3. Mathematical assertions are often phrased in terms of definite descriptions such as "the set of natural numbers" or "the spectrum of a". We enable definite descriptions by extending the language of set theory to include terms of the form

$$\iota x\colon \pi(x, y_0, \ldots, y_n),$$

which denotes the unique x that satisfies $\pi(x, y_0, \ldots, y_n)$ for the tuple $(y_0, \ldots, y_n)$, provided there exists a unique such x for all tuples $(y_0, \ldots, y_n)$. If there is a unique such x in both $\mathfrak{N}_1$ and $\mathfrak{M}[G]$, and furthermore the formula $\pi(x, y_0, \ldots, y_n)$ is absolute, then for all values of $(y_0, \ldots, y_n)$ in $\mathfrak{N}_1$, the term $\iota x\colon \pi(x, y_0, \ldots, y_n)$ denotes the same set in $\mathfrak{M}[G]$ as it denotes in $\mathfrak{N}_1$; in this case we say that $\iota x\colon \pi(x, y_0, \ldots, y_n)$ is absolute.

If the property $\mathbf{p}$ is interpreted by the formula $\pi(x, y_0, \ldots, y_n)$, then the definite description "the x such that $\mathbf{p}$" is interpreted by the term $\iota x\colon \pi(x, y_0, \ldots, y_n)$, and we say that this definite description is absolute in case the term $\iota x\colon \pi(x, y_0, \ldots, y_n)$ is absolute. In practice, a definite description is only meaningful for certain types of parameters, e. g., "the Euler characteristic of X" is not defined for C*-algebras X. We therefore adopt the convention that definite descriptions denote the empty set for parameters that are not of the appropriate type. This convention is of no practical significance, as a careful use of definite descriptions is a basic part of natural mathematical thought.

We summarize the situation by saying that we may freely use definite descriptions, just as we do in ordinary mathematical discourse, and that an absolute definite description is one that denotes the same object in both $\mathfrak{N}_1$ and $\mathfrak{M}[G]$. A definite description "the x such that **p**" is absolute if **p** is an absolute property that is a definition of x in both $\mathfrak{M}[G]$ and $\mathfrak{N}_1$, and the parameters of **p** range over an absolute class, i. e., over all tuples satisfying a given absolute property.

1.6.4. "The class of groups" is not a definite description in the sense above. We can express individual classes by formulas, but we cannot encode all classes by hereditary sets; this would lead to Russell's paradox. We will instead say that "the class of groups" is a class description, and that a class description "the class of x such that **p**" is absolute in case **p** is absolute and every object x satisfying **p** for appropriate parameters in the Solovay model $\mathfrak{N}_1$ is itself in $\mathfrak{N}_1$. Note that if the class of objects x such that **p** is always a set in both $\mathfrak{M}[G]$ and $\mathfrak{N}_1$, then "the class of x such that **p**" is absolute iff "the set of x such that **p**" is absolute.

We treat category descriptions in the same way, but with some sensitivity to the category-theoretic viewpoint. A category description consists of properties that specify the class of objects, the class of morphisms, the composition of morphisms, etc. To be termed absolute, a category description should satisfy the following conditions: First, its constituent properties should be absolute. Second, for all appropriate parameters from the Solovay model $\mathfrak{N}_1$, every morphism between objects in $\mathfrak{N}_1$ should itself be in $\mathfrak{N}_1$. Third, for all appropriate parameters from $\mathfrak{N}_1$, every object should be in $\mathfrak{N}_1$ *up to isomorphism*; we do not ask that these isomorphisms be canonical in any way. Thus the two categories need not be equal, but they must be weakly equivalent. Note that the class description "the class of x such that **p**" is absolute iff the category description "the category of x such that **p** with only identity morphisms" is absolute. Thus, definite descriptions of sets can be viewed as a special case of class descriptions, which, in turn, can be viewed as a special case of category descriptions.

We remark that, because the axiom of choice fails in the Solovay model $\mathfrak{N}_1$, when working with categories in the Solovay model, the appropriate notion of equivalence of categories is anaequivalence [17].

Below, **p** and **q** denote arbitrary absolute properties, **t** and **s** denote arbitrary absolute definite descriptions, and **n** denotes an arbitrary numeral. All other symbols denote *variables*, and symbols in parentheses denote *bound variables* that do not name objects for which the stated property holds or fails. The following are absolute:

1. **s** where the variable (x) has value **t**
2. **p** where the variable (x) has value **t**
3. not **p**
4. **p** and **q**
5. **p** or **q**
6. if **p** then **q**

7. x is equal to y
8. X is a set
9. x is an element of the set X
10. the set of elements (x) of the set X such that **p**
11. the set X is a subset of the set Y
12. the union of sets X and Y
13. the intersection of sets X and Y
14. the set X excluding the elements of the set Y
15. the union of the sets in the set X
16. the set X is empty
17. there exists an element (x) in the set X such that **p**
18. for all elements (x) of the set X it is the case that **p**
19. f is a function
20. the domain of the function f
21. the codomain of the function f
22. the value of the function f at an element x of its domain
23. the function f is injective
24. the function f is surjective
25. the range of the function f
26. the surjective function that maps each element (x) of the set X to **t**
27. the inclusion function of the subset X into the set Y
28. the identity function on the set X
29. the composition of composable functions f and g
30. the inverse of invertible function f
31. the set of natural numbers

32. the natural number **n**
33. the sum of natural numbers n and m
34. the product of natural number n and m
35. the natural number n is less than the natural number m
36. the set of functions from the set of natural numbers to the set X
37. the set of n-tuples of elements of the set X for the natural number n
38. the m-th element of the n-tuple $\overrightarrow{x}$ for the natural number m less than the natural number n
39. the set of n-tuples whose m-th element for the natural number m less than the natural number n, is an element of the m-th element of the n-tuple $\overrightarrow{X}$ of sets
40. the set X is finite
41. the set X is countable
42. the set of finite subsets of X
43. the set of countable subsets of X
44. the set of functions from the countable set X to the set Y
45. R is a binary relation
46. the domain of the binary relation R
47. the codomain of the binary relation R
48. the binary relation R is symmetric
49. the binary relation R is antisymmetric
50. the binary relation R is transitive
51. the binary relation R is reflexive
52. the binary relation R is an equivalence relation
53. the set of equivalence classes of the equivalence relation R
54. the function taking each element of the domain of the equivalence relation R to its equivalence class
55. G is a group

56. R is a ring
57. F is a field
58. A is an algebra
59. V is a vector space
60. X is a basic space, i. e., X is a set equipped with a topological basis
61. the function f from the basic space X to the basic space Y is continuous
62. the closure of the subset Y of the basic space X
63. f is a net in the set X
64. the net f in the basic space X converges to the point x
65. the basic space X is separable
66. the basic space X is Polish, i. e., it is a set equipped with a separable completely metrizable topology
67. the set of continuous maps from the Polish space X to the Polish space Y
68. the set of open subsets of the Polish space X
69. the set of closed subsets of the Polish space X
70. the set of Borel subsets of the Polish space X
71. the set of Borel functions from the Polish space X to the Polish space Y
72. the ring of integers
73. the field of real numbers
74. the field of complex numbers
75. the function f between Euclidean spaces is smooth
76. $\mathfrak{M}$ is a smooth manifold
77. the function f between smooth manifolds is smooth
78. the extended real line, i. e., the closed interval $[-\infty, +\infty]$
79. the Lebesgue integral as a function from Borel sets of real numbers to the extended real line

80. the set X of real numbers is Lebesgue measurable
81. the extended complex plane
82. the function f is holomorphic
83. the function f is entire
84. X is a metric space
85. the basic space obtained from the metric space X, i. e., the set of points of the metric space X equipped with the topological basis of open balls
86. the metric space X is complete
87. $\mathcal{X}$ is a Banach space
88. the dual space of the separable Banach space $\mathcal{X}$
89. $\mathcal{X}$ is a Banach algebra
90. $\mathcal{H}$ is a Hilbert space
91. the subset X of the Hilbert space $\mathcal{H}$ is an orthonormal basis
92. x is a bounded operator on the Hilbert space $\mathcal{H}$
93. A is a concrete C*-algebra of operators on a Hilbert space $\mathcal{H}$
94. the space of ultraweakly continuous functionals on the concrete C*-algebra A
95. A is an abstract C*-algebra
96. μ is a state on the abstract C*-algebra A
97. π is a $*$-homomorphism from the abstract C*-algebra A to the abstract C*-algebra B
98. ρ is a $*$-homomorphic action of the abstract C*-algebra A on the Hilbert space $\mathcal{H}$
99. $\mathcal{H}$ is a separable Hilbert space
100. the set of all separable concrete C*-algebras on the separable Hilbert space $\mathcal{H}$
101. the set of all von Neumann algebras on the separable Hilbert space $\mathcal{H}$
102. the state space of the separable C*-algebra A
103. the C*-algebra A is approximately finite dimensional

104. the von Neumann algebra M on a separable Hilbert space $\mathcal{H}$ is approximately finite dimensional

105. the separable C*-algebra A is type I

106. the von Neumann algebra M on the separable Hilbert space $\mathcal{H}$ is type I

107. the von Neumann algebra M on the separable Hilbert space $\mathcal{H}$ is type II

108. the von Neumann algebra M on the separable Hilbert space $\mathcal{H}$ is type III

109. the factor M on the separable Hilbert space $\mathcal{H}$ is type I_n for the natural number n

110. the factor M on the separable Hilbert space $\mathcal{H}$ is type I_∞

111. the factor M on the separable Hilbert space $\mathcal{H}$ is type II_1

112. the factor M on the separable Hilbert space $\mathcal{H}$ is type II_∞

113. the factor M on the separable Hilbert space $\mathcal{H}$ is type III_λ for the real number λ

114. the category of countable graphs and graph morphisms

115. the category of countable groups and group homomorphisms

116. the category of second countable locally compact Hausdorff spaces and continuous maps

117. the category of second countable locally compact groups and continuous group homomorphisms

118. the category of second countable compact Hausdorff spaces and continuous maps

119. the category of complete separable metric spaces and contractive maps

120. the category of Polish spaces and continuous maps

121. the category of standard Borel spaces and measurable functions

122. the category of separable Banach spaces and bounded linear maps

123. the category of separable Hilbert spaces and bounded linear maps

124. the category of separable C*-algebras and bounded linear maps

125. the category of separable C*-algebras and $*$-homomorphisms

126. the category of separable C*-algebras and C*-morphisms

127. the category of von Neumann algebras on separable Hilbert spaces and ultraweakly continuous linear maps

128. the category of von Neumann algebras on separable Hilbert spaces and ultraweakly continuous unital $*$-homomorphisms

129. the category of Borel equivalence relations on Polish spaces and Borel reductions

130. the class of ordinals

131. the ordinal α is less than the ordinal β

132. the set of countable ordinals

133. the smallest uncountable ordinal

Example 1.6.5 (terms and equations). We begin with item 22: "the value of the function f at an element x of its domain" is an absolute definite description; symbolically, "$f(x)$" is absolute. Item 1 explains that substitution preserves absoluteness, so nested terms such as "$f(g(h(x)))$" are absolute. Next, we can deduce that being the **m**-th element of a pair, or any **n**-tuple is absolute, so "the pair whose first element is x_0 and whose second element is x_1" is an absolute definite description; symbolically "(x_0, x_1)" is absolute, and similarly for all **n**-tuples. We now combine these observations to deduce that arbitrary terms formed of nested function symbols of arbitrary arity, e. g., "$f(g(x, y), h(y))$", are absolute. Finally, since the substitution of absolute definite descriptions into an absolute property yields another absolute property, we deduce that any equation is absolute.

Example 1.6.6 (Associativity of addition). The example above shows that the equation $(n+m)+k = n+(m+k)$ is absolute, so we might be tempted to jump to the conclusion that addition of natural numbers is associative in the Solovay model $\mathfrak{N}_1$, because it is associative normally, but we have said nothing about the variables n, m, k and $+$! However, the property "if $+$ is the addition of natural numbers and n is a natural number and m is a natural number and k is a natural number, then $(n+m)+k = n+(m+k)$" is absolute; since it holds for *all* objects in $\mathfrak{M}[G]$, it holds for *all* objects in $\mathfrak{N}_1$. Thus, the associativity of addition is verified in the Solovay model $\mathfrak{N}_1$.

Example 1.6.7 (quantification). Consider the equation "$x \cdot x \cdot x = 2$". It is absolute, together with the specifications that x denotes a real number, that $\cdot$ denotes muliplication, and that 2 denotes the number two. This equation has a solution in the Solovay model $\mathfrak{N}_1$, just as it does normally, because "there exists an element x of the set of real numbers such that $x \cdot x \cdot x = 2$" is absolute. We may similarly verify the solution of $x \cdot x \cdot x = 2$ is unique in the Solovay model $\mathfrak{N}_1$, and thereby establish that the definite description "the cube root of two" is absolute.

It is crucial that we quantify over a set! The property "X is a nonmeasurable set of real numbers" is absolute, but "there exists an X such that X is a nonmeasurable set of real numbers" is certainly not. The property "there exists an element of the set of sets of real numbers X such that X is a nonmeasurable set of real numbers" quantifies over the set of sets of real numbers, but it is not absolute because the definite description "the set of sets of real numbers" is not absolute.

Example 1.6.8 (Fermat's last theorem). In order to show that the inequality "$a^n + b^n \neq c^n$", together with its usual specifications, is absolute, we need only to show that "the exponentiation of natural numbers" is an absolute definite description. The property "exp is a function, and the domain of exp is the Cartesian square of the set of natural numbers, and the codomain of exp is the set of natural numbers, and for all natural numbers n, $exp(n, 0) = 1$, and for all natural numbers n and m, $exp(n, m + 1) = exp(n, m) \cdot n$" is absolute, so we can verify that there is a unique such object in the Solovay model $\mathfrak{N}_1$ by quantifying over "the set of functions from the Cartesian square of the set of natural numbers to the set of natural numbers"; this is an absolute definite description by item 44. Therefore, "if n is greater than two, then $a^n + b^n \neq c^n$" is absolute, so we have verified that Fermat's last theorem holds in the Solovay model $\mathfrak{N}_1$.

