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KLEIN–MASKIT COMBINATION THEOREM FOR ANOSOV SUBGROUPS: AMALGAMS

SUBHADIP DEY AND MICHAEL KAPOVICH

ABSTRACT. The classical Klein–Maskit combination theorems provide sufficient conditions to construct new Kleinian groups using old ones. There are two distinct but closely related combination theorems: The first deals with amalgamated free products, whereas the second deals with HNN extensions. This article gives analogs of both combination theorems for Anosov subgroups.

CONTENTS

1. INTRODUCTION

In the theory of Kleinian groups (discrete isometry groups of $\mathbb{H}^3$), the *Klein–Maskit combination theorems* provide techniques to construct new Kleinian groups. The history of combination theorems dates back to Klein's 1883 paper [22], which gave sufficient conditions for a subgroup Γ of $G = \mathrm{Isom}(\mathbb{H}^3)$ generated by two discrete subgroups Γ_1 and Γ_2 of G to be discrete and isomorphic to the free product $\Gamma_1 \star \Gamma_2$. Subsequently, Maskit [30, 31, 32, 34], dealing with the cases of amalgamated free products and HNN extensions, gave far-reaching generalizations of the *Klein combination theorem*. Maskit's combination theorems also furnish sufficient conditions for the combined group to have certain geometrical properties, such as *convex-cocompactness* or *geometric-finiteness*. See Maskit's book [33] for an account of those results. Later on, Ivascu [15] and, more recently, Li-Ohshika-Wang [24, 25] extended the Klein–Maskit combination theorems to the setting of discrete isometry groups of higher dimensional hyperbolic spaces. The combination theorems were further generalized in the context of group actions on Gromov-hyperbolic spaces; see [1, 11, 27, 28, 29, 35, 41]. We refer to [37] for a recent survey of combination theorems.

In the recent years, *Anosov subgroups* of semisimple Lie groups have emerged as a higher-rank generalization of convex-cocompact Kleinian groups. Our motivation for this article is to provide suitable analogs of the Klein–Maskit combination theorems in the context of Anosov subgroups. See Theorem A and Theorem B below, where we state our versions of the *first* and *second* Klein–Maskit combination theorems, respectively. We note that the general strategy of our proof of these results is quite different from Maskit's proof and its subsequent generalization by Ivascu, and Li-Ohshika-Wang. The main distinction comes from the fact that Kleinian groups act as *discrete convergence groups* on the ideal boundary of the hyperbolic space. In contrast, such convergence property of general discrete subgroups is absent in the higher rank setting. However, the special class of discrete subgroups, called *regular subgroups*, which includes *Anosov subgroups*, exhibits a (more technical) form of convergence action on suitable partial flag varieties. A significant part of the technical core of this paper is devoted to developing machinery to verify that our combined group retains the higher rank convergence property. After that, the main difficult step is to construct an equivariant *boundary map* and demonstrate its continuity. Furthermore, in contrast to our arguments, Maskit's proof and its subsequent generalizations for Kleinian groups relied upon proving that certain limit points are conical and we currently do not know how to do this for higher rank *regular antipodal* subgroups without constructing boundary maps in the, more special, Anosov setting.

Previously, in our paper with Bernhard Leeb [6], we gave an earlier form of our Combination Theorem for Anosov subgroups; in that paper, using the *local-to-global principle* for *Morse quasigeodesics*, we proved a version of the Klein Combination Theorem for Anosov subgroups. Moreover, we conjectured a *sharper* form of the Combination Theorem in that paper, which was recently confirmed in [5]. The discussions in [5, 6] were limited only to the case of free products. The purpose of the present article is to deal with the case of *amalgams* (amalgamated free products and HNN extensions). Nevertheless, our work partly builds on our earlier work in [5]. The generalization to amalgamated free products and HNN extensions presented in this paper requires overcoming many challenges not present in the case of free products.

1.1. **Main results.** Let G be a real semisimple Lie group of noncompact type and with a finite center. We will assume some mild conditions on G (see Assumption 3.1). Let $X = G/K$ be the associated symmetric space, where K is a maximal compact subgroup of G. Let σ_{mod} be a model spherical chamber in the Tits building $\partial_{\mathrm{Tits}}X$ of X and let $\iota : \sigma_{\mathrm{mod}} \to \sigma_{\mathrm{mod}}$ be the opposition involution. We consider the class of τ_{mod}-*Anosov subgroups* of G, where τ_{mod} is an ι-invariant face of σ_{mod}. For a discrete subgroup Γ of G, the τ_{mod}-*limit set* of Γ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$, the partial flag manifold associated to the face τ_{mod}, is denoted by $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$. See §3 for the definitions.

Given discrete subgroups Γ_A, Γ_B of G, a pair of compact subsets $A, B \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ with nonempty interiors is called an *interactive pair* for the triple $(\Gamma_A, \Gamma_B; \mathrm{H})$, where $\mathrm{H} = \Gamma_A \cap \Gamma_B$, if the interiors A° and B° of A and B are disjoint, and H leaves the sets A and B *precisely invariant*. This means that for all $\eta \in \mathrm{H}$, $\eta A = A$ and $\eta B = B$, and for all $\alpha \in \Gamma_A \setminus \mathrm{H}$ and $\beta \in \Gamma_B \setminus \mathrm{H}$, we have $\alpha B \subset A^\circ$ and $\beta A \subset B^\circ$. We borrow this concept from Maskit's work. Cf. Figure 1 for an illustration.

The first main result of this paper, which provides an analog of the first Klein–Maskit combination theorem [33, Theorem VII.C.2], is as follows:

Theorem A (Amalgamated free products). *Let Γ_A and Γ_B be τ_{mod}-Anosov subgroups of G. Assume that $\mathrm{H} := \Gamma_A \cap \Gamma_B$ is quasiconvex (see Definition 2.5) in Γ_A or in Γ_B. Suppose that there exists an* interactive pair (A, B) *in* $\mathrm{Flag}(\tau_{\mathrm{mod}})$ *for* $(\Gamma_A, \Gamma_B; \mathrm{H})$ *such that the following conditions are satisfied:*

(i) *The interiors of A and B are* antipodal *to each other (see Definition 3.3).*

(ii) *The pairs of subsets $(A, \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_B)\backslash\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}))$ and $(B, \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A)\backslash\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}))$ of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ are antipodal to each other.*

Then, the subgroup $\Gamma = \langle \Gamma_A, \Gamma_B \rangle$ of G is τ_{mod}-Anosov and is naturally isomorphic to the abstract amalgamated free product $\Gamma_A \star_{\mathrm{H}} \Gamma_B$.
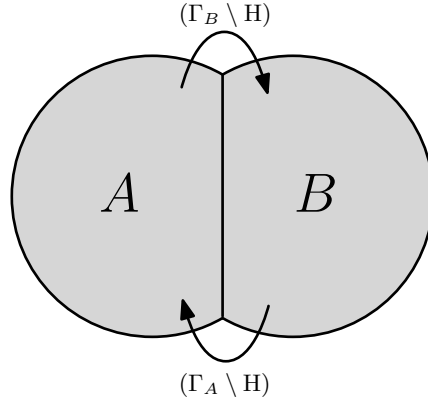

FIGURE 1. An interactive pair.

Given a discrete subgroup M of G and an element $f \in G$, a triple $(A, B_{\pm})$ of compact subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ with nonempty interiors is called an *interactive triple* for the quadruple $(\mathrm{M}; \mathrm{H}_{\pm}; f)$, where

$$\mathrm{H}_{+} := \mathrm{M} \cap (f\mathrm{M}f^{-1}) \quad \text{and} \quad \mathrm{H}_{-} := (f^{-1}\mathrm{M}f) \cap \mathrm{M}, \tag{1.1}$$

if the interiors A°, B_{-}°, and B_{+}° of A, B_{-}, and B_{+}, respectively, are nonempty and pairwise disjoint, $B_{-} \cap B_{+} = \emptyset$, $f^{\pm 1}(A) \subset B_{\pm}$, $f^{\pm 1}(B_{\pm}) \subset B_{\pm}^{\circ}$, and $\mathrm{H}_{\pm}$ leaves $B_{\pm}$ *precisely invariant*. This means $\mathrm{H}_{\pm}B_{\pm} = B_{\pm}$ and $\mu(B_{\pm}) \subset A^{\circ}$ for all $\mu \in \mathrm{M} \setminus \mathrm{H}_{\pm}$. Cf. Figure 2.

Our second main result, stated below, gives an analog of the second Klein–Maskit combination theorem [33, Theorem VII.E.5].

Theorem B (HNN extensions). *Let M be a τ_{mod}-Anosov subgroup of G, let $f \in G$, and let $\mathrm{H}_{\pm}$ be as in (1.1). Assume that H_{+} or H_{-} is quasiconvex in M. Suppose that there exists an interactive triple $(A, B_{\pm})$ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$ for $(\mathrm{M}; \mathrm{H}_{\pm}; f)$ such that the following conditions are satisfied:*

(i) *The pairs of subsets $(A^{\circ}, B_{\pm}^{\circ})$, (B_{+}, B_{-}) of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ are antipodal to each other.*

(ii) *$\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{M}) \setminus \Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}_{\pm})$ is antipodal to $B_{\pm}$.*

Then, the subgroup $\Gamma = \langle \mathrm{M}, f \rangle$ of G is τ_{mod}-Anosov and is naturally isomorphic to the abstract HNN extension $\mathrm{M}\star_{\phi}$ of M by ϕ, where the isomorphism $\phi : \mathrm{H}_{-} \to \mathrm{H}_{+}$ is given by $\phi(\eta) = f\eta f^{-1}$, for all $\eta \in \mathrm{H}_{-}$.
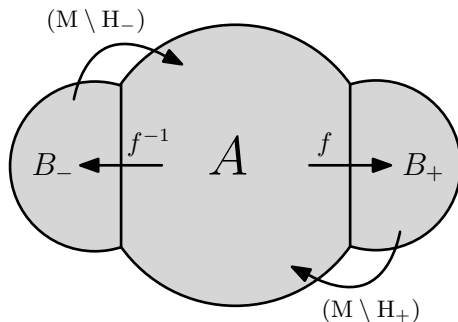
FIGURE 2. An interactive triple.

See §4, resp. §5, for the proof of Theorem A, resp. Theorem B.

Remark 1.1. In the statement of Theorem A, one may relax the second condition (ii) if it is known that $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) \cap \partial A = \Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}) = \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_B) \cap \partial B$. Similarly, in the statement of Theorem B, one may relax the second condition (ii) if it is known that $\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{M}) \cap \partial A = \Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}_+) \cup \Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}_-)$. See Remarks 4.4 and 5.3.

Remark 1.2. The "converse" of Theorem B (and Theorem A) does not hold in general: More precisely, let M be a τ_{mod}-Anosov subgroup of G and $f \in G$. Suppose $\mathrm{H}_+ := \mathrm{M} \cap (f\mathrm{M}f^{-1})$ or $\mathrm{H}_- := (f^{-1}\mathrm{M}f) \cap \mathrm{M}$ is quasiconvex in M. If the subgroup $\Gamma = \langle \mathrm{M}, f \rangle$ of G is τ_{mod}-Anosov and is naturally isomorphic to $\mathrm{M}*_\phi$, where $\phi : \mathrm{H}_- \to \mathrm{H}_+$ is an isomorphism given by $\phi(\eta) = f\eta f^{-1}$, $\eta \in \mathrm{H}_-$, it does not necessarily imply the existence of an interactive triple $(A, B_\pm)$ as stated in Theorem B. Here we discuss an example of such a failure when $G = \mathrm{PSL}(2, \mathbf{C})$.

Example 1.3. Start with a compact hyperbolic 3-manifold M with a single totally geodesic boundary component of genus 2 and let M_1, M_2 be two copies of M. Pick essential separating annuli $A_i \subset \partial M_i$ for $i = 1, 2$, and glue M_1, M_2 along A_1, A_2. The result is a compact 3-manifold N with two homeomorphic boundary components S_1, S_2, both of genus 2. The manifold N has an essential properly embedded annulus A (corresponding to A_1, A_2) with the boundary circles $C_i \subset S_i$, $i = 1, 2$. The manifold N is irreducible and contains no essential annuli other than A. By Thurston's Hyperbolization Theorem (see [16], [39] or [38, Theorem A']), N admits a convex-cocompact hyperbolic structure. Glue the two boundary surfaces of N by a homeomorphism $\phi : S_1 \to S_2$ such that $\phi(C_1)$ is not homotopic to C_2 in S_2. Then, the resulting closed manifold K is irreducible, Haken, and atoroidal, and hence admits a hyperbolic structure, again by Thurston's theorem. The surfaces S_1, S_2 are incompressible and $\pi_1(S_1), \pi_1(S_2)$ are quasiconvex in $\pi_1(K)$.

Although $\pi_1(K) = \pi_1(N)\star_{\phi_*}$ is a cocompact (hence Anosov) subgroup of $G = \mathrm{PSL}(2, \mathbf{C})$, the quadruple $(\pi_1(N); \pi_1(S_1), \pi_1(S_2); f)$, where $f \in G$ corresponds to ϕ, does not admit an interactive triple in $\partial_\infty \mathbb{H}^3$: To see this, note that $C_1 \subset S_1$ is homotopic $C_2 \subset S_2$ in N, and so $\mathrm{H}_+ = \pi_1(S_1)$ and $\mathrm{H}_- = \pi_1(S_2)$ violate Corollary 5.4, a conclusion arising from the existence of an interactive triple.

Before concluding the introduction, let us illustrate the hypotheses and conclusions of Theorem A and Theorem B in the following examples:

1.2. Illustrative examples.

Let S be a closed, orientable surface of genus $g \geq 2$ and let $\Gamma := \pi_1(S)$. Labourie's [23] pioneering work showed that *Hitchin representations* $\rho : \Gamma \to \mathrm{PSL}(d, \mathbf{R})$ form a rich class of σ_{mod}-Anosov representations. Let us denote by $\xi_\rho : \partial_\infty \Gamma \to \mathrm{Flag}(\sigma_{\mathrm{mod}})$ the associated Γ-equivariant boundary map onto the limit set $\Lambda_{\rho(\Gamma)} \subset \mathrm{Flag}(\sigma_{\mathrm{mod}})$. According to Fock-Goncharov [9], an important characteristic of the Hitchin representations is a certain *positivity property* of the limit map ξ_ρ; cf. §6.

Given an essential *separating* simple closed curve $s \subset S$, let $[\eta]$ denote the conjugacy class in Γ representing s. The curve s cuts the surface into two subsurfaces, S_A and S_B. The group Γ can be written as an amalgamated free product

$$\Gamma = \Gamma_A \star_{\mathrm{H}} \Gamma_B,$$

where $\Gamma_A = \pi_1(S_A)$, $\Gamma_B = \pi_1(S_B)$, $\mathrm{H} = \langle \eta \rangle$, for some $\eta \in [\eta]$, equipped with natural monomorphisms $\phi_A : \mathrm{H} \to \Gamma_A$ and $\phi_B : \mathrm{H} \to \Gamma_B$ induced by the inclusion homeomorphisms $s \hookrightarrow \partial S_A$ and $s \hookrightarrow \partial S_B$. The image of $\mathrm{H} = \langle \eta \rangle$ under a Hitchin representation $\rho : \Gamma \to \mathrm{PSL}(d, \mathbf{R})$ is a cyclic σ_{mod}-Anosov subgroup of $\mathrm{PSL}(d, \mathbf{R})$. Let $\sigma_\pm \in \Lambda_{\rho(\Gamma)} = \xi_\rho(\partial_\infty \Gamma)$ denote the attractive/repulsive fixed points of $\rho(\eta)$ in $\mathrm{Flag}(\sigma_{\mathrm{mod}})$. The 2-point set $\{\sigma_+, \sigma_-\}$ cuts $\Lambda_{\rho(\Gamma)}$ in a pair of closed arcs c_A and c_B; we chose the names in such a way that $\Lambda_{\rho(\Gamma_A)} \subset c_A$ and $\Lambda_{\rho(\Gamma_B)} \subset c_B$. See the left side of Figure 3.

Let Ω be the set of all points in $\mathrm{Flag}(\sigma_{\mathrm{mod}})$ antipodal to both $\sigma_\pm$; this is the intersection of a pair of opposite maximal Schubert cells in $\mathrm{Flag}(\sigma_{\mathrm{mod}})$:

$$\Omega = C(\sigma_+) \cap C(\sigma_-).$$

There are two distinguished connected components of Ω, denoted by A° and B°, whose closures A and B, respectively, contain c_A and c_B. Both A and B are preserved by H because H preserves c_A and c_B. Using the positivity property of ξ and the fact that for all $\alpha \in \rho(\Gamma_A \setminus \mathrm{H})$ and $\beta \in \rho(\Gamma_B \setminus \mathrm{H})$, $\alpha c_B \subset c_A \setminus \{\sigma_\pm\}$ and $\beta c_A \subset c_B \setminus \{\sigma_\pm\}$, it follows that

$$\alpha B \subset A^\circ \quad \text{and} \quad \beta A \subset B^\circ,$$

see [14, Corollary 3.9]. Moreover, the interiors A° and B° are antipodal to each other, see [14, Proposition 2.5(3)]. Therefore, (A, B) is an interactive pair for $(\rho(\Gamma_A), \rho(\Gamma_B); \rho(\mathrm{H}))$, which satisfies the hypothesis of Theorem A.

Let us describe a continuous family of Hitchin representations $\rho_t : \Gamma \to \mathrm{PSL}(d, \mathbf{R})$ obtained by *bending* the given Hitchin representation ρ along s, a type of deformation that generalizes the classical Dehn twists along simple closed geodesics in a hyperbolic surface. This family is parametrized by $t \in Z_0(\eta)$, where $Z_0(\eta) \cong \mathbf{R}^{d-1}$ is the identity component of the centralizer of η in $\mathrm{PSL}(d, \mathbf{R})$, and the σ_{mod}-Anosov property of this family can be verified by a simple application of Theorem A: Each connected component of Ω is also preserved by $Z_0(\eta)$. In particular, $Z_0(\eta)$ preserves A and B. For $t \in Z^0(\eta)$, let

$$\rho_t : \Gamma_B \to \mathrm{PSL}(d, \mathbf{R}), \quad \rho_t(\beta) = t\rho(\beta)t^{-1}.$$

Clearly, $\rho_t|_{\mathrm{H}} = \rho|_{\mathrm{H}}$. Moreover, we observe that (A, B) is still an interactive pair for the triple $(\rho(\Gamma_A), \rho_t(\Gamma_B); \rho(\mathrm{H}))$. Therefore, Theorem A directly verifies that

$$\rho_t : \Gamma = \Gamma_A \star_{\mathrm{H}} \Gamma_B \to \langle \rho(\Gamma_A), t\rho(\Gamma_B)t^{-1} \rangle < \mathrm{PSL}(d, \mathbf{R}),$$

is injective, and its image is a σ_{mod}-Anosov subgroup of G.

Similarly, if s is an essential *non-separating* simple closed curve in S, then the fundamental group $\Gamma = \pi_1(S)$ can be written as an HNN extension

$$\Gamma = \mathrm{M} \star_\phi, \quad \phi : \mathrm{H}_- \xrightarrow{\cong} \mathrm{H}_+,$$

where M is the fundamental group of the surface S' obtained by cutting S along s, $\mathrm{H}_- = \langle \eta \rangle$ and $\mathrm{H}_+ = \langle \eta' \rangle$ are the images of the homomorphisms of the fundamental groups induced by the two inclusion homeomorphisms $s \hookrightarrow \partial_- S'$ and $s \hookrightarrow \partial_+ S'$, and $\phi : \mathrm{H}_- \to \mathrm{H}_+$ is the isomorphism induced by the conjugation by some element $f \in \Gamma$.

