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Measure-Valued Branching Diffusions with Singular Interactions

by

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Summary: The usual super-Brownian motion is a measure-valued process that arises as a high density limit of a system of branching Brownian particles in which the branching mechanism is critical. In this work we consider analogous processes that model the evolution of a system of two such populations in which there is inter-species competition or predation.

We first consider a competition model in which inter-species collisions may result in casualties on both sides. Using a Girsanov approach, we obtain existence and uniqueness of the appropriate martingale problem in one dimension. In two and three dimensions we establish existence only. However, we do show that, in three dimensions, any solution will not be absolutely continuous with respect to the law of two independent super-Brownian motions. Although the supports of two independent super-Brownian motions collide in dimensions four and five, we show that there is no solution to the martingale problem in these cases.

We next study a predation model in which collisions only affect the "prey" species. Here we can show both existence and uniqueness in one, two and three dimensions. Again, there is no solution in four and five dimensions. As a tool for proving uniqueness, we obtain a representation of martingales for a super-process as stochastic integrals with respect to the related orthogonal martingale measure.

We also obtain existence and uniqueness for a related single population model in one dimension in which particles are killed at a rate proportional to the local density. This model appears as a limit of a rescaled contact process as the range of interaction goes to infinity.

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0 Introduction

Critical branching measure-valued diffusions or superprocesses arise as limits of branching particle system undergoing random migration and critical (or near critical) reproduction. These processes give a rich class of solutions to higher dimensional non-linear stochastic p.d.e.'s. Their qualitative and limiting behaviour is fairly well understood (e.g. Dawson-Perkins (1991)), and their potential theory is linked with the behaviour of solutions to a (deterministic) non-linear p.d.e. (e.g. Dynkin (1992a), Le Gall (1993)). A precise mathematical treatment is made possible by the fundamental independence of the branching particles. In modelling populations or genotype frequencies it is natural to introduce interactions between the branching particles. These interactions invalidate almost all of the mathematical tools used in the study of superprocesses (and their close cousins). One major exception is the Girsanov theorem of Dawson (1978) which allows one to handle certain interactions in the immigration or emigration terms, which corresponds to 0th order terms in the Markov generator governing the migration. In Section 2 we derive a version of Dawson's result which is particularly well-suited to our needs (Theorems 2.3 and 2.5). Recently in Perkins (1993) interactions have been incorporated in the migration mechanism by means of a new type of strong equation. Neither of these two approaches are applicable in general to the most natural kind of "point interactions" in which an interaction only occurs if particles collide.

In this work we initiate a study of what should be the easiest case: point interactions in the immigration/emigration term. Consider two independent super-Brownian motions (i.e., the spatial migrations are governed by Brownian motions in $\mathbb{R}^d$). Now view these two populations as competing species so that inter-species "collisions" may result in casualties on either side. More precisely when different species come within an infinitesimal distance of each other, there is an infinitesimal probability that either of the colliding individuals is killed. In Section 3 we formulate a measure-valued martingale problem $(M_{\lambda L})$ for this model. The Girsanov theorem mentioned above is used to prove existence of solutions in dimensions three or less by means of a limiting argument (Theorem 3.6). In one dimension the Girsanov theorem applies directly to show there is a unique solution to $(M_{\lambda L})$. This solution is absolutely continuous (in law) with respect to a pair of independent super-Brownian motions (Theorem 3.9). The same approach is also used to prove existence and uniqueness in a martingale problem for a branching measure-valued diffusion (again in one spatial dimension) in which particles are killed at a rate proportional to the local density (Theorem 3.10). This model was conjectured by Rick Durrett, and shown in Mueller and Tribe (1993), to be the limit of a rescaled contact process as the interaction range goes to infinity. We also show that in 3 dimensions, solutions to $(M_{\lambda L})$ will be singular (in law) with respect to a pair of independent super-Brownian motions and so the Girsanov theorem cannot be used to prove uniqueness in law (see Theorem 3.11.) Hence the fundamental question of uniqueness in law to $(M_{\lambda L})$ in dimensions 2 or 3 remains unresolved (see Conjecture 3.7).

Obviously non-trivial solutions to $(M_{\lambda L})$ can only exist if inter-species collisions do occur. Two independent super-Brownian motions collide if and only if $d < 6$ (see Thm.3.6 and Prop.5.11 of Barlow-Evans-Perkins (1991), hereafter abbreviated as [BEP]). Our interacting processes can be dominated by a pair of independent super-Brownian motions (see Theorem 2.1) and therefore non-trivial solutions to $(M_{\lambda L})$ can only be expected if $d \leq 5$. In fact in Section 5 we show that solutions can only exist if $d \leq 3$ (Theorem 5.3), and therefore our existence result is sharp.

In Section 4 we study an easier kind of singular "interaction". When an inter-species "collision" occurs there is an infinitesimal probability of the type-1 particle being killed but the type-2 particle is not affected by the encounter. Hence this is not really an interactive model but rather a super-Brownian motion run in a random and unfriendly environment of a second super-Brownian motion. We formulate a martingale problem $(M^1_{\lambda L})$ for this pair of processes and, for dimensions 3 or less, establish existence, uniqueness and the Markov property of the solution (Theorem 4.9, Corollary 4.12). Again solutions will not exist for $d > 3$ (Theorem 5.3). The first step in this construction is to show that a super-Brownian in $\mathbb{R}^d (d \leq 3)$ is sufficiently regular to be the Revuz measure of a time-inhomogeneous continuous additive functional (CAF) of a Brownian motion (Theorem 4.1, Proposition 4.7(a)). Kill Brownian motion according to this random CAF to construct a nice Markov process B^k, with a random law. The law of the unique solution to $(M^1_{\lambda L})$ may be described as follows: The second population is a super-Brownian motion, and the conditional law of the first population given the second is that of the B^k-superprocess where the second population provides the Revuz measure used in the construction of B^k (see (4.18) in Theorem 4.9).

Our original motivation for studying this simple model was the hope that an iterative procedure in which one successively reverses the roles of the two populations would shed some light on the uniqueness question for the truly interactive model studied in Section 3. The fact that such a program can be carried out in a related model in which collisions reduce the masses of the colliding particles (Barlow-Perkins (199?)) suggests that this may still be feasible. The simple model considered here seems to present some non-trivial problems of its own. Some delicate path properties of super-Brownian motion (Proposition 4.7) are needed to carry out the construction of the random CAF in the above. In addition, a representation of super-Brownian martingales as stochastic integrals with respect to the associated orthogonal martingale measure plays a critical role in the uniqueness proof. This result, which holds for a broad class of superprocesses and is of independent interest, is presented in Section 1 (see Theorem 1.2).

We now gather together some notation which will be used throughout this article.

Notation. If E is a Polish space $\mathcal{E}$ or $\mathcal{B}(E)$ denotes its Borel σ-field. Let $M_F(E)$ (respectively $M_1(E)$) denote the space of finite (respectively, probability) measures on $(E, \mathcal{E})$, equipped with the topology of weak convergence. Let $\Omega = \Omega_E = C([0,\infty), M_F(E))$ denote the space of continuous $M_F(E)$-valued paths with the compact-open topology and let $\mathcal{F} = \mathcal{F}_E$ denote its Borel σ-field. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the canonical right continuous filtration on $(\Omega, \mathcal{F})$. Put $\theta_t : \Omega \to \Omega$, $t \geq 0$, for the usual shift maps, and, unless otherwise indicated, $X_t(\omega) = \omega(t)$ will denote the coordinate variables on Ω. Let $\mathcal{P}(\mathcal{F}_t)$ denote the σ-field of $(\mathcal{F}_t)$-predictable sets in $[0,\infty) \times \Omega$.

Write $C_b(E)$ for the Banach space of bounded continuous real-valued functions on E. If E is locally compact, $C_\ell(E)$ (respectively, $C_0(F)$) is the subspace of functions which have a finite limit at infinity (respectively, approach zero at infinity). Write $b\mathcal{E}$ for the set of bounded $\mathcal{E}$-measurable real-valued functions. Set $(b\mathcal{E})_+$ (respectively, $C_\ell^+(E)$, $C_b^+(E)$) to be the cone of non-negative functions in $b\mathcal{E}$ (respectively, $C_\ell(E)$, $C_b(E)$). Finally,

$$C_\ell^2(\mathbb{R}^d) = \{\phi \in C_\ell(\mathbb{R}^d) : \phi \text{ has continuous first and second partial derivatives,}$$
$$\Delta\phi \in C_\ell(\mathbb{R}^d)\}.$$

Write $\mu(f)$ for $\int f d\mu$.

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1 The Predictable Representation Property

We begin by recalling the martingale characterization of a class of superprocesses from Fitzsimmons (1988, 1992). Let $Y = (D, \mathcal{D}, \mathcal{D}_{t+}, \theta_t, Y_t, P_y)$ be the canonical realization of a Hunt process (quasi-left continuous, Borel right process) on a Polish state space E. Here, $\mathcal{D}$ is the Borel σ-field of D, the space of càdlàg E-valued paths, $Y_t(y) = y(t),\ y \in D$, and $\mathcal{D}_t = \sigma(Y_s : s \leq t)$. Let B denote the class of finely continuous functions in $b\mathcal{E}$ and write U^α for the α-resolvent of Y. The domain of the weak infinitesimal generator, G, of Y is $D(G) = U^\alpha(B) \subset B$ (independent of $\alpha > 0$) and for $f \in B$,

$$G(U^1 f) = U^1 f - f \ \in B.$$

It follows from Fitzsimmons (1988, Thm 4.1) and (1992, Thm. 1.5) that for each $m \in M_F(E)$, there is a unique probability $\mathbb{P}_m$ on $\Omega = (\Omega_E, \mathcal{F})$ that solves the following martingale problem (which we label as (M_m)):

$$X_0 = m,\ \mathbb{P}_m - \text{a.s.},$$
$$X_t(\phi) = X_0(\phi) + Z_t(\phi) + \int_0^t X_s(G\phi)ds,$$

$\forall t \geq 0$, $\mathbb{P}_m$ - a.s., $\forall \phi \in D(G)$; where $Z_t(\phi)$ is an a.s. continuous $(\mathcal{F}_t)$-martingale such that

$$\langle Z(\phi)\rangle_t = \int_0^t X_s(\phi^2)ds,$$

$\forall t \geq 0$, $\mathbb{P}_m$-a.s. The probability $\mathbb{P}_m$ is usually called the law of the $(Y, -\lambda^2/2)$-superprocesses starting at m. As we will restrict ourselves to finite variance branching mechanisms scaled as above we simply call $\mathbb{P}_m$ the law of the Y-superprocess starting at m. If Y is a Feller process with a locally compact state space, the above result holds with G the strong infinitesimal generator of Y on its domain $D(G) \subset C_\ell(E)$ (Ethier-Kurtz (1986, p. 404)). When G is the generator of Brownian motion on $\mathbb{R}^d$ (we write $G = \Delta/2$), $\mathbb{P}_m$ is the law of super-Brownian motion.

The set $D(G)$ is bounded-pointwise dense in $b\mathcal{E}$, and so Z_t extends trivially to an orthogonal martingale measure $\{Z_t(\phi) : \phi \in b\mathcal{E},\ t \geq 0\}$. Recall that $\mathcal{P}$ denotes the predictable σ-field for the filtration $(\mathcal{F}_t)$. As in Walsh (1986, Ch. 2) (and by a trivial localization argument), we may define $Z_t(\phi) = \int_0^t \int \phi(s,\omega,x)dZ(s,x)$ for any $\mathcal{P} \times \mathcal{E}$-measurable function $\phi : [0,\infty) \times \Omega \times E \to \mathbb{R}$ such that

$$\int_0^t \int \phi(s,\omega,x)^2 X_s(dx)ds < \infty, \ \forall t > 0, \ \mathbb{P}_m - \text{a.s.}$$

We denote the above class of integrands by $L^2_{loc}(X,\mathbb{P}_m)$ and write $\phi \in L^2(X,\mathbb{P}_m)$ (respectively, $L^2_\infty(X,\mathbb{P}_m)$) if, in addition,

$$\mathbb{P}_m(\int_0^t \int \phi(s,\omega,x)^2 X_s(dx)ds) < \infty, \ \forall t > 0,$$

(respectively,

$$\mathbb{P}_m(\int_0^\infty \int \phi(s,\omega,x)^2 X_s(dx)ds) < \infty).)$$

For $\phi \in L^2_{loc}(X,\mathbb{P}_m)$ (respectively, $L^2(X,\mathbb{P}_m)$) $Z_t(\phi)$ is a continuous local martingale (respectively, square integrable martingale) such that $\langle Z(\phi)\rangle_t = \int_0^t \int \phi(s,\omega,x)^2 X_s(dx)ds$.

We now prove the predictable representation property for X under $\mathbb{P}_m$. Recall that Y is a Hunt process with a Polish state space and $X = (\Omega,\mathcal{F},\mathcal{F}_t,X_t,\mathbb{P}_m)$ is the canonical realization of the Y-superprocess.

Theorem 1.1. *If $V \in L^2(\Omega,\mathcal{F},\mathbb{P}_m)$, there is an f in $L^2_\infty(X,\mathbb{P}_m)$ such that*

$$V = \mathbb{P}_m(V) + \int_0^\infty \int f(s,\omega,x)dZ(s,x), \ \mathbb{P}_m - a.s.$$

In particular, every square integrable $(\mathcal{F}_t)$-martingale, M_t, under $\mathbb{P}_m$ may be written as

$$M_t = \mathbb{P}_m(M_0) + \int_0^t \int f(s,\omega,x)dZ(s,x), \ \forall t \geq 0, \ \mathbb{P}_m - a.s.$$

for some $f \in L^2(X,\mathbb{P}_m)$.

Proof. It suffices to prove the second assertion.

As the martingale problem (M_m) is well-posed, we see from Theorem 2 and Proposition 2 of Jacod (1977) that for each $n \in \mathbb{N}$ there exist a finite set of functions $\phi_n^1,\ldots,\phi_n^{N(n)} \in D(G)$ and a finite set of $\mathcal{P}$ - measurable processes $h_n^1,\ldots,h_n^{N(n)}$ such that

$$f_n(s,\omega,x) \equiv \sum_i h_n^i(s,\omega)\phi_n^i(x) \in L^2(X,\mathbb{P}_m)$$

and

$$M_t = \mathbb{P}_m(M_0) + \lim_{n\to\infty} \int_0^t \int f_n(s,\omega,x) dZ(s,x)$$

in $L^2((\Omega, \mathcal{F}, \mathbb{P}_m)$ for each $t \geq 0$. Hence for each $t \geq 0$ we have that

$$\lim_{n,n'\to\infty} \mathbb{P}_m(\int_0^t \int [f_n(s,\omega,x) - f_{n'}(s,\omega,x)]^2 \, X_s(dx)\, ds)$$

$$= \lim_{n,n'\to\infty} \mathbb{P}_m([\int_0^t \int f_n(s,\omega,x)dZ(s,x) - \int_0^t \int f_{n'}(s,\omega,x)dZ(s,x)]^2) = 0.$$

Thus there exists $f \in L^2(X, \mathbb{P}_m)$ such that for each $t \geq 0$

$$\lim_{n\to\infty} \mathbb{P}_m([\int_0^t \int f_n(s,\omega,x)dZ(s,x) - \int_0^t \int f(s,\omega,x)dZ(s,x)]^2)$$

$$= \lim_{n\to\infty} \mathbb{P}_m(\int_0^t \int [f_n(s,\omega,x) - f(s,\omega,x)]^2 \, X_s(dx)\, ds) = 0$$

as required. □

Remark 1.2. (a) The above representation is reminiscent of the multiple stochastic integrals of Dynkin (1988). In fact the integrals are quite different. Dynkin was motivated by different questions and his multiple integrals were not martingales in the upper limit of integration.

(b) An analogous representation theorem for martingales with respect to the excursion fields of a one-dimensional Brownian motion is given in Rogers-Walsh (1991). The martingales there are represented as stochastic integrals with respect to the local time sheet. Le Gall (1991, 1993) has shown there is a close connection between the branching structure of X and the excursions of one-dimensional Brownian motion. It would be interesting if one could derive Theorem 1.1 from Theorem 2.1 of Rogers-Walsh (1991). In fact the above result seems to be the simpler one, so perhaps the converse question would be more natural.

2 On Dawson's Girsanov Theorem

We consider a bivariate version of the Girsanov theorem of Dawson (1978). The key ideas may be found in Dawson (1978) but we derive a result which is well-suited for our needs and may be used to verify the hypothesis of Theorem 5.1 of Dawson (1978).

Let ξ_i be a Hunt process with Polish state space E_i, α-resolvent U_i^α, and weak infinitesimal generator G_i for $i = 1, 2$ (see the previous section). Let $m_i \in M_F(F_i)$, and let $\mathbb{P}^i_{m_i}$ denote the law of the ξ_i-superprocess on $(\Omega_i, \mathcal{F}_i) = (\Omega_{E_i}, \mathcal{F}_{E_i})$.

Definition. We say that a pair of a.s.-continuous $(\mathcal{F}'_t)$-adapted $M_F(\mathbb{R}^d)$-valued processes (X^1, X^2) on some filtered space $(\Omega', \mathcal{F}', \mathcal{F}'_t, \mathbb{P}')$ satisfies (M_{m_1,m_2}) if

$$X^i_t(\phi) = m_i(\phi) + Z^i_t(\phi) + \int_0^t X^i_s(G_i\phi)ds - A^i_t(\phi)$$

$\forall t \geq 0$, $\mathbb{P}' -$ a.s, $\forall \phi \in D(G_i)$, $i = 1, 2$; where the $Z^i_t(\phi)$ are continuous $(\mathcal{F}'_t)$-martingales such that $Z^i_0(\phi) = 0$ a.s. and

$$\langle Z^i(\phi_i), Z^j(\phi_j)\rangle_t = \delta_{ij} \int_0^t X^i_s(\phi_i^2)ds$$

$\forall t \geq 0$, $\mathbb{P}' -$ a.s., $\forall \phi_i \in D(G_i)$, $\forall \phi_j \in D(G_j)$, and the A^i_t are (a.s.) non-decreasing, continuous, $(\mathcal{F}'_t)$-adapted, $M_F(E_i)$-valued processes starting at zero.

If (X^1, X^2) satisfies (M_{m_1,m_2}) with $A^1 = A^2 = 0$ then X^i has law $\mathbb{P}^i_{m_i}$ (see (M_m)) and Theorem 1.1 shows X^1 and X^2 are independent. (This could also be derived directly as for (M_m).) We call X^1 and X^2 independent ξ_1- and ξ_2-superprocesses with respect to $(\mathcal{F}'_t)$ in this case.

The next result was proved in [BEP, Thm 5.1] for $E_1 = E_2 = \mathbb{R}^d$ and $G_1 = G_2 = \Delta/2$. The proof extends with only notational changes to the present setting as well as to the case when ξ_i are Feller processes with strong infinitesimal generators G_i.

Theorem 2.1. *Let (X^1, X^2) satisfy (M_{m_1,m_2}) on some $(\Omega', \mathcal{F}', \mathcal{F}'_t, \mathbb{P}')$ and let*

$$(\Omega, \mathcal{F}, \mathcal{F}_t) = (\Omega' \times \Omega_1 \times \Omega_2, \mathcal{F}' \times \mathcal{F}_1 \times \mathcal{F}_2, \mathcal{F}'_t \times \mathcal{F}^1_t \times \mathcal{F}^2_t).$$

Let $\pi : \Omega \to \Omega'$ be the projection map. There is a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and $M_F(E_i)$-valued processes Y^1, Y^2 on $(\Omega, \mathcal{F}, \mathbb{P})$ such that:

(a) If $W \in b\mathcal{F}'$ then $\mathbb{P}(W \circ \pi | \mathcal{F}_t) = \mathbb{P}'(W|\mathcal{F}'_t) \circ \pi$, $\mathbb{P}$-a.s.

(b) Y^1 and Y^2 are independent ξ_1- and ξ_2-superprocesses with respect to $(\mathcal{F}_t)$ and $Y_0^i = m_i$, $\mathbb{P}$-a.s., $i = 1, 2,$.

(c) $X_t^i \circ \pi \le Y_t^i$, $\forall t \ge 0$, $\mathbb{P}$-a.s., $i = 1, 2$.

(d) If $Z_t^{Y^i}(\phi)$ is the martingale part of $Y_t^i(\phi)$ for $\phi \in D(G_i)$ then

$$\langle Z^{Y^i}(\phi_i), Z^j(\phi_j) \circ \pi \rangle_t = \delta_{ij} \int_0^t X_s^i \circ \pi(\phi_i^2) ds$$

$\forall t \ge 0$, $\mathbb{P} -$ a.s., $\forall \phi_i \in D(G_i)$, $\forall \phi_j \in D(G_j)$.

Remark 2.2. (a) The probability $\mathbb{P}$ is constructed as follows. If $(\omega', \omega_1, \omega_2)$ denotes a point in Ω, the ω' marginal is $\mathbb{P}'$ and conditional on ω', (ω_1, ω_2) are independent ξ_1- and ξ_2-superprocesses, respectively with zero initial conditions and time-inhomogeneous immigration given by $(A^1(dt, dx)(\omega'),$ $A^2(dt, dx)(\omega'))$ (see Dynkin 1993, Thm. 3.1, 4.1)). The process Y^i is given by $Y_t^i(\omega', \omega_1, \omega_2) = X_t^i(\omega') + \omega_i(t)$.

(b) Part (a) of the theorem implies $((X^1, X^2, A^1, A^2), \mathcal{F}'_t, \mathbb{P}')$ and $((X^1, X^2, A^1, A^2) \circ \pi,\ \mathcal{F}_t,\ \mathbb{P})$ have the same adapted distribution in the sense of Hoover-Keisler (1984). This means that all the random variables obtained from $(X, A) \equiv (X^1, X^2, A^1, A^2)$ by the operations of composition with bounded continuous functions and conditional expectation with respect to $(\mathcal{F}'_t)$ have the same law under $\mathbb{P}'$ as the corresponding random variables obtained from $(X, A) \circ \pi$ and $(\mathcal{F}_t)$ under $\mathbb{P}$. In particular (X, A) and $(X, A) \circ \pi$ have the same law on their respective spaces and $(X, A) \circ \pi$ will also satisfy (M_{m_1, m_2}) on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. In the future we may, and shall, study (X, A) through its clone $(X, A) \circ \pi$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and hence will simply assert the existence of a dominating pair of independent superprocesses (Y^1, Y^2).

For the rest of this section we work on the product space $(\Omega^2, \mathcal{F}^2) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ with its canonical right-continuous filtration $(\mathcal{F}_t^2)$, shift operators θ_t^2, $t \ge 0$, and coordinate variables $X^i(\omega_1, \omega_2)(t) = \omega_i(t)$, $X(t) = (X^1(t), X^2(t))$.

Definition. If $m_i \in M_F(E_i)$ and $g_i : [0,\infty) \times \Omega^2 \times E_i \to \mathbb{R}$ is $\mathcal{P}(\mathcal{F}_t^2) \times \mathcal{E}_i$-measurable for $i = 1, 2$, we say that a probability $\mathbb{P}$ on $(\Omega^2, \mathcal{F}^2)$ solves the martingale problem (M_{g_1,g_2}) if

$$X_0^i = m_i, \ \mathbb{P} - \text{a.s.},$$
$$\int_0^t \int |g_i(s,\omega,x)\phi(x)| X_s^i(dx)ds < \infty$$

and

$$X_t^i(\phi) = X_0^i(\phi) + Z_t^{i,g_i}(\phi) + \int_0^t X_s^i(G_i\phi)ds + \int_0^t \int g_i(s,\omega,x)\phi(x)X_s^i(dx)ds$$

$\forall t \geq 0$, $\mathbb{P}$-a.s., $\forall \phi \in D(G_i)$, $i = 1, 2$; where $Z_t^{i,g_i}(\phi)$ is an a.s. continuous $(\mathcal{F}_t^2)$-martingale under $\mathbb{P}$ such that

$$\langle Z^{i,g_i}(\phi_i), Z^{j,g_j}(\phi_j)\rangle_t = \delta_{ij} \int_0^t X_s^i(\phi_i^2)ds$$

$\forall t \geq 0$, $\mathbb{P}$-a.s., $\forall \phi_i \in D(G_i)$, $\forall \phi_j \in D(G_j)$.

For such a $\mathbb{P}$, Z^{i,g_i} extends to an orthogonal martingale measure and, as for ordinary superprocesses, one may define $\int_0^t \int \phi(s,\omega,x)dZ^{i,g_i}(s,x)$ for $\mathcal{P}(\mathcal{F}_t^2) \times \mathcal{E}_j$-measurable ϕ satisfying

$$I(\phi)(t) \equiv \int_0^t \int \phi(s,x,\omega)^2 X_s^i(dx)ds < \infty, \ \forall t > 0, \ \mathbb{P} - \text{a.s.}$$

This stochastic integral is a continuous $(\mathcal{F}_t^2)$-local martingale under $\mathbb{P}$ with square function $I(\phi)(t)$.

For $\phi \in D(G_i)$ we will use (M_{g_1,g_2}) to define $Z_t^{i,g_i}(\phi)(\omega)$ on $\{(t,\omega) : \int_0^t \int |g_i(s,\omega,x)\phi(x)|X_s^i(dx)ds < \infty\}$ and set $Z_t^{i,g_i}(\phi) = 0$ on the complement of this set. In this way $Z_t^{i,g_i}(\phi)$ is canonically defined on path space.