Example 1.6.9 (Fuglede's theorem: If x and y are bounded operators on a (possibly nonseparable) Hilbert space $\mathcal{H}$ and y is normal, then $xy = yx$ implies that $xy^* = y^*x$). We deduce that the following are absolute:

- $\mathcal{H}$ is a Hilbert space
- x and y are bounded operators on $\mathcal{H}$
- $yy^* = y^*y$
- $xy = yx$ implies that $xy^* = y^*x$
- if x and y are bounded operators on the Hilbert space $\mathcal{H}$, and y is normal, then $xy = yx$ implies that $xy^* = y^*x$

Example 1.6.10 (Kaplansky's density theorem: If $A \subseteq \mathcal{B}(\mathcal{H})$ is a concrete C*-algebra, then the unit ball of A is strongly dense in the unit ball of the strong closure of A). We deduce that the following are absolute:

- A is a concrete C*-algebra on the Hilbert space $\mathcal{H}$
- x is a contraction on the Hilbert space $\mathcal{H}$
- a is in the unit ball of A
- ξ is in $\mathcal{H}$ and $\|(a - x)\xi\| \leq 1$
- T is a finite subset of $\mathcal{H}$ such that there exists an element a of the concrete C*-algebra A on the Hilbert space $\mathcal{H}$ such that for all ξ in T it is the case that $\|(a - x)\xi\| \leq 1$

- T is a finite subset of $\mathcal{H}$ such that there exists an element a of the unit ball of the concrete C*-algebra A on the Hilbert space $\mathcal{H}$ such that for all ξ in T it is the case that $\|(a-x)\xi\| \leq 1$
- if A is a concrete C*-algebra on the Hilbert space $\mathcal{H}$, and x is a contraction on $\mathcal{H}$ with the property that every SOT neighborhood of x contains an element of A, then every SOT neighborhood of X contains an element of the unit ball of A

Example 1.6.11 (Gelfand duality for separable commutative C*-algebras: If A is a separable commutative C*-algebra then $A \cong C_0(\hat{A})$). We deduce that the following are absolute:

- A is a separable commutative C*-algebra
- the space of homomorphic states of the separable commutative C*-algebra A

(The space of homomorphic states of an arbitrary commutative C*-algebra is not an absolute definite description; see example 1.9.1.) We verify that the spectrum of a separable commutative C*-algebra is a Polish space. We deduce that the following are absolute:

- the set of continuous functions from the space of homomorphic states on the separable commutative C*-algebra A to the set of complex numbers that vanish at infinity
- if A is a separable commutative C*-algebra, then $C_0(\hat{A})$ is a C*-algebra

We verify that if A is a separable commutative C*-algebra, then $C_0(\hat{A})$ is separable and complete. We deduce that the following are absolute:

- the set of $*$-isomorphisms from the separable commutative C*-algebra A to $C_0(\hat{A})$
- the separable commutative C*-algebra A is isomorphic to $C_0(\hat{A})$

Example 1.6.12 (If V is a closed subspace of the Hilbert space $\mathcal{H}$, then there exists an orthogonal projection operator p such that $p\mathcal{H} = V$). We deduce that the following are absolute:

- V is a closed subspace of the Hilbert space $\mathcal{H}$
- there exists an element of the closed subspace V of the Hilbert space $\mathcal{H}$ closest to the element ξ of $\mathcal{H}$

We verify that if V is a closed subspace of $\mathcal{H}$, and ξ is an element of $\mathcal{H}$, then there exists a unique element of V closest to ξ. We deduce that the following are absolute:

- the element of the closed subspace V of the Hilbert space $\mathcal{H}$ closest to the element ξ of $\mathcal{H}$
- the function taking each element of the Hilbert space $\mathcal{H}$ to the closest element of the closed subspace V

- the function taking each element of the Hilbert space $\mathcal{H}$ to the closest element of the closed subspace V is an orthogonal projection operator on $\mathcal{H}$ whose image is V

We verify that if V is a closed subspace of the Hilbert space $\mathcal{H}$, then the function p taking each element of $\mathcal{H}$ to the closest element of V is an orthogonal projection operator on $\mathcal{H}$ such that $p\mathcal{H} = V$; thus, such an orthogonal projection operator exists.

1.7 Cheat sheet

This section lists theorems in the theory of operator algebras that hold in the Solovay model $\mathfrak{N}_1$. A theorem that is verifiable by a straightforward absoluteness argument is terminated with a period. A theorem whose verification requires the scrutiny of a substantial part of its usual proof for applications of the axiom of choice is terminated with two periods.. A theorem whose verification requires a proof different from its usual proof is terminated with three periods... A theorem which fails ordinarily, but which holds in the Solovay model $\mathfrak{N}_1$ is punctuated with a exclamation mark! Theorems punctuated in these last two ways are addressed in appendix C.

The following theorems hold in the Solovay model $\mathfrak{N}_1$:

Let X be a metric space.

1.7.1. The metric space X has a completion..

1.7.2. If X is complete, then the intersection of a countable family of dense open sets is dense.

Let X be a complete separable metric space.

1.7.3 (cf. [13] theorem 15.6). If X is uncountable, then there is a bijection from $\mathbb{I}$ to X such that the preimage any Borel set is Borel, and the image of any Borel set is Borel.

1.7.4 (cf. [13] theorem 17.41). If m is an atomless Borel probability measure on X, then there is a bijection from $\mathbb{I}$ to X such that the preimage of any Borel set is Borel, the image of any Borel set is Borel, and the pushforward of Lebesgue measure on the unit interval is m.

1.7.5. If m is a Borel probability measure on X, then for every subset $S \subseteq X$, there are Borel subsets $B_0, B_1 \subseteq X$ such that $B_0 \subseteq S \subseteq B_1$ and $m(B_0) = m(B_1)$!

Let m, m_0, and m_1 be (totally defined) probability measures on sets T, T_0, and T_1, respectively, with each set injectable into $\mathbb{R}$.

1.7.6. The measure m is a pushforward of Lebesgue measure on $\mathbb{I}$, i. e., there is a function $f\colon \mathbb{I} \to T$ such that $m(X)$ is equal to the Lebesgue measure of $f^{-1}(X)$ for each $X \subseteq T$!

1.7.7. If $f\colon T_0 \times T_1 \to \mathbb{C}$ is a function such that

1. $\int_{t_0 \in T_0} \int_{t_1 \in T_1} |f(t_0, t_1)| \, dm_1 \, dm_0 < \infty$,

2. $\int_{(t_0,t_1)\in T_0\times T_1} |f(t_0,t_1)|\, d(m_0\times m_1) < \infty$, or

3. $\int_{t_1\in T_1}\int_{t_0\in T_0} |f(t_0,t_1)|\, dm_0\, dm_1 < \infty$,

then

$$\begin{aligned}\int_{t_0\in T_0}\int_{t_1\in T_1} f(t_0,t_1)\, dm_1\, dm_0 &= \int_{(t_0,t_1)\in T_0\times T_1} f(t_0,t_1)\, d(m_0\times m_1)\\ &= \int_{t_1\in T_1}\int_{t_0\in T_0} f(t_0,t_1)\, dm_0\, dm_1!\end{aligned}$$

Let X be a topological space, let $Y\subseteq X$ be a subspace, and let $f\colon X\to Z$ be a function to another topological space.

1.7.8 (cf. [20] proposition 1.3.6). A point belongs to the closure of the set Y iff there is a net in Y converging to that point...

1.7.9 (cf. [20] proposition 1.4.3). The function f is continuous at $x\in X$ iff for every net (x_λ) converging to X, the net $(f(x_\lambda))$ converges to $f(x)$...

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces.

1.7.10. Any linear function from $\mathcal{X}$ to $\mathcal{Y}$ is bounded!

1.7.11. A surjective linear function from $\mathcal{X}$ to $\mathcal{Y}$ is open!

1.7.12. If $\{T_\lambda\}$ is a family of (bounded) linear functions from $\mathcal{X}$ to $\mathcal{Y}$ such that $\{T_\lambda x\}$ is bounded for all $x\in X$, then $\{\|T_\lambda\|\}$ is bounded.

Let $\mathcal{X}$ be a separable Banach space.

1.7.13. Every bounded complex-valued linear function defined on a subspace of $\mathcal{X}$ extends to a linear function of the same norm on all of $\mathcal{X}$.

1.7.14. The unit ball of $\mathcal{X}^*$ is Polish and compact, in the weak* topology.

1.7.15. If $K\subseteq\mathcal{X}^*$ is bounded and weak*-closed, then every element of K is in the weak*-closed convex hull of the extreme points of K, and is the barycenter of a Borel probability measure concentrated on the extreme points of K.

Let $\mathcal{H}$ be a Hilbert space. (We do not define Hilbert spaces to be separable.)

1.7.16. If φ is a functional on $\mathcal{H}$, then there exists a vector $\xi\in\mathcal{H}$ such that $\varphi(\eta)=\langle\xi|\eta\rangle$ for all $\eta\in\mathcal{H}$.

1.7.17. If ψ is a bounded sesquilinear form on $\mathcal{H}$, then there exists a bounded operator x on $\mathcal{H}$ such that $\psi(\xi,\eta)=\langle\xi|x\eta\rangle$ for all $\xi,\eta\in\mathcal{H}$.

1.7.18. The function $p\mapsto p\mathcal{H}$ is a bijection between projection operators on $\mathcal{H}$ and its closed subspaces.

1.7.19. Every monotonically decreasing net of positive operators on $\mathcal{H}$ has a greatest lower bound, and converges to it ultrastrongly.

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a nondegenerate concrete C*-algebra.

1.7.20. If φ is a vector functional on A, then there exist $\xi, \eta \in \mathcal{H}$ such that $\|\xi\|^2 = \|\varphi\| = \|\eta\|^2$ and $\varphi(a) = \langle \eta | a\xi \rangle$ for all $a \in A$.

1.7.21. If φ is a vector functional on A, then there exist unique positive vector functionals φ_+ and φ_- on A such that $\varphi = \varphi_+ - \varphi_-$ and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$.

1.7.22. If φ is a vector functional on A, then there exists a unique positive functional $|\varphi|$ on A such that $\| \, |\varphi| \, \| = \|\varphi\|$ and $|\varphi(a)|^2 \leq \|\varphi\| |\varphi|(a^*a)$ for all $a \in A$.

1.7.23. The ultraweak closure of A is equal to its double commutant A''..

1.7.24. If x is in the ultraweak closure of A, and $\|x\| \leq 1$, then x is in the ultraweak closure of the unit ball of A.

1.7.25 (cf. [19] theorem 2.4.3). If $\mathcal{H}$ is separable, then A is a von Neumann algebra iff it closed under limits of ascending sequences.

1.7.26 (cf. [19] section 3.12). The centralizers of A form a C*-algebra that is isomorphic to the C*-algebra of multipliers of A.

Let $B \subseteq \mathcal{B}(\mathcal{H})$ be a concrete C*-algebra that is closed under limits of ascending sequences, e. g., a von Neumann algebra; see [19, section 4.5].

1.7.27. The support projection of every self-adjoint operator in B is itself in B.

1.7.28. The projections of B are closed under countable meets and joins.

1.7.29. For every operator $b \in B$, there exists a unique partial isometry u such that u^*u is the support projection of $|x| = (x^*x)^{\frac{1}{2}}$ and $x = u|x|$.

Let A be an abstract C*-algebra.

1.7.30. The positive elements of A of norm strictly less than 1 form an approximate unit.

1.7.31. Every normal element of A has the continuous functional calculus.

1.7.32 (GNS). For each state μ on A, there exists a representation $\gamma_\mu \colon A \to \mathcal{B}(\mathcal{H}_\mu)$ and a cyclic vector ξ_μ such that $\langle \xi_\mu | \gamma_\mu(a) \xi_\mu \rangle = \mu(a)$ for all $a \in A$..

1.7.33. If $\pi \colon A \to \mathcal{B}(\mathcal{H})$ is a representation with cyclic vector η_0, $\mu = \langle \eta_0 | \pi(\cdot) \eta_0 \rangle$, and $\gamma_\mu \colon A \to \mathcal{B}(\mathcal{H}_\mu)$ is the GNS representation for μ, then there exists a unique unitary operator u from $\mathcal{H}$ to $\mathcal{H}_\mu$ such that $u\eta_0 = \xi_\mu$ and $u\pi(a) = \gamma_\mu(a)u$ for all $a \in A$..

1.7.34. For each state μ on A, the GNS representation $\gamma_\mu \colon A \to \mathcal{B}(\mathcal{H}_\mu)$ is irreducible iff μ is pure..

1.7.35. The ultraweak closure of A in its universal representation is an enveloping von Neumann algebra of A, i. e., every $*$-homomorphism from A into a von Neumann algebra factors uniquely through this ultraweak closure via an ultraweakly continuous $*$-homomorphism...

Let A be a separable C*-algebra.

1.7.36. The universal representation of A is faithful.

1.7.37. If A is commutative, then the spectrum $\hat{A}$ is a locally compact Polish space such that $A \cong C_0(\hat{A})$.

1.7.38. The convex hull of the pure states of A, together with 0, is weak* dense in the quasi-state space of A. Every state of A is the barycenter of a Borel probability measure on its pure state space.

1.7.39 (cf. [19] theorem 6.8.7). The following are equivalent:

(i) A is a C*-algebra of type I,

(ii) $\mathscr{B}(A)$ is a Borel $*$-algebra of type I.

(iv) A has a composition series in which each quotient has continuous trace.

(v) The image of every irreducible representation of A includes the compact operators.

(vi) Two irreducible representations of A are unitarily equivalent iff they have the same kernel.

(vii) The Borel structure on $\hat{A}$ generated by the Jacobson topology is standard.

(viii) Pedersen's Davies Borel structure on $\hat{A}$ is countably separated.

(ix) The Mackey Borel structure on $\hat{A}$ is countably separated.

(x) Every factor representation of A is type I.

(xi) A has no factor representations of type II.

