UNIVERSITY OF CALIFORNIA,
IRVINE

Logarithmic capacity of G_δ subsets of $[0, 1]$

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Fernando Quintino

Dissertation Committee:
Professor Anton Gorodetski, Chair
Professor Svetlana Jitomirskaya
Professor Abel Klein

2021

DEDICATION

Thus says the LORD: "Let not the wise man boast in his wisdom, let not the mighty man boast in his might, let not the rich man boast in his riches, but let him who boasts boast in this, that he understands and knows me, that I am the LORD who practices steadfast love, justice, and righteousness in the earth. For in these things I delight, declares the LORD."
Jeremiah 9:24, 25 ESV

To my wife, Brooke. I am grateful to do life with you. Without your support and patience throughout this program I would not have been able to finish.

To my family. I am thankful for the support of my parents, Fernando and Guadalupe, who encouraged me to pursue higher education. I am grateful for two awesome brothers, Aldo and Joshua, who made growing up an adventure. To my mother-in-law, Rhonda, who has always made me feel welcome. To my brother-in-law, Jordan, who encourages me. To my father-in-law, Ken, who always made me laugh. To my grandma, who always cared for me.

To all those who have supported me in many ways throughout graduate school. To my uncles, aunts, cousins, friends, and my church family. To my Tio Javier and Tia Coty, who gave us a home during graduate school.

Finally, to my Lord and Savior, Jesus Christ who gave me the strength to finish:

"To him who sits on the throne and to the Lamb be blessing and honor and glory and might forever and ever!"
Revelation 5:13 ESV

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ACKNOWLEDGMENTS

I would like to thank my advisor Anton Gorodetski for taking me in as his student. I am grateful for his mentorship, his guidance, for the many discussions that we have had over the years that have lead me to give better presentations, write better papers, and become a better mathematician.

I would like to thank Victor Kleptsyn for being my collaborator, for the numerous discussions that we have had over the years, for reading multiple drafts of Chapter 5 and offering many helpful suggestions (and for the idea of the proof of Lemma 5.3).

I would like to thank Svetlana Jitomirskaya for helping me get into the PhD program at UCI and for always supporting me. I have become a better mathematician because of your teaching style.

I am thankful for my undergraduate professors at California State University of Fullerton. Thank you Bogdan Suceavă and Scott Annin for your support and mentorship.

The author was supported in part by NSF grants DMS-1700143 (PI - M. Foreman) and DMS-1855541 (PI - A. Gorodetski).

CURRICULUM VITAE

Fernando Quintino

EDUCATION

Doctor of Philosophy in Mathematics **2021**
University of California, Irvine *Irvine, California*

Master of Science in Mathematics **2017**
University of California, Irvine *Irvine, California*

Bachelor of Arts in Mathematics **2015**
California State University of Fullerton *Fullerton, California*

Associate of Arts in Mathematics **2011**
Saddleback Community College *Mission Viejo, California*

RESEARCH EXPERIENCE

Graduate Research Assistant **2017 – 2021**
University of California, Irvine *Irvine, California*

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Teaching Associate **Summer Session II 2020**
University of California, Irvine *Irvine, California*

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University of California, Irvine *Irvine, California*

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Instructor **Summer 2017**
College for Kids, Saddleback Community College *Mission Viejo, California*

Courses: Pre-Algebra (3 sessions), Algebra 1A (2 sessions), Algebra 1B

Supplemental Instructor **2012 – 2015**
California State University of Fullerton *Fullerton, California*

Courses: Pre-Calculus, Calculus II, Multivariable Calculus

ONLINE VIDEOS EXPERIENCE

Math Pre-Requisite Review Program
University of California, Irvine
`https://www.math.uci.edu/~prerequisite-videos/`

Developed videos, notes, problem sets, and solutions on pre-requisite material for undergraduates:

Numerical Analysis I and Optimization I
`https://www.math.uci.edu/~prerequisite-videos/105a110a.html`

Introduction to Partial Differential Equations
`https://www.math.uci.edu/~prerequisite-videos/112a.html`

Elementary Analysis
`https://www.math.uci.edu/~prerequisite-videos/140a.html`

YouTube channel
Vector Quintino Mathematics
`https://www.youtube.com/channel/UCm1BNM2dswmHRW9LO5DOPZw`

COMMUNITY INVOLVEMENT

CSU Fullerton GRAM Scholars Day **May 21, 2019**

We hosted CSUF GRAM Scholars at UCI. These students work closely with Dr. Scott Annin and Dr. Anael Verdugo. The NSF program Graduate Readiness and Access in Mathematics (GRAM) is a project in which carefully selected undergraduates are chosen to receive special mentoring, training, mathematical skills, professional development, and more, with an eye towards bolstering their graduate school aspirations. I served as a panelist answering student's questions on graduate school and gave them a tour of the campus.

Student Research Symposium **April 28, 2018**

Organized by Pacific Math Alliance and PUMP: Preparing Undergraduates through Mentoring towards PhDs. The purpose of this diversity conference is to inform mathematics students (and in particular URMs) about Master's and PhD programs, and other careers in the mathematical sciences, and to offer them a chance to present their research in a friendly setting. I served as a panelist at the Student Research Symposium.

Math CEO **Winter 2018**

We hosted a group of 80 to 100 disadvantaged Latino middle school students on the UCI campus, where we introduced them to interesting mathematics. We traveled to the Math CEO student's middle school for Family College Night in Santa Ana to meet with their parents. I was able to tell them my story and answer any questions they might have about college.

2016 SACNAS (The National Diversity in STEM Conference) **October 2016**

Three days of cutting-edge science, training, mentoring, and cultural activities for students and scientists at all levels. I represented the Mathematics department at the UCI booth.

LANGUAGE SKILLS

Bilingual English and Spanish.

PUBLICATIONS

Logarithmic capacity of random G_δ-sets **2021**
Preprint
https://arxiv.org/abs/2012.01593

Phase transition of capacity for the uniform G_δ-sets **2021**
Potential Analysis
https://doi.org/10.1007/s11118-020-09896-8

CONFERENCES PRESENTATIONS

Phase transition of capacity for uniform G_δ's **September 2019**
Random Matrix Products and Anderson Localization
Banff International Research Station , Banff, Alberta, Canada

Interpolating Legendre Multiplier Sequences **January 2014**
AMS Session
Joint Mathematics Meetings, Baltimore Convention Center, Baltimore, MD

TALKS

Logarithmic capacity of random G-delta sets **January 29, 2021**
Research Seminar
University of California, Irvine

Phase transition of capacity for uniform G_δ-sets and another counterexample to Nevanlinna's conjecture **January 15, 2020**
Harmonic Analysis Seminar
University of California, Irvine

Phase transition of capacity for uniform G_δ-sets **October 11, 2019**
Ergodic Schrodinger Operators
University of California, Irvine

An Introduction to Dynamical Systems **October 7, 2019**
Invited speaker for undergraduates
California State University of Fullerton

A Crash Course on Capacity **October 4, 2019**
Ergodic Schrodinger Operators
University of California, Irvine

Furstenberg's Theorem on Random Product of Matrices part I **April 13**
Ergodic Schrodinger Operators Seminar
University of California, Irvine

Furstenberg's Theorem on Random Product of Matrices part II **April 20**
Ergodic Schrodinger Operators Seminar
University of California, Irvine

Furstenberg's Theorem on Random Product of Matrices part III **April 27, 2018**
Ergodic Schrodinger Operators Seminars
University of California, Irvine

Kingman's Subadditive ergodic theorem part II **October 27, 2017**
Ergodic Schrodinger Operators Seminar
University of California, Irvine

Kingman's Subadditive ergodic theorem part I **October 20, 2017**
Ergodic Schrodinger Operators Seminar
University of California, Irvine

Preparation for Putnam 2013 **Fall 2013**
Problem Solving Seminar
California State University of Fullerton

SUMMER GRADUATE SCHOOLS

Summer School on Teichmller Theory and its Connections to Geometry, Topology and Dynamics — **August 20 – 24, 2018**
The Fields Institute for Research in Mathematical Sciences
Research in Mathematical Sciences, Toronto, Ontario, M5T 3J1, Canada

The Third Great Lakes Mathematical Physics Meeting — **June 22 – 24, 2018**
The MSU Institute for Mathematical and Theoretical Physics
East Lansing, MI

Houston Summer School on Dynamical Systems — **May 16 – 24, 2018**
University of Houston, Houston, TX

Houston Summer School on Dynamical Systems — **May 17 – 25, 2017**
University of Houston, Houston, TX

AWARDS

Featured during Hispanic Heritage Month at the UCI Physical Sciences — **2020**

Most Promising Future Faculty Award nominee by former students — **2020**

Faculty Mentor Fellowship Honorable Mention — **2018 – 2019**

NSM College Recognition for Outstanding Undergraduate Research — **2014**

Ranked 69 out of 450 teams in the William Lowell Putnam Mathematical Competition — **2013**

Scored a 10 in the William Lowell Putnam Mathematical Competition — **2013**

Cheryl and Carl Carrera Mathematics Scholarship — **2013**

CSU Edison Transfer Scholorship — **2012**

ABSTRACT OF THE DISSERTATION

Logarithmic capacity of G_δ subsets of $[0,1]$

By

Fernando Quintino

Doctor of Philosophy in Mathematics

University of California, Irvine, 2021

Professor Anton Gorodetski, Chair

We study the logarithmic capacity of G_δ subsets of the interval $[0,1]$. Let S be of the form

$$S = \bigcap_{m} \bigcup_{k \geq m} I_k,$$

where each I_k is an open interval with center c_k in $(0,1)$ and with length l_k that decrease to 0 as $k \to \infty$. We provide sufficient conditions for S to have full capacity, i.e. $\mathrm{Cap}(S) = \mathrm{Cap}([0,1])$. We consider the case when the intervals decay exponentially and are placed in $[0,1]$ randomly with respect to some given distribution. We prove that the random G_δ sets generated by such distribution satisfy our sufficient conditions almost surely. Hence, the random G_δ sets have full capacity almost surely. This study is motivated by the G_δ set of exceptional energies in the parametric version of the Furstenberg theorem on random matrix products. Additionally, we also study the family of G_δ sets $\{S(\alpha)\}_{\alpha>0}$ that are generated by setting the intervals to $l_k = e^{-k^\alpha}$. We observe a sharp transition at $\alpha = 1$ from full capacity to zero capacity by varying $\alpha > 0$.

Our re-distribution construction can be considered as a generalization of a method applied by Ursell in his construction of a counter-example to a conjecture by Nevanlinna. Also, we propose a simple Cauchy-Schwartz inequality-based proof of related theorems by Lindeberg

and by Erdös and Gillis.

Chapter 1

Introduction

For an in depth study of Potential Theory we refer the read to *Potential Theory in the Complex Plane* [40] ([47, Appendix A] is a good source too). In this chapter, we will summarize the basic definitions and results from [40].

1.1 The energy of a measure

The collection of all finite signed Borel measures on $\mathbb{C}$ form a vector space over $\mathbb{R}$.

Definition 1.1. Given two finite singed Borel measures μ and ν on $\mathbb{C}$ we define their *interaction* by

$$I(\nu, \mu) := \iint (-\log |z - w|)\, d\nu(z)\, d\mu(w).$$

The interaction is a bilinear form on the vector space of finite singed Borel measures on $\mathbb{C}$ with the following properties:

1. $I(\nu, \mu) = I(\mu, \nu)$,

2. $I(\nu, \mu) > 0$, if ν and μ are probability measures and the union of their support has diameter of at most 1,

3. $I(\nu, \mu + \mu') = I(\nu, \mu) + I(\nu, \mu')$ and $I(\nu, c\mu) = cI(\nu, \mu)$.

This bilinear form is a generalization of the *energy* of measure μ:

Definition 1.2. Given a Borel measure μ on $\mathbb{C}$ we define the *energy* of μ by

$$I(\mu) := \iint (-\log|z - w|)\, d\mu(z)\, d\mu(w).$$

That is, $I(\mu) = I(\mu, \mu)$. We can think of the energy of a measure as *self-interacting.* To the best of our knowledge, the bilinear form of the energy was first introduced in [30] and is convenient in computations. The function $-\log|z - w|$ can be replaced by a non-negative lower semi continuous function in Definition 1.1 (see [2, p. 109]) and Definition 1.2 (see [2, Chapter 4]).

In physics, μ is a charge distribution on $\mathbb{C}$ and $p_\mu(z) := \int(-\log|z-w|)\, d\mu(w)$ is the potential energy at z due to z (see [40, p. 56]). Then $I(\mu)$ is the total energy of μ. For a deeper understanding of the physics behind logarithmic capacity see [2, Chapter 3].

Traditionally, a *polar* set is defined as a set on which some non-constant subharmonic function takes values $-\infty$ (see, for example, [26]). In the study of potential theory ([40, p. 56]) we work with the following definition:

Definition 1.3. We say that a subset E of $\mathbb{C}$ is *polar* if $I(\mu) = \infty$ for every non-trivial finite Borel measure with compact support contained in E.

The two definitions are the same (see [33, p. 1]). Given any subharmonic function u on a domain D in $\mathbb{C}$ such that $u \not\equiv -\infty$, then $\{z \in D : u(z) = -\infty\}$ is a G_δ polar set ([40, Theorem 3.5.1]). Hence, every subset of $\{z \in D : u(z) = -\infty\}$ is polar. Deny's theorem (see [12]) is a converse to this statement: every G_δ polar set is of the form $\{u = -\infty\}$ for some subharmonic function u. It can be shown that every Borel polar set is a subset of a G_δ polar set. For other links between these two definitions see [40, Section 3.5].

Trivial examples of polar sets are singleton sets and subsets of polar sets. Not only are polar sets the smallest of sets in capacity, polar sets are sets of measure zero when the energy of the measure is finite:

Theorem 1.1 ([40, Theorem 3.2.3])**.** *Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. If $I(\mu) < \infty$, then $\mu(E) = 0$ for every Borel polar set E.*

Proof. By way of contradiction, let us suppose that E is a Borel polar set with $\mu(E) > 0$. Since μ is a regular measure, we may find a compact subset K of E with positive measure. Restricting μ to the set K gives us $\nu := \mu|_K$ a finite Borel measure with compact support contained in E. Let d be the diameter of K. Since ν is a non-trivial measure and E is a polar set, then $I(\nu) = \infty$. However,

$$\begin{aligned}
I(\nu) &= \int\int -\log|z-w|d\nu(z)d\nu(w) \\
&= \int_K\int_K -\log|z-w|d\mu(z)d\mu(w) \\
&= \int_K\int_K -\log\frac{|z-w|}{d}d\mu(z)d\mu(w) - (\log d)(\mu(K))^2 \\
&\le \int\int -\log\frac{|z-w|}{d}d\mu(z)d\mu(w) - (\log d)(\mu(K))^2 \\
&= I(\mu) + (\log d)(\mu(\mathbb{C}))^2 - (\log d)(\mu(K))^2 < \infty.
\end{aligned}$$

□

By showing that $I(\mu) < \infty$ (with the corresponding measure), we get that every Borel polar set $E \subset \mathbb{C}$ has 2-dimensional Lebesgue measure zero, every Borel polar set $E \subset \mathbb{R}$ has 1-dimensional Lebesgue measure zero, and every Borel polar set has α-dimensional Hausdorff measure zero for each $\alpha > 0$ (see [40, pp. 56–57]).

Given a countable collection of Borel polar sets E_n and a finite Borel measure μ with compact support in the countable union of E_n, then the only measures with finite energy $I(\mu) < \infty$ are trivial measures by Theorem 1.1 (since $\mu(E_n) = 0$ for every n):

Theorem 1.2 ([40, Corollary 3.2.5]). *Every countable union of Borel polar sets is polar.*

As a corollary, we have that every countable set is polar. A natural question that follows is whether there are any uncountable sets that are polar. It turns out that the 1/3 Cantor set is not polar. Nonetheless, it is possible to generalize Cantor sets in the following way to get an uncountable set that is polar.

Example 1.1 (Generalized Cantor sets). Let $s = (s_n)_{n\geq 1}$ with $0 < s_n < 1$ for all n. In the usual way of constructing the 1/3 Cantor set, begin with the unit interval. At stage 1, remove the middle s_1 of the unit interval and call what remains $C(s_1)$. At stage n, remove s_n from the middle of each interval of $C(s_1, s_2, \ldots, s_{n-1})$ and call what remains $C(s_1, s_2, \ldots, s_n)$. By induction we have constructed a descending chain of closed sets (See Figure 1.1):

$$C(s_1) \supset C(s_1, s_2) \supset \cdots \supset C(s_1, s_2, \ldots, s_n) \supset \cdots .$$

Hence,

$$C(s) := \bigcap_{n\in\mathbb{N}} C(s_1, \ldots, s_n)$$

is compact and non-empty. Since every point is a boundary point, we indeed have a generalized Cantor set. If we allow $s_n := 1/3$, then we get the usual $C_{1/3}$, which is not polar. On the other hand, if we take $s_n := 1 - (1/2)^{2^n}$, we get a set that is polar and uncountable (see

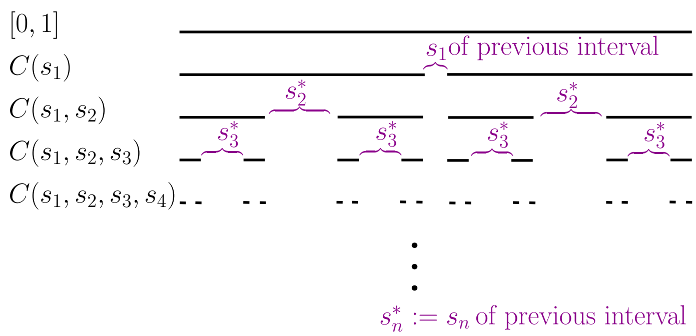

Figure 1.1: Generalized Cantor set

[40, Theorem 5.3.7]).

Definition 1.4. Let K be a subset of $\mathbb{C}$. We define the *Robin's constant* of K as

$$V(K) := \inf\{I(\nu) : \nu \in \mathcal{P}(K)\},$$

where the infimum is taken over the set of Borel probability measures with compact support contained in K. If there exists a probability measure $\nu \in \mathcal{P}(K)$ such that $I(\nu) = V(K)$, then ν is said to be the *equilibrium measure* for K.

When K is compact, there is a unique equilibrium measure for K. But before we prove the existence of such a measure, we need a useful lemma:

Lemma 1.1 ([40, Lemma 3.3.3])**.** *Let K be a compact subset of $\mathbb{C}$ and let μ_n be a sequence of Borel probability measures on K with weak* limit μ. Then,*

$$I(\mu) \leq \liminf_{n\to\infty} I(\mu_n).$$

Proof. Let d be the diameter of K. We have that

$$\begin{aligned}\liminf_{n\to\infty} \int_{K\times K} \left(-\log\frac{|z-w|}{d}\right) \mu_n(z)\, \mu_n(w) &\geq \liminf_{n\to\infty} \int_{K\times K} (\min\{-\log\frac{|z-w|}{d}, m\})\, \mu_n(z)\, \mu_n(w) \\ &= \int_{K\times K} (\min\{-\log\frac{|z-w|}{d}, m\})\, \mu(z)\, \mu(w),\end{aligned}$$

where the last holds since $\min\{-\log|z-w|, m\}$ is continuous for every $m > 1$. By taking $m \to \infty$ we get

$$\liminf_{n\to\infty} \int_{K\times K} \left(-\log\frac{|z-w|}{d}\right) \mu_n(z)\,\mu_n(w) \geq \int_{K\times K} \left(-\log\frac{|z-w|}{d}\right) \mu(z)\,\mu(w),$$

□

The existence of equilibrium measures is immediate. Indeed, assume $V(K)$ is finite (otherwise any atom on K is an equilibrium measure). Let μ_n be a sequence of Borel probability measures on K such that $I(\mu_n) \to V(K)$ as $n \to \infty$. Since K is compact, there exists a subsequence that converges to a weak* limit μ. By Lemma 1.1, we have that

$$I(\mu) \leq \liminf_{n\to\infty} I(\mu_n) = V(K).$$

Since μ has compact support in K, then we also have that $V(K) \leq I(\mu)$. Thus, we have:

Proposition 1 ([40, Theorem 3.3.2]). *If K is a compact subset of $\mathbb{C}$, then there exists an equilibrium measure for K.*

When K is compact, then the equilibrium measure for K is unique. Uniqueness is not as easy as existence. The proof can be found in [40, p. 75].

1.2 Logarithmic capacity

Definition 1.5. The *logarithmic capacity* of a subset $E \subset \mathbb{C}$ is defined by minimizing the energy:

$$\mathrm{Cap}(E) = \exp(-\inf\{I(\mu)\}),$$

where the infimum is taken over the set of Borel probability measures whose support is a compact subset of E (we interpret $e^{-\infty}$ as 0). We can view capacity as

$$\operatorname{Cap}(E) = e^{-V(E)}.$$

There are three immediate observations:

- $\operatorname{Cap}(E) = 0$ if and only if E is polar.
- If $A \subset B$, then it follows that $\operatorname{Cap}(A) \leq \operatorname{Cap}(B)$.
- $\operatorname{Cap}(\alpha E + \beta) = |\alpha| \operatorname{Cap}(E)$.

The first statement tells us that capacity gauges how far away a set is from being *polar*. The last statement holds by considering the map $z \mapsto \alpha z + \beta$ (see [40, Theorem 5.1.2] for the proof).

In general, computing the exact capacity of sets is non-trivial. For less complicated sets such as a line or a circle, it can be done. A line of length d has capacity $d/4$ and a circle of radius r has capacity r. See [40, Table 5.1] for a table of the capacities of other sets. For more complicated sets, we settle for estimates. See [14] for a paper on bounding the capacity of a finite union of intervals and [44] for estimates on the union of two intervals. [41] is another paper on estimates of the logarithmic capacity of compact sets.

It is tempting to think that capacity works like a measure. However, capacity is not continuous and not additive.

Example 1.2 (Capacity is not continuous in two dimensions)**.** Take the following descending chain of sets (see Figure 4.7):

$$O_n := \{z \in \mathbb{C} : 1 - \frac{1}{n} < \Im(z) < 1 \text{ and } 0 < \Re(z) < 1\}.$$

The intersection $\cap_n O_n = \emptyset$ is a polar set. On the other hand, $\mathrm{Cap}(O_n) > 1/4$ because each O_n contains a translation of the unit interval with capacity $1/4$. This shows that capacity is not continuous. Also, notice that the unit square has finite capacity while containing infinitely many translations of the unit line. Hence, capacity is also not additive.

Theorem 2 in [38] shows that capacity is not subadditive: there exists two compact sets A, B such that

$$\mathrm{Cap}(A) + \mathrm{Cap}(B) < \mathrm{Cap}(A \cup B).$$

See Section 4.6 for another example showing that capacity is not continuous on a bounded interval. Nonetheless, capacity is continuous on compact sets:

Theorem 1.3 ([40, Theorem 5.1.3])**.**

1. *Let $\{K_n\}$ be a descending sequence of compact subsets of $\mathbb{C}$:*

$$K_1 \supset K_2 \supset \cdots .$$

   Then

$$\lim_{n\to\infty} \mathrm{Cap}(K_n) = \mathrm{Cap}(K_\infty),$$

   where $K_\infty = \cap_n K_n$.

2. *Let $\{B_n\}$ be a ascending sequence of Borel subsets of $\mathbb{C}$:*

$$B_1 \subset B_2 \subset \cdots .$$

   Then

$$\lim_{n\to\infty} \mathrm{Cap}(B_n) = \mathrm{Cap}(B_\infty),$$

where $B_\infty = \cup_n B_n$.