Let $\rho : \Gamma \to \mathrm{PSL}(d, \mathbf{R})$ be a Hitchin representation with associated Γ-equivariant limit map $\xi_\rho : \partial_\infty \Gamma \to \mathrm{Flag}(\sigma_{\mathrm{mod}})$. The attractive/repulsive fixed point sets $\{\sigma_+, \sigma_-\}$ and $\{\sigma'_+, \sigma'_-\}$ of η and η', respectively, cut the limit set $\Lambda_{\rho(\Gamma)} = \xi_\rho(\partial_\infty \Gamma)$ in four closed arcs c_{A_+}, c_{A_-}, c_{B_+}, and c_{B_-} (see the right side of Figure 3). We have $\rho(f)\{\sigma_\pm\} = \{\sigma'_\pm\}$, $\rho(f)(c_{A_+} \cup c_{A_-} \cup c_{B_+}) \subset c_{B_+}$, and $\rho(f)^{-1}(c_{A_+} \cup c_{A_-} \cup c_{B_-}) \subset c_{B_-}$.
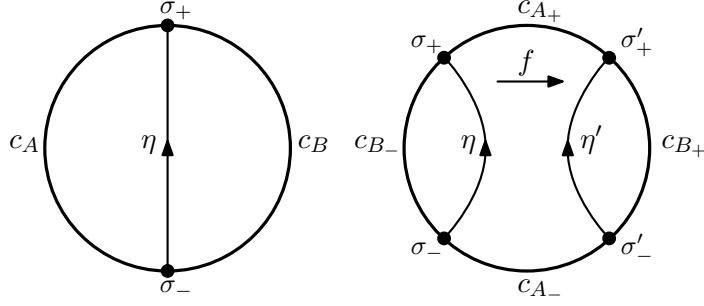

FIGURE 3

Define $B_-, B_+ \subset \mathrm{Flag}(\sigma_{\mathrm{mod}})$ to be the closure of connected component of $C(\sigma_+) \cap C(\sigma_-)$ containing the arc c_{B_-} and c_{B_+}, respectively. Lastly, define A to be the union of A_+ and A_-, where $A_\pm$ are the closures of the connected component of $C(\sigma_\pm) \cap C(\sigma_\pm)$ containing the arcs $c_{A_\pm}$, respectively. Then, using the positivity of ξ_ρ, one can show that $(A, B_\pm)$ is an interactive triple for $(\rho(\mathrm{M}); \rho(\mathrm{H}_\pm); \rho(f))$ (compare with the amalgamated free product case discussed above), thus verifying the hypothesis of Theorem B.

As before, we obtain a continuous family of Hitchin representations $\rho_t : \Gamma \to \mathrm{PSL}(d, \mathbf{R})$, parametrized by $t \in Z_0(\eta)$, by bending ρ along s: Observe that for any $t \in Z_0(\eta)$, $(A, B_\pm)$ is an interactive triple for $(\rho(\mathrm{M}); \rho(\mathrm{H}_\pm); \rho(f) \cdot t)$. Therefore, Theorem B directly verifies that

$$\rho_t : \Gamma = \mathrm{M} \star_\phi \to \langle \rho(\mathrm{M}), \rho(f) \cdot t \rangle < \mathrm{PSL}(d, \mathbf{R})$$

is injective with a σ_{mod}-Anosov image.

Organization of the paper. In §2, we present some preliminary material and results on amalgamated free products and HNN extensions of hyperbolic groups. This section presents some results (e.g., Lemma 2.9, Propositions 2.13 and 2.14) crucial in the proof of our main theorems. Then, in §3 we review some background material on discrete groups acting on symmetric spaces of noncompact type and present some lemmas (see §3.1), which are also frequently used in the proof of our main results to verify regularity and flag-convergence of certain sequences. In §4, resp. §5, we prove Theorem A, resp. Theorem B. Finally, in §6, using

the Combination Theorems, we give a proof of a recent result of Guichard-Labourie-Wienhard [14] stating that Θ-positive representations are Θ-Anosov.

2. Preliminaries on amalgams of hyperbolic groups

2.1. **Amalgamated free products.** Let Γ_A, Γ_B, and H be abstract groups together with monomorphisms $\phi_A : H \to \Gamma_A$, $\phi_B : H \to \Gamma_B$. The *free product of Γ_A and Γ_B amalgamated along* H, denoted by $\Gamma_A \star_H \Gamma_B$, has a presentation

$$\langle S_A, S_B \mid R_A, R_B, \phi_A(\eta)\phi_B(\eta)^{-1}, \eta \in H\rangle,$$

where $\langle S_A \mid R_A\rangle$ and $\langle S_B \mid R_B\rangle$ are presentations of Γ_A and Γ_B, respectively. We will identify H with $\phi_A(H) < \Gamma_A$ and $\phi_B(H) < \Gamma_B$ via the monomorphisms ϕ_A and ϕ_B, respectively.

Let $\Gamma = \Gamma_A \star_H \Gamma_B$. A *normal form* of an element $\gamma \in \Gamma$ is an expression

$$\gamma = \gamma_1\gamma_2\cdots\gamma_l, \tag{2.1}$$

such that the following conditions are satisfied:

(i) Each γ_i lies either in Γ_A, or in Γ_B. Moreover, if $l \geq 2$, then none of the letters γ_i belong to H.
(ii) No two successive letters γ_i, γ_{i+1} above lie in the same group Γ_A or Γ_B.

Unless H is trivial, the normal form of γ given by (2.1) is not necessarily unique. However, any other normal form of γ is obtained by a sequence of finite moves consisting of replacing a consecutive pair $\gamma_i\gamma_{i+1}$ in (2.1) by $(\gamma_i\eta_i)(\eta_i^{-1}\gamma_{i+1})$, where $\eta_i \in H$. This is a consequence of the Normal Form Theorem; see [26, Theorem IV.2.6]. It follows that any two normal forms of γ have the same number of letters. For $\gamma \in \Gamma \setminus H$, the *relative length* of γ, denoted by $\mathrm{rl}(\gamma)$, is the number of letters in a(ny) normal form of γ. If $\gamma \in H$, then define $\mathrm{rl}(\gamma) = 0$.

Definition 2.1 (Alternating sequences). A sequence (ω_k) in Γ is called *type A alternating* if there exists a pair of sequences, (α_n) in $\Gamma_A \setminus H$ and (β_n) in $\Gamma_B \setminus H$, such that

$$\omega_k = \begin{cases} \alpha_1\beta_1\cdots\beta_{m-1}\alpha_m, & k = 2m-1, \\ \alpha_1\beta_1\cdots\alpha_m\beta_m, & k = 2m. \end{cases} \tag{2.2}$$

Similarly, a sequence (ω_k) in Γ is called *type B alternating* if there exists a pair of sequences, (α_n) in $\Gamma_A \setminus H$ and (β_n) in $\Gamma_B \setminus H$, such that

$$\omega_k = \begin{cases} \beta_1\alpha_1\cdots\alpha_{m-1}\beta_m, & k = 2m-1, \\ \beta_1\alpha_1\cdots\beta_m\alpha_m, & k = 2m. \end{cases}$$

2.2. **HNN extensions.** Let M be an abstract group and $H_\pm < M$ be a pair of subgroups. Suppose that $\phi : H_- \to H_+$ is an isomorphism. The *HNN extension of* M *by* ϕ, denoted by $M\star_\phi$, has a presentation

$$\langle S_M, f \mid R_M, f\eta f^{-1}(\phi(\eta))^{-1}, \eta \in H_- \rangle,$$

where $\langle S_M \mid R_M \rangle$ is a presentation of M.

Every element $\gamma \in \Gamma = M\star_\phi$ can be written in the form

$$\gamma = \mu_0 f^{\epsilon_1}\mu_1 \cdots f^{\epsilon_n}\mu_n, \tag{2.3}$$

for some $n \geq 0$, where each μ_i is in M, and each ϵ_i is either 1 or -1, such that the following conditions are satisfied:

(i) For $i = 1, \ldots, n$, if $\epsilon_i = -1$ and $\mu_i \in H_+$, then $\epsilon_{i+1} = -1$.
(ii) For $i = 1, \ldots, n$, if $\epsilon_i = 1$ and $\mu_i \in H_-$, then $\epsilon_{i+1} = 1$.

Such an expression (2.3) is called a *normal form* of γ.

Although the normal form of an element is not unique, *Britton's Lemma* (see [26, p.181]) establishes certain uniqueness properties of the decomposition (2.3) of γ. In particular, the *relative length* $\mathrm{rl}(\gamma)$ of γ, i.e., the total number of letters f and f^{-1} appearing in (2.3), is unique.

Definition 2.2 (Alternating sequences)**.** A sequence (ω_n) in $M\star_\phi$ is called *alternating* if there exist sequences (μ_n) in M and (ϵ_n) in $\{\pm1\}$, satisfying

$$\epsilon_n = -1, \mu_n \in H_+ \implies \epsilon_{n+1} = -1, \quad \text{and} \quad \epsilon_n = 1, \mu_n \in H_- \implies \epsilon_{n+1} = 1 \tag{2.4}$$

for all $n \in \mathbf{N}$, and some element $\mu_0 \in M$ such that

$$\omega_n = \mu_0 f^{\epsilon_1}\mu_1 f^{\epsilon_2}\mu_2 \cdots f^{\epsilon_{n-1}}\mu_{n-1}f^{\epsilon_n}. \tag{2.5}$$

Remark 2.3. The condition (2.4) simply implies that the word in (2.5) is a normal form. It is also useful to think about an alternating sequence (ω_n) in $M\star_\phi$ as an infinite string of letters:

$$\mu_0, f^{\epsilon_1}, \mu_1, f^{\epsilon_2}, \mu_2, f^{\epsilon_3}, \mu_3 \ldots,$$

where μ_i and ϵ_i satisfy the condition (2.4).

2.3. **Bass-Serre trees.** In this paper, we will be using the formalism of *Bass-Serre trees* T associated with amalgamated free products and HNN extensions. A detailed treatment of this material can be found in Serre's book [40]; see also [7] for a more topological viewpoint on the construction.

(i) Consider an amalgamated free product $\Gamma = \Gamma_A \star_H \Gamma_B$. The vertex set $V(T)$ of the tree T is the set of cosets $\gamma\Gamma_A, \gamma\Gamma_B, \gamma \in \Gamma$. Accordingly, the vertex set $V(T)$ is bicolored, with one color corresponding to the cosets $\gamma\Gamma_A$ and the other color corresponding to the cosets $\gamma\Gamma_B$. The group Γ acts on $V(T)$ by the left multiplication.

Edges of T are defined so that T is a bipartite graph, i.e., vertices of the same color are never connected by an edge. The cosets $\gamma_1\Gamma_A, \gamma_2\Gamma_B$ are connected by an edge whenever there exists $\alpha \in \Gamma_A$ such that

$$\gamma_1\Gamma_B = \gamma_2\alpha\Gamma_B,$$

equivalently, there exists $\beta \in \Gamma_B$ such that

$$\gamma_1 \beta \Gamma_A = \gamma_2 \Gamma_A.$$

For instance, the vertices represented by the cosets Γ_A, Γ_B are connected by an edge $e \in E(T)$ since there exists an element $\eta \in H < \Gamma_A$ such that $\Gamma_B = \eta \Gamma_B$. Moreover, all elements $\alpha \in \Gamma_A$ such that $\alpha \Gamma_B = \Gamma_B$ necessarily belong to H. We, thus, label the edge e by the coset $1_\Gamma \cdot H = H$.

The left multiplication by the elements of Γ preserves the incidence relation between the vertices of T. Accordingly, the edges of T are labeled by the cosets γH, $\gamma \in G$. The Γ-stabilizer of the vertex $\gamma \Gamma_A$ equals $\gamma \Gamma_A \gamma^{-1}$, while the Γ-stabilizer of the vertex $\gamma \Gamma_B$ equals $\gamma \Gamma_B \gamma^{-1}$ and the Γ-stabilizer of an edge labeled γH equals $\gamma H \gamma^{-1}$.

(ii) Consider an HNN extension $\Gamma = M \star_{\phi: H_- \to H_+}$. The vertex set of T consists of left cosets of M in Γ. The edge set of T consists of edges of two types: The left $H_\pm$-cosets in Γ. The edge γH_+ connects γM to $\gamma f M$, while the edge γH_- connects γM and $\gamma f^{-1} M$. The Γ-action on T is defined by: $\gamma : \gamma' M \mapsto \gamma \gamma' M$. The Γ-action is transitive on the set of vertices and edges of T.

2.4. **Hyperbolic groups.** Let Γ be a finitely-generated group equipped with a left-invariant word metric, denoted by d_Γ. We use the notation $|\cdot|$ to denote the word length of elements of Γ, i.e., $|\gamma| = d_\Gamma(1_\Gamma, \gamma)$. Recall that Γ is called *(word) hyperbolic* if there exists $\delta \geq 0$ such that (Γ, d_Γ) is a δ-*hyperbolic* metric space: That is, for all $f, g, h, w \in \Gamma$,

$$(f, g)_w \geq \min\{(f, h)_w, (g, h)_w\} - \delta,$$

where $(f, g)_w$ denotes the *Gromov product* of f and g with respect to w:

$$(f, g)_w = \frac{1}{2}(d_\Gamma(f, w) + d_\Gamma(g, w) - d_\Gamma(f, g)).$$

It is a basic fact that the property of being hyperbolic does not depend on the choice of a word metric d_Γ, but the constant δ possibly depends on the chosen metric d_Γ.

Let Γ be a hyperbolic group equipped with a word metric d_Γ. A *(discrete) geodesic* in Γ is an isometric embedding $c : I \to \Gamma$ of an interval $I \subset \mathbf{Z}$. Such a geodesic $c : I \to \Gamma$ is called a *segment*, *ray*, or *line* when I is bounded, I is only bounded below, or $I = \mathbf{Z}$, respectively.

The *Gromov boundary* of Γ, denoted by $\partial_\infty \Gamma$, is the set of equivalence classes of asymptotic geodesic rays in Γ, which gives a natural compactification of (Γ, d_Γ),

$$\overline{\Gamma} = \Gamma \sqcup \partial_\infty \Gamma.$$

The topology of $\overline{\Gamma}$ is understood as follows: A pair of sequences (γ_n) and (γ'_n) in Γ are said to *fellow travel* if

$$(\gamma_n, \gamma'_n)_{1_\Gamma} \to \infty, \quad \text{as } n \to \infty.$$

A sequence (γ_n) in Γ *converges* to a point $\varepsilon \in \partial_\infty \Gamma$, which is denoted by

$$\gamma_n \xrightarrow{\text{Cay}} \varepsilon,$$

if (γ_n) fellow-travels the image sequence $(c(n))$ of a(ny) geodesic ray $c : \mathbf{N} \to \Gamma$ asymptotic to ε.

The following result can be checked easily using the definitions above:

Lemma 2.4. *Fellow traveling sequences in Γ have the same accumulation set in $\partial_\infty \Gamma$.*

2.5. **Nearest point projections to quasiconvex subgroups.** Let Γ be a hyperbolic group equipped with a left-invariant word metric d_Γ.

Definition 2.5 (Quasiconvexity).

(1) A subset $Y \subset \Gamma$ is called *quasiconvex* if there exists $K \geq 0$ such that for all $y_1, y_2 \in Y$, any geodesic segment in Γ connecting y_1 and y_2 lies in the K-neighborhood of Y.

(2) A subgroup H $< \Gamma$ is called a *quasiconvex* if H, as a subset of Γ, is quasiconvex.

For a quasiconvex subset $Y \subset \Gamma$, we choose a nearest point projection map

$$\mathrm{pr}_Y : \Gamma \to Y.$$

Note that nearest point projections are not necessarily unique, but since Y is quasiconvex, any two such maps are a finite distance apart.

Lemma 2.6. *Let $Y \subset \Gamma$ be a quasiconvex subset. For every $\gamma \in \Gamma$, $\mathrm{pr}_Y(\gamma) \in Y$ lies in a uniform neighborhood of any geodesic segment in Γ connecting γ to Y.*

Proof. Let $K \geq 0$ be a quasiconvexity constant for Y and $\delta \geq 0$ be a hyperbolicity constant for (Γ, d_Γ). Let $[y_0, \gamma]$ be a geodesic segment in Γ from $y_0 \in Y$ to γ.
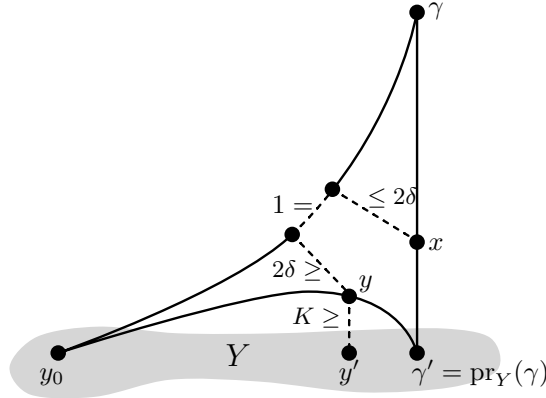

FIGURE 4

Consider a geodesic triangle in Γ with vertices y_0, γ, and $\gamma' := \mathrm{pr}_Y(\gamma)$, whose edge connecting y_0 to γ is $[y_0, \gamma]$. Since geodesic triangles in Γ are 2δ-thin, we have

$$[y_0, \gamma] \subset N_{2\delta}([y_0, \gamma'] \cup [\gamma, \gamma']),$$

where $N_r(\cdot)$ denotes the r-neighborhood in (Γ, d_Γ). In particular, there exist points $x \in [\gamma, \gamma']$ and $y \in [y_0, \gamma']$ such that $d_\Gamma(x, [y_0, \gamma]) \leq 2\delta$, $d_\Gamma(y, [y_0, \gamma]) \leq 2\delta$, and $d_\Gamma(x, y) \leq 4\delta + 1$. See Figure 4 for an illustration of the points in the Cayley graph of Γ. Since $[y_0, \gamma']$ lies in the K-neighborhood of Y, there exists $y' \in Y$, which is at most K distance away from y.

Since γ' is also a nearest point in Y to x, $d_\Gamma(x, \gamma') \leq d_\Gamma(x, y') \leq K + 4\delta + 1$. Therefore, γ' is at most $K + 6\delta + 1$ distance away from $[y_0, \gamma]$. $\qquad\square$

For a subset $Y \subset \Gamma$, let $\partial_\infty Y \subset \partial_\infty \Gamma$ denote the set of all accumulation points of Y in $\overline{\Gamma} = \Gamma \sqcup \partial_\infty \Gamma$.

Corollary 2.7. *Let $Y \subset \Gamma$ be a quasiconvex subset. A sequence (γ_n) in Γ has an accumulation point in $\partial_\infty Y$ if and only if $(\mathrm{pr}_Y(\gamma_n))$ is unbounded in Y.*

Proof. For the "if" part, suppose that $(\mathrm{pr}_Y(\gamma_n))$ is unbounded. After passing to a subsequence, we assume that $|\mathrm{pr}_Y(\gamma_n)| \to \infty$, as $n \to \infty$. Applying Lemma 2.6, it follows that (γ_n) and $(\mathrm{pr}_Y(\gamma_n))$ fellow travel. Thus, by Lemma 2.4, (γ_n) has an accumulation point in $\partial_\infty Y$.