(xii) A has no factor representations of type III.

1.7.40 (cf. [19] proposition 6.3.2). If A is type I, then the Mackey Borel structure and Pedersen's Davies Borel structure coincide with the Borel structure on $\hat{A}$ generated by the Jacobson topology.

Let $M, N \subseteq \mathcal{B}(\mathcal{H})$ be von Neumann algebras on a separable Hilbert space.

1.7.41. Every functional on M is ultraweakly continuous!

1.7.42. If M and N are approximately finite dimensional factors, both of type I_n for some $n \in \mathbb{N} \cup \{\infty\}$, of type II_k for some $k \in \{1, \infty\}$, or of type III_λ for some $\lambda \in (0, 1]$, then $M \cong N$.

1.8 Cardinality

Recall that for two sets X and Y, we write $X \preccurlyeq Y$ in case there is an injection from X into Y; this defines a preorder on the class of all sets. The proof of the Schröder-Bernstein theorem does not use the axiom of choice, so iff $X \preccurlyeq Y$ and $Y \preccurlyeq X$, then $X \approx Y$, i. e., there exists a bijection between X and Y. We write $X \prec Y$ if $X \preccurlyeq Y$, but $X \not\approx Y$.

Example 1.8.1. If $X \subseteq \mathbb{R}$, then $X \preccurlyeq \mathbb{N}$ or $X \approx \mathbb{R}$. Indeed, if X is not countable, then it contains a perfect subset. Every perfect subset of $\mathbb{R}$ is equinumerous to $\mathbb{R}$, so $\mathbb{R} \succcurlyeq X \succcurlyeq \mathbb{R}$.

Thus, in the sense above, the continuum hypothesis holds in $\mathfrak{N}_1$. However, $\mathbb{R} \not\approx \omega_1$. In the Solovay model, two sets need not be comparable.

Example 1.8.2. If every set of real numbers is Lebesgue measurable, then $\omega_1 \not\preccurlyeq \mathbb{R}$ and $\mathbb{R} \not\preccurlyeq \omega_1$. Recall that ω_1 denotes the least uncountable ordinal. If $\mathbb{R} \preccurlyeq \omega_1$, i. e., there exists an injection of $\mathbb{R}$ into ω_1, then $\mathbb{R}$ can be well-ordered, which implies the existence of a nonmeasurable set, by Vitali's theorem. Similarly, if $\omega_1 \preccurlyeq \mathbb{R}$, i. e., there exists an injection of ω_1 into $\mathbb{R}$, then ω_1 is equinumerous to an uncountable subset of $\mathbb{R}$, so $\omega_1 \approx \mathbb{R}$, and again $\mathbb{R}$ can be well-ordered, implying the existence of a nonmeasurable set.

The behavior of quotients is perhaps the most startling aspect of the cardinality hierarchy in the Solovay model $\mathfrak{N}_1$. In the absence of the axiom of choice, the existence of a surjection of X onto Y does not imply the existence of an injection of Y into X. In fact, a quotient of a set X may be *strictly larger* than X itself.

Example 1.8.3. In the Solovay model, $\mathbb{R}/\mathbb{Q} \succ \mathbb{R}$. If (q_n) is an enumeration of the rationals, then straightforward analysis shows that the function $t \mapsto \sum_{q_n<t} \frac{1}{n!}$ is an injection of $\mathbb{R}$ into $\mathbb{R}/\mathbb{Q}$. We show that $\mathbb{R}/\mathbb{Q} \not\preccurlyeq \mathbb{R}$ by constructing a nonprincipal ultrafilter on $\mathbb{R}/\mathbb{Q}$, and then showing that there exist no nonprincipal ultrafilters on $\mathbb{R}$. The subsets of $\mathbb{R}/\mathbb{Q}$ are in bijective correspondence with the translation invariant subsets of $\mathbb{R}$. Each subset of $\mathbb{R}$ is measurable, and if it is translation invariant, then it must have zero measure or infinite measure. Thus, the subsets of $\mathbb{R}/\mathbb{Q}$ that correspond to subsets of $\mathbb{R}$ of infinite measure form a nonprincipal ultrafilter on $\mathbb{R}/\mathbb{Q}$. Every ultrafilter in the Solovay model is countably complete, for otherwise we would immediately obtain a nonprincipal ultrafilter on $\mathbb{N}$. In particular, every ultrafilter on $\mathbb{R}$ is countably complete. For each natural number n, we can cover $\mathbb{R}$ by countably many closed intervals of length 2^{-n}, so for each natural number n we can choose an interval I_n of length 2^{-n} from the given ultrafilter. The family (I_n) has the finite intersection property, so its intersection is nonempty, diameter zero, and in the ultrafilter. Thus, the given ultrafilter is principal.

Every element of $\mathbb{R}/\mathbb{Q}$ is a countable subset of $\mathbb{R}$, so we have also shown that there are more than continuum many countable sets of real numbers.

1.9 Counterexamples

The axiom of choice is false in the Solovay model $\mathfrak{N}_1$, so everything that is equivalent to the axiom of choice is also false in $\mathfrak{N}_1$, e.g., Zorn's lemma, the well-ordering principle, Tychonoff's theorem [14], existence of vector space bases [3], etc.

Example 1.9.1 (Hahn-Banach theorem). The only functional on the C*-algebra $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$ is the zero functional. Indeed, any functional on $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$ yields a functional on $\ell^\infty(\mathbb{N})$ that vanishes on all finitely supported elements of $\ell^\infty(\mathbb{N})$, in contradiction to principle 8 of section 1.4.

Example 1.9.2 (Choquet's theorem). Let λ denote Lebesgue measure; the C*-algebra $L^\infty(\mathbb{I}, \lambda)$ has at least one state, but it has no pure states. One obvious state on $L^\infty(\mathbb{I}, \lambda)$ is integration against Lebesgue measure. To show that $L^\infty(\mathbb{I}, \lambda)$ has no pure states, suppose that μ is such a state. As usual, μ is a unital $*$-homomorphism $L^\infty(\mathbb{I}, \lambda) \to \mathbb{C}$ (1.7.34). Composing μ with the obvious quotient map $\ell^\infty(\mathbb{I}) \to L^\infty(\mathbb{I}, \lambda)$, we obtain a unital $*$-homomorphism $\ell^\infty(\mathbb{I}) \to \mathbb{C}$, which yields a nonprincipal ultrafilter on $\mathbb{I}$. As we discuss in example 1.8.3, no such object exists in the Solovay model $\mathfrak{N}_1$.

Example 1.9.3 (choosing representatives). We have already seen that it is sometimes impossible to choose representatives for a given equivalence relation, e. g., for $\mathbb{R}$ modulo $\mathbb{Q}$ (example 1.8.3). We now provide another example, that is closer to the material of the second chapter. For any C*-algebra A, the spectrum $\hat{A}$ is usually defined as the set of unitary equivalence classes of irreducible representations of A ([19, remark 4.1.1] [2, remark II.6.5.13]). This definition is evidently informal since the unitary equivalence classes of irreducible representations of a nonzero C*-algebra are proper classes, which cannot be the elements of a set. Under the axiom of choice, it is always possible to choose representative irreducible representations from these equivalence classes; I don't know whether this is always possible in the Solovay model $\mathfrak{N}_1$. It is impossible to choose such representatives that are GNS representations if the C*-algebra A is separable and not of type I, because such a choice yields an injection from $\hat{A}$ to $\mathcal{S}(A)$, which impossible (proposition 2.8.2).

Chapter 2

The continuum-weak topology

In this chapter, working in the Solovay model $\mathfrak{N}_1$, defined in section 1.4, we define the continuum-weak topology and the class of V*-algebras, and we show that the enveloping V*-algebra of a separable C*-algebra of type I is isomorphic to an ℓ^∞-direct sum of type I factors. Our primary reference is Pedersen's pair of books [19] [20]. I also recommend Blackadar's *Operator Algebras* [2] for a more comprehensive presentation. Some notation and terminology is reviewed in appendix A.

2.1 Basic definitions

2.1.1. A continuum in a set X is a function $\mathbb{I} \to X$.

Definition 2.1.2. Let $\mathcal{H}$ be a Hilbert space. The continuum-weak topology on $\mathcal{B}(\mathcal{H})$ is given by functionals of the form

$$x \mapsto \int_0^1 \langle \eta_t | x \xi_t \rangle \, dt \tag{I}$$

for families $(\eta_t \in \mathcal{H})$ and $(\xi_t \in \mathcal{H})$ such that the functions $(\|\eta_t\|^2 \colon t \in \mathbb{I})$ and $(\|\xi_t\|^2 \colon t \in \mathbb{I})$ are integrable with respect to Lebesgue measure.

2.1.3. When working with bounded operators, we will always use the closure line $\overline{(\cdot)}$ to denote closure in the continuum-weak topology.

2.1.4. Every probability measure on a set $T \preccurlyeq \mathbb{R}$ is a pushforward of Lebesgue measure on the unit interval; see 1.7.6. It follows that whenever (η_t) and (ξ_t) are families of vectors such that the functions $(\|\eta_t\|^2)$ and $(\|\xi_t\|^2)$ are integrable with respect to a probability measure m on some set $T \preccurlyeq \mathbb{R}$, the functional $x \mapsto \int_{t \in T} \langle \eta_t | x \xi_t \rangle \, dm$ is of the form (I). Clearly, the same is also true of any finite measure m on a set $T \preccurlyeq \mathbb{R}$. It follows that functionals of the form (I) are closed under addition and scalar multiplication, so *every* continuum-weakly continuous functional is of the form (I).

2.1.5. The continuum-weak topology is finer than the ultraweak topology, but coarser than the norm topology. We will see in section 2.4 that the continuum-weak topology coincides with the ultraweak topology for separable Hilbert spaces $\mathcal{H}$. It follows that in general the continuum-weak topology is incomparable with the strong operator topology. Furthermore, it is in general incomparable with the ultrastrong topology; for example, if $\mathcal{H} = \ell^2(\mathbb{R})$, then the net of projection operators corresponding to finite subsets of $\mathbb{R}$ converges to the identity in the ultrastrong topology, but diverges in the continuum-weak topology. By analogy with the ultrastrong topology, we can also define the continuum-strong topology as the topology given by seminorms $x \mapsto \sqrt{\int_0^1 \|x\xi_t\|^2\,dt}$ for $(\|\xi_t\|^2)$ integrable with respect to Lebesgue measure. In the usual way, one can show that the continuum-strong topology is finer than both the ultrastrong topology and the continuum-weak topology.

2.1.6. The adjoint operation is continuum-weakly continuous, operator addition is jointly continuum-weakly continuous, and operator multiplication is continuum-weakly continuous in each variable.

Definition 2.1.7. A concrete C*-algebra $E \subseteq \mathcal{B}(\mathcal{H})$ is a V*-algebra in case it is closed in the continuum-weak topology.

2.1.8. Every von Neumann algebra is a V*-algebra.

2.2 Continuum amplification

2.2.1. Let $E \subseteq \mathcal{B}(\mathcal{H})$ be a V*-algebra. Clearly, a functional $\varphi : E \to \mathbb{C}$ is continuum-weakly continuous iff it is a vector functional in the canonical representation of E on the Hilbert space $L^2(\mathbb{I}, \mathcal{H})$. Note that, in general, the isometry $u\colon L^2(\mathbb{I}) \otimes \mathcal{H} \to L^2(\mathbb{I}, \mathcal{H})$ given by $u(f \otimes \xi)(t) = f(t)\xi$ is not surjective; for example, if $\mathcal{H} = \ell^2(\mathbb{I})$ then the function taking every real number in $\mathbb{I}$ to the corresponding orthonormal basis element is square-integrable, but it is orthogonal to the image of u.

Proposition 2.2.2. *If (φ_s) is an indexed family of continuum-weakly continuous functionals on E such that $(\|\varphi_s\|\colon s \in \mathbb{I})$ is integrable, then the functional $\varphi : x \mapsto \int_0^1 \varphi_s(x)\,ds$ is also continuum-weakly continuous.*

Proof. Each continuum-weakly continuous functional is a vector functional for the canonical representation $\pi : E \to \mathcal{B}(L^2(\mathbb{I}, \mathcal{H}))$. Therefore, we can apply $\mathbf{AC_{ae}}$ to choose, for almost all $s \in \mathbb{I}$, vectors $\xi^s = [\xi^s_t\colon t \in \mathbb{I}]$ and $\eta^s = [\eta^s_t\colon t \in \mathbb{I}]$ in $L^2(\mathbb{I}, \mathcal{H})$, such that $\|\xi^s\|^2 = \|\varphi_s\| = \|\eta^s\|^2$, and $\varphi_s : x \mapsto \langle \eta^s | \pi(x) \xi^s \rangle$, i. e.,

$$\int_0^1 \|\xi^s_t\|^2\,dt = \|\varphi_s\| = \int_0^1 \|\eta^s_t\|^2\,dt$$

and

$$\varphi_s(x) = \int_0^1 \langle \eta^s_t | x \xi^s_t \rangle\,dt.$$

Applying Tonelli's theorem, we find that the function $(s,t) \mapsto \|\xi_t^s\|^2$ is integrable on $\mathbb{I} \times \mathbb{I}$, as is $(s,t) \mapsto \|\eta_t^s\|^2$, so by Fubini's theorem,

$$\int_0^1 \varphi_s(x)\, ds = \int_0^1 \left(\int_0^1 \langle \eta_t^s | x \xi_t^s \rangle \, dt \right) ds = \int_{(s,t)\in \mathbb{I}\times\mathbb{I}} \langle \eta_t^s | x\xi_t^s \rangle \, d(s,t)$$

for all $x \in E$. Thus, $\varphi\colon x \mapsto \int_0^1 \varphi_s(x)\, ds$ is continuum-weakly continuous by remark 2.1.4. □

Lemma 2.2.3. *Let (φ_n) be a norm-convergent sequence of functionals on E converging to φ. If φ_n is continuum-weakly continuous for each n, then so is φ.*