The next series of results answers the question how can we modify a set without changing its capacity. This will allow us to work with a more convenient set. The first modification of a set is the removal of a polar set. We are sure that the following theorem is known, but could not find it in the literature.

Theorem 1.4. *Let* K *be a compact set in* $\mathbb{C}$ *and* P *be a Borel polar set in* $\mathbb{C}$. *Then*

$$\operatorname{Cap}(K) = \operatorname{Cap}(K \setminus P).$$

Proof of Theorem 1.4. If K is polar, then the statement is trivial. Assume K is not polar. Let μ be the equilibrium measure for K. Since $I(\mu) < \infty$, then $\mu(P) = 0$ (see [40, Theorem 3.2.3]). If we set $E := K \setminus P$, then

$$1 = \mu(K) = \mu(E),$$

and

$$I(\mu) = \iint_{E\times E} (-\log|z-w|)\, \mu(z)\mu(w).$$

Since μ is a Borel measure, then μ is regular. Therefore, there exists a sequence of compact sets $C_n \subset E$ such that

$$\mu(C_n) \to \mu(E) = 1 \quad \text{as} \quad n \to \infty.$$

Additionally, by the dominated convergence theorem, we have

$$I(\mu|_{C_n}) \to \iint_{E\times E} (-\log|z-w|)\, \mu(z)\mu(w) = I(\mu) \quad \text{as} \quad n \to \infty.$$

Hence,

$$I\left(\frac{1}{\mu(C_n)}\,\mu|_{C_n}\right) \to I(\mu) \quad \text{as} \quad n \to \infty.$$

Since each $\frac{1}{\mu(C_n)}\,\mu|_{C_n}$ is a Borel probability measure with compact support contained in E, then

$$\inf\{I(\nu)\} \leq I\left(\frac{1}{\mu(C_n)}\,\mu|_{C_n}\right) \to I(\mu) \quad \text{as} \quad n \to \infty,$$

where the infimum is taken over Borel probability measures with compact support contained in E. Since $E \subset K$ and μ is the equilibrium measure for K, then we get the result. □

As we mentioned above, if $K \subset \mathbb{C}$, then $\mathrm{Cap}(K) = \mathrm{Cap}(K + \beta)$ for any $\beta \in \mathbb{C}$. Given two sets $K, L \subset \mathbb{C}$, define

$$K + L := \{z + w : z \in K, w \in L\}.$$

If K is a Borel polar set and K is countable, then $K + L$ is a countable union of sets of the form $K + l$ where $l \in L$. Hence, $\mathrm{Cap}(K + L) = \mathrm{Cap}(K) = \mathrm{Cap}(L) = 0$. The following theorem provides the converse when K is compact:

Theorem 1.5 ([33, Theorem 1.1])**.** *Let K, L be compact subsets of $\mathbb{C}$. Then the following are equivalent:*

1. $\mathrm{Cap}(K + L) = 0$ *whenever* $\mathrm{Cap}(L) = 0$,
2. *K is countable.*

Chapter 2

Initial motivation: exceptional energies in the 1D Anderson Model

The initial motivation for studying G_δ sets is the set of exceptional energies in the parametric version of Furstenberg's Theorem studied by Gorodetski and Kleptsyn in [25]. In this chapter, we will explore the connection of the set of exceptional energies with ergodic theory and Anderson localization.

2.1 Furstenberg's Theorem and positivity of the Lyapunov exponent

In this section, we will formulate Furstenberg's Theorem and follow the argument found in [9, Section 4.3] to apply the theorem to a group of matrices that play a special roll in the Anderson model. In this process, we will show the positivity of the Lyapunov exponent.

Denote the group of 2×2 matrices with real coefficients and determinant 1 by $SL(2, \mathbb{R})$.

Theorem 2.1. *(Furstenberg-Kesten Theorem [19]) Assume $T : X \to X$ is a μ-preserving transformation and $A : X \to SL(2,\mathbb{R})$ is a measurable map such that:*

$$\int \log \|A(x)\|\, d\mu(x) < \infty.$$

Then for μ-almost every $x \in X$, the following limit exists:

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|,$$

where

$$A^n(x) = A(T^{n-1}x)\cdots A(x) \quad (\textit{for } n > 0).$$

The function $\lambda : X \to [0,\infty)$ is T-invariant, μ-integrable, and its integral is given by

$$\Lambda = \int \lambda\, d\mu = \lim_{n\to\infty} \frac{1}{n} \int \log \|A^n\|\, d\mu = \inf_{n\geq 1} \frac{1}{n} \int \log \|A^n\|\, d\mu.$$

When T is ergodic, then λ is constant almost everywhere. Thus, $\lambda = \Lambda$ almost everywhere. Furstenberg-Ketsen Theorem follows from the subadditive ergodic theorem (see [50, Section 3.2] for the proof).

Furstenberg-Kesten Theorem can be stated in terms of products of random i.i.d. matrices (see [3, p. 14] for details). We say that a probability measure μ on $SL(2,\mathbb{R})$ satisfies the *integrability condition* if

$$\int_{SL(2,\mathbb{R})} \log \|M\|\, d\mu(M) < \infty. \tag{2.1}$$

Given random independent matrices $Y_1, Y_2, \ldots$ with distribution μ, Furstenberg-Kesten implies that

$$\lambda := \lim_{n\to\infty} \frac{1}{n} \log \|Y_n \cdots Y_1\| \quad \text{exists a.s. and is constant.}$$

λ is called the *Lyapunov exponent*. As any matrix in $SL(2,\mathbb{R})$ has norm ≥ 1, the Lyapunov exponent is non-negative. Furstenberg's theorem gives us sufficient conditions to establish $\lambda > 0$:

Theorem 2.2 (Furstenberg [18]). *Let μ be a probability measure on $SL(2,\mathbb{R})$ that satisfies the integrability condition (2.1). Let G_μ be the smallest closed subgroup which contains the support of μ. Assume that*

F.1 *G_μ is non compact,*

F.2 *there is no finite set $\emptyset \neq L \subset \mathbb{RP}^1$ such that $ML = L$ for all $M \in G_\mu$.*

Then $\lambda > 0$.

Proofs of Theorem 2.2 may be found in [4], [3], and [50, Section 6.3] (the latter two contain other related results as well).

Let us see how we can apply Furstenberg's theorem to a specific set of matrices that play a special role in the Anderson model. Consider a probability measure μ on $\mathbb{R}$ with more than 1 point in its support and the map $g : \mathbb{R} \to SL(2,\mathbb{R})$ defined by

$$v \mapsto \begin{pmatrix} E - v & -1 \\ 1 & 0 \end{pmatrix},$$

where E is some fixed real number. The push-forward of μ by the map g, forms a probability measure $\tilde{\mu} := \mu_* g$ on $SL(2,\mathbb{R})$. Notice that our matrices are of the form

$$M_x = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}.$$

Since the support of μ contains at least two points, then $\operatorname{supp} \tilde{\mu} \subset G_{\tilde{\mu}}$ does also. Then there exists $M_a, M_b \in G_{\tilde{\mu}}$. where $a \neq b$. Furthermore,

$$A := M_a M_b^{-1} = \begin{pmatrix} 1 & a-b \\ 0 & 1 \end{pmatrix} \in G_{\tilde{\mu}}.$$

By considering

$$A^n = \begin{pmatrix} 1 & a-b+n \\ 0 & 1 \end{pmatrix} \in G_{\tilde{\mu}},$$

$G_{\tilde{\mu}}$ is not compact, which is assumption **F.1**.

Suppose condition **F.2** is not satisfied. Then there exists a finite and non-empty set $L \subset \mathbb{RP}^1$ that is invariant under left multiplication by $G_{\tilde{\mu}}$. The line $e_1 = (1,0)^T \in \mathbb{RP}^1$ is a fixed point of the map A from above. Take any line $x \in \mathbb{RP}^1$, then $A^n x$ converges to e_1. Hence, $e_1 \in L$. Suppose there is another element x_1 in L that is distinct from e_1. We have that $Ax_1 \neq e_1$ since $A^{-1}e_1 = e_1$. Since $Ax_1 \in L$, there must exist another element $x_2 \in L$ that is distinct from e_1 such that $Ax_1 = x_2$. Continuing this argument and using the fact that L is finite, we have that there exists $x_1, \ldots, x_k \in L \setminus \{e_1\}$ such that

$$Ax_1 = x_2, \quad Ax_2 = x_3, \quad \ldots, \quad Ax_k = x_1.$$

This shows us that $\{A^n x_1\}$ is a sequence of points in $L \setminus \{e_1\}$. Thus, there is a convergent subsequence with limit point in $L \setminus \{e_1\}$, a contradiction since $A^n x$ converges to e_1 for all

$x \in \mathbb{RP}^1$. Hence, $L = \{e_1\}$. Let us consider another matrix in $G_{\tilde{\mu}}$,

$$B := M_a^{-1} M_b = \begin{pmatrix} 1 & 0 \\ a-b & 1 \end{pmatrix}.$$

This matrix creates a contradiction since $Be_1 \neq e_1$. Therefore, condition (2) is satisfied. If the integrability condition is satisfied, then Furstenberg's theorem, tells us the following:

Theorem 2.3 ([9, Theorem 4.3])**.** *In the setting above, for every fixed E, we have*

$$\lambda(E) = \lim_{n\to\infty} \frac{1}{n} \log \|g(v_n) \cdots g(v_1)\| > 0 \quad \text{exists a.s.}$$

2.2 The parameterized version of Furstenberg's Theorem and the set of exceptional energies

In this section, we will introduce a parameter E into Furstenberg's Theorem (similarly to Theorem 2.3). We will then take a look at the set of exceptional energies from the parametrized version of Furstenberg's theorem that was studied by Gorodetski and Kleptsyn in [25]. The set of exceptional energies is the main motivation for our work.

Let (Ω, μ) be a probability space, $J \subset \mathbb{R}$ be a compact interval of parameters, and $F : \Omega \times J \to SL(2, \mathbb{R})$ be a bounded measurable function such that for every any $\omega \in \Omega$, $a \mapsto F_a(\omega) \in SL(2, \mathbb{R})$ is a continuous map. For any $\bar{\omega} = (\omega_n)_{n=1}^{\infty} \in \Omega^{\mathbb{N}}$, define

$$T_{n,a,\bar{\omega}} = F_a(\omega_n) F_a(\omega_{n-1}) \cdots F_a(\omega_1).$$

Then, for every $a \in J$, there exists $\Omega_a \subset \Omega^{\mathbb{N}}$ with $\mu(\Omega_a) = 1$ such that for any $\bar{\omega} \in \Omega_a$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = \lambda_F(a) \quad \text{exists.}$$

As we mentioned above, this follows from Furstenberg-Kesten. A collection of $SL(2,\mathbb{R})$ matrices $\{M_\alpha\}_{\alpha\in A}$ is called *uniformly hyperbolic* if there exists a constant $\eta > 1$ such that for any finite sequence of matrices $M_{\alpha_1}, \ldots, M_{\alpha_n}$, we have $\|M_{\alpha_1} \cdots M_{\alpha_n}\| > \eta^n$. In the setting above, if we restrict $\{F_a(\omega)\}_{\omega\in\Omega}$ to be uniformly hyperbolic, then for $\mu^{\mathbb{N}} - a.e.\, \bar{\omega} \in \Omega^{\mathbb{N}}$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = \lambda_F(a) > 0,$$

for all $a \in J$. Gorodetski and Kleptsyn in [25] worked in the case complimentary to uniform hyperbolicity. The following are the assumptions in which Gorodetski and Kleptsyn worked under:

P.1 **(Furstenberg condition)** Denote by μ_a the measure $\mu_a = (F_a)_*(\mu)$. We assume that for each $a \in J$ the measure μ_a on $SL(2,\mathbb{R})$ satisfies the (individual) Furstenberg non-degeneracy condition, that is, its support is not contained in any compact subgroup of $SL(2,\mathbb{R})$, and there is no $\operatorname{supp}\mu_a$-invariant finite union of proper subspaces of $\mathbb{R}^2$.

P.2 **(C^1-boundedness)** The maps $F_a(\omega)$ are C^1-smooth in the parameter $a \in J$, with uniformly bounded C^1-norm, i.e. there exists $M > 0$ such that for all $\omega \in \Omega$ and all $a \in J$

$$\|F_a(\omega)\|, \left\|\frac{d}{da} F_a(\omega)\right\| \leq M.$$

P.3 **(Non-uniform hyperbolicity)** For each $a \in J$ the collection of matrices $\{F_a(\omega)\}_{\omega\in\Omega}$ is *not* uniformly hyperbolic.

P.4 (Monotonicity) There exists $\delta > 0$ such that

$$\frac{d}{da}\arg(F_a(\omega)\bar{v}) > \delta > 0$$

for all $a \in J, \omega \in \Omega, \bar{v} \in \mathbb{R}^2\backslash\{0\}$. In other words, as we increase the parameter, the image of any given vector $\bar{v}$ spins in the positive direction with a speed that is bounded from below.

We can now state their main result:

Theorem 2.4 (**PFT:** Parametric version of Furstenberg Theorem)**.** *Under the assumptions* ***P.1-P.4*** *above, for $\mu^{\mathbb{N}}$-almost every $\bar{\omega} \in \Omega^{\mathbb{N}}$ the following holds:*

- ***(Regular upper limit)*** *For every $a \in J$ we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = \lambda_F(a) > 0.$$

- ***(G_δ-vanishing)*** *The set*

$$S_0(\bar{\omega}) := \left\{ a \in J \mid \liminf_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = 0 \right\}$$

is a (random) dense G_δ-subset of the interval J.

- ***(Hausdorff dimension)*** *The (random) set of parameters with exceptional behaviour,*

$$S_e(\bar{\omega}) := \left\{ a \in J \mid \liminf_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| < \lambda_F(a) \right\},$$

has zero Hausdorff dimension:

$$\dim_H S_e(\bar{\omega}) = 0.$$

Questions on switching the quantifiers in Theorem 2.3 is very natural in Anderson localization proofs. Theorem 2.4 shows that for almost all ω's, the Lyapunov limit fails to exist for energies E from a dense G_δ subset of J:

$$0 = \liminf_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| < \limsup_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = \lambda_F(a).$$

In other words, the sets of exceptional energies $S_e(\bar{\omega})$ and $S_0(\bar{\omega})$ are two dense G_δ subsets of J for wich the Lyapunov limit fails to exist by switching the quantifiers. Additionally, Theorem 2.4 tells us that $S_e(\bar{\omega})$, which contains $S_0(\bar{\omega})$, has zero Hausdorff dimension (see [17, p. 21] for details of the Hausdorff dimension). Capacity is a finer measurement than the Hausdorff dimension in the sense that any set $E \subset \mathbb{C}$ that has zero capacity has zero Hausdorff dimension. A question of interest is the following one:

Question 2.1. What is the capacity of both $S_e(\bar{\omega})$ and $S_0(\bar{\omega})$?

Both $S_e(\bar{\omega})$ and $S_0(\bar{\omega})$ are of the form

$$S = \bigcap_{m} \bigcup_{k\geq m} I_k, \tag{2.2}$$

where each I_k is an open interval of length l_k with center at $c_k \in (0,1)$, and the sequence $\{l_k\}$ decreases exponentially to 0 as $k \to \infty$. This motivates the study of the capacity of G_δ sets of the form (2.2). Additionally, the answer to Question 2.1 can be used to compare the exceptional energies with the *Forbidden Energies* in [11] by comparing their logarithmic capacities.

2.3 Anderson Localization

2.3.1 1D Anderson Model

An important application of the parametric version of Furstenberg's Theorem is the 1D Anderson Model. The goal of the section is to introduce the Anderson model. We will need to first recall some basic definitions and results from functional analysis that are tailored to our setting. For an in-depth study on functional analysis, we refer the reader to [43]. For a basic explanation of spectral concepts and fundamental results of the the discrete one-dimensional Schrodinger operators, we refer the reader to Section 2.1 and Section 2.2 of [9] ([8] is a good resource too). [29] is also an introduction on random Schrodinger operators that contains information about the physics behind the operators.

We say an operator (a linear map) $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is *bounded* if there exists a constant $C \in \mathbb{R}$ such that $|Hx| \leq C|x|$ for every $x \in \ell^2(\mathbb{Z})$. The smallest possible C is denoted as $\|H\|$ and is the norm of H. We then define the *resolvent* set of H as

$$\rho(H) = \{E \in \mathbb{C} : E - H \text{ is a bijection with a bounded inverse}\}.$$

The *spectrum* of H is defined as $\sigma(H) := \mathbb{C} \setminus \rho(H)$. One of the properties of H being a self-adjoint operator is that $\sigma(H) \subset \mathbb{R}$. If there exists $E \in \mathbb{R}$ and a non-zero sequence $u \in \ell^2(\mathbb{Z})$ such that $Hu = Eu$, then $E \in \sigma(H)$. Such a sequence is called an *eigenvector* (or eigenfunction) and the corresponding E is called an *eigenvalue* (or in the Anderson model terminology, E is called *energy*). The set of all eigenvalues is called the *point spectrum* of H and is a subset of $\sigma(H)$.

A second benefit of working with a self adjoint operator is that we may apply functional calculus. Given a Borel bounded function $g : \mathbb{R} \to \mathbb{R}$, there exists a unique bounded operator

$g(H)$ on $\ell^2(\mathbb{Z})$. By application of Riesz-Markov theorem, for every $\psi \in \ell^2(\mathbb{Z})$, there exists a unique Borel measure μ_ψ such that

$$\langle \psi, g(H)\psi \rangle = \int_{\sigma(H)} g \, d\mu_\psi.$$

The measure μ_ψ is called the *spectral measure associated with the vector* ψ and may be decomposed uniquely as

$$\mu_\psi = \mu_{\psi,ac} + \mu_{\psi,sc} + \mu_{\psi,pp},$$

where $\mu_{\psi,ac}$ is the absolutely continuous piece (with respect to Lebesgue measure), $\mu_{\psi,sc}$ is the singular continuous piece (its support is on a set of Leb. measure zero and gives zero measure to singletons), and $\mu_{\psi,pp}$ is the pure point piece (sum of Dirac measures). Using this decomposition, we can define

$$\begin{aligned} \ell^2(\mathbb{Z})_{ac} &= \{\psi \in \ell^2(\mathbb{Z}) : \mu_\psi = \mu_{\psi,ac}\}, \\ \ell^2(\mathbb{Z})_{sc} &= \{\psi \in \ell^2(\mathbb{Z}) : \mu_\psi = \mu_{\psi,sc}\}, \\ \ell^2(\mathbb{Z})_{pp} &= \{\psi \in \ell^2(\mathbb{Z}) : \mu_\psi = \mu_{\psi,pp}\}. \end{aligned}$$

It turns out that each space is a closed subspace and invariant under H. Additionally, we have the following direct sum

$$\ell^2(\mathbb{Z}) = \ell^2(\mathbb{Z})_{ac} \oplus \ell^2(\mathbb{Z})_{sc} \oplus \ell^2(\mathbb{Z})_{pp}.$$

The setup of the Anderson model begins with a probability measure μ on $\mathbb{R}$ such that its topological support is bounded and contains at least two points. The Anderson-Bernoulli model is the particular case when the support contains exactly two points. We can consider the product space $\Omega = (\operatorname{supp} \mu)^{\mathbb{Z}}$. For every $\omega \in \Omega$, we define the potential $V_\omega : \mathbb{Z} \to \mathbb{R}$, by

$V_\omega(n) = \omega_n$. Note that V_ω is bounded since $\operatorname{supp}\mu$ is bounded. The family of Schrödinger operators on $\ell^2(\mathbb{Z})$, are defined as

$$[H_\omega u](n) = u(n+1) + u(n-1) + V_\omega(n)u(n). \tag{2.3}$$

For simplicity of notation, we write $H = H_\omega$ and $V = V_\omega$. We say that H has *purely pure point spectrum* if $\ell^2(\mathbb{Z})_{pp} = \ell^2(\mathbb{Z})$. *Anderson localization* is precisely, when for μ-almost every $\omega \in \Omega$, the following two properties hold:

1. (purely pure point spectrum) there exist a basis $\{u_k\}_{k\in\mathbb{N}}$ of eigenvectors on $\ell^2(\mathbb{Z})$,
2. (exponential decay) for every $k \in \mathbb{N}$, there exists $C, \gamma > 0$ such that $|u_k(n)| \le Ce^{-\gamma|n|}$ for all n.

2.3.2 The 1D Anderson Model and Furstenberg's theorem

In this section, will motivate the study of the Lyapunov exponent in the Anderson model and we will apply Furstenberg's theorem to the 1D Anderson Model to obtain positivity of the Lyapunov exponent.

To understand the connection between the Anderson model and the Lyapunov exponent, let us recall that a sequence $u \in \ell^2(\mathbb{Z})$ is an eigenvector of H if and only if $Hu = Eu$ for some energy $E \in \mathbb{R}$. Componentwise, that would mean,

$$u(n+1) + u(n-1) + V_\omega(n)u(n) = Eu(n), \quad n \in \mathbb{Z}. \tag{2.4}$$

The difference equation (2.4) is known as the *time-independent Schrödinger equation.* Equation (2.4) tells us that $u(n+1)$ depends only on $u(n)$ and $u(n-1)$. Therefore, u and E

satisfies the difference equation if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \begin{pmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}. \tag{2.5}$$

By defining,

$$\Pi_{n,E,\omega} = \begin{pmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.6}$$

then

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \Pi_{n,E,\omega} \times \cdots \times \Pi_{1,E,\omega} \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}, \tag{2.7}$$

for $n \geq 1$. We can obtain a similar equation for $n \leq 0$ by multiplying by the inverse of $\Pi_{n,E,\omega}$ in equation (2.5). Thus, all solutions to the difference equation form a 2 dimensional vector space and we have a way of generating every solution uniquely simply by determining $u(0)$ and $u(1)$. It should be noted that by first determining $u(0)$ and $u(1)$ we will obtain a sequence u that satisfies the difference equation, but u may not live in $\ell^2(\mathbb{Z})$. Conversely, any eigenvector and eigenvalue will satisfy the difference equation. Equation (2.7) gives us motivation to consider the growth of the product of matrices

$$T_{n,E,\omega} = \Pi_{n,E,\omega} \times \cdots \times \Pi_{1,E,\omega}.$$

Notice that the matrices $\Pi_{n,E,\omega}$ are of the form

$$\begin{pmatrix} E - v & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

In Section 2.1, we applied Furstenberg's theorem to matrices of the form (2.8) to obtain Theorem 2.3: for any $E \in \mathbb{R}$, for almost every $\omega \in (\operatorname{supp} \mu)^{\mathbb{Z}}$, there is a limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_{n,E,\omega}\| = \lambda_F(E) > 0. \tag{2.9}$$

It turns out that Anderson localization proofs begin with the positivity of $\lambda_F(E)$ from (2.9) and by various techniques concluded the desired result.

2.3.3 Different proofs of Anderson Localization

In this section, we will see why different proofs of Anderson Localization begin with the positivity of the Lyapunov exponent. We will end the section with a discussion on different proofs of Anderson Localization.

In Section 2.3.2, we established the positiveness of the Lyapunov exponent ($\lambda > 0$) for every E. Using the positiveness of λ and the fact that our matrices have bounded norm, we may apply the following result:

Theorem 2.5 (Oseledec-Ruelle Theorem, [8, Theorem 2.8])**.** *Suppose $T_n \in SL(2, \mathbb{R})$ obey*

$$\lim_{n\to\infty} \frac{1}{n} \|T_n\| = 0,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_n \cdots T_1\| = L > 0.$$

Then there exists a one-dimensional subspace $V \subset \mathbb{R}^2$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_n \cdots T_1 x\| = -L \text{ for } x \in V \setminus \{0\},$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_n \cdots T_1 x\| = L \text{ for } x \notin V.$$

Thus, our solutions to the difference equation

$$u(n+1) + u(n-1) + V_\omega(n)u(n) = Eu(n),$$

either decay or increase exponentially. Solutions to the difference equations that also obey

$$|u(n)| \leq C(1+|n|)^\delta, \quad \text{for all } n \in \mathbb{Z}$$

for some constants C, δ are called *generalized eigenfunctions* and the corresponding E *generalized eigenvalue.* It turns out that the closure of the set of generalized eigenvalues of H is the spectrum of H. Thus, we can focus on such E with polynomially bounded solutions. Focusing on such u's, we can exclude the case of exponential growth. Thus, every generalized eigenfucntion decays exponentially.

