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SCATTERING FOR A GENERALIZED DISPERSION MODEL

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Berkeley, California

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BOUNDS FOR THE CORRECTION TO THE BORN TERM
AND APPLICATIONS TO p - p SCATTERING
FOR A GENERALIZED DISPERSION MODEL

Judith Binstock
(Ph. D. Thesis)

May 3, 1967

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ABSTRACT

The unitarizing corrections to the Born term, as given by the Chew-Arndt, MacGregor-Arndt (j_0 even), and Scotti-Wong models, are shown to be bounded above and below. The bounds follow from the mathematical forms of the models, to all of which forms a theorem (proved in this paper) applies. The correction bounds are given for several p-p amplitudes for the range 0 to 300 MeV. The results are shown to explain why the apparently different models mentioned above give similar Born terms after subtraction of the correction from the experimentally determined (p-p) amplitude. A generalization of these models for the unitarizing correction is proposed, which has properties leading to partial-wave dispersion relations, and which has a specified relationship between the asymptotic behavior and the fluctuation of the sign of the left-hand discontinuity. Upper and lower bounds are found for the correction term prescribed by this generalized model (which is not restricted, in its application, to nucleon-nucleon scattering). Also given is a theorem for the full partial-wave amplitude, relating the number of sign changes of the left-hand discontinuity to the amplitude in the physical region. The theorem is applied as a consistency condition to some low partial-wave p-p amplitudes.

1. INTRODUCTION

Part 2 consists of a treatment of various models for the unitarizing correction to the Born term. Part 3 treats the full partial-wave amplitude. It is concerned with the relation between the number of sign changes of the left-hand discontinuity and the amplitude in the physical region. Each part contains its own introduction to the material.

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2. Bounds for the Correction to the Born Term and Applications to p - p Scattering for a Generalized Dispersion Model*

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The unitarizing corrections to the Born term, as given by the Chew-Arndt, MacGregor-Arndt (j_0 even), and Scotti-Wong models, are shown to be bounded above and below. The bounds follow from the mathematical forms of the models, to all of which forms a theorem (proved in this paper) applies. The correction bounds are given for several p - p amplitudes for the range 0 to 300 MeV. The results are shown to explain why the apparently different models mentioned above give similar Born terms after subtraction of the correction from the experimentally determined (p - p) amplitude. A generalization of these models for the unitarizing correction is proposed, which has properties leading to partial-wave dispersion relations, and which has a specified relationship between the asymptotic behavior and the fluctuation of the sign of the left-hand discontinuity. Upper and lower bounds are found for the correction term prescribed by this generalized model (which is not restricted, in its application, to nucleon-nucleon scattering).

I. INTRODUCTION

In nucleon-nucleon scattering, some attempts (1-3) utilizing dispersion relations have been made to fit the experimental phase shifts with a sum of one-meson-exchange terms plus a small correction term. The sum of one-meson-exchange terms is referred to as the Born term. In these attempts the only restrictions observed in choosing the form of the correction term are proper threshold behavior and reasonable asymptotic behavior in the lab kinetic-energy variable T . That these restrictions are not enough to place bounds on the correction term has been shown by Moravesik (4).¹ It was therefore not obvious

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¹ However, Moravesik's objection in (4) to the Scotti-Wong model (1) can easily be obviated by considering the unsubtracted dispersion relation to include an additional constant representing the contribution from the contour at infinity, so that the asymptotic behavior of the model may be acceptable. Also, his objection to the Kantor approximation (3) does not apply if this approximation is considered as a limiting case; because even though the assumptions rule out zero as the contribution of the left-hand integral of the correction term, they do not rule out an arbitrarily small value.

why three apparently dissimilar schemes for the calculation of the unitarizing correction term do in fact give similar results, as was pointed out by Arndt (2). In Section III of this paper it is shown that the three models in question are really special cases of a more general type of model, to which the theorem in Appendix A applies. The generalized model, not restricted to nucleon-nucleon scattering (but restricted to amplitudes connecting states whose quantum numbers exclude bound states and resonances in the direct channel, and also excluding s -waves), is given in Section II. In that section it is shown that the five properties of the generalized model are sufficient to determine bounds on the contribution to the correction from the unphysical cut. These bounds are plotted in Appendix B for the p - p amplitudes 3P_1 , 3P_2 , 1D_2 , E_2 , 3F_2 , 3F_3 , 3F_4 , 1G_4 , and E_4 for the three different models considered in Section III. As a result, the similarity in the structures of the different p - p corrections (calculated from the three different schemes) and the smallness of the real part of each of the corrections (relative to the real part of the experimental amplitude) are shown to be predetermined by the form of the correction scheme.

II. A GENERALIZED SCATTERING MODEL

First consider the case of nucleon-nucleon scattering. For each I -spin state, 0 and 1, we can write $h_l(T) = B_l(T) + u_l(T)$, where $h_l(T)$ represents any one of the five partial-wave amplitudes ${}^J h_L^i$ ($i = 1, \dots, 4$), or ${}^J h^i$ ($i = 5$). Here l equals the orbital angular momentum L when $i = 1, \dots, 4$; and l equals the total angular momentum J when $i = 5$. By unitarity,

$${}^J h_L^1 = \frac{E}{2ip} (e^{2i\epsilon_J} - 1), \quad (1a)$$

$${}^J h_L^2 = \frac{E}{2ip} (e^{2i\epsilon_{JJ}} - 1), \quad (1b)$$

$${}^J h_L^3 = \frac{E}{2ip} [(\cos 2\epsilon_J) \exp(2i\delta_{J-1,J}) - 1], \quad (1c)$$

$${}^J h_L^4 = \frac{E}{2ip} [(\cos 2\epsilon_J) \exp(2i\delta_{J+1,J}) - 1], \quad (1d)$$

$${}^J h^5 = \frac{E}{2p} \sin 2\epsilon_J \exp[i(\delta_{J+1,J} + \delta_{J-1,J})], \quad (1e)$$

where $E = \frac{1}{2}\sqrt{s}$, $p = [s/4 - M^2]^{1/2}$, $\sqrt{s}$ = total center-of-mass (c.m.) energy in the p - p channel, $\sqrt{t}$ and $\sqrt{u}$ are the total c.m. energies in the two crossed channels; and M = proton mass. The nuclear-bar convention (δ , ϵ) is used in (1c)–(1e) for the phase shifts δ and the coupling parameters ϵ . Equations (1a) and (1b) are singlet and triplet uncoupled amplitudes, respectively, for the $L = J$ states; (1c) and (1d) are triplet coupled amplitudes for the $L = J - 1$ and

$L = J + 1$ states, respectively; $(1c)$ is the triplet coupling amplitude. The phase shifts δ are real below the inelastic threshold of $T = 290$ MeV, where $T = 2p^2/M = \text{lab kinetic energy}$. Above the inelastic threshold, $\text{Im } \delta \geq 0$. With this notation², $h_l(T) \rightarrow_{T \rightarrow 0} \mathcal{O}(T^l)$. If the five independent helicity amplitudes ϕ_i (defined in ref. 7) each have a Mandelstam representation, and if the pion pole is the only singularity in positive t or u below the two-pion threshold, then Eqs. (4.23), (4.25), and (4.26) of ref. 7 give the following results. Each of the amplitudes in (1) has, after removal of the one-pion pole contribution, a left cut in the T plane corresponding to the range of T when $1 + [(2m_\pi)^2/MT]$ runs from 1 to -1 . In other words, this left cut in the T plane runs from $-\infty$ to $-L$, where $-L = -2m_\pi^2/M$. The right cut begins at $T = 0$. Since $B_l(T)$, the Born term, includes the one-pion pole contribution plus a remainder that is generally assigned no left cut above the two-pion threshold, it is reasonable for one to assume a $u_l(T)$ with a left cut from $-\infty$ to $-L$ and a right cut from 0 to T_c . (T_c is an arbitrary cutoff, and will be taken as 400 MeV in Appendix B.) The rest of the right cut is assumed to be carried by $B_l(T)$. This model may be regarded as a crude approximation to a Reggeized Born term for which the imaginary part approaches the imaginary part of the amplitude at high energies. Also, if $B_l(T) \rightarrow_{T \rightarrow 0} \mathcal{O}(T^l)$, then $u_l(T)$ will have similar threshold behavior.

Now let us choose a model for the correction term $u_l(T)$ that incorporates the threshold behavior and analytic structure just discussed for nucleon-nucleon scattering. As before, we write $h_l(T) = B_l(T) + u_l(T)$, where $h_l(T)$ represents ${}^J h_L^i$ (which labels the amplitudes coupling states of the same orbital angular momentum L , $L = l$) or ${}^J h^i$ (which labels amplitudes coupling states of different L , $J = l$). This model is for elastic amplitudes that do not involve s -waves and that do not have bound states or resonances in the direct channel. The properties that define this model are:

- (i) $u_l(T) \rightarrow_{T \rightarrow 0} \mathcal{O}(T^l)$, the standard threshold behavior;
- (ii) $T^{k+1} \text{Re } u_l(T) \rightarrow_{T \rightarrow \infty} 0$, [the same k that appears in property (v)];
- (iii) $u_l(T)$ has a left cut from $-a$ to $-b$ ($a > b > 0$, and a need not be finite), a right cut from 0 to T_c , and is otherwise real analytic;
- (iv) the right-hand discontinuity of $u_l(T)$ is $2i \text{Im } h_l(T)$ (Thus $B_l(T)$ is real from 0 to T_c and carries the full discontinuity above T_c);
- (v) the left-hand discontinuity multiplied by $1/2i$, denoted by $f_l(T)$, has $l + k$ sign changes³ over the left cut. (Note that in conjunction with the other 4 properties this property corresponds to a minimum number of sign changes when $h_l(T)$ represents ${}^J h_L^i$; and also when $h_l(T)$ represents ${}^J h^i$, if the μ_n 's defined in

² The expression $h_l(T) \rightarrow_{T \rightarrow 0} \mathcal{O}(T^l)$ is taken to mean that $\lim_{T \rightarrow 0} h_l(T)/T^l = \text{a finite constant}$.