Proof. Without loss of generality, we can assume that for all n, $\|\varphi_{n+1} - \varphi_n\| \leq 2^{-n}$. Writing $\psi_n = \varphi_{n+1} - \varphi_n$, we have that $\|\psi_n\| \leq 2^{-n}$, so $\sum_n \|\psi_n\| \leq 2$. Applying proposition 2.2.2, and the fact that, by 1.7.6, every probability measure on a countable set is a pushforward of Lebesgue measure on $\mathbb{I}$, we find that $\varphi = \sum_n \psi_n$ is continuum-weakly continuous. □

Lemma 2.2.4. *Each self-adjoint continuum-weakly continuous functional φ on E has a Jordan decomposition $\varphi = \varphi_+ - \varphi_-$, where φ_+ and φ_- are positive continuum-weakly continuous functionals, and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$. Thus, every continuum-weakly continuous function on E is a linear combination of continuum-weakly continuous states.*

Proof. Each self-adjoint continuum-weakly continuous functional φ is a vector functional for the canonical representation of E on $L^2(\mathbb{I}, \mathcal{H})$. Every self-adjoint vector functional on a concrete C*-algebra has a Jordan decomposition into vector functionals (1.7.21). □

Lemma 2.2.5. *Let μ be a continuum-weakly continuous state on E. There exists a family $(\xi_t) \in \mathcal{L}^2(\mathbb{I}, \mathcal{H})$ such that $\mu : x \mapsto \int_0^1 \langle \xi_t | x\xi_t \rangle dt$, and $\int_0^1 \|\xi_t\|^2\, dt = 1$*

Proof. The state μ is a vector functional for the canonical representation of E on $L^2(\mathbb{I}, \mathcal{H})$. □

Lemma 2.2.6. *Let $E \subseteq \mathcal{B}(\mathcal{H})$ and $F \subseteq \mathcal{B}(\mathcal{K})$ be V*-algebras. A (bounded) linear function $\pi : E \to F$ is continuum-weakly continuous iff the pullback of every vector state is continuum-weakly continuous.*

Proof. The forward direction is trivial. Therefore, it remains to show that if the pullback of every vector state is continuum-weakly continuous, then the pullback of every continuum-weakly continuous functional is continuum-weakly continuous. Each such functional is a linear combination of continuum-weakly continuous states, and each such state is an integral of vector states, so proposition 2.2.2 is sufficient to establish the claim. □

Lemma 2.2.7. *Let $M \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ be von Neumann algebras. An ultraweakly continuous linear function $\pi\colon M \to N$ is continuum-weakly continuous.*

Proof. This is a corollary of the preceding lemma. □

Proposition 2.2.8. *Let $E \subseteq \mathcal{B}(\mathcal{H})$ be a V*-algebra, and let $\rho : E \to \mathcal{B}(L^2(\mathbb{I}, \mathcal{H}))$ be its canonical representation on $L^2(\mathbb{I}, \mathcal{H})$. Then, ρ is a continuum-weakly homeomorphic $*$-isomorphism of E onto $\rho(E)$, which is itself a V*-algebra.*

Proof. Without loss of generality, assume that $E = \mathcal{B}(\mathcal{H})$. Fix $f \in L^2(\mathbb{I})$ of norm 1, and let $u_f : \mathcal{H} \to L^2(\mathbb{I}, \mathcal{H})$ be the isometry defined by $u_f \xi = [f(t)\xi \colon t \in \mathbb{I}]$. Thus, $\pi : x \mapsto u_f^* x u_f$ is a continuum-weakly continuous map $\mathcal{B}(\mathbb{I}, \mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\pi \circ \rho$ is the identity on $\mathcal{B}(\mathcal{H})$. The canonical representation ρ is itself continuum-weakly continuous because the pullback of every vector functional is trivially continuum-weakly continuous. The rest of the proof is elementary general topology. □

2.3 Projections in a V*-algebra

Lemma 2.3.1. *Let $\mathcal{H}$ be a Hilbert space. If (x_n) is a sequence in $\mathcal{B}(\mathcal{H})$ that converges to x in the ultraweak topology, then it converges to x in the continuum-weak topology.*

Proof. The set $\{\|x_n\|\}$ is bounded by some positive real number C, by an application of the uniform boundedness principle, since $\mathcal{B}(\mathcal{H})$ is isometrically isomorphic to the dual of $\mathcal{B}^1(\mathcal{H})$, the Banach space of trace class operators on $\mathcal{H}$. It follows that for all families $(\xi_t \in \mathcal{H} \colon t \in \mathbb{I})$ and $(\eta_t \in \mathcal{H} \colon t \in \mathbb{I})$ in $\mathcal{L}^2(\mathbb{I}, \mathcal{H})$,

$$\int \langle \eta_t | x_n \xi_t \rangle \, dt \to \int \langle \eta_t | x \xi_t \rangle \, dt,$$

by an application of the dominated convergence theorem, since $|\langle \eta_t | x_n \xi_t \rangle| \leq C \cdot \|\xi_t\| \cdot \|\eta_t\|$. □

Lemma 2.3.2. *Let E and F be V*-algebras, and let $\psi : E \to F$ be a continuum-weakly continuous positive map. Then, ψ is sequentially normal, in the sense that if $(x_n \in E \colon n \in \mathbb{N})$ is a descending sequence of positive operators in E converging ultraweakly to 0, then $(\psi(x_n))$ is a descending sequence of positive operators in F converging ultraweakly to 0.*

Proof. The various modes of convergence coincide for monotone sequences of positive operators (1.7.19) [2, corollary I.3.2.6]. In particular, if the greatest lower bound of (x_n) is 0, then $x_n \to 0$ ultraweakly, and therefore continuum-weakly, by lemma 2.3.1. By assumption, it follows that $\psi(x_n) \to \psi(0) = 0$ continuum-weakly, and therefore ultraweakly. □

Lemma 2.3.3. *Let $E \subseteq \mathcal{B}(\mathcal{H})$ be a V*-algebra. If $x \in E$ is positive, then its support projection $[x]$ is also in E.*

Proof.

$$[x] = \overset{uw}{\lim_{n \to \infty}} x^{\frac{1}{n}}$$

□

Proposition 2.3.4. *The projections of E are closed under countable meets and joins.*

Proof. The projections of E are closed under binary meets because

$$p \wedge q = \overset{uw}{\lim_{n\to\infty}} (pq)^n.$$

They are closed under binary joins because $p \vee q = [p+q]$. It follows that the projections of E are closed under countably infinite meets and joins because E is closed under limits of ascending and descending sequences. □

Lemma 2.3.5. *The projections of E are an approximate unit for E, for the norm topology.*

Proof. Recall (1.7.30) [2, proposition II.4.1.3] that for any C*-algebra, its positive elements of norm strictly less than 1 form an approximate unit. It follows that in E, the positive elements r satisfying $r^2 = \alpha r$ for $\alpha \in (0,1)$ form an approximate unit. It is then straightforward to show that the projections themselves form an approximate unit. □

Lemma 2.3.6. *If E is nondegenerate, then for all $\xi \in \mathcal{H}$, there is a projection $p \in E$ such that $p\xi = \xi$. (Example 2.3.9 shows that the identity operator need not be in E.)*

Proof. By lemma 2.3.5 above, the projections of E converge to the identity on $\mathcal{H}$ in the strong operator topology, and in particular $\lim_{p\nearrow 1} \|\xi - p\xi\| = 0$. Using the axiom of dependent choices, we can obtain an increasing sequence $(p_n \colon n \in \mathbb{N})$ of projections in E such that $\lim_{n\to\infty} \|\xi - p_n\xi\| = 0$. The ascending sequence (p_n) converges ultraweakly to some projection p, which is therefore in E. Therefore, $\lim_{n\to\infty} \|p\xi - p_n\xi\| = 0$. We conclude that that $p\xi = \xi$. □

Proposition 2.3.7. *Let E be a V*-algebra, and let $\varphi : E \to \mathbb{C}$ be a continuum-weakly continuous functional. There exist projections $p, q \in E$ such that $\varphi(pxq) = \varphi(x)$ for all $x \in E$.*

Proof. By proposition 2.2.8, we may assume that φ is a vector functional, so the existence of p and q then follow by lemma 2.3.6, above. □

Proposition 2.3.8. *Let X be a set. Then*

$$F = \{f \in \ell^\infty(X) \colon \operatorname{supp}(f) \not\succcurlyeq \mathbb{I}\}$$

is a V-algebra in its canonical representation on $\ell^2(X)$.*

Proof. It is easy to see that F is a $*$-algebra. The continuum-weak closure of F is a V*-algebra $\overline{F} \subseteq \ell^\infty(X)$. Suppose that $\overline{F} \neq F$. It follows that there is a positive function $f_\infty \in \overline{F} \setminus F$. By definition of F, $\operatorname{supp}(f_\infty) \succcurlyeq \mathbb{I}$, i. e., there is an injection $i \colon \mathbb{I} \to \operatorname{supp}(f_\infty)$. We now define a continuum-weakly continuous functional on $\ell^\infty(X)$ by

$$\varphi \colon h \mapsto \int_0^1 \langle e_{i(t)} | h e_{i(t)} \rangle \, dt = \int_0^1 h(t)\, dt.$$

Our choice of i guarantees that $\varphi(f_\infty) > 0$. However, for all $f \in F$, we have that $i(\mathbb{I}) \cap \mathrm{supp}(f) \not\approx \mathbb{I}$, but $i(\mathbb{I}) \cap \mathrm{supp}(f) \preccurlyeq \mathbb{I}$ because i is an injection. So, by **PSP**, $i(\mathbb{I}) \cap \mathrm{supp}(f) \preccurlyeq \mathbb{N}$. It follows that $\varphi(f) = 0$ for all $f \in F$, contradicting our assumption that $f_\infty \in \overline{F}$. □

Example 2.3.9. In particular, $\{f \in \ell^\infty(\mathbb{R})\colon \mathrm{supp}(f) \preccurlyeq \mathbb{N}\}$ is a V*-algebra on $\ell^2(\mathbb{R})$, so a V*-algebra need not be unital.

2.4 Separable Hilbert spaces

2.4.1. Let $\mathcal{H}$ be a separable Hilbert space. Every continuum-weakly continuous functional on $\mathcal{B}(\mathcal{H})$ is ultraweakly continuous, by 1.7.41 and lemma 2.2.4, so the ultraweak topology and the continuum-weak topology coincide on $\mathcal{B}(\mathcal{H})$.

Proposition 2.4.2. *Every nondegenerate V*-algebra on a separable Hilbert space is a von Neumann algebra.*

Proof. This is a corollary of remark 2.4.1, above. □

2.4.3. We will need the disintegration of ultraweakly continuous states on a direct integral of von Neumann algebras on separable Hilbert spaces. Rather than asking the reader to review Borel fields of Hilbert spaces, we will reprove these results, in part, in order to demonstrate the simplicity of the direct integral in the Solovay model $\mathfrak{N}_1$. We will work with a probability measure m on an index set $T \preccurlyeq \mathbb{R}$.

2.4.4. Let $(\mathcal{H}_t\colon t \in T)$ be a family of separable infinite-dimensional Hilbert spaces. We define the direct integral Hilbert space $\int^\oplus \mathcal{H}_t\, dm$ to consist of equivalence classes of square-integrable functions $(\xi_t \in \mathcal{H}_t\colon t \in T)$. Applying $\mathbf{AC_{ae}}$, we can choose isomorphisms $\mathcal{H}_t \cong \ell^2(\mathbb{N})$ for almost every $t \in T$, so $\int^\oplus \mathcal{H}_t\, dm \cong L^2(T, \ell^2(\mathbb{N})) \cong \ell^2(\mathbb{N})$. This shows that $\int^\oplus \mathcal{H}_t\, dm$ is separable and complete.

2.4.5. Let $(M_t \subseteq \mathcal{B}(\mathcal{H}_t)\colon t \in T)$ be a family of von Neumann algebras. We define the direct integral von Neumann algebra $\int^\oplus M_t\, dm$ to consists of equivalence classes of essentially bounded functions $(x_t \in M_t\colon t \in T)$. It is straightforward to show that the canonical action of $\int^\oplus M_t\, dm$ on the direct integral Hilbert space $\int^\oplus \mathcal{H}_t\, dm$ is well-defined and faithful. It is then straightforward to show that $\int^\oplus M_t\, dm$ is closed under limits of monotone countable sequences, and is therefore a von Neumann algebra; see 1.7.25.

Lemma 2.4.6. *Let $\varphi\colon \int^\oplus M_t\, dm \to \mathbb{C}$ be an ultraweakly continuous functional. There exists an integrable family of ultraweakly continuous functionals $(\varphi_t \in (M_t)_*\colon t \in T)$ such that*

$$\varphi : [x_t \in M_t\colon t \in T] \mapsto \int_{t \in T} \varphi_t(x_t)\, dm.$$

Proof. Since φ is ultraweakly continuous, there exist two square-summable sequences of vectors in $\int_t \mathcal{H}_t \, dm$, which may be written as $([\xi_t^n \colon t \in \mathcal{H}] \colon n \in \mathbb{N})$ and $([\eta_t^n \colon t \in \mathcal{H}] \colon n \in \mathbb{N})$, such that

$$\varphi : [x_t \colon t \in T] \mapsto \sum_n \int_t \langle \xi_t^n | x_t \eta_t^n \rangle \, dm$$

The square-summability of the two sequences means that $\sum_n \int_t \|\xi_t^n\|^2 \, dm < \infty$ and $\sum_n \int_t \|\eta_t^n\|^2 \, dm < \infty$. It follows by Tonelli's theorem that $(\xi_t^n \colon n \in \mathbb{N})$ and $(\eta_t^n \colon n \in \mathbb{N})$ are square-summable for almost every $t \in T$. Furthermore, since the elements of $\int^{\oplus} M_t dm$ are essentially bounded by definition, Fubini-Tonelli implies that

$$\varphi : [x_t \colon t \in T] \mapsto \int_t \sum_n \langle \xi_t^n | x_t \eta_t^n \rangle \, dm.$$

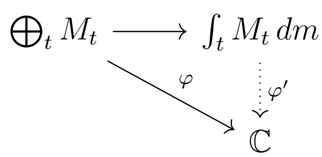
□

Lemma 2.4.7. *Every continuum-weakly continuous functional φ on the von Neumann algebra $\bigoplus_t M_t$ is of the form*

$$(x_t : t \in T) \mapsto \int_t \varphi_t(x_t) \, dm$$

for some integrable family $(\varphi_t \in (M_t)_* \colon t \in T)$ of ultraweakly continuous functionals, and some probability measure m on T.