For the "only if" part, we prove the contrapositive statement. Let us assume that $(\mathrm{pr}_Y(\gamma_n))$ is bounded. We show that (γ_n) has no accumulation points in $\partial_\infty Y$. We argue by contradiction: Suppose that $\varepsilon \in \partial_\infty Y$ is an accumulation point of (γ_n). After extraction, we may assume that $\gamma_n \xrightarrow{\mathrm{Cay}} \varepsilon$. Let (y_n) be any sequence in Y such that $y_n \xrightarrow{\mathrm{Cay}} \varepsilon$. Consequently, we must have $(\gamma_n, y_n)_{1_\Gamma} \to \infty$, as $n \to \infty$. However, Lemma 2.6 can be restated as

$$\sup_{\gamma \in \Gamma, y \in Y} (\gamma, y)_{\mathrm{pr}_Y(\gamma)} < \infty.$$

Since $(\mathrm{pr}_Y(\gamma_n))$ is bounded, the above implies

$$\sup_{m,n} (\gamma_n, y_m)_{1_\Gamma} < \infty,$$

a contradiction. $\qquad\square$

Lemma 2.8. *Let $\mathrm{H} < \Gamma$ be a quasiconvex subgroup. For any sequence (γ_n) in Γ, consider the sequence $(\hat{\gamma}_n)$ given by $\hat{\gamma}_n = \mathrm{pr}_\mathrm{H}(\gamma_n)^{-1}\gamma_n$.*

(i) Regarded as a sequence in the compact space $\overline{\Gamma} = \Gamma \sqcup \partial_\infty \Gamma$, $(\hat{\gamma}_n)$ has no accumulation points in $\partial_\infty \mathrm{H}$.

(ii) Suppose that (γ_n) diverges away from H, i.e., $d_\Gamma(\mathrm{H}, \gamma_n) \to \infty$, as $n \to \infty$. Then, the accumulation set of (γ_n^{-1}) in $\partial_\infty \Gamma$ is the same as that of $(\hat{\gamma}_n^{-1})$.

Proof. We first observe that for all $\gamma \in \Gamma$, the identity element is a nearest point in H to $\hat{\gamma} := \mathrm{pr}_\mathrm{H}(\gamma)^{-1}\gamma$: Indeed, for all $\eta \in \mathrm{H}$,

$$d_\Gamma(\eta, \hat{\gamma}) = d_\Gamma(\mathrm{pr}_\mathrm{H}(\gamma)\eta, \gamma) \geq d_\Gamma(\mathrm{pr}_\mathrm{H}(\gamma), \gamma), \tag{2.6}$$

where the inequality follows from the fact that $\mathrm{pr}_\mathrm{H}(\gamma)$ is a nearest point in H to γ. Moreover, $d_\Gamma(1_\Gamma, \hat{\gamma}) = d_\Gamma(\mathrm{pr}_\mathrm{H}(\gamma), \gamma)$. Hence, (2.6) implies that $d_\Gamma(\mathrm{H}, \hat{\gamma}) \geq d_\Gamma(1_\Gamma, \hat{\gamma})$.

Therefore, for any sequence (γ_n) in Γ, $\{\mathrm{pr}_\mathrm{H}(\hat{\gamma}_n) \mid n \in \mathbf{N}\} \subset \mathrm{H}$ is bounded. Part (i) now follows by applying Corollary 2.7.

We prove part (ii): Since $\mathrm{pr}_\mathrm{H}(\gamma_n)$ lies uniformly close to any geodesic in Γ connecting 1_Γ to γ_n (see Lemma 2.6) and $d_\Gamma(\mathrm{H}, \gamma_n) \to \infty$, it follows that

$$|\gamma_n| - |\mathrm{pr}_\mathrm{H}(\gamma_n)| \to \infty,$$

as $n \to \infty$. Thus

$$\begin{aligned}
(\gamma_n^{-1}, \hat{\gamma}_n^{-1})_{1_\Gamma} &= \frac{1}{2}(|\gamma_n^{-1}| + |\hat{\gamma}_n^{-1}| - d_\Gamma(\gamma_n^{-1}, \hat{\gamma}_n^{-1})) \\
&= \frac{1}{2}(|\gamma_n| + |\hat{\gamma}_n| - |\gamma_n \hat{\gamma}_n^{-1}|) \\
&= \frac{1}{2}(|\gamma_n| + |\hat{\gamma}_n| - |\mathrm{pr}_\mathrm{H}(\gamma_n)|) \to \infty,
\end{aligned}$$

as $n \to \infty$. In other words, (γ_n^{-1}) and $(\hat{\gamma}_n^{-1})$ fellow travel. By Lemma 2.4, (γ_n^{-1}) and $(\hat{\gamma}_n^{-1})$ have the same accumulation sets in $\partial_\infty \Gamma$. $\qquad\square$

An equivalent statement of the above result, which we will often use, is as follows:

Lemma 2.9. *Let* $H < \Gamma$ *be a quasiconvex subgroup. For any sequence* (γ_n) *in* Γ*, consider the sequence* $(\hat{\gamma}_n)$ *given by* $\hat{\gamma}_n = \gamma_n \operatorname{pr}_H(\gamma_n^{-1})$.

(i) $(\hat{\gamma}_n^{-1})$ *has no accumulation points in* $\partial_\infty H$.

(ii) *If* (γ_n^{-1}) *diverges away from* H*, then the accumulation sets of* (γ_n) *and* $(\hat{\gamma}_n)$ *in* $\partial_\infty \Gamma$ *coincide.*

2.6. **Amalgams of hyperbolic groups.** For the rest of this section, we restrict our discussion to amalgams (amalgamated free products and HNN extensions) of hyperbolic groups. The Bestvina-Feighn Combination Theorem, [3], provides some sufficient conditions for the hyperbolicity of amalgams. We review this theorem in the weakly malnormal case (although their actual result is much more general):

Definition 2.10. A subgroup H of a group Γ is said to be *weakly malnormal* if, for every $\gamma \in \Gamma \setminus H$, the subgroup $\gamma H \gamma^{-1} \cap H$ is finite.

Theorem 2.11 (Bestvina-Feighn, [3]). *Let* Γ_A*,* Γ_B*, and* M *be hyperbolic groups.*

(i) *If* $H < \Gamma_A$ *and* $H < \Gamma_B$ *are quasiconvex, weakly malnormal subgroups, then* $\Gamma_A \star_H \Gamma_B$ *is hyperbolic.*

(ii) *If* $H_\pm < M$ *are isomorphic (by* $\phi : H_- \to H_+$*), quasiconvex, weakly malnormal subgroups such that for all* $\mu \in M$*,* $H_- \cap \mu H_+ \mu^{-1}$ *is finite, then* $M\star_\phi$ *is hyperbolic.*

See also [21, Theorems 1 & 2].

This theorem has several addendums that we will use in what follows:

Theorem 2.12 (Mitra, [36]). *Under the assumptions, the subgroups* Γ_A *and* Γ_B *(resp.* M*) in Theorem 2.11 are quasiconvex in* $\Gamma_A \star_H \Gamma_B$ *(resp.* $M\star_\phi$*).*

2.7. **Boundary of an amalgam.** Our next goal is to describe the Gromov boundaries of the amalgamated free products $\Gamma = \Gamma_A \star_H \Gamma_B$ and HNN extensions $\Gamma = M\star_{\phi:H_- \to H_+}$ under certain extra assumptions. We will be assuming that the groups Γ_A, Γ_B, and M are hyperbolic, H is weakly malnormal and quasiconvex in Γ_A, Γ_B (in the amalgamated free product case) and, $H_\pm$ are weakly malnormal and quasiconvex in M and the intersection $H_- \cap \mu H_+ \mu^{-1}$ is finite, for all $\mu \in M$ (in the HNN extension case). Under these assumptions, Γ is hyperbolic (see Theorem 2.11). Moreover, in the case of HNN extensions, we let $f \in \Gamma$ denote the *stable letter*, the element corresponding to the subgroup isomorphism $\phi : H_- \to H_+$:

$$f \eta f^{-1} = \phi(\eta), \quad \eta \in H_-.$$

Our description of the boundary follows [20, 7.3], to which we refer the reader for details and proofs. We will describe the boundary (mainly) in the case of amalgamated free products since the HNN extension case is similar.

Let T denote the Bass-Serre tree associated with the amalgamated free product $\Gamma = \Gamma_A \star_H \Gamma_B$ (or the HNN extension); see §2.3. The group Γ acts on T with vertex-stabilizers (the

vertex-subgroups Γ_v) that are conjugates of Γ_A, Γ_B (or M in the HNN extension case) and edge-stabilizers (*edge-subgroups* Γ_e) which are conjugates of H.

Define a *tree of topological spaces* as follows: To each $v \in V(T), e \in E(T)$, we associate the Gromov boundary $\partial_\infty \Gamma_v$, $\partial_\infty \Gamma_e$. Whenever v is a vertex of an edge e, we have the inclusion homomorphism $\Gamma_e \to \Gamma_v$, which induces a topological embedding $f_{ev} : \partial_\infty \Gamma_e \to \partial_\infty \Gamma_v$. This data yields a tree of topological spaces. The tree gives rise to a topological space $\partial_I \Gamma$, the *topological realization* of the tree of topological spaces, by taking the push-out of the maps f_{ev}: The topological space $\partial_I \Gamma$ is a union of Gromov boundaries $\partial_\infty \Gamma_v$, $v \in V(T)$. More precisely, it is the quotient of the disjoint union of these boundaries by the equivalence relation defined as follows. For every edge $e = [v, w]$ of T, we have $\xi \in \partial_\infty \Gamma_v$ is equivalent to $\eta \in \partial_\infty \Gamma_w$ whenever there exists $\zeta \in \partial_\infty \Gamma_e$ such that $f_{ev}(\zeta) = \xi, f_{ew}(\zeta) = \eta$. The group Γ acts on $\partial_I \Gamma$ via the projection of the natural Γ-action on

$$\coprod_{v \in V(T)} \partial_\infty \Gamma_v.$$

In particular, $\partial_I \Gamma$ is either the union of the Γ-orbits of $\partial_\infty \Gamma_A \cup \partial_\infty \Gamma_B$ (in the amalgamated free product case) or $\partial_\infty M$ (in the HNN extension case).

The weak malnormality assumption on the amalgam implies that whenever $e \neq e'$ are distinct edges of T, $\partial_\infty \Gamma_e \cap \partial_\infty \Gamma_{e'} = \emptyset$ in $\partial_I \Gamma$. Accordingly, whenever the distance between vertices v, w is > 1,

$$\partial_\infty \Gamma_v \cap \partial_\infty \Gamma_w = \emptyset. \tag{2.7}$$

Moreover, the weak malnormality assumption also implies that all vertex-subgroups Γ_v and edge-subgroups Γ_e are quasiconvex in Γ. Hence, one obtains a natural Γ-equivariant inclusion map $\partial_I \Gamma \to \partial_\infty \Gamma$. It turns out that this map is injective and continuous. However, in general, this map need not be a topological embedding. In what follows, we will identify $\partial_I \Gamma$ with its image in $\partial_\infty \Gamma$.

The Gromov boundary $\partial_\infty \Gamma$ of Γ is the disjoint union Γ-invariant subsets

$$\partial_\infty \Gamma = \partial_I \Gamma \sqcup \partial_{II} \Gamma,$$

where $\partial_{II} \Gamma := \partial_\infty \Gamma \setminus \partial_I \Gamma$. Elements of $\partial_I \Gamma$, resp. $\partial_{II} \Gamma$, are called *type I*, resp. *type II*, (ideal) boundary points of Γ.

The second part of the boundary, $\partial_{II} \Gamma$, of $\partial_\infty \Gamma$ admits a Γ-equivariant continuous bijection to $\partial_\infty T$. (For instance, $\partial_\infty \langle f \rangle$ is the 2-point subset of $\partial_{II} \Gamma$ corresponding to the fixed-point set of f in $\partial_\infty T$.)

Proposition 2.13. *For every point $\varepsilon \in \partial_{II} \Gamma$, there exists an alternating sequence (ω_n) in Γ converging (in $\overline{\Gamma}$) to ε such that the following holds: If a sequence (γ_n) in Γ converges to ε, then there exists a function $F : \mathbf{N} \to \mathbf{N}$ diverging to infinity and $n_1 \in \mathbf{N}$ such that for all integers $n \geq n_1$, there exists a normal form of γ_n containing $\omega_{F(n)}$ as a left subword.*

We prove this result in §2.8.

Proposition 2.14. *Fix a word metric d_Γ on Γ. For all $\omega \in \Gamma$ satisfying $\mathrm{rl}(\omega) \geq 3$, there exists a constant $D = D(\omega) \geq 0$ such that the following holds: If $\gamma \in \Gamma$ is any element such that some normal form of γ contains some normal form of ω as a left subword, then*

(i) in the case $\Gamma = \Gamma_A \star_H \Gamma_B$, we have $d_\Gamma(\mathrm{pr}_{\Gamma_A}(\gamma), 1_\Gamma) \leq D$ and $d_\Gamma(\mathrm{pr}_{\Gamma_B}(\gamma), 1_\Gamma) \leq D$.

(ii) in the case $\Gamma = \mathrm{M} \star_\phi$, *we have* $d_\Gamma(\mathrm{pr}_\mathrm{M}(\gamma), 1_\Gamma) \leq D$.

Using this result, it can be shown that alternating sequences cannot have type I accumulation points in the boundary of Γ. See §2.9 for a proof of Proposition 2.14.

Notation 2.15. We set up some notation for the rest of this section: We let d_Γ denote an arbitrary word metric on Γ. Moreover, we reserve the notation d to denote a word metric on Γ induced by a finite symmetric generating set S of Γ, where, (i) in the case of $\Gamma = \Gamma_A \star_\mathrm{H} \Gamma_B$, S is the union of some chosen finite symmetric generating sets of Γ_A and Γ_B, and (ii) in the case of $\Gamma = \mathrm{M} \star_\phi$, S is the union of some chosen finite symmetric generating set of M and $\{f, f^{-1}\}$. Recall that the identity map $(\Gamma, d) \to (\Gamma, d_\Gamma)$ is a quasiisometry. Finally, we reserve the notation d_T to denote the distance function on the corresponding Bass-Serre tree T induced by declaring that all the edges of T are of unit length.

2.8. **Proof of Proposition 2.13.** The following result demonstrates the existence of an alternating sequence (ω_k) converging to $\varepsilon \in \partial_\mathrm{II}\Gamma$ in the statement of Proposition 2.13.

Lemma 2.16. *There exists $D_0 \geq 0$ such that for every $\varepsilon \in \partial_\mathrm{II}\Gamma$, there exists a D_0-alternating sequence (ω_n) in (Γ, d_Γ) converging to ε.*

In the statement above, "D_0-alternating" means that (ω_n) is alternating and lies within distance D_0 from any geodesic ray in (Γ, d_Γ) emanating from 1_Γ asymptotic to ε.

In the proof of Lemma 2.16, we work with the word-metric d; see Notation 2.15. The general case, i.e., when d_Γ is an arbitrary word metric, would then follow by applying the Morse lemma.

Proof of Lemma 2.16 in the case of amalgamated free products. Let us consider a uniform quasi-geodesic $c : \mathbf{N} \cup \{0\} \to (\Gamma, d)$ emanating from $c(0) = 1_\Gamma$ asymptotic to ε; such a ray is described by a sequence (s_i) in S such that for all $k \in \mathbf{N}$,

$$c(k) = s_1 \cdots s_k.$$

Note that the sequence $(c(k))$ cannot entirely lie in $\Gamma_A \cup \Gamma_B$ since, otherwise, the sequence $(c(k))$ would converge to a type I ideal boundary point. Let i_1 be the largest number such that $c(i_1)$ lies in $\Gamma_A \cup \Gamma_B$. For the same reason as above, the sequence $(c(i_1)^{-1}c(k))_{k \in \mathbf{N}}$ cannot lie in $\Gamma_A \cup \Gamma_B$. Thus, let $i_2 > i_1$ be the largest number such that $c(i_1)^{-1}c(i_2)$ lies in $\Gamma_A \cup \Gamma_B$. Similarly, let $i_3 > i_2$ be the largest number such that $c(i_2)^{-1}c(i_3)$ lies in $\Gamma_A \cup \Gamma_B$. Proceeding inductively, we find a sequence $(c(i_k)^{-1}c(i_{k+1}))_k$ which alternates between Γ_A and Γ_B; the elements of that sequence are the letters for our alternating sequence (ω_k):

$$\omega_k = c(i_1)[c(i_1)^{-1}c(i_2)] \cdots [c(i_{k-1})^{-1}c(i_k)] = c(i_k).$$

Since the sequence (ω_k) lies in the quasigeodesic ray c, it converges to ε. $\qquad \square$

Proof of Lemma 2.16 in the case of HNN extensions. For $\varepsilon \in \partial_\mathrm{II}\Gamma$, pick any uniform quasi-geodesic ray $c : \mathbf{N} \cup \{0\} \to \Gamma$ that emanates from $c(0) = 1_\Gamma$ and is asymptotic to ε. Such a ray is described by a sequence (s_i) in S such that for all $k \in \mathbf{N}$,

$$c(k) = s_1 \cdots s_k.$$

We find an infinite string

$$\hat{\mathrm{S}} : \quad \mu_0, f^{\epsilon_1}, \mu_1, f^{\epsilon_1}, \mu_2, \ldots, \tag{2.8}$$

which has the property that $\mu_i \in \mathrm{M}$, $\epsilon_i = \pm 1$, and, for every n,

$$\mathrm{rl}(\mu_0 f^{\epsilon_1} \mu_1 \cdots f^{\epsilon_{n-1}} \mu_{n-1} f^{\epsilon_n}) = n. \tag{2.9}$$

Let $i_0 \in \mathbf{N} \cup \{0\}$ be the largest number such that $c(i_0) \in \mathrm{M}$; set $\mu_0 = c(i_0)$. Let $i_1 \geq i_0$ be the largest number such that $c(i_0)^{-1} c(i_1) \in \{f, f^{-1}\}$; define ϵ_1 in the obvious way. Let $i_2 \geq i_1$ to be the largest number such that $c(i_1)^{-1} c(i_2) \in \mathrm{M}$; set $\mu_1 = c(i_1)^{-1} c(i_2)$. Let $i_3 \geq i_2$ be the largest number such that $c(i_2)^{-1} c(i_2) \in \{f, f^{-1}\}$; define ϵ_2 accordingly. We proceed inductively to obtain the above string. It is a straightforward check that $\mathrm{rl}(\mu_0 f^{\epsilon_1} \mu_1 \cdots f^{\epsilon_{n-1}} \mu_{n-1} f^{\epsilon_n}) = n$.

We observe that if $\epsilon_i = -1$ and $\mu_i \in \mathrm{H}_+$ (resp. $\epsilon_i = 1$ and $\mu_{i+1} \in \mathrm{H}_-$), then (2.9) forces $\epsilon_i = -1$ (resp. $\epsilon_i = 1$). Thus, (ω_k), where $\omega_k := \mu_0 f^{\epsilon_1} \mu_1 \cdots f^{\epsilon_{n-1}} \mu_{n-1} f^{\epsilon_n}$, is an alternating sequence. Since the sequence (ω_k) lies in the quasigeodesic ray c, it converges to ε. $\qquad\square$

The same construction used to prove Lemma 2.16 can be applied to "uniform quasigeodesic segments" in Γ to show the following result:

Definition 2.17. For $D \geq 0$, a *D-normal form* of $\gamma \in \Gamma$ satisfying $\mathrm{rl}(\gamma) \geq 1$ is a normal form of γ such that all left subwords in that normal form lie in a D-neighborhood of any geodesic segment in (Γ, d_Γ) connecting 1_Γ and γ.

Lemma 2.18. *There exists $D_0 \geq 0$ such that every element $\gamma \in \Gamma$ satisfying $\mathrm{rl}(\gamma) \geq 1$ has a D_0-normal form.*

Now, we prove Proposition 2.13.

Proof of Proposition 2.13. We give a proof in the amalgamated free product case; the HNN extension case is similar.