³ A change of sign for $f_l(T)$ occurs when its value changes from definitely positive to definitely negative, or vice versa.

(7) are all of the same sign and do not sum to zero. This is shown in the discussions following (7) and (8a).)

It then follows that $u_l(T)/T^l$ has a dispersion relation

$$\frac{u_l(T)}{T^l} = \frac{1}{\pi} \int_a^{-b} \frac{f_l(T') dT'}{T'^l(T' - T)} + \frac{1}{\pi} \int_0^{T_c} \frac{\text{Im } h_l(T') dT'}{T'^l(T' - T)} + C, \quad (2)$$

by properties (i), (iii), and (iv), where C is a real constant representing the contribution from the contour at infinity. By (2) and (ii),

$$T^{k+1} \text{Re } u_l(T) = \frac{T^{l+k+1}}{\pi} \left[\int_a^{-b} \frac{f_l(T') dT'}{T'^l(T' - T)} + \mathcal{O} \int_0^{T_c} \frac{\text{Im } h_l(T') dT'}{T'^l(T' - T)} + \pi C \right] \rightarrow_{T \rightarrow \infty} 0. \quad (3)$$

Expansion of $(T' - T)^{-1}$ in (3) then gives

$$\left. \begin{aligned} \int_a^{-b} \frac{f_l(T') dT'}{T'^{l-n}} &= - \int_0^{T_c} \frac{\text{Im } h_l(T') dT'}{T'^{l-n}}, \quad (n = 0, \dots, l+k), \\ C &= 0. \end{aligned} \right\} \quad (4)$$

The result is

$$u_l(T) = \frac{T^l}{\pi} I_{l+k}^{ab}(T) + \frac{T^l}{\pi} \int_0^{T_c} \frac{\text{Im } h_l(T') dT'}{T'^l(T' - T)}, \quad (5)$$

by (2) and (4), where

$$I_{l+k}^{ab}(T) = \int_a^{-b} \frac{f_l(T') dT'}{T'^l(T' - T)}. \quad (6)$$

Now define

$$\mu_n = \int_0^{T_c} \frac{\text{Im } h_l(T') dT'}{T'^{l-n}}, \quad (n = 0, \dots, l+k). \quad (7)$$

Each set of μ_n 's may be described by one of the following three cases:

- (a) $\mu_n \geq 0$, and $\sum_n \mu_n \neq 0$;
- (b) $\mu_n \leq 0$, and $\sum_n \mu_n \neq 0$;
- (c) either the μ_n 's are not all of the same sign, or else $\mu_n = 0$. By unitarity, we have case (a) for $h_l(T) = {}^J h_L$. For p - p scattering, we have case (b), by calculation (8), if $h_l(T) = {}^J h^b(J = 2 \text{ or } 4, T_c = 400 \text{ MeV})$. Then by (4), (6), (7), and property (v), the theorem in Appendix A applies. The bounds on $I_{l+k}^{ab}(T)$ are as given by Eq. (A.3) and the discussion following it:

$$\begin{aligned} \frac{1}{T+a} \sum_{n=0}^{l+k} \sum_{r=0}^n \frac{\binom{n}{r}^{\mu_r}}{a^r} \left(1 + \frac{T}{a}\right)^{-n} &\left(\begin{matrix} \leq \\ \geq \end{matrix}\right) I_{l+k}^{ab}(T) \\ &\left(\begin{matrix} \leq \\ \geq \end{matrix}\right) \frac{1}{T+b} \sum_{n=0}^{l+k} \sum_{r=0}^n \frac{\binom{n}{r}^{\mu_r}}{b^r} \left(1 + \frac{T}{b}\right)^{-n}, \quad T \geq 0 \end{aligned} \quad (8a)$$

The upper inequalities hold for case (a); the lower ones hold for case (b). Also by the theorem, $l + k$ is the minimum number of sign changes of $f_l(T)$ when case (a) or case (b) holds (and therefore when $h_l(T)$ represents any ${}^J h_L$ or the p - p amplitude ${}^J h^b$ ($J = 2$ or 4 , $T_c = 400$ MeV)). Assuming an upper limit of $-L$ on the extent of the left cut for the term $u_l(T)$, where $b \geq L > 0$, we also have the less-restrictive equation

$$0 \left(\begin{matrix} \leq \\ \geq \end{matrix} \right) I_{l+k}^{ab}(T) \left(\begin{matrix} \leq \\ \geq \end{matrix} \right) \sum_{n=0}^{l+k} \sum_{r=0}^n \binom{n}{r} \mu_r L^{-r-1} \left(1 + \frac{T}{L} \right)^{-n-1}, \quad T \geq 0, \quad (8a')$$

from the corollary to the theorem. As before, the upper inequalities hold for case (a); the lower inequalities hold for case (b). For case (c), we have

$$\frac{1}{T+c} \sum_{n=0}^{l+k} \sum_{r=0}^n \frac{\binom{n}{r} \mu_r}{c^r} \left(1 + \frac{T}{c} \right)^{-n} \left(\begin{matrix} \leq \\ \geq \end{matrix} \right) I_{l+k}^{ab}(T) \quad (8b)$$

$$\left(\begin{matrix} \leq \\ \geq \end{matrix} \right) \frac{1}{T+d} \sum_{n=0}^{l+k} \sum_{r=0}^n \frac{\binom{n}{r} \mu_r}{d^r} \left(1 + \frac{T}{d} \right)^{-n}, \quad T \geq 0.$$

The constants $c, d \in (a, b)$ and are chosen from the set $\{a, b, x_1, \dots, x_l\}$, where x_1 to x_l are solutions of $\sum_{r=0}^l \binom{l}{r} \mu_r x^{l-r} = 0$. Then if $\text{Im } h_l(T)$ is assumed to be experimentally known in the region $0 \leq T \leq T_c$ it follows that bounds are established on $u_l(T)$ when $T \geq 0$.

Now define a correction $u_l(T)$ to be of type (m) , with m some specified integer, if $u_l(T)$ has the properties (i) to (v) for $k = m$. We next show that the Chew-Arndt model (2) is of type (0), the MacGregor-Arndt model (j_0 even) (3) is of type (m) , where $m \leq -1$, and the Scotti-Wong model (1) with cutoff is of type (-2) . Also, the Kantor approximation (3) is shown to be a limiting case of the Chew-Arndt and Scotti-Wong models.

III. SPECIAL CASES OF THE MODEL GIVEN IN SECTION II

A. THE SCOTTI-WONG MODEL

In the Scotti-Wong (or S-W) model (1), with cutoff,⁴

$$h_l(T) = B_l(T) + u_l(T), \quad (9)$$

$$u_l(T) = \left(\frac{T}{T - T_0} \right)^{l-1} \frac{T}{\pi} \int_0^{T_c} \frac{dT' ((T' - T_0)/T')^{l-1} \text{Im } h_l(T')}{T'(T' - T)} \quad (10)$$

Now regard this as a limiting case where $(l-1)$ distinct poles

⁴ This model with cutoff T_c is considered in ref. 2. In ref. 1 a different sort of approximation is used, instead of a cutoff.

$T_i (i = 1, \dots, l-1)$ approach T_0 , where $T_0 < 0$. Then

$$u_l(T) = \lim_{\text{all } T_i \rightarrow T_0} \frac{T^{l-1}}{\prod_{i=1}^{l-1} (T - T_i)} \frac{T}{\pi} \int_0^{T_c} \frac{dT' \prod_{i=1}^{l-1} (T' - T_i) \operatorname{Im} h_l(T')}{T'^l (T' - T)}. \quad (10')$$

It follows that property (i) is satisfied, as is also (ii) for $k = -2$. The right-hand discontinuity is $2i \operatorname{Im} h_l(T)$, and so (iv) holds. The left-hand discontinuity of $u_l(T)$ is a sum of $(l-1)$ delta functions:

$$\lim_{\epsilon \rightarrow 0+} \frac{u_l(T + i\epsilon) - u_l(T - i\epsilon)}{2i} \Big|_{T < 0} = \lim_{\text{all } T_i \rightarrow T_0} \sum_{i=1}^{l-1} a_i \delta(T - T_i) \quad (11a)$$

$$a_i = \frac{-T_i^l}{\prod_{j \neq i} (T_i - T_j)} \int_0^{T_c} \frac{dT'}{T'^l} \prod_{j \neq i} (T' - T_j) \operatorname{Im} h_l(T'). \quad (11b)$$