Thus, we have shown that the desired property holds "for every energy E and for almost every ω". We want the desired property to hold "for almost every ω and for every energy E". Once that is accomplished, we can apply the following result which is known as "Shnol Theorem" (see [45], [23], [24]).

Theorem 2.6 (Shnol Theorem)**.** *Let $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be an operator of the form*

$$Hu(n) = u(n-1) + u(n+1) + V(n)u(n),$$

with bounded potential $\{V(n)\}_{n\in\mathbb{Z}}$. If every polynomially bounded solution to $Hu = Eu$ is in fact exponentially decreasing, then H has pure point spectrum, with exponentially decay-

ing eigenfunctions. Similar statement holds for operators on $\ell^2(\mathbb{Z})$ with Dirichlet boundary conditions.

Let

$$\varepsilon_\omega = \{E \in \mathbb{R} : \lambda(E) > 0, \exists \text{ solutions } u_\pm \sim e^{-\lambda(E)|n|},\, n \to \pm\infty\}.$$

Since for every energy E, for almost every ω, we have that $\lambda(E) > 0$ and there exist solutions $u_\pm \sim e^{-\lambda(E)|n|}$ as $n \to \pm\infty$, then we may apply Fubini's to concluded

$$\text{Leb}(\mathbb{R} \setminus \varepsilon_\omega) = 0,$$

for μ-almost every $\omega \in \Omega$. Thus, we have that the desired property holds for "almost every ω and for Lebesgue $a.e$ energy E". The question now is what happens on the set $\mathbb{R} \setminus \varepsilon_\omega$ of zero Lebesgue measure. It is possible to have singular continuous spectrum. In the case when the single-site distribution has a non-trivial absolutely continuous component, one can use a method called *spectral averaging* to conclude Anderson localization (see [8, Theorem 3.3] and [9, Section 4.4] for details). Among proofs of this kind are found in [46] and [48]. This method does not apply to the Anderson Bernoulli model and does not extend to higher dimensions. Another proof that is strictly one-dimensional is the Kunz-Souillard approach used in [12], [31], [10].

There is another method which is applicable to higher dimensions and does not require μ to have a non-trivial absolutely continuous component or to even be purely absolutely continuous. This approach is called *multi-scale analysis* (MSA). The MSA approach uses an inductive procedure where in the base case, the result of Theorem 2.3 is used. Proofs based on MSA are [20], [21], [7], [13], [22], [15].

A more geometrical proof was done by Gorodetski and Kleptsyn in [25]. The parametric version of Furstenberg's Theorem 2.4 is applicable to the Anderson model by taking $F_a(\omega_n) = \Pi_{n,a,\vec{\omega}}$ (see (2.6)). Then,

$$T_{n,E,\bar{\omega}} = \Pi_{n,E,\bar{\omega}} \times \cdots \times \Pi_{1,E,\bar{\omega}}.$$

Assumptions **P.1**-**P.4** apply to the 1D Anderson Model. Assumption **P.1** is exactly Furstenberg's assumptions **F.1** and **F.2**, which have already been verified in the 1D Anderson Model (see Section 2.1). Condition **P.2** is satisfied since $\operatorname{supp} \mu$ is compact. Johnson showed in [28] that the set of energies E for which the collection of matrices $\{\Pi_{n,E,\omega}\}_{\omega \in (\operatorname{supp} \mu)^{\mathbb{Z}}}$ is uniformly hyperbolic, is equal to the resolvent set of H_ω for $\mu^{\mathbb{Z}}-$ almost every ω. Hence, **P.3** is satisfied. For condition **P.4** (monotonicity) does not hold in general for matrices $\Pi_{n,E,\omega}$, but it does hold for a product of two consecutive matrices $\Pi_{n,E,\omega}\Pi_{n+1,E,\omega}$. See Section 1.3 in [30] for further details.

Using similar techniques as in the proof of PFT, Gorodetski and Kleptsyn proved:

Theorem 2.7 ([25, Theorem 1.11])**.** *Under the assumptions **P.1-P.4** we have:*

- *For almost all $\vec{\omega} \in \Omega^{\mathbb{N}}$, for all $a \in J$ the following holds. If*

$$\limsup_{n \to +\infty} \frac{1}{n} \log |T_{n,a,\bar{\omega}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)| < \lambda_F(a), \tag{2.10}$$

  then in fact $|T_{n,a,\bar{\omega}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)|$ tends to zero exponentially as $n \to \infty$. Namely,

$$\lim_{n \to +\infty} \frac{1}{n} \log |T_{n,a,\bar{\omega}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)| = -\lambda_F(a).$$

- *For almost all $\vec{\omega} \in \Omega^{\mathbb{Z}}$, for all $a \in J$ the following holds. If for some $\bar{v} \in \mathbb{R}^2 \setminus \{0\}$ we have*

$$\limsup_{n \to +\infty} \frac{1}{n} \log |T_{n,a,\bar{\omega}}\bar{v}| < \lambda_F(a), \quad \textit{and} \quad \limsup_{n \to +\infty} \frac{1}{n} \log |T_{-n,a,\bar{\omega}}\bar{v}| < \lambda_F(a), \tag{2.11}$$

where

$$T_{-n,a,\bar{\omega}} := F_a(\omega_{-n})^{-1} \dots F_a(\omega_{-1})^{-1} F_a(\omega_0)^{-1},$$

then both $|T_{n,a,\bar{\omega}}\bar{v}|, |T_{-n,a,\bar{\omega}}\bar{v}|$ *in fact tend to zero exponentially. Namely,*

$$\lim_{n\to+\infty} \frac{1}{n} \log |T_{n,a,\bar{\omega}}\bar{v}| = -\lambda_F(a), \quad \textit{and} \quad \lim_{n\to+\infty} \frac{1}{n} \log |T_{-n,a,\bar{\omega}}\bar{v}| = -\lambda_F(a).$$

Theorem 2.11 and Theorem 2.6 imply Anderson localization.

2.3.4 The history of Anderson localization

In this section, we will briefly narrate the history of Anderson localization from a mathematical perspective.

The history of Anderson localization began in 1958 when Anderson introduced the model in [1]. Anderson and his thesis advisor Vleck and Mott shared the Nobel Prize in Physics for their work on disordered systems in 1977 (see [32]).

The first mathematical proof of Anderson localization came around 1980 by Hunz and Souillard in [31]. The approach is strictly one-dimensional and is used in [10] and [12].

A few years later, Simon and Wolff (see [48]) pioneered a method called *spectral averaging*. This method is also strictly one-dimensional. The method requires the single-site distribution to have a non-trivial absolutely continuous component. Thus, the Anderson-Bernoulli model is excluded.

The following year, a complete proof that covered the Anderson-Bernoulli model was finally given by Carmona, Klein, and Martinelli in [7] using *Multi-Scale Analysis* (MSA). MSA was originally introduced by Frohlich and Spencer in [20] and developed further in many papers (see Chapter 2.3.3). The MSA approach is extendable to higher dimensions.

About 20 years later, there came a series of proofs of Anderson localization in 1D without the use of MSA for single-site distribution with compact support. One of those works was by Bucaj, Damanik, Fillman, Gerbuz, Vandenboom, Wang, and Zhang in [5]. Soon afterwards, Gorodetski and Kleptsyn in [25] provided a purely geometrical proof of Anderson Localization 1D using dynamical systems. At the same time, Jitomirskaya and Zhu provided a short proof of Anderson localization in [27].

Recently, Rangamani recovered the result proved by Carmona, Klein, and Martinelli in 1987 without multi-scale analysis in [42]. In both papers the authors only required the single-site distribution μ to have more than one point in its support and

$$\int |x|^t \, d\mu(x) < \infty \quad \text{for some } t > 0.$$

Chapter 3

A connection between the h-volume of a set and the capacity of a set

Our focus in this chapter will be on Nevanlinna's conjecture, which aimed to be a link between the *h-volume* of a set and the capacity of a set.

Part of the work in this chapter was published in *Potential Analysis* and were obtain in collaboration with Victor Kleptsyn (see [30]).

3.1 Nevanlinna's conjecture

In the early 1900's, Erdös and Gillis [16], Lindeberg [34], Ursell [49], Carleson [6], and Nevanlinna [36] were working on connecting the notion of the *h-volume* of a set with the *logarithmic capacity* of a set.

A function h that is defined in some right neighborhood of 0 is called a *measuring function* provided that h is continuous, positive, increasing, concave, and $h(0) = 0$. The *h-volume* of a set $E \subset \mathbb{R}$ is defined as

$$m_h(E) := \lim_{\varepsilon \to 0+} \inf_{\{(x_j, r_j)_{j\in\mathbb{N}}\} \in \mathcal{I}(E,\varepsilon)} \sum_j h(r_j),$$

where the infimum is taken over the set $\mathcal{I}(E, \varepsilon)$ of covers of E by balls of diameter less than ε:

$$\mathcal{I}(E,\varepsilon) = \left\{ (x_j, r_j)_{j\in\mathbb{N}} \mid \bigcup_j U_{r_j}(x_j) \supset E, \quad \forall j \quad r_j < \varepsilon \right\}.$$

While the notion of *h-volume* of a set may be generalized to subsets of $\mathbb{R}^n$, we will restrict the notions to subsets of $\mathbb{R}$. The *h-volume* of a set E is reduced to the α-Hausdorff measure of the set E by taking $h(r) = r^\alpha$ (see [17, p. 21] for details of the α-Hausdorff measure).

In [16], the following conjecture, going back to Nevanlinna's paper [36] (and in a sense, complementary to the results of Myrberg [35]), was mentioned:

Conjecture 1 ([36]; see also [16, (C), p. 186])**.** *If for the function h the integral $\int_0^\bullet \frac{h(t)}{t}\,dt$ diverges and for a closed set E the h-volume $m_h(E)$ is finite, then* $\mathrm{Cap}(E) = 0$.

A series of papers showed that the conjecture holds for $h_0(r) := \frac{1}{|\log r|}$ and so, for any function that dominates $h_0(r)$:

Theorem 3.1 (Erdös and Gillis [16, p. 187], generalizing Lindeberg [34, p. 27])**.** *If for a set E one has $m_{h_0}(E) < +\infty$, then* $\mathrm{Cap}(E) = 0$.

Remark 3.1. In [16], Theorem 3.1 is stated in terms of *transfinite diameter* for compact sets. The transfinite diameter of a compact set E is defined as the limit of $\zeta_n(E)$ of geometric means of pairwise distances of n points in the set,

$$\zeta_n(E) = \max_{z_1,\dots,z_n \in E} \left(\prod_{i\neq j} |z_i - z_j| \right)^{1/(n(n-1))}. \tag{3.1}$$

For a compact set its transfinite diameter coincides with its logarithmic capacity (see [47, Theorem B.1] and [40, Theorem 5.5.2]). A heuristic explanation is that taking a logarithm transforms (3.1) to a sum similar to a double-integral of $\log|z-w|$ over the square of a uniform point measure diffused on the points z_j (with the exception of the diagonal).

This result generalizes the previous one by Lindeberg [34, p. 27], where zero capacity was established under the assumption of a *zero* logarithmic measure. An alternate proof of Theorem 3.1 was later provided by Carleson in [7, Theorem 2]. In Section 4.5, we will show that the result holds for non-closed sets and is a corollary of the Cauchy-Schwarts inequality. To see how Theorem 3.1 plays a role in computing the capacity of G_δ sets, see Corollary 4.1.

In his 1938 paper [49], Ursell disproved this conjecture, showing that it is false for all functions h except those, for which the conjecture is implied by Theorem 3.1 above:

Theorem 3.2 (H.D. Ursell [49]). *If* $\liminf h(t)|\log t| = 0$, *then there exists a set* $E = E_h$ *such that*

$$m_h(E) < \infty, \quad \mathrm{Cap}(E) > 0.$$

In Section 4.2, we use the same construction that we use for the proof of Theorem 4.1 (that can be seen as an extension of Ursell's approach) to show that for non-closed sets E this conjecture fails even stronger:

Theorem 3.3 ([30]). *Let h be a measuring function, such that* $\frac{1}{|\log r|} \neq O(h(r))$ *as* $r \to 0+$. *Then there exists a* G_δ*-dense subset* $S \subset [0,1]$ *with* $m_h(S) = 0$ *and* full *capacity* $\mathrm{Cap}(S) = \mathrm{Cap}([0,1])$.

For further study, we re fer the reader to [37, p. 161] and [40, p. 83].

Chapter 4

Phase transition of logarithmic capacity for the uniform G_δ-sets

In our first study of G_δ subsets of the interval $[0,1]$ we will consider a family of dense G_δ subsets of $[0,1]$, defined as intersections of unions of small uniformly distributed intervals, and study their logarithmic capacity. Changing the speed at which the lengths of generating intervals decrease, we observe a sharp phase transition from full to zero capacity. Such a G_δ set can be considered as a toy model for the set of exceptional energies in the parametric version of the Furstenberg theorem on random matrix products.

Our re-distribution construction can be considered as a generalization of a method applied by Ursell in his construction of a counter-example to a conjecture by Nevanlinna. Also, we propose a simple Cauchy-Schwartz inequality-based proof of related theorems by Lindeberg and by Erdös and Gillis.

The results of this chapter were published in *Potential Analysis* and were obtain with Victor Kleptsyn (see [30]).

4.1 Introduction

4.1.1 The setting

Our main focus will be the study of "uniform" G_δ-sets on the interval $[0,1]$. That is, given a (sufficiently fast) decreasing sequence $r_n \to 0$, for every n we consider a union of n equally spaced intervals of length r_n:

$$V_n := \bigcup_{j=0}^{n-1} J_{j,n}, \tag{4.1}$$

where $J_{j,n}$ is the open interval of length r_n centered at $c_{j,n} = \frac{j+(1/2)}{n}$:

$$J_{j,n} := \left(c_{j,n} - \frac{r_n}{2}, c_{j,n} + \frac{r_n}{2}\right), \quad j = 0, 1, \ldots, n-1. \tag{4.2}$$

See Fig. 4.1.

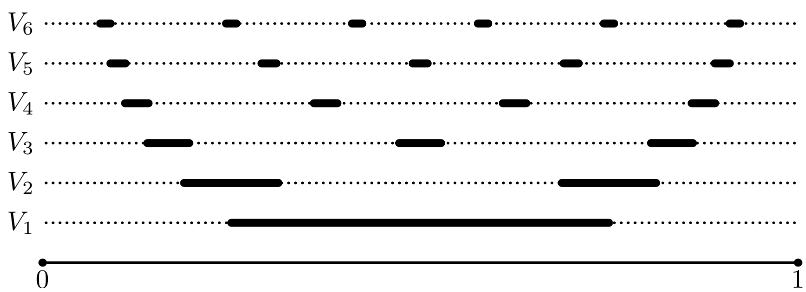

Figure 4.1: Sets V_n

Then we define the *uniform G_δ-set* S, corresponding to the sequence r_n, by

$$S := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} V_n; \tag{4.3}$$

it is immediate to see that S is indeed a G_δ-subset of $[0,1]$.

Our goal is now to study the properties of the set S. Once r_n goes to 0 faster than any power of n, this set is of zero Hausdorff dimension. However, this does not imply anything for its capacity — and one can consider the logarithmic capacity as a "finer" instrument to describe its properties.

Such an example is interesting for us for two reasons. First, considering different decrease speed for the lengths r_n, we observe a sharp phase transition: while for a fast decrease this set is of zero capacity, for a slower one it turns out to be of *full* capacity (that is, equal to the capacity of $[0, 1]$ itself). Second, such a situation, a G_δ-set generated by exponentially small intervals, can be considered as a model case for the set of exceptional energies in the parametric version of the Furstenberg theorem. The latter was our original motivation. We will summarize the motivation (see Chapter 2 for the full details).

In the paper [25, Section 1.2], the authors have considered the parametric version of a Furstenberg theorem, which describes the behaviour of the product

$$T_{n,\omega,a} = A_{\omega_n}(a) \dots A_{\omega_1}(a)$$

of random i.i.d. matrices $A_\cdot(a) \in SL(2, \mathbb{R})$, depending on a parameter a, taking values in some interval $J \subset \mathbb{R}$.

Under some assumptions, including the individual Furstenberg theorem for every parameter value, it was shown in [25, Theorem 1.5], that though almost surely for Lebesgue-almost all $a \in J$ one has

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_{n,\omega,a}\| = \lambda_F(a) > 0,$$

for the parameters from some random exceptional subset of parameters $S_e(\omega)$ this equality is violated. Moreover, for the parameters belonging to some (smaller) G_δ-set $S_0(\omega)$ one gets

$$\liminf_{n\to\infty} \frac{1}{n} \log \|T_{n,\omega,a}\| = 0.$$

The set $S_e(\omega)$ (and thus $S_0(\omega)$) in [25] were shown to have zero Hausdorff dimension. However, the question of their capacity is still open.

Due to their nature, these sets are very similar to those considered in this paper: they are obtained as countable intersection of unions of exponentially small intervals, that are placed in a (more or less) equidistributed way. Our theorem thus can be seen as a strong indication for that the exceptional sets of parameters for random matrix products are also of full capacity.

4.1.2 Statement of results

Recall that the sets V_n in (4.1) are unions of n intervals of length r_n. At the moment, we require only $r_n < \frac{1}{n}$ so that the intervals are pairwise disjoint; we will discuss possible speeds of decrease for the sequence r_n later.

Our first result is an easier version of Theorem 4.2. It is given to demonstrate the technique and part of the proof will be used later on.

Theorem 4.1 (Subexponential uniform G_δ)**.** *If the sequence r_n decreases subexponentially, then the corresponding uniform G_δ set S, defined by (4.3), has full capacity. That is, if $|\log r_n| = o(n)$, then*

$$\mathrm{Cap}(S) = \mathrm{Cap}([0,1]) > 0.$$

Remark 4.1. As the reader will see, in the proof of this theorem we will not use the fact that *all* the possible denominators n are used in the construction of the set S. Thus, the same conclusion holds for the set $S' := \bigcap_{m=1}^{\infty} \bigcup_{j=m}^{\infty} V_{n_j}$, provided that on the subsequence n_j one has $|\log r_{n_j}| = o(n_j)$.

Theorem 4.1 is already interesting because it shows that there exists a uniform G_δ set of full capacity. However, for a good model for the exceptional energies we need the speed of the intervals to decay much faster. We thus modify it to a more powerful, though more technically complicated, version. This upgraded version is stronger and observes the "phase transition".

Theorem 4.2 (Phase transition)**.** *For* $r_n = e^{-n^\alpha}$,

1. *if* $\alpha > 2$, *then* $\mathrm{Cap}(S) = 0$,
2. *if* $\alpha < 2$, *then* $\mathrm{Cap}(S) = \mathrm{Cap}([0,1])$.

A good question is what happens when $\alpha = 2$. We expect that S will still have full capacity, but to establish that, one would have to adjust the averaged re-distribution procedure (see Proposition 4), probably making the proof even more technical.

It is interesting to note that part (1) of Theorem 4.2 is a partial case of a more general statement (Theorem 3.1), going back to Erdös and Gillis [16] and to Lindeberg [34] (see Chapter 3 for full details).

A particular case of Theorem 3.1 is obtained by considering a set of the form

$$\widetilde{S} = \bigcap_m \bigcup_{k \geq m} I_k,$$

where I_k are intervals of length r'_k. Such a construction includes any uniform G_δ set S by enumerating all the intervals $J_{i,n}$ and then adding them one by one instead of by groups of V_n.

It is immediate to notice that if the series $\sum_n \frac{1}{|\log r'_n|} = \sum_n h_0(r'_n)$ converges, the m_{h_0}-volume of the set $\widetilde{S}$ vanishes, thus implying the following corollary (from which the first part of Theorem 4.2 immediately follows):

Corollary 4.1. *If the series* $\sum_n \frac{1}{|\log r'_n|}$ *converges, then the set* $\widetilde{S}$ *is of zero capacity.*

The following remark is quite natural, but requires a formal proof, so we put it as a proposition.

Proposition 2. *If* X *is a subset of interval* J *such that* $\mathrm{Cap}(X) = \mathrm{Cap}(J)$*, then given any subinterval* $J' \subset J$*, one has* $\mathrm{Cap}(X \cap J') = \mathrm{Cap}(J')$*.*

Corollary 4.2. *In the same setting as Theorem 4.1 or Theorem 4.2 for* $\alpha < 2$*, given any interval* $J \subset [0,1]$*, we have*

$$\mathrm{Cap}(J \cap S) = \mathrm{Cap}(J).$$

4.1.3 Plan of the paper

We start with introducing the re-distribution technique and prove Theorem 4.1 in Section 4.2; we then apply the same technique to show Theorem 3.3. We also prove Proposition 2 in the same section (thus ensuring that "full capacity" in inherited by restrictions on the subintervals). Section 4.3 is devoted to the proof of Proposition 3 that describes the energy of the re-distributed measure.

Due to a faster decrease of the intervals, we have to modify the proof of Theorem 4.1, adapting it to the second part of Theorem 4.2; this is done in Section 4.4.

Though the statement of Corollary 4.1 is a particular case of Theorem 3.1 of Lindeberg and Erdös and Gillis, we note that it can be easily obtained as a corollary of the Cauchy-Schwartz inequality. Namely, with help of it one can obtain an upper bound for the capacity of a union of intervals; under the assumption of Corollary 4.1 this bound converges to zero as $m \to \infty$. Moreover, the same argument allows to get another proof of this theorem, which is, to the best of our knowledge, not yet known. We present this (short) proof in Section 4.5, thus completing the proof of Theorem 4.2.

In the proof of Theorem 4.1, there is a tempting shortcut that cannot be taken. If the capacity was continuous for a descending family of open subsets of $[0, 1]$, the arguments of the proof would be much simpler. As we found no examples in the literature demonstrating such non-continuity for open subsets of $[0, 1]$, we present such an example in Section 4.6.

4.2 Subexponential decay

In this section, we will demonstrate the technique needed to prove Theorem 4.2 in a simpler setting by proving Theorem 4.1.

Both proofs are based on the idea of *re-distribution*. That is, given a measure μ that is supported on an interval or on a finite union of intervals, and given a smaller union of intervals $Y \subset X$, we can try finding a new measure μ', supported on Y, close to μ and with the energy $I(\mu')$ close to $I(\mu)$. Then Theorem 4.1 will be proven by iterating such a re-distribution on a "finer" and "finer" V_n's.

The natural way to do so is to "move" the charge, given by the measure μ, to the closest interval of Y, re-distributing it uniformly on each of these intervals; see Fig. 4.2.

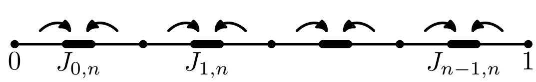

Figure 4.2: The idea of a re-distribution

However, for "good" (absolutely continuous with continuous density) measures μ and for the set $Y = V_n$ that is composed of equally spaced intervals of the same lengths, this operation can be approximated by a simpler one, the one of taking the conditional measure. As it is easier to work with, we will proceed with it.