Theorem 2.3. *Let $m_i \in M_F(E_i)$ $(i = 1, 2,)$ and assume g_i is $\mathcal{P}(\mathcal{F}_t^2) \times \mathcal{E}_i$-measurable and satisfies*

$$(2.1) \qquad \int_0^t \int g_i(s,\omega,x)^2 X_s^i(dx)ds < \infty, \ \forall t > 0, \ \mathbb{P}_{m_1}^1 \times \mathbb{P}_{m_2}^2 - a.s..$$

Let

$$g = (g_1, g_2)$$

and

$$R_t^g = \exp(\sum_{i=1}^2 \int_0^t \int g_i(s,\omega,x) Z^{i,0}(ds,dx) - \frac{1}{2}\int_0^t \int g_i(s,\omega,x)^2 X_s^i(dx)ds),$$

where $Z^{i,0}$ *is defined by* $(M_{0,0})$.

a) *If* $\mathbb{P}$ *is a solution of* (M_{g_1,g_2}) *such that*

$$\sum_{i=1}^2 \int_0^t \int g_i(s,\omega,x)^2 X_s^i(dx)ds < \infty, \tag{2.2}$$

$\mathbb{P}$*-a.s.,* $\forall t > 0$*, then*

$$\frac{d\mathbb{P}}{d\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}}|_{\mathcal{F}_t^2} = R_t^g \tag{2.3}$$

In particular, there is at most one solution of (M_{g_1,g_2}) *satisfying (2.2).*

b) *If* $g_i \le c$ *for* $i = 1, 2$*, for some* $c \in \mathbb{R}$*, then* R_t^g *is a* $\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}$*-martingale and (2.3) defines the unique solution* $\mathbb{P}$ *of* (M_{g_1,g_2}).

Note: The bound in (b) is only one-sided.

Proof. Let

$$T_n = \inf\{t : \sum_{i=1}^2 \int_0^t (\int (g_i(s,\omega,x)^2 + 1) X_s^i(dx) + 1) ds \ge n\}.$$

Observe that $T_n \le n$.

(a) Assume $\mathbb{P}$ solves (M_{g_1,g_2}) and satisfies (2.2). Let

$$\tilde{R}^g_{t\wedge T_n} = \exp(\sum_{i=1}^2 - \int_0^{t\wedge T_n} \int g_i(s,\omega,x) Z^{i,g_i}(ds,dx)$$

$$-\frac{1}{2}\int_0^{t\wedge T_n} \int g_i(s,\omega,x)^2 X_s^i(dx)ds).$$

For n fixed, $\tilde{R}^g_{t\wedge T_n}$ is a uniformly integrable martingale (under $\mathbb{P}$) and $d\mathbb{Q}_n = \tilde{R}^g_{T_n} d\mathbb{P}$ defines a probability on $\mathcal{F}^2$. Some elementary stochastic calculus shows that for $\phi \in D(G_i)$, $Z^{i,0}_{t\wedge T_n}(\phi)\tilde{R}^g_{t\wedge T_n}$ is a $\mathbb{P}$-local martingale, and therefore $Z^{i,0}_{t\wedge T_n}(\phi)$ is a continuous $\mathbb{Q}_n$-local martingale. As $\mathbb{Q}_n << \mathbb{P}$ we also have

$$\langle Z^{i,0}_{\cdot\wedge T_n}(\phi_i), Z^{j,0}_{\cdot\wedge T_n}(\phi_j)\rangle_t = \delta_{ij}\int_0^{t\wedge T_n} X^i_s(\phi_i^2)ds,$$

$\forall t \geq 0$, $\mathbb{Q}_n$-a.s, $\forall \phi_i \in D(G_i)$, $\forall \phi_j \in D(G_j)$. The bound $\int_0^{T_n} X^i_s(1)ds \leq n$ shows $Z^{i,0}_{\cdot\wedge T_n}(\phi_i)$ is a $\mathbb{Q}_n$-martingale $\forall \phi_i \in D(G_i)$. Therefore $\mathbb{Q}_n$ solves $(M_{0,0})$ "up to T_n". Let $\tilde{\mathbb{Q}}_n$ be the unique probability on $(\Omega^2, \mathcal{F}^2)_n$ such that $\tilde{\mathbb{Q}}_n|_{\mathcal{F}^2_{T_n}} = \mathbb{Q}_n|_{\mathcal{F}_{T^2_n}}$ and the $\tilde{\mathbb{Q}}_n$-conditional law of $X_{T_n+\cdot}$ given $\mathcal{F}^2_{T_n}$ is $\mathbb{P}^1_{X^1_{T_n}} \times \mathbb{P}^2_{X^2_{T_n}}$. It is now easy to see that $\tilde{\mathbb{Q}}_n$ solves $(M_{0,0})$ and this implies $\tilde{\mathbb{Q}}_n = \mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}$ (see the remark prior to Theorem 2.1). Therefore

$$\mathbb{Q}_n(T_n < t) = \tilde{\mathbb{Q}}_n(T_n < t) = \mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}(T_n < t) \to 0 \text{ as } n \to \infty,\ \forall t > 0,$$

by (2.1) and we have (note that (2.2) shows that $\tilde{R}^g_t$ is well-defined under $\mathbb{P}$)

$$\text{(2.4)}\quad \begin{aligned} \mathbb{P}(\tilde{R}^g_t) &\geq \mathbb{P}(\tilde{R}^g_{t\wedge T_n} 1(T_n \geq t)) \\ &= \mathbb{P}(\tilde{R}^g_{t\wedge T_n}) - \mathbb{P}(\tilde{\mathbb{R}}^g_{T_n} 1(T_n < t)) \\ &= 1 - \mathbb{Q}_n(T_n < t) \to 1 \text{ as } n \to \infty. \end{aligned}$$

This shows $\tilde{R}^g_t$ is a $\mathbb{P}$-martingale and hence there is a unique measure $\mathbb{Q}$ on $\mathcal{F}^2$ such that $d\mathbb{Q}|_{\mathcal{F}^2_t} = \tilde{R}^g_t d\mathbb{P}|_{\mathcal{F}^2_t}$ $\forall t \geq 0$. Repeating the above arguments, but now without the T_n's, one sees that $\mathbb{Q} = \mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}$, because $\mathbb{Q}$ solves $(M_{0,0})$. The only point on which we need to comment is the fact that $Z^{i,0}_t(\phi)$ is a $\mathbb{Q}$-martingale, as opposed to just a $\mathbb{Q}$-local martingale. Let $\{S_n\}$ be a sequence of stopping times reducing $Z^{i,0}_t(1)$. Then

$$\mathbb{Q}(X^i_t(1)) \leq \liminf_{n\to\infty} \mathbb{Q}(X^i_{t\wedge S_n}(1)) = m_i(1).$$

This shows that $\langle Z^{i,0}(\phi)\rangle_t$ is square integrable under $\mathbb{Q}$ and hence that $Z^{i,0}_t(\phi)$ is a $\mathbb{Q}$-martingale for $\phi \in D(G_i)$. Therefore we conclude that $\forall t > 0$

$$d\mathbb{P}|_{\mathcal{F}^2_t} = (\tilde{R}^g_t)^{-1} d(\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2})|_{\mathcal{F}^2_t} = R^g_t d(\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2})|_{\mathcal{F}^2_t}.$$

(b) Assume first $g_i \leq 0$. Let $g_{i,n}(s,\omega,x) = 1(s \leq T_n)g_i(s,\omega,x)$ and $g_n = (g_{1,n}, g_{2,n})$. Then

$$R^g_{t\wedge T_n} = R^{g_n}_t \tag{2.5}$$

$\forall t \geq 0$, $\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}$-a.s. As in (a) $d\mathbb{Q}_n = R^{g_n}_{T_n} d(\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2})$ is a probability on $\mathcal{F}^2$ and a standard Girsanov argument shows that $\mathbb{Q}_n$ solves $(M_{g_{1,n},g_{2,n}})$. The only non-obvious point is again the fact that $Z^{i,g_{i,n}}_t(\phi_i)$ is a $\mathbb{Q}_n$-martingale (not just a local martingale) for $\phi_i \in D(G_i)$. To see this, note that $X^i_t(1) \leq m_i(1) + Z^{i,g_{i,n}}_t(1)$ and argue as before. By Theorem 2.1 and Remark 2.2 we may assume (by passing to a larger space) that there are processes (Y^1, Y^2) with law $\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}$ and (X^1, X^2) with law $\mathbb{Q}_n$ such that $X^i_t \leq Y^i_t$, $\forall t \geq 0$, a.s. Therefore

$$\mathbb{Q}_n(T_n < t) \leq \mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}(T_n < t) \to 0 \text{ as } n \to \infty,\ \forall t > 0,$$

the last by (2.1). Now argue as in (2.4) to see that $\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}(R^g_t) = 1$ and hence R^g_t is a martingale (recall (2.5)).

It is now straightforward to show that (2.3) defines a solution $\mathbb{P}$ of (M_{g_1,g_2}). Turning to uniqueness, let $\mathbb{P}$ be any solution of (M_{g_1,g_2}). Theorem 2.1 shows that by passing to a larger space we may assume there are processes (X^1, X^2) with law $\mathbb{P}$ and dominating processes (Y^1, Y^2) with law $\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}$. Condition (2.1) guarantees that $\mathbb{P}$ satisfies (2.2). Part (a) now shows that $\mathbb{P}$ is given by (2.3).

Consider now $g_i \leq c$ and let $f_i = g_i - c \leq 0$. Condition (2.1) continues to hold with f_i in place of g_i. The previous case shows the unique solution $\mathbb{P} = \mathbb{P}^f$ of (M_{f_1,f_2}) is given by (2.3) with $f = (f_1, f_2)$ in place of g. Let

$$R_t = \exp(\sum_{i=1}^2 cZ^{i,f_i}_t(1) - \frac{c^2}{2}\int_0^t X^i_s(1)ds)$$

$$U_n = \inf\{t : X^1_t(1) + X^2_t(1) \geq n\} \wedge n.$$

Let $\mathbb{P}$ be a solution of (M_{g_1,g_2}). Then

$$X^i_{t\wedge U_n}(1) \leq m_i(1) + Z^{i,g_i}_{t\wedge U_n}(1) + c\int_0^{t\wedge U_n} X^i_s(1)ds,$$

$\forall t \geq 0$, $\mathbb{P}$-a.s. Take means and use Gronwall's and Fatou's lemmas to conclude

$$\mathbb{P}(X^i_t(1)) \leq m_i(1)e^{ct},\ \forall t \geq 0$$

and therefore

$$(2.6)\qquad \mathbb{P}(\int_0^t (X_s^1 + X_s^2)(c^2)ds) < \infty$$

The latter inequality plays the role of (2.2) and allows us to argue just as in (a) with $\mathbb{P}_1^{m_1} \times \mathbb{P}_2^{m_2}$ replaced by the equivalent law $\mathbb{P}^f$ and R_t^g replaced by R_t, to conclude that

$$\frac{d\mathbb{P}}{d\mathbb{P}^f}|_{\mathcal{F}_t^2} = R_t.$$

A simple calculation leads to (2.3), giving uniqueness in (M_{g_1,g_2}).

Now argue just as in Lemma 10.1.2.1 of Dawson (1992) to see that R_t is a $\mathbb{P}^f$-martingale. It is then easy to check that $d\mathbb{P}_{|\mathcal{F}_t^2} = R_t d\mathbb{P}^f_{|\mathcal{F}_t^2}$ solves (M_{g_1,g_2}). The uniqueness established above gives (2.3) and hence shows R_t^g is a $\mathbb{P}_{m_1}^1 \times \mathbb{P}_{m_2}^2$-martingale. □

Remark. Part (b) of the above, or more precisely its counterpart on $(\Omega_1, \mathcal{F}_1)$, appeared in the penultimate draft of Dawson (1992) but unfortunately failed to make the final cut.

Definition. If F is a Borel subset of $M_F(E_1) \times M_F(E_2)$ and $\{\mathbb{Q}_m : m \in F\}$ is a family of probabilities on $(\Omega^2, \mathcal{F}^2, \mathcal{F}_t^2)$, we say $(\Omega^2, \mathcal{F}^2, \mathcal{F}_t^2, \theta_t^2, X_t, (\mathbb{Q}_m)_{m \in F})$ is an F-valued diffusion iff

(i) $\mathbb{Q}_m(X_t \in F, \forall t \geq 0, X_0 = m) = 1$, $\forall m \in F$.

(ii) $m \mapsto \mathbb{Q}_m$ is a Borel measurable map from F to $M_1(\Omega^2)$.

(iii) For any $(\mathcal{F}_t^2)$-stopping time T such that $\mathbb{Q}_m(T < \infty) = 1$,

$$\mathbb{Q}_m(X \circ \theta_T^2 \in A | \mathcal{F}_T^2)(\omega) = \mathbb{Q}_{X_T(\omega)}(A),\ \mathbb{Q}_m - \text{a.s.},\ \forall A \in \mathcal{F}^2.$$

An analogous definition may be made for F a Borel subset of $M_F(E_1)$ and $\{\mathbb{Q}_m : m \in F\}$ probabilities on $\Omega = C([0,\infty), M_F(E_1))$.

Definition. If $\mathcal{C} \subset b\mathcal{E} \times b\mathcal{E}$, the bounded pointwise closure of $\mathcal{C}$ is the smallest class $\bar{\mathcal{C}}$ in $b\mathcal{E} \times b\mathcal{E}$ which contains $\mathcal{C}$ and such that $(\phi, \psi) \in \bar{\mathcal{C}}$ whenever $(\phi_n, \psi_n) \in \bar{\mathcal{C}}$ and $\phi_n \stackrel{bp}{\to} \phi$, $\psi_n \stackrel{bp}{\to} \psi$.

Lemma 2.4. *There is a countable set $D_i \subset D(G_i)$ such that the bounded pointwise closure of $\{(\phi, G_i\phi) : \phi \in D_i\}$ contains $\{(\phi, G_i\phi) : \phi \in D(G_i)\} =$ $\mathrm{graph}(G_i)$.*

Proof. Let D'_i be a countable set in $C_b(E_i)$ whose bounded pointwise closure is $b\mathcal{E}_i$ (recall E_i is Polish). Since D'_i is contained in the ξ_i-finely continuous functions in $b\mathcal{E}_i$, clearly

$$D_i \equiv \{U_i^1\phi : \phi \in D'_i\} \subset D(G_i).$$

Let $\bar{\mathcal{C}}_i$ denote the bounded pointwise closure of $\{(\phi, G_i\phi) : \phi \in D_i\}$ and let

$$\bar{D}_i = \{\phi \in b\mathcal{E}_i : (U_i^1\phi, U_i^1\phi - \phi) \in \bar{\mathcal{C}}_i\}.$$

If $\phi_n \in \bar{D}_i$ and $\phi_n \stackrel{bp}{\to} \phi$, then $U_i^1\phi_n \stackrel{bp}{\to} U_i^1\phi$ and therefore $(U_i^1\phi, U_i^1\phi - \phi) \in \bar{\mathcal{C}}_i$. Therefore $\bar{D}_i$ is closed under bounded pointwise convergence, and since $D'_i \subset \bar{D}_i$ we conclude that $\bar{D}_i = b\mathcal{E}_i$. This shows that $(U_i^1\phi, U_i^1\phi - \phi) \in \bar{\mathcal{C}}_i$ for all ϕ in $b\mathcal{E}_i$ and, as this set contains $\mathrm{graph}(G_i)$, we are done. □

Theorem 2.5. *Assume $\Gamma_i : M_F(E_1) \times M_F(E_2) \times E_i \to \mathbb{R}$, $i = 1, 2$, are Borel maps such that $\Gamma_i \le c$, $i = 1, 2$, and let $g_i(s, \omega, x) = \Gamma_i(X_s(\omega), x)$. Let F be a Borel subset of $M_F(E_1) \times M_F(E_2)$ such that*

$$(2.1) \text{ holds } \forall m = (m_1, m_2) \in F. \tag{2.7}$$

$$\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}(X_t \in F, \forall t \ge 0) = 1,\ \forall (m_1, m_2) \in F. \tag{2.8}$$

For each $m \in F$ there is a unique solution $\mathbb{P}^g_m$ of (M_{g_1,g_2}) given by (2.3). Moreover, $(\Omega^2, \mathcal{F}^2, \mathcal{F}^2_t, \theta^2_t, X_t, (\mathbb{P}^g_m)_{m\in F})$ is an F-valued diffusion.

Proof. The existence of a unique solution $\mathbb{P}^g_m$ of (M_{g_1,g_2}) which also satisfies $\mathbb{P}^g_m(X_t \in F,\ \forall t \ge 0) = 1$ (the set in question is universally measurable so we are working with completions here) follows from Theorem 2.3, (2.8) and $\mathbb{P}^g_m|_{\mathcal{F}^2_t} << \mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}|_{\mathcal{F}^2_t}$. Since $\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}(\Phi)$ is Borel measurable on $M_F(E_1) \times M_F(E_2)$ for $\Phi \in b\mathcal{F}^2$ and

$$\mathbb{P}^g_m(\psi) = \mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}(\psi R^g_t),\ \forall \psi \in b\mathcal{F}^2_t,$$

it is easy to see that $\mathbb{P}_m^g(\Phi)$ is $\mathcal{B}(F)$-measurable in m, $\forall \Phi \in b\mathcal{F}^2$.

Let T be a bounded $(\mathcal{F}_t^2)$-stopping time and let $\hat{\mathbb{P}}_T(\omega)$ denote a $\mathbb{P}_m^g$-regular conditional distribution of $X \circ \theta_T^2$ given $\mathcal{F}_T^2$. Let $D_i \ni 1$ be as in Lemma 2.4, $\mathcal{F}_s^0 = \sigma(X_u : u \leq s)$ and C_s denote a countable set in $b\mathcal{F}_s^0$ whose bounded pointwise closure is $b\mathcal{F}_s^0$. Let

$$\Lambda = \{\omega \in \Omega^2 : \sum_{i=1}^2 \int_0^t \int |g_i(s,\omega,x)| X_s^i(dx) ds < \infty, \forall t > 0\}.$$

Our definition of $Z^{i,g_i}(\phi_i)$ and the equality $g_i(s, \theta_T^2\omega, x) = g_i(s+T, \omega, x)$ show that (drop dependence on g_i)

$$(2.9)\quad Z_t^i(\phi_i) \circ \theta_T = Z_{t+T}^i(\phi_i) - Z_T^i(\phi_i),\ \forall t \geq 0, \phi_i \in D(G_i),\ i = 1, 2,\ \forall \omega \in \Lambda.$$

If $\phi_i \in D_i$, $\psi \in C_s$ and $s < t$, then, since $\mathbb{P}_m^g(\Lambda) = 1$ by (2.7), we have

$$\begin{aligned} & \mathbb{P}_m^g((Z_t^i(\phi_i) - Z_s^i(\phi_i)) \circ \theta_T(\psi \circ \theta_T) | \mathcal{F}_T^2) \\ = \ & \mathbb{P}_m^g(\mathbb{P}_m^g(Z_{t+T}^i(\phi_i) - Z_{s+T}^i(\phi_i) | \mathcal{F}_{T+s}^2) \psi \circ \theta_T | \mathcal{F}_T^2) = 0,\ \mathbb{P}_m^g - \text{a.s.} \end{aligned}$$

Therefore

$$\hat{\mathbb{P}}_T(\omega)((Z_t^i(\phi_i) - Z_s^i(\phi_i))\psi) = 0,\ \forall \psi \in C_s,\ \mathbb{P}_m^g - \text{a.a. } \omega$$

and so

$$(2.10)\qquad \hat{\mathbb{P}}_T(\omega)(Z_t^i(\phi_i) - Z_s^i(\phi_i) | \mathcal{F}_s^0) = 0$$

for all rationals such that $0 \leq s < t$, $\forall \phi_i \in D_i, i = 1, 2$, $\mathbb{P}_m^g$-a.a. ω.

Clearly $\Lambda \subset (\theta_T^2)^{-1}(\Lambda)$ and so we may fix ω outside a $\mathbb{P}_m^g$-null set such that $\hat{\mathbb{P}}_T(\omega)(\Lambda) = 1$ and (2.10) holds. Our definition of $Z_t^i(\phi)$ shows that on Λ, $Z_t^i\ (\phi)$ is continuous in t, $\forall \phi \in D(G_i)$, and the equality in (M_{g_1,g_2}) holds $\forall \phi \in D(G_i)$. Therefore these last two conclusions hold $\hat{\mathbb{P}}_T(\omega)$-a.s. Since $Z_s^i(\phi_i)$ is $\mathcal{F}_s^0$-measurable, we may take limits from above in $s \in \mathbb{Q}$ in (2.10) to see that $Z^i(\phi_i)$ is an a.s.-continuous $(\mathcal{F}_t^2)$-martingale under $\hat{\mathbb{P}}_T(\omega)$, $\forall \phi_i \in D_i$. Use the pathwise construction of quadratic variation, (2.9) and $\mathbb{P}_m^g|_{\mathcal{F}_t^2} << \mathbb{P}_{m_1}^1 \times \mathbb{P}_{m_2}^2|_{\mathcal{F}_t^2}$ to see that for $\mathbb{P}_m^g$-a.e. ω

$$(2.11)\qquad \langle Z^i(\phi_i), Z^j(\phi_j)\rangle_t = \delta_{ij} \int_0^t X_s^i(\phi_j^2) ds,$$

$\forall t \geq 0$, $\hat{\mathbb{P}}_T(\omega) -$ a.s., $\forall \phi_i \in D_i$, $i = 1, 2$. If our fixed ω also satisfies (2.11), then the above shows that $\hat{\mathbb{P}}_T(\omega)$ solves (M_{g_1,g_2}) with $(m_1, m_2) = (X^1_T(\omega), X^2_T(\omega))$ provided we restrict the class of test functions ϕ_i to D_i. Use Gronwall's lemma as in (2.6) to see that

$$\hat{\mathbb{P}}_T(\omega)\Big(\int_0^t X^1_s(1) + X^2_s(1) ds\Big) \leq (m_1(1) + m_2(1)) \int_0^t e^{cs} ds < \infty. \tag{2.12}$$

Since D_i is bounded pointwise dense in $b\mathcal{E}_i$, we may now extend $Z^i_t(\phi)$ to an almost surely continuous, orthogonal martingale measure as usual (all now with respect to $\hat{\mathbb{P}}_T(\omega)$). Take bounded pointwise limits in $(\phi, G_i\phi)$ to see that (M_{g_1,g_2}) holds (under $\hat{\mathbb{P}}_T(\omega)$) for all ϕ_i in $D(G_i)$. (Note that if $\phi_n \stackrel{bp}{\to} \phi$, then (2.12) shows $Z^i_t(\phi_n) \to Z^i_t(\phi)$ in $L^2(\hat{\mathbb{P}}_T(\omega))$ and we can take limits in (2.11).) Uniqueness of solutions to (M_{g_1,g_2}) shows that $\hat{\mathbb{P}}_T(\omega) = \mathbb{P}^g_{X_T(\omega)}$, which proves the strong Markov property for bounded T. For an arbitrary stopping time T such that $T < \infty$, $\mathbb{P}^g_m$-a.s., a standard truncation argument completes the proof. $\square$

By taking $E_2 = \{0\}$ and $g_2 = 0$ and $m_2 = 0$ in the above we get a corresponding theorem for a solution $\mathbb{P}$ (a probability on $(\Omega, \mathcal{F}, \mathcal{F}_t) = (\Omega_1, \mathcal{F}_1, \mathcal{F}^1_t)$) of the martingale problem (M_g) defined as follows:

$$X_0 = m, \ \mathbb{P} - \text{a.s.},$$
$$X_t(\phi) = X_0(\phi) + Z^g_t(\phi) + \int_0^t X_s(G_1\phi) ds + \int_0^t \int g(s, \omega, x) X_s(dx) ds,$$

$\forall t \geq 0$, $\mathbb{P}$-a.s., $\forall \phi \in D(G_1)$; where $Z^g_t(\phi)$ is an a.s. continuous $(\mathcal{F}_t)$-martingale under $\mathbb{P}$ such that

$$\langle Z^g(\phi)\rangle_t = \int_0^t X_s(\phi^2) ds,$$

$\forall t \geq 0$, $\mathbb{P}$-a.s.

Here $g : [0, \infty) \times \Omega \times E_1 \to \mathbb{R}$ is $\mathcal{P}(\mathcal{F}_t) \times \mathcal{E}_1$-measurable. We let the reader formulate the obvious version of Theorem 2.3, but state the analogue of Theorem 2.5 for future reference.

Corollary 2.6. *Assume $g(s, \omega, x) = \Gamma(X_s(\omega), x)$ for some Borel $\Gamma : M_F(E_1) \times E_1 \to \mathbb{R}$ such that $\Gamma \leq c$. Let F_1 be a Borel subset of $M_F(E_1)$*

such that

$$\int_0^t \int \Gamma(X_s, x)^2 X_s(dx)ds < \infty,\ \forall t > 0,\ \mathbb{P}_m - a.s.,\ \forall m \in F_1 \tag{2.13}$$

$$\mathbb{P}_m(X_t \in F_1, \forall t \geq 0) = 1,\ \forall m \in F_1. \tag{2.14}$$

For each $m \in F_1$ *there is a unique solution* $\mathbb{P}_m^g$ *of* (M_g) *given by*

$$\frac{d\mathbb{P}_m^g}{d\mathbb{P}_m}|_{\mathcal{F}_t} = \exp\{\int_0^t \int \Gamma(X_s, x) Z^{1,0}(ds, dx) - \frac{1}{2}\int_0^t \int \Gamma(X_s, x)^2 X_s(dx)ds\}.$$

The process $(\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, (\mathbb{P}_m^g)_{m \in F_1})$ *is an* F_1*-valued diffusion.*

Remark 2.7. If ξ_i are Feller processes with locally compact state spaces E_i and strong infinitesimal generators G_i on $D(G_i) \subset C_\ell(E_i)$, all the results of this section hold with some minor simplifications in the proofs. □

3 A Two-Type Martingale Problem for Singular Interactions

We specialize the notation of the last section and take $E_i = \mathbb{R}^d$ and $G_i = \Delta/2$, the strong infinitesimal generator of d-dimensional Brownian motion B on its domain $D(\Delta/2) \subset C_\ell(\mathbb{R}^d)$. Hence $\Omega = C([0,\infty), M_F(\mathbb{R}^d))$, $\mathcal{F} = \mathcal{B}(\Omega)$, $(\mathcal{F}_t)$ and $(\mathcal{F}_t^2)$ are the canonical right-continuous filtrations on Ω and Ω^2, respectively, $(\theta_t^2)_{t\geq 0}$ are the shift operators on Ω^2 and X and (X^1, X^2) are the coordinate variables on Ω and Ω^2, respectively. Now $\mathbb{P}_m$ denotes the law of super-Brownian motion on $(\Omega, \mathcal{F})$, starting at m. Let $p_t(x)$ be the standard Brownian transition density (that is, p_t is the density of a Gaussian distribution with mean 0 and variance t).