Thus there are $l-1$ of the a_i , of alternating sign when the T_i are arranged in the order $T_1 < T_2 < \dots < T_{l-1}$. Therefore the number of sign changes of the left-hand discontinuity is $l+k$ ($k = -2$), satisfying property (v). Property (iii) is also satisfied because the left cut is from T_1 to T_{l-1} (which shrinks to a point T_0), and the right cut is from 0 to T_c . The term $u_l(T)$ as given in (10') is real analytic except for these cuts. Hence the S-W model is of type (-2) , when $l \geq 2$. The shrinking of the left cut corresponds to $-a \rightarrow T_0$, $-b \rightarrow T_0$, ($T_0 < 0$). Thus (7) and (8a, b) yield

$$u_l(T) = \frac{T}{\pi} \frac{1}{T - T_0} \sum_{n=0}^{l-2} \sum_{r=0}^n \frac{\binom{n}{r} \mu_r |T_0|^{-r}}{(1 + T/|T_0|)^n} + \frac{T^l}{\pi} \int_0^{T_c} \frac{\operatorname{Im} h_l(T') dT'}{T'^l (T' - T)} \quad (12)$$

$$\mu_n = \int_0^{T_c} \frac{\operatorname{Im} h_l(T') dT'}{T'^{l-n}}. \quad (13)$$

This S-W model includes the Kantor approximation (3), when $T_0 \rightarrow -\infty$. Equation (12) holds for $l \geq 1$, where the double sum is zero if $l = 1$, and where $h_l(T)$ may represent any partial-wave amplitude, $^J h_L^i$ or $^J h^i$.

The requirements $T_0 \leq -L$ and $L = 2m_\pi^2/M$ give for p - p scattering the less restrictive bounds

$$\frac{T^l}{\pi} \int_0^{T_c} \frac{\operatorname{Im} h_l(T') dT'}{T'^l (T' - T)} \left(\begin{matrix} \leq \\ \geq \end{matrix} \right) u_l(T) \left(\begin{matrix} \leq \\ \geq \end{matrix} \right) \frac{T^l}{\pi} \frac{1}{T + L} \sum_{n=0}^{l-2} \sum_{r=0}^n \frac{\binom{n}{r} \mu_r}{L(1 + T/L)^n} + \frac{T^l}{\pi} \int_0^{T_c} \frac{\operatorname{Im} h_l(T') dT'}{T'^l (T' - T)}, \quad (12')$$

from (8a'). The upper inequalities hold for case (a), the lower ones for case (b). Both (12) and (12') hold for $l \geq 1$. The double sum is zero when $l = 1$.

B. THE MACGREGOR-ARNDT MODEL

The MacGregor-Arndt approximation (2) is the following:

$$h_l(T) = B_l(T) + u_l(T) \quad (14)$$

$$\left. \begin{aligned} u_l(T) &= \frac{T^l}{\pi} \left[\int_{-\infty}^{-L} \frac{f_l(T') dT'}{T'^l(T' - T)} + \int_0^{T_c} \frac{\text{Im } h_l(T') dT'}{T'^l(T' - T)} \right] \\ u_l(T) &\rightarrow_{T \rightarrow \infty} 0 \\ L &= 2m_\pi^2/M \end{aligned} \right\} \quad (15)$$

$$f_l(T') = (2 + 2L/T')^{3/2} (2L/T') (1 + 2L/T')^{j_0} \sum_{j=0}^{l-1} \alpha_j T'^j$$

Consider the case when j_0 is even. The right-hand discontinuity is $2i \text{Im } h_l(T')$ for $0 < T' < T_c$; the left-hand discontinuity is $2if_l(T')$ for $-\infty < T' < -L$. Thus properties (iii) and (iv) are satisfied. Note that $u_l(T) \rightarrow_{T \rightarrow 0} \mathcal{O}(T^l)$ by construction, giving property (i). The second line of Eq. (15) corresponds to property (ii) with $k \leq -1$. With j_0 even, $f_l(T')$ has at most $l-1$ sign changes over the region $T' = -\infty$ to $T' = -L$. This is also the minimum number of sign changes for cases (a) and (b), as was discussed following Eqs. (6) and (7). Thus property (v) is satisfied. The result is that the M-A approximation, with j_0 even, is for cases (a) and (b) a type (-1) correction with (8a) applying.

$$\begin{aligned} \frac{T^l}{\pi} \int_0^{T_c} \frac{dT' \text{Im } h_l(T')}{T'^l(T' - T)} \left(\begin{matrix} \leq \\ \geq \end{matrix} \right) u_l(T) \left(\begin{matrix} \leq \\ \geq \end{matrix} \right) \frac{T^l}{\pi} \int_0^{T_c} \frac{dT' \text{Im } h_l(T')}{T'^l(T' - T)} \\ + \frac{T^l}{\pi} \frac{1}{(T + L)} \sum_{n=0}^{l-1} \sum_{r=0}^n \frac{\binom{n}{r} \mu_r L^{-r}}{(1 + T/L)^n} \end{aligned} \quad (16)$$

The upper inequalities hold if $h_l(T) = {}^J h_L^J$, and the lower ones if $h_l(T) = {}^J h^J$ ($J = 2, 4$ and $T_c = 400$ MeV, for p - p scattering).

$$\mu_n = \int_0^{T_c} \frac{\text{Im } h_l(T') dT'}{T'^{n+l-n}}, \quad (n = 0, \dots, l-1) \quad (17)$$

For case (c) the M-A approximation, with j_0 even, is a type (m) correction, $m \leq -1$.

When j_0 is an odd integer, property (v) does not hold and so the theorem does not apply. No bounds are derived in this case.

C. THE CHEW-ARNDT MODEL

In the Chew-Arndt model (2),

$$h_l(T) = B_l(T) + u_l(T) \quad (18)$$

$$u_l(T) = \frac{1}{\pi} \int_0^{T_c} \frac{dT' T' Q_l(1 + i/MT')}{(T' - T) T' Q_l(1 + i/MT')} \text{Im } h_l(T'), \quad (19)$$

where $Q_l(1 + \bar{l}/MT)$ is a Legendre polynomial of the second kind. Now check for properties (i) through (v). The threshold behavior of $u_l(T)$ goes as $T^{-1}Q_l(1 + \bar{l}/MT)$, which corresponds to (i). The term $u_l(T)$ goes asymptotically as $T^{-2}Q_l(1 + \bar{l}/MT)$, which gives $u_l(T) \rightarrow_{T \rightarrow \infty} \mathcal{O}(T^{-2} \log T)$. This asymptotic behavior agrees with (ii) for $k = 0$. The right-hand discontinuity is $2i \operatorname{Im} h_l(T)$, giving (iv). The left-hand discontinuity divided by $2i$ is

$$\lim_{\epsilon \rightarrow 0+} \frac{u_l(T + i\epsilon) - u_l(T - i\epsilon)}{2i} = \frac{P_l(1 + \bar{l}/MT)}{2T} \int_0^{T_c} \frac{T' \operatorname{Im} h_l(T') dT'}{(T' - T)Q_l(1 + \bar{l}/MT')}, \quad (20)$$

for $T < -\bar{l}/2M$. This discontinuity has l sign changes over the left cut, $-\infty < T < -\bar{l}/2M$, corresponding to (v) for $k = 0$. The right cut runs from 0 to T_c , and so (iii) also holds. Now the correction is of type (0) and so (8) applies.

$$\left. \begin{aligned} \frac{T^l}{\pi} \int_0^{T_c} \frac{\operatorname{Im} h_l(T') dT'}{T'^l(T' - T)} \left(\begin{array}{l} \leq \\ \geq \end{array} \right) u_l(T) \left(\begin{array}{l} \leq \\ \geq \end{array} \right) \frac{T^l}{\pi} \int_0^{T_c} \frac{\operatorname{Im} h_l(T') dT'}{T'^l(T' - T)} \\ + \frac{T^l}{\pi} \frac{1}{T + \bar{l}/2M} \sum_{n=0}^l \sum_{r=0}^n \frac{\binom{n}{r} \mu_r (\bar{l}/2M)^{-r}}{(1 + 2MT/\bar{l})^n} \end{aligned} \right\} \quad (21)$$

$$\bar{l} \geq 4m_\pi^2$$

$$\mu_r = \int_0^{T_c} \frac{\operatorname{Im} h_l(T') dT'}{T'^{l-r}}. \quad (22)$$

The upper set of inequalities holds when $h_l(T) = {}^J h_L^i$ (nucleon-nucleon); the lower set holds when $h_l(T) = {}^J h^5$ ($J = 2, 4$, $T_c = 400$ MeV, for p - p). The largest T for which $u_l(T)$ may have a left cut, $T = -L = -2m_\pi^2/M$, corresponds to a minimum $\bar{l}$ of $4m_\pi^2$. Then by (8a'), we find that

$$\left. \begin{aligned} \frac{T^l}{\pi} \int_0^{T_c} \frac{\operatorname{Im} h_l(T') dT'}{T'^l(T' - T)} \left(\begin{array}{l} \leq \\ \geq \end{array} \right) u_l(T) \left(\begin{array}{l} \leq \\ \geq \end{array} \right) \frac{T^l}{\pi} \int_0^{T_c} \frac{\operatorname{Im} h_l(T') dT'}{T'^l(T' - T)} \\ + \frac{T}{\pi} \frac{1}{T + L} \sum_{n=0}^l \sum_{r=0}^n \frac{\binom{n}{r} \mu_r L^{-r}}{(1 + T/L)^n} \end{aligned} \right\} \quad (21')$$

with $L = 2m_\pi^2/M$ and the inequalities subject to the conditions cited following (22). Note that the Kantor approximation (3) is included as the limiting case of $\bar{l} \rightarrow \infty$.