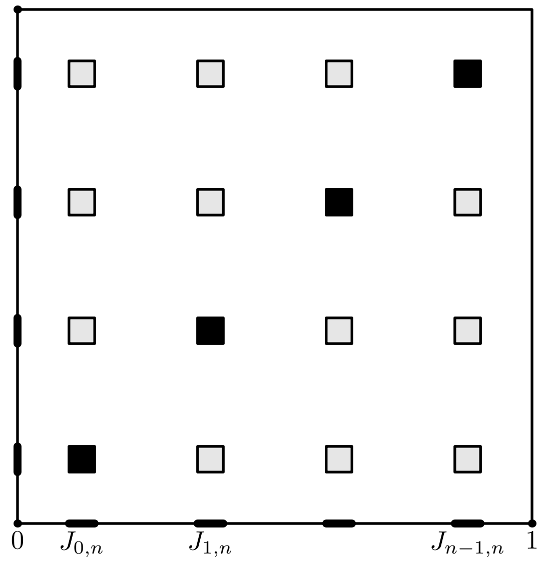

Figure 4.3: Self-interaction (dark squares) and outer-interaction (light ones) parts of the energy integral for the re-distributed measure $R(\mu|V_n)$.

Definition 4.1. Given a finite measure μ on set $[0,1]$ and measurable set Y with positive measure, we define the *re-distribution* of μ on Y to be the conditional measure

$$R(\mu|Y) = \frac{1}{\mu(Y)}\, \mu|_Y \, .$$

Now, let μ be an absolutely continuous measure on $[0,1]$ with continuous density. Let us see how its re-distribution on some V_n changes its energy. The energy of a measure is given by a double integral (see Definition 1.2), and the energy of the re-distribution $R(\mu|V_n)$ can be naturally decomposed into two parts: for the variables x and y belonging to the same interval $J_{i,n}$ and to two different ones; see Fig. 4.3.

It turns out (and this is a statement of Lemma 4.3 below) that the second part tends to the initial energy $I(\mu)$. Meanwhile, the first ("self-interaction") part behaves as

$$\frac{|\log r_n|}{n} \cdot \left(\int f^2 \, dx + o(1) \right);$$

see Lemma 4.2 below. Adding this together, one will get the following proposition.

Proposition 3. *Let* $\mu = f(x)\,dx$*, where* $f \in C([0,1])$*, and* $\mu_n := R(f|V_n)$*. Then*

$$I(\mu_n) = I(\mu) + o(1) + \left(\int_0^1 f^2(x) + o(1)\right) \frac{|\log r_n|}{n}. \tag{4.4}$$

We postpone the proof of Proposition 3 until Section 4.3, and we will now use it to prove Theorem 4.1. First, note that under the assumptions of this theorem we can omit the self-interaction term:

Corollary 4.3. *If* $|\log r_n| = o(n)$*, then* $I(\mu_n) \to I(\mu)$ *as* $n \to \infty$*.*

Using this corollary, we immediately get a first full-capacity statement.

Corollary 4.4. *If* $|\log r_n| = o(n)$*, then we have*

$$\mathrm{Cap}\left(\bigcup_{n=m}^{\infty} V_n\right) = \mathrm{Cap}([0,1]) \quad \textit{for every } m \in \mathbb{N}.$$

Proof. Consider the measure $\mu_{[0,1]} = f_{[0,1]}(x)dx$, where

$$f_{[0,1]}(x) = \frac{1}{\pi\sqrt{x(1-x)}}.$$

It is known (see e.g. [47, Eq. (A.53)]) that this measure minimizes the energy for probability measures supported on $[0,1]$:

$$I(\mu_{[0,1]}) = \inf\{I(\mu) \mid \mu \in \mathcal{P}([0,1])\},$$

and hence that $\mathrm{Cap}([0,1]) = e^{-I(\mu_{[0,1]})}$.

Formally, we cannot apply Corollary 4.3 to this measure, as its density function is not continuous at the endpoints of $[0,1]$. To avoid this problem, note that there exists a family of probability measures $\mu^\delta = f_\delta(x)\,dx$ on $[0,1]$ with $f_\delta \in C([0,1])$, such that $I(\mu^\delta) \to I(\mu_{[0,1]})$

as $\delta \to 0$.

Indeed, consider a family of cut-off densities

$$\widehat{f}_\delta(x) = \begin{cases} \frac{x}{\delta} \cdot f_{[0,1]}(\delta), & x \in [0, \delta), \\ f_{[0,1]}(x), & x \in [\delta, 1-\delta], \\ \frac{1-x}{\delta} \cdot f_{[0,1]}(1-\delta), & x \in (1-\delta, 1], \end{cases}$$

the corresponding (non-probability) measures $\widehat{\mu}_\delta := \widehat{f}_\delta(x)\, dx$ on $[0, 1]$, and let

$$Z_\delta := \widehat{\mu}_\delta([0,1]) = \int_0^1 f_\delta(x) dx$$

be the corresponding normalization constants. Then (for instance, by dominated convergence theorem) we have

$$I(\widehat{\mu}_\delta) \to I(\mu_{[0,1]}), \quad Z_\delta \to 1$$

as $\delta \to 0$ (here we apply Definition 1.2 to non-probability measures $\widehat{\mu}_\delta$). Hence, for the family of probability measures $\mu^\delta := \frac{1}{Z_\delta}\widehat{\mu}_\delta$ we also have

$$I(\mu^\delta) = \frac{1}{Z_\delta^2} I(\widehat{\mu}_\delta) \to I(\mu_{[0,1]}), \quad \delta \to 0.$$

Now, let $m \in \mathbb{N}$ be fixed. For any $\varepsilon > 0$ the above arguments imply that there exists $\delta > 0$ such that $I(\mu^\delta) < I(\mu_{[0,1]}) + \varepsilon/2$. Fix such $\delta > 0$ and consider the family of re-distributed measures $\mu_n^\delta := R(\mu^\delta | V_n)$. As the measure μ^δ has a continuous density, due to Corollary 4.3 we have

$$I(\mu_n^\delta) \to I(\mu^\delta), \quad n \to \infty.$$

In particular, there exists $n \geq m$ such that

$$I(\mu_n^\delta) \leq I(\mu^\delta) + \varepsilon/2 \leq I(\mu_{[0,1]}) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we thus get that

$$\inf\left\{ I(\mu) \mid \mu \in \mathcal{P}\left(\bigcup_{n=m}^{\infty} V_n\right)\right\} \leq I(\mu_{[0,1]}),$$

and hence the desired

$$\mathrm{Cap}\left(\bigcup_{n=m}^{\infty} V_n\right) = \mathrm{Cap}([0,1]) \quad \text{for every } m \in \mathbb{N}.$$

□

It is known that capacity is continuous with respect to any increasing sequence of Borel sets of $\mathbb{C}$ and decreasing sequence of *compact* subsets of $\mathbb{C}$. Our sequence of sets $(\bigcup_{n=m}^{\infty} V_n)_{m\in\mathbb{N}}$ is decreasing, but is not closed.

This is where it would be tempting to conclude by continuity. If the capacity *was* continuous for a decreasing family of open subsets of $[0,1]$, Corollary 4.4 would immediately imply Theorem 4.1.

For decreasing families of (open) subsets of $\mathbb{C}$, it is known that such continuity does not take place; however, all the examples that we found in the literature were essentially two-dimensional. This naturally motivates a question of whether it holds for the subsets of a bounded interval. However, it turns out that it is not the case; we construct a counter-example in Section 4.6.

Thus, we continue the proof of Theorem 4.1 by iterating the re-distributions procedure.

Namely, we have the following

Lemma 4.1. *Let $|\log r_n| = o(n)$, and $U \subset [0,1]$ be a finite union of intervals, and a measure $\nu = f(x)\,dx$ be a measure with a piecewise-continuous density, supported in U. Then for any $\varepsilon > 0$ and any m there exist $n \geq m$ and a measure ν' with a piecewise-continuous density, such that*

$$I(\nu') < I(\nu) + \varepsilon,$$

and the support of ν' is contained in $U \cap V_n$.

Proof. As in the proof of Corollary 4.4, there exists a family $\nu^\delta = f_\delta(x)\,dx$ of probability measures, supported on U, such that $f_\delta \in C([0,1])$ and such that $I(\nu^\delta) \to I(\nu)$ as $\delta \to 0$. Indeed, if intervals $(a_i, b_i) \subset U$ are the intervals of continuity of the density $f(x)$, we consider a new (non-probability) density

$$\widehat{f}_\delta(x) = \begin{cases} \frac{x-a_i}{\delta} \cdot f_{[0,1]}(a_i + \delta), & x \in [a_i, a_i + \delta), \\ f(x), & x \in [a_i + \delta, b_i - \delta], \\ \frac{b_i - x}{\delta} \cdot f_{[0,1]}(b_i - \delta), & x \in (b_i - \delta, b_i]; \end{cases}$$

see Fig. 4.4. Then, define

$$\widehat{\nu}^\delta = \widehat{f}_\delta(x)\,dx, \quad Z_\delta = \widehat{\nu}^\delta([0,1]), \quad \nu^\delta = \frac{1}{Z_\delta}\widehat{\nu}^\delta.$$

As before, we get

$$Z_\delta \to 1, \quad I(\widehat{\nu}^\delta) \to I(\nu) \quad \text{as} \quad \delta \to 0,$$

and hence $I(\nu^\delta) = \frac{1}{Z_\delta^2} I(\widehat{\nu}^\delta) \to I(\nu)$.

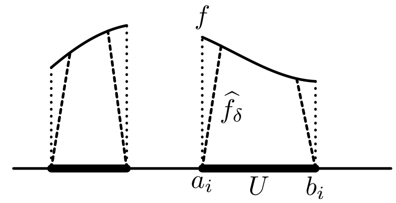

Figure 4.4: Transforming the density $f(x)$ into a continuous one.

Now, if $\varepsilon > 0$ is given, take such a measure ν^δ that $I(\nu^\delta) < I(\nu) + \frac{\varepsilon}{2}$. Applying Proposition 3 to the re-distributions $\nu_n^\delta := R(\nu^\delta | V_n)$ of this measure, we get that $I(\nu_n^\delta) = I(\nu^\delta) + o(1)$. Hence, for some $n \geq m$ we have

$$I(\nu_n^\delta) < I(\nu^\delta) + \frac{\varepsilon}{2} < I(\nu) + \varepsilon;$$

by construction, the measure ν_n^δ is supported on $V_n \cap U$ and has a piecewise continuous density. □

Note that Lemma 4.1 suffices to prove Theorem 4.1:

Proof of Theorem 4.1. Fix an arbitrary $\varepsilon > 0$. We are going to construct a Borel probability measures ν_n, satisfying $I(\nu_n) < I(\mu_{[0,1]}) + \varepsilon$ and concentrating on the set S. Start (as in the proof of Corollary 4.4) with a measure ν_0 with a continuous density on $[0,1]$, satisfying $I(\nu_0) < I(\mu_{[0,1]}) + \frac{\varepsilon}{2}$.

Recursively applying Lemma 4.1, we construct a sequence ν_k of measures with a piecewise continuous density, and an increasing sequence of numbers n_k, such that the measure ν_k is supported on $V_{n_1} \cap \cdots \cap V_{n_k}$ and that $I(\nu_k) < I(\nu_{k-1}) + \frac{\varepsilon}{2^{k+1}}$.

Then, we have

$$I(\nu_k) < I(\mu_{[0,1]}) + \frac{\varepsilon}{2} + \sum_{j=1}^{k} \frac{\varepsilon}{2^{j+1}} < I(\mu_{[0,1]}) + \varepsilon.$$

Now, denote $C_k := \overline{V}_{n_1} \cap \cdots \cap \overline{V}_{n_k}$; note that this set differs from the intersection of the corresponding open sets V_{n_j} by at most a finite number of endpoints.

The family C_k is a decreasing family of compact sets, on which measures ν_k are respectively supported. Hence, any weak limit point ν_∞ of the sequence ν_k is supported on $C_\infty := \bigcap_k C_k$.

Recall that passing to the weak limit does not increase the energy (see, e.g., [40, Lemma 3.3.3]). Indeed, for a $*$-convergent sequence $\mu_j \to \mu$ of measures on $[0,1]$ one has

$$I(\mu) = \lim_{C\to\infty} \int F_C(x,y)\, d\mu(x)\, d\mu(y), \tag{4.5}$$

where $F_C(x,y) = \min(-\log|x-y|, C)$. Thus for any $z < I(\mu)$ there exists C such that the integral on the right-hand side of (4.5) is at least z. For such C,

$$\liminf_{j\to\infty} I(\mu_j) \geq \liminf_{j\to\infty} \int F_C(x,y)\, d\mu_j(x)\, d\mu_j(y) =$$
$$= \int F_C(x,y)\, d\mu(x)\, d\mu(y) \geq z,$$

and as $z < I(\mu)$ was arbitrary, we get the desired

$$\liminf_{j\to\infty} I(\mu_j) \geq I(\mu).$$

In fact, that is exactly the argument that is used to show the capacity is continuous on decreasing families of compact subsets.

Applying the above argument to our convergent subsequence $\mu_j := \nu_{k_j} \to \nu_\infty$, we get

$$I(\nu_\infty) \leq \lim_j I(\nu_{k_j}) < I(\mu_{[0,1]}) + \varepsilon.$$

On the other hand, ν_∞ is supported on $C_\infty \subset S \cup D$, where $D := \bigcup_k (\partial V_k)$ is a countable set of endpoints. As $I(\nu_\infty)$ is finite, this measure does not have any atoms hence $\nu_\infty(D) = 0$, and thus the measure ν_∞ is in fact supported on S. Hence, for an arbitrary $\varepsilon > 0$ there exists a measure ν_∞, supported on S, such that

$$I(\nu_\infty) < I(\mu_{[0,1]}) + \varepsilon,$$

and thus $\mathrm{Cap}(S) = \mathrm{Cap}([0,1])$. □

Also, note that the same construction allows to establish Theorem 3.3.

Proof of Theorem 3.3. Indeed, assume that the relation $\frac{1}{|\log r|} = O(h(r))$ as $r \to 0+$ does not hold. Then there exists a sequence $r_j \to 0$ along which

$$h(r_j) = o\left(\frac{1}{|\log r_j|}\right) \text{ as } j \to \infty.$$

Extracting a subsequence if necessary, we can assume that

$$h(r_j) \cdot |\log r_j| < 4^{-j-1}.$$

Choose now integer numbers $n_j = \left[\sqrt{\frac{|\log r_j|}{h(r_j)}}\right]$, roughly speaking, inserting n_j multiplicatively in the middle between $|\log r_j|$ and $\frac{1}{h(r_j)}$. Then (for all sufficiently large j) we have

$$n_j h(r_j) < 2^{-j}, \quad \frac{n_j}{|\log r_j|} > 2^j. \tag{4.6}$$

Consider now the G_δ-set

$$\widetilde{S} = \bigcap_{m=1}^{\infty} \bigcup_{j=m}^{\infty} V_{n_j},$$

where V_{n_j} are still defined by (4.1)–(4.2); in other words, we are now using only the denominators n_j with the corresponding radii r_j. The first of inequalities in (4.6) then implies that this set is of zero h-volume, as the series $\sum_j n_j h(r_j)$ converges. On the other, the second inequality in (4.6) ensures that $|\log r_j| = o(n_j)$. Hence the same technique as in the proof of Theorem 4.1 is applicable, showing that the set $\widetilde{S}$ is actually of full capacity on $[0,1]$. $\square$

We conclude the section with the proof of Proposition 2 (Corollary 4.2 follows).

Proof of Proposition 2. For any interval $[a,b]$, let $\mu_{[a,b]}$ be the probability measure with the least energy on this interval, that is,

$$\mu_{[a,b]} = \rho_{[a,b]}(x)\,dx, \quad \rho_{[a,b]}(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}.$$

By assumption of the full capacity, there exists a sequence of measures ν_n, supported on X, such that $I(\nu_n) \to I(\mu_J)$. Upon extracting a subsequence, we can assume that this sequence of measures converges weakly. Again using the fact that passing to the weak limit does not increase the energy, we get

$$I\left(\lim_{n\to\infty} \nu_n\right) \le \lim_{n\to\infty} I(\nu_n) = I(\mu_J); \tag{4.7}$$

as μ_J is the unique minimum of the energy function on $\mathcal{P}(J)$, we thus have $\nu_n \to \mu_J$ as $n \to \infty$. Moreover, the inequality in (4.7) turns into an equality. An equality in (4.7) is equivalent to the uniform integrability of the function $-\log|x-y|$ w.r.t. these measures, that is, to

$$\forall \varepsilon > 0 \quad \exists r > 0 : \quad \forall n \quad \iint_{|x-y|<r} \big|\log|x-y|\big|\, d\nu_n(x)\, d\nu_n(y) < \varepsilon.$$

(If it does not take place for some $\varepsilon > 0$, the sides of the inequality in (4.7) differ by at least ε, and vice versa.)

Now, for every $\delta > 0$, take a continuous positive function $f_\delta \in C(J)$, supported on J', such that the measures $f_\delta\, dx|_{J'}$ are probability ones and converge to $\mu_{J'}$, and so do their energies:

$$I(f_\delta\, dx|_{J'}) \to I(\mu_{J'}); \tag{4.8}$$

it can be done in the same way as the cut-off is done on the first step of the proof of Corollary 4.4. These measures can then be re-written as

$$f_\delta(x)\, dx|_{J'} = \frac{f_\delta(x)}{\rho_J(x)}\, \rho_J(x)\, dx = \frac{f_\delta(x)}{\rho_J(x)}\, \mu_J;$$

denote then $\widetilde{f}_\delta(x) := \frac{f_\delta(x)}{\rho_J(x)}$.

Consider the measures

$$\widehat{\mu}_{\delta,n} := \widetilde{f}_\delta(x)\nu_n,$$

and their normalized versions

$$\mu_{\delta,n} = \frac{1}{Z_{\delta,n}}\widehat{\mu}_{\delta,n}, \quad Z_{\delta,n} := \widehat{\mu}_{\delta,n}(J).$$

For each δ, the measures $\widehat{\mu}_{\delta,n}$ converge weakly as $n \to \infty$ to $\widetilde{f}_\delta(x)\mu_J = f_\delta(x)\, dx|_{J'}$; as the limit measure is a probability one, we have

$$Z_{\delta,n} = \int \widetilde{f}_\delta(x) d\nu_n(x) \xrightarrow[n\to\infty]{} \int \widetilde{f}_\delta(x) d\mu_J = 1.$$

Now, as the function $\widetilde{f}_\delta$ is bounded, the function $-\log|x-y|$ is still uniformly integrable w.r.t. these measures, and hence

$$I(\widehat{\mu}_{\delta,n}) \xrightarrow[n\to\infty]{} I(f_\delta\, dx|_{J'}).$$

Thus, we also have

$$I(\mu_{\delta,n}) = \frac{1}{Z_{\delta,n}^2} I(\widehat{\mu}_{\delta,n}) \xrightarrow[n\to\infty]{} I(f_\delta\, dx|_{J'})$$

Now, passing to the limit as $\delta \to 0$ and using (4.8), we get

$$\lim_{\delta\to 0}\lim_{n\to\infty} I(\mu_{\delta,n}) = I(\mu_{J'}).$$

As the measures $\mu_{\delta,n}$ are supported on $X \cap J'$, and $\mu_{J'}$ is the least energy probability measure on J', we get the desired

$$\mathrm{Cap}(X \cap J') = \mathrm{Cap}(J').$$

□

4.3 Energy of the re-distributed measure

This section is devoted to the proof of Proposition 3.

Proof of Proposition 3. First, note that the normalization constant $\mu(V_n)$ satisfies

$$\mu(V_n) = nr_n \cdot (1 + o(1)).$$

Indeed, for any $\varepsilon > 0$ due to the uniform continuity of $f(x)$ for all sufficiently large n we have $|f(x) - f(c_{i,n})| < \varepsilon$ for all $x \in J_{i,n}$. Hence,

$$\left| \int_{J_{i,n}} f(x)\,dx - f(c_{i,n}) r_n \right| < \varepsilon r_n;$$

summing over $i = 0, \dots, n-1$ and dividing by nr_n, we get

$$\left| \frac{1}{nr_n}\mu(V_n) - \frac{1}{n}\sum_{i=0}^{n-1} f(c_{i,n}) \right| < \varepsilon.$$

Now, $\frac{1}{n}\sum_{i=0}^{n-1} f(c_{i,n}) \to \int_{[0,1]} f(x)\,dx = 1$; as $\varepsilon > 0$ was arbitrary, we thus get the desired

$$\frac{1}{nr_n}\mu(V_n) = 1 + o(1).$$

Now, multiplying (4.4) by $(1+o(1))$ does not change its right-hand side, so we can consider (non-probability) measure $\frac{1}{nr_n}\mu|_{V_n}$ instead of $R(\mu|V_n) = \frac{1}{\mu(V_n)}\mu|_{V_n}$. It is also useful to extend the definition of the energy, considering it as a bilinear form (see Definition 1.1). Let us recall some of the immediate properties:

1. $I(\nu) = I(\nu,\nu)$,
2. $I(\nu,\mu) = I(\mu,\nu)$,
3. $I(\nu,\mu) > 0$, if μ and ν are supported on $[0,1]$,
4. $I(\nu,\mu+\mu') = I(\nu,\mu) + I(\nu,\mu')$; $I(\nu,c\mu) = cI(\nu,\mu)$.

The measure $\frac{1}{nr_n}\mu|_{V_n}$ can be written as

$$\frac{1}{nr_n}\mu|_{V_n} = \frac{1}{n}\sum_{i=0}^{n-1}\mu_{i,n},$$

where $\mu_{i,n} := \frac{1}{r_n}\mu|_{J_{i,n}}$. Thus, we can decompose $I\left(\frac{1}{nr_n}\mu|_{V_n}\right)$ as

$$\begin{aligned} I\left(\frac{1}{nr_n}\mu|_{V_n}\right) &= \frac{1}{n^2}\sum_{i,j=0}^{n-1} I(\mu_{i,n},\mu_{j,n}) \\ &= \frac{1}{n^2}\sum_{i} I(\mu_{i,n}) + \frac{1}{n^2}\sum_{i\neq j} I(\mu_{i,n},\mu_{j,n}). \end{aligned}$$

Proposition 3 now follows from the next two Lemmas, 4.2 and 4.3, estimating the diagonal and off-diagonal sums respectively. □

Lemma 4.2.

$$\frac{1}{n^2}\sum_{i=0}^{n-1} I(\mu_{i,n}) = \frac{|\log r_n|}{n}\left(\int_0^1 f^2(x)dx + o(1)\right). \tag{4.9}$$

Lemma 4.3.

$$\frac{1}{n^2}\sum_{i\neq j} I(\mu_{i,n},\mu_{j,n}) = I(\mu) + o(1). \tag{4.10}$$

Proof of Lemma 4.2. Let us first estimate $I(\mu_{i,n})$ for an individual i, comparing it with the energy of the uniform measure $\frac{1}{r_n}dx|_{J_{i,n}}$. Indeed,

$$I(\mu_{i,n}) = \iint_{J_{i,n}} (-\log|x-y|)f(x)f(y)\,\frac{dx}{r_n}\frac{dy}{r_n},$$

and hence

$$(\min_{J_{i,n}} f(x))^2 \cdot I\left(\frac{1}{r_n}dx|_{J_{i,n}}\right) \leq I(\mu_{i,n}) \leq \left(\max_{J_{i,n}} f(x)\right)^2 \cdot I\left(\frac{1}{r_n}dx|_{J_{i,n}}\right). \tag{4.11}$$

Rescaling and a change of variables immediately shows that

$$I\left(\frac{1}{r_n}dx|_{J_{i,n}}\right) = \log r_n + I(\,dx|_{[0,1]}) = \log r_n\cdot(1+o(1)). \tag{4.12}$$

Fix an arbitrarily small $\varepsilon > 0$; for all sufficiently large n, the function $f(x)^2$ then oscillates less than $\varepsilon/2$ on any of the intervals $J_{i,n}$. Thus, from (4.11) and (4.12), for all sufficiently large n we get

$$\frac{1}{|\log r_n|}I(\mu_{i,n}) \in (f^2(c_{i,n})-\varepsilon, f^2(c_{i,n})+\varepsilon).$$

Summing over i and dividing by n, we get

$$\left|\frac{1}{n|\log r_n|}\sum_i I(\mu_{i,n}) - \frac{1}{n}\sum_i f^2(c_{i,n})\right| < \varepsilon.$$

The second sum converges to the Riemann integral $\int_0^1 f^2(x)dx$; as $\varepsilon > 0$ was arbitrary, we get

$$\frac{1}{n|\log r_n|}\sum_i I(\mu_{i,n}) = \int_0^1 f^2(x)dx + o(1).$$

Multiplying by $\frac{|\log r_n|}{n}$, we get the desired (4.9). □

Before proceeding with Lemma 4.3, let us estimate the interaction energy for uniformly distributed measures on the subintervals, comparing it to the interaction energy between point charges at their centers.