We continue with the proof of Lemma 2.16 above (the amalgamated free product case). Let (γ_n) be any sequence in Γ converging to $\varepsilon \in \partial_{\mathrm{II}} \Gamma$. Since (γ_n) fellow travels the uniform quasigeodesic c, we may pick a divergent sequence (t_n) in $\mathbf{N} \cup \{0\}$ and, for each n, a uniform quasigeodesic segment

$$c_n : [0, l_n] \cap \mathbf{Z} \to \Gamma, \quad c_n(0) = 1_\Gamma, c_n(l_n) = \gamma_n,$$

such that for all $n \in \mathbf{N}$, $c_n|_{[0,t_n] \cap \mathbf{Z}} = c|_{[0,t_n] \cap \mathbf{Z}}$. See Figure 5 for an illustration of c and c_n in (the Cayley graph) of Γ. We apply the same procedure used in the proof of Lemma 2.16 on c_n to yield a D-normal form for γ_n.
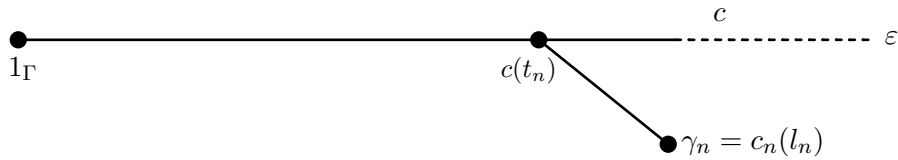

FIGURE 5

Lemma 2.19. *There exists $n_1 \in \mathbf{N}$ such that for all $n \geq n_1$, the prescribed normal form of γ_n contains ω_1 as a leftmost subword.*

Proof. We argue by contradiction: Suppose that the assertion is false. Then, a divergent sequence (n_i) exists in $\mathbf{N}$ such that for all i, the leftmost letter λ_i of the prescribed normal form of γ_{n_i} is different from ω_1. Since $c_{n_i}|_{[0,t_{n_i}]\cap\mathbf{Z}} = c|_{[0,t_{n_i}]\cap\mathbf{Z}}$, for all i large enough it must hold that $\lambda_i = c_{n_i}(r_i)$, where $r_i > t_{n_i}$. However, $\lambda_i \in \Gamma_A \cup \Gamma_B$. Since c_{n_i} are uniform quasigeodesics with $c_{n_i}(0) = 1_\Gamma \in \Gamma_A \cup \Gamma_B$, it follows that $c_{n_i}([0, r_i] \cap \mathbf{Z})$, which contains $c([0, t_{n_i}] \cap \mathbf{Z})$ as a subset, lies in a uniform neighborhood of $\Gamma_A \cup \Gamma_B$. However, since $t_{n_i} \to \infty$, $\mathrm{rl}(c(t_{n_i}))$ goes to infinity. Thus, $c(t_{n_i})$ cannot stay in a uniform neighborhood of $\Gamma_A \cup \Gamma_B$ (cf. Proposition 2.14), yielding a contradiction. $\qquad\square$

We now finish the proof of Proposition 2.13 by induction. Suppose that $c(i_1) = \omega_1$. Since $c_n|_{[0,t_n]\cap\mathbf{Z}} = c|_{[0,t_n]\cap\mathbf{Z}}$, by the claim above, it holds that for all sufficiently large n, say $n \geq n_1$, $c_n(i_1) = c(i_1) = \omega_1$. Applying the same argument to the quasigeodesic ray/segments

$$\bar{c} : [0, \infty) \cap \mathbf{Z} \to \Gamma, \qquad c(t) = \omega_1^{-1}c(t + i_1),$$
$$\bar{c}_n : [0, l_n - i_1] \cap \mathbf{Z} \to \Gamma, \quad c_n(t) = \omega_1^{-1}c_n(t + i_1),$$

shows that there exists $n_2 > n_1$ such that for all $n \geq n_2$, the prescribed normal form of γ_n contains ω_2 as a leftmost subword. Arguing inductively, it follows that there exists an increasing sequence (n_k) of natural numbers such that for all $n \geq n_k$, the prescribed normal form of γ_n contains ω_k as a leftmost subword. Thus, the desired function $F : \mathbf{N} \to \mathbf{N}$ in Proposition 2.13 can be defined as $F(n) := k$, if $n \in [n_k, n_{k+1})$. $\qquad\square$

2.9. **Proof of Proposition 2.14.** Let us first consider the case of amalgamated free products: For $\gamma \in \Gamma = \Gamma_A \star_\mathrm{H} \Gamma_B$, consider a normal form of γ:

$$\gamma = \gamma_1 \gamma_2 \cdots \gamma_l.$$

If $\gamma_1 \in \Gamma_A \setminus \mathrm{H}$, then normal form given above yields a finite sequence of points in T:

$$\Gamma_B, \quad \Gamma_A, \quad \gamma_1\Gamma_B, \quad \gamma_1\gamma_2\Gamma_A, \quad \gamma_1\gamma_2\gamma_3\Gamma_B, \quad \ldots, \quad \gamma_1\gamma_2\cdots\gamma_l\Gamma_*, \tag{2.10}$$

where $* = A$ if l is even, or $* = B$ if l is odd. We observe that any two consecutive points in the above are adjacent vertices in T (cf. §2.3), and the sequence does not *backtrack*, i.e., for any point in (2.10), the vertices on the left and right to it are different: For even i, let us examine the portion of the path

$$\gamma_1 \cdots \gamma_{i-1}\Gamma_B, \quad \gamma_1 \cdots \gamma_i\Gamma_A, \quad \gamma_1 \cdots \gamma_{i+1}\Gamma_B.$$

Note that $\gamma_i \in \Gamma_B \setminus \mathrm{H}$ and $\gamma_{i+1} \in \Gamma_A \setminus \mathrm{H}$. Applying $(\gamma_1 \cdots \gamma_i)^{-1}$ to the above, we obtain

$$\Gamma_B = \gamma_i^{-1}\Gamma_B, \quad \Gamma_A, \quad \gamma_{i+1}\Gamma_B.$$

However, since $\gamma_{i+1} \notin \Gamma_B$, $\Gamma_B \neq \gamma_{i+1}\Gamma_B$. A similar analysis can be done when i is odd.

Therefore, if we connect each pair of consecutive vertices in (2.10) by the unique edge in T determined by the pair, we obtain a geodesic path in T.

Lemma 2.20. *If $\gamma \in \Gamma = \Gamma_A \star_\mathrm{H} \Gamma_B$ lies in the coset represented by $v \in V(T)$, then $|d_T(v, \Gamma_A) - \mathrm{rl}(\gamma)| \leq 1$, $|d_T(v, \Gamma_B) - \mathrm{rl}(\gamma)| \leq 1$.*

Proof. This claim is easily checked when $\gamma \in \mathrm{H}$. So, suppose that $\gamma \notin \mathrm{H}$. Let $\gamma_1\gamma_2 \cdots \gamma_l$ be a normal form of γ such that $\gamma_1 \in \Gamma_A \setminus \mathrm{H}$; the case $\gamma_1 \in \Gamma_B \setminus \mathrm{H}$ follows by a similar argument. Note that v is one of the two rightmost entries in the sequence (2.10). Moreover,

the discussion above shows that the sequence (2.10) is a geodesic sequence of vertices in T. Thus, $\mathrm{rl}(\gamma) \geq d_T(v, \Gamma_A) \geq \mathrm{rl}(\gamma) - 1$ and $\mathrm{rl}(\gamma) + 1 \geq d_T(v, \Gamma_B) \geq \mathrm{rl}(\gamma)$. $\qquad\square$

Similar discussion holds in the case of HNN extensions: For $\gamma \in \Gamma = \mathrm{M}\star_\phi$, let

$$\gamma = \mu_0 f^{\epsilon_1} \mu_1 \cdots f^{\epsilon_n} \mu_n$$

be a normal form of γ. The above normal form produces a finite sequence in T:

$$\mathrm{M}, \quad \mu_0 f^{\epsilon_1}\mathrm{M}, \quad \mu_0 f^{\epsilon_1}\mu_1 f^{\epsilon_2}\mathrm{M}, \quad \ldots, \quad \gamma\mathrm{M} = \mu_0 f^{\epsilon_1}\mu_1 \cdots f^{\epsilon_n}\mathrm{M}.$$

Similarly to the amalgamated free product case discussed above, one can check that the above sequence does not backtrack and that consecutive vertices in the path are adjacent in T. This yields the following:

Lemma 2.21. *For all $\gamma \in \Gamma = \mathrm{M}\star_\phi$, $d_T(v, \mathrm{M}) = \mathrm{rl}(\gamma)$.*

Proof. The proof is similar to that of Lemma 2.20. We omit the details. $\qquad\square$

We prove Proposition 2.14.

Proof of Proposition 2.14. We discuss the proof in the case of amalgamated free products $\Gamma = \Gamma_A \star_\mathrm{H} \Gamma_B$; the case of HNN extensions $\Gamma = \mathrm{M}\star_\phi$ is similar.

Consider a normal form of γ which contains some normal form of ω as a left subword:

$$\gamma = \underbrace{\gamma_1 \cdots \gamma_m}_{=\omega} \gamma_{m+1} \cdots \gamma_l.$$

We further assume that $\gamma_1 \in \Gamma_A \setminus \mathrm{H}$; the case $\gamma_1 \in \Gamma_B \setminus \mathrm{H}$ follows similarly. The above normal form induces a geodesic path in T (see the paragraph before Lemma 2.20) from the vertex Γ_B to the vertex $v_\gamma := \gamma\Gamma_*$, where $* = A$ if l is even, or $* = B$ if l is odd. This path also contains the vertex $v_\omega := \omega\Gamma_*$, where $* = A$ if m is even, or $* = B$ if m is odd. The vertex Γ_A also lies in that path, see (2.10). Thus, any path in T connecting v_γ to Γ_A or Γ_B must contain v_ω.

For the rest of the proof, let $*$ denote either A or B. Let $c : [0, n] \cap \mathbf{Z} \to \Gamma$ be a shortest geodesic in (Γ, d)[1] such that $c(0) \in \Gamma_*$ and $c(n) = \gamma$. By definition, $c(0)$ is a closest point in Γ_* to $c(k)$, for any k in the domain of c. For $k \in [0, n] \cap \mathbf{Z}$, let $\bar{c}(k) := \{c(k)\Gamma_A, c(k)\Gamma_B\}$.

Claim. *For $k = 0, \ldots, n - 1$, we have that $\bar{c}(k) \cap \bar{c}(k + 1) \neq \emptyset$.*

Proof. We observe that $c(k + 1) = c(k) \cdot s_k$, for some generator $s_k \in S \subset \Gamma_A \cup \Gamma_B$. Let $c(k) = \tilde{\gamma}_1 \cdots \tilde{\gamma}_r$ be a normal form of $c(k)$. Assume that $\tilde{\gamma}_r \in \Gamma_A$; the other possibility $\tilde{\gamma}_r \in \Gamma_B$ can be similarly treated. If $s_k \in \Gamma_A$, then $c(k)\Gamma_A = \tilde{\gamma}_1 \cdots \tilde{\gamma}_{r-1}\Gamma_A \in \bar{c}(k + 1)$ since $c(k + 1) = \tilde{\gamma}_1 \cdots \tilde{\gamma}_{r-1}(\tilde{\gamma}_r s_k) \in \tilde{\gamma}_1 \cdots \tilde{\gamma}_{r-1}\Gamma_A$. Similarly, if $s_k \in \Gamma_B$, then $c(k)\Gamma_B \in \bar{c}(k + 1)$. The claim follows. $\qquad\square$

Let $V' = \bigcup_{k=0}^n \bar{c}(k) \subset V(T)$. Consider the induced subtree of $T' \subset T$ determined by V'. By the claim above, T' is connected. Since $c(0) \in \Gamma_*$, we get that $\Gamma_* \in V'$. Obviously, $v_\gamma \in V'$ as well. Since T' is connected, by the first paragraph of this proof, $v_\omega \in V'$. Therefore, there exists some k_0 in the domain of c such that $v_\omega \in \bar{c}(k_0)$. In other words, $c(k_0)$ lies in (the coset represented by) v_ω. Since $c(0)$ is also a closest point in Γ_* to $c(k_0)$, it follows that $c(0)$ is uniformly close to $\mathrm{pr}_{\Gamma_*}(v_\omega)$. However, since $\mathrm{rl}(\omega) \geq 3$, by Lemma 2.20, $d_T(v_\omega, \Gamma_*) \geq 2$. Thus,

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[1] See Notation 2.15 for our notation.

by (2.7),[2] $\partial_\infty \Gamma_* \cap \partial_\infty v_\omega = \emptyset$ and, hence, $\mathrm{pr}_{\Gamma_*}(v_\omega)$ is bounded in Γ_* (see Corollary 2.7). This completes the proof of this result when $d_\Gamma = d$.

In the general case, i.e., when d_Γ is an arbitrary word metric, then the result follows by the fact that the identity map $(\Gamma, d) \to (\Gamma, d_\Gamma)$ is a quasiisometry.   □

3. Preliminaries on discrete isometry groups of symmetric spaces

We recall some preliminary facts on symmetric spaces, mainly to set up some notations. Then, we refer to [2, Appendix 5] for a quick discussion and to [8] for a more detailed exposition.

Let G be a real semisimple Lie group of noncompact type with a finite center. We impose some mild assumptions on G; see Assumption 3.1 below. Let $X = G/K$ denote the (globally) symmetric space of G, where K is a maximal compact subgroup of G. Then, X is a non-positively curved G-homogeneous space such that X has no compact or Euclidean de Rham factors, i.e. X is a symmetric space of *noncompact type*.

The ideal boundary $\partial_\infty X$ is the set of equivalence classes of asymptotic rays in X on which G acts naturally. The point stabilizers in G of the action $G \curvearrowright \partial_\infty X$ are called *parabolic subgroups* of G. The ideal boundary $\partial_\infty X$ carries a natural G-invariant spherical building structure, called the *Tits building* of X, and denoted by $\partial_{\mathrm{Tits}} X$. The top-dimensional simplices in $\partial_{\mathrm{Tits}} X$ are called *chambers*.

In this paper, we impose the following assumption on G, which are standing assumptions in the papers of Kapovich-Leeb-Porti [17, 18, 19] we rely upon in this work:

Assumption 3.1. The group G has a finitely many connected components such that for the associated symmetric space $X = G/K$, the spherical Tits building $\partial_{\mathrm{Tits}} X$ is *thick*: That is, every *panel* (i.e., a codimension one simplex) in $\partial_{\mathrm{Tits}} X$ is contained in at least three distinct chambers.

Remark 3.2. Assumption 3.1 is satisfied if G is connected, for instance, when $G = \mathrm{SL}(n, \mathbb{R})$. However, there are cases where this condition is not met. For instance, if $X = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ and G is the *full* isometry group of X, the associated spherical Tits building is not thick. Such semisimple Lie groups are excluded from this theory.

Each chamber in $\partial_\infty X$ is also a fundamental domain for the action $G \curvearrowright \partial_\infty X$. The stabilizer in G of a chamber σ (which is the same as the stabilizer in G of any interior point of σ) is called a *minimal parabolic subgroup* of G. The group G also acts on the set of chambers in $\partial_{\mathrm{Tits}} X$ transitively, so any two minimal parabolic subgroups are conjugate. Therefore, the space of all chambers in $\partial_{\mathrm{Tits}} X$ can be identified with the analytic manifold

$$\mathrm{Flag}(\sigma_{\mathrm{mod}}) := G/P_{\sigma_{\mathrm{mod}}},$$

where σ_{mod} is a chosen chamber in $\partial_{\mathrm{Tits}} X$ and $P_{\sigma_{\mathrm{mod}}}$ is the minimal parabolic subgroup stabilizing σ_{mod}.

More generally, for a face $\nu_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$, the space of simplices ν in $\partial_{\mathrm{Tits}} X$ of *type* ν_{mod}, i.e., those simplices ν in $\partial_{\mathrm{Tits}} X$ which can be brought to ν_{mod} by the action $G \curvearrowright \partial_{\mathrm{Tits}} X$, can be identified with the analytic manifold

$$\mathrm{Flag}(\nu_{\mathrm{mod}}) := G/P_{\nu_{\mathrm{mod}}},$$

_____

[2]Note that $\partial_\infty \Gamma_{v_\omega} = \partial_\infty v_\omega$.

where $P_{\nu_{\mathrm{mod}}}$ is the parabolic subgroup of G stabilizing ν_{mod}.

A pair of simplices $\nu_{\pm}$ in $\partial_{\mathrm{Tits}}X$ is called *antipodal* if there is a complete geodesic line in X, which is forward (resp. backward) asymptotic to some interior point of the simplex ν_+ (resp. ν_-). If $\nu_{\mathrm{mod}}^{\pm} \subset \sigma_{\mathrm{mod}}$ denote the types of $\nu_{\pm}$, then they satisfy

$$\nu_{\mathrm{mod}}^+ = \iota(\nu_{\mathrm{mod}}^-),$$

where $\iota : \sigma_{\mathrm{mod}} \to \sigma_{\mathrm{mod}}$ denotes the *opposition involution*. For a simplex ν_- in $\partial_{\mathrm{Tits}}X$ of type ν_{mod}^-, the set of simplices in $\partial_{\mathrm{Tits}}X$ (regarded as points in $\mathrm{Flag}(\nu_{\mathrm{mod}}^+)$, where $\nu_{\mathrm{mod}}^+ = \iota(\nu_{\mathrm{mod}}^-)$) antipodal to ν_-,

$$C(\nu_-) := \{\nu_+ \in \mathrm{Flag}(\nu_{\mathrm{mod}}^+) \mid \nu_+ \text{ is antipodal to } \nu_-\} \subset \mathrm{Flag}(\nu_{\mathrm{mod}}^+),$$

is an open dense $P_{\nu_{\mathrm{mod}}^-}$-homogeneous subset of $\mathrm{Flag}(\nu_{\mathrm{mod}}^+)$.

Definition 3.3 (Antipodality). A pair (A, B), where $A \subset \mathrm{Flag}(\nu_{\mathrm{mod}}^+)$ and $B \subset \mathrm{Flag}(\nu_{\mathrm{mod}}^-)$, is said to be *antipodal* to each other (or, A is *antipodal* to B) if, for all $\nu_+ \in A$ and all $\nu_- \in B$, ν_+ is antipodal to ν_-.

3.1. **Regular sequences.** Let $\nu_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ be a face. A sequence (g_n) in G is said to ν_{mod}-*converge* to a point $\nu \in \mathrm{Flag}(\nu_{\mathrm{mod}})$, which is denoted by

$$g_n \xrightarrow{\mathrm{flag}} \nu,$$

if every subsequence of (g_n) has a further subsequence (g_{n_k}) such that there exists $\nu_- \in \mathrm{Flag}(\iota\nu_{\mathrm{mod}})$ such that

$$g_{n_k}|_{C(\nu_-)} \to \nu, \quad \text{uniformly on compacts.}$$

In this situation, (g_n) is called a ν_{mod}-*convergent* sequence in G and $\nu \in \mathrm{Flag}(\nu_{\mathrm{mod}})$ is called the ν_{mod}-*limit point* of (g_n). For a ν_{mod}-convergent sequence (g_n) in G, its ν_{mod}-limit point ν is unique. See [18, §4] for more details.

For a discrete subgroup Γ of G, the set of all ν_{mod}-limit points in $\mathrm{Flag}(\nu_{\mathrm{mod}})$ of ν_{mod}-convergent subsequences of Γ is called the ν_{mod}-*limit set* of Γ, which is denoted by

$$\Lambda_{\nu_{\mathrm{mod}}}(\Gamma).$$

The limit set $\Lambda_{\nu_{\mathrm{mod}}}(\Gamma)$ is a Γ-invariant compact subset of $\mathrm{Flag}(\nu_{\mathrm{mod}})$; however; it may be empty, even if Γ is infinite.