IV. SUMMARY

The three models for the correction term $u_l(T)$ considered in this paper have been shown to be special cases of a more general model. This generalized model

for $u_l(T)$, described in Section II, has bounds that are given by (5), (7), and (8a, b). Furthermore, for all of the p - p amplitudes considered in Appendix B we also have (8a'). For this case the bounded regions for corrections $u_l(T)$ (for models of type (m)) are nested—the bounded region for m includes the bounded region for m' , where $-l \leq m' \leq m$. Also, the corrections of type (m) all have one of their bounds in common. This is the lower bound for the amplitudes given by (1a)–(1d), and the upper bound for the coupling amplitude given by (1e). The figures in Appendix B show that the bounds are sufficiently limiting to force models of type (m) (for the p - p amplitudes considered) to all yield similar results, if m is small enough. As particular cases when m is small enough, $m = -2, -1$, or 0 correspond to the three models considered—S-W with cutoff, M-A (j_0 even), and C-A, respectively.

APPENDIX A. BOUNDS ON $I_l^{ab}(T)$

THEOREM. *An integral of the form*

$$I_l^{ab}(T) = \int_a^{-b} \frac{f_l(T') dT'}{(T' - T)T^{l+1}}, \quad (\text{A.1})$$

($a > b > 0$, a not necessarily finite, $T \geq 0$), with $(l + 1)$ finite moments μ_n given and with $f_l(T')$ having l sign changes³ over the region of integration, is bounded above

and below. The $(l + 1)$ moments

$$\mu_n = - \int_{-a}^{-b} \frac{f_l(T') dT'}{T'^{l-n}} \quad (n = 0, \dots, l) \quad (\text{A.2})$$

need not all have the same sign. The three possible cases are: (a) $\mu_n \geq 0$, $\sum_n \mu_n \neq 0$; (b) $\mu_n \leq 0$, $\sum_n \mu_n \neq 0$; (c) either the μ_n 's are not all of the same sign, or else $\mu_n = 0$ for all n . If all of the $(l + 1)$ μ_n 's have the same sign and do not sum to zero,⁵ which is case (a) or (b), then l is the minimum number of sign changes that $f_l(T')$ may have. For these two cases, the bounds can be written

$$\frac{1}{a+T} \sum_{n=0}^l \sum_{r=0}^n \binom{n}{r} \frac{\mu_r}{a^r} \left(1 + \frac{T}{a}\right)^{-n} \begin{pmatrix} \leq \\ \geq \end{pmatrix} I_1^{ab}(T) \quad (\text{A.3})$$

$$\begin{pmatrix} \leq \\ \geq \end{pmatrix} \frac{1}{b+T} \sum_{n=0}^l \sum_{r=0}^n \binom{n}{r} \frac{\mu_r}{b^r} \left(1 + \frac{T}{b}\right)^{-n}.$$

The upper inequalities hold for case (a); the lower set hold for case (b). For case (c), (A.3) holds when a and b are replaced by c and d , where $c, d \in (a, b)$ and are chosen from the set $\{a, b, x_1, \dots, x_l\}$, where x_1 to x_l are solutions of $\sum_{r=0}^l \binom{l}{r} \mu_r x^{l-r} = 0$. All symbols (except x_1 to x_l) stand for real quantities.

Proof: Set

$$f_l(T') = c \phi^2(T') \prod_{n=l+2}^{2l+1} (T' - T_n), \quad (\text{A.4})$$

where c is a constant. This form specifies $f_l(T')$ to be a general real function having at most l sign changes in the region of integration. Now solve the isoperimetric problem corresponding to the extremizing of $I_1^{ab}(T)$.

Form

$$H \equiv \frac{f_l(T')}{(T' - T) T'^l} + \sum_{n=0}^l \frac{\lambda_n f_l(T')}{T'^{l-n}} \quad (\text{A.5})$$

from (A.1) and (A.2). The λ_n correspond to Lagrangian multipliers. Then (A.4) gives

$$H = \frac{\phi^2(T') c}{T'^l} \prod_{n=l+2}^{2l+1} (T' - T_n) \left[\frac{1}{T' - T} + \sum_{n=0}^l \lambda_n T'^n \right]. \quad (\text{A.6})$$

Rewrite

$$\left[\frac{1}{T' - T} + \sum_{n=0}^l \lambda_n T'^n \right] = \frac{\lambda_l \prod_{n=0}^{l+1} (T' - T_n)}{T' - T}, \quad (\text{A.7})$$

⁵ In the applications of the theorem in this paper (specifically, the derivation of the correction bounds for the p - p amplitudes considered in Appendix B), the μ_n 's are all of the same sign and do not sum to zero.

where these T_n are the $(l+2)$ roots of the expression. (If λ_l were zero there would be fewer roots in (A.7), but this situation is equivalent to that in (A.7), with some T_n outside the range of T' and with some other constant in place of λ_l .) Then (A.6) and (A.7) yield

$$H = \frac{c\phi^2(T')\lambda_l \prod_{n=0}^{2l+1} (T' - T_n)}{T'^l (T' - T)}. \quad (\text{A.6}')$$

Take

$$\frac{\partial H}{\partial \phi} = 0 \quad (\text{A.8})$$

(since this is the Euler-Lagrange equation when H has no explicit dependence on $d\phi(T')/dT'$), and square the result. Then the coefficient of $f_l(T')/T'^l$ has $(2l+1)$ roots. So

$$\frac{f_l(T')}{T'^l} = \sum_{i=1}^{2l+1} a_i \delta(T' - T_i) \quad (\text{A.4}')$$

Now rewrite (A.1) and (A.2), using (A.4').

$$I_l^E(T) = \sum_{i=1}^{2l+1} \frac{a_i}{T_i - T} \quad (\text{A.9})$$

$$\mu_n = - \sum_{i=1}^{2l+1} a_i T_i^n, \quad (n = 0, \dots, l) \quad (\text{A.10})$$

(a_i is any real number, and $-a < T_i < -b < 0$.) Then the finding of the extrema of $I_l^E(T)$ corresponds to the extremizing of $I_l^E(T)$ as given by (A.9), varying the $2(2l+1)$ parameters a_i and T_i , subject to the constraint (A.10).

However, this prescription can be simplified. Consider the equations corresponding to variation of a_i and T_i :

$$G \equiv \sum_{i=1}^{2l+1} \frac{a_i}{T_i - T} + \sum_{n=0}^l \lambda_n \sum_{i=1}^{2l+1} a_i T_i^n; \quad (\text{A.11})$$

$$\frac{\partial G}{\partial a_i} = \frac{1}{T_i - T} + \sum_{n=0}^l \lambda_n T_i^n = 0, \quad (\text{A.12})$$

for those a_i not held fixed at zero;

$$\frac{\partial G}{\partial T_i} = -\frac{1}{(T_i - T)^2} + \sum_{n=0}^l \lambda_n n T_i^{n-1} = 0, \quad \text{for } a_i \neq 0. \quad (\text{A.13})$$

Because of the assumption of l sign changes, there must be at least $(l+1)$ of the a_i which are not zero. Let $(i = 1, \dots, l+1)$ label these a_i . They will be varied. The remaining l of the a_i are not restricted, and they may be either varied or held fixed, without affecting the analysis to follow. Suppose also that one of

the T_i (associated with a nonidentically-zero a_i) is varied. Call this T_1 . No restrictions on T_i , with $i \neq 1$, are made (except that $-a < T_i < -b$); they also may be either varied or held fixed without affecting the following analysis.

From (A.12), we obtain

$$\begin{aligned} \lambda_0 + \lambda_1 T_1 + \lambda_2 T_1^2 + \cdots + \lambda_l T_1^l &= -\frac{1}{T_1 - T} \\ \lambda_0 + \lambda_1 T_2 + \lambda_2 T_2^2 + \cdots + \lambda_l T_2^l &= -\frac{1}{T_2 - T} \\ &\vdots \\ \lambda_0 + \lambda_1 T_{l+1} + \lambda_2 T_{l+1}^2 + \cdots + \lambda_l T_{l+1}^l &= -\frac{1}{T_{l+1} - T} \end{aligned} \quad (\text{A.14})$$

Equation (A.14) corresponds to the variation of the first $(l+1)$ a_i , as defined previously. From (A.13), we know that

$$\lambda_1 + 2\lambda_2 T_1 + 3\lambda_3 T_1^2 + \cdots + l\lambda_l T_1^{l-1} = \frac{1}{(T_1 - T)^2}. \quad (\text{A.15})$$

This corresponds to the variation of T_1 . The $T_1, T_2, \dots, T_{l+1}$ in (A.14) may all be regarded as distinct, since they are positions of delta functions. As already pointed out, the assumption that there be l sign changes of $f_l(T')$ requires that there be at least $l+1$ distinct T_i , which were chosen to be T_i ($i = 1, \dots, l+1$).