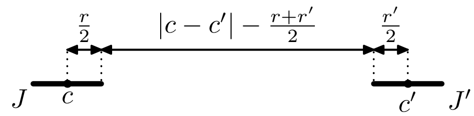

Figure 4.5: Two intervals J, J' and their centers.

Lemma 4.4. *Let $J, J' \subset [0,1]$ be two disjoint intervals with centers c, c' and with lengths r, r' respectively (see Fig. 4.5). Then the interaction energy between the uniform measures on these intervals satisfies*

$$-\log|c-c'| < I\left(\frac{1}{r}dx|_J, \frac{1}{r'}dx|_{J'}\right) < (-\log|c-c'|) + \Delta,$$

where $\Delta = \min(2, (-\log(1-\frac{r+r'}{2|c-c'|})))$.

Proof. The lower bound is implied by the Jensen's inequality: as the function $F(x,y) = -\log|x-y|$ is convex on the rectangle $J \times J'$,

$$\iint_{J\times J'} F(x,y)\,\frac{dx}{r}\,\frac{dy}{r'} > F(c,c') = -\log|c-c'|.$$

Now, for any $x \in J$, $y \in J'$ we have

$$-\log|x-y| = -\log|c-c'| - \log\frac{|x-y|}{|c-c'|}$$
$$= -\log|c-c'| - \log\left(1 - \frac{|c-c'|-|x-y|}{|c-c'|}\right)$$

and the upper bound by $(-\log(1-\frac{r+r'}{2|c-c'|}))$ follows as it is the maximal possible value of the second term.

To get a uniform upper bound by 2, consider first the interaction between a uniform measure and a point charge. Note that for any $y \in J'$ we have

$$\int_J (-\log|x-y|)\frac{dx}{r} = -\log|c-y| - \frac{|c-y|}{r} \cdot \int_{-\frac{r/2}{|c-y|}}^{\frac{r/2}{|c-y|}} \log(1+s)\, ds =$$

$$= -\log|c-y| - \frac{|c-y|}{r} \cdot \int_0^{\frac{r/2}{|c-y|}} \log(1-s^2)\, ds;$$

as the function $-\log(1-s^2)$ is monotone increasing, the maximal value of its average will be if it is averaged on the largest possible interval, that is, over $[0,1]$ (that corresponds to $|c-y| = r/2$, in other words, y being on the boundary of J). In this case, a straightforward computation shows that the second term is equal to

$$\int_{-1}^{1} (-\log(1+s))\, \frac{ds}{2} = 1 - \log 2 < 1.$$

Thus, for any $y \in J'$ we have

$$\int_J (-\log|x-y|)\frac{dx}{r} < -\log|c-y| + 1.$$

Finally, averaging with respect to $y \in J'$, we get

$$\iint_{J\times J'} (-\log|x-y|)\frac{dx}{r}\frac{dy}{r'} < \int_{J'} (-\log|c-y|)\, \frac{dy}{r'} + 1 < \log|c-c'| + 2.$$

□

Proof of Lemma 4.3. Fix an arbitrary small $\delta > 0$, and let $M := \max_{[0,1]} f(x)$. Let us decompose the sum on the left-hand side of (4.10) into two parts, depending on whether the centers $c_{i,n}$ and $c_{j,n}$ are closer than δ to each other:

$$\frac{1}{n^2}\sum_{i\neq j} I(\mu_{i,n},\mu_{j,n}) = \frac{1}{n^2}\sum_{0<|c_{i,n}-c_{j,n}|<\delta} I(\mu_{i,n},\mu_{j,n}) + \frac{1}{n^2}\sum_{|c_{i,n}-c_{j,n}|\geq\delta} I(\mu_{i,n},\mu_{j,n}).$$

Note that the first sum can be bounded by an arbitrarily small constant by choosing an appropriate $\delta > 0$. Indeed, note first that

$$I(\mu_{i,n},\mu_{j,n}) < M^2 I\left(\frac{1}{r_n}dx|_{J_{i,n}}, \frac{1}{r_n}dx|_{J_{j,n}}\right).$$

Taking $\delta < 1/e^2$ and thus ensuring $-\log|c_{i,n}-c_{j,n}| > 2$ once $|c_{i,n}-c_{j,n}| < \delta$, we get

$$\begin{aligned}\frac{1}{n^2}\sum_{0<|c_{i,n}-c_{j,n}|<\delta} I(\mu_{i,n},\mu_{j,n}) &< \frac{1}{n^2}M^2\sum_{0<|c_{i,n}-c_{j,n}|<\delta} I\left(\frac{1}{r_n}dx|_{J_{i,n}}, \frac{1}{r_n}dx|_{J_{j,n}}\right)\\ &< 2\frac{M^2}{n^2}\sum_{0<|c_{i,n}-c_{j,n}|<\delta}(-\log|c_{i,n}-c_{j,n}|)\end{aligned}$$

Now, for each i we have

$$\frac{1}{n}\sum_{\substack{j:\\ 0<|c_{i,n}-c_{j,n}|<\delta}}(-\log|c_{i,n}-c_{j,n}|) \leq \frac{2}{n}\sum_{k=1}^{[\delta n]}(-\log\frac{k}{n}) < 2\int_0^\delta(-\log s)\,ds, \tag{4.13}$$

as the function $(-\log s)$ is decreasing on $[0,1]$; see Fig. 4.6, left. Averaging (4.13) over i, we get

$$\frac{1}{n^2}\sum_{0<|c_{i,n}-c_{j,n}|<\delta} I(\mu_{i,n},\mu_{j,n}) < 4M^2\int_0^\delta(-\log s)\,ds.$$

As the integral on the right-hand side tends to 0 as $\delta\to 0$, for any $\varepsilon > 0$ we have

$$\exists\delta_0>0: \quad \forall\delta<\delta_0\,\forall n\in\mathbb{N} \quad \frac{1}{n^2}\sum_{0<|c_{i,n}-c_{j,n}|<\delta} I(\mu_{i,n},\mu_{j,n}) < \varepsilon. \tag{4.14}$$

Now, for any fixed $\delta > 0$, the function $f(x)f(y)(-\log|x-y|)$ is uniformly continuous on the

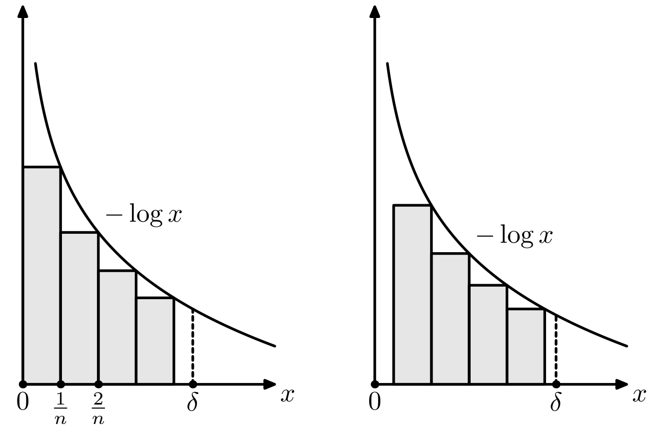

Figure 4.6: Comparing integral sums and the integral for the $-\log x$ function: nonshifted (left) and shifted (right) sums.

subset $\{|x-y| \geq \delta\}$, and hence

$$\frac{1}{n^2} \sum_{|c_{i,n}-c_{j,n}|\geq \delta} I(\mu_{i,n}, \mu_{j,n}) \xrightarrow[n\to\infty]{} \int_{\{|x-y|\geq\delta\}} f(x)f(y)(-\log|x-y|)\, dx\, dy. \tag{4.15}$$

The integral on the right-hand side of (4.15) tends to $I(\mu)$ as $\delta \to 0$. Hence, for any sufficiently small δ it is ε-close to $I(\mu)$. Fixing such $\delta < \delta_0$, from (4.15) for all sufficiently large n we get

$$\left| \frac{1}{n^2} \sum_{|c_{i,n}-c_{j,n}|\geq\delta} I(\mu_{i,n}, \mu_{j,n}) - I(\mu) \right| < 2\varepsilon,$$

and joining it with (4.14),

$$\left| \frac{1}{n^2} \sum_{i\neq j} I(\mu_{i,n}, \mu_{j,n}) - I(\mu) \right| < 3\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we get the desired

$$\frac{1}{n^2} \sum_{i\neq j} I(\mu_{i,n}, \mu_{j,n}) = I(\mu) + o(1).$$

This completes the proof of Lemma 4.3, and hence of Proposition 3. □

4.4 Phase transition

Let us move on to prove Theorem 4.2. The key ingredient in the sub-exponential case was that the re-distribution of μ_n on a single level, V_n, of a given measure μ gave us a close approximation of $I(\mu)$. If $r_n = e^{-n^\alpha}$, then Proposition 3 yields

$$I(\mu_n) = I(\mu) + o(1) + \left(\int_0^1 f^2(x) + o(1)\right) n^{\alpha-1}.$$

For $1 \leq \alpha < 2$, a simple re-distribution does not suffice, as the self-interaction term has an asymptotics of $n^{\alpha-1}$ and hence does not tend to zero. The re-distribution thus will have to be done on multi-levels. Namely, let

$$F_m := \{n = m, \dots, 2m-1 \,:\, n \text{ is prime}\},$$

that is, the set of prime numbers in $[m, 2m-1]$, and denote by $N_m = \#F_m$ its cardinality.

Notice that V_p and V_q are disjoint for distinct $p, q \in F_m$. Indeed, this follows from the fact that the centers $c_{k,p} = \frac{2k+1}{2p}$ are distinct for $p \in F_m$, and that

$$\left|\frac{a}{2p} - \frac{b}{2q}\right| = \left|\frac{aq - bp}{2pq}\right| \geq \frac{1}{2m^2} > e^{-m^\alpha}.$$

Let μ_n be the re-distribution of μ on V_n, where $n \in F_m$. Given a collection of positive numbers $\{p_n\}_{n \in F_m}$ such that

$$\sum_{n \in F_m} p_n = 1,$$

consider a *averaged re-distribution*:

$$\mu^m = \widetilde{R}_m(\mu) := \sum_{n \in F_m} p_n \mu_n,$$

that is a convex combination of measures μ_n, supported on a finite union

$$\widehat{V}_m := \bigcup_{n \in F_m} V_n.$$

The averaging allows to regain control on the self-interaction term. That is, the energy of the averaged measure μ^m satisfies

$$I(\mu^m) = \sum_{n \in F_m} p_n^2 I(\mu_n) + \sum_{i \neq j} p_i p_j I(\mu_i, \mu_j). \tag{4.16}$$

Take p_i to be uniform: let $p_i = \frac{1}{N_m}$ for every $i \in F_m$. We have $I(\mu_n) = O(n^{\alpha-1})$, and due to the Prime Number Theorem $N_m \sim \frac{m}{\log m}$ as $m \to \infty$. Hence, the first term in (4.16) can be estimated as

$$\sum_{n \in F_m} p_n^2 I(\mu_n) = \frac{1}{N_m^2} \sum_{n \in F_m} I(\mu_n) \leq \frac{1}{N_m} \max_{n \in F_m} I(\mu_n) = \frac{O(m^{\alpha-1})}{m/\log m} = O\left(\frac{\log m}{m^{2-\alpha}}\right) = o(1), \tag{4.17}$$

as $\alpha < 2$.

On the other hand, we claim that the interaction energy between different μ_n's is close to the one of the initial measure μ:

Lemma 4.5. *Let $\mu = f(x)\,dx$ be a measure with a continuous density on $[0,1]$. Then for $n, n' \in F_m$, $n \neq n'$ we have*

$$I(\mu_n, \mu_{n'}) = I(\mu) + o(1)$$

(uniformly on the choice of n and n') as $m \to \infty$.

Postponing its proof till the end of this section, note that it immediately imples

Proposition 4. *Let $\mu = f(x)\,dx$ be a measure with a continuous density on $[0,1]$. Then for the family of its averaged re-distributions $\mu_m = \widetilde{R}_m(\mu)$ we have*

$$I(\mu^m) = I(\mu) + o(1).$$

Proof. Due to (4.16), the energy $I(\mu^m)$ is the sum of two terms; the first one is $o(1)$ due to (4.17), while the second is $I(\mu) + o(1)$ due to Lemma 4.5. □

We then get

Lemma 4.6. *Let $r_n = e^{-n^\alpha}$, where $\alpha < 2$. Let $U \subset [0,1]$ be a finite union of intervals, and a measure $\nu = f(x)\,dx$ be a measure with a piecewise-continuous density, supported in U. Then for any $\varepsilon > 0$ and any k there exist $m \geq k$ and a measure ν' with a piecewise-continuous density, such that*

$$I(\nu') < I(\nu) + \varepsilon,$$

and the support of ν' is contained in $U \cap \widehat{V}_m$.

Proof. As in the proof of Lemma 4.1, we can find a measure $\nu_\delta = f_\delta(x)\,dx$ with continuous density on $[0,1]$, such that $\operatorname{supp} \nu_\delta \subset \operatorname{supp} \nu$ and that $I(\nu_\delta) < I(\nu) + \frac{\varepsilon}{2}$. Applying Proposition 4 to $\mu = \nu_\delta$ concludes the proof. □

Proof of part (2) of Theorem 4.2. We now deduce Theorem 4.2 from Lemma 4.6 in exactly the same way, as earlier we have deduced Theorem 4.1 from Lemma 4.1. Namely, for any

$\varepsilon > 0$ we iterate the re-distribution procedure, obtaining a family of measures ν_k with piecewise continuous density on $[0, 1]$, for which we control both the supports and the energy.

To do so, we start with the measure ν_0 that is supported on $[0, 1]$ and that satisfies $I(\nu_0) < I(\mu_{[0,1]}) + \frac{\varepsilon}{2}$. Now, if a measure ν_{k-1} is already constructed, due to Lemma 4.6 there exists a measure ν_k with

$$I(\nu_k) < I(\nu_{k-1}) + \frac{\varepsilon}{2^{k+1}} \text{ and } \operatorname{supp} \nu_k \subset \operatorname{supp} \nu_{k-1} \cap \widehat{V}_{m_k}$$

for some $m_k > k$. Any accumulation point ν_∞ of the measures ν_k is thus supported on a intersection of closures

$$\bigcap_k cl\left(\widehat{V}_{m_k}\right) \subset S \cup D,$$

where D is a countable set of endpoints of V_n's, and satisfies

$$I(\nu_\infty) < (I(\mu_{[0,1]}) + \frac{\varepsilon}{2}) + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = I(\mu_{[0,1]}) + \varepsilon.$$

As a finite energy measure, the measure ν_∞ does not charge a countable set D, and is thus supported on S. As $\varepsilon > 0$ was arbitrary, we thus get

$$\inf_{\nu \in \mathcal{P}(S)} I(\nu) = I(\mu_{[0,1]}),$$

and hence the desired $\operatorname{Cap}(S) = \operatorname{Cap}([0, 1])$. □

We conclude this section with the proof of Lemma 4.5.

Proof of Lemma 4.5. As in the proof of Lemma 4.3, fix an arbitrarily small $\delta > 0$, and decompose

$$I(\mu_n, \mu_{n'}) = \frac{1}{n \cdot n'} \sum_{i=0}^{n-1} \sum_{j=0}^{n'-1} I(\mu_{i,n}, \mu_{j,n'})$$

into two parts, depending on the distance $|c_{i,n} - c_{j,n'}|$:

$$I(\mu_n, \mu_{n'}) = \frac{1}{nn'} \sum_{|c_{i,n}-c_{j,n'}|<\delta} I(\mu_{i,n}, \mu_{j,n'}) + \frac{1}{nn'} \sum_{|c_{i,n}-c_{j,n'}|\geq\delta} I(\mu_{i,n}, \mu_{j,n'}). \tag{4.18}$$

The sum over intervals whose centers are closer than δ from each other can be made arbitrarily small by a choice of δ and by taking sufficiently large m. Indeed, for any fixed j we have

$$\frac{1}{n} \sum_{i:\,|c_{i,n}-c_{j,n'}|<\delta} (-\log|c_{i,n} - c_{j,n'}|) < -\frac{2}{n} \log \min_i |c_{i,n} - c_{j,n'}| + 2\int_0^\delta (-\log s)\, ds, \tag{4.19}$$

see Fig. 4.6, right. Due to the estimates above the minimal distance $\min_j |c_{i,n} - c_{j,n'}|$ is at least $\frac{1}{2m^2}$, so the first summand does not exceed $\frac{2}{m}\log(2m^2)$ and hence tends to 0. The second can be made arbitrarily small due to the integrability of the function log at 0. Finally, averaging (4.19) over j, we get the desired (arbitrarily small) bound for the first summand in (4.18).

On the other hand, for any fixed δ, the function $f(x)f(y)(-\log|x-y|)$ is continuous on the set $|x-y| \geq \delta$, and the second summand in (4.18) behaves like its Riemann sum. Hence, we have

$$\frac{1}{nn'} \cdot \sum_{|c_{i,n}-c_{j,n'}|\geq\delta} I(\mu_{i,n}, \mu_{j,n'}) \to \iint_{|x-y|\geq\delta} f(x)f(y)(-\log|x-y|)\, dx\, dy$$

uniformly in $n, n' \in F_m$ as $m \to \infty$.

For any $\varepsilon > 0$, take δ sufficiently small so that the integral on the right-hand side is $\frac{\varepsilon}{2}$-close to $I(\mu)$, and that the first summand in (4.18) does not exceed $\frac{\varepsilon}{2}$ for all sufficiently large m. Then, we have

$$|I(\mu_n, \mu_{n'}) - I(\mu)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and as $\varepsilon > 0$ is arbitrary, this concludes the proof. □

4.5 Zero capacity: Lindeberg and Erdös-Gillis theorems

This section is devoted to the Cauchy-Schwartz based proof of Corollary 4.1, as well as of Lindeberg and Erdös-Gillis' Theorem 3.1, and hence of the first part of Theorem 4.2. The key step is the following.

Lemma 4.7. *Let $J'_1, J'_2, \ldots \subset [0,1]$ be a sequence of intervals of length $|J'_k| =: r'_k$, such that the series $\sum_{k=1}^{\infty} \frac{1}{|\log r_k|}$ converges. For any probability measure μ supported on $\bigcup_{k=1}^{\infty} J'_k$, we have*

$$I(\mu) \geq \frac{1}{\sum_{k=m}^{\infty} 1/|\log r'_k|}.$$

Proof. We transform the union $\bigcup_{k=1}^{\infty} J'_k$ into a disjoint one by setting

$$\tilde{V}_1 := J'_1, \quad \tilde{V}_k := J'_k \setminus \bigcup_{i=1}^{k-1} J'_i.$$

Let μ be any probability measure supported on $\bigcup_{k=1}^{\infty} J'_k$; denote $p_k := \mu(\tilde{V}_k)$. Then $\sum_k p_k = \mu(\bigcup_{k=1}^{\infty} J'_k) = 1$. Without loss of generality, we can assume $p_k > 0$ for all k, otherwise removing the corresponding J'_k.

Let $\mu'_k := \frac{1}{p_k} \mu|_{\tilde{V}_k}$ be the corresponding conditional measures. Then,

$$\mu = \sum_k p_k \mu'_k,$$

and thus

$$I(\mu) = \sum_{k,l} p_k p_l I(\mu'_k, \mu'_l) \geq \sum_k p_k^2 I(\mu'_k).$$

Now, the measure μ'_k is supported on J'_k, that is an interval of length r'_k, and hence $I(\mu'_k) \geq |\log r'_k|$. Thus,

$$I(\mu) \geq \sum_k p_k^2 |\log r'_k|.$$

Applying Cauchy-Schwartz inequality, we get

$$\left(\sum_k p_k^2 |\log r'_k|\right) \left(\sum_k \frac{1}{|\log r'_k|}\right) \;\geq\; \left(\sum_k \sqrt{p_k^2 |\log r'_k| \cdot \frac{1}{|\log r'_k|}}\right)^2 \;=\; \left(\sum_k p_k\right)^2 \;=\; 1,$$

and hence,

$$I(\mu) \geq \sum_k p_k^2 |\log r'_k| \geq \frac{1}{\sum_k \frac{1}{|\log r'_k|}}. \tag{4.20}$$

□

This lemma immediately implies Corollary 4.1. Indeed, for any m the set $\tilde{S}$ is contained in $\bigcup_{k \geq m} J'_k$, and hence,

$$\mathrm{Cap}(\tilde{S}) \leq \mathrm{Cap}\left(\bigcup_{k \geq m} J'_k\right) \leq \exp\left(-\frac{1}{\sum_{k=m}^{\infty} 1/|\log r'_k|}\right).$$

As m is arbitrary, and the tail sum of a convergent series tends to zero, passing to the limit as $m \to \infty$ we get the desired

$$\mathrm{Cap}(\tilde{S}) = 0.$$

Proof of Theorem 3.1. Assume that $m_{h_0}(E) = R < \infty$. Then for an arbitrarily small $\varepsilon > 0$ there exists its cover $\bigcup_j I_j \supset E$ by intervals of length at most ε, such that $\sum_j h_0(|I_j|) = \sum_j \frac{1}{|\log |I_j||} < 2R$. Estimate (4.20) then implies that for any probability measure μ on E one has $I(\mu) \geq \frac{1}{2R}$. Moreover, actually (4.20) is a lower bound for the part of the integral ε-close to the diagonal (as x and y can be restricted to belong to the same interval):

$$\iint_{|x-y|<\varepsilon} \log|x-y|\, d\mu(x)\, d\mu(y) > \frac{1}{2R}. \tag{4.21}$$

Recall now that $\varepsilon > 0$ was arbitrary; if there was a measure μ on E with $I(\mu) < \infty$, the left hand side of (4.21) would tend to zero as $\varepsilon \to 0$. On the other hand, the right-hand side is a constant. This contradiction shows that for any measure μ on E one has $I(\mu) = +\infty$, and thus that $\mathrm{Cap}(E) = 0$. □

Remark 4.2. Actually, the statements of Lemma 4.7 and Theorem 3.1 hold in any dimension, with balls replacing the intervals and their diameters taken instead of lengths, and the proofs are the same word for word.