A key ingredient to our approach to singular interactions is the collision local time of two measure-valued processes, introduced in [BEP].

Definition. For $\epsilon > 0$ define $g_\epsilon : M_F(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ and $L^\epsilon : \Omega^2 \to \Omega$ by $g_\epsilon(\mu, x) = \int p_\epsilon(x-y)\mu(dy)$ and

$$L_t^\epsilon(X^1, X^2)(\phi) = \int_0^t \int g_\epsilon(X_s^1, x)\phi(x)X_s^2(dx)ds, \ \phi \in b\mathcal{B}(\mathbb{R}^d).$$

A pair of continuous $M_F(\mathbb{R}^d)$-valued processes (Y^1, Y^2) on some $(\Omega', \mathcal{F}', \mathbb{P}')$ have collision local time $L_t(Y^1, Y^2)$ iff $L_t(Y^1, Y^2)$ is an a.s. continuous $M_F(\mathbb{R}^d)$-valued process such that $L_t^\epsilon(Y^1(\omega), Y^2(\omega))(\phi) \stackrel{\mathbb{P}'}{\to} L_t(Y^1, Y^2)(\phi)$ as $\epsilon \downarrow 0$, $\forall t \geq 0$ and $\phi \in C_\ell(\mathbb{R}^d)$.

Remarks 3.1. (a) The definition in [BEP] uses another, symmetric, definition of $L^\epsilon(X^1, Y^2)$. However, as is remarked in [BEP, Sec.1], these two different definitions of $L^\epsilon(X^1, Y^2)$ lead to equivalent definitions of $L(Y^1, Y^2)$.

(b) If $L_t(Y^1, Y^2)$ exists it is clearly unique up to evanescent sets and non-decreasing in t a.s. That is, almost surely, $\forall s < t$, $L_t(Y^1, Y^2) - L_s(Y^1, Y^2) \in M_F(\mathbb{R}^d)$. Therefore $L(Y^1, Y^2)((s,t] \times A) = L_t(Y^1, Y^2)(A) - L_s(Y^1, Y^2)(A)$ extends to an a.s. unique measure $L(Y^1, Y^2)(dt, dx)$ on $\mathcal{B}([0,\infty) \times \mathbb{R}^d)$ which is supported by the intersection of the closed graphs of Y^1 and Y^2 (see Section 1 of [BEP]). Intuitively, $L(Y^1, Y^2)$ measures the space-time distribution of the collisions between the two populations Y^1 and Y^2.

Proposition 3.2. *Let Y^1, Y^2 be continuous $M_F(\mathbb{R}^d)$-valued processes on some $(\Omega', \mathcal{F}', \mathbb{P}')$ which have a collision local time $L_t(Y^1, Y^2)$. Let $\mathbb{P}_Y$ be the law of $Y = (Y^1, Y^2)$ on $(\Omega^2, \mathcal{F}^2)$. There is an $(\mathcal{F}_t^2)$-predictable mapping $\tilde{L} : [0, \infty) \times \Omega^2 \to M_F(\mathbb{R}^d)$ which depends only on $\mathbb{P}_Y$ and satisfies*

(i) $\tilde{L}_t(Y^1(\omega), Y^2(\omega)) = L_t(Y^1, Y^2)(\omega)$, $\forall t \geq 0$, $\mathbb{P}'$-a.s.

(ii) $\tilde{L}_t \circ \theta_s^2 = \tilde{L}_{t+s} - \tilde{L}_s$, $\forall s, t \geq 0$, $\mathbb{P}_Y$-a.s.

(iii) $\tilde{L}_t(\theta_s^2 \omega) = \lim_{k\to\infty} L_t^{\eta_k}(\theta_s^2 \omega)$, $\forall s, t \geq 0$, $\mathbb{P}_Y$-a.a. ω for some sequence $\eta_k \downarrow 0$.

Proof. A diagonalization argument shows there is a countable dense set D in $C_\ell^+(\mathbb{R}^d)$ and a sequence $\eta_k \downarrow 0$ (depending only on $\mathbb{P}^Y$) such that

$$L_t^{\eta_k}(Y^1(\omega), Y^2(\omega))(\phi) \to L_t(Y^1, Y^2)(\omega)(\phi),$$

$\forall t \in \mathbb{Q} \cap [0, \infty)$, $\forall \phi \in D$, $\mathbb{P}'$-a.s. As the limit is a.s. continuous in t and the approximating processes are non-decreasing, an elementary argument shows that

$$\sup_{t \leq T} |(L_t^{\eta_k}(Y^1(\omega), Y^2(\omega)) - L_t(Y^1, Y^2))(\phi)| = 0$$

$\forall T > 0$, $\forall \phi \in D$, $\mathbb{P}'$-a.s. This implies

$$L_t^{\eta_k}(Y^1(\omega), Y^2(\omega)) \to L_t(Y^1, Y^2)(\omega) \text{ in } M_F(\mathbb{R}^d) \tag{3.1}$$

$\forall t \geq 0$, $\mathbb{P}'$-a.s. Let

$$\Lambda = \{\omega \in \Omega^2 : L_t^{\eta_k}(\omega) \text{ converges in } M_F(\mathbb{R}^d) \text{ as } k \to \infty,\ \forall t \geq 0\}.$$

Note that $\mathbb{P}_Y(\Lambda) = 1$ by (3.1). Define

$$\tilde{L}_t(\omega) = \begin{cases} \lim_{k\to\infty} L_t^{\eta_k}(\omega), & \text{if the limit exists in } M_F(\mathbb{R}^d), \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $\tilde{L}$ is $(\mathcal{F}_t^2)$-predictable because L^{η_k} is. Clearly, (i) is immediate from (3.1). If $\omega \in \Lambda$ and $s, t \geq 0$ then

$$L_t^{\eta_k}(\theta_s^2 \omega) = L_{t+s}^{\eta_k}(\omega) - L_s^{\eta_k}(\omega) \tag{3.2}$$

shows that $\theta_s^2\omega \in \Lambda$ and therefore, by the definition of $\tilde{L}$,

$$\begin{aligned} \tilde{L}_t(\theta_s^2\omega) &= \lim_{k\to\infty} L_t^{\eta_k}(\theta_s^2\omega) \\ &= \lim_{k\to\infty} L_{t+s}^{\eta_k}(\omega) - L_s^{\eta_k}(\omega) \quad \text{(by (3.2))} \\ &= \tilde{L}_{t+s}(\omega) - \tilde{L}_s(\omega), \end{aligned}$$

the last because $\omega \in \Lambda$. This gives (ii) and the first line in the above gives (iii). $\square$

Remark 3.3. (a) If Y is as above and $\mathcal{F}_t^Y = \cap_n \sigma(Y_s : s \le t + n^{-1})$, then by the above we may, and shall, assume $L_t(Y^1, Y^2)$ is $(\mathcal{F}_t^Y)$-predictable. When $Y = X$ on $(\Omega', \mathcal{F}') = (\Omega^2, \mathcal{F}^2)$ the two notations L and $\tilde{L}$ can be confusing. Our convention will be to write $L_t(X^1, X^2)$ for $\tilde{L}_t(X^1, X^2)$ and hence treat $L_t(X^1, X^2)$ as a predictable function on $[0, \infty) \times \Omega^2$. Note, however, the function depends on the underlying probability $\mathbb{P}$ on $(\Omega^2, \mathcal{F}^2)$.

(b) In the above argument the sequence $\{\eta_k\}$ may be taken as an appropriate subsequence of any given sequence $\{\epsilon_n\}$ decreasing to zero. This allows us to construct a single sequence, and hence a single $\tilde{L}$, which satisfies the conclusions of the above theorem for a pair of given $M_F(\mathbb{R}^d)^2$-valued processes (Y^1, Y^2) and $(\tilde{Y}^1, \tilde{Y}^2)$, each possessing a collision local time.

Let $\mathcal{M}(m_1, m_2)$ denote the set of a.s. continuous $M_F(\mathbb{R}^d)^2$-valued processes which satisfy (M_{m_1,m_2}) (now with $G_i = \Delta/2$). Note the underlying probability space is allowed to vary. If $\underline{\Omega}' = (\Omega', \mathcal{F}', \mathbb{P}')$ is given let

$$\mathcal{M}(\underline{\Omega}') = \mathcal{M}(\underline{\Omega}', m_1, m_2) = \{(Y^1, Y^2) \in \mathcal{M}(m_1, m_2) : Y^1, Y^2 \text{ are defined on } \underline{\Omega}'\}.$$

In this case the filtration associated with (Y^1, Y^2) in (M_{m_1,m_2}) is still allowed to vary.

If $m_1, m_2 \in M_F(\mathbb{R}^d)$ satisfy a mild finite energy condition ((5.1) below), $d \le 5$, and $(Y^1, Y^2) \in \mathcal{M}(m_1, m_2)$, then $L_t(Y^1, Y^2)$ exists [BEP, Thm. 5.9]. If (Y^1, Y^2) are independent super-Brownian motions (i.e. $A^i \equiv 0$ in (M_{m_1,m_2})) and $m_i \ne 0$ then $L_t(Y^1, Y^2)$ is non-trivial [BEP, Remark 5.12]. For our purposes it will be the uniform (in $\mathcal{M}$) results which will be important.

Notation. Let $B(x, r)$ be the open ball in $\mathbb{R}^d$ of radius r and centered at x. If $m \in M_F(\mathbb{R}^d)$, let $D(m, r) = \sup\{m(B(x, r)) : x \in \mathbb{R}^d\}$. Set $M_F^s(\mathbb{R}^d) =$

$\{m \in M_F(\mathbb{R}^d) : \int_0^1 r^{1-d} D(m,r)dr < \infty\}$. Clearly, $M_F^s(\mathbb{R}^d)$ is a Borel subset of $M_F(\mathbb{R}^d)$.

Lemma 3.4. *Assume $d \le 3$, $m_i \in M_F^s(\mathbb{R}^d)$, $i = 1, 2$, and $\psi \in C_b(\mathbb{R}^d)$. Then* $\forall T > 0$

$$\lim_{\epsilon\downarrow 0} \sup_{(\tilde{X}^1,\tilde{X}^2)\in\mathcal{M}(m_1,m_2)} \| \sup_{t\le T} |L_t^\epsilon(\tilde{X}^1(\omega), \tilde{X}^2(\omega))(\psi) - L_t(\tilde{X}^1, \tilde{X}^2)(\omega)(\psi)| \wedge 1\|_1$$

$$= 0$$

Proof. If L^ϵ is replaced by

$$\tilde{L}_t^\epsilon(\tilde{X}^1, \tilde{X}^2)(\phi) = \int_0^t \int\int p_\epsilon(x_2 - x_1)\phi((x_1 + x_2)/2)\tilde{X}_s^1(dx_1)\tilde{X}_s^2(dx_2)ds,$$

this result is contained in [BEP, Thm. 5.10]. Let $(\tilde{X}^1, \tilde{X}^2) \in \mathcal{M}(m_1, m_2)$ and assume without loss of generality there are (Y^1, Y^2) with law $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$ such that $\tilde{X}^i \le Y^i$ a.s. (Theorem 2.1 and Remark 2.2). If $\psi \in C_b(\mathbb{R}^d)$ and $T > 0$

$$\sup_{t\le T} |\tilde{L}_t^\epsilon(\tilde{X}^1, \tilde{X}^2)(\psi) - L_t^\epsilon(\tilde{X}^1, \tilde{X}^2)(\psi)| \tag{3.3}$$

$$\le \int_0^T \int\int p_\epsilon(x_2 - x_1)|\psi(x_2) - \psi((x_1 + x_2)/2)|Y_s^1(dx_1)Y_s^2(dx_2)ds$$

$$\le \int_0^T \int\int p_\epsilon(x_2-x_1)1(|x_2-x_1| < \delta)|\psi(x_2)-\psi((x_1+x_2)/2)|Y_s^1(dx_1)Y_s^2(dx_2)ds$$

$$+2\|\psi\|_\infty \epsilon^{-d/2} \exp\{-\delta^2/2\epsilon\} \int_0^T Y_s^1(1)Y_s^2(1)ds.$$

Using the fact that $L_T^\epsilon(Y^1(\omega), Y^2(\omega))(1) \stackrel{L^1}{\to} L_T(Y^1, Y^2)(1)$ as $\epsilon \downarrow 0$ ([BEP, Thm. 5.9]), it is easy to make the first term arbitrarily small in $\| \quad \|_1$ for all sufficiently small ϵ by fixing δ sufficiently small (use the fact that most of the mass of $L_T(Y^1, Y^2)$ may be supported on a compact set on which ψ is uniformly continuous). The last term in (3.3) clearly approaches zero in L^1 as $\epsilon \downarrow 0$ for our fixed δ. Therefore the left side of (3.3) converges in L^1 to zero as $\epsilon \downarrow 0$ uniformly in $(\tilde{X}^1, \tilde{X}^2)$ and the proof is complete. □

Definition. Let ρ be a bounded metric on $M_F(\mathbb{R}^d)$ which induces the weak topology, and let $\underline{\Omega}' = (\Omega', \mathcal{F}', \mathbb{P}')$. Identify processes in $\mathcal{M}(\underline{\Omega}', m_1, m_2)$ which agree up to a $\mathbb{P}'$-evanescent set and define a metric d on $\mathcal{M}(\underline{\Omega}')$ by

$$d((Y^1, Y^2), (W^1, W^2)) = \sum_{n=1}^{\infty} \mathbb{P}'(\sup_{t \le n} \rho(Y_t^1, W_t^1) + \rho(Y_t^2, W_t^2))2^{-n}.$$

Let $\mathcal{C}(\underline{\Omega}')$ be the set of measurable maps $L : \Omega' \to C([0, \infty), \mathbb{R})$ and identify maps which agree up to $\mathbb{P}'$-null sets. Hence processes $L(\omega', t)$ which are $a.s.$ continuous in t are considered to be elements of $\mathcal{C}(\underline{\Omega}')$. Define a metric d' on $\mathcal{C}(\underline{\Omega}')$ by

$$d'(L_1, L_2) = \sum_{n=1}^{\infty} \mathbb{P}'(\sup_{t \le n} |L_1(t) - L_2(t)| \wedge 1)2^{-n}.$$

Lemma 3.5. *Assume $d \le 3$, $m_i \in M_F^s(\mathbb{R}^d)$, $i = 1, 2$, and $\psi \in C_b(\mathbb{R}^d)$. Then $(Y^1, Y^2) \to L(Y^1, Y^2)(\psi)$ is a continuous mapping from $\mathcal{M}(\underline{\Omega}', m_1, m_2)$ to $\mathcal{C}(\underline{\Omega}')$.*

Proof. Let $\psi_\epsilon(x_1, x_2) = p_\epsilon(x_2 - x_1)\psi(x_2)$ for $\epsilon > 0$, and define $T_\epsilon : \Omega^2 \to C([0, \infty), \mathbb{R})$ by

$$T_\epsilon(\mu^1, \mu^2)(t) = \int_0^t \int \int \psi_\epsilon d(\mu_s^1 \times \mu_s^2) ds$$

If $(\mu_n^1, \mu_n^2) \to (\mu^1, \mu^2)$ in Ω^2, then clearly $T_\epsilon(\mu_n^1, \mu_n^2)(t) \to T_\epsilon(\mu^1, \mu^2)(t)$ pointwise. It is easy to see $\{T_\epsilon(\mu_n^1, \mu_n^2) : n \in \mathbb{N}\}$ are uniformly equicontinuous and therefore $T_\epsilon(\mu_n^1, \mu_n^2) \to T_\epsilon(\mu^1, \mu^2)$ in $C([0, \infty), \mathbb{R})$. That is, T_ϵ is continuous. It is now easy to check that

$$(Y^1, Y^2) \mapsto L_\cdot^\epsilon(Y^1, Y^2)(\psi) = T_\epsilon(Y^1(\omega), Y^2(\omega))$$

is continuous as a mapping from $\mathcal{M}(\underline{\Omega}')$ to $\mathcal{C}(\underline{\Omega}')$. Lemma 3.4 shows that $L_\cdot(Y^1, Y^2)(\psi)$ is a uniform limit of these continuous maps and therefore is also a continuous map from $\mathcal{M}(\underline{\Omega}')$ to $\mathcal{C}(\underline{\Omega}')$. □

Recall the "competing species" model described in the Introduction. Casualities may occur in either population when type 1 and type 2 particles collide. We are ready to state a martingale problem for this model. Let λ denote a non-negative parameter which gives the intensity of killing when particles collide. A probability $\mathbb{P}$ on $(\Omega^2, \mathcal{F}^2)$ solves $(M_{\lambda L})$ if and only if

$$X_0^i = m_i$$

$\mathbb{P}$-a.s., $i = 1, 2$,

$$X_t^i(\phi) = X_0^i(\phi) + Z_t^i(\phi) + \int_0^t X_s^i(\Delta\phi/2)ds - \lambda L_t(X^1, X^2)(\phi),$$

$\forall t \geq 0$, $\mathbb{P}$-a.s. $\forall \phi \in D(\Delta/2)$, $i = 1, 2$; where $Z_t^i(\phi)$ is an a.s. continuous $(\mathcal{F}_t^2)$-martingale under $\mathbb{P}$ such that

$$\langle Z^i(\phi_i), Z^j(\phi_j)\rangle_t = \delta_{ij} \int_0^t X_s^i(\phi_i^2)ds,$$

$\forall t \geq 0$, $\mathbb{P}$-a.s., $\forall \phi_i, \phi_j \in D(\Delta/2)$.

Implicit in $(M_{\lambda L})$ is the existence of $L_t(X^1, X^2)$. A pair of a.s. continuous $M_F(\mathbb{R}^d)$-valued processes (Y^1, Y^2) on some $(\Omega', \mathcal{F}', \mathbb{P}')$ solves $(M_{\lambda L})$ iff their law $\mathbb{P}_Y$ on $(\Omega^2, \mathcal{F}^2)$ is a solution.

Theorem 3.6. *If $d \leq 3$ and $m_1, m_2 \in M_F^s(\mathbb{R}^d)$, then a solution to $(M_{\lambda L})$ exists $\forall \lambda \geq 0$.*

Proof. If $\epsilon > 0$ then clearly (2.1) is satisfied by

$$g_1(s, \omega, x) = -\lambda g_\epsilon(X_s^2, x) \leq 0 \text{ and } g_2(s, \omega, x) = -\lambda g_\epsilon(X_s^1, x) \leq 0.$$

Therefore Theorem 2.3(b) implies that (use the notation from (M_{g_1,g_2}))

$$\begin{aligned} \frac{d\mathbb{P}_\epsilon}{d\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}}\Big|_{\mathcal{F}_t^2} &= \exp\{-\lambda \int_0^t \int g_\epsilon(X_s^2, x) Z^{1,0}(ds, dx) - \lambda \int_0^t \int g_\epsilon(X_s^1, x) Z^{2,0}(ds, dx) \\ &- \frac{\lambda^2}{2} \int_0^t \int g_\epsilon(X_s^2, x)^2 X_s^1(dx)ds - \frac{\lambda^2}{2} \int_0^t \int g_\epsilon(X_s^1, x)^2 X_s^2(dx)ds\} \end{aligned}$$

defines the unique solution $\mathbb{P}_\epsilon$ to the martingale problem (M_ϵ) defined as follows:

$$X_0^i = m_i \text{ a.s.}$$

$$X_t^1(\phi) = X_0^1(\phi) + \tilde{Z}_t^{1,\epsilon}(\phi) + \int_0^t X_s^1(\Delta\phi/2)ds - \lambda L_t^\epsilon(X^2, X^1)(\phi),$$

$\forall t \geq 0$, $\mathbb{P}_\epsilon$-a.s., $\forall \phi \in D(\Delta/2)$,

$$X_t^2(\phi) = X_0^2(\phi) + \tilde{Z}_t^{2,\epsilon}(\phi) + \int_0^t X_s^2(\Delta\phi/2)ds - \lambda L_t^\epsilon(X^1, X^2)(\phi),$$

$\forall t \geq 0$, $\mathbb{P}_\epsilon -$ a.s., $\forall \phi \in D(\Delta/2)$; where $\tilde{Z}_t^{i,\epsilon}(\phi)$ is an a.s. continuous $(\mathcal{F}_t^2)$-martingale (under $\mathbb{P}_\epsilon$) such that

$$\langle \tilde{Z}^{i,\epsilon}(\phi_i), \tilde{Z}^{j,\epsilon}(\phi_j)\rangle_t = \delta_{ij} \int_0^t X_s^i(\phi_i^2)ds,$$

$\forall t \geq 0$, $\mathbb{P}_\epsilon$-a.s., $\forall \phi_i, \phi_j \in D(\Delta/2)$.

Therefore (X^1, X^2) on $(\Omega^2, \mathcal{F}^2, \mathbb{P}_\epsilon)$ belongs to $\mathcal{M}(m_1, m_2)$ and by Theorem 2.1 we may work on a larger space ($(\Omega', \mathcal{F}') = (\Omega^4, \mathcal{F}^4)$ will do) with a filtration $(\mathcal{F}_t')$ and a probability $\mathbb{P}_\epsilon'$ carrying processes $(X^{1,\epsilon}, X^{2,\epsilon})$ which satisfy (M_ϵ) and independent $(\mathcal{F}_t')$-super-Brownian motions $(Y^{1,\epsilon}, Y^{2,\epsilon})$ starting at m_1 and m_2, respectively such that $X_t^{i,\epsilon} \leq Y_t^{i,\epsilon}$, $\forall t \geq 0$, $\mathbb{P}_\epsilon'$-a.s.

Choose $\epsilon_n \downarrow 0$ and let

$$\mathbb{P}_n(\cdot) = \mathbb{P}'_{\epsilon_n}((X^{1,\epsilon_n} X^{2,\epsilon_n}, L^{\epsilon_n}(X^{2,\epsilon_n}, X^{1,\epsilon_n}), L^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n})) \in \cdot).$$

We claim $\{\mathbb{P}_n\}$ is a tight sequence of probabilities on $(\Omega^4, \mathcal{F}^4)$. If $T, \delta, \eta > 0$ and $\phi \in C_b^+(\mathbb{R}^4)$, then

$$\limsup_{n\to\infty} \mathbb{P}'_{\epsilon_n}(\sup\{|L_t^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n})(\phi) - L_s^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n})(\phi)| : s,t \leq T, |s-t| \leq \delta\} > \eta)$$

$$\leq \limsup_{n\to\infty} \mathbb{P}'_{\epsilon_n}(\sup\{|L_t^{\epsilon_n}(Y^{1,\epsilon_n}, Y^{2,\epsilon_n})(\phi) - L_s^{\epsilon_n}(Y^{1,\epsilon_n}, Y^{2,\epsilon_n})(\phi)| : s,t \leq T, |s-t| \leq \delta\} > \eta)$$

$$\leq \mathbb{P}_{m_1} \times \mathbb{P}_{m_2}(\sup\{|L_t(X^1, X^2)(\phi) - L_s(X^1, X^2)(\phi)| : s,t \leq T, |s-t| \leq \delta\} > \eta/2)$$

$$\to 0 \text{ as } \delta \downarrow 0,$$

where the second inequality follows from Lemma 3.4. Theorem 8.2 of Billingsley (1968) implies $\{L_\cdot^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n})(\phi) : n \in \mathbb{N}\}$ is tight in $C = C([0,\infty), \mathbb{R})$, $\forall \phi \in C_\ell(\mathbb{R}^d)$ (i.e. their laws under $\mathbb{P}'_{\epsilon_n}$ are tight). By Dawson (1991, Thm. 4.6.1) (the result given there for càdlàg $M_F(\mathbb{R}^d)$-valued processes carries over to continuous $M_F(\mathbb{R}^d)$-valued processes), $\{L^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n}) : n \in \mathbb{N}\}$ is tight in $C([0,\infty), M_F(\bar{\mathbb{R}}^d))$, where $\bar{\mathbb{R}}^d$ is the one-point compactification of $\mathbb{R}^d$. The domination of $L^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n})$ by $L^{\epsilon_n}(Y^{1,\epsilon_n}, Y^{2,\epsilon_n})$ shows that each limit point (a law on $C([0,\infty), M_F(\bar{\mathbb{R}}^d))$) is in fact supported by $C([0,\infty), M_F(\mathbb{R}^d)) = \Omega$. The tightness of $\{L^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n}) : n \in \mathbb{N}\}$ in Ω now follows.