Now look for solutions of (A.14) plus (A.15) for T_1 . Note that in the limit $T_1 \rightarrow T_i$, for $i \in \{2, \dots, l+1\}$, the difference between the first and the i th equations of (A.14) yields (A.15). So there are at least l roots of T_1 for the system (A.14) plus (A.15). In fact, these two equations are equivalent to $\prod_{i=2}^{l+1} (T_1 - T_i) = 0$, because there are also no more than l roots of T_1 in the system. Proof of this statement follows.

The last l equations of (A.14) give λ_1 to λ_l in terms of $\lambda_0, T_2, \dots, T_{l+1}$, in the form of the ratio of two determinants. The finiteness of λ_1 to λ_l is assured because the denominator may be written in the form $(T_2 \cdots T_{l+1}) \prod_{i,j=2, i < j}^{l+1} (T_i - T_j)$, and the numerator must have a factor $\prod_{i,j=2, i < j}^{l+1} (T_i - T_j)$ in it (because the i th and j th rows are identical at $T_i = T_j$). (Also, we have $(-a < T_i < -b < 0)$ because the positions of the delta functions must be within the region of integration in order to contribute to the integral.) Then subtraction of (l multiplied by the first equation in (A.14)) from $((T_1 - T)$ multiplied by (A.15)) yields an equation equivalent to one obtained by setting equal to zero a polynomial of order l in T_1 . The coefficients in this polynomial are all functions of $\lambda_0, T_2, \dots, T_{l+1}$.

Therefore, (A.14) plus (A.15) has precisely l roots in T_1 , the roots being $T_2, \dots, T_{l+1}$. In other words (A.9) is extremized when one delta function in (A.10) approaches another. The result is that the extrema of (A.9), subject to

(A.10), occur when all $T_i \rightarrow x$, $-a < T_i < -b$. The situation is equivalent to one in which only $(l+1)$ of the $(2l+1)$ a_i 's are not identically zero. So solve

$$I_l^B(T) = \sum_{i=1}^{l+1} \frac{a_i}{T_i - T} \quad (\text{A.16})$$

$$\mu_n = - \sum_{i=1}^{l+1} a_i T_i^n, \quad (n = 0, \dots, l), \quad (\text{A.17})$$

where all the $T_i \rightarrow x$, $-a < T_i < -b$. Then vary x over the range of integration of the integral, from $-a$ to $-b$, to obtain the bounds of $I_l^B(T)$. These bounds hold when the number of sign changes of $f_l(T')$ is l .

At this step in the proof, before solving (A.16) and (A.17), it is possible to verify the statement in the theorem that if all of the $(l+1)\mu_n$'s have the same sign and do not sum to zero, then l is the minimum number of sign changes that $f_l(T')$ may have. Suppose the same steps in the entire preceding proof are applied to the case of $I_{l+1}^B(T)$ (with the form given in (A.1)) having $(l+2)$ moments μ_n ($n = 0, \dots, l+1$) all of the same sign. Only now $f_{l+1}(T')$ is required to have l sign changes.

The result corresponding to (A.17) is

$$\mu_n = - \sum_{i=1}^{l+1} a_i T_i^n, \quad (n = 0, \dots, l+1). \quad (\text{A.18})$$

Consistency requires the augmented determinant, D , to be zero, for nonzero a_i . The T_i are distinct, since they are delta-function positions.

$$D = \begin{vmatrix} \mu_0 & a_1 & a_2 & \dots & a_{l+1} \\ \mu_1 & a_1 T_1 & a_2 T_2 & \dots & a_{l+1} T_{l+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{l+1} & a_1 T_1^{l+1} & a_2 T_2^{l+1} & \dots & a_{l+1} T_{l+1}^{l+1} \end{vmatrix} \quad (\text{A.19})$$

$$= \left(\prod_{i=1}^{l+1} a_i \right) \begin{vmatrix} \mu_0 & 1 & 1 & \dots & 1 \\ -\mu_1 & |T_1| & |T_2| & \dots & |T_{l+1}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{l+1} \mu_{l+1} & |T_1|^{l+1} & |T_2|^{l+1} & \dots & |T_{l+1}|^{l+1} \end{vmatrix} \cdot (-1)^{(l+1)(l+2)/2}.$$

Since the a_i were selected as nonzero, and all the μ_i are positive and do not sum to zero, it is sufficient to show that the coefficients of μ_i in the last determinant are each of the same sign and are each not zero. In other words, it is sufficient

to show that for distinct $|T_i|$ ($-a < T_i < -b < 0$), if

$$D_k = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ |T_1| & |T_2| & \cdots & |T_{l+1}| \\ \vdots & \vdots & & \vdots \\ |T_1|^{k-1} & |T_2|^{k-1} & \cdots & |T_{l+1}|^{k-1} \\ |T_1|^{k+1} & |T_2|^{k+1} & \cdots & |T_{l+1}|^{k+1} \\ \vdots & \vdots & & \vdots \\ |T_1|^{l+1} & |T_2|^{l+1} & \cdots & |T_{l+1}|^{l+1} \end{vmatrix}, \quad (k = 0, \dots, l+1), \quad (\text{A.20})$$

then all D_k have the same sign, and $D_k \neq 0$. This is now shown.

Since D_k has all the zeroes of D_{l+1} , where $D_{l+1} = \prod_{i,j=1,l+1, i < j}^{l+1} (|T_i| - |T_j|)$, D_k can be written

$$D_k = D_{l+1} F_{l+1-k}(|T_1|, \dots, |T_{l+1}|). \quad (\text{A.21})$$

Now $F_{l+1-k}(|T_1|, \dots, |T_l|, \dots, |T_j|, \dots, |T_{l+1}|)$ is symmetric under the interchange $i \leftrightarrow j$ ($i, j \in \{1, \dots, l+1\}$), because D_k/D_{l+1} has that property. Also, the sum of the exponents of the different $|T_i|$'s in each term of D_k is fixed at $(l+1)(l+2)/2 - k$. So F_{l+1-k} is a symmetrized sum of terms with $(l+1-k)$ as the total number of different $|T_i|$'s in each term. No $|T_i|$ will have an exponent higher than one because the exponent of $|T_i|$ in any term of D_k is at most one more than the exponent of $|T_i|$ in D_{l+1} . It follows that

$$\begin{aligned} F_{l+1-k} &= |T_{l+1}| |T_l| \cdots |T_{k+1}| + (\text{other terms needed for complete} \\ &\quad \text{symmetry}), \quad \text{if } k = 0, \dots, l \\ &= P_* |T_{l+1}| |T_l| \cdots |T_{k+1}|, \quad \text{if } k = 0, \dots, l \\ &= 1, \quad \text{if } k = l+1. \end{aligned} \quad (\text{A.22})$$

Here P_* is a symmetrizing operator that picks out the $\binom{l+1}{k}$ different ways of selecting $(l+1-k)$ different $|T_i|$'s from the $(l+1)$ available. Since $|T_i| > b > 0$, it follows that $F_{l+1-k} > 0$ for all k . Also D_{l+1} is not equal to 0 by the distinctness of the $|T_i|$'s. Thus the D_k 's all have the same sign and are nonzero, affirming that D cannot be zero and therefore that $f_{l+1}(T')$ cannot have l sign changes and at the same time satisfy the conditions of the theorem. The same proof holds for $f_{l+1}(T')$ and r sign changes, when $r < l$. This completes the proof that l is the minimum number of sign changes of $f_l(T')$, where $f_l(T')$ is as described in the theorem.

The next step is to solve (A.16) and (A.17). Write

$$I_l^E(T) = \int_0^\infty \frac{dT'}{T'} \left[\frac{\prod_{i=1}^{l+1} (T_i - T')}{\prod_{i=1}^{l+1} (T_i - T)} - 1 \right] \frac{g_l(T')}{(T' - T)}, \quad (\text{A.23})$$

where $l \geq 0$, $-a \leq T_i \leq -b$. (Since $I_l^E(T)$ satisfies (A.1), a comparison of (A.23) with (A.1) shows that there exists a function $g_l(T')$ which satisfies (A.23).) This has no right cut. The left-hand discontinuity multiplied by $1/2i$ is a sum of delta functions at T_i :

$$\frac{f_l(T')}{T'^l} = \sum_{i=1}^{l+1} a_i \delta(T' - T_i). \quad (\text{A.24})$$

The result is simply (A.16) and (A.17) (after application of (A.1) and (A.2)). So the limit all $T_i \rightarrow x$, ($-a \leq x \leq b$), should give extrema of $I_l^E(T)$ for fixed x . In this limit,

$$\begin{aligned} I_l^E(T) &= \int_0^\infty \frac{dT'}{T'^l} \left[\frac{(x - T')^{l+1}}{(x - T)^{l+1}} - 1 \right] \frac{g_l(T')}{(T' - T)} \\ &= \int_0^\infty \frac{dT'}{T'^l} \frac{g_l(T')}{(T' - T)} \left[\frac{(x - T')^{l+1} - (x - T)^{l+1}}{(x - T)^{l+1}} \right]. \end{aligned} \quad (\text{A.25})$$

But

$$(x - T')^{l+1} - (x - T)^{l+1} = (T - T') \sum_{p=0}^l (x - T')^p (x - T)^{l-p}. \quad (\text{A.26})$$