Proof of the first part of Theorem 4.2. Take the sequence J'_k to be an enumeration of the family $J_{k,n}$. Then,

$$\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} J'_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} V_n;$$

for each n there are n intervals J'_k of length r_n (that is, $J_{0,n}, \dots, J_{n-1,n}$), and hence

$$\sum_{k=m}^{\infty} \frac{1}{|\log r'_k|} = \sum_{n=m}^{\infty} \frac{n}{|\log r_n|} = \sum_{n=m}^{\infty} \frac{1}{n^{\alpha-1}} \tag{4.22}$$

as $r_n = e^{-n^\alpha}$. As for $\alpha > 2$ the series (4.22) converges, $\mathrm{Cap}(S) = 0$ due to Theorem 4.1. □

4.6 Non-continuity of capacity on bounded interval

As mentioned previously, in the proof of Theorem 4.1, there is a tempting shortcut that cannot be taken. It is already known that capacity does not satisfy limit properties that a measure does. In particular, it is not continuous under descending collection of sets. Recall Example 1.2, one can take the collection of open bounded sets

$$O_n := \left\{ z \in \mathbb{C} : 1 - \frac{1}{n} < \Im(z) < 1 \text{ and } 0 < \Re(z) < 1 \right\}.$$

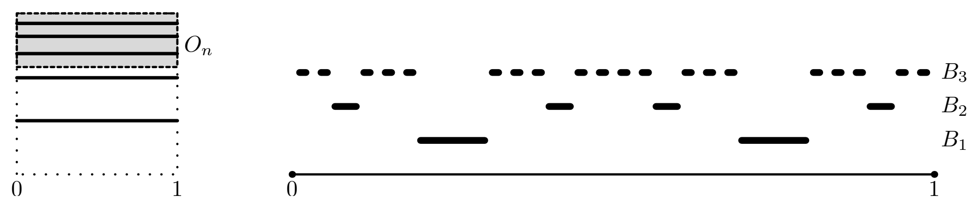

Figure 4.7: Discontinuity of capacity on $[0,1]^2$ and on $[0,1]$

Then, each O_n contains a translation of the interval $(0,1)$; see Fig. 4.7, left. Hence, $\mathrm{Cap}(O_n) \geq 1/4$. If capacity was continuous on descending open sets, we would have that

$$1/4 \leq \lim_{n\to\infty} \mathrm{Cap}(O_n) = \mathrm{Cap}\left(\bigcap_{n\in\mathbb{N}} O_n\right) = \mathrm{Cap}(\emptyset) = 0.$$

A question that appears naturally is whether capacity was continuous under a descending collection of open sets contained in $[0,1]$. If so, Corollary 4.4 would imply that $\mathrm{Cap}(S) = \mathrm{Cap}([0,1])$, where S is a G_δ-set of the form (4.3). Unfortunately, the answer to the continuity question on $[0,1]$ is negative, as one can see from the following example.

Example 4.1. There exist pairwise disjoint open sets $B_1, B_2, \dots$ contained in $[0,1]$ with capacity bounded away from 0. In other words, there exists $\varepsilon > 0$ such that

$$\mathrm{Cap}(B_n) \geq \varepsilon,$$

for any $n \in \mathbb{N}$.

Example 4.2. There exists a descending sequence $W_1 \supset W_2 \supset \dots$ of open sets contained in $[0,1]$ such that

$$\operatorname{Cap}\left(\bigcap_{n\in\mathbb{N}} W_n\right) = 0 < \varepsilon \leq \operatorname{Cap}\left(W_k\right),$$

for some ε and every $k \in \mathbb{N}$.

Construction of Example 4.2 out of Example 4.1 is immediate: take

$$W_m := \bigcup_{n\geq m} B_n,$$

where B_n's are given by Example 4.1. Indeed, one then has $\bigcap_n W_n = \emptyset$, $W_1 \supset W_2 \dots$ by construction, as well as $\operatorname{Cap}(W_n) \geq \operatorname{Cap}(B_n) \geq \varepsilon$. This example shows the discontinuity of example on descending sequence of open subsets of $[0,1]$: one has $\operatorname{Cap}(\bigcap_n W_n) = 0$ while $\lim_{n\to\infty} \operatorname{Cap}(W_n) \geq \varepsilon$. Let us pass to the construction proving Example 4.1.

To construct the desired sets B_n, consider the unions $V_n = \cup_i J_{i,n}$ given by (4.1), taking the decreasing speed for the lengths $r_n := 2^{-n}$. Take a subsequence n_k of indices to be defined by $n_1 = 2^{10}, n_k = 2^{n_{k-1}+1}$, and define (see Fig. 4.7, right)

$$B_k := V_{n_k} \setminus \bigcup_{i=1}^{k-1} \overline{V}_{n_i}.$$

The sets B_k are then open and disjoint by construction. To show that they satisfy the conclusion of the proposition, it suffices to find probability measures ν_k, supported on B_k, such that the energies $I(\nu_k)$ are uniformaly bounded. That is, there exists C such that for all k one has $I(\nu_k) \leq C$. This implies $\operatorname{Cap}(B_k) \geq e^{-C}$, and thus the conclusion of the proposition holds with $\varepsilon = e^{-C}$.

To do so, first consider the uniform measures ν_k° on V_{n_k}, letting $\nu_k^\circ := R(\mathrm{Leb}\,|V_{n_k})$, where

Leb is the Lebesgue measure on $[0,1]$. Due to Proposition 3,

$$I(\nu_k^\circ) = I(\text{Leb}) + \frac{n_k \log 2}{n_k} + o(1) = 3/2 + \log 2 + o(1).$$

Now, let

$$\nu_k := \frac{\nu_k^\circ|_{B_k}}{\nu_k^\circ(B_k)}.$$

Then,

$$I(\nu_k) = \frac{1}{\nu_k^\circ(B_k)^2} I(\nu_k^\circ),$$

so it suffices to check that $\nu_k^\circ(B_k)$ stays bounded away from zero. In fact, we will show that $\nu_k^\circ(B_k) \geq 1/2$. This will follow from a purely geometrical observation:

Lemma 4.8.

$$\text{Leb}(V_{n_k} \cap X_k) = \text{Leb}(X_k) \cdot \text{Leb}(V_{n_k}),$$

where

$$X_k := [0,1] \setminus \bigcup_{i=1}^{k-1} V_{n_i}.$$

Proof. Note that all the endpoints of V_{n_i}, $i = 1, \ldots, k-1$ are of the form

$$\frac{2j+1}{2n_i} \pm \frac{r_{n_i}}{2} = \frac{2j+1}{2^{n_{i-1}+2}} \pm \frac{1}{2^{n_i+1}},$$

and hence can be represented as

$$\frac{a}{2^{n_{k-1}+1}} = \frac{a}{n_k}.$$

Hence, X_k is (up to a finite number of points) a union of intervals of the form

$$\left(\frac{a}{2^{n_{k-1}+1}}, \frac{a+1}{2^{n_{k-1}+1}} \right) = \left(\frac{a}{n_k}, \frac{a+1}{n_k} \right). \tag{4.23}$$

We have

$$X_k = \bigcup_{a \in \mathcal{A}} \left(\frac{a}{n_k}, \frac{a+1}{n_k} \right) \cup P,$$

where P consists of a finite number of points (See Fig. 4.8).

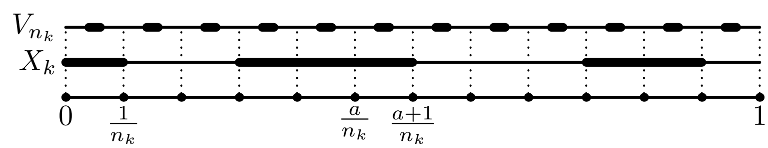

Figure 4.8: The set V_{n_k} and the decomposition into dyadic intervals

On each interval of the form (4.23), the set V_{n_k} cuts the same measure:

$$\mathrm{Leb} \left(V_{n_k} \cap \left[\frac{a}{n_k}, \frac{a+1}{n_k} \right] \right) = r_{n_k}.$$

Hence,

$$\mathrm{Leb}(V_{n_k} \cap X_k) = r_{n_k} \cdot \#\mathcal{A} = (r_{n_k} \cdot n_k) \left(\frac{\#\mathcal{A}}{n_k} \right) = \mathrm{Leb}(V_{n_k}) \cdot \mathrm{Leb}(X_k).$$

□

Due to this lemma, $\nu_k^\circ(B_k) = \mathrm{Leb}(X_k)$. On the other hand,

$$\mathrm{Leb}(X_k) \geq 1 - \sum_{i=1}^{k-1} \mathrm{Leb}(V_{n_i}) \geq 1/2.$$

We have obtained the desired $\nu_k^\circ(B_k) \geq 1/2$, and hence

$$I(\nu_k) \leq 4(3/2 + \log 2 + o(1)),$$

thus concluding the construction.

Chapter 5

Logarithmic capacity of random G_δ sets

In our second study of G_δ subsets of the interval $[0,1]$ we will consider more general G_δ sets and provide sufficient conditions such for G_δ subsets to have full capacity, i.e. $\mathrm{Cap}(S) = \mathrm{Cap}([0,1])$. We will also consider the case when the intervals decay exponentially and are placed in $[0,1]$ randomly with respect to some given distribution. The random G_δ sets generated by such distribution satisfy our sufficient conditions almost surely and hence, have full capacity almost surely. This study is motivated by the G_δ set of exceptional energies in the parametric version of the Furstenberg theorem on random matrix products. We also study the family of G_δ sets $\{S(\alpha)\}_{\alpha>0}$ that are generated by setting the decreasing speed of the intervals to $l_k = e^{-k^\alpha}$. We observe a sharp transition from full capacity to zero capacity by varying $\alpha > 0$.

The results of this chapter have been submitted (see [39]).

5.1 Introduction

5.1.1 The setting

Our focus will be on the capacity of G_δ's of the form:

$$S = \bigcap_m \bigcup_{k \geq m} I_k, \tag{5.1}$$

where each I_k is an open interval of length l_k with center at $c_k \in (0,1)$. The sequence $\{l_k\}$ is taken to approach 0 as $k \to \infty$. It is immediate that S is a G_δ subset of $[0,1]$. Under certain assumptions, we will show that the set S has full capacity on the unit interval:

Definition 5.1. Let $J \subset \mathbb{C}$. A set $E \subset J$ is said to have *full capacity* on J if

$$\mathrm{Cap}(E) = \mathrm{Cap}(J).$$

The capacity of an interval J is $\mathrm{Cap}(J) = \frac{|J|}{4}$ (see, e.g. [40, p. 135], [47, Example A.17]).

Our results and methods can be extended to higher dimensions, but we do not elaborate on that here. In this paper, we will be focused on G_δ subsets of an interval of the real line. We are mostly interested in one-dimension because our motivation came from the one-dimensional random G_δ sets of exceptional energies in the parametric version of the Furstenberg theorem. We will summarize this motivation in Section 5.1.3 (see Chapter 2 for full details).

5.1.2 Main results

Our first main result is devoted to the random setting, being a "toy model" for the exceptional energies in the parametric version of the Furstenberg theorem (see Section 5.1.3). Namely, the set of exceptional parameters in [25] is generated by exponentially small intervals, that are asymptotically distributed with respect to some (dynamically defined) measure.

A *random* G_δ set is obtained by viewing the centers $\{c_k\}$ as random variables. As a toy model, it is reasonable to consider first the set generated by random intervals, that are placed independently (with the same "reasonable" distribution of their centers) - instead of some complicated definition coming from the random dynamical systems.

Theorem 5.1. *Let $\{c_k\}$ be i.i.d. with an absolutely continuous distribution on any interval J with almost everywhere positive and uniformly bounded density function. Take $l_k = e^{-\lambda k}$ for some fixed $\lambda > 0$ and let S be the corresponding G_δ-set* (5.1)*. Then, almost surely S has full capacity on the unit interval:*

$$\mathrm{Cap}(S) = \mathrm{Cap}([0,1]) > 0.$$

Remark 5.1. Full capacity is a property that is inherited when restricted to subintervals (see [30, Proposition 1.6]): If E is a subset of interval J such that $\mathrm{Cap}(E) = \mathrm{Cap}(J)$, then given any subinterval $J' \subset J$, one has $\mathrm{Cap}(E \cap J') = \mathrm{Cap}(J')$.

Remark 5.2. In Theorem 5.1 (and Theorem 5.3 below), the interval $[0,1]$ may be replaced with any bounded interval J due to the fact that $\mathrm{Cap}(\beta \cdot J) = \beta \cdot \mathrm{Cap}([0,1])$ for some $\beta > 0$. Without loss of generality, we will only be working on the interval $[0,1]$. So, we will take dx to be the restriction to the unit interval: $dx|_{[0,1]}$.

Now, take the centers $\{c_k\}$ from Theorem 5.1 and let us vary the lengths of the intervals as a function of the parameter $\alpha \in (0,1]$: $l_k = e^{-\lambda k^\alpha}$. It turns out that at $\alpha = 1$ the capacity undergoes a (sharp) phase transition:

Theorem 5.2 (Random phase transition). *Let $\{c_k\}$ be i.i.d. with an absolutely continuous distribution with density function that is bounded and positive almost everywhere with respect to the Lebesgue measure. Let S be generated by $l_k := e^{-\lambda k^\alpha}$ for $\lambda > 0$ and $\alpha > 0$. Then*

1. $\mathrm{Cap}(S) = \mathrm{Cap}([0,1]) > 0$ *for* $0 < \alpha \leq 1$ *almost surely,*
2. $\mathrm{Cap}(S) = 0$ *for* $1 < \alpha$.

Remark 5.3. The phase transition in Theorem 5.2 (and in Theorem 5.4 below) is analogous to the one observed in [30] (see eq. (5.6) and Theorem 5.6 in Section 5.1.3). It is interesting to note that in the present paper we actually establish the full capacity at the critical point $\alpha = 1$, while for the setting in [30] full capacity at the critical value $\alpha = 2$ was only a conjecture.

Theorem 5.1 and Theorem 5.2 follow from our deterministic results. Our first main deterministic result provides sufficient conditions for a G_δ set S, defined by (5.1), to have full capacity on the unit interval:

Theorem 5.3 (Sufficient conditions for full capacity). *Assume that the intervals $\{I_k\}$ from* (5.1) *have exponentially decreasing lengths $l_k = e^{-\lambda k}$ for some fixed $\lambda > 0$ and satisfy the assumptions **A.1** - **A.3** below. Then the G_δ set S has full capacity on the unit interval:*

$$\mathrm{Cap}(S) = \mathrm{Cap}([0,1]) > 0.$$

The assumptions that are imposed in this theorem, roughly speaking, state that these intervals are sufficiently uniformly placed and sufficiently well-spaced (both in terms of "average" and minimal distances between their centers).

First, we will pack intervals $\{I_k\}$ into groups with the indices from

$$\mathcal{A}_n := \{n, \dots, 2n-1\}, \tag{5.2}$$

and then pack these into larger groups:

$$\mathcal{A}_{n,q(n)} := \mathcal{A}_n \cup \mathcal{A}_{2n} \cup \dots \cup \mathcal{A}_{2^q n}, \tag{5.3}$$

where $q \in \mathbb{N}$. We will assume the following:

A.1 (distribution) The centers are *distributed* with respect to some density function $\varphi(x) \in L^1([0,1], dx)$, where $\varphi(x) > 0$ a.e.. Namely, for every $f \in C([0,1])$, we have

$$\frac{1}{\#(\mathcal{A}_n)} \sum_{k \in \mathcal{A}_n} f(c_k) \to \int_0^1 f(x)\varphi(x)\, dx \quad \text{as} \quad n \to \infty.$$

Also, there exists a sequence $q(n)$ of integer numbers, such that $q(n) \to \infty$ as $n \to \infty$ and that the following two conditions hold:

A.2 (log-**average spacing)** For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all n large enough for any $n', n'' \in \{n, 2n, \dots, 2^{q(n)}n\}$ we have

$$\frac{1}{\#(\mathcal{A}_{n'} \times \mathcal{A}_{n''})} \sum (-\log|c_i - c_j|) < \varepsilon,$$

where the sum is over $i \in \mathcal{A}_{n'}, j \in \mathcal{A}_{n''}$ such that $i \neq j$ and $|c_i - c_j| < \delta$.

A.3 (gap control) For every $\varepsilon > 0$, for all large enough n, we have that for every $i, j \in \mathcal{A}_{n,q(n)}$ with $i \neq j$ the following holds:

$$\frac{l_i + l_j}{2|c_i - c_j|} < \varepsilon.$$

Applying Theorem 5.3, we obtain our next result, a deterministic phase transition for the capacity. Again, take the centers $\{c_k\}$ that assumptions **A.1**-**A.3** are satisfied for the choice of lengths $l_k = e^{-\lambda k}$ for some fixed $\lambda > 0$ and varying the speed at which the lengths of intervals decrease, we observe a sharp phase transition in the deterministic setting:

Theorem 5.4 (Deterministic phase transition)**.** *Let the centers $\{c_k\}$ be the same centers from Theorem 5.3. Let S be generated by $l_k := e^{-\lambda k^\alpha}$ for $\alpha > 0$. Then*

1. $\mathrm{Cap}(S) = \mathrm{Cap}([0,1]) > 0$ *for* $0 < \alpha \leq 1$,
2. $\mathrm{Cap}(S) = 0$ *for* $1 < \alpha$.

The next theorem states that both the random theorems (Theorem 5.1 and Theorem 5.2) follow from the deterministic theorems (Theorem 5 and Theorem 5.4):

Theorem 5.5. *Let $\{c_k\}$ be i.i.d. with an absolutely continuous distribution on any interval J with almost everywhere positive and uniformly bounded density function. Take $l_k = e^{-\lambda k}$ for some fixed $\lambda > 0$. Then, almost surely assumptions **A.1**-**A.3** are satisfied.*

5.1.3 Motivation and historical background

In this section, we will summarize the motivation behind our project and the historical background. For full details see Chapter 2.

Gorodetski and Kleptsyn in [25, Section 1.2] studied the set of exceptional energies in the parametric version of the Furstenberg theorem. Consider

$$T_{n,\omega,a} := A_{\omega_n}(a) \ldots A_{\omega_1}(a)$$

where matrices $A_{\omega_k}(a) \in SL(2,\mathbb{R})$ are i.i.d., depending on a parameter a, taking values in some interval $J \subset \mathbb{R}$. Furstenberg's theorem implies that for every $a \in J$, for almost every

ω, we get

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_{n,\omega,a}\| = \lambda_F(a) > 0. \tag{5.4}$$

Questions on switching the quantifiers in the limit appear naturally in spectral theory, specifically, in Anderson localization proofs.

In [25, Theorem 1.5], the authors proved that almost surely switching the quantifiers leads to the occurrence of a different kind of behavior. Namely, under some technical assumptions, it was shown that for almost every ω, there exists some random *exceptional energies* subset of parameters $S_e(\omega) \subset J$ such that (5.4) does not hold. Additionally, there also exists a smaller set of parameters G_δ-set $S_0(\omega)$ such that for all $a \in S_0(\omega)$, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \|T_{n,\omega,a}\| = 0.$$

Both these sets are random G_δ's of the form (5.1).

Additionally, in [25] it was shown that the set $S_e(\omega)$ (and thus $S_0(\omega)$) have zero Hausdorff dimension. Capacity is a finer measurement than the Hausdorff dimension in the sense that any set $E \subset \mathbb{C}$ that has zero capacity must have zero Hausdorff dimension. The question as to what is the capacity of both $S_e(\omega)$ and $S_0(\omega)$ is still open. If one can show that those sets satisfy assumptions **A.1-A.3** (and this is what we conjecture), our Theorem 5.3 will imply that these sets have full capacity, that is $\mathrm{Cap}(S_e(\omega)) = \mathrm{Cap}(S_0(\omega)) = \mathrm{Cap}(J)$, in the same way as we get full capacity in the "toy model" Theorem 5.2.

The capacity of such G_δ's is also interesting because it showcases a phase transition. That is, a drastic transition from zero capacity to full capacity precisely when the series

$$\sum_k \frac{1}{|\log l_k|} \tag{5.5}$$

transitions from convergent to divergent. As we mentioned above, capacity gauges how far away a set is from being polar. Hence, as we change the speed of intervals so that the series (5.5) transitions from convergent to divergent, S goes from being polar to being as far away as possible from polar, there is no middle ground. This transition was first noticed by Kleptsyn and Quintino (see [30]) in the case when the centers $\{c_k\}$ are equidistributed in the following way: for every n we consider n equally spaced centers:

$$c_{j,n} = \frac{2j+1}{2n} \quad \text{for every} \quad j = 0, \ldots, n-1,$$

and with the restriction that the corresponding interval $J_{j,n}$ have the same length r_n for $j = 0, \ldots, n-1$. The *uniform G_δ-set* $\widetilde{S}$, corresponding to the sequence r_n, is given by

$$\widetilde{S} := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{j=0}^{n-1} J_{j,n}. \tag{5.6}$$

Any uniform G_δ set $\widetilde{S}$ may be written in the generic setting (5.1) by ordering $J_{j,n}$ and relabeling. They noticed that there is a "phase transition" in which $\widetilde{S}$ goes from having zero capacity to full capacity:

Theorem 5.6 (Phase transition [30, Theorem 1.2]). *For $r_n = e^{-n^\alpha}$,*

1. *if $\alpha > 2$, then* $\mathrm{Cap}(\widetilde{S}) = 0$,
2. *if $\alpha < 2$, then* $\mathrm{Cap}(\widetilde{S}) = \mathrm{Cap}([0,1])$.

We refer to $\alpha = 2$ as the critical case because it is precisely when the sum (5.5) transitions from convergent to divergent. Note that there are n intervals of length e^{-n^α} in (5.6), and that is why the critical case is $\alpha = 2$ in Theorem 5.6 and not $\alpha = 1$. Also, note that full capacity of $\widetilde{S}$ in the critical case in [30] was *conjectured*, but not proved; contrary to this, in the setting of the present paper the analogous statement for the critical $\alpha = 1$ is *established* (see Remark 5.3).

The zero capacity part in all the theorems above goes back to the works in the first half of twentieth century: a 1918 paper by Lindeberg [34] and 1937 by Erdös and Gillis [16] (see Chapter 3 for full details and in particular Theorem 3.1). A corollary of Theorem 3.1 is (see [30, Corollary 1.4] for the proof):

Corollary 5.1. *Let S be defined by* (5.1)*. If the series $\sum_k \frac{1}{|\log l_k|}$ converges, then the set S is of zero capacity.*

In the same 1937 paper, Erdös and Gillis [16, (C), p. 186] have mentioned a conjecture, going back to Nevanlinna's paper [36], that aimed at generalizing Theorem 3.1 to other h-volume settings. This conjecture was disproved by Ursell [49]; the re-distribution construction that was used in [30] and that we are using in the present paper can be seen as an extension of his technique.

5.1.4 Sketch of the proof and plan of the paper

In this section, we will give a sketch of the proofs and end with the plan of the paper.

The statement in the phase transition theorems 5.2 and 5.4 for $\alpha > 1$ is a result from [30] and does not require assumptions **A.1** - **A.3** (see Corollary 5.1 above).

Due to monotonocity of capacity, the statement in the phase transition theorems 5.2 and 5.4 for $0 < \alpha < 1$ follows by establishing full capacity for $\alpha = 1$. For the deterministic phase transition this is Theorem 5.3. For the random phase transition the result follows from Theorem 5.3 by showing that assumptions **A.1-A.3** hold almost surely, that is Theorem 5.5.

In Section 5.2, we will show that the centers from the random phase transition satisfy assumptions **A.1** - **A.3** for $\alpha = 1$ (Theorem 5.5). Hence, the random phase transition holds.

Thus, the main task is to show that S has full capacity for $\alpha = 1$ (Theorem 5.3). The method that we will employ to show full capacity is the *re-distribution technique* under assumptions **A.1-A.3** for $\alpha = 1$.