A sequence (g_n) in G is ν_{mod}-*regular* if every subsequence of (g_n) contains a ν_{mod}-convergent subsequence. Moreover, if (g_n) is ν_{mod}-regular, then the inverse sequence (g_n^{-1}) is $\iota\nu_{\mathrm{mod}}$-regular. Similarly, a subgroup Γ of G is called ν_{mod}-*regular* if every sequence in Γ is ν_{mod}-regular. Such subgroups are necessarily discrete. Clearly, ν_{mod}-regular subgroups are also $\iota\nu_{\mathrm{mod}}$-regular.

The following results help verify the regularity and flag-convergence of certain sequences considered in this paper.

Let (g_n) be a sequence in G.

Lemma 3.4. *If there exists a compact subset $A \subset \mathrm{Flag}(\nu_{\mathrm{mod}})$ with a nonempty interior such that the sequence $(g_n A)$ of compact subsets of $\mathrm{Flag}(\nu_{\mathrm{mod}})$ converges to a point $\nu \in \mathrm{Flag}(\nu_{\mathrm{mod}})$, then (g_n) is ν_{mod}-regular and $g_n \xrightarrow{\mathrm{flag}} \nu$.*

See [5, Lemma 1.10].

Let d be a distance function on $\mathrm{Flag}(\nu_{\mathrm{mod}})$ compatible with the manifold topology of $\mathrm{Flag}(\nu_{\mathrm{mod}})$. Let (A_n) a sequence of nonempty subsets of $\mathrm{Flag}(\nu_{\mathrm{mod}})$. We say that (A_n) *shrinks*, if the diameter of A_n converges to zero as $n \to \infty$. Moreover, we say that (A_n) *converges* a point $\nu \in \mathrm{Flag}(\nu_{\mathrm{mod}})$, written as $A_n \to \nu$, if the the diameters of the subsets $A_n \cup \{\nu\}$ converge to zero as $n \to \infty$. Since $\mathrm{Flag}(\nu_{\mathrm{mod}})$ is compact, these notions do not depend on the chosen distance function.

Lemma 3.5. *If there exists a compact subset $A \subset \mathrm{Flag}(\nu_{\mathrm{mod}})$ with a nonempty interior such that $(g_n A)$ shrinks, then (g_n) is ν_{mod}-regular.*

See [5, Corollary 1.11].

The next result can be extracted from the proof of [5, Lemma 3.4]. For completeness, we give a proof.

Lemma 3.6. *If there exist compact subsets $B, B' \subset \mathrm{Flag}(\nu_{\mathrm{mod}})$ with nonempty interior such that $B' \subset B^\circ$, $B \subset C(\nu)$ for some $\nu \in \mathrm{Flag}(\nu_{\mathrm{mod}})$, and for all $n \in \mathbf{N}$, $g_{n+1}g_n^{-1}(B) \subset B'$, then (g_n) is ν_{mod}-regular.*

Proof. After replacing the sequence (g_n) by (h_n), where $h_n := g_n g_1^{-1}$, we have that

$$h_n(B) \subset B \quad \text{and} \quad h_{n+1}h_n^{-1}(B) \subset B'. \tag{3.1}$$

Suppose, to the contrary, that (g_n) is not ν_{mod}-regular. Then (h_n) is also not ν_{mod}-regular. Moreover, by the right side of (3.1), it follows that the sequence (h_n) has no accumulation points in G. Therefore, after extraction, (h_n) is η_{mod}-*pure* for some face $\eta_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ such that $\nu_{\mathrm{mod}} \not\subset \eta_{\mathrm{mod}}$. By [17, Proposition 9.5], after further extraction, there exist $\eta_+ \in \mathrm{Flag}(\eta_{\mathrm{mod}})$ and $\eta_- \in \mathrm{Flag}(\iota\eta_{\mathrm{mod}})$, and a surjective algebraic map $\phi : C_{\mathrm{Fu}}(\eta_-) \to \mathrm{St}_{\mathrm{Fu}}(\eta_+)$ such that $h_{n_k}|_{C_{\mathrm{Fu}}(\eta_-)} \to \phi$ uniformly on compacts. Here $\mathrm{St}_{\mathrm{Fu}}(\eta_+) := \pi^{-1}(\eta_+)$ and $C_{\mathrm{Fu}}(\eta_-) := \pi^{-1}(C(\eta_-))$, where π denotes the unique G-equivariant projection $\mathrm{Flag}(\sigma_{\mathrm{mod}}) \to \mathrm{Flag}(\eta_{\mathrm{mod}})$ mapping $\sigma_{\mathrm{mod}} \mapsto \eta_{\mathrm{mod}}$.

By (3.1), we obtain that $h_{n_k}\tilde{B} \subset \tilde{B}$, where

$$\tilde{B} = \bigcup_{\nu \in B} \mathrm{St}_{\mathrm{Fu}}(\nu).$$

Consider any point $\eta \in C(\eta_-) \subset \mathrm{Flag}(\eta_{\mathrm{mod}})$ such that $\mathrm{St}_{\mathrm{Fu}}(\eta) \cap (\tilde{B})^\circ$ is nonempty.[3] By [5, Lemma 1.8],

$$\lim_{k \to \infty} \max_{\sigma \in \mathrm{St}_{\mathrm{Fu}}(h_{n_k}\eta)} d(h_{n_k}h_{n_{k-1}}^{-1}\sigma, \sigma) = 0. \tag{3.2}$$

Since, for all $k \in \mathbf{N}$, $h_{n_k}(\tilde{B})^\circ \subset (\tilde{B})^\circ$, $\mathrm{St}_{\mathrm{Fu}}(h_{n_k}\eta) \cap (\tilde{B})^\circ \neq \emptyset$. Moreover, since $\nu_{\mathrm{mod}} \not\subset \eta_{\mathrm{mod}}$, by [5, Lemma 1.3], $\mathrm{St}_{\mathrm{Fu}}(h_{n_k}\eta)$ is not contained in $C_{\mathrm{Fu}}(\nu)$ for any $\nu \in \mathrm{Flag}(\nu_{\mathrm{mod}})$. As $B \subset C(\nu)$ for some $\nu \in \mathrm{Flag}(\nu_{\mathrm{mod}})$, it follows that $\mathrm{St}_{\mathrm{Fu}}(h_{n_k}\eta) \not\subset \tilde{B}$. However, by [5, Lemma 1.2], $\mathrm{St}_{\mathrm{Fu}}(h_{n_k}\eta)$ is connected. So, $\mathrm{St}_{\mathrm{Fu}}(h_{n_k}\eta)$ intersects $\partial\tilde{B}$. Pick $\sigma_k \in \mathrm{St}_{\mathrm{Fu}}(h_{n_k}\eta) \cap \partial\tilde{B}$ for each $k \in \mathbf{N}$. By (3.2), we have that

$$\lim_{k \to \infty} d(h_{n_k}h_{n_{k-1}}^{-1}\sigma_k, \sigma_k) = 0. \tag{3.3}$$

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[3]Note that since $C_{\mathrm{Fu}}(\nu_-)$ is open dense in $\mathrm{Flag}(\sigma_{\mathrm{mod}})$ and $\tilde{B}$ has nonempty interior in $\mathrm{Flag}(\sigma_{\mathrm{mod}})$, $C_{\mathrm{Fu}}(\nu_-) \cap \tilde{B}$ is nonempty.

Since $h_{n+1}h_n^{-1}(B) \subset B'$ (by (3.1)), we have that $h_m h_n^{-1}(B) \subset B'$ for all $m > n$. In particular, $h_{n_k}h_{n_{k-1}}^{-1}(B) \subset B'$, for all $k \in \mathbf{N}$, which implies that

$$h_{n_k}h_{n_{k-1}}^{-1}(\tilde{B}) \subset \tilde{B}' := \bigcup_{\nu \in B'} \mathrm{St}_{\mathrm{Fu}}(\nu).$$

Since $\sigma_k \in \tilde{B}$, by above, we get $h_{n_k}h_{n_{k-1}}^{-1}(\sigma_k) \in \tilde{B}'$. However, $\tilde{B}' \subset \tilde{B}^{\circ}$, which shows that

$$d(h_{n_k}h_{n_{k-1}}^{-1}(\sigma_k), \sigma_k) \geq d(\tilde{B}', \partial\tilde{B}) > 0, \quad \text{for all } k \in \mathbf{N},$$

contradicting (3.3). $\qquad\square$

Lemma 3.7. *Suppose that (g_n) is ν_{mod}-regular. If there exist compact subsets $A, B \subset \mathrm{Flag}(\nu_{\mathrm{mod}})$ with nonempty interior such that for all $n \in \mathbf{N}$, $g_n B \subset A$, then all the ν_{mod}-limit-points of (g_n) lie in A.*

Proof. Let $\nu \in \mathrm{Flag}(\nu_{\mathrm{mod}})$ be a ν_{mod}-limit point of (g_n). Then, there exists $\nu_- \in \mathrm{Flag}(\iota\nu_{\mathrm{mod}})$ and a subsequence (g_{n_k}) such that $g_{n_k}|_{C(\nu_-)}$ converges to the constant map $C(\nu_-) \to \{\nu\}$ uniformly on compacts, as $k \to \infty$. Let $\nu_1 \in C(\nu_-) \cap B$. Then, the sequence $(g_{n_k}\nu_1)$, which lies in A, converges to ν. Thus, $\nu \in A$. $\qquad\square$

Lemma 3.8. *Suppose that $g_n \xrightarrow{\mathrm{flag}} \nu \in \mathrm{Flag}(\nu_{\mathrm{mod}})$. Let $A_- \subset \mathrm{Flag}(\iota\nu_{\mathrm{mod}})$ be a subset containing all the $\iota\nu_{\mathrm{mod}}$-limit points of (g_n^{-1}). If $A \subset \mathrm{Flag}(\nu_{\mathrm{mod}})$ is any compact subset antipodal to A_-, then $(g_n A)$ converges to $\{\nu\}$.*

Proof. Suppose not. Then, there exists a sequence (ν_k) in A and a subsequence (g_{n_k}) of (g_n) such that $(g_{n_k}\nu_k)$ converges to some point $\nu' \in \mathrm{Flag}(\nu_{\mathrm{mod}})$ different from ν. After further extraction of (g_{n_k}), we may assume that $(g_{n_k}^{-1})$ $\iota\nu_{\mathrm{mod}}$-converges to some point $\nu_- \in A_-$. Since, by hypothesis, $A \subset C(\nu_-)$, by [19, Lemma 4.18], we get that $g_{n_k}(A) \to \nu$, as $n \to \infty$. Since, for each k, $\nu_k \in A$, we also obtain that $g_{n_k}\nu_k \to \nu$, which implies that $\nu' = \nu$. This is a contradiction. $\qquad\square$

The following result follows from Lemma 3.8:

Lemma 3.9. *Suppose that (g_n) is ν_{mod}-regular. Let $A_- \subset \mathrm{Flag}(\iota\nu_{\mathrm{mod}})$ be a subset containing all the $\iota\nu_{\mathrm{mod}}$-limit points of (g_n^{-1}). If $A \subset \mathrm{Flag}(\nu_{\mathrm{mod}})$ is any compact subset antipodal to A_-, then $(g_n A)$ shrinks.*

3.2. **Anosov subgroups.** We fix once and for all an ι-invariant face τ_{mod} of σ_{mod}. We focus on the special class of discrete subgroups of G, called τ_{mod}-*Anosov subgroups*. This class of subgroups has several different characterizations. For our purpose, we use the following characterization of τ_{mod}-Anosov subgroups:

Definition 3.10 (Asymptotically embedded)**.** A subgroup Γ of G is τ_{mod}-*asymptotically embedded* if

(i) Γ, as an abstract group, is hyperbolic, and
(ii) there exists a Γ-equivariant antipodal map[4] $\xi : \partial_\infty\Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$, which preserves the *convergence dynamics*: That is, for every sequence (γ_n) in Γ and every point $\varepsilon \in \partial_\infty\Gamma$,

$$\text{if } \gamma_n \xrightarrow{\mathrm{Cay}} \varepsilon, \text{ then } \gamma_n \xrightarrow{\mathrm{flag}} \xi(\varepsilon). \tag{3.4}$$

___

[4]That is, ξ maps every distinct pair of points to a pair of antipodal points.

The image of the map ξ in the above definition is precisely $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$, the limit set of Γ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$: Clearly, the image of ξ is contained in $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$. To show that opposite inclusion, let $\tau \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$. By definition, there exists a sequence (γ_n) in Γ such that $\gamma_n \xrightarrow{\mathrm{flag}} \tau$. After extraction, (γ_n) converges in $\bar{\Gamma}$ to some point $\varepsilon \in \partial_\infty \Gamma$. By (3.4), $\xi(\varepsilon) = \tau$.

Remark 3.11. The above definition of τ_{mod}-asymptotically embedded subgroups, which is a slight variation of the one given by Kapovich-Leeb-Porti in [18, Definition 5.12], is more straightforward to verify in this paper due to a certain "ping-pong" type arguments we use here. The equivalence between these two definitions can be checked as follows (cf. [5, §5]):

Suppose that $\Gamma < G$ is τ_{mod}-asymptotically embedded in the sense of Definition 3.10. By the first condition (i), Γ is hyperbolic.

We first verify that Γ is τ_{mod}-regular: If (γ_n) is any sequence in Γ without repeated entries, then, after extraction, $\gamma_n \xrightarrow{\mathrm{Cay}} \varepsilon$, for some $\varepsilon \in \partial_\infty \Gamma$. Thus, by condition (ii), (γ_n) τ_{mod}-converges to $\xi(\varepsilon)$. Thus, (γ_n) is τ_{mod}-regular.

We next verify that ξ is an embedding. Indeed, since $\partial_\infty \Gamma$ is a compact and $\mathrm{Flag}(\tau_{\mathrm{mod}})$ is Hausdorff, it is enough to establish that ξ is an injective continuous map: Since ξ is an antipodal map, ξ is injective. To verify continuity, pick $x \in X$ and consider the map

$$\phi : \overline{\Gamma} \to \bar{X}^{\tau_{\mathrm{mod}}} = X \sqcup \mathrm{Flag}(\tau_{\mathrm{mod}}),$$

whose restriction to $\partial_\infty \Gamma$ is ξ and whose restriction to Γ is the orbit map $\gamma \mapsto \gamma \cdot x$. The space $X \sqcup \mathrm{Flag}(\tau_{\mathrm{mod}})$ is topologized with the topology of τ_{mod}-convergence. The restriction of this topology to X or $\mathrm{Flag}(\tau_{\mathrm{mod}})$ coincides with their respective manifold topologies. By (3.4), ϕ is continuous and, hence, so is the restriction $\xi = \phi|_{\partial_\infty \Gamma}$.

Therefore, ξ is a Γ-equivariant homeomorphism between $\partial_\infty \Gamma$ and its image, $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$. Finally, since ξ is an antipodal map, $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is an antipodal subset of $\mathrm{Flag}(\tau_{\mathrm{mod}})$. Therefore, Γ is τ_{mod}-*antipodal* in the sense of [18]. Hence, Γ is τ_{mod}-asymptotically embedded in the sense of [18, Definition 5.12].

The other direction is straightforward.

3.3. **Interactive pairs and triples.** Let τ_{mod} be an ι-invariant face of σ_{mod}. Following Maskit [33], we define the notion of "interactive pairs" and "interactive triples."

3.3.1. *Interactive pairs.* Let Γ_A and Γ_B be discrete subgroups of G, and let $\mathrm{H} := \Gamma_A \cap \Gamma_B$. We denote this data by the triple $(\Gamma_A, \Gamma_B; \mathrm{H})$.

Definition 3.12 (Interactive pair)**.** A pair (A, B) of compact subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}}) = G/P_{\tau_{\mathrm{mod}}}$ is called an *interactive pair* for $(\Gamma_A, \Gamma_B; \mathrm{H})$ if the following conditions are satisfied:

(i) The interiors A° of A and B° of B are nonempty and disjoint.
(ii) H leaves the sets A and B *precisely invariant*, i.e., $\mathrm{H}A = A$, $\mathrm{H}B = B$, and, for all elements $\alpha \in \Gamma_A \setminus \mathrm{H}$ and $\beta \in \Gamma_B \setminus \mathrm{H}$, we have that $\alpha B \subset A^\circ$ and $\beta A \subset B^\circ$.

See Figure 1 for an illustration.

Proposition 3.13. *If* $(\Gamma_A, \Gamma_B; \mathrm{H})$ *admits an interactive pair* (A, B) *in* $\mathrm{Flag}(\tau_{\mathrm{mod}})$, *then the natural homomorphism*

$$\rho : \Gamma_A \star_{\mathrm{H}} \Gamma_B \to \langle \Gamma_A, \Gamma_B \rangle < G$$

from the abstract amalgamated free product $\Gamma_A \star_H \Gamma_B$ to G in injective. In particular, the subgroup $\Gamma := \langle \Gamma_A, \Gamma_B \rangle$ of G is naturally isomorphic to $\Gamma_A \star_H \Gamma_B$.

Proof. Let $\gamma \in \Gamma_A \star_H \Gamma_B$ be any nontrivial element. We want to show that $\rho(\gamma)$ is nontrivial. Clearly, if $\gamma \in (\Gamma_A \cup \Gamma_B) \setminus \{1_\Gamma\}$, then $\rho(\gamma) = \gamma$ and, hence, $\rho(\gamma)$ is nontrivial. So, we may assume that $\gamma \notin \Gamma_A \cup \Gamma_B$. Choose a normal form $\gamma = \gamma_1 \cdots \gamma_l$ of γ (see §2.1). Since $\gamma \notin \Gamma_A \cup \Gamma_B$, we have $l \geq 2$. Assume that $\gamma_l \in \Gamma_B$ (the other possibility that $\gamma_l \in \Gamma_A$ can be similarly analyzed). Then, $\rho(\gamma)A \subset A^\circ \cup B^\circ$ and, in particular, $\rho(\gamma)A \neq A$. Thus, $\rho(\gamma)$ is a nontrivial element of G. $\qquad\square$

Compare with [33, Theorem VII.A.10].

Remark 3.14. With a little more effort, one can also show that the image of the homomorphism ρ in Proposition 3.13 is discrete. However, we do not need this result to prove our main theorems.

3.3.2. *Interactive triples.* Let M be a discrete subgroup of G, and let $f \in G$. Define

$$H_- := (f^{-1}Mf) \cap M, \quad H_+ := M \cap (fMf^{-1}).$$

Clearly, $fH_- f^{-1} = H_+$. We denote this data by the quadruple $(M; H_\pm; f)$.

Definition 3.15 (Interactive triples)**.** A triple $(A, B_\pm)$ of compact subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ is called an *interactive triple* for $(M; H_\pm; f)$ if the following conditions are met:

(i) The interiors A°, B_-°, B_+° of A, B_-, B_+, respectively, are nonempty and pairwise disjoint. Furthermore, $B_- \cap B_+ = \emptyset$.

(ii) $H_\pm$ leaves $B_\pm$ *precisely invariant*, i.e., $H_\pm B_\pm = B_\pm$ and $\mu(B_\pm) \subset A^\circ$ for all $\mu \in M \setminus H_\pm$.

(iii) $f^{\pm 1}(A) \subset B_\pm$ and $f^{\pm 1}(B_\pm) \subset B_\pm^\circ$.

See Figure 2 for an illustration.