Therefore

$$\begin{aligned} I_l^E(T) &= - \int_0^\infty \frac{dT'}{T'^l} g_l(T') \sum_{p=0}^l \frac{(x - T')^p}{(x - T)^{p+1}} \\ &= \int_0^\infty \frac{dT'}{T'^l} g_l(T') \sum_{p=0}^l \sum_{r=0}^p \frac{\binom{p}{r} T'^r}{[1 + (T/|x|)]^p |x|^r (T + |x|)} \\ &= \sum_{p=0}^l \sum_{r=0}^p \frac{\binom{p}{r}}{[1 + (T/|x|)]^p |x|^r (T + |x|)} \left\{ \int_0^\infty \frac{dT'}{T'^{l-r}} g_l(T') \right\}, \end{aligned} \quad (\text{A.27})$$

where $T \geq 0$. Note that from (A.1) and (A.2) we have

$$I_l^E(T) = \sum_{i=0}^l \frac{\mu_i}{T^{l+1}} + \text{something of higher order in } 1/T. \quad (\text{A.28})$$

But (A.23) gives

$$I_l^E(T) = \sum_{i=0}^l \int_0^\infty \frac{dT'}{T'^l} g_l(T') \frac{T'^i}{T^{l+1}} + \text{something of higher order in } 1/T. \quad (\text{A.29})$$

By (A.28) and (A.29),

$$\mu_i = \int_0^\infty \frac{dT' g_l(T')}{T'^{l-i}}, \quad (i = 0, \dots, l). \quad (\text{A.30})$$

So (A.27) and (A.30) give

$$I_l^E(T) = \sum_{p=0}^l \sum_{r=0}^p \frac{\binom{p}{r}}{[1 + (T/|x|)]^p} \frac{\mu_r}{|x|^r} \frac{1}{(T + |x|)}, \quad (\text{A.31})$$

where $-a \leq x \leq -b < 0$. This is clearly bounded, and so $I_l^E(T)$ is bounded.

In the case when the μ_r are not all of the same sign, it is necessary to look at the solutions of

$$\frac{\partial I_l^E(T)}{\partial x} = 0,$$

as well as at the end points $x = -a$ and $x = -b$, to determine the bounds on $I_l^E(T)$. However, suppose all the μ_r are positive. Then

$$\frac{\partial I_l^E(T)}{\partial x} = \frac{(l+1)|x|^l}{(T-x)^{l+2}} \left\{ \sum_{r=0}^l \binom{l}{r} \frac{\mu_r}{|x|^r} \right\} > 0, \quad (\text{A.32})$$

and so the maximum occurs at $x = -b$. The minimum occurs at $x \rightarrow -a$. If all the μ_r 's were negative, the inequalities would be reversed. The result is Eq. (A.3).

COROLLARY. *If $I_l^{ab}(T)$ satisfies the conditions given in the statement of the theorem, with all of the $(l+1)\mu_r$'s having the same sign, and if $a' \geq a$ and $b \geq b' > 0$, then (A.3) holds when a and b are replaced by a' and b' respectively. This follows trivially from (A.32).*

APPENDIX B. BOUNDS ON $u_l(T)$ AND $B_l(T)$ FOR p - p SCATTERING

The bounds given by (5), (7), and (8a') are applied to corrections $u_l(T)$ of type (m) ($m = -2, -1, 0$). The bounds on $B_l(T) = h_l(T) - u_l(T)$ are also shown.

Correction	is of	Type
C-A ($\bar{l} \geq 4$)		0
M-A (j_0 even)		-1
S-W ($T_0 \leq -2m_\pi^2/M$)		-2

(These corrections are considered in Section III.) In the range 0 to 400 MeV, $h_l(T)$ is taken to be $h_l(T)^{\text{exp}}$, and $T_e = 400$ MeV. The experimental data are taken from refs. 2 and 8. The experimental errors are not shown.

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FIG. 1. Bounds on $\frac{180}{\pi} \frac{\text{Re } u_l(T)}{(1 + 2M/T)^{1/2}} \approx$ correction to δ_l or ϵ_l of the Born term, compared to

$$\left. \begin{matrix} \delta^{\text{Exp}} \\ \epsilon^{\text{Exp}} \end{matrix} \right\} \approx \frac{180}{\pi} \frac{\text{Re } h_l^{\text{Exp}}(T)}{(1 + 2M/T)^{1/2}} \approx \left\{ \begin{matrix} \delta_l^{\text{Exp}} \\ \epsilon_l^{\text{Exp}} \end{matrix} \right.$$

(a) 2P_1 ; (b) 2P_2 .

FIG. 2. Same as for Fig. 1. (a) 1D_2 ; (b) E_2 .

FIG. 3. Same as for Fig. 1. (a) 3F_2 ; (b) 3F_3

FIG. 4. Same as for Fig. 1. (a) 3F_4 ; (b) 1G_4

FIG. 5. Same as for Fig. 1, but for E_4 instead

FIG. 6. Bounds on

$$\frac{180}{\pi} \frac{\text{Re } B_l(T)}{(1 + 2M/T)^{1/2}} = \frac{180}{\pi} \frac{\text{Re } [h_l^{\text{Exp}}(T) - u_l(T)]}{(1 + 2M/T)^{1/2}} \approx \delta_l \text{ or } \epsilon_l \text{ of Born term,}$$

where $\text{Re } u_l(T)$ has the bounds graphed in Figs. 1 to 5. (a) 3P_1 ; (b) 3P_2 .

FIG. 7. Same as for Fig. 6. (a) $1D_2$; (b) E_2

FIG. 8. Same as for Fig. 6. (a) $3F_2$; (b) $3F_2$

FIG. 9. Same as for Fig. 6. (a) F_4 ; (b) G_4 .

FIG. 10. Same as for Fig. 6, but for E_4 instead.

APPENDIX C. DISCUSSION OF THE C-A AND M-A MODELS

Since both the C-A and M-A models are taken from an unpublished reference (2), they are further discussed here.

1. C-A

Assuming a Mandelstam representation and a strip approximation write, for the singlet p - p amplitude,

$$h_i(T) = B_i(T) + \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dt' h_i^{1,4}(T, t') Q_i \left(1 + \frac{t'}{MT}\right) / MT, \quad (C.1)$$

where

$$h_i^{1,4}(T, t) \equiv \frac{1}{\pi} \int_0^{T_c} dT' [h_{si}(T', t) + (-1)^i h_{su}(T', t)] / (T' - T), \quad (C.2)$$

and h_{si} and h_{su} are the usual double-spectral functions. Make the additional assumption that the integral in (C.1) can be approximated as

$$\int_{4m_\pi^2}^{\infty} dt' h_i^{1,4}(T, t') Q_i \left(1 + \frac{t'}{MT}\right) = C h_i^{1,4}(T, \bar{t}) Q_i \left(1 + \frac{\bar{t}}{MT}\right), \quad (C.3)$$

for some $\bar{t}$, where C is independent of T . Then from real analyticity of $h_i(T)$ it follows that

$$\text{Im } h_i(T) = C [h_{si}(T, \bar{t}) + (-1)^i h_{su}(T, \bar{t})] / \pi MT, \quad 0 < T < T_c. \quad (C.4)$$

(For this model, $B_i(T)$ is real for $0 < T < T_c$.) Then (C.1) through (C.4) may be combined to yield

$$h_i(T) = B_i(T) + u_i(T) \quad (C.5)$$

$$u_i(T) = \frac{1}{\pi} \int_0^{T_c} \frac{dT'}{T' - T} \frac{T'}{T} \frac{Q_i(1 + \bar{t}/MT)}{Q_i(1 + \bar{t}/MT')} \text{Im } h_i(T'). \quad (C.6)$$

(C.5) and (C.6) correspond to Eqs. (18) and (19) of Section III. This model is applied by Arndt (2) to all five independent p - p amplitudes.

2. M-A

Again assume both a Mandelstam representation and a strip approximation for the p - p helicity amplitudes, so that only that part of the double-spectral function within the strip $0 < T' < T_c$ contributes to the amplitude. Then by real analyticity of $h_i(T)$ we may write

$$h_i(T) = B_i(T) + u_i(T) \quad (C.7)$$

$$u_i(T) = \frac{1}{\pi} \int_{-\infty}^{-L} \frac{f_i(T')}{T' - T} dT' + \frac{1}{\pi} \int_0^{T_c} \frac{\text{Im } h_i(T')}{T' - T} dT', \quad (C.8)$$

where $L = 2m^2/M$ and $\text{Im } B_l(T) \equiv 0$ within the strip. (L corresponds to the 2π threshold in the crossed channels.) The left-hand discontinuity $f_l(T)$ must have the property

$$f_l(T) \rightarrow_{T \rightarrow -L} \mathcal{O}(T + L)^{3/2}. \quad (\text{C.9})$$

This is derived in ref. 2. Then, requiring normal threshold behavior of $u_l(T)$, an expansion of $(T' - T)^{-1}$ in (C.8) yields

$$u_l(T) = \frac{T^l}{\pi} \left[\int_{-\infty}^{-L} \frac{f_l(T') dT'}{T'^l(T' - T)} + \int_0^{T^c} \frac{\text{Im } h_l(T') dT'}{T'^l(T' - T)} \right] \quad (\text{C.8}')$$

$$\frac{T^n}{\pi} \left[\int_{-\infty}^{-L} \frac{f_l(T') dT'}{T'^{n+1}} + \int_0^{T^c} \frac{\text{Im } h_l(T') dT'}{T'^{n+1}} \right] = 0, \quad (n = 0, \dots, l-1). \quad (\text{C.10})$$

The additional requirement

$$u_l(T) \rightarrow_{T \rightarrow \infty} 0, \quad (\text{C.10}')$$

when combined with (C.8'), includes (C.10) as a consequence. A simple choice for an approximation to $f_l(T)$, which allows the first integral in (C.10) to converge, is

$$f_l(T) = (2 + 2L/T)^{3/2} (2L/T^l) (1 + 2L/T)^{j_0} \sum_{j=0}^{l-1} \alpha_j T'^j. \quad (\text{C.11})$$

Then (C.7), (C.8'), (C.10'), and (C.11) are equivalent to Eqs. (14) and (15) of Section III.