Consider next the tightness of $\{X^{2,\epsilon_n} : n \in \mathbb{N}\}$, i.e., of their laws on $(\Omega, \mathcal{F})$. Let $\phi \in D(\Delta/2)$. The domination $X^{2,\epsilon_n} \le Y^{2,\epsilon_n}$ shows that $\{\int_0^\cdot X_s^{2,\epsilon_n}(\Delta\phi/2)ds : n \in \mathbb{N}\}$ is tight in $C([0,\infty), \mathbb{R})$. The same domination, Burkholder's inequality and Theorem 12.3 of Billingsley (1968) show that $\{\tilde{Z}^{2,\epsilon_n}(\phi) : n \in \mathbb{N}\}$ is tight in C. The tightness of $\{X_\cdot^{2,\epsilon_n}(\phi) : n \in \mathbb{N}\}$ in C now follows from (M_{ϵ_n}) and the tightness of $\{L_\cdot^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n})(\theta) : n \in \mathbb{N}\}$ proved above. Now proceed as for $\{L_\cdot^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n}) : n \in \mathbb{N}\}$ to conclude that $\{X_\cdot^{2,\epsilon_n} : n \in \mathbb{N}\}$ is tight in Ω. The tightness of $\{\mathbb{P}_n\}$ follows.

By Skorokhod's representation theorem (Ethier-Kurtz (1986, p. 102)), we may redefine $(X^{1,\epsilon_n}, X^{2,\epsilon_n}, L^{\epsilon_n}(X^{2,\epsilon_n}, X^{1,\epsilon_n}), L^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n}))$ as adapted processes on a common filtred probability space $\underline{\Omega}' = (\Omega', \mathcal{F}', \mathcal{F}'_t, \mathbb{P}')$ such that this 4-tuple converges $\mathbb{P}'$-a.s. to $(\tilde{X}^1, \tilde{X}^2, A^1, A^2)$. Clearly each A^i is an a.s. non-decreasing, continuous $M_F(\mathbb{R}^d)$-valued process. Routine arguments show that $\forall \phi_i \in D(\Delta/2)$

$$Z_t^i(\phi_i) = \tilde{X}_t^i(\phi_i) - m_i(\phi_i) - \int_0^t \tilde{X}_s^i(\Delta\phi_i/2)ds + \lambda A_t^i(\phi_i) \tag{3.4}$$

is a continuous $(\mathcal{F}'_t)$-martingale such that

$$\langle Z^i(\phi_i), Z^j(\phi_j)\rangle_t = \delta_{ij} \int_0^t \tilde{X}_s^i(\phi_i^2)ds \tag{3.5}$$

(the bounds $X^{i,\epsilon_n} \le Y^{i,\epsilon_n}$ give the necessary integrability conditions). Therefore $(\tilde{X}^1, \tilde{X}^2) \in \mathcal{M}(\underline{\Omega}', m_1, m_2)$. If $\phi \in D(\Delta/2)$, then

$$\begin{aligned} & d'(L_\cdot^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon})(\phi), L_\cdot(\tilde{X}^1, \tilde{X}^2)(\phi)) \\ \le\ & d'(L_\cdot^{\epsilon_n}(X^{1,\epsilon_n}, X^{2,\epsilon_n})(\phi), L_\cdot(X^{1,\epsilon_n}, X^{2,\epsilon_n})(\phi)) \\ & + d'(L_\cdot(X^{1,\epsilon_n}, X^{2,\epsilon_n})(\phi), L_\cdot(\tilde{X}^1, \tilde{X}^2)(\phi)) \\ & \to 0 \text{ as } n \to \infty, \end{aligned}$$

where we have used Lemma 3.4 to handle the first term and Lemma 3.5 to handle the second. This proves $A^1_t = A^2_t = L_t(\tilde{X}^1, \tilde{X}^2)$. Therefore (3.4) and (3.5) show that the law of $(\tilde{X}^1, \tilde{X}^2)$ on $(\Omega^2, \mathcal{F}^2)$ is a solution of $(M_{\lambda L})$. $\square$

Here then is the fundamental conjecture which we have not been able to prove in dimensions $d = 2, 3$.

Conjecture 3.7. If $d \le 3$ and $m_1, m_2 \in M^s_F(\mathbb{R}^d)$ the solution to $(M_{\lambda L})$ is unique.

In the rest of this section we assume $d = 1$ and show how Dawson's Girsanov theorem proves the conjecture in this case.

Let $U : M_F(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ be the Borel measurable mapping

$$U(\mu, x) = \begin{cases} \lim_{n\to\infty} \frac{n}{2}\mu([x - \frac{1}{n}, x + \frac{1}{n}]), & \text{if it exists} \\ 0, & \text{otherwise.} \end{cases}$$

Also consider the $\mathcal{P}(\mathcal{F}^2_t)\times\mathcal{B}(\mathbb{R})$-measurable canonical "densities" $u_i(t, \omega, x) = U(X^i_t(\omega), x)$, $i = 1, 2,$. It is easy to check that

$$\Omega_{ac} = \{\omega \in \Omega : \omega(t) << dx,\ \forall t > 0\}(dx \text{ is Lebesgue measure})$$

is a universally measurable subset of Ω. Clearly, if $\omega \in \Omega_{ac}$, then $\omega_t(dx) = U(\omega(t), x)dx$, $\forall t > 0$.

Lemma 3.8. *Suppose that $d = 1$. Assume $\tilde{X} = (\tilde{X}^1, \tilde{X}^2)$ satisfies (M_{m_1,m_2}) on some $(\Omega', \mathcal{F}', \mathcal{F}'_t, \mathbb{P}')$. Then $\tilde{X}^i_t(dx) = U(\tilde{X}^i_t, x)dx$, $\forall t > 0$, $i = 1, 2$ $\mathbb{P}'$-a.s. and*

$$L_t(\tilde{X}^1, \tilde{X}^2)(\phi) = \int_0^t \int \phi(x)U(\tilde{X}^1_s, x)U(\tilde{X}^2_s, x)dxds, \tag{3.6}$$

$\forall t \ge 0$, $\forall \phi \in b\mathcal{B}(\mathbb{R})$, $\mathbb{P}'$-a.s.

Proof. By Theorem 2.1 (see also the ensuing Remarks) we may assume without loss of generality there are a pair of independent $(\mathcal{F}'_t)$-super Brownian motions (Y^1, Y^2) starting at m_1, m_2, such that $\tilde{X}^i_t \le Y^i_t$, $\forall t \ge 0$, $\mathbb{P}'$-a.s. The measure Y^i_s has a density $w_i(s, x) = U(Y^i_s, x)$ which is jointly continuous on

$(0,\infty)\times\mathbb{R}$, $\mathbb{P}'$-a.s. (see Konno-Shiga (1988, Thm. 1.4) or Reimers (1989, Thm. 7.1)). Therefore $\tilde{X}^1_t(dx)=v_i(t,x)dx$, $\forall t>0$, $\mathbb{P}'$-a.s., where

$$v_i(t,\omega,x)=U(\tilde{X}^i_t(\omega),x)\le w_i(t,\omega,x),$$

$\forall t>0$, $\forall x\in\mathbb{R}$, $\mathbb{P}'$-a.s. Let

$$v_i^\epsilon(t,x)=\int p_\epsilon(x-y)v_i(t,y)dy$$

and

$$w_i^\epsilon(t,x)=\int p_\epsilon(x-y)w_i(t,y)dy.$$

Observe that

$$\lim_{\epsilon\downarrow 0} v_i^\epsilon(t,x)=v_i(t,x) \tag{3.7}$$

and

$$\lim_{\epsilon\downarrow 0} w_i^\epsilon(t,x)=w_i(t,x), \tag{3.8}$$

Lebesgue-a.a. x, $\forall t>0$, $\mathbb{P}'$-a.s.

By continuity, it suffices to prove (3.6) for a fixed $t>0$ and $\phi\in C_b^+(\mathbb{R})$. Choose $\epsilon_n\downarrow 0$ such that $L_t^{\epsilon_n}(\tilde{X}^1,\tilde{X}^2)(\phi)\to L_t(\tilde{X}^1,\tilde{X}^2)(\phi)$, $\mathbb{P}'$-a.s. (see Lemma 3.4). Since

$$L_t^{\epsilon_n}(\tilde{X}^1,\tilde{X}^2)(\phi)=\int_0^t\int v_1^{\epsilon_n}(s,y)v_2(s,y)\phi(y)dyds,$$

Fatou's lemma together with (3.7) gives

$$L_t(\tilde{X}^1,\tilde{X}^2)(\phi)\ge\int_0^t\int v_1(s,y)v_2(s,y)\phi(y)dydx,\ \mathbb{P}'-\text{a.s.} \tag{3.9}$$

To complete the proof it suffices to show

$$\lim_{n\to\infty}\mathbb{P}'(\int_0^t\int v_1^{\epsilon_n}(s,y)v_2(s,y)\phi(y)dyds) \tag{3.10}$$

$$=\mathbb{P}'(\int_0^t\int v_1(s,y)v_2(s,y)\phi(y)dyds)<\infty.$$

Indeed this, together with Fatou's lemma, implies

$$\begin{aligned}\mathbb{P}'(L_t(\tilde{X}^1,\tilde{X}^2)(\phi)) &\le \liminf_{n\to\infty}\mathbb{P}'(L_t^{\epsilon_n}(\tilde{X}^1,\tilde{X}^2)(\phi))\\ &= \mathbb{P}'(\int_0^t\int v_1(s,y)v_2(s,y)\phi(y)dyds)\end{aligned}$$

and (3.9) would then give the required result (3.6). To prove (3.10) define a finite measure $\nu_{\tilde{X}^2}$ on $[0,t]\times\mathbb{R}^d$ by

$$\nu_{\tilde{X}^2}(A)=\mathbb{P}'(\int_0^t\int 1_A(s,x)v_2(s,x)dxds)$$

and similarly define ν_{Y^2}. In view of (3.7), (3.10) is equivalent to the uniform of integrability of $\{v_1^{\epsilon_n}:n\in\mathbb{N}\}$ with respect to $\nu_{\tilde{X}^2}$, which is implied by the uniform integrability of $\{w_1^{\epsilon_n}:n\in\mathbb{N}\}$ with respect to ν_{Y^2}. The latter is equivalent to (see (3.8))

$$\mathbb{P}'(\lim_{n\to\infty}\int w_1^{\epsilon_n}(s,x_2)d\nu_{Y^2}(s,x_2))=\mathbb{P}'(\int w_1(s,x_2)d\nu_{Y_2}(s,x_2)). \tag{3.11}$$

The left side of (3.11) equals

$$\lim_{n\to\infty}\mathbb{P}'(\int_0^t\int\int p_{\epsilon_n}(x_2-x_1)w_1(s,x_1)w_2(s,x_2)dx_1dx_2ds)$$

$$=\lim_{n\to\infty}\int_0^t\int\int\int\int p_{\epsilon_n}(x_2-x_1)p_s(x_1-z_1)p_s(x_2-z_2)dx_1dx_2m_1(dz_1)m_2(dz_2)ds$$

(by Konno-Shiga (1988, (2.14))

$$=\lim_{n\to\infty}\int_0^t\int\int p_{\epsilon_n+2s}(z_1-z_2)m_1(dz_1)m_2(dz_2)ds$$

$$=\int_0^t\int p_{2s}(z_1-z_2)m_1(dz_1)m_2(dz_2)ds,$$

the last by dominated convergence ($p_{\epsilon_n+2s}\le cs^{-1/2}$). Again (2.14) of Konno-Shiga (1988) shows the last integral equals $\mathbb{P}'(\int w_1(s,x_2)d\nu_{Y^2}(s,x_2))$. This gives (3.11) and we are done. □

Recall the notation $Z^{i,0}$ from (M_{g_1,g_2}) but now with $E_i=\mathbb{R}$ and $G_i=\Delta/2$.

Theorem 3.9. *Let $d = 1$, $\lambda \geq 0$ and*

$$F = \{(m_1, m_2) \in M_F(\mathbb{R})^2 : \int\int \log^+(1/|x_1 - x_2|) m_1(dx_1) m_2(dx_2) < \infty\}.$$

(a) $\forall m = (m_1, m_2) \in F$ there is a unique solution $\mathbb{P}_m$ to $(M_{\lambda L})$.
(b) $\forall m \in F$, $\forall t > 0$

$$\frac{d\mathbb{P}_m}{d\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}}|_{\mathcal{F}^2_t} = \begin{array}{ll} \exp & \{-\lambda(\int_0^t \int u_2(s,x) dZ^{1,0}(s,x) + \int_0^t \int u_1(s,x) dZ^{2,0}(s,x)) \\ & -(\lambda^2/2) \int_0^t \int u_2(s,x)^2 u_1(s,x) + u_1(s,x)^2 u_2(s,x) dx ds\}. \end{array}$$

(c) $(\Omega^2, \mathcal{F}^2, \mathcal{F}^2_t, \theta^2_t, X_t, (\mathbb{P}_m)_{m\in F})$ is an F-valued diffusion.

Proof. Lemma 3.8 shows that $\mathbb{P}$ solves $(M_{\lambda L})$ if and only if $\mathbb{P}$ solves (M_{g_1,g_2}) with $g_1(s,w,x) = -\lambda U(X^2_s(\omega), x)$ and $g_2(s,\omega,x) = -\lambda U(X^1_s(\omega), x)$. As $-\lambda U \leq 0$, the Theorem will follow from Theorem 2.5 once (2.7) and (2.8) are verified.

Letting $t_0 \downarrow 0$ in (2.14) of Konno-Shiga (1988), we have

$$(3.12) \qquad u_i(t,x) = \int p_t(x-y) m_i(dy) + \int_0^t \int p_{t-s}(x-y) dZ^{i,0}(s,y),$$

$\mathbb{P}^i_{m_i}$-a.s., $\forall (t,x)$. Therefore, if $(m_1, m_2) \in F$,

$$\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}(\int_0^t \int u_1(s,x)^2 u_2(s,x) dx ds)$$

$$= \int_0^t \int [(\int p_s(x-y) m_1(dy))^2 + \int_0^s \int p_{s-v}(x-y)^2 \mathbb{P}_{m_1}(u_1(v,y)) dy dv]$$
$$\times (\int p_s(x-w) m_2(dw)) dx ds$$

$$\leq \int_0^t \int [s^{-1/2} m_1(1)(\int p_s(x-y) m_1(dy)) + \int_0^s \int p_s(x-z)(2\pi)^{-1/2}(s-v)^{-1/2} dm_1(z) dv]$$
$$\times (\int p_s(x-w) m_2(dw)) dx ds$$

$$= m_1(1) \int\int\int_0^t s^{-1/2} p_{2s}(y_1 - y_2) ds\, m_1(dy_1) m_2(dy_2)$$
$$+ 2(2\pi)^{-1/2} \int\int\int_0^t s^{1/2} p_{2s}(y_1 - y_2) ds\, m_1(dy_1) m_2(dy_2)$$

$$\leq m_1(1)\int\int 1+\log^+(4t/(y_1-y_2)^2)m_1(dy_1)m_2(dy_2)+2(2\pi)^{-1/2}m_1(1)m_2(1)t$$

which is finite since $(m_1, m_2) \in F$. This proves (2.7). Turning to (2.8), recall that $u^i(s,x)$ is a (jointly) continuous density for X^i_s, $\forall s > 0$, $\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}$-a.s. (Reimers (1989)), and has compact support in x, $\forall s > 0$, $\mathbb{P}^1_{m_1} \times \mathbb{P}^2_{m_2}$-a.s. (Dawson-Iscoe-Perkins (1989, Thm. 1.2)). Condition (2.8) follows and the proof is complete. □

We close this section with a related martingale problem on Ω for a self-interacting population. For θ, $\lambda \geq 0$ and $m \in M_F(\mathbb{R})$ we will say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ solves the martingale problem $(M^{\theta,\lambda}_m)$ if the following holds:

$$X_0 = m,\ \mathbb{P}-\text{a.s.},$$

$$X_t(\phi) = X_0(\phi) + Z^{\theta,\lambda}_t(\phi) + \int_0^t X_s(\frac{\phi''}{6} + \theta\phi - \lambda U(X_s,\cdot)\phi)ds,$$

$\forall t \geq 0$, $\mathbb{P}$-a.s., $\forall \phi \in C^2_\ell(\mathbb{R})$; where $Z^{\theta,\lambda}(\phi)$ is a martingale such that

$$\langle Z^{\theta,\lambda}(\phi)\rangle_t = \int_0^t X_s(\phi^2)ds,$$

$\forall t \geq 0$, $\mathbb{P}$-a.s.

The presence of $U(X_s, x)$ in the above suggests we are only interested in $\mathbb{P}$ such that $\mathbb{P}(\Omega_{ac}) = 1$. We will see that this is in fact a consequence of $(M^{\theta,\lambda}_m)$.

Solutions to $(M^{\theta,\lambda}_m)$ were conjectured by Rick Durrett to arise as a limit of rescaled one-dimensional contact processes as the interaction range approaches infinity. The $-\lambda U(X_s,\cdot)$ term in $(M^{\theta,\lambda}_m)$ kills particles at a rate proportional to their local density. It arises from the approximating contact processes because of the suppression of "offspring" which jump onto an occupied site. The $\phi''/6$ term (as opposed to the usual $\phi''/2$) arises from this particular approximation. In Perkins (1989a) it is shown that a discrete time version of these contact processes do converge weakly to the unique solution of $(M^{\theta,\lambda}_m)$. Here we will only show how Corollary 2.6 give existence and uniqueness of solutions to $(M^{\theta,\lambda}_m)$. This result is due to Don Dawson who told one of us that his Girsanov approach will work in this setting. Mueller and Tribe (1993) study the properties of solutions of $(M_m^{\theta,\lambda})$ and confirm Durrett's conjecture.

Let $\mathbb{P}'_m$ denote the law of super-Brownian motion on $(\Omega, \mathcal{F})$ but now scale the Brownian motion to have generator $\Delta/6$. Also let Z_t denote the associated orthogonal martingale measure, i.e., $Z_t = Z_t^{0,0}$ in the above notation.

Theorem 3.10. *Let $\theta, \lambda \geq 0$ and*

$$F_1 = \{m \in M_F(\mathbb{R}) : \int\int \log^+(1/|x_1 - x_2|)dm(x_1)dm(x_2) < \infty\}.$$

(a) $\forall m \in F_1$ there is a unique solution $\mathbb{P}_m^{\theta,\lambda}$ to $(M_m^{\theta,\lambda})$
(b) $\forall m \in F_1$, $\mathbb{P}_m^{\theta,\lambda}(\Omega_{ac}) = 1$, $\mathbb{P}_m^{\theta,\lambda}|_{\mathcal{F}_t} << \mathbb{P}'_m|_{\mathcal{F}_t}$, $\forall t > 0$, and

$$\frac{d\mathbb{P}_m^{\theta,\lambda}}{d\mathbb{P}'_m}|_{\mathcal{F}_t} = \exp\{\int_0^t \int \theta - \lambda U(X_s, x)dZ(s,x) - \frac{1}{2}\int_0^t \int (\theta - \lambda U(X_s,x))^2 X_s(dx)ds\}$$

(c) $(\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, (\mathbb{P}_m^{\theta,\lambda})_{m \in F_1})$ is an F_1-valued diffusion.

Proof. The result will follow from Corollary 2.6 with $\Gamma(X_s, x) = \theta - \lambda U(X_s, x) \leq \theta$, once conditions (2.13) and (2.14) are verified. (Note that $\mathbb{P}_m^{\theta,\lambda}(\Omega_{ac}) = 1$ is immediate from the absolute continuity result.) As in the proof of Theorem 3.9, (2.14) is clear from the fact that X_t has a continuous density with compact support, $\forall t > 0$, $\mathbb{P}'_m$-a.s.

Let $m \in F_1$ and $u(t, \omega, x) = U(X_t(\omega), x)$. Condition (2.13) would clearly follow from

$$\mathbb{P}'_m(\int_0^T \int u(t,x)^3 dxdt) < \infty, \ \forall T > 0. \tag{3.13}$$

Equation (3.12) and Burkholder's inequality show that

$$\mathbb{P}'_m(u(t,x)^3) \leq c((\int p_t(x-y)dm(y))^3 + \mathbb{P}'_m((\int_0^t X_s(p_{t-s}(x-\cdot)^2)ds)^{3/2})). \tag{3.14}$$

The second term is bounded by

$$\begin{aligned}
&\mathbb{P}'_m((\int_0^t (t-s)^{-3/4} X_s(p_{t-s}(x-\cdot)^{1/2})ds)^{3/2}) \\
&\leq \mathbb{P}'_m(\int_0^t (t-s)^{-3/4} X_s(p_{t-s}(x-\cdot)^{1/2})^{3/2} ds) 2t^{1/8} \text{ (Jensen's inequality)} \\
&\leq \int_0^t (t-s)^{-3/4} \mathbb{P}'_m(X_s(p_{t-s}(x-\cdot))^{3/4} X_s(1)^{3/4}) ds 2t^{1/8} \text{ (Hölder's inequality)} \\
(3.15) \quad &\leq \int_0^t (t-s)^{-3/4} [\mathbb{P}'_m(X_s(p_{t-s}(x-\cdot)))]^{3/4} [\mathbb{P}'_m(X_s(1)^3)]^{1/4} ds 2t^{1/4} \text{ (Hölder again).}
\end{aligned}$$

Now

$$\begin{aligned}
\int [\mathbb{P}'_m(X_s(p_{t-s}(x-\cdot)))]^{3/4} dx &= \int [\int p_t(x-z) dm(z)]^{3/4} dx \\
&\leq \int \int p_t(x-z)^{3/4} dm(z) m(1)^{-1/4} dx \\
&\leq ct^{1/8} m(1)^{3/4}.
\end{aligned}$$

It is clear from this that (3.15) is integrable in (t, x) over $[0, T] \times \mathbb{R}$, $\forall T > 0$. From (3.13) and (3.14) it remains only to show

$$(3.16) \qquad \int_0^T \int (\int p_t(x-y) dm(y))^3 dx dt < \infty, \ \forall T > 0.$$

The left side is bounded by

$$\begin{aligned}
&\int_0^T \int \int \int p_t(x-y_1) p_t(x-y_2) dx dm(y_1) dm(y_2) t^{-1/2} dt m(1) \\
&= \int \int (\int_0^T t^{-1/2} p_{2t}(y_1-y_2) dt) dm(y_1) dm(y_2) m(1) \\
&\leq c \int \int 1 + \log^+(T/(y_1-y_2)^2) dm(y_1) dm(y_2) m(1)
\end{aligned}$$

which is finite because $m \in F_1$. □

We close this Section with a result which shows these Girsanov techniques will not work for $d = 3$. The proof is given at the end of the next section.

Theorem 3.11. *Assume $d = 3$, $\lambda > 0$ and $m_1, m_2 \in M_F^s(\mathbb{R}^3) \backslash \{0\}$. If $\mathbb{P}$ solves $(M_{\lambda L})$ then $\mathbb{P}|_{\mathcal{F}_1^2}$ is not absolutely continuous with respect to $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}|_{\mathcal{F}_1^2}$.*

4 Killing Super-Brownian Motion in a Random Environment

We study in this section the much simpler problem in which the first population X^1 may be killed when it comes in contact with the second population X^2, but X^2 is not affected by these encounters. Existence and uniqueness for the appropriate martingale problem is established in Theorem 4.9 and the fact that the resulting process is a diffusion on a suitable space of measures is proved in Theorem 4.11. As a preliminary to studying the uniqueness question, we first consider a similar martingale problem in which the killing measure-valued process X^2 is replaced by a deterministic measure-valued function. Uniqueness for this latter martingale problem is obtained in Theorem 4.5. We have omitted the proof of the companion existence result (see Remark 4.6.)

We continue to use the notation of the previous section.

Definition. Let $W = (T, B)$ denote space-time Brownian motion on the canonical space of paths $C([0,\infty), E)$, where $E = [0,\infty) \times \mathbb{R}^d$. Thus W is a Feller process (in the sense of Ethier-Kurtz (1986, Sec. 4.2)) with semigroup $\{P_t : t \geq 0\}$ and laws

$$Q_{\tau,y}(W \in A) = \Pi_y((\tau + \cdot, B_\cdot) \in A)$$

where Π_y is Wiener measure starting at y.

If $\mu \in \Omega \equiv C([0,\infty), M_F(\mathbb{R}^d))$ and $\eta > 0$, let

$$f_\eta^\mu(u,x) = g_\eta(\mu_u, x) = \int p_\eta(x-y)\mu_u(dy),\ (u,x) \in E,$$

and define a continuous additive functional (CAF) for W by

$$A_\eta^\mu(t) = \int_0^t f_\eta^\mu(W_s)ds.$$

Dependence on μ in the above quantities will often be suppressed. Let

$$\begin{aligned} h(\mu,\epsilon) &= \sup_{(\tau,x)\in E} \int_0^\epsilon f_s^\mu(\tau+s,x)ds,\ M(\mu) = \int_0^\infty \mu_s(1)ds, \\ \Phi &= \{\mu \in \Omega : \lim_{\epsilon\downarrow 0} h(\mu,\epsilon) = 0,\ \mu_t = 0 \text{ for sufficiently large } t\} \\ &\in \mathcal{F} \equiv \mathcal{B}(\Omega). \end{aligned}$$

Theorem 4.1. *If $\mu \in \Phi$, there is a CAF A^μ for W such that*

$$\lim_{\eta \to 0} \sup_{(\tau,x)\in E} Q_{\tau,x}(\sup_t (A^\mu_\eta(t) - A^\mu(t))^2) = 0.$$

If $h : [0,1] \to [0,\infty)$ is such that $h(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$ and $M \in \mathbb{N}$, the convergence in the above is uniform on $\Phi(h, M) = \{\mu \in \Phi : M(\mu) \le M,\ h(\mu,\epsilon) \le h(\epsilon),\ \forall \epsilon \le 1\}$.