We introduce this technique in Section 5.3.1. Namely, we will begin with the equilibrium measure ν_J on the interval J, then we will construct a probability measure ν_1 such that the energy $I(\nu_1)$ approximates the energy $I(\nu_J)$ and whose support is a subset of $\operatorname{supp}\nu_J$ and is a finite union of intervals $\{I_k\}$. Then we will construct another probability measure ν_2 such that the energy $I(\nu_2)$ approximates the energy $I(\nu_1)$ and whose support is a subset of $\operatorname{supp}\nu_1$ and is a finite union of intervals $\{I_k\}$. Inductively, repeating this procedure, we get a sequence of probability measures that have their energies that are arbitrarily close to $I(\nu_J)$ and such that their supports create a decreasing sequence of compact subsets. After passing to the weak-limit we obtain a measure supported on S (Proposition 5), thus proving the desired full capacity for the set S (see Section 5.3.2 for the proof). Proposition 6 states that the above technique is applicable when assumptions **A.1-A.3** are satisfied. Hence, Theorem 5.3 follows from Proposition 5 and Proposition 6.

Finally, in Section 5.4 we develop the tools to prove Proposition 6. In Section 5.4.5 we conclude with the proof of Proposition 6.

5.2 In the random setting A.1-A.3 are a.s. satisfied

This section is devoted to the proof of Theorem 5.5: we assume that the centers $\{c_k\}$ are i.i.d. random variables and show that if their distributions are nice, then assumptions **A.1-A.3** are satisfied.

The distribution immediately follows from the law of large numbers:

Lemma 5.1. *Under the assumptions of Theorem 5.5, assumption **A.1** is almost surely satisfied.*

Now, take $q(n) = [\log_2(\log n)]$. The uniform gap control can be obtained by a straightforward estimate of the probability of two random centers being close to each other:

Lemma 5.2. *Under the assumptions of Theorem 5.5, for $q(n) = [\log_2(\log n)]$, assumption **A.3** is almost surely satisfied.*

Proof. Let K be the upper bound for the density of the distribution, and let $\varepsilon > 0$ be fixed. For any $i \neq j$, $i, j \in \mathcal{A}_{n,q(n)}$, if

$$\frac{l_i + l_j}{2|c_i - c_j|} < \varepsilon$$

does not hold, it implies that

$$|c_i - c_j| \leq \frac{l_i + l_j}{2\varepsilon} < \frac{1}{\varepsilon} e^{-\lambda n},$$

and the probability of such an event (for any given i and j) does not exceed $\frac{2K}{\varepsilon} e^{-\lambda n}$. As there are less than $2^{q(n)+1} n < 2n^2$ possible indices i and j, the total probability that the condition is violated for a given n does not exceed $4n^4 \cdot \frac{2K}{\varepsilon} e^{-\lambda n}$. The series

$$\sum_n 4n^4 \cdot \frac{2K}{\varepsilon} e^{-\lambda n}$$

converges, and the application of the Borel-Cantelli Lemma concludes the proof. □

Finally, the log-averages of spaces also can be controlled quite directly:

Lemma 5.3. *Under the assumptions of Theorem 5.5, for $q(n) = [\log_2(\log n)]$, assumption **A.2** is almost surely satisfied.*

Proof. Given $\varepsilon > 0$ be given and set

$$G(X,Y) = (-\log|X-Y|)\mathbb{1}_{(0,\delta)}(|X-Y|),$$

where $\delta > 0$ and $G(X,X) = 0$. We have that

$$\frac{1}{\#(\mathcal{A}_{n'} \times \mathcal{A}_{n''})} \sum_{0<|c_i-c_j|<\delta} (-\log|C_i - C_j|) = \frac{1}{\#(\mathcal{A}_{n'} \times \mathcal{A}_{n''})} \sum_{i\in\mathcal{A}_{n'}, j\in\mathcal{A}_{n''}} G(C_i, C_j).$$

Suppose the law of large numbers holds for $G(X,Y)$: as $n \to \infty$ we have

$$\frac{1}{\#(\mathcal{A}_{n'} \times \mathcal{A}_{n''})} \sum_{i\in\mathcal{A}_{n'}, j\in\mathcal{A}_{n''}} G(C_i, C_j) \to \mathbb{E}\, G(C_1, C_2),$$

where for $n', n'' \in \{n, 2n, \ldots, 2^{q(n)}n\}$. Then we may find a δ such that **A.2** holds.

The law of large numbers holds by considering the difference:

$$H(x,y) := G(x,y) - c - \mathbb{E}[G(x,y) - c|y] - \mathbb{E}[G(x,y) - c|x],$$

where $c = \mathbb{E}\, G(x,y)$ and their average

$$\frac{1}{n'n''} S_{n',n''} = \frac{1}{n'n''} \sum_{i\neq j} H(x,y). \tag{5.7}$$

Now, consider the fourth power of $S_{n',n''}$ and take its expectation:

$$\mathbb{E}(S_{n',n''})^4 = \sum \mathbb{E}[H(c_{i_1}, c_{i_2})H(c_{j_1}, c_{j_2})H(c_{k_1}, c_{k_2})H(c_{l_1}, c_{l_2})]. \tag{5.8}$$

The function $H(x,y)$ has the property that $\mathbb{E}[H(x,y)|y] = \mathbb{E}[H(x,y)|x] = 0$. Hence, if a term has an independent random variable, say c_{i_1}, then

$$\mathbb{E}[H(c_{i_1}, c_{i_2})H(c_{j_1}, c_{j_2})H(c_{k_1}, c_{k_2})H(c_{l_1}, c_{l_2})] = 0.$$

On the other hand, when every random variable is depended on another random variable, we can count the non-vanishing terms. There are $(n'n'')$ terms of the form $\mathbb{E}[H(c_{i_1}, c_{i_2})^4]$. There are $(n'n'')^2$ terms of the form

$$\mathbb{E}[H(c_{i_1}, c_{i_2})^2 H(c_{j_1}, c_{j_2})^2].$$

There are at most $(n'n'')(n' + n'')4$ terms of the form

$$\mathbb{E}[H(c_{i_1}, c_{i_2})^2 H(c_{j_1}, x)H(c_{k_1}, y)],$$

where $x, y \in \{c_{i_1}\, c_{i_2}\}$. Lastly, there are at most $n'n''(n' + n'')^2$ terms of the form

$$\mathbb{E}[H(c_{i_1}, c_{i_2})H(c_{i_1}, c_{j_2})H(c_{k_1}, c_{i_2})H(c_{k_1}, c_{j_2})].$$

Since $\mathbb{E}[H(x, y)^4] < \infty$, then

$$\mathbb{E}\, S^4_{n',n''} \leq C' \max\{(n'n'')^2, n'^3 n'', n' n''^3\},$$

where $C' > 0$ is some constant. An application of the Chebyshev inequality implies that

$$\begin{aligned}\mathbb{P}\left(|S_{n',n''}| > \varepsilon(n'n'')\right) &\leq \mathbb{E}(S_n)^4/(\varepsilon(n'n''))^4 \\ &\leq \frac{C'}{\varepsilon^4} \max\left\{\frac{1}{(n'n'')^2}, \frac{1}{n'n''^3}, \frac{1}{n'^3 n''}\right\}.\end{aligned}$$

Since $n \leq n', n''$, then

$$\mathbb{P}\left(|S_{n',n''}| > \varepsilon(n'n'')\right) \leq \frac{C'}{\varepsilon^4}\frac{1}{n^4}.$$

We have that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{n'=n}^{2^{q(n)}n}\sum_{n''=n}^{2^{q(n)}n}\mathbb{P}\left(|S_{n',n''}| > \varepsilon(n'n'')\right) &\leq \frac{C'}{\varepsilon^4}\sum_{n=1}^{\infty}\frac{2^{2q(n)}}{n^2}\\ &\leq \frac{C'}{\varepsilon^4}\sum_{n=1}^{\infty}\frac{(\log n)^2}{n^2},\end{aligned}$$

which is finite. By Borel-Cantelli lemma, $|S_{n',n''}| > \varepsilon(n'n'')$ does not occur infinitely often with probability 1. Let ε_k be a sequence of positive numbers that decreases to 0 as $k \to \infty$. For each ε_k, $|S_{n',n''}| > \varepsilon_k(n'n'')$ does not occur infinitely often with probability 1. Since the countable intersection of sets of full measure has full measure, then for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, for every $n', n'' \in \{n, 2n, \ldots, 2^{q(n)}n\}$, we have $|S_{n',n''}| < \varepsilon(n'n'')$ with probability 1. That is, the average (5.7) goes to 0 as $n \to \infty$.

□

Together, lemmas 5.1, 5.2, 5.3 imply Theorem 5.5.

5.3 The re-distribution technique

5.3.1 Introducing the technique

Our main tool for establishing full capacity for a set S (what is needed for the proof of Theorem 5.3) will be the *re-distrubtion technique* that was introduced in [30]. We will recall the method in this section. The main property that allows it to work will be the following one:

Definition 5.2. We say that S (in the generic setting (5.1)) is *re-distributable* if the following holds: for every probability measure ν with piecewise bounded continuous density that is supported on a finite collection of intervals in $[0,1]$ and for every $\varepsilon > 0$ and every $m \in \mathbb{N}$, there exists another probability measure ν' with piecewise bounded continuous density such that

1. $I(\nu') < I(\nu) + \varepsilon$,

2. ν' is supported on $\operatorname{supp} \nu \cap V_n$ for some $n \geq m$,

where V_n is a finite union of I_k's with $k \geq n$.

The following proposition then allows us to establish full capacity:

Proposition 5. *If S is re-distributable, then S has full capacity on the unit interval:*

$$\mathrm{Cap}(S) = \mathrm{Cap}([0,1]).$$

Proposition 6. *Assume **A.1** - **A.3** for interval lengths $l_k = e^{-\lambda k}$ for some $\lambda > 0$. Then the set S is re-distributable.*

Section 5.4 is devoted to the proof of Proposition 6.

5.3.2 Proof of Proposition 5

In this section, we will prove that S has full capacity when S is re-distributable.

As we have mentioned in Section 5.1.4, the proof of Proposition 5.3 is obtained by inductively constructing a sequence of measures with smaller and smaller support. Let us make these arguments formal:

Proof of Proposition 5. The density function for the equilibrium measure for the unit interval is

$$f_{[0,1]}(x) = \frac{1}{\pi\sqrt{x(1-x)}},$$

for $x \in (0,1)$ and 0 otherwise (see e.g. [47, Eq. (A.53)]). Given $\varepsilon > 0$, there exists a continuous density function f such that

$$I(f(x)dx) < I(f_{[0,1]}(x)dx) + \varepsilon.$$

Let $d\nu_0(x) := f(x)\,dx$ with support $[0,1]$. Applying Definition 5.2 to ν_0, there exists ν_1 with support V_{n_1} and

$$I(\nu_1) < I(\nu_0) + \varepsilon/2^2.$$

Apply Definition 5.2 to ν_1, there exists ν_2 with support $V_{n_1} \cap V_{n_2}$ and

$$I(\nu_2) < I(\nu_1) + \varepsilon/2^3.$$

and $n_1 < n_2$. By induction and applying Definition 5.2, for each $m \in \mathbb{N}$ there exists a Borel probability measure ν_m that is supported on

$$C_m := V_{n_1} \cap \cdots \cap V_{n_m}.$$

We consider the telescoping sum:

$$I(\nu_m) - I(\nu_0) = \sum_{i=1}^{m} \big(I(\nu_i) - I(\nu_{i-1})\big) < \sum_{i=1}^{m} \frac{\varepsilon}{2^{i+1}}.$$

It follows that

$$I(\nu_m) < I(\nu_0) + \varepsilon.$$

As in [30], any weak* limit will work. Assume that ν_∞ is a weak* limit of $\{\nu_m\}$. Passing to a weak* limit can only decrease the energy (see [40, Lemma 3.3.3]):

$$I(\nu_\infty) \le \liminf_{m\to\infty} I(\nu_m) < I(\nu_0) + \varepsilon < I(f_{[0,1]}(x)\,dx) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$I(\nu_\infty) \le I(f_{[0,1]}(x)\,dx).$$

If ν_∞ has compact support contained in S, then we are done. The weak* limit only allows us to conclude that ν_∞ has compact support contained in

$$\bar{C}_\infty := \bigcap_m \mathrm{Cl}(C_m).$$

However, $\bar{C}_\infty$ differs from

$$C_\infty := \bigcap_m C_m \subset S,$$

by at most a countable set P (the collection of boundary points of each C_m). Since $I(\nu_\infty) < \infty$, then $\nu_\infty(P) = 0$ (see [40, Theorem 3.2.3]). By regularity of Borel measures, we may find a Borel probability measure with compact support contained in C_∞ that differs from $I(\nu_\infty)$ as small as we want. Hence,

$$I(f_{[0,1]}(x)\,dx) = \inf\{I(\nu) : \nu \in \mathcal{P}(C_\infty)\}.$$

Since $C_\infty \subset S$, then S has full capacity:

$$\mathrm{Cap}(S) = \mathrm{Cap}([0,1]).$$

□

5.4 Proving Proposition 6: A.1-A.3 imply re-distribution

5.4.1 Properties of assumptions A.1-A.3

In this section, we will discuss some of the properties of assumptions **A.1-A.3** that will be needed in the proofs.

The gap control property (assumption **A.3**) is aimed at controlling the gaps between two distinct intervals in a collection of intervals in a uniform way. Let $I, I' \subset (0,1)$ be two disjoint intervals with centers c, c', then the gap between I and I' is

$$\mathrm{dist}(I, I') = |c - c'| - \frac{1}{2}(|I| + |I'|) > 0. \tag{5.9}$$

We will control the gaps by controlling the ratio of the average of the lengths and the distance between their centers (see **A.3**).

Remark 5.4. Notice that by letting $\varepsilon < 1$ in **A.3**, we get (5.9). Hence, gap control implies that the intervals in

$$\{I_k : k \in 2n, \ldots, 2^{q(n)}n - 1\},$$

are pairwise disjoint. Each measure that we construct in Section 5.4.2 will be supported on pair-wise disjoint collection:

$$\{I_k : k \in \mathcal{A}_n\} = \{I_n, I_{n+1}, \ldots, I_{2n-1}\}.$$

In Section 5.4.5, we will construct a measure that is an average of measures from Section 5.4.2. Hence, the average measure will be supported on:

$$\{I_n, I_{n+1}, \ldots, I_{2n-1}\}$$
$$\{I_{2n}, I_{2n+1}, \ldots, I_{2^2n-1}\}$$
$$\vdots$$
$$\{I_{2^{q(n)-1}n}, I_{2^{q(n)-1}n+1}, \ldots, I_{2^{q(n)}n-1}\}.$$

Assumption **A.3** allows the collection of intervals above to be disjoint.

The distribution property (assumption **A.1**) requires the centers to be *distributed* with respect to some function $\varphi(x) \in L^1([0,1],\, dx)$, where $\varphi(x) > 0$ a.e.:

$$\frac{1}{\#(\mathcal{A}_n)} \sum_{k \in \mathcal{A}_n} f(c_k) \to \int_0^1 f(x)\varphi(x)\, dx \quad \text{as} \quad n \to \infty, \tag{5.10}$$

for every continuous function $f \in C([0,1])$. This definition is a generalization of equidistributed sequences.

Remark 5.5. Note that (5.10) will hold for piecewise continuous functions f since we may approximate such functions from above and below by continuous functions in L_1. Equation (5.10) extends to 2-dimensions: Let f be any piecewise continuous function. Then

$$\frac{1}{\#(\mathcal{A}_n \times \mathcal{A}_{n'})} \sum_{i \in \mathcal{A}_n, j \in \mathcal{A}_{n'}} f(c_i) f(c_j) \to \int_0^1 \int_0^1 f(x) f(y) \varphi(x) \varphi(y)\, dx dy, \tag{5.11}$$

as $m \to \infty$ and $n, n' \geq m$.

To show that S is re-distributable (see Definition 5.2), we will show that the following statement holds:

P.1 For each positive continuous function f on the interval $[0,1]$, there exists a sequence of probability measures $\{\mu^n\}$ so that each μ^n has a piecewise bounded continuous density with support contained in a finite union of disjoint I_k's with $k \geq n$ and with asymptotic behavior:

$$I(\mu^n) = \frac{I(f(x)\,dx)}{(\int_0^1 f(x)\,dx)^2} + o(1).$$

With the distribution assumption **A.1** and log-average spacing assumption **A.2**, we can see that the centers have the asymptotic behavior that is needed in **P.1**:

Lemma 5.4. *Under assumptions **A.1** and **A.2**, for every $f \in C([0,1])$, as $n \to \infty$ and $n \leq n', n''$, we have that*

$$\frac{1}{\#(\mathcal{A}_{n'} \times \mathcal{A}_{n''})} \sum_{i \neq j} (-\log|c_i - c_j|) f(c_i) f(c_j) \to I(f(x)\varphi(x)\,dx), \tag{5.12}$$

where the sum is taken over $(i,j) \in \mathcal{A}_{n'} \times \mathcal{A}_{n''}$ and $i \neq j$. Moreover,

$$I(f(x)\varphi(x)\,dx) < \infty.$$

Remark 5.6. The density function $\varphi(x)$ in assumption **A.1** is not to be confused with the continuous density function f in Definition 5.2 and in **P.1**. Once the centers are distributed with respect to $\varphi(x)$, the function $\varphi(x)$ is fixed. The continuous density function f in Definition 5.2 and in **P.1** is arbitrary.

Proof of Lemma 5.4. Given $\varepsilon > 0$, let $\delta > 0$ satisfy assumption **A.2**. For $s > 0$, define

$f_s(x) = -\log x$ for $x \geq s$ and 0 otherwise. Using Fatou's lemma, we get

$$\iint_{|x-y|<\delta} (-\log|x-y|)\varphi(x)\varphi(y)\,dx\,dy \leq \liminf_{s\to 0^+} \iint_{|x-y|<\delta} f_s(|x-y|)\varphi(x)\varphi(y)\,dx\,dy.$$

Using assumption **A.1** for $n', n'' \in \{n, 2n, \ldots, 2^{q(n)}n\}$, we have

$$\begin{aligned}\liminf_{s\to 0^+} \iint_{|x-y|<\delta} f_s(|x-y|)\varphi(x)\varphi(y)\,dx\,dy =& \liminf_{s\to 0^+} \lim_{n\to\infty} \sum_{0<|c_i-c_j|<\delta} \frac{f_s(|c_i-c_j|)}{\#(\mathcal{A}_{n'}\times\mathcal{A}_{n''})} \\ \leq& \lim_{n\to\infty} \sum_{0<|c_i-c_j|<\delta} \frac{(-\log|c_i-c_j|)}{\#(\mathcal{A}_{n'}\times\mathcal{A}_{n''})} \\ \leq& \varepsilon,\end{aligned}$$

where the last holds by assumption **A.2** for some $\delta > 0$. Hence, for every continuous function f, we have finite energy:

$$I(f(x)\varphi(x)\,dx) = \iint (-\log|x-y|)f(x)\varphi(x)f(y)\varphi(y)\,dx\,dy < \infty.$$

Note that

$$\begin{aligned}\sum_{i\neq j} \frac{(-\log|c_i-c_j|)f(c_i)f(c_j)}{\#(\mathcal{A}_{n'}\times\mathcal{A}_{n''})} - \sum_{i\neq j} \frac{(-\log|c_i-c_j|)f_\delta(c_i)f_\delta(c_j)}{\#(\mathcal{A}_{n'}\times\mathcal{A}_{n''})} \\ = \sum_{0<|c_i-c_j|<\delta} \frac{(-\log|c_i-c_j|)f(c_i)f(c_j)}{\#(\mathcal{A}_{n'}\times\mathcal{A}_{n''})}.\end{aligned}$$

Let $n \to \inf$, then by assumptions **A.1** and **A.2**, we have

$$\left| \lim_{n\to\infty} \sum_{i\neq j} \frac{(-\log|c_i-c_j|)f(c_i)f(c_j)}{\#(\mathcal{A}_{n'}\times\mathcal{A}_{n''})} - \iint f_\delta(|x-y|)f(x)f(y)\varphi(x)\varphi(y) \right| \\ \leq \varepsilon \cdot (\max|f|)^2.$$

By letting $\delta \to 0$, we have that

$$\left| \lim_{n\to\infty} \sum_{i\neq j} \frac{(-\log|c_i - c_j|)f(c_i)f(c_j)}{\#(\mathcal{A}_{n'} \times \mathcal{A}_{n''})} - I(f(x)\varphi(x)\,dx) \right| \leq \varepsilon \cdot \max|f|.$$

Since $\varepsilon > 0$ is arbitrary, we get (5.12).

□

5.4.2 Construction of a single-level re-distribution

Our first step in constructing the probability measures in **P.1** is to construct a *single-level re-distribution* probability measure. This section is devoted to the construction of such probability measures.

We begin with a "re-distribution" type of measure $f(x)\,dx|_{[0,1]}$ onto a single interval:

$$\mu_k = \frac{f(x)\,dx|_{I_k}}{|I_k|}.$$

We do not call this a re-distribution as in [30] because the measure is not necessarily a probability measure. We consider the average of μ_k's:

$$\mu_{\mathcal{A}_n} := \frac{1}{\#\mathcal{A}_n} \sum_{k\in\mathcal{A}_n} \mu_k,$$

where $\mathcal{A}_n$ are defined in (5.2) and satisfy the gap control property **A.3**. Recall that gap control implies that for large enough n, our collection of intervals are disjoint (see Remark 5.4). Notice that each measure $\mu_{\mathcal{A}_n}$ is not necessarily a probability measure. To correct that, let us define

$$V_n = \bigcup_{k\in\mathcal{A}_n} I_k.$$

Then, we consider the *single-level re-distribution* probability measure:

$$\hat{\mu}_n := \frac{\mu_{\mathcal{A}_n}}{\mu_{\mathcal{A}_n}(V_n)}, \tag{5.13}$$

which is supported on V_n and it's energy is

$$I(\hat{\mu}_n) = \frac{1}{(\mu_{\mathcal{A}_n}(V_n))^2} I(\mu_{\mathcal{A}_n}).$$

Thus, we are interested in the asymptotic behavior of $\mu_{\mathcal{A}_n}(V_n)$ and the asymptotic behavior of:

$$I\left(\mu_{\mathcal{A}_n}\right) = \frac{1}{(\#\mathcal{A}_n)^2} \sum_{k \in \mathcal{A}_n} I(\mu_k) + \frac{1}{(\#\mathcal{A}_n)^2} \sum_{i \neq j} I(\mu_i, \mu_j). \tag{5.14}$$

The first sum is referred to as the *self-interaction* sum because in $I(\mu_k) = I(\mu_k, \mu_k)$ the same measure is interacting with itself. The second sum is referred to as *outer-interaction* sum because we have two measures with disjoint supports interacting with each other in $I(\mu_k, \mu_j)$.