Proposition 3.16. *If $(M; H_\pm; f)$ admits an interactive triple $(A, B_\pm)$ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$, then the natural homomorphism*

$$M\star_\phi \to G$$

is injective. In particular, the subgroup $\Gamma := \langle M, f \rangle$ of G is naturally isomorphic to the HNN extension of M by the isomorphism $\phi : H_- \to H_+$ is given by $\phi(\eta) = f\eta f^{-1}$, for all $\eta \in H_-$.

Proof. The proof of this result is similar to the one of Proposition 3.13. Hence, we omit the details. $\qquad\square$

Compare with [33, Theorem VII.D.12].

4. Proof of Theorem A

In this section, we work under the hypothesis of Theorem A. To simplify our situation, we replace A and B by the closure of their respective interiors. It is easy to see that $(\mathrm{cl}(A^\circ), \mathrm{cl}(B^\circ))$ is still an interactive pair for $(\Gamma_A, \Gamma_B; H)$. However, the main advantage of the interactive pair $(\mathrm{cl}(A^\circ), \mathrm{cl}(B^\circ))$ is a stronger antipodality hypothesis:

Lemma 4.1. $\mathrm{cl}(A^\circ)$, *resp.* $\mathrm{cl}(B^\circ)$, *is antipodal to* B°, *resp.* A°.

Proof. Recall that A° is antipodal to B° under the hypothesis of Theorem A. Let $\tau_+ \in \mathrm{cl}(A^\circ)$ and $\tau_- \in B^\circ$ be any points. We show that $\tau_\pm$ are antipodal.

Fix an auxiliary distance function d on $\mathrm{Flag}(\tau_\mathrm{mod})$ compatible with its manifold topology. For $\tau \in \mathrm{Flag}(\tau_\mathrm{mod})$, let $\mathcal{E}(\tau)$ denote the (compact) subset of $\mathrm{Flag}(\tau_\mathrm{mod})$ consisting of all points which are not antipodal to τ. Let (τ_n) be a sequence in A° converging to τ_+ and let $B_\epsilon(\tau_-)$ be a closed metric ball in $\mathrm{Flag}(\tau_\mathrm{mod})$ centered at τ_- of radius $\epsilon > 0$ small enough such that $B_\epsilon(\tau_-) \subset B^\circ$. By hypothesis, τ_n and B° are antipodal. Hence, $d(\mathcal{E}(\tau_n), B_\epsilon(\tau_-)) > 0$, implying that $d(\mathcal{E}(\tau_n), \tau_-) \geq \epsilon$. Since the Hausdorff distance between $\mathcal{E}(\tau_n)$ and $\mathcal{E}(\tau_+)$ converges to zero as $n \to \infty$, it follows that $d(\mathcal{E}(\tau_+), \tau_-) \geq \epsilon > 0$. So, $\tau_- \notin \mathcal{E}(\tau_+)$. $\square$

Assumption 4.2. After replacing (A, B) by $(\mathrm{cl}(A^\circ), \mathrm{cl}(B^\circ))$ in the hypothesis of Theorem A, in place of (i), we will assume the following:

(i)' The pairs of subsets (A, B°) and (A°, B) of $\mathrm{Flag}(\tau_\mathrm{mod})$ are antipodal to each other.

This replacement does not affect the conclusion of Theorem A.

In §4.1, we will take additional advantage given by hypothesis (i)' above; see, for instance, the proof of Proposition 4.7.

Lemma 4.3. *Under the hypothesis of Theorem A*

$$\Lambda_{\tau_\mathrm{mod}}(\Gamma_A) \subset A, \quad \Lambda_{\tau_\mathrm{mod}}(\Gamma_B) \subset B, \quad and \quad \Lambda_{\tau_\mathrm{mod}}(\mathrm{H}) \subset A \cap B.$$

Proof. It will be enough to show that $\Lambda_{\tau_\mathrm{mod}}(\Gamma_A) \subset A$.

If $\mathrm{H} = \Gamma_A$, then Γ_A preserves A, and since A has a nonempty interior, all τ_mod-limit points of Γ must lie in A (see Lemma 3.7).

So, we can assume that H is a proper subgroup of Γ_A, i.e., there exists some element $\alpha \in \Gamma_A$ which is not an element of H. For any point $\tau_+ \in \Lambda_{\tau_\mathrm{mod}}(\Gamma_A)$, consider a sequence (α_n) in Γ_A such that $\alpha_n \xrightarrow{\mathrm{flag}} \tau_+$. We may (and will) also assume that no elements of (α_n) lie in H: Indeed, for every $n \in \mathbf{N}$, if $\alpha_n \in \mathrm{H}$, then replace the n-th entry α_n in (α_n) by $\alpha_n\alpha$. After replacing all such entries, the resulting sequence, again denoted by (α_n), does not share any elements with H but still satisfies $\alpha_n \xrightarrow{\mathrm{flag}} \tau_+$.

Since for all $n \in \mathbf{N}$, $\alpha_n B \subset A$, Lemma 3.7 implies that $\tau_+ \in A$. $\square$

Remark 4.4. If $\Lambda_{\tau_\mathrm{mod}}(\Gamma_A) \cap \partial A = \Lambda_{\tau_\mathrm{mod}}(\mathrm{H})$, then, by Lemma 4.3, $\Lambda_{\tau_\mathrm{mod}}(\Gamma_A) \setminus \Lambda_{\tau_\mathrm{mod}}(\mathrm{H})$ lies in A°. Then, $\Lambda_{\tau_\mathrm{mod}}(\Gamma_A) \setminus \Lambda_{\tau_\mathrm{mod}}(\mathrm{H})$ is antipodal to B; see Assumption 4.2. Similarly, if $\Lambda_{\tau_\mathrm{mod}}(\Gamma_B) \cap \partial B = \Lambda_{\tau_\mathrm{mod}}(\mathrm{H})$, then $\Lambda_{\tau_\mathrm{mod}}(\Gamma_B) \setminus \Lambda_{\tau_\mathrm{mod}}(\mathrm{H})$ is antipodal to A.

Consequently, when it is known that $\Lambda_{\tau_\mathrm{mod}}(\Gamma_A) \cap \partial A = \Lambda_{\tau_\mathrm{mod}}(\mathrm{H}) = \Lambda_{\tau_\mathrm{mod}}(\Gamma_B) \cap \partial B$, then the second condition (ii) in Theorem A is automatically satisfied.

Corollary 4.5. *Under the hypothesis of Theorem A, the subgroup H is weakly malnormal in both Γ_A and Γ_B.*

Proof. Since $\Lambda_{\tau_\mathrm{mod}}(\mathrm{H}) \subset A \cap B$, the interactive pair assumption implies that for all $\alpha \in \Gamma_A \setminus \mathrm{H}$ and $\beta \in \Gamma_B \setminus \mathrm{H}$,

$$\alpha(\Lambda_{\tau_\mathrm{mod}}(\mathrm{H})) \cap \Lambda_{\tau_\mathrm{mod}}(\mathrm{H}) = \beta(\Lambda_{\tau_\mathrm{mod}}(\mathrm{H})) \cap \Lambda_{\tau_\mathrm{mod}}(\mathrm{H}) = \emptyset.$$

If, say, $\alpha H \alpha^{-1} \cap H$ were infinite, it would contain an infinite order element $\eta \in H$;[5] hence,

$$\alpha^{-1}(\Lambda_{\tau_{\mathrm{mod}}}(\langle \eta \rangle)) = \Lambda_{\tau_{\mathrm{mod}}}(\langle \alpha^{-1} \eta \alpha \rangle)$$

would be nonempty subsets of $\Lambda_{\tau_{\mathrm{mod}}}(H)$. That would be a contradiction. $\square$

Corollary 4.6. *Under the hypothesis of Theorem A, the subgroup Γ of G generated by Γ_A and Γ_B in G is hyperbolic.*

Proof. If $H = \Gamma_A \cap \Gamma_B$ is quasiconvex in one of Γ_A or Γ_B, then it is quasiconvex in both of them: This follows from the general fact that if Γ is a τ_{mod}-Anosov subgroup of G and $H < \Gamma$ is a subgroup, then H is quasiconvex in Γ if and only if H is a τ_{mod}-Anosov subgroup of G. This general fact is a consequence of the τ_{mod}-*URU* characterization of τ_{mod}-Anosov subgroups; see [18, Equivalence Theorem 1.1 & Remark 1.2(i)].

Thus, under the hypothesis Theorem A, H is quasiconvex in Γ_A and Γ_B. Moreover, by Proposition 3.13, $\langle \Gamma_A, \Gamma_B \rangle$ is naturally isomorphic to $\Gamma_A \star_H \Gamma_B$. Then, together with Corollary 4.5, Theorem 2.11(i) implies that $\langle \Gamma_A, \Gamma_B \rangle$ is hyperbolic. $\square$

4.1. **Alternating sequences.** Recall the notion of alternating sequences from Definition 2.1. The main result of this subsection is as follows:

Proposition 4.7. *If (ω_n) is a type A alternating sequence in $\Gamma = \Gamma_A \star_H \Gamma_B$, then the nested sequence of compact subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$,*

$$\omega_1 B \supset \omega_2 A \supset \omega_3 B \supset \omega_4 A \supset \cdots ,$$

converges to a point $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$.

Similarly, if (ω_n) is a type B alternating sequence in Γ, then the nested sequence of compact subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$,

$$\omega_1 A \supset \omega_2 B \supset \omega_3 A \supset \omega_4 B \supset \cdots ,$$

converges to a point $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$.

For a (type A or B) alternating sequence $\underline{\omega} = (\omega_n)$, let us denote the point $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ obtained as a limit in the above by $\tau_{\underline{\omega}}$. As a direct corollary of the above and Lemma 3.4, we obtain the following:

Corollary 4.8. *If $\underline{\omega} = (\omega_n)$ is an alternating sequence in $\Gamma = \Gamma_A \star_H \Gamma_B$, then, for all $\gamma \in \Gamma$,*

$$\gamma \omega_n \xrightarrow{\mathrm{flag}} \gamma \tau_{\underline{\omega}}, \quad as\ n \to \infty.$$

4.1.1. *Regularity.* We begin by showing that alternating sequences are τ_{mod}-regular (see Corollary 4.11). Instead of directly proving the τ_{mod}-regularity of an alternating sequence, our approach is to study the dynamics of the corresponding inverse sequence and establish its τ_{mod}-regularity, which appears to be easier. This leads us to the following definition:

Definition 4.9 (Special sequences)**.** A sequence $(\tilde{\gamma}_n)$ in $\Gamma = \Gamma_A \star_H \Gamma_B$ is *special* if there exist sequences (α_n) in $\Gamma_A \setminus H$, (β_n) in $\Gamma_B \setminus H$, and an increasing function $F : \mathbf{N} \to \mathbf{N}$ such that

$$\tilde{\gamma}_n = \beta_{F(n)} \alpha_{F(n)} \beta_{F(n)-1} \alpha_{F(n)-1} \cdots \beta_1 \alpha_1. \tag{4.1}$$

---

[5]Every torsion subgroup of a hyperbolic group is finite; see [10, Ch. 8, Corollary 36].

The advantage of this definition is that subsequences also inherit the property of being special. This characteristic will be essentially used in the proof of the following lemma:

Lemma 4.10. *Special sequences in Γ are τ_{mod}-regular.*

Proof. Consider any special sequence $(\tilde{\gamma}_n)$ in Γ, and assume that the normal form of $\tilde{\gamma}_n$ is given by (4.1). Consider any subsequence of $(\tilde{\gamma}_n)$, again denoted by $(\tilde{\gamma}_n)$.

If the inverse sequence $(\beta_{F(n)}^{-1})$ corresponding to the leftmost letter sequence diverges away from H, then let $\hat{\beta}_{F(n)} := \hat{\beta}_{F(n)} \, \mathrm{pr}_{\mathrm{H}}(\hat{\beta}_{F(n)}^{-1})$. Since $(\hat{\beta}_{F(n)})$ is τ_{mod}-regular,[6] after extraction, there exists $\tau_\pm \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ so that

$$\hat{\beta}_{F(n_k)}^{\pm 1} \xrightarrow{\mathrm{flag}} \tau_\pm,$$

as $k \to \infty$. Note that $\tau_- \notin \Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H})$ (cf. Lemma 2.9(i)). Therefore, by the antipodality hypothesis (ii) of Theorem A, A is a compact subset of $C(\tau_-)$. Hence, $\hat{\beta}_{F(n_k)} A \xrightarrow{\mathrm{flag}} \tau_+$ as $k \to \infty$. Note that for all $n \in \mathbf{N}$, $\tilde{\gamma}_n B \subset \hat{\beta}_n A$. Thus, $\tilde{\gamma}_{n_k} B \xrightarrow{\mathrm{flag}} \tau_+$, as $k \to \infty$. By Lemma 3.4, $(\tilde{\gamma}_{n_k})$ is τ_{mod}-regular.

Otherwise, after passing to a further subsequence $(\tilde{\gamma}_{n_k})$ of $(\tilde{\gamma}_n)$ we can (and will) assume that there exists an element $\tilde{\beta} \in \Gamma_B \setminus \mathrm{H}$ and a sequence (η_k) in H such that

$$\beta_{F(n_k)} = \tilde{\beta}\eta_k, \quad \forall n \in \mathbf{N}. \tag{4.2}$$

Set $B' = \tilde{\beta}(A) \subset B^\circ$. Note that for all $k \in \mathbf{N}$, $\tilde{\gamma}_{n_{k+1}} \tilde{\gamma}_{n_k}^{-1} B \subset B'$. So, by Lemma 3.6, $(\tilde{\gamma}_{n_k})$ is τ_{mod}-regular.

Therefore, every subsequence of the original sequence contains a τ_{mod}-regular subsequence, showing that the original sequence is τ_{mod}-regular. This concludes the proof. $\square$

For an alternating sequence (ω_n), applying Lemma 4.10 to (ω_n^{-1}), we obtain the following result.

Corollary 4.11. *Alternating sequences in $\Gamma = \Gamma_A \star_{\mathrm{H}} \Gamma_B$ are τ_{mod}-regular.*

4.1.2. *Proof of Proposition 4.7.* Let us assume that (ω_n) is of type A; the other possibility can be similarly treated. Let (β_n) denote the rightmost letter sequence corresponding to (ω_{2n}), see Definition 2.1. Consider the special sequence (ω_{2n}^{-1}).

We first show that there exists a sequence (η_n) in H such that $(\hat{\omega}_{2n}^{-1})$, where $\hat{\omega}_{2n} = \omega_{2n}\eta_n^{-1}$, has a subsequence that is τ_{mod}-regular and τ_{mod}-converges to some point τ_- antipodal to A: By Lemma 2.9, there exist sequences (η_n) and $(\hat{\eta}_n)$ in H such that both the sequences $(\hat{\beta}_n)$ and $(\hat{\beta}_n^{-1})$ accumulate outside $\partial_\infty \mathrm{H}$ in $\bar{\Gamma}_B = \Gamma_B \sqcup \partial_\infty \Gamma_B$, where $\hat{\beta}_n = \hat{\eta}_n \beta_n \eta_n^{-1}$. Note that $(\hat{\omega}_{2n})$, where $\hat{\omega}_{2n} = \omega_{2n}\eta_n^{-1}$, is still alternating[7] and, hence, by Corollary 4.11, is a τ_{mod}-regular sequence. Passing to a subsequence, $(\hat{\beta}_{n_k})$ is (i) either a constant sequence, $\hat{\beta} \in \Gamma_B \setminus \mathrm{H}$, (ii) or

$$\hat{\beta}_{n_k}^{\pm 1} \xrightarrow{\mathrm{flag}} \tau_\pm \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_B) \setminus \Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}), \quad \text{as } k \to \infty. \tag{4.3}$$

---

[6]This follows by the assumption that Γ_B is τ_{mod}-Anosov and $(\hat{\beta}_{F(n)})$ is a divergent sequence in Γ_b.

[7]Indeed, the corresponding sequences in Γ_A and Γ_B for $(\hat{\omega}_n)$ can be taken to be $(\eta_{n-1}\alpha_n)$ and $(\beta_n\eta_n^{-1})$, respectively.

If (i) holds, then notice that $\hat{\omega}_{2n_k}^{-1} B \subset \hat{\beta}^{-1} A$. In this case, by Lemma 3.7, all the τ_{mod}-limit points of $(\hat{\omega}_{2n_k}^{-1})$ must lie in $\hat{\beta}^{-1} A \subset B^\circ$. By Assumption 4.2, $\hat{\beta}^{-1} A$ is antipodal to A. If the (ii) holds, then we observe that $\hat{\omega}_{2n_k}^{-1}(B) \subset \hat{\beta}_{n_k}^{-1}(A)$. Then, the sequence $(\hat{\beta}_{n_k}^{-1} A)$, and hence $(\hat{\omega}_{2n_k}^{-1} B)$, converges to the τ_{mod}-limit point τ_- (see (4.3)) of the sequence $\hat{\beta}_{n_k}^{-1}$. By the antipodality assumption (ii) in the hypothesis of Theorem A, τ_- is antipodal to A.

Let (η_n) be a sequence in H as in the preceding paragraph. Let $\hat{\omega}_{2n} = \omega_{2n} \eta_n^{-1}$. By Corollary 4.11, $(\hat{\omega}_{2n})$ is τ_{mod}-regular (cf. the preceding paragraph). By above, we can (and will) extract a subsequence $(\hat{\omega}_{2k_n})$ of $(\hat{\omega}_{2n})$, such that the inverse sequence $(\hat{\omega}_{2k_n}^{-1})$ τ_{mod}-converges to some point τ_-, which is antipodal to A. After further extraction, we may assume that $(\hat{\omega}_{2k_n})$ τ_{mod}-converges to some point $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$. Since $A \subset C(\tau_-)$ is compact, the sequence of subsets $(\hat{\omega}_{2k_n} A)$ converges to τ. However, for all $n \in \mathbf{N}$, $\hat{\omega}_{2n} A = \hat{\omega}_{2n} \eta_n A = \omega_{2n} A$. So, $\omega_{2k_n} A \to \tau$.

Thus, the nestedness of the subsets $(\omega_{2n} A)$ implies $\omega_{2n} A \to \tau$. Then, we also obtain $\omega_{2n-1}(B) \to \tau$ by using $\omega_{2n+1} B \subset \omega_n A$. This completes the proof. $\qquad\square$

4.2. **The boundary map.** Recall from §2.7 that there is a Γ-invariant decomposition of $\partial_\infty \Gamma$ as the disjoint union $\partial_{\mathrm{I}} \Gamma \sqcup \partial_{\mathrm{II}} \Gamma$, where $\partial_{\mathrm{II}} \Gamma$ admits an equivariant continuous bijection to the Gromov boundary of the Bass-Serre tree T associated with the amalgamated free product $\Gamma = \Gamma_A \star_{\mathrm{H}} \Gamma_B$.

In this subsection, we construct a Γ-equivariant *boundary map* from the Gromov boundary $\partial_\infty \Gamma$ of $\Gamma = \Gamma_A \star_{\mathrm{H}} \Gamma_B$ to $\mathrm{Flag}(\tau_{\mathrm{mod}})$,

$$\xi : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}}). \tag{4.4}$$

We define this map ξ separately on $\partial_{\mathrm{I}} \Gamma$ and $\partial_{\mathrm{II}} \Gamma$; see §4.2.1 for the definition of $\xi|_{\partial_{\mathrm{I}} \Gamma}$ and §4.2.2 for the definition of $\xi|_{\partial_{\mathrm{II}} \Gamma}$.

In §4.2.4, we verify that ξ is an antipodal map. Finally, in §4.2.3, we show that the map ξ is dynamics preserving.