Proof. The functional φ is a vector functional for the continuum amplification of the V*-algebra $\bigoplus_t M_t$ (proposition 2.2.8), so there exists a unique positive functional $|\varphi|$ such that $\| \, |\varphi| \, \| = \|\varphi\|$, and $|\varphi(x)|^2 \leq \|\varphi\| \cdot |\varphi|(x^* x)$ for all $x \in \bigoplus_t M_t$ (1.7.22). The restriction of $|\varphi|$ to $\ell^\infty(T) \subseteq \bigoplus_t M_t$ satisfies $|\varphi|(\sum_n p_n) = \sum_n |\varphi|(p_n)$ for all countable families (p_n) of pairwise disjoint projections in $\ell^\infty(T)$, for otherwise we would obtain a state on $\ell^\infty(\mathbb{N})$ that is not ultraweakly continuous. Thus, writing p_A for the projection in $\ell^\infty(T)$ corresponding to a subset $A \subseteq T$, we find that $m \colon A \mapsto |\varphi|(p_A) \cdot \|\varphi\|^{-1}$ is a probability measure on T. It is straightforward to show that if $(x_t) \in \bigoplus_t M_t$ vanishes almost everywhere, then $\varphi \colon (x_t) \mapsto 0$; thus, φ factors through the canonical map $(x_t) \mapsto [x_t]$:



Since $\int_t M_t \, dm$ is a von Neumann algebra on the separable Hilbert space $\int_t \mathcal{H}_t \, dm$, the functional φ' is automatically (1.7.41) ultraweakly continuous, and lemma 2.4.6 yields the desired family $(\varphi_t \in (M_t)_* \colon t \in T)$. □

2.5 Canonical continuum predual

2.5.1. If $M \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, then the canonical predual M_* of M consists of all ultraweakly continuous functionals $M \to \mathbb{C}$. It is a predual of M in the sense that $(M_*)^*$ is canonically isomorphic to M, but I don't know if it is unique up to isometric isomorphism, because the usual proof of this result relies on the axiom of choice. Of course, the predual is unique if $\mathcal{H}$ is separable.

Definition 2.5.2. If $E \subseteq \mathcal{B}(\mathcal{H})$ is a V*-algebra, then the canonical continuum predual $E_\bullet$ of E consists of all continuum-weakly continuous functionals $E \to \mathbb{C}$.

Proposition 2.5.3. *Let $E \subseteq \mathcal{B}(\mathcal{H})$ be a V*-algebra. The function $\iota\colon E \to (E_\bullet)^*$ defined by $x \mapsto (\varphi \mapsto \varphi(x))$ is an isometry onto the continuum-linear functionals, i. e., functionals $f \in (E_\bullet)^*$ such that*

$$f\left(\int_0^1 \varphi_t \, dt\right) = \int_0^1 f(\varphi_t)\, dt$$

for all integrable families $(\varphi_t \in E_\bullet \colon t \in \mathbb{I})$.

Proof. Without loss of generality, E is nondegenerate. By Kaplansky's density theorem, each ultraweakly continuous functional on E'' restricts to a continuum-weakly continuous functional on E of the same norm, so $(E'')_* \subseteq E_\bullet$. This implies that ι is an isometry, since certainly $\iota(x)(\varphi) \leq \|x\| \cdot \|\varphi\|$. The continuum-linearity of $\iota(x)$ for each $x \in E$ is trivial by definition of integration of continuum-weakly continuous functionals; see proposition 2.2.2.

It remains to show that every continuum-linear functional $f\colon E_\bullet \to \mathbb{C}$ is in the image of ι. Suppose otherwise, and consider the restriction of f to $(E'')_* \subseteq E_\bullet$. Since E'' is canonically isomorphic to the dual $((E'')_*)^*$, we obtain an operator $x_0 \in E''$ such that $\varphi(x_0) = f(\varphi)$ for all $\varphi \in (E'')_*$. If $x_0 \notin E$, then there is a continuum-weakly continuous functional $x \mapsto \int_0^1 \langle \eta_t | x \xi_t\rangle \, dt$ that vanishes on E, but not on x_0. The former property can be phrased as the equation $\int_0^1 \langle \eta_t | \cdot | \xi_t \rangle \, dt = 0$ in $E_\bullet$. Applying the continuum-linearity of f, we find that

$$\int_0^1 \langle \eta_t | x_0 \xi_t \rangle \, dt = \int_0^1 f(\langle \eta_t | \cdot | \xi_t \rangle)\, dt = f\left(\int_0^1 \langle \eta_t | \cdot | \xi_t\rangle \, dt\right) = 0.$$

This contradicts our choice of (ξ_t) and (η_t). □

Lemma 2.5.4. *Let $E \subseteq \mathcal{B}(\mathcal{H})$ be a unital V*-algebra, and let $\mathcal{S}_\bullet \subseteq E_\bullet$ be its space of continuum-weakly continuous states. For each $x \in E_{sa}$, we define $\hat{x}\colon \mathcal{S}_\bullet \to \mathbb{R}$ by $\hat{x}(\mu) = \mu(x)$. Then, $x \mapsto \hat{x}$ is a bijective isometric correspondence between the self-adjoint elements of E, and functions $\mathcal{S}_\bullet \to \mathbb{R}$ that are continuum-affine, i. e., that satisfy*

$$f\left(\int_0^1 \mu_t \, dt\right) = \int_0^1 f(\mu_t)\, dt$$

for all families $(\mu_t \in \mathcal{S}_\bullet \colon t \in \mathbb{I})$.

Proof. We observe that $\hat{x} = \iota(x)|_{\mathcal{S}_\bullet}$, so $\hat{x}$ is continuum-affine. Furthermore, the function $x \mapsto \hat{x}$ is isometric, because the norm of a self-adjoint operator can be computed as a supremum over vector states.

If $f \colon \mathcal{S}_\bullet \to \mathbb{R}$ is continuum-affine, then it is affine in the usual sense; therefore, f extends uniquely to a linear function $\tilde{f} \colon E_\bullet \to \mathbb{C}$, by lemma 2.2.4. The canonical continuum predual $E_\bullet$ is a Banach space by lemma 2.2.3, so $\tilde{f}$ is automatically bounded. By proposition 2.5.3, to show that $f = \hat{x}$ for some $x \in E$, it is sufficient to show that $\tilde{f}$ is continuum-linear, i. e.,

$$\tilde{f}\left(\int_0^1 \varphi_t \, dt\right) = \int_0^1 \tilde{f}(\varphi_t)\, dt$$

for each integrable family $(\varphi_t \in E_\bullet \colon t \in \mathbb{I})$. By lemma 2.2.4, we may assume that each φ_t is positive, and that $\| \int_0^1 \varphi_t \, dt\| = \int_0^1 \varphi_t(1)\, dt = 1$. The assignment $A \mapsto \int_{t\in A} \|\varphi_t\|\, dt$ is a probability measure m on $\mathbb{I}$. If we define $\hat{\varphi}_t = \|\varphi_t\|^{-1}\varphi_t$ for nonzero φ_t, and $\hat{\varphi}_t = \psi$ otherwise, for some fixed $\psi \in E_\bullet$, then it remains to show

$$\tilde{f}\left(\int_{t\in T} \hat{\varphi}_t \, dm\right) = \int_{t\in T} \tilde{f}(\hat{\varphi}_t)\, dm,$$

which is just the condition that f is continuum-affine on $\mathcal{S}_\bullet$, because m is a pushforward of Lebesgue measure on $\mathbb{I}$. □

2.6 The enveloping V*-algebra

Definition 2.6.1. If A is a C*-algebra, then its enveloping V*-algebra is $V^*(A) = \overline{\gamma_{\mathcal{S}}(A)}$, where $\gamma_{\mathcal{S}} \colon A \to \mathcal{B}(\bigoplus_{\mu\in\mathcal{S}(A)} \mathcal{H}_\mu)$ is the universal representation.

Lemma 2.6.2. *Every continuum-weakly continuous state on $V^*(A)$ is a vector state, so we have isometric isomorphisms:*

$$\begin{array}{ccccccc}
\mathcal{S}(A) & \overset{\cong}{\longleftrightarrow} & \mathcal{S}(\gamma_{\mathcal{S}}(A)) & \overset{\cong}{\longleftrightarrow} & \mathcal{S}_\bullet(V^*(A)) & \overset{\cong}{\longleftrightarrow} & \mathcal{S}_*(W^*(A)) \\
\mu & \longmapsto & \langle \xi_\mu | \cdot \xi_\mu \rangle & \longmapsto & \langle \xi_\mu | \cdot \xi_\mu \rangle & \longmapsto & \langle \xi_\mu | \cdot \xi_\mu \rangle
\end{array}$$

Proof. By construction every state of A factors through the universal representation $\gamma_{\mathcal{S}}$ via a vector state. Each vector state is ultraweakly continuous, and therefore extends uniquely to an ultraweakly continuous state on the ultraweak closure $W^*(A)$, and also to a unique continuum-weakly continuous state on the continuum-weak closure $V^*(A)$. □

Lemma 2.6.3. *The continuum-weak topology on $V^*(A)$ coincides with the ultraweak topology.*

Proof. This is a corollary of lemma 2.2.4 and lemma 2.6.2. □

Proposition 2.6.4. *If A is unital, and $x \in W^*(A)$ is self-adjoint, then $x \in V^*(A)$ iff the function $\hat{x} : \mathcal{S}(A) \to \mathbb{R}$ defined by $\hat{x} : \mu \mapsto \langle \xi_\mu | x \xi_\mu \rangle$ satisfies*

$$\hat{x}\left(\int \mu_t \, dt \right) = \int \hat{x}(\mu_t) \, dt$$

for all indexed families $(\mu_t \in \mathcal{S}(A) \colon t \in \mathbb{I})$.

Proof. The isomorphism $\mathcal{S}(A) \cong \mathcal{S}_\bullet(V^*(A))$ in lemma 2.6.2 respects integration in the sense that $\mu(a) = \int_0^1 \mu_t(a) \, dt$ for all $a \in A$ iff $\langle \xi_\mu | x | \xi_\mu \rangle = \int_0^1 \langle \xi_{\mu_t} | x \xi_{\mu_t} \rangle \, dt$ for all $x \in V^*(A)$, whenever $(\mu_t \colon t \in \mathbb{I})$ is a family of states, because the functional $x \mapsto \langle \xi_\mu | x \xi_\mu \rangle - \int_0^1 \langle \xi_{\mu_t} | x \xi_{\mu_t} \rangle \, dt$ is continuum-weakly continuous. Thus, identifying $\mathcal{S}(A)$ with $\mathcal{S}_\bullet(V^*(A))$, we can apply lemma 2.5.4 to find that $x \mapsto \hat{x}$ is an isometric isomorphism of $V^*(A)_{sa}$ onto the continuum-affine functions $\mathcal{S}(A) \to \mathbb{R}$. Of course, $x \mapsto \hat{x}$ is isometric on $W^*(A)_{sa}$, so the proposition follows. □

2.6.5. If A is a unital *separable* C*-algebra, then $a \mapsto \hat{a}$ is an isometric isomorphism between A_{sa} and the space of continuous affine functions on $\mathcal{S}(A)$. Furthermore, since $\mathcal{S}(A) \preccurlyeq \mathbb{R}$, every probability measure on $\mathcal{S}(A)$ is a pushforward of Lebesgue measure on $\mathbb{I}$, so $x \in V^*(A)$ iff $\hat{x}$ is strongly affine [16, section 2].

2.6.6. In proposition 2.6.4, we insist that A be unital because the state space of a nonunital C*-algebra need not be closed under integration. For example, the state space of $c_0(\mathbb{I})$ is not closed under integration because the integral of its homomorphic states over Lebesgue measure is zero.

Theorem 2.6.7. *Let A be a C*-algebra, let E be a V*-algebra, and let $\rho : A \to E$ be a $*$-homomorphism. There exists a unique continuum-weakly continuous $*$-homomorphism $\pi : V^*(A) \to E$ such that $\pi \circ \gamma_\mathcal{S} = \rho$:*

$$\begin{array}{ccc} A & \xrightarrow{\gamma_\mathcal{S}} & V^*(A) \\ & \searrow^{\rho} & \vdots\, {\scriptstyle \pi\, !} \\ & & E \end{array}$$

Proof. The universal property of the enveloping von Neumann algebra $W^*(A) = \gamma_\mathcal{S}(A)''$ yields an ultraweakly continuous $*$-homomorphism $\pi \colon W^*(A) \to E''$ such that $\pi \circ \gamma_\mathcal{S} = \rho$. By lemma 2.2.7, π is also continuum-weakly continuous, which implies that $\pi(V^*(A)) \subseteq E$; thus π makes the above diagram commute. The uniqueness of π is trivial. □

2.6.8. It follows from lemma 2.3.1 that every V*-algebra is closed under limits of ultraweakly convergent sequences. Therefore, $V^*(A)$ includes Davies's σ-envelope $A^\sim$ [6], and Pedersen's enveloping Borel $*$-algebra $\mathscr{B}(A)$ [18].

2.7 The atomic representation

2.7.1. In the Solovay model $\mathfrak{N}_1$, a C*-algebra may fail to have a pure state even if it has a faithful state. For example, Lebesgue measure λ yields a faithful state on the von Neumann algebra $L^\infty(\mathbb{I}, \lambda)$, but it is easy to show that this C*-algebra has no pure states (example 1.9.2).