In Section 5.4.3, we will discuss the asymptotic behavior of the outer-interaction and in Section 5.4.4 we will work on controlling the asymptotic behavior of the self-interaction sum. In Section 5.4.5, we will put the two together. We will finish the section with the asymptotic behavior of $\mu_{\mathcal{A}_n}(V_n)$:

Lemma 5.5. *If **A.1** and **A.3** hold, then*

$$\mu_{\mathcal{A}_n}(V_n) = \int f(x)\varphi(x)\,dx + o(1).$$

Proof. Since for large n the intervals in

$$\{I_k : k \in \mathcal{A}_n\}$$

are disjoint due to **A.3** (see Remark 5.4), then $\mu_k(V_n) = \frac{1}{|I_k|}\int_{I_k} f(x)\,dx$. Hence,

$$\mu_{\mathcal{A}_n}(V_n) = \frac{1}{\#\mathcal{A}_n}\sum_{k\in\mathcal{A}_n}\frac{1}{|I_k|}\int_{I_k} f(x)\,dx.$$

Due to the uniform continuity of f on the interval $[0,1]$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{if} \quad |x-y| < \delta.$$

Since the lengths of the intervals I_k approach 0, then there exists $N \in \mathbb{N}$ such that for every $k \geq N$ we have

$$|f(x) - f(c_i)| < \varepsilon \quad \text{if} \quad x \in I_k.$$

Therefore, for every $n \geq N$ and every $k \in \mathcal{A}_n$, we have

$$|f(x) - f(c_k)| < \varepsilon \quad \text{if} \quad x \in I_k.$$

It follows that for large enough n, we have

$$\left|\mu_{\mathcal{A}_n}(V_n) - \frac{1}{\#\mathcal{A}_n}\sum_{i\in\mathcal{A}_n} f(c_i)\right| < \varepsilon.$$

As the centers are distributed with respect to $\varphi(x)$ **A.1**, it follows that

$$\left|\mu_{\mathcal{A}_n}(V_n) - \left(\int f(x)\varphi(x)\,dx + o(1)\right)\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, then the result holds. □

5.4.3 Asymptotic behavior of outer-interaction

In this section, we are interested in the asymptotic behavior of the outer-interaction sum:

$$\frac{1}{(\#\mathcal{A}_n)^2}\sum_{i\neq j} I(\mu_i,\mu_j),$$

where the sum is taken over $i,j\in\mathcal{A}_n$ and $i\neq j$. It is the outer-interaction sum that gives the limit point in **P.1**:

Lemma 5.6. *We have*

$$\frac{1}{(\#\mathcal{A}_n)^2}\sum_{i\neq j} I(\mu_i,\mu_j) = I(f(x)\varphi(x)dx) + o(1), \tag{5.15}$$

where $i,j\in\mathcal{A}_n$.

To show Lemma 5.6, we want to estimate $(-\log|x-y|)$ by $(-\log|c-c'|)$ where x and y are in intervals with centers c and c', respectively. The next lemma allows us to do that.

Lemma 5.7. *Let f be any continuous function on $[0,1]$ and let J, J' be two disjoint intervals in $[0,1]$ with lengths r, r' and centers c, c', respectively. Define*

$$\mu := \frac{1}{r}\, f(x)\,dx|_J \quad \textit{and} \quad \mu' := \frac{1}{r'}\, f(x)\,dx|_{J'}\,.$$

Let $\varepsilon>0$. If

$$\frac{r+r'}{2|c-c'|} \leq (1-e^{-\varepsilon}), \tag{5.16}$$

then

$$|I(\mu,\mu') - (-\log|c-c'|)f(c)f(c')| \leq (2K(-\log|c-c'|)+K^2)\varepsilon,$$

where $K = \|f\|_\infty$.

Proof. Let us first prove that for $a, b > 0$, if $|a-b| \le b(1-e^{-\varepsilon})$, then

$$|\log a - \log b| \le \varepsilon.$$

The above inequality holds if and only if

$$-\varepsilon \le \log(a/b) \le \varepsilon,$$

if and only if

$$be^{-\varepsilon} \le a \le be^{\varepsilon},$$

if and only if

$$-b(1-e^{-\varepsilon}) \le a-b \le b(e^{\varepsilon}-1).$$

Since $(1-e^{-\varepsilon}) \le (e^{\varepsilon}-1)$, then

$$|\log a - \log b| \le \varepsilon$$

holds when

$$|a-b| \le b(1-e^{-\varepsilon}).$$

For any two disjoint intervals $J, J' \in [0,1]$ with centers c, c' and with lengths r, r' respectively, we have

$$||x-y|-|c-c'|| \leq |(x-c)+(c'-y)| \leq \frac{r+r'}{2}.$$

Since assumption (5.16) is equivalent to

$$\frac{r+r'}{2} \leq |c-c'|(1-e^{-\varepsilon}),$$

then

$$|(-\log|x-y|)-(-\log|c-c'|)| < \varepsilon.$$

Since f is uniformly continuous on $[0,1]$, then there exists $\delta > 0$ such that $|f(a)-f(b)| < \varepsilon$ if $|a-b| < \delta$. If $0 < r, r' < \delta$, we have that $|f(x)-f(c)| < \varepsilon$ and $|f(y)-f(c')| < \varepsilon$. We would like to combine the three inequalities.

Suppose $A, a, B, b \in \mathbb{R}$ and $\varepsilon_a, \varepsilon_b > 0$ such that

$$|A-a| < \varepsilon_a \quad \text{and} \quad |B-b| < \varepsilon_b.$$

We have that

$$|AB-ab| \leq |AB-Ab|+|Ab-ab| \leq (|A|\varepsilon_b+|b|\varepsilon_a).$$

One application of the above gives

$$|f(x)f(y)-f(c)f(c')| \leq 2K\varepsilon,$$

where $K = \|f\|_\infty$. A third application yields

$$|(-\log|x-y|)f(x)f(y) - (-\log|c-c'|)f(c)f(c')| < (2K\varepsilon(-\log|c-c'|) + K^2\varepsilon).$$

Integrating by

$$\frac{1}{r}\, dx|_J \quad \text{and} \quad \frac{1}{r'}\, dy|_{J'}$$

finishes the proof. □

Now that we can estimate $(-\log|x-y|)$ by $(-\log|c-c'|)$ where x and y are in intervals with centers c and c', respectively, we are ready to estimate

$$\frac{1}{(\#\mathcal{A}_n)^2}\sum_{i\neq j} I(\mu_i, \mu_j),$$

by

$$\frac{1}{(\#\mathcal{A}_n)^2}\sum_{i\neq j}(-\log|c_i - c_j|)f(c_i)f(c_j).$$

Let us go back to Lemma 5.6 and prove the asymptotic behavior of the outer-interaction:

Proof of Lemma 5.6. Let $\epsilon > 0$ be given and let $K = \|f\|_\infty$. The gap control **A.3** guarantees that there exists $N \in \mathbb{N}$ such that for every $n \geq N$ and every $i \neq j$, where $i, j \in \mathcal{A}_n$, we have

$$\frac{l_i + l_j}{2|c_i - c_j|} \leq (1 - e^{-\varepsilon}),$$

which is the condition (5.16) in Lemma 5.7. Since l_k decrease to 0 as $k \to \infty$, then for all large enough n we may apply Lemma 5.7 to get

$$|I(\mu_i,\mu_j) - (-\log|c_i - c_j|)f(c_i)f(c_j)| \leq (2K(-\log|c_i - c_j|) + K^2)\varepsilon,$$

for every $n \geq N$ and every $i \neq j$, where $i, j \in \mathcal{A}_n$. Adding this up for $i \neq j$ where $i, j \in \mathcal{A}_n$ and then dividing by $(\#\mathcal{A}_n)^2$, gives us:

$$\begin{aligned} \left| \frac{1}{(\#\mathcal{A}_n)^2} \sum_{i \neq j} I(\mu_i,\mu_j) - \frac{1}{(\#\mathcal{A}_n)^2} \sum_{i \neq j} (-\log|c_i - c_j|)f(c_i)f(c_j) \right| \\ \leq \frac{2K\varepsilon}{(\#\mathcal{A}_n)^2} \sum_{i \neq j} (-\log|c_i - c_j|) + K^2\varepsilon. \end{aligned}$$

We will apply Lemma 5.4 twice to the last two sums. We apply the lemma to the last sum by taking $f = 1$ in Lemma 5.4, and then we apply the lemma again for arbitrary f in Lemma 5.4 to get:

$$\begin{aligned} \left| \lim_{n\to\infty} \frac{1}{(\#\mathcal{A}_n)^2} \sum_{i \neq j} I(\mu_i,\mu_j) - I(f(x)\varphi(x)dx) \right| \\ \leq (2KI(\varphi(x)dx) + K^2)\varepsilon. \end{aligned}$$

Lemma 5.4 also informs us that the energy of $\varphi(x)dx$ is finite, hence $0 < (2KI(\varphi(x)dx) + K^2)\varepsilon < \infty$. As $\varepsilon > 0$ is arbitrary, it follows that

$$\frac{1}{(\#\mathcal{A}_n)^2} \sum_{i \neq j} (-\log|c_i - c_j|)f(c_i)f(c_j) \to I(f(x)\varphi(x)dx).$$

Lemma 5.6 holds.

□

5.4.4 Asymptotic behavior of self-interaction

In this section, we will to control the self-interaction:

Lemma 5.8. *If* $l_k = e^{-\lambda k}$ *where* $\lambda > 0$*, then*

$$\frac{1}{(\#\mathcal{A}_n)^2} \sum_{k \in \mathcal{A}_n} I(\mu_k) = O\left(\frac{1}{(\#\mathcal{A}_n)^2} \sum_{k \in \mathcal{A}_n} k + o(1) \right).$$

Proof. By shifting and a change of variables, we get

$$I(\frac{1}{l_k} dx|_{I_i}) = -\log l_k + I(dx|_{[0,1]}) = -\log l_k \cdot (1 + o(1)).$$

If $l_k = e^{-\lambda k}$, then adding the above over $k \in \mathcal{A}_n$ gives us our result. □

If the self-interaction sum vanishes in the limit, then we will be able to finish the proof with a single-level re-distribution. Let us see what the self-interaction tells us:

Lemma 5.9. *If* $\mathcal{A}_n := \{n, \ldots, p(n) - 1\}$ *where* $p(n)$ *is an integer-valued function such that* $p(n) \geq 2n$*, then*

$$\frac{1}{(\#\mathcal{A}_n)^2} \sum_{k \in \mathcal{A}_n} k = \frac{p(n) + n - 1}{2(p(n) - n)}.$$

If $p(n) >> n$*, then the right-hand side is close to* $\frac{1}{2}$*.*

Proof. Let

$$S = \sum_{k \in \mathcal{A}_n} k.$$

Then the arithmetic sum becomes

$$S = \frac{(p(n) + n - 1)(\#\mathcal{A}_n)}{2}.$$

Since $\#\mathcal{A}_n = p(n) - n$, then dividing by $(\#\mathcal{A}_n)^2$ we get what we want. □

Remark 5.7. Lemma 5.9 tells us that no matter how many intervals are included in our single-level of re-distribution, the self-interaction sum will never vanish.

But since we can bound the self-interaction sum uniformly for all n, we will be able to apply a multi-level re-distribution in Section 5.4.5. That is, we will take the average of measures $\hat{\mu}_n$ to handle the self-interaction sum.

5.4.5 The proof of Proposition 6

In this section, we will use a multi-level re-distribution to show **P.1** holds and prove Proposition 6.

Let us first see where the asymptotic behavior of a single-level re-distribution leads:

Proposition 7 (Single-level re-distribution)**.** *Let $l_k := e^{-\lambda k}$ and $\lambda > 0$. If assumptions* ***A.1-A.3*** *are satisfied, then*

$$I(\hat{\mu}_n) = \frac{I(f(x)\varphi(x)\,dx)}{(\int f(x)\varphi(x)\,dx)^2} + O\left(\frac{1}{(\int f(x)\varphi(x)\,dx)^2}\right) + o(1) = O(1),$$

where each $\hat{\mu}_n$ is the corresponding measure defined in (5.13)*.*

Proof. We have that

$$I(\hat{\mu}_n) = \frac{1}{(\mu_{\mathcal{A}_n}(V_n))^2} I(\mu_{\mathcal{A}_n}).$$

Breaking down the last, we get

$$I\left(\mu_{\mathcal{A}_n}\right) = \frac{1}{(\#\mathcal{A}_n)^2} \sum_{k\in\mathcal{A}_n} I(\mu_k) + \frac{1}{(\#\mathcal{A}_n)^2} \sum_{i\neq j} I(\mu_i, \mu_j).$$

Applying Lemma 5.6 to the outer-interaction sum and Lemma 5.9 to the self-interaction sum, and lastly, applying Lemma 5.5 to the normalization $\mu_{\mathcal{A}_n}(V_n)$ completes the proof.

$\square$

Remark 5.7 tells us that using a single level re-distribution will not render S to be re-distributable no matter how many intervals are included in $\{I_k : k \in \mathcal{A}_n\}$. We will need to take the average of $q(n)$ single-level re-distribution measures, where $q(n)$ is an integer-valued function such that $q(n) \to \infty$ as $n \to \infty$.

Let $\hat{\mu}_n$ be a single-level re-distribution as defined in (5.13). For each n, we consider a *multi-level re-distribution* probability measure:

$$\mu^m := \frac{1}{\#\mathcal{B}_m} \sum_{s \in \mathcal{B}_m} \hat{\mu}_{2^s m},$$

where

$$\mathcal{B}_m := \{0, \dots, q(m) - 1\}.$$

Since each $\hat{\mu}_n$ is supported on

$$V_n := \bigcup_{k \in \mathcal{A}_n} I_k = \bigcup_{k=n}^{2n-1} I_k,$$

then μ^m is supported on

$$V_n, V_{2n}, V_{2^2 n}, \dots, V_{2^{q(n)-1} n}.$$

See Remark 5.4 for details.

Our convex measure can now be partitioned into a new self-interaction sum and a new outer-interaction sum:

$$I(\mu^m) = \frac{1}{(\#\mathcal{B}_m)^2} \sum_{s\in\mathcal{B}_m} I(\hat{\mu}_{2^s m}) + \frac{1}{(\#\mathcal{B}_m)^2} \sum_{\substack{s,t\in\mathcal{B}_m \\ s\neq t}} I(\hat{\mu}_{2^s m}, \hat{\mu}_{2^t m}).$$

Proposition 7 tells us that

$$I(\hat{\mu}_n) = O(1).$$

Hence,

$$\begin{aligned}\frac{1}{(\#\mathcal{B}_m)^2} \sum_{s\in\mathcal{B}_m} I(\hat{\mu}_{2^s m}) =& \frac{1}{(\#\mathcal{B}_m)^2} \sum_{s\in\mathcal{B}_m} O(1) \\ \leq& \frac{O(1)}{(\#\mathcal{B}_m)} \to 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

That is, the self-interaction sum vanishes. The outer-interaction sum gives what we aim:

Lemma 5.10. *Assume* ***A.1-A.3****. As $m \to \infty$ we have that*

$$\frac{1}{(\#\mathcal{B}_m)^2} \sum_{s\neq t} I(\hat{\mu}_{2^s m}, \hat{\mu}_{2^t m}) \to \frac{I(f(x)\varphi(x)\,dx)}{(\int f(x)\varphi(x)\,dx)^2},$$

where the sum is over $s \neq t$ and $s,t \in \mathcal{B}_m$.

We will leave the proof of Lemma 5.10 to the end of the section. Note that the vanishing of the self-interaction sum and Lemma 5.10 gives us:

Proposition 8. *For every $f \in C([0,1])$, we have that*

$$I(\mu^m) = \frac{I(f(x)\varphi(x)\,dx)}{(\int f(x)\varphi(x)\,dx)^2} + o(1).$$

Notice that with Proposition 8 we can show that S is re-distributable when $\varphi(x) \equiv 1$. In order to remove $\varphi(x)$, we will need to apply Proposition 8 to a continuous approximation of

$1/\varphi(x)$ and take an appropriate subsequence:

Proposition 9. *Suppose for each continuous function f, there exists a sequence of probability measures $\{\mu^n\}$ so that each μ^n has a piecewise continuous density with support V_n, where V_n is a finite union of disjoint I_k's with $k \geq n$ with asymptotic behavior:*

$$I(\mu^n) = \frac{I(f(x)\varphi(x)dx)}{(\int f(x)\varphi(x)\,dx)^2} + o(1). \tag{5.17}$$

*Then property **P.1** holds.*

Let us go back to show that S is re-distributable using a multi-level re-distribution before we prove Proposition 9.

Proof of Proposition 6 . Since $\varphi(x) > 0$ almost everywhere, then combining Proposition 8 and Proposition 9 shows **P.1** holds.

Given any probability measure ν with piecewise continuous density that is supported on a finite collection of intervals in $[0, 1]$ and given any $\varepsilon > 0$, we may apply **P.1** to a continuous L^1 approximation of the density function of ν to show that there exists a probability measure ν' satisfying properties (1) and (2) in Definition 5.2. Thus, S is re-distributable. □

Now, let us analyze the asymptotic behavior of the new outer-interaction sum:

Proof of Lemma 5.10 . The goal is to show that

$$I(\hat{\mu}_n, \hat{\mu}_{n'}) = \frac{I(\mu_{\mathcal{A}_n}, \mu_{\mathcal{A}_{n'}})}{\mu_{\mathcal{A}_n}(V_n) \cdot \mu_{\mathcal{A}_{n'}}(V_{n'})} \to \frac{I(f(x)\varphi(x)\,dx)}{(\int f(x)\varphi(x)\,dx)^2},$$

as $m \to \infty$ and independently of our choice of $n, n' \in \{2^s m : s \in \mathcal{B}_m\}$. Once we accomplish this, then

$$\frac{1}{(\#\mathcal{B}_m)^2}\sum_{s\neq t} I(\hat{\mu}_{2^s m},\hat{\mu}_{2^t m}) \to \frac{I(f(x)\varphi(x)\,dx)}{(\int f(x)\varphi(x)\,dx)^2},$$

as $m\to\infty$.

Lemma 5.5 shows that

$$\frac{1}{\mu_{\mathcal{A}_n}(V_n)\cdot\mu_{\mathcal{A}_{n'}}(V_{n'})} \to \frac{1}{(\int f(x)\varphi(x)\,dx)^2},$$

as $m\to\infty$ and independently of our choice of $n, n'\in\{2^s m : s\in\mathcal{B}_m\}$.

Let us focus on $I(\mu_{\mathcal{A}_n},\mu_{\mathcal{A}_{n'}})$. Given $n\neq n'$ where $n,n'\in\{2^s m : s\in\mathcal{B}_m\}$, we have that

$$I(\mu_{\mathcal{A}_n},\mu_{\mathcal{A}_{n'}}) = \frac{1}{(\#\mathcal{A}_n)(\#\mathcal{A}_{n'})}\sum_{i\neq j} I(\mu_i,\mu_j), \tag{5.18}$$

where $(i,j)\in\mathcal{A}_n\times\mathcal{A}_{n'}$. By the gap control assumption **A.3**, we know that for all large enough m and every $i\neq j$, where $i,j\in\{m,\dots,2^{q(m)}m-1\}$, we have

$$\frac{l_i+l_j}{2|c_i-c_j|}\leq(1-e^{-\varepsilon}),$$

which is the needed condition (5.16) to apply Lemma 5.7 for all large enough m. Lemma 5.7 gives us

$$|I(\mu_i,\mu_j)-(-\log|c_i-c_j|)f(c_i)f(c_j)|\leq(2K(-\log|c_i-c_j|)+K^2)\varepsilon,$$

for every $i\neq j$ where $i,j\in\{m,\dots,2^{q(m)}m-1\}$ and for all large enough m. Adding this up over $(i,j) \in \mathcal{A}_n \times \mathcal{A}_{n'}$ and then dividing by $(\#\mathcal{A}_n)(\#\mathcal{A}_{n'})$ gives us:

$$\left|\frac{1}{(\#\mathcal{A}_n)(\#\mathcal{A}_{n'})}\sum_{i\neq j} I(\mu_i,\mu_j) - \frac{1}{(\#\mathcal{A}_n)(\#\mathcal{A}_{n'})}\sum_{i\neq j}(-\log|c_i-c_j|)f(c_i)f(c_j)\right|$$
$$\leq \frac{2K\varepsilon}{(\#\mathcal{A}_n)(\#\mathcal{A}_{n'})}\sum_{i\neq j}(-\log|c_i-c_j|)$$
$$+K^2\varepsilon.$$

We remark that the inequality holds independently of our choice of $n, n' \in \{2^s m : s \in \mathcal{B}_m\}$ for all large enough m. By the distribution assumption **A.1** and the log-average spacing assumption **A.2**, we can apply Lemma 5.4 twice to the last two sums above. One application yields:

$$\frac{1}{(\#\mathcal{A}_n)(\#\mathcal{A}_{n'})}\sum_{i\neq j}(-\log|c_i-c_j|)f(c_i)f(c_j) \to I(f(x)\varphi(x)\,dx),$$

as $m \to \infty$ with $n, n' \geq m$. For the second application we take $f = 1$ in Lemma 5.4 to get

$$\frac{1}{(\#\mathcal{A}_n)^2}\sum_{i\neq j}(-\log|c_i-c_j|) \to I(\varphi(x)\,dx) < \infty,$$

as $m \to \infty$ with $n, n' \geq m$. Therefore, for $n, n' \in \{2^s m : s \in \mathcal{B}_m\}$, we have

$$\left|\lim_{m\to\infty} I(\mu_{\mathcal{A}_n},\mu_{\mathcal{A}_{n'}}) - I(f(x)\varphi(x)\,dx)\right|$$
$$\leq 2K\varepsilon I(\varphi(x)\,dx) + K^2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $I(\varphi(x)\,dx) < \infty$, then for $n, n' \in \{2^s m : s \in \mathcal{B}_m\}$, we have

$$\left|\lim_{m\to\infty} I(\mu_{\mathcal{A}_n},\mu_{\mathcal{A}_{n'}}) - I(f(x)\varphi(x)\,dx)\right| = 0.$$

Therefore, as $m \to \infty$, then

$$I(\hat{\mu}_n, \hat{\mu}_{n'}) \to \frac{I(f(x)\varphi(x)\,dx)}{(\int f(x)\varphi(x)\,dx)^2},$$

where $n, n' \in \{2^s m : s \in \mathcal{B}_m\}$, which completes the proof. □

Proof of Proposition 9 . For every continuous function h, set

$$f(x) := \frac{h(x)}{\varphi(x)},$$

when $\varphi \neq 0$ and 0 otherwise. For each $\varepsilon > 0$, there exists a continuous function f' such that

$$\left| \frac{I(f_0(x)\varphi(x)\,dx)}{(\int_0^1 f_0(x)\varphi(x)\,dx)^2} - \frac{I(f(x)\varphi(x)\,dx)}{(\int_0^1 f(x)\varphi(x)\,dx)^2} \right| < \varepsilon/2.$$

Applying (5.17) to f_0, gives us that for each $\varepsilon > 0$, there exists N such that for every $n \geq N$ we have

$$\left| \frac{I(f_0(x)\varphi(x))}{(\int_0^1 f_0(x)\varphi(x)\,dx)^2} - I(\mu^n) \right| < \varepsilon/2,$$

where each μ^n is a probability measure with a piecewise continuous density with support in V_n, where V_n is a finite unions of disjoint I_k's with $k \geq n$. Since $f(x)\varphi(x) = h(x)$ a.e., then for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ as large as we need such that

$$\begin{aligned} \left| I(\mu^n) - \frac{I(h(x)\,dx)}{(\int_0^1 h(x)\,dx)^2} \right| \leq & \left| I(\mu^n) - \frac{I(f_0(x)\varphi(x))}{(\int_0^1 f_0(x)\varphi(x)\,dx)^2} \right| \\ & + \left| \frac{I(f_0(x)\varphi(x))}{(\int_0^1 f_0(x)\varphi(x)\,dx)^2} - \frac{I(f(x)\varphi(x))}{(\int_0^1 f(x)\varphi(x)\,dx)^2} \right| \\ < & \varepsilon. \end{aligned}$$

Hence, there exists a subsequence n_k and probability measures μ^{n_k} with piecewise continuous density supported in V_{n_k} such that

$$I(\mu^{n_k}) = \frac{I(h(x)\,dx)}{(\int_0^1 h(x)\,dx)^2} + o(1). \tag{5.19}$$

Each μ^{n_k} is a probability measure with a piecewise continuous density with support contained in V_{n_k}, that is a finite union of disjoint I_j's with $j \geq k$. Hence, these $\{\mu^{n_k}\}$ satisfy **P.1**. $\square$

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