4.2.1. *Definition of the boundary map for type I points.* For $\varepsilon \in \partial_{\mathrm{I}} \Gamma$, pick $\gamma \in \Gamma$ such that $\gamma^{-1} \varepsilon \in \partial_\infty \Gamma_A \cup \partial_\infty \Gamma_B$. If $\gamma^{-1} \varepsilon \in \partial_\infty \Gamma_A$ (resp. $\gamma^{-1} \varepsilon \in \partial_\infty \Gamma_B$), then define

$$\xi(\varepsilon) := \gamma \xi_A(\gamma^{-1} \varepsilon), \quad (\text{resp. } \xi(\varepsilon) := \gamma \xi_B(\gamma^{-1} \varepsilon)). \tag{4.5}$$

We first show that the map $\xi : \partial_{\mathrm{I}} \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ is well-defined.

Lemma 4.12. *For $\gamma \in \Gamma$, if $\gamma(\partial_\infty \Gamma_A) \cap \partial_\infty \Gamma_A \neq \emptyset$, then $\gamma \in \Gamma_A$. The same conclusion holds when A is replaced by B.*

Proof. Note that $\gamma(\partial_\infty \Gamma_A) = \partial_\infty(\gamma \Gamma_A \gamma^{-1})$. By (2.7), the nonemptyness of $\gamma(\partial_\infty \Gamma_A) \cap \partial_\infty \Gamma_A$ implies that $d_T(\gamma \Gamma_A, \Gamma_A) \leq 1$. Thus, $\gamma \in \Gamma_A$. $\qquad\square$

By Lemma 4.12, the element γ in the definition of ξ is unique up to the right multiplication by elements of Γ_A (resp. Γ_B). Since the maps ξ_A, ξ_B are equivariant for Γ_A, Γ_B, respectively, it follows that the definition of $\xi(\varepsilon)$ in (4.5) does not depend on the choice of $\gamma \in \Gamma$.

Finally, note that, by definition, we have the following result:

Lemma 4.13. *The map $\xi : \partial_{\mathrm{I}} \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ in (4.5) is Γ-equivariant.*

4.2.2. *Definition of the boundary map for type II points.* For $\varepsilon \in \partial_{\mathrm{II}}\Gamma$, we choose an alternating sequence (ω_n) given by Proposition 2.13 such that $\omega_n \xrightarrow{\mathrm{Cay}} \varepsilon$. If (ω_n) is of type A, resp. type B, then define

$$\xi(\varepsilon) := \bigcap_{n \in \mathbf{N}} \omega_{2n}A, \quad \text{resp. } \xi(\varepsilon) := \bigcap_{n \in \mathbf{N}} \omega_{2n}B. \tag{4.6}$$

(cf. Proposition 4.7).

A consequence of the following result is that ξ is well-defined on $\partial_{\mathrm{II}}\Gamma$:

Lemma 4.14. *For $\varepsilon \in \partial_{\mathrm{II}}\Gamma$ and a sequence (γ_n) in Γ, if $\gamma_n \xrightarrow{\mathrm{Cay}} \varepsilon$, then $\gamma_n \xrightarrow{\mathrm{flag}} \xi(\varepsilon)$.*

Proof. Let (ω_n) be an alternating sequence as above, which we suppose to be of type A (the type B case can be dealt with similarly). Then, by Proposition 2.13, there exists a function $F : \mathbf{N} \to \mathbf{N}$ diverging to infinity and $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$, we may (and will) choose a normal form of γ_n containing $\omega_{F(n)}$ as a left subword of that form. Furthermore, we may also assume that the function F takes only even values.

We split $(\gamma_n)_{n \geq n_0}$ into two subsequences: The first subsequence (γ_{k_n}) contains all elements of $(\gamma_n)_{n \geq n_0}$ with the rightmost letter contained in Γ_A, and the complementary subsequence (γ_{l_n}) includes all elements of $(\gamma_n)_{n \geq n_0}$ with rightmost letter contained in Γ_B. Notice that for all $n \in \mathbf{N}$,

$$\gamma_{k_n}B \subset \omega_{F(k_n)}A \quad \text{and} \quad \gamma_{l_n}A \subset \omega_{F(l_n)}A.$$

Since $\omega_{2n}A \to \xi(\varepsilon)$, we obtain that both sequences $(\gamma_{k_n}B)$ and $(\gamma_{l_n}A)$ of subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ converge to $\xi(\varepsilon)$. Therefore, by Lemma 3.4, $\gamma_{k_n} \xrightarrow{\mathrm{flag}} \xi(\varepsilon)$ and $\gamma_{l_n} \xrightarrow{\mathrm{flag}} \xi(\varepsilon)$, yielding the conclusion. $\square$

Together with Corollary 4.8, we obtain:

Corollary 4.15. *The map in (4.6) is Γ-equivariant.*

4.2.3. *The boundary map preserves convergence dynamics.* The following result is an analog of Lemma 4.14 for type I boundary points.

Lemma 4.16. *Let $\varepsilon \in \partial_{\mathrm{I}}\Gamma$ and let (γ_n) be a sequence in Γ. If $\gamma_n \xrightarrow{\mathrm{Cay}} \varepsilon$, then $\gamma_n \xrightarrow{\mathrm{flag}} \xi(\varepsilon)$.*

Proof. Using the equivariance of the Γ-action, it will be enough to prove the result when $\varepsilon \in \partial_\infty\Gamma_A \cup \partial_\infty\Gamma_B$. We argue by contradiction:

If the assertion is false, then there exists $\varepsilon \in \partial_\infty\Gamma_A \cup \partial_\infty\Gamma_B$ and a sequence (γ_n) in Γ such that

$$\gamma_n \xrightarrow{\mathrm{Cay}} \varepsilon, \text{ but } \xi(\varepsilon) \text{ is not a } \tau_{\mathrm{mod}}\text{-accumulation point of } (\gamma_n). \tag{4.7}$$

By Lemma 2.18, let us choose D-normal forms for each element of (γ_n), for some $D \geq 0$. After extraction, we may assume that the leftmost and rightmost letters of those forms come from the same group, say Γ_A and Γ_*, respectively, where $*$ is either A or B; let (α_n) be the sequence of those leftmost letters. We also assume that $* = A$; the other choice can be similarly analyzed.

Clearly, (α_n^{-1}) cannot diverge away from H: Otherwise, since (α_n) fellow travels (γ_n), $\alpha_n \xrightarrow{\mathrm{Cay}} \varepsilon$ and, if (α_n^{-1}) diverges away from H, then

$$\gamma_n B \subset \alpha_n B \to \tau,$$

where τ must be $\xi(\varepsilon)$. But, in this case, Lemma 3.4 shows that $\gamma_n \xrightarrow{\text{flag}} \tau$, a disagreement with (4.7).

Therefore, after passing to a subsequence, we may assume that the elements of (γ_n) have normal forms with a common leftmost letter α_1. Repeating the same argument to the sequence $(\alpha_1^{-1}\gamma_n)$ yields another subsequence whose elements have normal forms with a common leftmost letter β_1. Thus, the original sequence (γ_n) has a subsequence whose elements have normal forms with two leftmost common letters. Proceeding inductively, for every $l \in \mathbf{N}$, we can find a subsequence (γ_n') of the original sequence (γ_n) such that the elements of (γ_n') have normal forms with a common leftmost subword of length at least l. For $l = 3$, Proposition 2.14 shows that (γ_n') has bounded nearest-point projections to Γ_A and Γ_B. Thus, (γ_n') cannot have any accumulation points in the boundary of Γ_A and Γ_B. This contradicts our initial assumption that $\varepsilon \in \partial_\infty \Gamma_A \cup \partial_\infty \Gamma_B$. $\qquad\square$

Combining Lemmas 4.14 and 4.16, we obtain the following:

Corollary 4.17. *The map $\xi : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ preserves convergence dynamics.*

It follows that Γ is τ_{mod}-regular; cf. Remark 3.11.

4.2.4. *The boundary map is antipodal.*

Proposition 4.18. *The map $\xi : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ in (4.4) (obtained by combining (4.5) and (4.6)) is* antipodal: *That is for every pair of distinct points $\varepsilon_\pm \in \partial_\infty \Gamma$, the points $\tau_\pm := \xi(\varepsilon_\pm) \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ are antipodal to each other.*

We recall that *the action $G \curvearrowright G/P$ preserves antipodality*: That is, if $\hat{\tau}_\pm \in G/P$ is an antipodal pair, then, for all $g \in G$, $g\hat{\tau}_\pm$ is also an antipodal pair.

Lemma 4.19. *Let v, w be any vertices in the Bass-Serre tree T such that $d_T(v, w) \geq 2$. Then, $\xi(\partial_\infty \Gamma_v) = \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_v)$ is antipodal to $\xi(\partial_\infty \Gamma_w) = \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_w)$.*

Proof. By by equivariance, it is enough to assume that $w = \Gamma_A$ or $w = \Gamma_B$. We assume the former, i.e., $w = \Gamma_A$; the possibility of $w = \Gamma_B$ can be analyzed similarly. Since $d_T(v, w) \geq 2$, we have that $\partial_\infty \Gamma_A \cap \partial_\infty \Gamma_v = \emptyset$ (see (2.7)).

Suppose first that v is a coset of Γ_B; so let $\gamma \in \Gamma$ be any element such that $\gamma\Gamma_B = v$. It is easy to see that such an element γ must satisfy $\mathrm{rl}(\gamma) \geq 2$. We may also choose γ so that it has a normal form whose rightmost letter lies in $\Gamma_A \setminus \mathrm{H}$. If the leftmost letter α of that normal form also lies in $\Gamma_A \setminus \mathrm{H}$, then $\mathrm{rl}(\gamma) \geq 3$, and $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_{\alpha^{-1}v}) \subset \beta\alpha' A \subset B^\circ$, where β and α' are the second and third letters from the left in that normal form of γ, respectively. Since A and B° are antipodal to each other (see Assumption 4.2), and $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) \subset A$, we have that $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A)$ and $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_{\alpha^{-1}v})$ are antipodal to each other, hence so is the pair $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) = \alpha\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A)$ and $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_v) = \alpha\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_{\alpha^{-1}v})$. If the leftmost letter of the normal form of γ is some element $\beta \in \Gamma_B \setminus \mathrm{H}$, then, by a similar argument, it follows that $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_v) \subset B^\circ$, which is antipodal to $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) \subset A$.

Suppose now that v is a coset of Γ_A. In this case, we can choose an element $\gamma \in \Gamma$ such that $\gamma\Gamma_A = v$, $\mathrm{rl}(\gamma) \geq 1$, and the rightmost letter of a normal form of γ is an element of $\Gamma_B \setminus \mathrm{H}$. Adapting a similar argument as above, the result follows in this case as well. $\qquad\square$

Proof of Proposition 4.18. Combining the following cases, it would follow that the map $\xi :$ $\partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ is antipodal. Recall that ξ is Γ-equivariant (by Lemma 4.13 and Corollary 4.15).

Case 1. *Suppose that both points $\varepsilon_\pm \in \partial_\infty \Gamma$ are of type I.* Since ξ is Γ-equivariant, it is enough to assume that $\varepsilon_- \in \partial_\infty \Gamma_A \cup \partial_\infty \Gamma_B$. Let us also assume that $\varepsilon_- \in \partial_\infty \Gamma_A$; the case $\varepsilon_- \in \partial_\infty \Gamma_B$ can be treated similarly.

If $\varepsilon_+ \in \partial_\infty \Gamma_A \cup (\Gamma_A(\partial_\infty \Gamma_B))$, then finding a suitable element $\alpha \in \Gamma_A$, we obtain that $\alpha \varepsilon_+ \in \partial_\infty \Gamma_A \cup \partial_\infty \Gamma_B$. Since, by the hypothesis of Theorem A, $\xi(\partial_\infty \Gamma_A \cup \partial_\infty \Gamma_B) \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ is an antipodal subset, $\xi(\alpha \varepsilon_\pm)$ is an antipodal pair and, hence, so is $\xi(\varepsilon_\pm)$.

If $\varepsilon_+ \notin \partial_\infty \Gamma_A \cup (\Gamma_A(\partial_\infty \Gamma_B))$, then ε_+ lies in the boundary of a vertex group Γ_v of the Bass-Serre tree T such that $d_T(\Gamma_A, v) \geq 2$. By Lemma 4.19, $\xi(\varepsilon_\pm)$ are antipodal.

Case 2. *Suppose that both points $\varepsilon_\pm \in \partial_\infty \Gamma$ are of type II.* Consider a pair of alternating sequences $(\omega_n^\pm)$ converging (in $\overline{\Gamma}$) to $\varepsilon_\pm$ (see Lemma 2.16).

Lemma 4.20. *There exists $n \in \mathbf{N}$ such that $\omega_n^+ \notin \omega_n^- \mathrm{H}$.*

Proof. If this is false, then for all $n \in \mathbf{N}$, $\omega_n^+ \in \omega_n^- \mathrm{H}$. Consider the sequence $(\hat{\omega}_n^{-1})$, where $\hat{\omega}_n := \omega_n^- \mathrm{pr}_\mathrm{H}((\omega_n^-)^{-1})$. By Lemma 2.9, $(\hat{\omega}_n^{-1})$ has no accumulation points in $\partial_\infty \mathrm{H}$. Moreover, (ω_n^-) diverges away from H (cf. Proposition 2.14). Thus, $(\hat{\omega}_n(\partial_\infty \mathrm{H}))$ converges to ε_-, see Lemma 2.9(ii). It follows that the sequence of uniformly quasiconvex subsets $(\hat{\omega}_n(\mathrm{H}))$ of Γ converges to ε_-. Since we have assumed that $\omega_n^+ \in \omega_n^- \mathrm{H} = \hat{\omega}_n \mathrm{H}$, we obtain that $\omega_n^+ \xrightarrow{\mathrm{Cay}} \varepsilon_-$, which shows that $\varepsilon_+ = \varepsilon_-$. This is a contradiction. $\qquad\square$

Let $n_0 \in \mathbf{N}$ be the smallest number such that $\omega_{n_0}^+ \notin \omega_{n_0}^- \mathrm{H}$. Consider the alternating sequences $((\omega_{n_0}^+)^{-1} \omega_n^+)$ and $((\omega_{n_0}^+)^{-1} \omega_n^-)$ converging to $(\omega_{n_0}^+)^{-1} \varepsilon_+$ and $(\omega_{n_0}^+)^{-1} \varepsilon_-$, respectively. By choice of n_0, it is evident that the first element of those sequences lie in different groups Γ_A and Γ_B. Therefore, one of the points $\xi((\omega_{n_0}^+)^{-1} \varepsilon_\pm)$ lies in the interior of A while the other one lies in the interior of B (cf. Proposition 4.7) and thus they are antipodal. Consequently, $\xi(\varepsilon_\pm)$ are antipodal.

Case 3. *Suppose that $\varepsilon_- \in \partial_\infty \Gamma$ is of type I and $\varepsilon_+ \in \partial_\infty \Gamma$ is of type II.* By the same argument as in the third case in [5, §4.2], it follows that $\xi(\varepsilon_\pm)$ are antipodal. $\qquad\square$

4.3. **Proof of Theorem A.** By Proposition 3.13, the subgroup $\Gamma = \langle \Gamma_A, \Gamma_B \rangle$ of G is naturally isomorphic to $\Gamma_A \star_\mathrm{H} \Gamma_B$. We show that Γ is a τ_{mod}-Anosov subgroup or, equivalently, a τ_{mod}-asymptotically embedded subgroup (see Definition 3.10) of G:

   (i) By Corollary 4.6, Γ is hyperbolic.
  (ii) Finally, the boundary map $\xi : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ in Equation (4.4) is Γ-equivariant (by Lemma 4.13 and Corollary 4.15), antipodal (by Proposition 4.18), and preserves convergence dynamics (by Corollary 4.17).

   This concludes the proof of the Theorem A. $\qquad\square$

5. Proof of Theorem B

Throughout this section, we work under the hypothesis of Theorem B.

Assumption 5.1. In the proof of Theorem B, we will replace $(A, B_\pm)$ by $(\mathrm{cl}(A^\circ), \mathrm{cl}(B^\circ_\pm))$, which is again an interactive triple for $(\mathrm{M}; \mathrm{H}_\pm; f)$. In doing so, we may replace the condition (i) in the hypothesis of Theorem B by the following stronger one (cf. Lemma 4.1):

(i)' *The pairs of subsets $(A, B^\circ_\pm)$, $(A^\circ, B_\pm)$ of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ are antipodal. Moreover, B_- is antipodal to B_+.*

We begin the proof of Theorem B with the following observation:

Lemma 5.2. *Under the hypothesis of Theorem B,*

(i) $\Lambda_{\tau_{\mathrm{mod}}}(\langle f \rangle)$ *consists of two points, one of them lies in the interior of B_+ and the other one lies in the interior of B_-, and $\langle f \rangle$ is a cyclic τ_{mod}-Anosov subgroup of G.*

(ii) $\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{M}) \subset A$ *and* $\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}_\pm) \subset A \cap B_\pm$.

Proof. Since, for all $n \in \mathbf{N}$, $f^{n+1}f^{-n}(B_+) = f(B_+) \subset B^\circ_+$ (by the second condition of Definition 3.15), applying Lemma 3.6, it follows that $\langle f \rangle$ is τ_{mod}-regular. Since, for all $n \in \mathbf{N}$, $f^{-n}(f^{-1}B_-) \subset f^{-1}B_- \subset B^\circ_-$, all the τ_{mod}-limit points of the sequence $(f^{-n})_{n\in\mathbf{N}}$ lie in B°_-. Since B_+ is antipodal to B°_- (see Assumption 5.1), by Lemma 3.9, $(f^n B_+)_{n\in\mathbf{N}}$ shrinks. Therefore, since the sequence $(f^n B_+)_{n\in\mathbf{N}}$ is nested, $(f^n B_+)_{n\in\mathbf{N}}$ must converge to some point $\tau_+ \in B^\circ_+$. Similarly, the nested sequence $(f^{-n}B_-)_{n\in\mathbf{N}}$ of compact subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ converges to some point $\tau_- \in B^\circ_-$. In particular, the limit set of $\langle f \rangle$ is $\{\tau_\pm\}$, is antipodal, and has cardinality two. That $\langle f \rangle$ is τ_{mod}-Anosov follows from [18, Lemma 5.38]. This proves (i).

Proof (ii) is similar to that of Lemma 4.3. Hence, we omit the details. $\square$

Remark 5.3. Suppose that it is known that $\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{M}) \cap \partial A = \Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}_+) \cup \Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}_-)$. Then, by Lemma 5.2(ii), the subset $\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{M}) \setminus \Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}_+)$ lies in $A^\circ \cup B_-$ and, hence, it is antipodal to B_+; see Assumption 5.1. Similarly, $\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{M}) \setminus \Lambda_{\tau_{\mathrm{mod}}}(\mathrm{H}_-)$ is antipodal to B_-. Thus, in this situation, the second condition (ii) of Theorem B is automatically satisfied.

Corollary 5.4. *Under the hypothesis of Theorem B, the subgroups $\mathrm{H}_\pm$ are weakly malnormal in M, and, for all $\mu \in \mathrm{M}$, the intersection $\mathrm{H}_- \cap \mu \mathrm{H}_+ \mu^{-1}$ is finite.*

Proof. The proof is similar to the one of Corollary 4.5; we omit the details. $\square$

Corollary 5.5. *Under the hypothesis of Theorem B, the subgroup Γ of G generated by M and f in G is hyperbolic.*

Proof. Arguing similarly to the first paragraph of the proof of Corollary 4.6, it follows that $\mathrm{H}_\pm$ are both quasiconvex in M. Then, the claim follows from Theorem 2.11(ii), Proposition 3.16, and Corollary 5.4. Compare this with the second paragraph of the proof of Corollary 4.6. $\square$

For convenience, we introduce the following notation, which is frequently used in this section.