Proof. Set

$$F_\eta(\tau,x) = \int_0^\infty \int p_{s+\eta}(x-y)\mu_{\tau+s}(dy)ds = Q_{\tau,x}(A_\eta(\infty)),$$

and

$$F(\tau,x) = \int_0^\infty \int p_s(x-y)\mu_{\tau+s}(dy)ds.$$

Note that for each $\epsilon > 0$,

$$\begin{aligned} |(F - F_\eta)(\tau,x)| &\le \int_0^\epsilon \int p_s(x-y)\mu_{\tau+s}(dy)ds \\ &+ \int p_\eta(x-z)(\int_0^\epsilon \int p_s(z-y)\mu_{\tau+s}(dy)ds)dz \\ &+ \int_\epsilon^\infty \int |p_s(x-y) - p_{s+\eta}(x-y)|\mu_{\tau+s}(dy)ds \end{aligned}$$

and so

$$\sup_{(\tau,x)} |(F - F_\eta)(\tau,x)| \le 2h(\mu,\epsilon) + \sup_{u\ge\epsilon}\sup_z |p_u(z) - p_{u+\eta}(z)|\ M(\mu).$$

For $\mu \in \Phi$ it is now clear that

$$\lim_{\eta\to 0+} \|F - F_\eta\|_\infty = 0 \tag{4.1}$$

and the rate of convergence is uniform in $\mu \in \Phi(h, M)$.

For the moment fix $\mu \in \Phi$ and $(\tau,x) \in E$. If we set

$$\begin{aligned} e_\eta(t) &= Q_{\tau,x}(A_\eta(\infty)|W_s, s \le t) \\ &= A_\eta(t) + F_\eta(W_t) \text{ (Markov property)}, \end{aligned} \tag{4.2}$$

then since $A_\eta(\infty)$ is uniformly bounded, e_η is a non-negative martingale such that $\lim_{t\to\infty} e_\eta(t) = A_\eta(\infty)$, $Q_{\tau,x}$-a.s., and in $L^2(Q_{\tau,x})$. Doob's maximal L^2 inequality therefore gives

$$\begin{aligned}
& Q_{\tau,x}(\sup_t |e_\eta(t) - e_{\eta'}(t)|^2) \\
\leq\ & cQ_{\tau,x}((A_\eta(\infty) - A_{\eta'}(\infty))^2) \\
=\ & 2cQ_{\tau,x}(\int_0^\infty (f_\eta - f_{\eta'})(W_u) Q_{W_u}(\int_0^\infty (f_\eta - f_{\eta'})(W_t)dt)du) \\
=\ & 2cQ_{\tau,x}(\int_0^\infty [(f_\eta - f_{\eta'})(W_u)][(F_\eta - F_{\eta'})(W_u)]du) \\
\leq\ & 4c\|F\|_\infty \|F_\eta - F_{\eta'}\|_\infty,
\end{aligned}$$

where in the last line we have used Chapman-Kolmogorov to see $\|F_\eta\|_\infty \leq \|F\|_\infty$. Combine the above with (4.2) to see

$$\sup_{(\tau,x)\in E} Q_{\tau,x}(\sup_t |A_\eta(t) - A_{\eta'}(t)|^2) \leq c(\|F_\eta - F_{\eta'}\|_\infty^2 + \|F\|_\infty^2 \|F_\eta - F_{\eta'}\|_\infty^2). \tag{4.3}$$

Since $\|F_\eta\|_\infty \leq c(\eta)M(\mu)$ it is clear from (4.1) that $\|F\|_\infty$ is bounded uniformly in $\mu \in \Phi(h, M)$, and therefore (4.3) converges to zero as $\eta, \eta' \to 0$ uniformly in $\mu \in \Phi(h, M)$ (again use (4.1)). From this conclusion we can now carry through the general argument subsequent to line 3.10 of Blumenthal and Getoor (1968) in order to construct a CAF $A = A^\mu$ with the desired properties. □

For $\mu \in \Phi$ we introduce the sub-Markov semigroups on $b\mathcal{E}$

$$\bar{P}_t^{\eta,\mu}(f)(\tau, y) = \bar{P}_t^\eta(f)(\tau, y) = Q_{\tau,y}(\exp\{-A_\eta(t)\}f(W_t))$$

$$\bar{P}_t^\mu(f)(\tau, y) = \bar{P}_t(f)(\tau, y) = Q_{\tau,y}(\exp\{-A(t)\}f(W_t)).$$

$\bar{P}_t^\eta$ and $\bar{P}_t$ are the semigroups of the processes obtained by killing W according to the CAF's A_η and A, respectively. Let $\bar{W}^\eta$ and $\bar{W}$ denote these killed processes and let $\bar{Q}_{\tau,y}^\eta$ and $Q_{\tau,y}^\eta$ denote their laws on $C([0,\infty), E_\Delta)$. Here Δ, the cemetary point, is added to E as a discrete point to form E_Δ. The weak continuity of $(\tau, y) \to Q_{\tau,y}$ and the fact that $A_\eta(t)$ is a continuous functional of W show that $\bar{P}_t^\eta : C_0(E) \to C_0(E)$. The fact that μ_t has compact support in t shows that

$$\lim_{(\tau,y)\to\infty} \bar{P}_t^\eta 1(\tau, y) = 1$$

and hence $\{\bar{P}_t^\eta : t \geq 0\}$ is a semigroup on $C_\ell(E)$. Since $A_\eta(t) \leq ct$, $\bar{P}_t^\eta$ is a (non-conservative) Feller semigroup (i.e. strongly continuous) on $C_\ell(E)$. Let $\bar{G}_{\eta,\mu} = \bar{G}_\eta$ (respectively $\bar{G}_\mu = \bar{G}$ and G_0) denote the (strong) infinitesimal generators of $(\bar{P}_t^\eta)$ (respectively $(\bar{P}_t)$ and (P_t)) on their domains in $C_\ell(E)$. By Dynkin (1965, p. 298)

$$D(\bar{G}_\eta) = D(G_0) \text{ and } \bar{G}_\eta \phi = G_0\phi - f_\eta \phi. \tag{4.4}$$

If $\phi \in C_\ell^{1,2}(E) = \{\phi \in C_\ell(E) : \frac{\partial \phi}{\partial s}, \frac{\Delta \phi}{2} \in C_\ell(E)\}$, then $\phi \in D(G_0)$ and

$$G_0\phi(s,x) = \frac{\partial \phi}{\partial s}(s,x) + \frac{\Delta}{2}\phi(s,x).$$

(Δ applies only to the spatial variables).

Proposition 4.2. *Let $\mu \in \Phi$. Then:*

(a) $\lim_{\eta \to 0+} \sup_{t \geq 0} \|\bar{P}_t^\eta - \bar{P}_t\| = 0$ *($\| \quad \|$ denotes the operator norm on $C_\ell(E)$).*

(b) $\{\bar{P}_t : t \geq 0\}$ *is a Feller semigroup on $C_\ell(E)$.*

(c) $\forall f \in D(\bar{G})$, $\exists f_\eta \in D(\bar{G}_\eta)$ *such that* $(f_\eta, \bar{G}_\eta f_\eta) \to (f, \bar{G} f)$ *in* $C_\ell(E)^2$ *as* $\eta \downarrow 0$.

Proof. If $f \in b\mathcal{E}$,

$$\sup_{t \geq 0} \|\bar{P}_t^\eta f - \bar{P}_t f\|_\infty \leq \|f\|_\infty \sup_{(\tau,x)} Q_{\tau,x}(\sup_{t \geq 0} |\exp(-A_\eta(t)) - \exp(-A(t))|) \to 0$$

as $\eta \downarrow 0$ by Theorem 4.1. Claims (a) and (b) are now immediate. Claim (c) is then a consequence of Ethier-Kurtz (1986, Thm. 1.6.1) (which extends trivially to our continuous parameter setting). □

Let $\bar{X} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{X}_t, \bar{\mathbb{P}}_m^0)$ ($\bar{\mathbb{P}}_m^0 = \bar{\mathbb{P}}_m^{0,\mu}$ for $m \in M_F(E)$) denote the $\bar{W}$-superprocess on $\bar{\Omega} = C([0,\infty), M_F(E))$ with its Borel σ-field $\bar{\mathcal{F}}$, canonical right-continuous filtration $(\bar{\mathcal{F}}_t)$ and coordinate mappings $\bar{X}_t$. As usual m denotes the initial measure. Although the underlying Markov process is assumed to be conservative in the literature, it is easy to construct and characterize superprocesses in the non-conservative case through the same martingale problem. The details are given in the Appendix. Let $\bar{\mathbb{P}}^{\eta}_m = \bar{\mathbb{P}}^{\eta,\mu}_m$ denote the law of the $\bar{W}^{\eta}$-superprocess on $(\bar{\Omega}, \bar{\mathcal{F}})$ and let $\bar{\mathbb{P}}_m$ denote the law of the W-superprocess. Note that for $\eta > 0$, $\bar{\mathbb{P}}^{\eta,\mu}_m$ is defined for all $\mu \in \Omega$.

It follows from Theorem A.1 (in the appendix) and (4.4) that $\bar{\mathbb{P}}^{\eta,\mu}_m$ is the unique law on $(\bar{\Omega}, \bar{\mathcal{F}})$ such that $\forall \phi \in D(G_0)$ the following holds:

$$\bar{X}_t(\phi) = m(\phi) + \bar{Z}^{\eta,\mu}_t(\phi) + \int_0^t \bar{X}_s(G_0\phi - f^{\mu}_{\eta}\phi)ds,$$

$\forall t \geq 0$; where $\bar{Z}^{\eta,\mu}_t(\phi)$ is a continuous $(\bar{\mathcal{F}}_t)$-martingale under $\bar{\mathbb{P}}^{\eta,\mu}_m$ such that $\bar{Z}^{\eta,\mu}_0(\phi) = 0$ and

$$\langle \bar{Z}^{\eta,\mu}(\phi)\rangle_t = \int_0^t \bar{X}_s(\phi^2)ds,$$

$\forall t \geq 0$, $\bar{\mathbb{P}}^{\eta,\mu}_m$-a.s. Label the above martingale problem $(\bar{M}^{\eta}_m)$.

Notation. If $\mu, \nu \in M_F(\mathbb{R}^d)$ let

$$d(\mu,\nu) = \sup\{|\mu(\phi) - \nu(\phi)| : \phi \text{ Lipschitz continuous with Lipschitz constant at most one, } \|\phi\|_\infty \leq 1\}.$$

The metric d is the Vasershtein metric on $M_F(\mathbb{R}^d)$ and is a complete metric which induces the weak topology on $M_F(\mathbb{R}^d)$ (see Ethier-Kurtz (1986, p.150, problem 2)). Denote the uniform metric on Ω by $\rho(\mu,\mu') = \sup_{t\geq 0} d(\mu(t), \mu'(t))$.

Although normally we would equip Ω with the compact-open topology, in the next result we use the ρ-topology. Let Ω_ρ denote Ω equipped with the ρ-topology.

Proposition 4.3. *(a) $\forall \mu \in \Phi$, $m \in M_F(E)$, $\bar{\mathbb{P}}^{\eta,\mu}_m \xrightarrow{w} \bar{\mathbb{P}}^{0,\mu}_m$ as $\eta \downarrow 0$.*

(b) $\forall \eta > 0$, the map $(m,\mu) \mapsto \bar{\mathbb{P}}^{\eta,\mu}_m$ is continuous from $M_F(E) \times \Omega_\rho$ to $M_1(\bar{\Omega})$.

(c) The map $(m,\mu) \mapsto \bar{\mathbb{P}}^{0,\mu}_m$ is a Borel measurable map from $M_F(E) \times \Phi$ to $M_1(\bar{\Omega})$.

Proof. (a) To directly apply the convergence results in Ethier-Kurtz (1986) we note that $(\bar{M}^{\eta}_m)$is equivalent to the requirement that:

$$\exp\{-\bar{X}_t(\phi)\} - \int_0^t \exp\{-\bar{X}_s(\phi)\}[\bar{X}_s(-G_0\phi + f^{\mu}_{\eta}\phi + \phi^2/2)]ds$$

is an $(\bar{\mathcal{F}}_t)$-martingale under $\bar{\mathbb{P}}_m^{\eta,\mu}$ starting at $\exp\{-m(\phi)\}$, $\forall \phi \in D(G_0)_+ = \{\phi \in D(G_0) : \inf \phi > 0\}$. Label this latter martingale problem $(\tilde{M}_m^\eta)$.

To see that $(\bar{M}_m^\eta)$ implies $(\tilde{M}_m^\eta)$ involves only elementary stochastic calculus and the converse implication is only slightly more involved (because one must show $\bar{\mathbb{P}}_m^{\eta,\mu}(\bar{X}_t(1)) \le m(1) < \infty$ to see that $\bar{Z}_t^{\eta,\mu}(\phi)$ is square integrable). Theorem 2.1 allows us to bound $\bar{X}_t$ (under $\bar{\mathbb{P}}_m^{\eta,\mu}$) by a $(W, -\lambda^2/2)$ super-process. This together with Helly's characterization of compactness in the space of measures give the compact containment condition

(4.5) $\forall \delta, T > 0$, $\exists$ a compact set $K \subset M_F(E)$ such that $\inf_{0<\eta\le 1} \bar{\mathbb{P}}_m^{\eta,\mu}(\bar{X}_t \in K,\ \forall 0 \le t \le T) \ge 1 - \delta$.

Now use Proposition 4.2(c) as in the argument on p.407 of Ethier-Kurtz (1986) to verify condition (f) of Corollary 8.7 in Ch.4 of the same reference. This together with Theorem A.1 (which gives uniqueness for $(\tilde{M}_m^\eta)$) and (4.5) allow us to derive (a) from Corollary 8.16 in Ch.4 of Ethier-Kurtz (1986).

(b) Let $(m_n, \mu_n) \to (m, \mu)$ in $M_F(E) \times \Omega_\rho$. By (4.4), if $\phi \in D(\bar{G}_{\eta,\mu_n}) = D(G_0)$ then

$$
\begin{aligned}
|\bar{G}_{\eta,\mu_n}\phi(u,x) - \bar{G}_{\eta,\mu}\phi(u,x)| &\le \|\phi\|_\infty |g_\eta(\mu_n(u),x) - g_\eta(\mu(u),x)| \\
&\le c_\eta \|\phi\|_\infty d(\mu_n(u),\mu(u)) \qquad (4.6)\\
&\to 0 \text{ uniformly in}(u,x) \in E.
\end{aligned}
$$

The last step uses the uniform convergence of $\{\mu_n\}$. The compact containment condition (4.5) for $\{\bar{\mathbb{P}}_{m_n}^{\eta,\mu_n} : n \in \mathbb{N}\}$ follows as in (a). That is, we may define $\{Y_n\}$ with laws $\bar{\mathbb{P}}_{m_n}$ which bound $\{\bar{X}_n\}$ (with laws $\bar{\mathbb{P}}_{m_n}^{\eta,\mu_n}$) and use the weak continuity of $\mathbb{P}_{m_n}$ in m_n (see Dynkin (1989, Thm. 8.1)) to obtain the analogue of (4.5). The rest of the proof now proceeds as in (a) (use (4.6) in place of Proposition 4.2(c)).

(c) This is immediate from (a) and (b). □

Notation. Set $C_0^1([0,\infty)) = \{\phi \in C_0([0,\infty)) : \phi' \in C_0([0,\infty))\}$, $C_0^2(\mathbb{R}^d) = \{\phi \in C_0(\mathbb{R}^d) \cap C_\ell^2(\mathbb{R}^d) : \Delta\phi \in C_0(\mathbb{R}^d)\}$. The projection $\pi : M_F(E) \to M_F(\mathbb{R}^d)$ is given by

$$\pi(\mu)(A) = \mu([0,\infty) \times A).$$

Definition. If B is a Banach space and $\mathcal{A} : D(\mathcal{A}) \subset B \to B$ is a linear map, we say a set $D \subset D(\mathcal{A})$ is a core for $\mathcal{A}$ if and only if the closure of $\{(\phi, \mathcal{A}\phi) : \phi \in D\}$ in $B \times B$ contains $\{(\phi, \mathcal{A}\phi) : \phi \in D(\mathcal{A})\}$.

Lemma 4.4 *If* $D_0 = \{\phi(t,x) : \phi(t,x) = \phi_1(t)\phi_2(x),\ \phi_1 \in C_0^1([0,\infty)),\ \phi_2 \in C_0^2(\mathbb{R}^d)\} \cup \{1\}$, *then the linear span of* D_0 *is a core for* G_0.

Proof. This is a simple application of Ethier-Kurtz (1986, Prop. 1.3.3). □

We are at last in a position to state the martingale problem mentioned at the start of the section that models a randomly evolving population killed in the presence of a deterministically evolving second population. Recall that the primary reason we are studying this model is as a prelude to establishing uniqueness in the martingale problem describing a randomly evolving population killed in the presence of an independent super-Brownian motion.

Definition. Define the following additional σ-fields of subsets of Ω:

$$\mathcal{F}[\tau,\infty) = \sigma\{X_u : u \geq \tau\},\ \mathcal{F}_t^0 = \sigma\{X_u : u \leq t\},$$

$$\mathcal{F}[\tau,t] = \cap_n \sigma\{X_u : \tau \leq u \leq t + \frac{1}{n}\}\ (0 \leq \tau \leq t < \infty).$$

Let $\mu \in \Phi$, $\tau \geq 0$, $m \in M_F(\mathbb{R}^d)$ and D be a core for $D(\Delta/2)$ such that $1 \in D \subset C_\ell^2(\mathbb{R}^d)$. We say that a law $\mathbb{P}$ on $(\Omega, \mathcal{F}[\tau,\infty))$ solves the martingale problem (M_m^μ) if the following hold:

$$X_\tau = m,\ \mathbb{P}\text{-a.s.},$$

$$X_t(\phi) = X_\tau(\phi) + Z_t(\phi) + \int_\tau^t X_s(\frac{\Delta\phi}{2})ds - L_t(\phi),$$

$\forall t \geq \tau$, $\mathbb{P}$-a.s., $\forall \phi \in D$; where $\{Z_t(\phi) : t \geq \tau\}$ is an a.s. continuous $\mathcal{F}[\tau,t]$-martingale under $\mathbb{P}$ such that

$$\langle Z(\phi)\rangle_t = \int_\tau^t X_s(\phi^2)ds,$$

$\forall t \geq \tau$, $\mathbb{P}$-a.s., and L_t is an a.s. continuous $M_F(\mathbb{R}^d)$-valued process such that for some sequence $\eta_n \to 0$, $\forall t \geq \tau$ and $\forall \phi \in D$,

$$L_t(\phi) = \mathbb{P} - \lim_{n\to\infty} \int_\tau^t \int g_{\eta_n}(\mu_s, x)\phi(x)X_s(dx)ds.$$

Theorem 4.5. *Let $\mu \in \Phi$, $\tau \geq 0$, $m \in M_F(\mathbb{R}^d)$ and D be as in the previous definition. Let $\mathbb{P}$ be a law on $(\Omega, \mathcal{F}[\tau,\infty))$ that satisfies (M^μ_m). Then $\mathbb{P}(A) = \bar{\mathbb{P}}^{0,\mu}_{\delta_\tau \times m}(\pi(\bar{X}_{\cdot-\tau}) \in A)$, $\forall A \in \mathcal{F}[\tau,\infty)$, and, in particular, $\mathbb{P}$ is unique.*

Proof. Let

$$L^n_t(A) = \int_\tau^t X_s(g_{\eta_n}(\mu_s, \cdot)1_A)ds, \ t \geq \tau, \ A \in \mathcal{B}(\mathbb{R}^d).$$

As D is a core, D is dense in $C_\ell(\mathbb{R}^d)$. Let $\phi \in C_\ell(\mathbb{R}^d)$ and choose $\{\phi_m\} \subset D$ such that $\|\phi_m - \phi\|_\infty \to 0$. Then

$$\begin{aligned} |L^n_t(\phi) - L_t(\phi)| &\leq |L^n_t(\phi - \phi_m)| + |L^n_t(\phi_m) - L_t(\phi_m)| + |L_t(\phi_m - \phi)| \\ &\leq \|\phi_m - \phi\|_\infty (L^n_t(1) + L_t(1)) + |L^n_t(\phi_m) - L_t(\phi_m)|. \end{aligned}$$

First choose m large so the first term is small in probability uniformly in n and then choose n large so the second term is small in probability. This is possible because $L^n_t(\phi_m) \overset{\mathbb{P}}{\to} L_t(\phi_m)$ for all m and $L^n_t(1) \overset{\mathbb{P}}{\to} L_t(1)$. Therefore

$$L^n_t(\phi) \overset{\mathbb{P}}{\to} L_t(\phi), \ \forall \phi \in C_\ell(\mathbb{R}^d), \tag{4.7}$$

and we may choose a countable dense set D' in $C^+_\ell(\mathbb{R}^d) = \{f \in C_\ell(\mathbb{R}^d) : f \geq 0\}$ with $1 \in D'$ and a sequence $\{n_k\}$ such that

$$\lim_{k\to\infty} L^{n_k}_t(\phi) = L_t(\phi), \ \forall t \in \mathbb{Q} \cap [\tau, \infty), \ \forall \phi \in D', \ \mathbb{P} - \text{a.s.}$$

As the limit is a.s. continuous in t and $L^{n_k}_t(\phi)$ is non-decreasing in t, an elementary argument shows that

$$\lim_{k\to\infty} \sup_{\tau \leq t \leq T} |L^{n_k}_t(\phi) - L_t(\phi)| = 0, \ \forall \phi \in D', \ T > \tau, \ \mathbb{P} - \text{a.s.} \tag{4.8}$$

If $\rho = \sup\{s : \mu_s \neq 0\}$, then $\rho < \infty$ and

$$L^{n_k}_t(1) = L^{n_k}_\rho(1), \ \forall t \geq \rho \tag{4.9}$$

and therefore $L_t(1) = L_\rho(1)$, $\forall t \geq \rho$, $\mathbb{P}$-a.s. by the above. It now follows from (4.8) that

$$\lim_{k\to\infty} \sup_{\tau \leq t} |L^{n_k}_t(\phi) - L_t(\phi)| = 0, \ \forall \phi \in D', \ \mathbb{P} - \text{a.s.} \tag{4.10}$$

Equation (4.10) shows that $L_t(\phi)$ is non-decreasing in t and $L_\tau(\phi) = 0$ for all $\phi \in C_\ell^+(\mathbb{R}^d)$, $\mathbb{P}$-a.s. Hence there is a unique (up to null sets) random finite measure $\bar{L}$ on $[\tau, \infty) \times \mathbb{R}^d$ such that

$$\bar{L}([\tau, t] \times A) = L_t(A),\ \forall t \geq \tau,\ A \in \mathcal{B}(\mathbb{R}^d),\ \mathbb{P} - \text{a.s.}$$

If $\bar{L}^n$ is the corresponding measure for L_t^n, then by using (4.9) and (4.10) first to get a.s. tightness of $\{\bar{L}^{n_k}\}$ and then to see that $\bar{L}$ is the only limit point a.s., one obtains

$$\bar{L}^{n_k} \to \bar{L} \text{ in } M_F([\tau, \infty) \times \mathbb{R}^d) \text{ a.s.} \tag{4.11}$$

The conditions of (M_m^μ) imply that

$$\mathbb{P}(\int_\tau^T X_s(1)ds) \leq (T - \tau)m(1). \tag{4.12}$$

As D is bounded pointwise dense in $b\mathcal{B}(\mathbb{R}^d)$, this allows us to extend Z_t to an orthogonal martingale measure $\{Z_t(\phi) : t \geq \tau,\ \phi \in b\mathcal{B}(\mathbb{R}^d)\}$ such that $\{Z_t(\phi) : t \geq \tau\}$ is a continuous L^2 martingale with respect to the filtration $(\mathcal{F}[\tau, t])_{t \geq \tau}$ under $\mathbb{P}$ with

$$\langle Z(\phi) \rangle_t = \int_\tau^t X_s(\phi^2)ds,$$

$\forall t \geq \tau$, $\mathbb{P}$-a.s. As in Walsh (1986, Ch.2), we can define $\int_\tau^t \int \phi(s, x, \omega) Z(ds, dx)$ for the usual class of $\mathcal{P}(\mathcal{F}[\tau, t]) \times \mathcal{B}(\mathbb{R}^d)$-measurable integrands.