2.7.2. If A is a *separable* C*-algebra, then every state $\mu \in \mathcal{S}(A)$ is a mixture of pure states, in the sense that there is a probability measure m on the pure state space $\partial\mathcal{S}(A)$ such that $\mu(a) = \int_{\nu \in \partial\mathcal{S}} \nu(a)\, dm$ for all $a \in A$. This is a consequence of Choquet's theorem, which yields such a Borel probability measure on $\partial\mathcal{S}(A)$, which of course extends uniquely to a totally defined probability measure, since $\partial\mathcal{S}(A)$ is Polish; see 1.7.38, 1.7.5, and [19] proposition 4.3.2. We remark that Choquet's theorem has a constructive proof [21]. Clearly $\partial\mathcal{S}(A) \preccurlyeq \mathcal{S}(A) \preccurlyeq \mathbb{I}$, so every state on A is the integral of a continuum of pure states, in the sense that there is a function $t \mapsto \varphi_t \in \partial\mathcal{S}(A)$ such that $\mu(a) = \int_0^1 \varphi_t(a)\, dt$ for all $a \in A$.

2.7.3. Let A be a C*-algebra. The atomic representation $\gamma_\partial = \gamma_{\partial\mathcal{S}(A)}$ is the direct sum of all GNS representations γ_μ of A for $\mu \in \partial\mathcal{S}(A)$, i. e, $\gamma_\partial \colon a \mapsto \bigoplus_{\mu \in \partial\mathcal{S}(A)} \gamma_\mu(a)$. It follows from remark 2.7.2 that if A is separable, then the atomic representation if faithful. On the other hand, the atomic representation of $L^\infty(\mathbb{R}, \lambda)$ is evidently trivial (remark 2.7.1).

Lemma 2.7.4. *Let A be a C*-algebra such that every state $\mu \in \mathcal{S}(A)$ is the integral of a continuum of states in $\partial\mathcal{S}(A)$. Then the continuum-weakly continuous states on $\overline{\gamma_\partial(A)}$ are in bijective correspondence with the states of A via the assignment $\varphi \mapsto \varphi \circ \gamma_\partial$.*

Proof. The assignment is injective because $\gamma_\partial(A)$ is trivially dense in $\overline{\gamma_\partial(A)}$. The assignment is surjective because each state of A is the integral of a continuum of pure states, each of which is canonically a vector state of the representation γ_∂. □

Lemma 2.7.5. *Let E be a V*-algebra, and let $\mu : E \to \mathbb{C}$ be a continuum-weakly continuous state with GNS representation $(\gamma_\mu, \mathcal{H}_\mu, \xi_\mu)$. The map γ_μ is continuum-weakly continuous. Furthermore, if $A \subseteq E$ is a continuum-weakly dense C*-subalgebra, then $\overline{\gamma_\mu(A)\xi_\mu} = \mathcal{H}_\mu$.*

Proof. Let $\varphi : \mathcal{B}(\mathcal{H}_\mu) \to \mathbb{C}$ be a vector functional, i. e., $\varphi : y \mapsto \langle \zeta | y\eta \rangle$ for some $\eta, \zeta \in \mathcal{H}_\mu$. Since ξ_μ is a cyclic vector, there are sequences (e_n) and (f_n) from E such that $\gamma_\mu(e_n)\xi_\mu \to \eta$ and $\gamma_\mu(f_n)\xi_\mu \to \zeta$. By elementary functional analysis,

$$\langle \gamma_\mu(f_n)\xi_\mu | \cdot | \gamma_\mu(e_n)\xi_\mu \rangle \to \langle \zeta | \cdot | \eta \rangle,$$

in the norm topology, and therefore

$$\langle \gamma_\mu(f_n)\xi_\mu | \gamma_\mu(\cdot)\gamma_\mu(e_n)\xi_\mu \rangle \to \langle \zeta | \gamma_\mu(\cdot)\eta \rangle$$

in the norm topology. For each n, the functional $x \mapsto \langle \gamma_\mu(f_n)\xi_\mu | \gamma_\mu(x)\gamma_\mu(e_n)\xi_\mu \rangle = \mu(f_n^* x e_n)$ is continuum-weakly continuous. Thus, $\langle \zeta | \gamma_\mu(\cdot)\eta \rangle$ is the limit of a sequence of continuum-weakly continuous functionals in the norm topology, and is therefore itself continuum-weakly continuous, by lemma 2.2.3. We have shown that the pullback of every vector functional along γ_μ is continuum-weakly continuous; by lemma 2.2.6, γ_μ is itself continuum-weakly continuous.

Suppose that $A \subseteq E$ is a continuum-weakly dense C*-subalgebra, and $\eta \in \mathcal{H}_\mu$ is a vector such that $\langle \eta | \gamma_\mu(a)\xi_\mu \rangle = 0$ for all $a \in A$. We have already established that the functional $x \mapsto \langle \eta | \gamma_\mu(x)\xi_\mu \rangle$ is continuum-weakly continuous so it vanishes for all $x \in E$. We conclude that $\eta = 0$. Therefore, $\overline{\gamma_\mu(a)\xi_\mu} = \mathcal{H}_\mu$. □

Proposition 2.7.6 (cf. theorem 2.6.7)**.** *Let A be a C*-algebra such that every state $\mu \in \mathcal{S}(A)$ is the integral of a continuum of states in $\partial S(A)$, e. g., a separable C*-algebra. Then γ_∂ factors uniquely through $\gamma_{\mathcal{S}}$ via a continuum-weakly homeomorphic $*$-isomorphism:*

$$\begin{array}{ccc} A & \xrightarrow{\gamma_{\mathcal{S}}} & V^*(A) \\ & \searrow^{\gamma_\partial} & \updownarrow{\scriptstyle \cong\,!} \\ & & \overline{\gamma_\partial(A)} \end{array}$$

Proof. The enveloping V*-algebra is defined by its universal property, up to canonical isomorphism; this is established by a straightforward category-theoretic argument. In order to adapt this argument, we need only to prove the two claims diagramed below:

1. $$\begin{array}{ccc} A & \xrightarrow{\gamma_\partial} & \overline{\gamma_\partial(A)} \\ & \searrow^{\gamma_\partial} & \downarrow{\scriptstyle id\,!} \\ & & \overline{\gamma_\partial(A)} \end{array}$$

2. $$\begin{array}{ccc} A & \xrightarrow{\gamma_\partial} & \overline{\gamma_\partial(A)} \\ & \searrow^{\gamma_{\mathcal{S}}} & \downarrow{\scriptstyle !} \\ & & V^*(A) \end{array}$$

The first claim, that γ_∂ factors through itself uniquely, via the identity, is trivial. To prove the second claim, that $\gamma_{\mathcal{S}}$ factors through γ_∂ uniquely, we work with the universal continuum-weakly continuous representation of $\overline{\gamma_\partial(A)}$, i. e.,

$$\gamma_\bullet \colon x \mapsto \bigoplus_{\mu \in \mathcal{S}_\bullet} \gamma_\mu(x),$$

where $\mathcal{S}_\bullet$ denotes the space of continuum-weakly continuous states $\overline{\gamma_\partial(A)} \to \mathbb{C}$. Lemma 2.7.4 and lemma 2.7.5, above, now imply that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\quad\gamma_\partial\quad} & \overline{\gamma_\partial(A)} \\ \downarrow{\scriptstyle \gamma_{\mathcal{S}}} & & \downarrow{\scriptstyle \gamma_\bullet} \\ \bigoplus_{\mu \in \mathcal{S}} \mathcal{B}(\mathcal{H}_\mu) & \xleftrightarrow{\cong} & \bigoplus_{\mu \in \mathcal{S}_\bullet} \mathcal{B}(\mathcal{H}_\mu) \end{array}$$

Of course, $\gamma_\bullet(\overline{\gamma_\partial(A)}) \subseteq \overline{\gamma_\bullet(\gamma_\partial(A))}$, so taking continuum-weak closures of images we find that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\gamma_\partial} & \overline{\gamma_\partial(A)} \\ \big\downarrow{\scriptstyle \gamma_{\mathcal{S}}} & & \big\downarrow{\scriptstyle \gamma_\bullet} \\ \overline{\gamma_{\mathcal{S}}(A)} & \xleftrightarrow{\cong} & \overline{\gamma_\bullet(\gamma_\partial(A))} \end{array}$$

Thus $\gamma_{\mathcal{S}} \colon A \to \overline{\gamma_{\mathcal{S}}(A)} = V^*(A)$ factors through γ_∂ via a continuum-weakly continuous $*$-homomorphism. The uniqueness of the factorization is trivial. □

Lemma 2.7.7. *Let T be a locally compact Polish space. The continuum-weak closure of $C_0(T) \subseteq \mathcal{B}(\ell^2(T))$ is $\ell^\infty(T)$.*

Proof. Clearly, $\overline{C_0(T)} \subseteq C_0(T)'' = \ell^\infty(T)$. Let $\mathscr{B}(T) \subseteq \ell^\infty(T)$ denote the algebra of bounded Borel functions, i. e., of bounded Baire functions; these algebras are equal on any second countable locally compact Hausdorff space; see [20] proposition 6.2.9. We have the inclusion $\mathscr{B}(T) \subseteq \overline{C_0(T)}$, because V*-algebras are closed under limits of monotone sequences, by lemma 2.3.1. If $\overline{C_0(T)} \neq \ell^\infty(T)$, then $\overline{\mathscr{B}(T)} \neq \ell^\infty(T)$, so there is a continuum-weakly continuous function $\varphi \colon \ell^\infty(T) \to \mathbb{C}$ such that $\varphi(\mathscr{B}(T)) = 0$, but $\varphi(p) \neq 0$ for some projection $p \in \ell^\infty(T)$.

By lemma 2.4.7, there is a probability measure m and an integrable function $h \colon T \to \mathbb{C}$ such that $\varphi(f) = \int_{t \in T} f(t)h(t)\, dm$ for all $f \in \ell^\infty(T)$. By **LM** (1.7.5), there is a Borel projection $b \in \mathscr{B}(T)$ such that $b(t) = p(t)$ for almost every $t \in T$. It follows that $\varphi(p) = \int p(t)h(t)\, dm = \int b(t)h(t)\, dm = \varphi(b) = 0$, contradicting our choice of p. □

Proposition 2.7.8 (cf. theorem 2.6.7)**.** *Let X be a second countable locally compact Hausdorff space. Then $V^*(C_0(X)) \cong \ell^\infty(X)$:*

$$\begin{array}{ccc} C_0(X) & \xrightarrow{\gamma_{\mathcal{S}}} & V^*(C_0(X)) \\ & \searrow & \big\updownarrow{\scriptstyle \cong\, !} \\ & & \ell^\infty(X) \end{array}$$

Proof. A second countable locally compact Hausdorff space is Polish, because it is an open subset of its one-point compactification, which is second countable and therefore Polish. We recall that a locally compact Hausdorff space X is second countable iff $C_0(X)$ is separable, and then we apply proposition 2.7.6 and lemma 2.7.7, above. □

2.8 Separable C*-algebras of type I

2.8.1. Let A be a C*-algebra. There are many equivalent definitions of $\hat{A}$, the spectrum of A. The elements of $\hat{A}$ may be thought of as equivalence classes of pure states under unitary equivalence, or as equivalence classes of irreducible representations under unitary equivalence, or as equivalence classes of type I factor representations under quasiequivalence. Recall that a nondegenerate representation $\rho\colon A \to \mathcal{B}(\mathcal{H})$ is a (type I) factor representation iff $\rho(A)''$ is a factor (of type I) [19, remark 3.8.13], and that representations ρ_0 and ρ_1 are quasiequivalent iff there is an ultraweakly homeomorphic $*$-isomorphism $\pi : \rho_0(A)'' \to \rho_1(A)''$ such that $\pi \circ \rho_0 = \rho_1$ [19, remark 3.3.6].

Under the axiom of choice, we often work with the spectrum $\hat{A}$ by choosing representatives from these equivalence classes. In the Solovay model $\mathfrak{N}_1$, this isn't always possible. Therefore, for each unitary equivalence class C of pure states, we define γ_C to be the direct sum of all the GNS representations γ_μ for $\mu \in C$, i. e., $\gamma_C : a \mapsto \bigoplus_{\mu \in C} \gamma_\mu(a)$. The representation γ_C is a type I factor representation in the quasiequivalence class corresponding to C. We will sometimes write $\hat{A} = \{\gamma_C \colon C \in \partial\mathcal{S}(A)/\sim_u\}$.

Proposition 2.8.2. *Let A be a separable C*-algebra. Then, A is type I iff $\hat{A}$ has cardinality of at most the continuum.*

Proof. It is a standard theorem that a C*-algebra is type I iff the Mackey Borel structure on $\hat{A}$ is standard; the Mackey Borel structure consists of sets whose preimages under the GNS construction are Borel subsets of $\partial\mathcal{S}(A)$. A pair of pure states produce unitarily equivalent irreducible representations iff they themselves are unitary equivalent. This a Borel equivalence relation; see [7] proposition 7.

If A is type I, then $\hat{A}$ is standard, i. e., $\hat{A}$ equipped with the σ-algebra of Mackey Borel sets is isomorphic, as a measurable space, to a Polish space X equipped with its σ-algebra of Borel sets. Clearly, $\hat{A} \preccurlyeq \mathbb{R}$.

If $\hat{A}$ is not type I, then the Mackey Borel structure on $\hat{A}$ is not countably separated (1.7.39), so unitary equivalence on $\partial\mathcal{S}(A)$ is not Borel reducible to the identity relation on $\mathbb{R}$. By the Glimm-Effros dichotomy [10], it follows that $\hat{A} \succcurlyeq (\prod_{n=0}^{\infty} \mathbb{Z}_2 / \sum_{n=0}^{\infty} \mathbb{Z}_2) \succ \mathbb{R}$. □

2.8.3. Let A be a separable C*-algebra. Pedersen has shown that the canonical map $\gamma_{\mathcal{S}}(A)'' \to \gamma_\partial(A)''$ is isometric on the enveloping Borel $*$-algebra $\mathscr{B}(A)$ [18]. Since $\mathscr{B}(A) \subseteq V^*(A)$, as in remark 2.6.8, the canonical map $V^*(A) \to \overline{\gamma_\partial(A)}$ is also isometric on $\mathscr{B}(A)$. We will sometimes abuse notation by identifying $\mathscr{B}(A)$ with its image, and writing $\mathscr{B}(A) \subseteq \overline{\gamma_\partial(A)}$.