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ERRATA

The sentence below (8b) on page 6 should read:

The constants $c, d \in (a, b)$ and are chosen from the set $\{a, b, x_1, \dots, x_{l+k}\}$, where x_1 to x_{l+k} are solutions of

$$\sum_{r=0}^{l+k} \binom{l+k}{r} \mu_r x^{l+k-r} = 0.$$

APPENDIX D. A MORE GENERAL THEOREM

THEOREM. An integral of the form

$$I_{\ell}^{ab}(T) = \int_{-a}^{-b} \frac{f_{\ell}(T') dT'}{(T' - T)}, \quad (D.1)$$

($a > b > 0$, a not necessarily finite, $T \geq 0$), with at least $(\ell + 1)$ finite moments μ_n given and with $f_{\ell}(T')$ having ℓ sign changes over the region of integration, is bounded above and below. The m moments

$$\mu_n = - \int_{-a}^{-b} f_{\ell}(T') T'^n dT', \quad (n = 0, \dots, m) \quad m \geq \ell \quad (D.2)$$

need not all have the same sign. The three possible cases are:

(a) $\mu_n \geq 0$, $\sum_n \mu_n \neq 0$; (b) $\mu_n \leq 0$, $\sum_n \mu_n \neq 0$; (c) either the μ_n 's are not all of the same sign, or else $\mu_n = 0$ for all n . If all of the (m) μ_n 's have the same sign and do not sum to zero, which is case (a) or (b), then m is the minimum number of sign changes that $f_{\ell}(T')$ may have. For these two cases, the bounds can be written

$$\frac{1}{a+T} \sum_{n=0}^m \sum_{r=0}^n \binom{n}{r} \frac{\mu_r}{a^r} \left(1 + \frac{T}{a}\right)^{-n} \begin{pmatrix} \leq \\ \geq \end{pmatrix} I_{\ell}^{ab}(T) \quad (D.3)$$

$$\begin{pmatrix} \leq \\ \geq \end{pmatrix} \frac{1}{b+T} \sum_{n=0}^m \sum_{r=0}^n \binom{n}{r} \frac{\mu_r}{b^r} \left(1 + \frac{T}{b}\right)^{-n}.$$

The upper inequalities hold for case (a); the lower set hold for case

(b). For case (c), (D.3) holds when a and b are replaced by c and d ,

where $c, d \in (a, b)$ and are chosen from the set $\{a, b, x_1, \dots, x_m\}$ so as to maximize (minimize) the upper (lower) bounds of I_ℓ^{ab} . x_1 to x_m are solutions of $\sum_{r=0}^m \binom{m}{r} \mu_r x^{m-r} = 0$. All symbols (except x_1 to x_m) stand for real quantities.

Proof: A moment's reflection should reveal that if this theorem holds for the case $\ell = m$ then it must also hold for $\ell < m$. All possible cases where ℓ , the number of sign changes of $f_\ell(T)$ in the region of integration, is less than m can be regarded as limits of the $\ell = m$ case. So the same bounds (D.3) must hold for $\ell < m$ as do for $\ell = m$, although they cannot necessarily be reached. Since the statement of this theorem for the case $\ell = m$ is just the theorem given in Appendix A, it follows that the theorem is proved.

3. A THEOREM ON THE NUMBER OF SIGN CHANGES OF THE LEFT-HAND DISCONTINUITY OF PARTIAL-WAVE AMPLITUDES

I. INTRODUCTION

We shall now consider the following question. What gross features of the partial-wave amplitude in the unphysical region can be determined from measurements of the direct-channel phase shifts and elasticity factors? More specifically, what predictions do analyticity and unitarity alone allow us to make about the partial-wave amplitude in an energy region disjoint from the physical region, given a fair amount of scattering information in the physical region?

We answer the question in part, by giving a theorem which can be applied to the partial-wave amplitude for equal-mass elastic scattering. The theorem is a consistency condition relating ν -- the number of sign changes of the left-hand partial-wave discontinuity -- to the partial-wave amplitude in the physical region. Used as a test, the theorem can only rule out a value of ν .

II. THEOREM AND PROOF

Theorem. Let $f_\ell(T)$ be a partial-wave amplitude which is real-analytic in the T plane except for two cuts along the real axis from $-\infty$ to $-L$ and from 0 to $+\infty$. The subscript ℓ designates the threshold behavior $f_\ell(T) \xrightarrow[T \rightarrow 0]{} O(T^\ell)$. Let ν be the number of sign changes of its discontinuity along the left cut. Assume ν is finite.¹ Assume $f_\ell(T)$ is bounded by a polynomial in T as $|T| \rightarrow \infty$, and that $f_\ell(T)$ approaches definite (not necessarily finite) limits as $T \rightarrow -\infty \pm i\epsilon$. (By ϵ is meant an arbitrarily small positive number.) Then for any ℓ' such that $\ell' \geq \nu - \ell + 2$, we have the inequality

$$E(a_1, \nu) \leq \pi \operatorname{Re}[T, T_1, \dots, T_{\ell'}] - \rho \int_0^\infty \frac{dT'}{T' - T} \frac{\operatorname{Im} f_\ell(T')}{T'^{\ell'} \prod_{i=1}^{\ell'} (T' - T_i)} \leq E(a_2, \nu) \quad (1a)$$

for $T, T_1, \dots, T_{\ell'} \geq 0$ and all distinct, where

$$E(a_i, \nu) \equiv \sum_{n=0}^{\nu} \sum_{r=0}^n \frac{1}{a_i^n + T} \binom{n}{r} \frac{\mu_r}{(1 + T/a_i)^n a_i^r}, \quad i = 1, 2 \quad (1b)$$

$$\mu_r \equiv \int_0^\infty dT' T'^{r-\ell} \operatorname{Im} f_\ell(T') \prod_{i=1}^{\ell'} (T' - T_i), \quad (1c)$$

¹ W. S. Woolcock, Phys. Rev. 153, 1449 (1967) gives a proof that by assuming ν finite we rule out the possibility that the integral over the left cut, in the dispersion relation for the amplitude $f_\ell(T)$, may need additional (i.e. more than required for the integral over the right cut) subtractions in order to converge.

and $[T, T_1, \dots, T_{\ell}]$ is the divided difference² of the function $f_{\ell}(T)/T^{\ell}$. The constants $a_1, a_2 \in [\infty, L]$ and are chosen from the set $\{+\infty, L, x_1, \dots, x_{\nu}\}$ so as to maximize $E(a_2, \nu)$ and to minimize $E(a_1, \nu)$, where x_1 to x_{ν} are the roots of

$$\sum_{r=0}^{\nu} \binom{\nu}{r} \mu_r x^{\nu-r} = 0.$$

Proof: Let us define $f_{\ell}(T)$ to be the elastic equal-mass partial-wave scattering amplitude. For spinless particles this may be written

$$f_{\ell}(T) = \frac{S_{\ell}^{\text{el}} - 1}{2i} \sqrt{1 + 2M/T} \quad (2)$$

where $T = s/2M - 2M$ = the lab kinematic energy of the beam particle, s is the square of the sum of the center-of-mass energies of the two incoming particles, ℓ is the orbital angular momentum, and the magnitude of the S-matrix element (connecting the state of the incoming particles with angular momentum ℓ with the outgoing state of the same particles in the same angular momentum state) S_{ℓ}^{el} is by unitarity less than or equal to 1. The threshold behavior³ is

² See Abramowitz and Stegun, Handbook of Mathematical Functions, (Dover), 1965 p. 877. The divided difference $[T, T_1]$ of a function $g(T)$ is $[g(T) - g(T_1)]/(T - T_1)$. In general, the divided difference $[T, T_1, \dots, T_{\ell}]$ of a function $g(T)$ corresponds to ℓ' subtractions at the specified points $T_1, \dots, T_{\ell}$.

³ For a discussion of the threshold behavior of partial-wave amplitudes, see Marc Ross, ed., Quantum Scattering Theory, (Indiana University Press, Bloomington) page 80.

$$f_{\ell}(T) \xrightarrow[T \rightarrow 0]{} \mathcal{O}(T^{\ell}). \quad (3)$$

For particles with spin, $f_{\ell}(T)$ will represent one of the matrix elements of

$$\frac{J_S e^{l-1}}{2i} \sqrt{1 + 2M/T}, \quad (4)$$

where ℓ indicates that the element has the threshold behavior shown in Eq. (3). For this case, with spin, ℓ does not in general correspond to the orbital angular momentum. However, J does indicate the total angular momentum. From unitarity, each element of the matrix $J_S e^{l-1}$ is of magnitude less than or equal to unity. In what follows, we shall assume that the amplitude $f_{\ell}(T)$ is real-analytic except for a left cut from $T = -\infty$ to $T = -L$ and a right cut from $T = 0$ to $T = +\infty$. We therefore exclude from consideration those reactions with bound state poles.