By taking limits of $(\phi, \Delta\phi/2)$ $(\phi \in D)$ in $C_\ell(\mathbb{R}^d)^2$ we see that (M_m^μ) continues to hold $\forall \phi \in D(\Delta/2)$ (recall (4.12)). If $\phi(s, x) = \phi_1(s)\phi_2(x)$ for $\phi_1 \in C_0^1([0, \infty))$ and $\phi_2 \in C_0^2(\mathbb{R}^d)$ (see the notation prior to Lemma 4.2), then some easy stochastic calculus and (M_m^μ) (the latter for ϕ_2) give

$$\begin{aligned} X_t(\phi_t) = m(\phi_\tau) &+ \int_\tau^t \int \phi(s, x) dZ(s, x) \\ &+ \int_\tau^t X_s(G_0\phi_s)ds - \int_\tau^t \int \phi(s, x)\bar{L}(ds, dx), \end{aligned} \tag{4.13}$$

$\forall t \geq \tau$, $\mathbb{P}$-a.s. Equation (4.13) continues to hold for ϕ in the core D_0 introduced in Lemma 4.4 and hence, by taking limits, for all ϕ in $D(G_0)$. Let

$\phi \in D(\bar{G})$ and use Proposition 4.2(c) to choose $\phi_n \in D(\bar{G}_{\eta_n}) = D(G_0)$ (by (4.4)) such that

$$(\phi_n, \bar{G}_{\eta_n}\phi_n) \to (\phi, \bar{G}\phi) \text{ in } C_\ell(E)^2. \tag{4.14}$$

Apply (4.13) with $\phi = \phi_n$ and use (4.4) to conclude (write $Z_t(\phi)$ for the martingale term in (4.13))

$$\begin{aligned} X_t(\phi_n(t)) = m(\phi_n(\tau)) + Z_t(\phi_n) \quad &+ \int_\tau^t X_s((\bar{G}_{\eta_n}\phi_n)(s))ds \\ &+ \int_\tau^t X_s(g_{\eta_n}(\mu_s, \cdot)\phi_n(s))ds \\ &- \int_\tau^t \int \phi_n(s,x)\bar{L}(ds,dx), \end{aligned}$$

$\forall t \geq \tau$, $\mathbb{P}$-a.s. Let $n \to \infty$ through $\{n_k\}$ and use (4.11), (4.14), and Doob's inequality along with (4.12) to handle the martingale terms, to conclude

$$X_t(\phi_t) = m(\phi(\tau)) + Z_t(\phi) + \int_\tau^t X_s((\bar{G}\phi)(s))ds$$

$\forall t \geq \tau$, $\mathbb{P}$-a.s. If $\vec{X}_t = \delta_{\tau+t} \times X_{\tau+t}$ for $t \geq 0$, this becomes (let $\vec{Z}_t(\phi) = Z_{t+\tau}(\phi)$, $\vec{\mathcal{F}}_t = \mathcal{F}[\tau, \tau+t]$, $t \geq 0$)

$$\vec{X}_t(\phi) = (\delta_{\tau+t} \times m)(\phi) + \vec{Z}_t(\phi) + \int_0^t \vec{X}_s(\bar{G}\phi)ds,$$

$\forall t \geq 0$, $\mathbb{P}$-a.s., $\forall \phi \in D(\bar{G})$; where $\vec{Z}_t(\phi)$ is a continuous $(\vec{\mathcal{F}}_t)$-martingale such that $\vec{Z}_0(\phi) = 0$ and

$$\langle \vec{Z}(\phi)\rangle_t = \int_\tau^{t+\tau} X_s(\phi_s^2)ds = \int_0^t \vec{X}_s(\phi^2)ds,$$

$\forall t \geq 0$, $\mathbb{P}$-a.s. Theorem A.1 in the Appendix implies $\vec{X}$ has distribution $\bar{\mathbb{P}}^{0,\mu}_{\delta_\tau \times m}$. The result follows because $X_t = \pi(\vec{X}_{t-\tau})$, $\forall t \geq \tau$. $\square$

Remark 4.6. (a) A uniqueness result without a companion existence theorem is of questionable value. In fact it is true that $\mathbb{P} = \bar{\mathbb{P}}^{0,\mu}_{\delta_\tau \times m}\,(\pi(\bar{X}_{\cdot-\tau}) \in \cdot)$ solves (M_m^μ). Our proof of this apparently simple result is ridiculously complicated and we have not included it here. We will see that Theorem 3.6 will give solutions to (M_m^μ) for $\mathbb{P}_m$-a.a. μ ($\mathbb{P}_m$ continues to denote the law of super Brownian motion on $(\Omega, \mathcal{F})$) and this will suffice for our purposes.

(b) If $m \in M_F(\mathbb{R}^d)$ and $\mu \in \Phi$, let $\mathbb{P}_m^\mu(A) = \bar{\mathbb{P}}^{0,\mu}_{\delta_0 \times m}(\pi(\bar{X}) \in A)$ for $A \in \mathcal{F}$. Proposition 4.3(c) shows that $(m, \mu) \mapsto \mathbb{P}_m^\mu$ is a Borel measurable map from $M_F(\mathbb{R}^d) \times \Phi$ to $M_1(\Omega)$. Recall the notation $M_F^s(\mathbb{R}^d)$ from Section 3 (see Lemma 3.4)

Proposition 4.7. *Assume $d \le 3$ and $m \in M_F^s(\mathbb{R}^d)$. Then*
(a) $\mathbb{P}_m(\Phi) = 1$
(b) $X_t \in M_F^s(\mathbb{R}^d)$, $\forall t \ge 0$, $\mathbb{P}_m$-a.s.

Proof. (a) Clearly $t \mapsto X_t(1)$ has compact support $\mathbb{P}_m$-a.s. It remains to show

$$\lim_{\epsilon \downarrow 0} \sup_{x \in \mathbb{R}^d, \tau \ge 0} \int_0^\epsilon \int p_s(y - x) X_{s+\tau}(dx) ds = 0, \ \mathbb{P}_m - \text{a.s.} \tag{4.15}$$

If $d = 1$ this follows from the trivial bound $p_s(y - x) \le s^{-1/2}$, so let us assume $d = 2$ or 3. Let $\zeta(r) = r^2(1 + \log^+(1/r))^{4-d}$. Theorem 4.7 and Lemma 4.6 of [BEP] show there are constants $c_1, c_2 > 0$ and an $r_0(\omega) > 0$ $\mathbb{P}_m$-a.s. such that

$$D(X_t, r) \le c_1(D(m, c_2 r) + \zeta(r)), \ \forall t \ge 0 \text{ and } r \in (0, r_0). \tag{4.16}$$

Choose ω such that $r_0(\omega) > 0$ and fix $\tau \ge 0$, $x \in \mathbb{R}^d$. Let $\nu_s([0, r)) = X_{s+\tau}(B(x, r))$. Inequality (4.16) implies that

$$\nu_s([0, r)) \le c_3(\omega)(D(m, c_2 r) + \zeta(r)), \tag{4.17}$$

first for $0 < r < r_0(\omega)$ and then for all $r > 0$ by choosing $c_3(\omega)$ appropriately.

$$\int_0^\epsilon \int_{\mathbb{R}^d} p_s(y - x) X_{s+\tau}(dy) ds = \int_0^\epsilon \int_0^\infty p_s(r) \nu_s(dr) ds$$

$$= \int_0^\epsilon \int_0^\infty \nu_s([0, r))(2\pi)^{-d/2} s^{-1-d/2} r e^{-r^2/2s} dr ds \text{ (by parts)}$$

$$\le c_3(\omega)[\int_0^\epsilon \int_0^\infty D(m, c_2 r) r e^{-r^2/2s} s^{-1-d/2} dr ds$$

$$+ \int_0^\epsilon \int_0^\infty \zeta(r) r c^{-r^2/2s} s^{-1-d/2} dr ds] \text{ (by (4.17))}.$$

The second integral goes to zero as $\epsilon \downarrow 0$ because $d < 4$. The first integral equals

$$\int_0^\epsilon \int_0^\infty D(m, c_2 x\sqrt{s}) x e^{-x^2/2} s^{-d/2} dx ds$$

$$= \int_0^\infty \int_0^{c_2 x\sqrt{\epsilon}} D(m,u) u^{1-d} du x^{d-1} e^{-x^2/2} dx (2c_2^{d-2}).$$

This approaches zero as $\epsilon \downarrow 0$ because $m \in M_F^s(\mathbb{R}^d)$. As the above bounds are uniform in (τ, x), the proof of (4.15) is complete.

(b) This is immediate from (4.16). □

Remark 4.8. Corollary 4.8 of [BEP] in fact implies $X_t \in M_F^s(\mathbb{R}^d)$, $\forall t > 0$, $\mathbb{P}_m$-a.s., $\forall m \in M_F(\mathbb{R}^d)$ (for $d \le 3$).

We are ready to introduce the martingale problem $(M^1_{\lambda L})$ discussed in the Introduction. Recall that this is intended as a model for a pair of branching particle systems in which inter-species collisions may kill off the particle in the first population but have no effect on the particle from the second population.

Definition. Suppose that $\lambda \ge 0$ and $m_1, m_2 \in M_F(\mathbb{R}^d)$. We say that a probability $\mathbb{P}$ on $(\Omega^2, \mathcal{F}^2)$ solves the martingale problem $(M^1_{\lambda L})$ if the following holds $\forall \phi \in C^2_\ell(\mathbb{R}^d)$:

$$X_0^1 = m_1, \ \mathbb{P} - \text{a.s.},$$

$$X_t^1(\phi) = X_0^1(\phi) + Z_t^1(\phi) + \int_0^t X_s^1(\Delta\phi/2) ds - \lambda L_t(X^1, X^2)(\phi)$$

$\forall t \ge 0$, $\mathbb{P}$-a.s.,

$$X_0^2 = m_2, \ \mathbb{P} - \text{a.s.},$$

$$X_t^2(\phi) = X_0^2(\phi) + Z_t^2(\phi) + \int_0^t X_s^2(\Delta\phi/2) ds$$

$\forall t \ge 0$, $\mathbb{P}$-a.s.; where $Z_t^i(\phi)$ are a.s. continuous $(\mathcal{F}_t^2)$-martingales under $\mathbb{P}$ such that

$$\langle Z^i(\phi_i), Z^j(\phi_j)\rangle_t = \delta_{ij} \int_0^t X_s^i(\phi_i^2) ds$$

$\forall t \ge 0$, $\mathbb{P}$-a.s., $\forall \phi_i, \phi_j \in C^2_\ell(\mathbb{R}^d)$.

Theorem 4.9. *Assume $d \le 3$, $\lambda \ge 0$ and $m_1, m_2 \in M_F^s(\mathbb{R}^d)$. There is a unique probability $\mathbb{P}$ on $(\Omega^2, \mathcal{F}^2)$ that solves $(M^1_{\lambda L})$.*

In fact $\mathbb{P}$ is given by

$$\mathbb{P}(A \times B) = \int_F 1_B(\omega) \mathbb{P}_{m_1}^{\lambda\omega}(A) d\mathbb{P}_{m_2}(\omega), \ \forall A, B \in \mathcal{F}. \tag{4.18}$$

Proof. The existence of a $\mathbb{P}$ satisfying $(M_{\lambda L}^1)$ follows just as in Theorem 3.6.

Assume $\mathbb{P}$ satisfies $(M_{\lambda L}^1)$. Let $\mathbb{P}(X^2)(\cdot)$ be a regular conditional probability for X^1 given $\sigma(X^2)$. Note that the uniqueness of the martingale problem for super-Brownian motion (e.g. Ethier-Kurtz (1986, Ch.9 Thm. 4.2)) shows that $\mathbb{P}(X^2 \in \cdot) = \mathbb{P}_{m_2}$.

As usual, Z^i extends to an orthogonal martingale measure. If $f_i(s, \omega, x)$ is $\mathcal{P}(\mathcal{F}_t^2) \times \mathcal{B}(\mathbb{R}^d)$-measurable such that

$$\int_0^t \int f_i(s, \omega, x)^2 X_s^i(dx) ds < \infty, \ \forall t > 0, \ \mathbb{P} - \text{a.s.}, \tag{4.19}$$

then the stochastic integral

$$\int_0^t \int f_i(s, \omega, x) dZ^i(s, x) \equiv Z_t^i(f)$$

exists and is a continuous local martingale such that

$$\langle Z^i(f_i), Z^j(f_j) \rangle_t = \delta_{ij} \int_0^t \int f_i(s, \omega, x)^2 X_s^i(dx) ds. \tag{4.20}$$

Let $\phi \in C_\ell^2(\mathbb{R}^d)$. We claim $Z_t^1(\phi)$ is an $\mathcal{F}_t \times \mathcal{F}$-martingale. Fix $s < t$ and let $Y \in b\sigma(X^2)$, $W \in b\mathcal{F}_s^{X^1}$. Here $\mathcal{F}_s^{X^1} = \cap_n \sigma(X_u^1 : u \leq s + 1/n)$. By Theorem 1.1, $\exists f \in L_\infty^2(X^2, \mathbb{P}_{m_2})$ such that $Y = \mathbb{P}(Y) + \int_0^\infty \int f(s, X^2, x) dZ^2(s, x)$, $\mathbb{P}$-a.s. Therefore

$$\begin{aligned}
& \mathbb{P}((Z_t^1(\phi) - Z_s^1(\phi))YW) \\
= \ & \mathbb{P}((Z_t^1(\phi) - Z_s^1(\phi))W)\mathbb{P}(Y) \\
+ \ & \mathbb{P}(W\mathbb{P}((Z_t^1(\phi) - Z_s^1(\phi)) \int_0^\infty \int f(u, X^2, x) dZ^2(u, x) | \mathcal{F}_s^2)) \\
= \ & 0 + \mathbb{P}(W\mathbb{P}((Z_t^1(\phi) - Z_s^1(\phi))(Z_t^2(f) - Z_s^2(f)) | \mathcal{F}_s^2)) \\
= \ & \mathbb{P}(W\mathbb{P}(\langle Z^1(\phi), Z^2(f) \rangle_t - \langle Z^1(\phi), Z^2(f) \rangle_s | \mathcal{F}_s^2)) \\
= \ & 0 \quad (\text{by } (4.20)).
\end{aligned}$$

It follows that $\mathbb{P}(Z_t^1(\phi)|\mathcal{F}_s^0 \times \mathcal{F}) = Z_s^1(\phi)$, $\mathbb{P}$-a.s. Letting $s \downarrow u$ through rational values on both sides, we see that $Z_t^1(\phi)$ is an $\mathcal{F}_t \times \mathcal{F}$-martingale.

From Theorem 2.1 it is clear that $\mathbb{P}(\sup_{t\leq T} X_t^1(1)^p) < \infty$, $\forall p, T > 0$, and therefore it follows from $(M_{\lambda L}^1)$ and Burkholder's inequality that $\mathbb{P}(\sup_{t\leq T} |Z_t^1(\phi)|^p) < \infty$, $\forall p, T > 0$ and $\phi \in b\mathcal{B}(\mathbb{R}^d)$. Therefore, for $\phi \in C_\ell^2(\mathbb{R}^d)$

$$\begin{aligned} M_t(\phi) = Z_t^1(\phi)^2 - \int_0^t X_s^1(\phi^2)ds &= 2\int_0^t Z_s^1(\phi)dZ_s^1(\phi) \\ &= 2\int_0^t \int Z_s^1(\phi)\phi(x)dZ^1(s,x) \end{aligned}$$

is a square integrable $(\mathcal{F}_t^2)$-martingale and by modifying the previous argument we see it is also an $(\mathcal{F}_t \times \mathcal{F})$-martingale.

Let N_t be a $\mathbb{P}$-a.s. continuous $(\mathcal{F}_t \times \mathcal{F})$-martingale. We claim that for $\mathbb{P}_{m_2}$-a.a. ω^2, $(t,\omega^1) \to N(t,\omega^1,\omega^2)$ is a $\mathbb{P}(\omega^2)(\cdot)$-a.s. continuous $(\mathcal{F}_t)$-martingale with respect to $\mathbb{P}(\omega^2)$. Let $C_s \subset \mathcal{F}_s^0$ be a countable set whose bounded pointwise closure is $b\mathcal{F}_s^0$. If $s \leq t$ and $W \in C_s$, then

$$\begin{aligned} &\mathbb{P}((N_t - N_s)(W \circ X^1)Y) = 0, \ \forall Y \in b\sigma(X^2) \\ &\Rightarrow \mathbb{P}((N_t - N_s)W \circ X^1 | X^2) = 0 \ \mathbb{P}\text{-a.s.} \\ &\Rightarrow \mathbb{P}(\omega^2)((N_t - N_s)(\cdot,\omega^2)W) = 0 \text{ for } \mathbb{P}_{m_2} - \text{a.a. } \omega^2. \end{aligned}$$

Therefore we may fix ω^2 outside of $\mathbb{P}_{m_2}$-null set such that

$$\mathbb{P}(\omega^2)((N_t - N_s)(\cdot,\omega^2)W) = 0, \ \forall W \in C_s, \ \forall 0 \leq s \leq t \text{ rationals}, \tag{4.21}$$

and $t \mapsto N_t(\omega^1,\omega^2)$ is continuous for $\mathbb{P}(\omega^2)$-a.a. ω^1. Equation (4.21) extends immediately to all W in $b\mathcal{F}_s^0$, so that

$$\mathbb{P}(\omega^2)(N_n(\cdot,\omega^2)|\mathcal{F}_s^0)(\omega^1) = N_s(\omega^1,\omega^2), \ \forall s \in [0,n]\cap\mathbb{Q}, \ \forall n \in \mathbb{N}, \ \mathbb{P}(\omega^2)\text{−a.a. } \omega^1.$$

Fix $t \in [0,n]$ and choose rationals $s_m \downarrow t$. Take limits in the above to see that

$$\mathbb{P}(\omega^2)(N_n(\cdot,\omega^2)|\mathcal{F}_t)(\omega^1) = N_t(\omega^1,\omega^2), \ \mathbb{P}(\omega^2) - \text{a.a. } \omega^1, \ \forall t \leq n.$$

This proves the claim.

Let $D \subset C_\ell^2(\mathbb{R}^d)$ be a countable core for $\Delta/2$ on $D(\Delta/2)$ with $1 \in D$. For example, one may take $D = \{P_{\epsilon_n}\phi : \phi \in D', n \in \mathbb{N}\}$ where $1 \in D'$ is a countable dense set in $C_\ell(\mathbb{R}^d)$, (P_t) is the Brownian semigroup, and $\epsilon_n \downarrow 0$. Now apply the above result with $N_t = Z_t^1(\phi)$ or $M_t(\phi)$, $\phi \in D$, to conclude

(4.22) For $\mathbb{P}_{m_2}$-a.a. ω^2, $\forall \phi \in D$, $(t,\omega^1) \to Z_t^1(\phi)(\omega^1,\omega^2)$ and $M_t(\phi)(\omega^1,\omega^2)$ are a.s. continuous $(\mathcal{F}_t)$-martingales under $\mathbb{P}(\omega^2)$.

We may define an a.s. unique random finite measure $\bar{L}$ on $[0,\infty)\times\mathbb{R}^d$ such that $\bar{L}([0,t]\times A) = \lambda L_t(X^1,X^2)(A)$, $\forall t \geq 0$, $A \in \mathcal{B}(\mathbb{R}^d)$, $\mathbb{P}$-a.s.

For $\eta > 0$ define $\bar{L}^\eta(X^1,X^2) \in M_F([0,\infty)\times\mathbb{R}^d)$ by

$$\bar{L}^\eta([0,t]\times A) = \int_0^t X_s^1(g_\eta(\lambda X_s^2,\cdot)1_A)ds,\ t \geq 0, A \in \mathcal{B}(\mathbb{R}^d).$$

Argue as in the derivation cf (4.11) to see there is a sequence $\eta_k \downarrow 0$ such that

$$\bar{L}^{\eta_k} \to \bar{L} \text{ in } M_F([0,\infty)\times\mathbb{R}^d)\ \mathbb{P}-\text{a.s.}$$

From the above we may conclude that for $\mathbb{P}_{m_2}$ − a.a. ω^2, for $\mathbb{P}(\omega^2)$ − a.a. ω^1,

$$\bar{L}^{\eta_k}(\omega^1,\omega^2) \to \bar{L}(\omega^1,\omega^2) \text{ in } M_F([0,\infty)\times\mathbb{R}^d),$$

$$(4.23)\qquad \bar{L}(\omega^1,\omega^2)([0,t]\times A) = \lambda L_t(X^1,X^2)(A),\ \forall t \geq 0,\ A \in \mathcal{B}(\mathbb{R}^d)$$

$$\text{and } \lambda L_t(X^1,X^2) \text{ is continuous in } t.$$

Fix ω^2 outside a $\mathbb{P}_m$-null set such that $\lambda\omega^2 \in \Phi$ (use Proposition 4.7),

$$\omega_t^1(\phi) = m_1(\phi) + Z_t^1(\phi)(\omega^1,\omega^2) + \int_0^t \omega_s^1(\Delta\phi/2)ds - \lambda L_t(\omega^1,\omega^2)(\phi)$$

$\forall t \geq 0$, $\phi \in D$, for $P(\omega^2)$-a.a. ω^1, and so that ω^2 is not in the exceptional null sets from (4.22) and (4.23). The latter implies

$$\int_0^t \omega_s^1(g_{\eta_k}(\lambda\omega_s^2,\cdot)\phi(\cdot))ds \to \lambda L_t(\omega^1,\omega^2)(\phi),\ \forall\phi \in D,\ \mathbb{P}(\omega^2) - \text{a.a. } \omega^1.$$

We therefore have shown that $\mathbb{P}(\omega^2)$ solves the martingale problem $(M_{m_1}^{\lambda\omega^2})$. Theorem 4.5 implies that for ω^2 as above, $\mathbb{P}(\omega^2) = \mathbb{P}_{m_1}^{\lambda\omega^2}$, and as the latter is $\mathcal{F}$-measurable in ω^2 (see Remarks 4.6(b)), (4.18) follows and $\mathbb{P}$ is unique. □

Remark 4.10(a). Note the above proof shows directly (i.e. without Proposition 4.3) that $\mu \mapsto \mathbb{P}^{\mu}_{m_1}$ is $\bar{\mathcal{F}}$-measurable in $\mu \in \Phi$, where $\bar{\mathcal{F}}$ is the $\mathbb{P}_{m_2}$-completion of $\mathcal{F}$ and $m_i \in M^s_F(\mathbb{R}^d)$.

(b) The above proof goes through unchanged if instead of $-\lambda L_t(X^1, X^2)(\phi)$ in $(M^1_{\lambda L})$ we have $-\lambda \hat{L}_t(\omega^1, \omega^2)(\phi)$ where $\hat{L}_t(\omega^1, \omega^2)$ is a $\mathbb{P}$-a.s. continuous, non-decreasing $M_F(\mathbb{R}^d)$-valued process for which there is a sequence $\eta_k \downarrow 0$ such that

$$L^{\eta_k}_t(X^1, X^2)(\phi) \xrightarrow{\mathbb{P}} \hat{L}_t(\phi) \text{ as } k \to \infty,\ \forall t \geq 0,\ \forall \phi \in C^2_\ell(\mathbb{R}^d). \tag{4.24}$$

Suppressing dependence on λ, we let $\mathbb{P}^1_{m_1,m_2}$ denote the probability given by (4.18). Hence if $(m_1, m_2) \in M^s_F(\mathbb{R}^d)^2$, $\mathbb{P}^1_{m_1,m_2}$ is the unique solution of $(M^1_{\lambda L})$.

Theorem 4.11. *Suppose that $d \leq 3$. Let $\tilde{X}_t = (\tilde{X}^1_t, \tilde{X}^2_t)$ be a $\mathbb{P}'$-a.s. continuous, $(\mathcal{F}'_t)$-adapted $M_F(\mathbb{R}^d)^2$-valued process on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ equipped with a right-continuous filtration $(\mathcal{F}'_t)$. Assume $m_1, m_2 \in M^s_F(\mathbb{R}^d)$ and $\forall \phi \in C^2_\ell(\mathbb{R}^d)$ the following conditions (which we label as $(M'_{\lambda L})$) hold:*

$$\tilde{X}^1_t(\phi) = m_1(\phi) + Z^1_t(\phi) + \int_0^t \tilde{X}^1_s(\Delta\phi/2)ds - \lambda L_t(\tilde{X}^1, \tilde{X}^2)(\phi),$$

$\forall t \geq 0$, $\mathbb{P}'$-a.s. (in particular, $L(\tilde{X}^1, \tilde{X}^2)$ exists),

$$\tilde{X}^2_t(\phi) = m_2(\phi) + Z^2_t(\phi) + \int_0^t \tilde{X}^2_s(\Delta\phi/2)ds,$$

$\forall t \geq 0$, $\mathbb{P}'$-a.s.; where $Z^i_t(\phi)$ are a.s. continuous $(\mathcal{F}'_t)$-martingales under $\mathbb{P}'$ such that $Z^i_0(\phi_i) = 0$ and

$$\langle Z^i(\phi_i), Z^j(\phi_j)\rangle_t = \delta_{ij} \int_0^t \tilde{X}^i_s(\phi_i^2)ds,$$

$\forall t \geq 0$, $\mathbb{P}'$-a.s.

Then:

(a) $\tilde{X}_t \in M^s_F(\mathbb{R}^d)$, $\forall t \geq 0$, $\mathbb{P}'$-a.s.

(b) If T is any a.s. finite $(\mathcal{F}'_t)$-stopping time and $\psi \in b(\mathcal{F}^2)$, then

$$\mathbb{P}'(\psi(\tilde{X}_{T+\cdot})|\mathcal{F}'_T) = \mathbb{P}^1_{\tilde{X}_T}(\psi),\ \mathbb{P}' - a.s.$$

Proof. (a) By Theorem 2.1 and Remark 2.2 we may assume there are independent $(\mathcal{F}'_t)$-super Brownian motions (Y^1, Y^2) such that $Y^i = m_i$ and $\tilde{X}^i_t \le Y^i_t$, $\forall t \ge 0$, a.s. Proposition 4.7(b) implies $(Y^1_t, Y^2_t) \in M^s_F(\mathbb{R}^d)$, $\forall t \ge 0$, a.s. and (a) follows.