2.8.4. For all $\gamma \in \hat{A}$, the von Neumann algebra $\gamma(A)''$ is a type I factor, so an operator is in the center of $\bigoplus \gamma(A)''$ iff it is a scalar on each direct summand. In this way we obtain a $*$-isomorphism between the center of $\bigoplus \gamma(A)''$ and the von Neumann algebra of bounded functions $\hat{A} \to \mathbb{C}$. Pedersen calls such a function Davies Borel in case it corresponds to an element of $\mathscr{B}(A) \subseteq \overline{\gamma_\partial(A)}$ under this isomorphism; see [19] section 4.7. Of course, a subset $B \subseteq \hat{A}$ is said to be Davies Borel iff its indicator function is Davies Borel, i. e., $(1 \colon \gamma \in B;\ 0 \colon \gamma \in \hat{A} \setminus B) \in \mathscr{B}(A)$.

Theorem 2.8.5 (cf. theorem 2.6.7). *Let A be a separable C^*-algebra of type I. Then there exists a unique continuum-weakly homeomorphic $*$-isomorphism $V^*(A) \cong \bigoplus_{\gamma \in \hat{A}} \gamma(A)''$:*

$$\begin{array}{ccc} A & \xrightarrow{\gamma_S} & V^*(A) \\ & \searrow^{\gamma_\partial} & \Big\updownarrow{\scriptstyle \cong\, !} \\ & & \bigoplus \gamma(A)'' \end{array}$$

Proof. We're proving a stronger version of proposition 2.7.6; it remains to show that $\overline{\gamma_\partial(A)} = \bigoplus \gamma(A)''$. Suppose that

$$(x_\gamma^\infty \colon \gamma \in \hat{A}) \in \left(\bigoplus_\gamma \gamma(A)'' \right) \setminus \overline{\gamma_\partial(A)}$$

is an operator of norm less than 1. It follows that there is a continuum-weakly continuous functional φ on $\bigoplus_\gamma \gamma(A)''$ such that $\varphi(\overline{\gamma_\partial(A)}) = \{0\}$, but $\varphi(x_\gamma^\infty \colon \gamma) \neq 0$. Each summand $\gamma(A)''$ is canonically representable on the separable Hilbert space of Hilbert-Schmidt operators in $\gamma(A)''$, and the representation of $\bigoplus_\gamma \gamma(A)''$ on the direct sum of these Hilbert spaces yields the same continuum-weak topology as before, by an application of lemma 2.2.6. Thus, φ is still continuum-weakly continuous in this new representation, so lemma 2.4.7 shows that there is a family $(\varphi_\gamma \in (\gamma(A)'')_*)$ of ultraweakly continuous functionals, and a probability measure m on $\hat{A}$ such that

$$\varphi : (x_\gamma) \mapsto \int_\gamma \varphi_\gamma(x_\gamma)\, dm$$

for all $(x_\gamma) \in \bigoplus_\gamma \gamma(A)''$.

Fix a sequence $(a_n \in A_1 \colon n \in \mathbb{N})$ that is dense in the unit ball $A_1 \subseteq A$. By Kaplansky's density theorem, $(\gamma(a_n) \in A)$ is dense in the unit ball of $\gamma(A)''$ for each $\gamma \in \hat{A}$. Symbolically:

$$\forall \gamma \in \hat{A} \colon \exists n \in \mathbb{N} \colon |\varphi_\gamma(x_\gamma^\infty - \gamma(a_n))| \leq \epsilon$$
$$\exists f \colon \hat{A} \longrightarrow \mathbb{N} \colon \forall^{ae} \gamma \in \hat{A} \colon |\varphi_\gamma(x_\gamma^\infty - \gamma(a_{f(\gamma)}))| \leq \epsilon$$
$$\exists f \colon \hat{A} \xrightarrow{Borel} \mathbb{N} \colon \forall^{ae} \gamma \in \hat{A} \colon |\varphi_\gamma(x_\gamma^\infty - \gamma(a_{f(\gamma)}))| \leq \epsilon$$

The first step is a straightforward application of $\mathbf{AC_{ae}}$, and the second is a straightforward application of **DC**, and **LM** in the guise of 1.7.5; equipped with the Davies Borel structure, $\hat{A}$ is a standard Borel space, so each subset $f^{-1}(n) \subseteq \hat{A}$ differs from some Borel subset on a set of measure zero.

For each natural number n, let $p_n \in \ell^\infty(\hat{A}) \cap \mathscr{B}(A) \subseteq \ell^\infty(\hat{A}) \cap \overline{\gamma_\partial(A)}$ be the projection corresponding to the Davies Borel set $f^{-1}(n)$. Trivially, the sets $f^{-1}(n)$ are pairwise disjoint, and their union is $\hat{A}$, so $\sum_n p_n = 1$ in the ultraweak topology; similarly, the series

$\sum_n \gamma_\partial(a_n)p_n$ is ultraweakly convergent because $\|a_n\| \leq 1$ for all n. Furthermore,

$$\begin{aligned}\left|\varphi\left((x_\gamma^\infty : \gamma) - \sum_n \gamma_\partial(a_n)p_n\right)\right| &= \left|\varphi(x_\gamma^\infty - \gamma(a_{f(\gamma)}) : \gamma \in \hat{A})\right| \\ &= \left|\int_\gamma \varphi_\gamma(x_\gamma^\infty - \gamma(a_{f(\gamma)}))\, dm\right| \\ &\leq \int_\gamma |\varphi_\gamma(x_\gamma^\infty - \gamma(a_{f(\gamma)}))|\, dm \leq \int_\gamma \epsilon\, dm = \epsilon\end{aligned}$$

Thus, we can ensure that $\varphi(\sum_n \gamma_\partial(a_n)p_n) \neq 0$. Of course, $\sum_n \gamma_\partial(a_n)p_n \in \overline{\gamma_\partial(A)}$ since each term is a product of elements in $\overline{\gamma_\partial(A)}$, and the series converges continuum-weakly, by lemma 2.3.1. This contradicts our choice of φ. □

Appendix A

Notation and terminology

The symbol $\mathbb{I}$ denotes the unit interval. The symbol $\mathbb{T}$ denotes either the unit circle in the complex plane, or the quotient $\mathbb{R}/\mathbb{Z}$, as appropriate to the context. A topological space is Polish iff it is homeomorphic to a complete separable metric space.

Let $f\colon T \to S$ be a function. The expression $(f(t) \in S\colon t \in T)$ also denotes the function f; the expression $\{f(t) \in S\colon t \in T\}$ denotes its range; and the expression $[f(t) \in S\colon t \in T]$ denotes its equivalence class under the relevant equivalence relation. Where there is no danger of ambiguity, we will sometimes denote these objects simply by $(f(t))$, $\{f(t)\}$, and $[f(t)]$.

Let X be a locally compact Hausdorff space. The support of a function $f\colon X \to \mathbb{C}$ is the set $\mathrm{supp}(f) = \{x \in X\colon f(x) \neq 0\}$, and f is compactly supported iff the closure $\overline{\mathrm{supp}(f)}$ is compact. A function $f\colon X \to \mathbb{C}$ is in $\ell^\infty(X)$ iff it is bounded; it is in $\ell^2(X)$ iff it is square-summable; it is in $\ell^1(X)$ iff it is absolutely summable; it is in $c_0(X)$ iff $\{|f(x)| \geq \epsilon\}$ is finite for all $\epsilon > 0$; it is in $C_0(X)$ iff it is continuous and $\{|f(x)| \geq \epsilon\}$ is compact for all $\epsilon > 0$; it is in $C_c(X)$ iff it is continuous and compactly supported; and it is in $\mathscr{B}(X)$ iff its real and imaginary parts can be obtained from the continuous compactly supported real-valued functions on X by taking limits of ascending and descending sequences. A subset $S \subseteq X$ is Baire iff the function $f\colon X \to \{0,1\}$ such that $f^{-1}(1) = S$ is in $\mathscr{B}(X)$ [20, remark 6.2.10].

Let T be a set. A complex-valued measure on T is a function $m\colon \{S \subseteq T\} \to \mathbb{C}$ that is countably additive in the sense that the series $\sum_n m(S_n)$ converges absolutely to $m(\bigcup_n S_n)$ for all countable families $(S_n \subseteq T)$ of pairwise disjoint subsets. A finite measure is a complex-valued measure m such that $m(S) \geq 0$ for all $S \subseteq T$; it is a probability measure if furthermore $m(T) = 1$. If T is equipped with a finite measure m, and $\mathcal{X}$ is a Banach space, then $\mathcal{L}^1(T, \mathcal{X}) = \{f\colon T \to \mathcal{X}\colon \int_{t\in T} \|f(t)\|\, dm < \infty\}$ is the set of integrable functions, $\mathcal{L}^2(T, \mathcal{X}) = \{f\colon T \to \mathcal{X}\colon \int_{t\in T} \|f(t)\|^2\, dm < \infty\}$ is the set of square-integrable functions, and $\mathcal{L}^\infty(T, \mathcal{X}) = \{f\colon T \to \mathcal{X}\colon \inf_{m(S)=m(T)} \sup_{t\in S} \|f(t)\| < \infty\}$ is the set of essentially bounded functions; and $L^1(T, \mathcal{X})$, $L^2(T, \mathcal{X})$, and $L^\infty(T, \mathcal{X})$ are the corresponding Lebesgue spaces. The unit interval $\mathbb{I}$ will often be implicitly equipped with Lebesgue measure.

Let A be a C*-algebra. A functional on A is a bounded linear function $A \to \mathbb{C}$. If μ is a state on A, then $\gamma_\mu\colon A \to \mathcal{B}(\mathcal{H}_\mu)$ denotes the GNS representation of A for the state μ. If T

is a set of states, then γ_T denotes the direct sum representation $a \mapsto \bigoplus_{\mu \in T} \gamma_\mu(a)$. The space of all states of A is denoted $\mathcal{S}(A)$, and $\gamma_{\mathcal{S}(A)}$ is often abbreviated $\gamma_{\mathcal{S}}$; this is the universal representation. The space of all *pure* states is denoted $\partial\mathcal{S}(A)$, and $\gamma_{\partial\mathcal{S}(A)}$ is often abbreviate γ_∂; this is the atomic representation. Also, $W^*(A)$ is the closure of $\gamma_{\mathcal{S}}(A)$ in the ultraweak topology, and $V^*(A)$ is the closure of $\gamma_{\mathcal{S}}(A)$ in the continuum-weak topology, which is to be defined.

Let A be a concrete C*-algebra on some Hilbert space $\mathcal{H}$. Then, A is nondegenerate iff $A\mathcal{H}$ is norm dense in $\mathcal{H}$. A vector functional on A is a functional of the form $a \mapsto \langle \xi | a\eta \rangle$ for some $\xi, \eta \in \mathcal{H}$. A vector state on A is a state that is a vector functional, or equivalently, a state of the form $a \mapsto \langle \xi | a\xi \rangle$ for some $\xi \in \mathcal{H}$. The Banach space of all ultraweakly continuous functionals is denoted A_*; the subspace of all ultraweakly continuous states is denoted $\mathcal{S}_*(A)$, and $\gamma_{\mathcal{S}_*(A)}$ is often abbreviated γ_*. The Banach space of all continuum-weakly continuous functionals is denoted $A_\bullet$; the subspace of all continuum-weakly continuous states is denoted $\mathcal{S}_\bullet(A)$, and $\gamma_{\mathcal{S}_\bullet(A)}$ is often abbreviated $\gamma_\bullet$. The expression $\overline{A}$ always denotes the closure of A in the continuum-weak topology.

A V*-algebra is a concrete C*-algebra that is closed in the continuum-weak topology, which is to be defined. The term is intended to suggest an analogy with W*-algebras, i. e., von Neumann algebras. The term "V*-algebra" was previously used for a different class of operator algebras [1]; our appropriation of this term might be justified by the fact that this class of operator algebras coincides with that of C*-algebras [1, theorem 4.3]. The term "v*-algebra" is also used in universal algebra [29].

A diagram with a dotted arrow expresses that there is a morphism which makes that diagram commute. A diagram with an exclamation mark expresses that the morphism making that diagram commute is unique. A diagram with neither of these features simply expresses that that diagram commutes.

The axioms of Zermelo-Fraenkel set theory are those listed in chapter 1 of Jech's *Set Theory* [11]. We use boldface and lightface to distinguish between properties of hereditary sets, and the set-theoretic formulas that express them. For example, **ZF** is Zermelo-Fraenkel set theory, which consists of propositions; ZF is the corresponding set of set-theoretic formulas.

The axiom of choice **AC** is the proposition that for any family of nonempty sets $(X_t \colon t \in I)$ indexed by an arbitrary set I, there exists a function $f \colon I \to \bigcup_{t \in I} X_t$ such that $f(t) \in X_t$ for all $t \in I$; the theory **ZF** + **AC** is abbreviated **ZFC**. The axiom of choices almost everywhere $\mathbf{AC_{ae}}$ is the proposition that for any family of nonempty sets $(X_t \colon t \in \mathbb{R})$ indexed by the real line $\mathbb{R}$, there exists a function $f \colon \mathbb{R} \to \bigcup_{t \in \mathbb{R}} X_t$ such that $f(t) \in X(t)$ for almost all $t \in \mathbb{R}$, in the sense of Lebesgue measure. The axiom of dependent choices **DC** is the proposition that in any directed graph where every vertex is the source of some edge, there is an infinite walk starting from any given vertex.

The Lebesgue measurability axiom **LM** is the proposition that every subset $X \subseteq \mathbb{R}$ is Lebesgue measurable, in the sense that there exists a G_δ subset $Y \subseteq \mathbb{R}$ whose symmetric difference with X can be covered by a countable family of intervals, the sum of whose lengths can be made arbitrarily small. The Baire property axiom **BP** is the proposition that every subset $X \subseteq \mathbb{R}$ has the Baire property, in the sense that there exists an open set U whose symmetric difference with X can be covered by a countable family of closed nowhere dense sets. The perfect set property axiom **PSP** is the proposition that every subset $X \subseteq \mathbb{R}$ has the perfect set property, in the sense that it is either countable, or contains a perfect subset of $\mathbb{R}$, i. e., a nonempty subset of $\mathbb{R}$ that has no isolated points in the subspace topology.