In addition to (3), there are two more properties of the amplitude that are of interest:

$$|f_{\ell}(T)| \leq \sqrt{1 + 2M/T} \quad \text{for } T > 0, \quad (5)$$

by (2), (4), and unitarity; and

$\text{Im } f_\ell(T)$ has ν sign changes over the left cut, from $T = -\infty$ to $T = -L$, by definition of ν .
(The discontinuity of $f_\ell(T)$ is $2i \text{Im } f_\ell(T)$ from real-analyticity.)

We shall proceed to derive a dispersion relation in T for $[T, T_1, \dots, T_\ell]$, the divided difference² of the function $f_\ell(T)/T^\ell$. This corresponds to the function $f_\ell(T)/T^\ell$ subtracted ℓ' times, at the points $T_1, \dots, T_\ell$, which we shall assume to be positive and distinct. According to the Sugawara-Kanazawa theorem,⁴ if a function $[T, T_1, \dots, T_\ell]$ has the following four properties:

- (a) $[T, T_1, \dots, T_\ell]$ is analytic in T except for cuts on the real axis from $-\infty$ to $-L$ and from C_1 to $+\infty$;
- (b) $[T, T_1, \dots, T_\ell]$ is bounded by a polynomial in T as $|T| \rightarrow \infty$;
- (c) $[T, T_1, \dots, T_\ell]$ has finite limits $F(\infty \pm i\epsilon)$ as $T \rightarrow \infty \pm i\epsilon$;
- (d) $[T, T_1, \dots, T_\ell]$ approaches definite (not necessarily finite) limits as $T \rightarrow -\infty \pm i\epsilon$;

then

$$\lim_{|T| \rightarrow \infty} [T, T_1, \dots, T_\ell] = \begin{cases} [\infty + i\epsilon, T_1, \dots, T_\ell], & \text{for } \text{Im } T > 0 \\ [\infty - i\epsilon, T_1, \dots, T_\ell], & \text{for } \text{Im } T < 0 \end{cases}$$

and

⁴ This statement of the theorem is from G. Barton, Dispersion Techniques in Field Theory (W. A. Benjamin, New York) 1965, page 81. See footnote 1 for a proof of this theorem.

$$[T, T_1, \dots, T_{\ell}] = \frac{1}{\pi} \left\{ \int_{C_1}^{\infty} + \int_{-\infty}^{-L} \right\} \frac{dT'}{T' - T} \Delta[T', T_1, \dots, T_{\ell}] + \overline{[\infty, T_1, \dots, T_{\ell}']}$$

$$\Delta[T, T_1, \dots, T_{\ell}] \equiv ([T + i\epsilon, T_1, \dots, T_{\ell}] - [T - i\epsilon, T_1, \dots, T_{\ell}])/2i$$

$$\overline{[T, T_1, \dots, T_{\ell}]} \equiv ([T + i\epsilon, T_1, \dots, T_{\ell}] + [T - i\epsilon, T_1, \dots, T_{\ell}])/2. \quad (6)$$

By ϵ is meant an arbitrarily small positive number. It is clear that if these four properties apply to the amplitude $f_{\ell}(T)$, then they also apply to its subtracted form $[T, T_1, \dots, T_{\ell}]$. Property (c) follows from unitarity. (See (5).) The other three properties are assumed to hold for $f_{\ell}(T)$, so that the dispersion relation (6) may be written for $[T, T_1, \dots, T_{\ell}]$.

We may now derive conditions on the scattering amplitude, given ν . Since $[T, T_1, \dots, T_{\ell}]$ is the divided difference of $f_{\ell}(T)/T^{\ell}$,

$$\begin{aligned} [T, T_1, \dots, T_{\ell}] &\xrightarrow{T \rightarrow \infty} \text{constant} \times f_{\ell}(T)/T^{\ell+\ell'} \\ &\xrightarrow{T \rightarrow \infty} \text{constant} \times 1/T^{\ell+\ell'}, \end{aligned} \quad (7)$$

where the second line follows from (5). Then the expansion in powers of $1/T$ of the term $(T' - T)^{-1}$ appearing in the integrands in the dispersion relation (6) for $[T, T_1, \dots, T_{\ell}]$ results from (7) in

$$\int_{-\infty}^{-L} dT' T'^r \text{Im } f_{\ell}(T') \prod_{i=1}^{\ell'} (T' - T_i)^{-1} = -\mu_r, \quad r = 0, \dots, \ell + \ell' - 2 \quad (8)$$

$$\mu_r \equiv \int_0^{\infty} dT' T'^r \operatorname{Im} f_{\ell}(T') \prod_{i=1}^{\ell'} (T' - T_i)^{-1} \quad (9)$$

$$[\infty, T_1, \dots, T_{\ell'}] = 0. \quad (10)$$

The real-analyticity of $f_{\ell}(T)$ has been used. Note that by (7) and (9) all the μ_r are finite. Now choose ℓ' such that

$$\ell' \geq \nu - \ell + 2. \quad (11)$$

Then given (8) and (11) the theorem in Appendix A of Part II states that

$$E(a_1, \nu) \leq \int_{-\infty}^{-L} \frac{dT' \operatorname{Im} f_{\ell}(T')}{(T' - T) T'^{\ell} \prod_{i=1}^{\ell'} (T' - T_i)} \leq E(a_2, \nu), \quad T \geq 0 \quad (12)$$

where

$$E(a_1, \nu) \equiv \sum_{n=0}^{\nu} \sum_{r=0}^n \binom{n}{r} \frac{1}{a_1 + T} \frac{\mu_r}{(1 + T/a_1)^n a_1^r}, \quad i = 1, 2. \quad (13)$$

The constants $a_1, a_2 \in [\infty, L]$ and are chosen from the set $\{\infty, L, x_1, \dots, x_{\nu}\}$ so as to maximize $E(a_2, \nu)$ and to minimize $E(a_1, \nu)$, where x_1 to x_{ν} are the roots of

$$\sum_{r=0}^{\nu} \binom{\nu}{r} \mu_r x^{\nu-r} = 0 \quad (14)$$

and $-L$ is the upper limit of the unphysical cut of $f_{\ell}(T)$. Eqs. (9) through (14) are equivalent to the statement of the theorem, which is labeled (1).

III. DISCUSSION

In evaluating the integrands in Eqs. (1a) and (1c), which appear in the statement of the theorem, we do not need the scattering amplitude $f_\ell(T)$ above some arbitrary cut-off, say T_c . The subtractions at T_1 provide a rapid cut-off at large T , and the amplitude $f_\ell(T)$ itself is subject to a unitarity bound (stated in Eq. (5)). The bounds on the contribution to the integrals may therefore be calculated.

The results from this method may be compared with the following inequalities found by Jin and Martin:⁵

$$v \geq \ell - 2 \quad \text{if } \ell \text{ is odd}$$

$$v \geq \ell \quad \text{if } \ell \text{ is odd and the upper part of the} \\ \text{unphysical cut is due entirely to the exchange} \\ \text{of one meson}$$

$$v \geq \ell - 1 \quad \text{if } \ell \text{ is even.}$$

Here v is the number of sign changes of the discontinuity of the partial-wave amplitude over the left cut. It is not measured relative to the sign of the discontinuity over the right cut.

Note that as a consequence of the theorem in Appendix D, that if a given value of v is ruled out in this way, then all lesser values of v are also excluded.

⁵ Y. S. Jin and A. Martin, Phys. Rev. 135B, 1369 (1964).

IV. APPLICATION TO p - p SCATTERING

Application of the theorem in this section as a consistency condition yields the following results for p - p scattering:⁶

partial-wave amplitude	results of application of this theorem ⁷	inequalities found by Jin and Martin ⁵
1S_0	$\nu \geq 2$	$\nu \geq 0$
3P_1	$\nu \geq 2$	$\nu \geq 1$
3P_2	$\nu \geq 2$	$\nu \geq 1$

⁶ Phase shifts were supplied by R. A. Arndt, Lawrence Radiation Laboratory, Livermore, California, private communication, 1965.

⁷ Since the theorem is only a consistency condition, each possible value of ν must be checked separately. This column reflects the fact that the calculations have not been attempted past $\nu = 1$, although such calculations are of course possible. Better lower limits on ν can be obtained, with additional effort.

4. SUMMARY

A generalized dispersion model for the unitarizing correction to the Born term is given. (Only equal-mass, elastic scattering is considered.) The upper and lower bounds on this correction are explicitly given, in terms of the experimentally available direct-channel phase shifts. This model is shown to include as special cases the Chew-Arndt, MacGregor-Arndt (j_0 even), Scotti-Wong, and Kantor models. The main features of the model are analyticity (except for cuts from unitarity and crossing), specifications both of the number of fluctuations of the left-hand-discontinuity (LHD) of the correction term and of the asymptotic behavior of the correction term, and unitarity of the full partial-wave amplitude.

A similar exploitation of the analyticity and unitarity of the full partial-wave amplitude is shown to yield a consistency condition relating the number of fluctuations of the LHD of the full partial-wave amplitude to the direct-channel phase shifts. The consistency condition is shown to have power as well as rigor, in that its application to $p - p$ scattering data yields new information.

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