(b) For $\phi \in C^2_\ell(\mathbb{R}^d)$ define $\tilde{Z}^i(\phi) : [0,\infty) \times \Omega^2 \to \mathbb{R}$, $i = 1, 2$, by

$$\tilde{Z}^1_t(\phi)(\omega^1, \omega^2) = \omega^1_t(\phi) - \omega^1_0(\phi) - \int_0^t \omega^1_s(\Delta\phi/2)ds + \lambda \tilde{L}_t(\omega^1, \omega^2)(\phi)$$

$$\tilde{Z}^2_t(\phi)(\omega^1, \omega^2) = \omega^2_t(\phi) - \omega^2_0(\phi) - \int_0^t \omega^2_s(\Delta\phi/2)ds.$$

Here $\tilde{L}_t$ is as in Proposition 3.2 but with respect to the law $\mathbb{P}_{\tilde{X}}$ of $(\tilde{X}^1, \tilde{X}^2)$ on $(\Omega^2, \mathcal{F}^2)$. By an easy truncation argument, it suffices to consider bounded T. Let $\hat{\mathbb{P}}_T(\omega')$ be a regular conditional probability for $\theta_T(\tilde{X}) = \tilde{X}_{T+\cdot}$ given $\mathcal{F}'_T$. Proposition 3.2 (ii) implies

$$\tilde{Z}^i_t(\phi)(\theta^2_T\omega) = \tilde{Z}^i_{t+T}(\omega) - \tilde{Z}^i_T(\omega),\ \forall t \ge 0,\ \mathbb{P}_{\tilde{X}} - \text{a.s.},\ i = 1, 2. \tag{4.25}$$

Let $s \le t$ and C_s be a countable set which is bounded pointwise dense in $b((\mathcal{F}^0_s)^2)$. If $\psi \in C_s$, then

$$\begin{aligned}
&\mathbb{P}'([\tilde{Z}^i_t(\phi)(\theta^2_T(\tilde{X}(\omega'))) - \tilde{Z}^i_s(\phi)(\theta^2_T(\tilde{X}(\omega')))]\psi(\theta^2_T(\tilde{X}(\omega')))|\mathcal{F}'_T)\\
=\ &\mathbb{P}'(\mathbb{P}'(\tilde{Z}^i_{t+T}(\phi)(\tilde{X}(\omega')) - \tilde{Z}^i_{s+T}(\phi)(\tilde{X}(\omega')))|\mathcal{F}'_{T+s})\psi(\theta^2_T(\tilde{X}(\omega')))|\mathcal{F}'_T)\\
=\ &\mathbb{P}'(\mathbb{P}'(Z^i_{t+T}(\phi) - Z^i_{s+T}(\phi)|\mathcal{F}'_{T+s})\psi(\theta^2_T(\tilde{X}))|\mathcal{F}'_T) \text{ (by } (M'_{\lambda L}))\\
=\ &0.
\end{aligned}$$

This implies

$$\hat{\mathbb{P}}_T(\omega')((\tilde{Z}^i_t(\phi) - \tilde{Z}^i_s(\phi))\psi) = 0,\ \forall \psi \in C_s,\ \forall \text{rationals } s \le t,\ \mathbb{P}' - \text{a.s.}.$$

As in the proof of Theorem 4.9, this implies that for $\mathbb{P}'$-a.a. ω', $\tilde{Z}^i(\phi)$ is an a.s. continuous $(\mathcal{F}^2_t)$-martingale under $\hat{\mathbb{P}}_T(\omega')$. Similarly if $\phi_i \in C^2_\ell(\mathbb{R}^d)$ for $\mathbb{P}'$-a.a. ω'

$$M_t(\phi_i, \phi_j) = \tilde{Z}^i_t(\phi_i)\tilde{Z}^j_t(\phi_j) - \delta_{ij}\int_0^t \omega^i(\phi_i^2)ds$$

is an a.s. continuous $(\mathcal{F}^2_t)$-martingale under $\hat{\mathbb{P}}_T(\omega')$.

By Proposition 3.2 (iii) there is a sequence $\eta_k \to 0$ such that

$$\tilde{L}_t(\theta_T(\tilde{X}(\omega'))) = \lim_{k\to\infty} L_t^{\eta_k}(\theta_T(\tilde{X}(\omega'))),\ \forall t \geq 0,\ \mathbb{P}' - \text{a.s.}$$

Therefore for $\mathbb{P}'$-a.s. ω'

$$\tilde{L}_t(\omega^1, \omega^2) = \lim_{k\to\infty} L_t^{\eta_k}(\omega^1, \omega^2),\ \forall t \geq 0,\ \hat{\mathbb{P}}_T(\omega') - \text{a.s.}$$

Theorem 4.9 and Remark 4.10(b) imply that for $\mathbb{P}'$-a.a. ω', $\hat{\mathbb{P}}_T(\omega') = \mathbb{P}^1_{X_T(\omega')}$ because $\hat{\mathbb{P}}_T(\omega')$ solves $(M^1_{\lambda L})$ (modified as in (4.24)) and $X_{T(\omega')} \in M_F^s(\mathbb{R}^d)^2$, $\mathbb{P}'$-a.s., by (a). The result follows because $(m_1, m_2) \mapsto \mathbb{P}^1_{m_1,m_2}$ is Borel measurble on $M_F(\mathbb{R}^d)^2$ by Remark 4.6(b) and (4.18). □

Corollary 4.12. *If $d \leq 3$ and $\lambda \geq 0$, then $(\Omega^2, \mathcal{F}^2, \mathcal{F}^2_t, \theta^2_t, X_t, (\mathbb{P}^1_m)_{m\in M_F^s(\mathbb{R}^d)^2})$ is an $(M_F^s(\mathbb{R}^d))^2$-valued diffusion.*

Proof. The Borel measurability of $(m_1, m_2) \mapsto \mathbb{P}^1_{m_1,m_2}$ was noted at the end of the above proof. The result is now immediate from the previous theorem. □

If $d = 1$ and $(m_1, m_2) \in F$ (as in Theorem 3.9), then one could argue exactly as in Theorem 3.9 to obtain existence and uniqueness of $\mathbb{P}^1_{m_1,m_2}$ satisfying $(M^1_{\lambda L})$ as well as the Girsanov-type formula

$$\frac{d\mathbb{P}^1_{m_1,m_2}}{d\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}}|_{\mathcal{F}^2_t} = \exp\{-\lambda \int_0^t \int u_2(s,x) dZ^{1,0}(s,x) - \lambda^2/2 \int_0^t \int u_2(s,x)^2 u_1(s,x) dx ds\}.$$

We show below that this absolute continuity result fails for $d = 3$ and conjecture that it also fails for $d = 2$. Hence our alternative approach to uniqueness in $(M^1_{\lambda L})$ seems to be necessary.

We require a pair of preliminary results, the first of which (Lemma 5.1) will be proved in the next section to give a self-contained treatment of the non-existence results treated there. The second is the following path property of super-Brownian motion.

Lemma 4.13. *If $\gamma(r) = r^2(\log^+ 1/r)^{-(1/2)-\eta}$ for some $\eta > 0$ and $d > 1$, then*

$$\lim_{r\downarrow 0} \frac{X_t(B(x,r))}{\gamma(r)} = +\infty \text{ for } X_t - a.a.\ x, \mathbb{P}_m - a.s.,\ \forall t > 0,\ m \in M_F(\mathbb{R}^d).$$

We omit the proof as a more precise result will be given in Perkins and Taylor (199?).In that work an integral test on γ will be given for $d > 2$ to decide whether or not the above lim inf is $+\infty$ or 0 for X_t-a.a. x, $\mathbb{P}_m$-a.s. The sufficient condition for $+\infty$ will also apply for $d = 2$. In fact we will only need a much cruder result for $d = 3$ in Theorem 4.14 below.

Theorem 4.14. *Let $d = 3$, $\lambda > 0$ and assume $m_1, m_2 \in M_F^s(\mathbb{R}^3)$. If*

$$\tilde{\mathbb{P}}^1_{m_1,m_2}(A) = \mathbb{P}^1_{m_1,m_2}(\{L_1(X^1, X^2) \neq 0\} \cap A)$$

then $\tilde{\mathbb{P}}^1_{m_1,m_2}|_{\mathcal{F}_1^2}$ and $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}|_{\mathcal{F}_1^2}$ are mutually singular measures. In particular, $\mathbb{P}^1_{m_1,m_2}|_{\mathcal{F}_1^2}$ is not absolutely continuous with respect to $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}|_{\mathcal{F}_1^2}$ whenever $m_i \neq 0$ for $i = 1, 2$.

Proof. The last assertion is immediate from the first since $\tilde{\mathbb{P}}^1_{m_1,m_2} \neq 0$ when $m_i \neq 0$ for $i = 1, 2$. Recall that for $\phi \in D(\Delta/2)$,

$$Z_t^{i,0}(\phi) = X_t^i(\phi) - X_0^i(\phi) - \int_0^t X_s^i(\Delta\phi/2)ds$$

(see (M_{g_1,g_2}) in Section 3). For $\phi \in C_\ell^2(\mathbb{R}^d)$ we define $Z_t^1(\phi)$ by $(M^1_{\lambda L})$, that is,

$$Z_t^1(\phi) = Z_t^{1,0}(\phi) + \lambda L_t(X^1, X^2)(\phi), \tag{4.26}$$

where $L_t(X^1, X^2) = \tilde{L}_t(X^1, X^2)$ is as in Proposition 3.2 and where the underlying measure may be $\mathbb{P}^1_{m_1,m_2}$ or $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$ (Remark 3.3 (b) gives the existence of an $(\mathcal{F}_t^2)$-predictable map L which works for both measures simultaneously). Following Walsh (1986) we can define stochastic integrals $Z_t^{1,0}(\phi)$ (respectively, $Z_t^1(\phi)$) with respect to $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$ (respectively $\mathbb{P}^1_{m_1,m_2}$) for all $\mathcal{P}(\mathcal{F}_t^2) \times \mathcal{B}(\mathbb{R}^d)$-measurable ϕ satisfying

$$\mathbb{P}^1_{m_1,m_2}\Big(\int_0^t X_s^1(\phi(s,\cdot)^2)ds\Big) + \mathbb{P}_{m_1} \times \mathbb{P}_{m_2}\Big(\int_0^t X_s^1(\phi(s,\cdot)^2)ds\Big) < \infty, \ \forall t > 0. \tag{4.27}$$

These integrals are a.s.-continuous square-integrable martingales with square function $\int_0^t X_s^1(\phi(s,\cdot)^2)ds$. If, in addition to (4.27),

$$\sup_{s \le t, x \in \mathbb{R}^d} |\phi(s,\omega,x)| < \infty, \ \mathbb{P}^1_{m_1,m_2} + \mathbb{P}_{m_1} \times \mathbb{P}_{m_2} - \text{a.s.}, \ \forall t > 0, \tag{4.28}$$

then $L_t(X^1,X^2)(\phi) = \int_0^t \int \phi(s,\omega,x)L(X^1,X^2)(ds,dx)$ exists and is a.s. continuous in t (for both $\mathbb{P}^1_{m_1,m_2}$ and $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$). We can now use (4.26) to define $Z_t^{1,0}(\phi)$ with respect to $\mathbb{P}^1_{m_1,m_2}$ for any $\mathcal{P}(\mathcal{F}_t) \times \mathcal{B}(\mathbb{R}^d)$-measurable ϕ satisfying (4.27) and (4.28). To distinguish this stochastic integral from the $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$-stochastic integral with respect to $Z^{1,0}$, we denote the former by $\tilde{Z}_t^{1,0}(\phi)$. We now show these two stochastic integrals are consistent. More precisely we claim that

$$\text{(4.29)}\quad \begin{array}{l} \forall\phi \text{ which is } \mathcal{P}(\mathcal{F}_t^2) \times \mathcal{B}(\mathbb{R}^d)\text{-measurable and satisfying (4.27) and (4.28)} \\ \exists H_\phi : [0,\infty) \times \Omega^2 \to \mathbb{R} \text{ which is } (\mathcal{F}_t^2)\text{-predictable such that} \\ \\ Z_t^{1,0}(\phi) = H_\phi(t,X),\ \forall t \geq 0,\ \mathbb{P}_{m_1} \times \mathbb{P}_{m_2}\text{-a.s., and} \\ \tilde{Z}_t^{1,0}(\phi) = H_\phi(t,X),\ \forall t \geq 0,\ \mathbb{P}^1_{m_1,m_2}\text{-a.s.} \end{array}$$

Consider first

$$\phi \in \mathcal{S} = \{\sum_{i=1}^n f_i(s,\omega)\phi_i(x) : f_i \text{ bounded, left-continuous and } (\mathcal{F}_t^2)-\text{adapted}, \phi_i \in C^2_\ell(\mathbb{R}^d)\}.$$

Then the construction of the above stochastic integrals gives

$$Z_t^{1,0}(\phi) = \sum_{i=1}^n \int_0^t f_i(s)dZ_s^{1,0}(\phi_i),\ \forall t \geq 0,\ \mathbb{P}_{m_1} \times \mathbb{P}_{m_2} - \text{a.s.}$$

Similarly, using (4.26) one gets

$$\tilde{Z}_t^{1,0}(\phi) = \sum_{i=1}^n \int_0^t f_i(s)dZ_s^{1,0}(\phi_i),\ \forall t \geq 0,\ \mathbb{P}^1_{m_1,m_2} - \text{a.s.}$$

By approximating f_i by the usual sequence of step functions $\{f_i^n : n \in N\}$ and taking an appropriate subsequence one constructs a $\mathcal{P}(\mathcal{F}_t^2)$-measurable function H_ϕ such that the conclusion of (4.29) holds. If $\mathcal{C}$ denotes the class of ϕ in $b(\mathcal{P}(\mathcal{F}_t^2) \times \mathcal{B}(\mathbb{R}^d))$ for which (4.29) holds, then it is easy to see that $\mathcal{C}$ is closed under bounded pointwise limits. As $\mathcal{S} \subset \mathcal{C}$ (by the above), Theorem 1.21 of Dellacherie-Meyer (1978) implies $\mathcal{C}$ contains all ϕ in $b(\mathcal{P}(\mathcal{F}_t^2) \times \mathcal{B}(\mathbb{R}^d))$. Now consider ϕ as in (4.29) and let $\phi_n = (\phi \vee (-n)) \wedge n$. Then there is a subsequence such that

$$Z_t^{1,0}(\phi_{n_k}) \to Z_t^{1,0}(\phi)$$

uniformly in t on compacts $\mathbb{P}^1_{m_1,m_2}$-a.s. and

$$\tilde{Z}^{1,0}_t(\phi_{n_k}) = Z^1_t(\phi_{n_k}) - \lambda L_t(X^1, X^2)(\phi_{n_k}) \to \tilde{Z}^{1,0}_t(\phi)$$

uniformly in t on compacts $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$-a.s. Now define

$$H_\phi(t, X) = \begin{cases} \lim_{k\to\infty} H_{\phi_{n_k}}(t, X), & \text{if it exists,} \\ 0, & \text{otherwise,} \end{cases}$$

where $H_{\phi_{n_k}}$ as in (4.29) exists since ϕ_{n_k} is bounded. Clearly H_ϕ is as in (4.29) and the claim is proved. As a result we shall write $Z^{1,0}_t(\phi)$ for both $\tilde{Z}^{1,0}_t(\phi)$ and $Z^{1,0}_t(\phi)$ (and both $\mathbb{P}^1_{m_1,m_2}$ and $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$).

Let

$$G_n(t, X) = \int_{2^{-n/2}}^t \int X^2_s(B(x, 2^{-n}))Z^{1,0}(ds, dx)$$

or, more precisely, $G_n(t, X) = H_{\phi_n}(t, X)$ where

$$\phi_n(s, \omega, x) = \omega^2_s(B(x, 2^{-n}))\ 1(s \geq 2^{-n/2}).$$

Using Theorem 2.1 it is trivial to see that ϕ_n satisfies (4.27) and (4.28), so H_{ϕ_n} exists.

Let $\zeta(r) \equiv r^2(1 + \log(1/r))$. Theorem 4.7 of [BEP] shows that

(4.30)
$D(X^2_t, r) \leq c_1(D(m_2P_t, c_2r)+\zeta(r))$ for $0 < r < r_1(\omega)$ and some $r_1(\omega) > 0$, $\mathbb{P}_{m_1}\times\mathbb{P}_{m_2}$–a.s..

Under $\mathbb{P}_{m_1}\times\mathbb{P}_{m_2}$, (4.30) and a simple calculation show that for $n \geq n_0(\omega)$, where n_0 is a.s. finite, the continuous martingale G_n satisfies

$$(4.31)\qquad \begin{aligned} \langle G_n\rangle_t &= \textstyle\int_{2^{-n/2}}^t \int X^2_s(B(x, 2^{-n}))^2 X^1_s(dx)ds \\ &\leq c_2\zeta(2^{-n})2^{-3n}\hat{L}^n_t(X^1, X^2)(1) \end{aligned}$$

where

$$\hat{L}^n_t(X^1, X^2)(1) = \int_0^t \int X^2_s(B(x, 2^{-n}))X^1_s(dx)ds2^{3n}.$$

Now it is easy to see that

$$\begin{aligned} \hat{L}^n_t(X^1, X^2)(1) &\leq c_3L^{2^{-2n}}_t(X^1, X^2)(1) \\ &\xrightarrow{L^1} c_3L_t(X^1, X^2)(1) \text{ as } n \to \infty, \end{aligned}$$

where in the last time the L^1 convergence is with respect to $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$ and we have used [BEP, Theorem 5.9] (an integration by parts shows that $m_i \in M_F^s(\mathbb{R}^d)$ implies the hypothesis of that result are satisfied). The L^1 boundedness of $\{\hat{L}_1^n : n \in \mathbb{N}\}$ and a Borel-Cantelli argument show that if $\eta > 0$ is fixed, then $\langle G_n \rangle_1 \leq c_2 2^{-3n} \zeta(2^{-n}) n^{1+\eta}$ for sufficiently large n $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$-a.s. This and a well-known estimate (Rogers-Williams (1987, IV. 37.12)) imply

$$
\begin{aligned}
\sup_{t \leq 1} |G_n(t)| &\leq 2^{-3n/2}(\zeta(2^{-n}))^{1/2} n^{\frac{1}{2}+\eta} \\
&\leq c_4 2^{-5n/2} n^{1+\eta}
\end{aligned}
\tag{4.32}
$$

for sufficiently large n, $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$-a.s., $\forall \eta > 0$. Now let

$$
A = \{\omega \in \Omega^2 : |G_n(1)| \leq c_4 2^{-5n/2} n^{1+\eta} \text{ for sufficiently large } n\} \in \mathcal{F}_1^2.
$$

With respect to $\mathbb{P}^1_{m_1,m_2}$, our definition of $Z^{1,0}$ (i.e. (4.26)) gives

$$
\begin{aligned}
G_n(t) &= \int_{2^{-n/2}}^t \int X_s^2(B(x, 2^{-n})) Z^1(ds, dx) \\
&- \lambda \int_{2^{-n/2}}^t \int X_s^2(B(x, 2^{-n})) L(X^1, X^2)(ds, dx) \\
&\equiv G_{n,1}(t) - G_{n,2}(t).
\end{aligned}
$$

The process $G_{n,1}$ is a continuous square integrable martingale under $\mathbb{P}^1_{m_1,m_2}$ and if $Y^i \geq X^i$ are independent super Brownian motions (working now on a larger space – see Theorem 2.1) then

$$
\langle G_{n,1} \rangle_t = \int_{2^{-n/2}}^t \int X_s^2(B(x, 2^{-n}))^2 X_s^1(dx) ds \leq \int_{2^{-n/2}}^t Y_s^2(B(x, 2^{-n}))^2 Y_s^1(dx) ds
$$

$$
\leq c_2 \zeta(2^{-n}) 2^{-3n} \hat{L}_t^n(Y^1, Y^2)(1) \text{ for } n \text{ large a.s. (see (4.31))}
$$

Repeating the above argument gives

$$
|G_{n,1}(1)| \leq c_4 2^{-5n/2} n^{1+\eta} \text{ for sufficiently large } n, \ \mathbb{P}^1_{m_1,m_2} - \text{a.s.} \tag{4.33}
$$

Take mean values to see that $Y^2 = X^2$ a.s. Let γ be as in Lemma 4.13 and set

$$
\phi(s, x, X^2) = 1\{\liminf_{r \downarrow 0} X_s^2(B(x, r)) \gamma(r)^{-1} < +\infty\} = 0
$$

for X^2_s-a.a. x, a.s., $\forall s > 0$ (by Lemma 4.13). Lemma 5.1 now gives

$$\phi(s, x, X^2) = 0,\ L(Y^1, X^2) - \text{a.a. } (s, x),\ \text{a.s.} \tag{4.34}$$

Recall from Proposition 3.2 (iii) and our construction of L (see Remark 3.3 (b)) that for some $\epsilon_n \downarrow 0$

$$L_t(Y^1, X^2) = \lim_{n\to\infty} L_t^{\epsilon_n}(Y^1, X^2),\ \forall t \geq 0,\ \text{a.s.}$$

and

$$L_t(X^1, X^2) = \lim_{n\to\infty} L_t^{\epsilon_n}(X^1, X^2),\ \forall t \geq 0,\ \text{a.s.}$$

Hence clearly $L(X^1, X^2) \leq L(Y^1, X^2)$ (as random measures on $[0, \infty) \times \mathbb{R}^d$) a.s. and so (4.34) implies

$$\phi(s, x, X^2) = 0,\ L(X^1, X^2) - \text{a.a. } (s, x),\ \mathbb{P}^1_{m_1,m_2} - \text{a.s.}$$

Therefore

$$\liminf_{n\to\infty} G_{n,2}(1)\gamma(2^{-n})^{-1}$$

$$\geq \int_0^1 \int \liminf_{n\to\infty} 1(s \geq 2^{-n/2})\, \lambda X_s^2(B(x, 2^{-n}))\gamma(2^{-n})^{-1} L(X^1, X^2)(ds, dx)\ \text{(Fatou)}$$

$$= \infty 1\{L_1(X^1, X^2)(1) > 0\},\ \mathbb{P}^1_{m_1,m_2} - \text{a.s.}$$

(with the convention $\infty \cdot 0 = 0$). This and (4.33) show that

$$\lim_{n\to\infty} G_n(1)\gamma(2^{-n})^{-1} = -\infty,\ \mathbb{P}^1_{m_1,m_2} - \text{a.s. on } \{L_1(X^1, X^2) \neq 0\}. \tag{4.36}$$

In particular $\tilde{\mathbb{P}}^1_{m_1,m_2}(A) = 0$, while (4.32) shows $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}(A) = 1$. This proves the result. □

Open Problem. Is $\mathbb{P}^1_{m_1,m_2}(X^1 \in \cdot) << \mathbb{P}_{m_1}$ on $\mathcal{F}_t$?

Proof of Theorem 3.11. We assume $\mathbb{P}|_{\mathcal{F}_1^2} << \mathbb{P}_{m_1} \times \mathbb{P}_{m_2}|_{\mathcal{F}_1^2}$ and proceed by modifying the proof of Theorem 4.14 to obtain a contradiction. We use the notation of the proof of Theorem 4.14.

Argue just as in the proof of the inequality (4.33) to see that

$$|G_{n,1}(1)| \leq c_1 2^{-5n/2} n^{1+\eta} \text{ for sufficiently large } n,\ \mathbb{P} - \text{a.s.} \tag{4.37}$$

Lemmas 4.13 and 5.1 imply

$$\phi(s,x,X^2)=0,\ L(Y^1,X^2)-\text{a.a. }(s,x),\ \mathbb{P}_{m_1}\times\mathbb{P}_{m_2}-\text{a.s.}$$

The absolute continuity assumption and Remark 3.3(b) therefore show that

$$\phi(s,x,X^2)=0,\ L(Y^1,X^2)-\text{a.a. }(s,x),\ \mathbb{P}-\text{a.s.},$$

which in turn gives (set $0\cdot\infty=0$)

$$\liminf_{n\to\infty} G_{n,2}(1)\gamma(2^{-n})^{-1}$$

$$\geq \int_0^1\int \liminf_{n\to\infty} 1(s\geq 2^{-n/2})\,\lambda X_s^2(B(x,2^{-n}))\gamma(2^{-n})^{-1}L(X^1,X^2)(ds,dx)\ \text{(Fatou)}$$

$$=\infty 1\{L_1(X^1,X^2)(1)>0\},\ \mathbb{P}-\text{a.s.}$$

This and (4.37) imply that

$$\mathbb{P}(A\cap\{L_1(X^1,X^2)(1)>0\})=0$$

and therefore

$$\mathbb{P}(A^c\cap\{L_1(X^1,X^2)(1)>0\})=\mathbb{P}(L_1(X^1,X^2)(1)>0)>0. \tag{4.38}$$

Note that the last inequality must hold since otherwise $\mathbb{P}|_{\mathcal{F}_1^2}=\mathbb{P}_{m_1}\times\mathbb{P}_{m_2}|_{\mathcal{F}_1^2}$ by $(M_{\lambda L})$ and we know $\mathbb{P}_{m_1}\times\mathbb{P}_{m_2}(L_1(X^1,X^2)(1)>0)>0$ for $m_i\neq 0$ (see [BEP, Prop. 5.11].) Now (4.38) and the fact that $\mathbb{P}_{m_1}\times\mathbb{P}_{m_2}(A)=1$ (see (4.32)) contradict our absolute continuity assumption and so the proof is complete. □

5 Non-existence for $d > 3$

Recall from Section 1 (or [BEP]) that two independent "d-dimensional" super-Brownian motions have a non-trivial collision local time for $d \le 5$. It is therefore natural to consider the interactive martingale problems $(M_{\lambda L})$ and $(M^1_{\lambda L})$ for $d \le 5$ and not just $d \le 3$. The construction of solutions to $(M_{\lambda L})$ in Section 3 relied on a convergence result for $L(X^1, X^2)$ which was uniform in $(X^1, X^2) \in \mathcal{M}_{m_1,m_2}$ (Lemma 3.4) and which could only be proved for $d \le 3$. The treatment of $(M^1_{\lambda L})$ in Section 4 was based on constructing a CAF, A, for each Brownian path in the X^1 population. The existence of A required $d \le 3$. In either case the restriction to $d \le 3$ seemed to be an artifact of the proof. In this section we show that in fact $(M_{\lambda L})$ and $(M^1_{\lambda L})$ cannot be solved for $d = 4$ or 5. This work is joint with Martin Barlow.