Appendix B

The axiom of choices almost everywhere

Definition B.0.1. The axiom of choices almost everywhere, $\mathbf{AC_{ae}}$, is the following choice principle: for every family of nonempty sets $(X_t \colon t \in \mathbb{R})$ indexed by the real numbers $\mathbb{R}$, there is a function $f \colon \mathbb{R} \to \bigcup_{t\in\mathbb{R}} X_t$ such that $f(t) \in X_t$ for almost all $t \in \mathbb{R}$, in the sense of Lebesgue measure.

Theorem B.0.2. *The axiom of choices almost everywhere is satisfied by the transitive model* $\mathfrak{N}_1$ *obtained by applying Solovay's construction* [27] *to a countable transitive model* $\mathfrak{M}$ *of the axiom of constructibility.*

The argument below was suggested to me by **John Steel**. It is possibly the same argument as in [12, section 1], but the authors do not make their assumptions clear.

We briefly recall the situation: $\mathfrak{M}[G]$ is a countable transitive model of set theory; $\mathfrak{M}$ is an inner submodel of $\mathfrak{M}[G]$ satisfying the axiom of constructibility; κ is an ordinal that is a strongly inaccessible cardinal in $\mathfrak{M}$, and that is $\aleph_1$ in $\mathfrak{M}[G]$; $\mathfrak{N}_1$ is another inner submodel of $\mathfrak{M}[G]$, which consists of sets that are hereditarily definable from a sequence of ordinals in $\mathfrak{M}[G]$; G is an $\mathfrak{M}$-generic ultrafilter on the Boolean algebra densely generated [11, Theorem 14.10] by the Levý collapse partial order $\mathrm{Col}(\aleph_0, < \kappa)$ [11, Theorem 15.22]; and $\mathfrak{M}[G]$ is the smallest transitive model that contains G and includes $\mathfrak{M}$.

proof of theorem B.0.2. We work in $\mathfrak{M}[G]$. Every element x of $\mathfrak{N}_1$ is $\mathfrak{M}$-$\mathbb{R}$ definable [27, lemma III.2.8] in the sense that there is a 3-ary set-theoretic formula ϕ, a set m in $\mathfrak{M}$, and a real number r such that x is the unique set satisfying $\phi(x, m, r)$ [27, remark III.1.3]. The reflection principle implies that there is an ordinal α such that x is the unique set satisfying $V_\alpha \models \phi(x, m, r)$. On the other hand, any set that can be characterized in this way is obviously $\mathfrak{M}$-$\mathbb{R}$-definable since for all ordinals α, all formulae ϕ, and all sets m in $\mathfrak{M}$, the 3-tuple $\{\alpha, \phi, m\}$ is also in m. Thus a set x is $\mathfrak{M}$-$\mathbb{R}$-definable iff there exist an ordinal α, a set-theoretic formula ϕ, a set m in $\mathfrak{M}$, and real number r such that x is the unique set satisfying $V_\alpha \models \phi(x, m, r)$. For each $\mathfrak{M}$-$\mathbb{R}$-definable x, there is a least such ordinal α^x.

Then, in the lexicographical ordering, there is a least set-theoretic formula ϕ^x such that x is the unique set satisfying $V_{\alpha^x} \models \phi^x(x, m, r)$ for some m in $\mathfrak{M}$ and r in $\mathbb{R}$. Finally, because constructibility is absolute [11, corollary 13.15], $\mathfrak{M}$ consists of the constructible sets, so we have definable well-ordering on $\mathfrak{M}$ [11, theorem 13.18]; thus, for each $\mathfrak{M}$-$\mathbb{R}$-definable x, there is a least element m^x of $\mathfrak{M}$ with respect to this well-ordering, such that x is the unique set satisfying $V_{\alpha^x} \models \phi^x(x, m^x, r)$.

Now, fix a family of sets $(X_t \colon t \in \mathbb{R})$ in $\mathfrak{N}_1$. Since $\mathfrak{N}_1$ is transitive, the elements of $\bigcup_{t\in\mathbb{R}} X_t$ are also in $\mathfrak{N}_1$, and in particular, are $\mathfrak{M}$-$\mathbb{R}$-definable. For each t, let (α_t, ϕ_t, m_t) be the minimum of $\{(\alpha^x, \phi^x, m^x) \colon x \in X_t\}$, ordered lexicographically from left to right, so that there necessarily exists a real number r such that there is a unique x satisfying $V_{\alpha_t} \models \phi_t(x, m_t, r)$; let Y_t be the set of all such r. Observe, that we have defined the family (Y_t) from the family (X_t), which is $\mathfrak{M}$-$\mathbb{R}$-definable, so (Y_t) is itself $\mathfrak{M}$-$\mathbb{R}$-definable. It now follows [27, remark III.1.12] that there is a Borel function $h \colon \mathbb{R} \to \mathbb{R}$ such that $h(t) \in Y(t)$ for almost all $t \in \mathbb{R}$. Therefore, we define $f(t)$ to be the unique x such that $V_{\alpha_t} \models \phi_t(x, m_t, h(t))$ if such an x exists, as it does for almost all t, and some arbitrary fixed $x_0 \in X_0$, otherwise. By construction, $f(t) \in X_t$ for almost all t. Furthermore, f is $\mathfrak{M}$-$\mathbb{R}$-definable, since h is $\mathfrak{M}$-$\mathbb{R}$-definable because it is Borel [27, lemma III.2.8, lemma III.2.12], and x_0 is $\mathfrak{M}$-$\mathbb{R}$-definable because it is in $\mathfrak{N}_1$. Each real number is definable from a sequence of ordinals, because it is obviously definable form a sequence of bits. Each set m in $\mathfrak{M}$ is definable from a sequence of ordinals, because $\mathfrak{M}$ has a definable well-ordering. Thus, f is definable from a sequence of ordinals, and since each its domain and codomain are both hereditarily definable from a sequence of ordinals, it follows that f is likewise hereditarily definable form a sequence of ordinals, i. e., f is in $\mathfrak{N}_1$. □

Appendix C

Proofs for section 1.7

Proof of 1.7.5. All countable subsets of X are Borel, so we may assume that m is atomless. Apply 1.7.4. □

Proof of 1.7.6. Without loss of generality, we assume that (T, m) is atomless. Clearly $T \not\preceq \mathbb{N}$, so by **PSP**, $T \approx \mathbb{I}$. Thus, it is sufficient that every atomless probability measure on $\mathbb{I}$ is the pushforward of Lebesgue measure on $\mathbb{I}$, which follows immediately from 1.7.4 and 1.7.5. □

Proof of 1.7.7. By 1.7.6, we may assume that $T_0, T_1 = \mathbb{I}$, and m_0 and m_1 are Lebesgue measure. It follows from 1.7.5 that f is measurable, and we can verify via absoluteness that Fubini-Tonelli holds for any Lebesgue-measurable function $\mathbb{I} \times \mathbb{I} \to \mathbb{C}$. □

Proof of 1.7.8. Clearly, if there is a net in Y converging to some point x, then that point is in the closure of Y. On the other hand, if x is in the closure of Y, then the following net in Y converges to x: the index set consists of all pairs (U, y) where U is a neighborhood of x, and y is any point in $Y \cap U$, $(U_0, y_0) \leq (U_1, y_1)$ iff $U_0 \subseteq U_1$, and the value of the net at index (U, y) is y. □

Proof of 1.7.9. If f is continuous at x, then by definition, the preimage of every neighborhood of $f(x)$ contains a neighborhood of x. Thus if (x_λ) is a net in X converging to $x \in X$, then eventually $f(x_\lambda)$ is in any given neighborhood of $f(x)$. On the other hand, if f is not continuous at X, then by definition there is a neighborhood V of $f(x)$ whose preimage contains no neighborhood of x; in this case, x is in the closure of the complement of this preimage so by 1.7.8 there is a net (x_λ) converging to x such that the net $f(x_\lambda)$ does not converge to $f(x)$ because it is never in V. □

Proof of 1.7.10. This is a known consequence of the fact that every subset of $\mathbb{I}$ is Lebesgue measurable [8]. □

Proof of 1.7.11. We combine the open mapping theorem, verified via absoluteness, with 1.7.10. □

Proof of 1.7.35. Without the axiom of choice, a representation is not necessarily the direct sum of cyclic representations; thus, the usual proof does not apply. Recall that $\gamma_\mathcal{S}\colon A \to \mathcal{B}(\mathcal{H}_\mathcal{S})$ denotes the universal representation of A. Fix another representation $\rho\colon A \to \mathcal{B}(\mathcal{H})$.

For vectors $\xi, \eta \in \mathcal{H}$, let $\omega_{\xi,\eta}\colon \gamma_\mathcal{S}(A) \to \mathbb{C}$ be given by $\gamma_\mathcal{S}(a) \mapsto \langle \xi | \rho(a)\eta \rangle$, which is easily seen to be well-defined by considering the cyclic representation of A on the subspace $\overline{\rho(A)\eta} \subseteq \mathcal{H}$. Let $\tilde{\omega}_{\xi,\eta}$ be its unique ultraweak extension to $W^*(A)$, the ultraweak closure of $\gamma_\mathcal{S}(A)$. For each operator $x \in W^*(A)$, the expression $\tilde{\omega}_{\xi,\eta}(x)$ defines a sesquilinear form on $\mathcal{H}$; it is bounded:

$$\|\tilde{\omega}_{\xi,\eta}(x)\| \leq \|\tilde{\omega}_{\xi,\eta}\| \cdot \|x\| \leq \|\omega_{\xi,\eta}\| \cdot \|x\| \leq \|\xi\| \cdot \|\eta\| \cdot \|x\|$$

The second inequality follows by Kaplansky's density theorem. We define $\pi(x)$ to be the unique bounded operator such that $\langle \xi | \pi(x)\eta \rangle = \tilde{\omega}_{\xi,\eta}(x)$. Since $\tilde{\omega}_{\xi,\eta}(x)$ is linear in x, we obtain a bounded linear map $\pi : W^*(A) \to \mathcal{B}(\mathcal{H})$. Note that $\pi \circ \gamma_\mathcal{S} = \rho$, since $\langle \xi | \pi(\gamma_\mathcal{S}(a))\eta \rangle = \tilde{\omega}_{\xi,\eta}(\gamma_\mathcal{S}(a)) = \omega_{\xi,\eta}(\gamma_\mathcal{S}(a)) = \langle \xi | \rho(a)\eta \rangle$ for all $\xi, \eta \in \mathcal{H}$. The map π is ultraweakly continuous since it pulls every vector functional back to a vector functional; this implies uniqueness.

It is easy to verify that π is unital and self-adjoint. For all $\xi, \eta \in \mathcal{H}$,

$$\langle \xi | \pi(1)\eta \rangle = \tilde{\omega}_{\xi,\eta}(1) = \omega_{\xi,\eta}(1) = \langle \xi | \eta \rangle.$$

Thus, $\pi(1) = 1$. If $x \in W^*(A)$ is self-adjoint, then we fix a net $(b_\lambda \in \gamma_\mathcal{S}(A)_{sa})$ that ultraweakly converges to x, and compute for all $\xi, \eta \in \mathcal{H}$:

$$\langle \pi(x)\xi | \eta \rangle = \overline{\langle \eta | \pi(x)\xi \rangle} = \overline{\tilde{\omega}_{\eta,\xi}(x)} = \lim_\lambda \overline{\omega_{\eta,\xi}(b_\lambda)} = \lim_\lambda \langle \xi | \rho(b_\lambda)\eta \rangle = \cdots = \langle \xi | \pi(x)\eta \rangle$$

We establish that π is a homomorphism in the usual way: we note that it is a homomorphism on $\gamma_\mathcal{S}(A)$, and then extend this property to $W^*(A)$ first on the first factor, and then on the second. For all $a_0, a_1 \in A$,

$$\pi(\gamma_\mathcal{S}(a_0)\gamma_\mathcal{S}(a_1)) = \pi(\gamma_\mathcal{S}(a_0a_1)) = \rho(a_0a_1) = \rho(a_0)\rho(a_1) = \pi(\gamma_\mathcal{S}(a_0))\pi(\gamma_\mathcal{S}(a_1))$$

For all $x \in W^*(A)$ and all $b \in \gamma_\mathcal{S}(A)$, there is a net $(b_\lambda \in \gamma_\mathcal{S}(A))$ that converges to x ultraweakly, and therefore, for all $\xi, \eta \in \mathcal{H}$,

$$\langle \xi | \pi(x)\pi(b)\eta \rangle = \lim_\lambda \langle \xi | \pi(b_\lambda)\pi(b)\eta \rangle = \lim_\lambda \langle \xi | \pi(b_\lambda b)\eta \rangle = \langle \xi | \pi(xb)\eta \rangle$$

Finally, for all $x_0, x_1 \in W^*(A)$, there is a net $(b_\lambda \in \gamma_\mathcal{S}(A))$ that converges to x_1 ultraweakly, and therefore, for all $\xi, \eta \in \mathcal{H}$,

$$\langle \xi | \pi(x_0)\pi(x_1)\eta \rangle = \lim_\lambda \langle \xi | \pi(x_0)\pi(b_\lambda)\eta \rangle = \lim_\lambda \langle \xi | \pi(x_0 b_\lambda)\eta \rangle = \langle \xi | \pi(x_0x_1)\eta \rangle.$$

□

Proof of 1.7.41. A functional on M is ultraweakly continuous iff it is countably additive, by [28] corollary III.3.11. A functional is countably additive iff it is countably additive on every von Neumann subalgebra of M isomorphic to $\ell^\infty(\mathbb{N})$, and **BP** implies that this is the case [27] [22] [23, 29.37 and 29.38]. This proof was suggested to me by **Alexandru Chirvasitu**. □

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