First note that if $d > 5$, then by Theorem 2.1 and the a.s.-non-intersection of the graphs of two independent super-Brownian motions (see [BEP, Thm. 3.6, Remark 5.12(a))), the only possible solution to $(M_{\lambda L})$ and $(M^1_{\lambda L})$ is $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$ for which $L_t(X^1, X^2) \equiv 0$. If m_1 and m_2 have disjoint closed supports, clearly $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$ is a solution of $(M_{\lambda L})$ and $(M^1_{\lambda L})$. Therefore these martingale problems are only of interest for $d \le 5$ which we assume for the rest of this section.

We continue with the notation of Sections 3 and 4. In particular, $\mathbb{P}_m$ continues to denote the law of super-Brownian motion on $(\Omega, \mathcal{F}) = (C([0,\infty), M_F(\mathbb{R}^d))$, Borel sets). Also, $X_t(\omega) = \omega(t)$ and (X^1_t, X^2_t) denote the coordinate mappings on $(\Omega, \mathcal{F})$ and $(\Omega^2, \mathcal{F}^2)$, respectively.

Notation. $g_0 : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ is given by

$$g_0(x) = \begin{cases} 1, & d = 1 \\ \ln^+(1/|x|), & d = 2 \\ |x|^{2-d}, & d > 2 \end{cases} .$$

Lemma 5.1. *Let $d \le 5$ and assume $m_1, m_2 \in M_F(\mathbb{R}^d)$ satisfy*

$$\text{(5.1)} \quad \begin{aligned} &\int\int g_0(x_1 - x_2) dm_1(x_1) dm_2(x_2) < \infty, \text{ if } d < 5 \\ &\int\int (x_1 - x_2)^{-4} dm_1(x_1) dm_2(x_2) < \infty, \text{ if } d = 5. \end{aligned}$$

If $\phi : [0,t] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ *is bounded and* $\mathcal{B}([0,t] \times \mathbb{R}^d) \times \mathcal{F}$*-measurable, then*

$$\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}(\int_0^t \int \phi(s,x,X^2) L(X^1,X^2)(ds,dx)|X^2) \tag{5.2}$$

$$= \int_0^t \int \int \phi(s,x_2,X^2) p_s(x_2 - x_1) dm_1(x_1) X_s^2(dx_2) ds \ \ a.s.$$

Proof. Theorem 5.9 of [BEP] and the bound (3.3) imply that for $\psi \in C_b(\mathbb{R}^d)$

$$\lim_{\epsilon \downarrow 0} \| \sup_{u \le t} |L_u^\epsilon(X^1,X^2)(\psi) - L_u(X^1,X^2)(\psi)| \|_1 = 0$$

(the L_1 norm is taken with respect to $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}$). An elementary argument now shows that if $\psi \in C_\ell([0,t] \times \mathbb{R}^d)$ then

$$\lim_{\epsilon \downarrow 0} \| \int_0^t \int \psi(s,x) L^\epsilon(X^1,X^2)(ds,dx) \tag{5.3}$$

$$- \int_0^t \int \psi(s,x) L(X^1,X^2)(ds,dx) \|_1 = 0$$

(for example, one can first extract a subsequence along which one has weak convergence in $M_F([0,t] \times \mathbb{R}^d)$ a.s. as in (4.11)). Let $\psi(s,x)$ be as above. A standard bootstrapping argument shows that w.p.1

$$\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}(\int_0^t \int \psi(s,x) L^\epsilon(X^1,X^2)(ds,dx)|X^2)$$

$$= \mathbb{P}_{m_1} \times \mathbb{P}_{m_2}(\int_0^t \int \int \psi(s,x_2) p_\epsilon(x_2 - x_1) X_s^1(dx_1) X_s^2(dx_2) ds | X^2)$$

$$= \int_0^t \int \mathbb{P}_{m_1} \times \mathbb{P}_{m_2}(\int p_\epsilon(x_2 - x_1) X_s^1(dx_1)|X^2) \psi(s,x_2) X_s^2(dx_2) ds$$

$$= \int_0^t \int \int p_{\epsilon+s}(x_2 - x_1) m_1(dx_1) \psi(s,x_2) X_s^2(dx_2) ds. \tag{5.4}$$

Note that

$$\mathbb{P}_{m_1} \times \mathbb{P}_{m_2}(\int_0^t \int \int p_{\epsilon+s}(x_2 - x_1) X_s^2(dx_2) m_1(dx_1) ds)$$

$$= \int_0^t \int \int p_{\epsilon+2s}(x_2 - x_1) m_2(dx_2) m_1(dx_1) ds$$

$$\to \int_0^t \int \int p_{2s}(x_2 - x_1) m_2(dx_2) m_1(dx_1) ds \text{ (use (5.1) and dominated convergence)}$$

$$(5.5) \qquad = \mathbb{P}_{m_1} \times \mathbb{P}_{m_2} \Big(\int_0^t \int \int p_s(x_2 - x_1) X_s^2(dx_2) m_1(dx_1) ds \Big) < \infty.$$

This shows that $\{p_{\epsilon+s}(x_2 - x_1) : \epsilon > 0\}$ is a uniformly integrable family with respect to $X_s^2(dx_2)m_1(dx_1)dsd(\mathbb{P}_{m_1} \times \mathbb{P}_{m_2})$ and, as ψ is bounded, this allows us to take a limit as $\epsilon \downarrow 0$ inside the integral sign in (5.4) and conclude

$$L^1 - \lim_{\epsilon \downarrow 0} \int_0^t \int \int p_{\epsilon+s}(x_2 - x_1) m_1(dx_1) \psi(s, x_2) X_s^2(dx_2) ds$$

$$= \int_0^t \int \int p_s(x_2 - x_1) m_1(dx_1) \psi(s, x_2) X_s^2(dx_2) ds.$$

This together with (5.3) allows us to take L^1-limits on both sides of (5.4) to obtain the required result with $\psi(s, x)$ in place of $\phi(s, x, X^2)$.

Observation (5.5), together with the above for $\psi = 1$, shows that both sides of (5.2) are integrable for any bounded ϕ. Therefore (5.2) is preserved under bounded pointwise limits. Moreover, (5.2) holds for $\phi(s, x, X^2) = \sum_{i=1}^n \psi_i(s, x) S_i(X^2)$ for bounded measurable S_i and $\psi_i \in C_\ell([0, t] \times \mathbb{R}^d)$ by the above. Now pass to the bounded pointwise closure of this class of functions to complete the proof. □

Lemma 5.2. *The integral* $\int 1(|x - y| \le 1)|x - y|^{-2} X_s(dx)$ *takes the value* ∞ *for* X_s*-a.a.* y, $\forall s > 0$, $\mathbb{P}_m$*-a.s.*

Proof. Let $h - m(A)$ denote the Hausdorff h-measure of A where $h(r) = r^2 \log^+(\log^+(1/r))$. If $d = 1$ the results is trivial because $X_s(dx) = u(s, x)dx$ for some jointly continuous density u. Assume $d \ge 2$. Let $S(X_t)$ be the closed support of X_t. Fix ω outside a $\mathbb{P}_m$-null set such that $S(X_t)$ is compact $\forall t > 0$, and

$$(5.6) \qquad X_t(A) \ge c(d) h - m(A \cap S(X_t)), \ \forall A \in \mathcal{B}(\mathbb{R}^d), \ \forall t > 0$$

(see Perkins (1989, Theorems 1 and 2) and Dawson-Iscoe-Perkins (1989, Thm. 1.2)). Assume now the desired conclusion fails for ω as above and some $s > 0$. Since $S(X_s)$ is compact this means

$$X_s(\{y : \int |x-y|^{-2} X_s(dx) < \infty\}) > 0$$

and therefore for large enough N, $X_s(\Lambda_N) > 0$ where

$$\Lambda_N = \{y \in S(X_s) : \int |x-y|^{-2} X_s(dx) \leq N\}.$$

Therefore

$$\int_{\Lambda_N} \int_{\Lambda_N} |x-y|^{-2} X_s(dx) X_s(dy) < \infty,$$

and so Λ_N has positive two-dimensional capacity. By Taylor (1961), this implies $x^2 - m(\Lambda_N) = \infty$. On the other hand

$$h - m(\Lambda_N) = h - m(\Lambda_N \cap S(X_s)) \leq c(d)^{-1} X_s(\Lambda_N) < \infty$$

and this contradicts $x^2 - m(\Lambda_N) = \infty$. $\square$

Theorem 5.3. *Let $d = 4$ or 5 and assume $m_1, m_2 \in M_F(\mathbb{R}^d)\backslash\{0\}$ satisfy (5.1). If $\lambda > 0$, $(M^1_{\lambda L})$ and $(M_{\lambda L})$ have no solutions.*

Proof. Let (X^1, X^2) satisfy $(M^1_{\lambda L})$,$\lambda > 0$. Theorem 2.1 allows us to enlarge our probability space so that it supports a super-Brownian motion $Y^1 \geq X^1$. Theorem 2.1(d) shows that if Z^{Y^1} is the orthogonal martingale measure associated with Y^1, then $\langle Z^{Y^1}(\phi_1), Z^2(\phi_2)\rangle_t = 0$, $\forall \phi_i \in D(\Delta/2)$. It follows that Y^1 and X^2 are independent super-Brownian motions starting at m_1 and m_2, respectively (see [BEP, Thm 1.2] or use Theorem 1.1 above). Lemma 5.7 of [BEP] with $\alpha = 0$ and $\psi = 1$ implies that

$$\int_0^t \int \int |x_1 - x_2|^{2-d} X^2_s(dx_2) L(X^1, X^2)(ds, dx_1) < \infty, \ \forall t \geq 0, \text{ a.s.}$$

If

$$A(X^2) = \{(s, x_1) : \int |x_1 - x_2|^{2-d} X^2_s(dx_2) = \infty\},$$

and we continue to write $L(X^1, X^2)$ for the induced random measure on $[0, \infty) \times \mathbb{R}^d$, then the above implies

$$L(X^1, X^2)(A(X^2)) = 0 \text{ a.s.} \tag{5.7}$$

On the other hand, Lemma 5.2 implies (write $1_{A^c}(s,x,X^2)$ for $1_{A(X^2)^c}(s,x)$)

$$\int_0^\infty 1_{A^c}(s,x,X^2)X_s^2(dx)ds = 0 \text{ a.s.}$$

$$\Rightarrow \int_0^\infty \int 1_{A^c}(s,x,X^2)L(Y^1,X^2)(ds,dx) = 0 \text{ a.s. (Lemma 5.1)}$$

$$\Rightarrow L(X^1,X^2)(A(X^2)^c) = 0 \text{ a.s.,}$$

the last because $L(X^1,X^2) \le L(Y^1,X^2)$ a.s. (as in the proof of Theorem 4.14). This, together with (5.7) implies $L(X^1,X^2)=0$ a.s. Therefore, $(M^1_{\lambda L})$ implies (X^1,X^2) is a pair of independent super-Brownian motions ([BEP, Thm 1.2]). The fact that $L(X^1,X^2)=0$ a.s. contradicts Proposition 5.11 of [BEP], and hence there can be no solution to $(M^1_{\lambda L})$.

Assume now (X^1,X^2) satisfies $(M_{\lambda L})$. By enlarging our probability space as in Theorem 2.1 we may also assume there is a pair of independent super-Brownian motions (Y^1,Y^2) such that $Y^i \ge X^i$ a.s. Theorem 2.1 (d) shows that (Y^1,X^2) satisfies (M_{m_1,m_2}) with $A^1=0$ and $A^2_t = \lambda L_t(X^1,X^2)$. We may therefore apply Lemma 5.7 of [BEP] (with $\alpha=0$ and $\psi=1$) to conclude

$$\int_0^t \int\int |x_1-x_2|^{2-d} Y_s^1(dx_1) L(X^1,X^2)(ds,dx_2) < \infty, \ \forall t \ge 0, \text{ a.s.}$$

If

$$B(Y^1) = \{(s,x_2) : \int |x_1-x_2|^{2-d} Y_s^1(dx_1) = \infty\}$$

then the above implies

$$L(X^1,X^2)(B(Y^1)) = 0 \text{ a.s.}$$

On the other hand by applying Lemmas 5.1 and 5.2 as in the previous argument one gets

$$L(X^1,X^2)(B(Y^1)^c) = 0 \text{ a.s.}$$

(again we use $L(X^1,X^2) \le L(Y^1,Y^2)$ a.s.). The proof is completed as above. □

Remarks 5.4. (a) Although $L_t(X^1,X^2)$ exists if $(X^1,X^2) \in \mathcal{M}(m_1,m_2)$ and $d \le 5$, the above result shows that the uniform convergence result, Lemma 3.4, must fail for $d>3$. If it held, then the proof of Theorem 3.6 would produce solutions to $(M_{\lambda L})$ (and $M^1_{\lambda L}$), contradicting the above result.

(b) Note that (5.1) is needed to ensure that $L_t(X^1, X^2)$ exists ([BEP, Thm. 5.9]) and hence $(M^1_{\lambda L})$ and $(M_{\lambda L})$ make sense.

(c) The rather slick argument above hides the intuitive reason for the non-existence of solutions for $d > 3$: the only collisions that occur are between particles whose family trees will die out in an infinitesimal time due to the critical branching. Hence killing off some of these particles has no effect on the population.

A. Appendix: Superprocesses for Non-conservative Markov Processes

In this appendix we prove existence and uniqueness to the standard martingale problem for superprocesses when the underlying Markov process is not conservative. In all the references we know of, the underlying semigroup is assumed to satisfy $P_t 1 = 1$. If the underlying process W is killed by the CAF $A_t = \int_0^t b(W_s)ds$ ($b \geq 0$ bounded) this effectively introduces an emigration term and is standard. We are interested in CAF's with more singular Revuz measures. We make no attempt at maximal generally because we are only interested in a fairly particular case.

Let $W = (D, \mathcal{D}, \mathcal{D}_{t+}, \theta_t, W_t, P_y)$ be the canonical realization of a Feller process on a locally compact state space E and with semigroup P_t on $C_\ell(E)$. Let A_t be a CAF for W and define a sub-Markov semigroup $\{\bar{P}_t : t \geq 0\}$ on $C_\ell(E)$ by

$$\bar{P}_t f(x) = P_x(e^{-A_t} f(W_t)).$$

We assume

$$(A.1) \qquad \{\bar{P}_t : t \geq 0\} \text{ is a Feller}$$

$$\text{(i.e. strongly continuous) semigroup on } C_\ell(E).$$

If $E_\Delta = E \cup \{\Delta\}$ (Δ is added as a discrete point) and e is an independent exponential time, then

$$\bar{W}_t = \begin{cases} W_t & \text{if } A_t < e \\ \Delta & \text{if } A_t \geq e \end{cases}$$

is a strong Markov process with semigroup $\bar{P}_t$. Here $\bar{P}_t$ is extended trivially to a semigroup on $C_\ell^\Delta(E_\Delta) = \{f \in C_\ell(E_\Delta) : f(\Delta) = 0\}$. Finally, we introduce the semigroup $\{P_t^\Delta : t \geq 0\}$ on $C_\ell(E_\Delta)$ given by

$$P_t^\Delta f(x) = \bar{P}_t f(x) + P_x(1 - e^{-A_t}) f(\Delta).$$

Thus $\{P_t^\Delta : t \geq 0\}$ is the semigroup of the strong Markov process W_t^Δ which is $\bar{W}_t$ but now viewed as an E_Δ-valued process with Δ a trap. It is easy to see that (A.1) implies W_t^Δ is an E_Δ-valued Feller process. Let $\bar{G}$ and G^Δ denote the strong infinitesimal generators of $\bar{P}_t$ and P_t^Δ, respectively. We consider $D(\bar{G})$ as a subset of $C_\ell^\Delta(E_\Delta)$ and $D(G^\Delta) \subset C_\ell(E_\Delta)$.

Let $X = (\Omega^\Delta, \mathcal{F}^\Delta, \mathcal{F}_t^\Delta, X_t, (\mathbb{P}_m)_{m\in M_F(E_\Delta)})$ be the canonical realization of the W^Δ-superprocess on $\Omega^\Delta = C([0,\infty), M_F(E_\Delta))$ with its Borel σ-field $\mathcal{F}^\Delta$ and canonical right-continuous filtration $(\mathcal{F}_t^\Delta)$. The process X is a Feller process (Dynkin 1989, Sec.8)) and $\mathbb{P}_m$ is the unique law on Ω^Δ satisfying the following martingale problem (which we label as (M^Δ)):

$$X_0 = m,\ \mathbb{P}_m - \text{a.s.},$$
$$X_t(\phi) = X_0(\phi) + Z_t(\phi) + \int_0^t X_s(G^\Delta\phi)ds,$$

$\forall t \geq 0$, $\mathbb{P}_m$-a.s., $\forall \phi \in D(G^\Delta)$; where $Z_t(\phi)$ is a continuous $(\mathcal{F}_t^\Delta)$-martingale under $\mathbb{P}_m$ such that

$$\langle Z(\phi)\rangle_t = \int_0^t X_s(\phi^2)ds,$$

$\forall t \geq 0$, $\mathbb{P}_m$-a.s.

(see Ethier-Kurtz (1986, Sec.9.4)).

If $\mu \in M_F(E^\Delta)$ let μ^r denote the restriction of μ to the Borel sets in E. We call X_t^r (under $\mathbb{P}_m$) the $\bar{W}$-superprocess starting at m^r. The next theorem gives a natural martingale characterization of the law of this process on $\bar{\Omega} = C([0,\infty), M_F(E))$. Let $\bar{X}_t(\omega) = \omega(t)$ denote the coordinate variables on $\bar{\Omega}$ with its Borel σ-field $\bar{\mathcal{F}}$ and natural right continuous filtration $(\bar{\mathcal{F}}_t)$.

Theorem A.1. *For all $m \in M_F(E)$ there is a unique law $\bar{\mathbb{P}}_m$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ that solves the following martingale problem $(\bar{M})$:*

$$\bar{X}_0 = m\ \bar{\mathbb{P}}_m - a.s.,$$
$$\bar{X}_t(\phi) = \bar{X}_0(\phi) + \bar{Z}_t(\phi) + \int_0^t \bar{X}_s(\bar{G}\phi)ds,$$

$\forall t \geq 0$, $\forall \phi \in D(\bar{G})$; where $\bar{Z}_t(\phi)$ is a continuous $(\bar{\mathcal{F}}_t)$-martingale under $\bar{\mathbb{P}}_m$ such that

$$\langle \bar{Z}\rangle_t = \int_0^t \bar{X}_s(\phi^2)ds,$$

$\forall t \geq 0$, $\bar{\mathbb{P}}_m$-a.s.

Moreover $\bar{X} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{X}_t, (\bar{\mathbb{P}})_{m \in M_F(E)})$ *is an* $M_F(E)$*-valued diffusion and*

$$(A.2) \qquad \bar{\mathbb{P}}_m(\cdot) = \mathbb{P}_m(X^r \in \cdot)$$

Proof. If $\phi \in D(\bar{G}) \subset C_\ell^\Delta(E_\Delta)$, then $P_t^\Delta \phi = \bar{P}_t \phi$ and therefore $\phi \in D(G^\Delta)$ and $G^\Delta \phi = \bar{G}\phi$. It is now clear from (M^Δ) that $\bar{\mathbb{P}}_m$ given by (A.2) is a solution of $(\bar{M})$.

Turning to uniqueness, let $\bar{\mathbb{P}}_m$ be a solution of $(\bar{M})$. If $\psi_\epsilon(x) = \int_0^\epsilon \bar{P}_s 1(x) ds/\epsilon$, then Ethier-Kurtz (1986, Prop. 1.1.5 (a)) shows that $\psi_\epsilon \in D(\bar{G})$ and

$$\bar{G}\psi_\epsilon(x) = (\bar{P}_\epsilon 1(x) - 1)\epsilon^{-1} 1_E(x) = P_x(e^{-A_\epsilon} - 1)\epsilon^{-1} 1_E(x) \equiv -g_\epsilon(x).$$

Clearly $\psi_\epsilon \to 1$ uniformly on E as $\epsilon \downarrow 0$ by strong continuity. As usual, $\bar{Z}$ extends to an orthogonal martingale measure $\{\bar{Z}_t(\psi) : \psi \in b\mathcal{E}\}$ under $\bar{\mathbb{P}}_m$. By Doob's inequality,

$$\sup_{t \le T} |\bar{Z}_t(\psi_\epsilon) - \bar{Z}_t(1)| \xrightarrow{L^2} 0 \text{ as } \epsilon \downarrow 0,$$

and clearly $\bar{X}_t(\psi_\epsilon) \to \bar{X}_t(1)$, $\forall t \ge 0$, $\bar{\mathbb{P}}_m$-a.s. Put $\phi = \psi_\epsilon$ in $(\bar{M})$ and let $\epsilon \downarrow 0$ to see that for some $\epsilon_n \downarrow 0$,

$$(A.3) \qquad \int_0^t \bar{X}_s(g_{\epsilon_n}) ds \to C_t, \ \forall t \ge 0, \ \bar{\mathbb{P}}_m - \text{a.s. as } n \to \infty$$

where $C_t = m(1) + \bar{Z}_t(1) - \bar{X}_t(1)$ is a.s. a continuous non-decreasing process. Now enlarge our probability space, to $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}})$ say, so that it supports an independent $(\hat{\mathcal{F}}_t)$-Brownian motion B_t. Let S_t be the pathwise unique solution of

$$S_t = \int_0^t \sqrt{S_u} dB_u + C_t$$

(Barlow-Perkins (1983, Thm. 3.2)). Let $X_t = \bar{X}_t + S_t \delta_\Delta \in M_F(E_\Delta)$. If $\phi \in D(G^\Delta)$ and $\bar{\phi}(x) = \phi(x) 1_E(x)$, then Ethier-Kurtz (1986, Prop. 1.1.5(a)) implies

$$\bar{\phi}_\epsilon(x) = \int_0^\epsilon \bar{P}_s \bar{\phi}(x) dx/\epsilon \in D(\bar{G})$$

and

$$
\begin{aligned}
\bar{G}\bar{\phi}_\epsilon &= (\bar{P}_\epsilon\bar{\phi}(x) - \bar{\phi}(x))/\epsilon \\
&= (P_\epsilon^\Delta \phi(x) - \phi(x))/\epsilon - P^x(1 - e^{-A_\epsilon})\phi(\Delta)\epsilon^{-1}1_E(x) \\
&\equiv G^{\Delta,\epsilon}\phi(x) - g_\epsilon(x)\phi(\Delta),
\end{aligned} \tag{A.4}
$$

where $G^{\Delta,\epsilon}\phi(x) \to G^\Delta\phi(x)$ uniformly on E_Δ as $\epsilon \downarrow 0$. Let ϕ_ϵ extend $\bar{\phi}_\epsilon$ to E_Δ by $\phi_\epsilon(\Delta) = \phi(\Delta)$. Combining $(\bar{M})$ with $\phi = \bar{\phi}_\epsilon$ and (A.4) gives

$$
\begin{aligned}
X_t(\phi_\epsilon) &= \bar{X}_t(\bar{\phi}_\epsilon) + S_t\phi(\Delta) \\
&= m(\phi_\epsilon) + \bar{Z}_t(\bar{\phi}_\epsilon) + \phi(\Delta)\int_0^t \sqrt{S_u}\,dB(u) \\
&+ \int_0^t \bar{X}_s(G^{\Delta,\epsilon}\phi)ds - \phi(\Delta)\int_0^t \bar{X}_s(g_\epsilon)ds + \phi(\Delta)C_t,
\end{aligned}
$$

$\forall t \geq 0$, $\hat{\mathbb{P}}-$ a.s.. Note that $\phi_\epsilon \to \phi$ and $\bar{\phi}_\epsilon \to \bar{\phi}$ uniformly on E_Δ and E, respectively. Now let $\epsilon = \epsilon_n \downarrow 0$ in the above and use (A.3) to conclude

$$
X_t(\phi) = m(\phi) + Z_t(\phi) + \int_0^t X_s(G^\Delta\phi)ds,\ \forall t \geq 0,\ \hat{\mathbb{P}} - \text{a.s.},
$$

where $Z_t(\phi) = \bar{Z}_t(\bar{\phi}) + \phi(\Delta)\int_0^t \sqrt{S_u}dB_u$ is a continuous $(\hat{\mathcal{F}}_t)$-martingale such that

$$
\langle Z(\phi)\rangle_t = \int_0^t X_u(\phi^2)du,\ \forall t \geq 0,\ \hat{\mathbb{P}} - \text{a.s.}
$$

Comparing this with (M_Δ) we get $\hat{\mathbb{P}}(X \in \cdot) = \mathbb{P}_m(\cdot)$. This in turn implies $\bar{\mathbb{P}}_m(\cdot) = \mathbb{P}_m(X^r \in \cdot)$ because $\bar{X}_t = (X_t)^r$ in the above. Hence $\bar{\mathbb{P}}_m$ is unique.

The strong Markov property of $\bar{X}$ is a standard consequence of the uniqueness in $(\bar{M})$ (e.g. see the proof of Theorem 2.5). The Borel measurability of $m \mapsto \bar{\mathbb{P}}_m$ is clear from that of $m \mapsto \mathbb{P}_m$. □

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