Notation 5.6. If $\epsilon = 1$, then H_ϵ will denote H_+ and B_ϵ will denote B_+. Similarly, if $\epsilon = -1$, then H_ϵ will denote H_- and B_ϵ will denote B_-.

5.1. **Alternating sequences.** Recall the notion of alternating sequences from Definition 2.2. The main result of this subsection is as follows:

Proposition 5.7. *Let (ω_n) be an alternating sequence in $\Gamma = \mathrm{M} \star_\phi$ in the normal forms given by* (2.5):

$$\omega_n = \mu_0 f^{\epsilon_1} \mu_1 f^{\epsilon_2} \mu_2 \cdots f^{\epsilon_{n-1}} \mu_{n-1} f^{\epsilon_n}. \tag{5.1}$$

Then, the nested sequence of compact subsets $(\omega_n(A \cup B_{\epsilon_n}))_n$ of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ converges to a point.

We remind our reader that we are using the notation introduced in Notation 5.6.

By the above proposition, it follows that an alternating sequence $\underline{\omega} = (\omega_n)$ in Γ τ_{mod}-converges to the point

$$\tau_{\underline{\omega}} := \bigcap_{n \in \mathbf{N}} \omega_n(A \cup B_{\epsilon_n}). \tag{5.2}$$

As a corollary, we obtain:

Corollary 5.8. *If $\underline{\omega} = (\omega_n)$ is an alternating sequence in Γ, then the sequence $(\omega_n A)$ of compact subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ converges to the τ_{mod}-limit point $\tau_{\underline{\omega}}$ of (ω_n).*

However, we remark that the sequence $(\omega_n A)$ is possibly not nested. For example, consider an alternating sequence (ω_n) given by $\omega_n = (\eta f)^n$ for $n \in \mathbf{N}$, where $\eta \in \mathrm{H}_+$. Then, for all $m \geq n+2$, we have $\omega_m(A) \cap \omega_n(A) = (\eta f)^n(((\eta f)^{m-n}A) \cap A) \subset f(\eta f)^n(B^\circ \cap A) = \emptyset$.

Corollary 5.9. *If $\underline{\omega} = (\omega_n)$ is an alternating sequence in Γ, then, for all $\gamma \in \Gamma$,*

$$\gamma\omega_n \xrightarrow{\mathrm{flag}} \gamma\tau_{\underline{\omega}}, \quad \text{as } n \to \infty.$$

Proof. By Corollary 5.8, it follows that $(\gamma\omega_n A)_n$ converges to $\gamma\tau_{\underline{\omega}}$. Then, Lemma 3.4 yields the conclusion. $\square$

5.1.1. *Regularity.*

Definition 5.10 (Special sequences)**.** A sequence $(\tilde{\gamma}_n)$ in $\Gamma = \mathrm{M} \star_\phi$ is called *special* if there exist sequences (ϵ_n) in $\{\pm 1\}$, (μ_n) in M satisfying

$$\epsilon_n = 1, \mu_n \in \mathrm{H}_+ \implies \epsilon_{n+1} = 1 \quad \text{and} \quad \epsilon_n = -1, \mu_n \in \mathrm{H}_- \implies \epsilon_{n+1} = -1,$$

and an element $\mu_0 \in \mathrm{M}$, and an increasing function $F : \mathbf{N} \to \mathbf{N}$ such that for all $n \in \mathbf{N}$,

$$\tilde{\gamma}_n = f^{\epsilon_{F(n)}} \mu_{F(n)-1} f^{\epsilon_{F(n)-1}} \cdots \mu_1 f^{\epsilon_1} \mu_0. \tag{5.3}$$

Note that special sequences are subsequences of the inverse sequence of some alternating sequence in Γ. Compare this with Definition 2.2.

Lemma 5.11. *Special sequences in Γ are τ_{mod}-regular.*

Proof. Let $(\tilde{\gamma}_n)$ be a special sequence as above. We will assume that $\mu_0 = 1_{\mathrm{M}}$; such a change is a bounded perturbation of the original sequence. Hence the property of being τ_{mod}-regular remains unaffected. To show that $(\tilde{\gamma}_n)$ is τ_{mod}-regular, it would be enough to show that every subsequence of $(\tilde{\gamma}_n)$ contains a τ_{mod}-regular subsequence.

So, consider any subsequence of $(\tilde{\gamma}_n)$, again denoted by $(\tilde{\gamma}_n)$. After extraction,[8] we may assume that $\epsilon_{F(n)-1}$ are all the same, say

$$\epsilon_{F(n)-1} = 1, \quad \text{for all } n \in \mathbf{N}.$$

---

[8]Note that, by definition, subsequences of special sequences are special.

Case 1. *Suppose that after passing to a subsequence, it holds for all $n \in \mathbf{N}$ that $\mu_{F(n)-1} \in H_+$.*
Passing to another subsequence, we will also assume that $F(n+1) - F(n) \geq 10$, for all $n \in \mathbf{N}$.
Since $\mu_{F(n)-1} \in H_+$ and $\epsilon_{F(n)-1} = 1$, we have

$$\epsilon_{F(n)} = 1, \quad \text{for all } n \in \mathbf{N}.$$

(see Definition 5.10). So,

$$\begin{aligned}
\tilde{\gamma}_n &= f\mu_{F(n)-1}f\mu_{F(n)-2}f^{\epsilon_{F(n)-2}}\cdots\mu_1 f^{\epsilon_1} \\
&= ff(f^{-1}\mu_{F(n)-1}f\mu_{F(n)-2})f^{\epsilon_{F(n)-2}}\cdots\mu_1 f^{\epsilon_1}.
\end{aligned} \tag{5.4}$$

Moreover,

$$\tilde{\gamma}_{n+1}\tilde{\gamma}_n^{-1} = ff(f^{-1}\mu_{F(n)-1}f\mu_{F(n)-2})f^{\epsilon_{F(n)-1}}\cdots f^{\epsilon_{F(n-1)+1}}\mu_{F(n-1)}.$$

Observe that if $\epsilon_{F(n-1)+1} = -1$, then $\mu_{F(n-1)} \notin H_+$, since we observed in the preceding paragraph that $\epsilon_{F(n-1)} = 1$. Else, $\epsilon_{F(n-1)+1} = 1$. Consequently, in both cases (i.e., $\epsilon_{F(n-1)+1} = 1$ or -1), it holds that

$$\tilde{\gamma}_{n+1}\tilde{\gamma}_n^{-1}(B_+) \subset f(B_+) \subset B_+^\circ.$$

Since the above is true for all $n \in \mathbf{N}$, Lemma 3.6 applies to show that $(\tilde{\gamma}_n)$ is τ_{mod}-regular.

Case 2. *Suppose that, after passing to a subsequence, it holds for all $n \in \mathbf{N}$ that $\mu_{F(n)-1} \notin H_+$.*
Consider the sequence $(\hat{\gamma}_n)$, where

$$\hat{\gamma}_n := \mu_{F(n)-1}f^{\epsilon_{F(n)-1}}\cdots\mu_1 f^{\epsilon_1} = f^{-\epsilon_{F(n)}}\tilde{\gamma}_n.$$

We show that $(\hat{\gamma}_n)$ is τ_{mod}-regular since this would imply that $(\tilde{\gamma}_n)$ is τ_{mod}-regular.
For all $n \in \mathbf{N}$,

$$\hat{\gamma}_n A \subset \mu_{F(n)-1}B_+ \subset A. \tag{5.5}$$

If the sequence $(\mu_{F(n)-1}^{-1})$ remains at a bounded distance away from H_+, then after further extraction, we may assume that $(\mu_{F(n)-1})$ lies in a single coset μH_+, for some $\mu \notin H_+$. So, it holds that

$$\hat{\gamma}_{n+1}\hat{\gamma}_n^{-1}(A) \subset \mu B_+ \subset A^\circ,$$

for all $n \in \mathbf{N}$. With Lemma 3.6, the above implies that $(\hat{\gamma}_n)$ is τ_{mod}-regular.

Otherwise, after further extraction, $(\mu_{F(n)-1}^{-1})$ diverges away from H_+. Consider the sequence $(\hat{\mu}_{F(n)-1})$, where $\hat{\mu}_{F(n)-1} := \mu_{F(n)-1}\mathrm{pr}_{H_+}(\mu_{F(n)-1}^{-1})$. By Lemma 2.9, $(\hat{\mu}_{F(n)-1}^{-1})$ accumulates in $\partial_\infty M \setminus \partial_\infty H_+$. Thus, all τ_{mod}-flag accumulation points of $(\hat{\mu}_{F(n)-1}^{-1})$ are antipodal to B_+ (see the second condition in the hypothesis of Theorem B). By Lemma 3.9, the sequence $(\hat{\mu}_{F(n)-1}B_+)$ of compact subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ shrinks. Since $\hat{\mu}_{F(n)-1}B_+ = \mu_{F(n)-1}B_+$, $(\mu_{F(n)-1}B_+)$ shrinks. Consequently, it follows from (5.5) that $(\hat{\gamma}_n A)$ shrinks. By Lemma 3.5, $(\hat{\gamma}_n)$ is τ_{mod}-regular. Hence, $(\tilde{\gamma}_n)$ is τ_{mod}-regular. $\qquad\square$

By definition, the inverse sequence corresponding to an alternating sequence is special so that Lemma 5.11 directly implies:

Corollary 5.12. *Alternating sequences in $\Gamma = M\star_\phi$ are τ_{mod}-regular.*

5.1.2. *Proof of Proposition 5.7.* It is a straightforward verification that for all $n \in \mathbf{N}$, $\mu_n(B_{\epsilon_{n+1}}) \subset A \cup B_{\epsilon_n}$, yielding that

$$\omega_{n+1}(A \cup B_{\epsilon_{n+1}}) = \omega_n \mu_n f^{\epsilon_{n+1}}(A \cup B_{\epsilon_{n+1}}) \subset \omega_n \mu_n B_{\epsilon_{n+1}} \subset \omega_n(A \cup B_{\epsilon_n}). \tag{5.6}$$

Therefore, the sequence $(\omega_n(A \cup B_{\epsilon_n}))_n$ is nested.

Thus, to prove that $(\omega_n(A \cup B_{\epsilon_n}))_n$ converges to a point, it will be enough to show that this sequence contains a subsequence that shrinks. This is what we show.

Case 1. *Suppose that for infinitely many $n \in \mathbf{N}$, $\mu_{n-1} \in \mathrm{H}_{\epsilon_n}$.* Let $\mathbf{P} \subset \mathbf{N}$ denote an infinite subset such that for all $n \in \mathbf{P}$, $\mu_{n-1} \in \mathrm{H}_{\epsilon_n}$ and, for all $n \in \mathbf{P}$, ϵ_n are the same, say $\epsilon_n = 1$. Therefore, for all $n \in \mathbf{P}$, $\epsilon_{n-1} = 1$ (see Definition 2.2).

Let us first show that $(\omega_{n-1} B_+)_{n \in \mathbf{P}}$ shrinks: We observe that for $n \in \mathbf{P}$,

$$\omega_{n-1}^{-1}(\mu_0 A) \subset B_-,$$

which shows that all the τ_{mod}-limit points of $(\omega_{n-1}^{-1})_{n \in \mathbf{P}}$ lie in B_- (see Lemma 3.7). By Corollary 5.12, we know that $(\omega_{n-1})_{n \in \mathbf{P}}$ is τ_{mod}-regular. Since B_+ is antipodal to B_-, by Lemma 3.5 $(\omega_{n-1} B_+)_{n \in \mathbf{P}}$ shrinks.

Since, for all $n \in \mathbf{P}$, $\epsilon_n = 1$, we get that $B_{\epsilon_n} = B_+$. Moreover, since $\mu_{n-1} \in \mathrm{H}_+$,

$$\omega_n(A \cup B_{\epsilon_n}) = \omega_{n-1} \mu_{n-1} f(A \cup B_+) \subset \omega_{n-1} \mu_{n-1} B_+ = \omega_{n-1} B_+.$$

Therefore, by the previous paragraph, $(\omega_n A)_{n \in \mathbf{P}}$ shrinks.

Therefore, we may assume now the complementary case to the above one, which is that for at most finitely many $n \in \mathbf{N}$, $\mu_{n-1} \in \mathrm{H}_{\epsilon_n}$. In fact, it would be enough to assume:

Case 2. *For all $n \in \mathbf{N}$, $\mu_{n-1} \notin \mathrm{H}_{\epsilon_n}$.* Suppose that for infinitely many $n \in \mathbf{N}$, $\epsilon_n = 1$ (the case $\epsilon_n = -1$ can be dealt with in a similar way). Let $\mathbf{P} \subset \mathbf{N}$ be a subset such that for all distinct $n, n' \in \mathbf{P}$, $|n - n'| \geq 10$, and $\epsilon_n = 1$, for all $n \in \mathbf{P}$. For $n \in \mathbf{P}$, let us consider the normal form of ω_{n+1} given by

$$\omega_{n+1} = \mu_0 f^{\epsilon_1} \mu_1 \cdots f^{\epsilon_{n-2}} \mu_{n-2} f^{\epsilon_{n-1}} \hat{\mu}_{n-1} f \hat{\mu}_n f^{\epsilon_{n+1}}, \tag{5.7}$$

where

$$\hat{\mu}_{n-1} = \mu_{n-1} \mathrm{pr}_{\mathrm{H}_+}(\mu_{n-1}^{-1}) \quad \text{and} \quad \hat{\mu}_n = f^{-1}(\mathrm{pr}_{\mathrm{H}_+}(\mu_{n-1}^{-1}))^{-1} f \mu_n. \tag{5.8}$$

We observe that since $\mu_{n-1} \notin \mathrm{H}_+$, $\hat{\mu}_{n-1}$ is also not an element of H_+, for all $n \in \mathbf{P}$.

For $n \in \mathbf{P}$, let

$$\hat{\omega}_n := \mu_0 f^{\epsilon_1} \mu_1 \cdots f^{\epsilon_{n-2}} \mu_{n-2} f^{\epsilon_{n-1}} \hat{\mu}_{n-1} f = \omega_{n+1}(\hat{\mu}_n f^{\epsilon_{n+1}})^{-1}. \tag{5.9}$$

Since $(\hat{\omega}_n)_{n \in \mathbf{P}}$ is a subsequence of an alternating sequence, by Corollary 5.12, it is τ_{mod}-regular. One directly checks (cf. (5.6)),

$$\omega_{n+1}(A \cup B_{\epsilon_{n+1}}) \subset \hat{\omega}_n A, \quad \text{for all } n \in \mathbf{P}. \tag{5.10}$$

After extraction, we may assume that for all $n \in \mathbf{P}$, ϵ_{n-1} is constant, $\epsilon = \pm 1$. After another extraction, we also assume that $(\hat{\mu}_{n-1}^{-1})_{n \in \mathbf{P}}$ either (i) diverges away from $\mathrm{H}_{-\epsilon}$, or (ii) remains in a fixed coset $\hat{\mu} \mathrm{H}_{-\epsilon}$, for some $\hat{\mu} \in \mathrm{M}$. So, for $n \in \mathbf{P}$,

$$f \hat{\omega}_n^{-1}(\mu_0 A) = \hat{\mu}_{n-1}^{-1} f^{-\epsilon} \mu_{n-2}^{-1} \cdots f^{-\epsilon_1} \mu_0^{-1}(\mu_0 A).$$

In the first case (i), since we are assuming that $(\hat{\mu}_{n-1})_{n\in\mathbf{P}}$ diverges away from $H_{-\epsilon}$, by Lemma 2.8, it follows that $(\tilde{\mu}_{n-1})_{n\in\mathbf{P}}$ is τ_{mod}-regular and its τ_{mod}-limit points lie only in $\Lambda_{\tau_{\mathrm{mod}}}(M)\setminus\Lambda_{\tau_{\mathrm{mod}}}(H_{-\epsilon})$, where $\tilde{\mu}_{n-1} := \mathrm{pr}_{H_{-\epsilon}}(\hat{\mu}_{n-1})^{-1}\hat{\mu}_{n-1}$. Since $\Lambda_{\tau_{\mathrm{mod}}}(M)\setminus\Lambda_{\tau_{\mathrm{mod}}}(H_{-\epsilon})$ is antipodal to $B_{-\epsilon}$, by Lemma 3.9, $(\tilde{\mu}_{n-1}^{-1}B_{-\epsilon})_{n\in\mathbf{P}} = (\hat{\mu}_{n-1}^{-1}B_{-\epsilon})_{n\in\mathbf{P}}$ shrinks. So, $(f\hat{\omega}_{n-1}^{-1}(\mu_0 A))_{n\in\mathbf{P}}$ shrinks as well. By definition of $\hat{\mu}_{n-1}$ in (5.8), all the τ_{mod}-limit points of $(\hat{\mu}_{n-1}^{-1})_{n\in\mathbf{P}}$ lie in $\Lambda_{\tau_{\mathrm{mod}}}(M) \setminus \Lambda_{\tau_{\mathrm{mod}}}(H_+)$. Thus, after further extraction, we may assume that

$$\hat{\mu}_{n-1}^{-1} \xrightarrow{\mathrm{flag}} \tau_- \in \Lambda_{\tau_{\mathrm{mod}}}(M) \setminus \Lambda_{\tau_{\mathrm{mod}}}(H_+), \quad \text{as } n \to \infty \text{ in } \mathbf{P}.$$

Therefore, it holds that $f\hat{\omega}_{n-1}^{-1}(\mu_0 A) \to \tau_-$, as $n \to \infty$ in $\mathbf{P}$. Thus, by Lemma 3.4 the sequence $(f\hat{\omega}_{n-1}^{-1})_{n\in\mathbf{P}}$ τ_{mod}-flag converges to τ_-. Since τ_- is antipodal to B_+, by Lemma 3.9, $(\hat{\omega}_{n-1}f^{-1}B_+)_{n\in\mathbf{P}}$ shrinks. Since $fA \subset B_+$, it follows that $(\hat{\omega}_{n-1}A)_{n\in\mathbf{P}}$ shrinks. Thus, by (5.10), $(\omega_{n+1}(A \cup B_{\epsilon_{n+1}}))_{n\in\mathbf{P}}$ shrinks.

In the second case (ii), suppose first that $\epsilon = -1$. Therefore, by the assumption of Case 2 and (5.8), we have that $\hat{\mu} \notin H_+$. Thus,

$$f\hat{\omega}_n^{-1}(\mu_0 A) \subset \hat{\mu}_{n-1}^{-1}B_+ \subset \hat{\mu}B_+ \subset A^\circ,$$

for all $n \in \mathbf{P}$. So, in this case, $(f\hat{\omega}_{n-1}^{-1})_{n\in\mathbf{P}}$ has no τ_{mod}-limit points in B_+, since, by above, all of them lie in the interior of A (cf. Lemma 3.7). Therefore, since $(\hat{\omega}_{n-1}f^{-1})_{n\in\mathbf{P}}$ is τ_{mod}-regular,[9] by Lemma 3.9, $(\hat{\omega}_{n-1}f^{-1}B_+)_{n\in\mathbf{P}}$ shrinks. Thus, by (5.10), $(\omega_{n+1}(A \cup B_{\epsilon_{n+1}}))_{n\in\mathbf{P}}$ shrinks.

Still assuming (ii), suppose now that $\epsilon = 1$. If $\hat{\mu} \notin H_-$, then proceeding as in the previous paragraph, it follows that $(\omega_{n+1}(A \cup B_{\epsilon_{n+1}}))_{n\in\mathbf{P}}$ shrinks. Else, we must have $\hat{\mu}_{n-1} \in H_-$, for all $n \in \mathbf{P}$. Observing a different normal form of $\hat{\omega}_n$,

$$\hat{\omega}_n := \mu_0 f^{\epsilon_1}\mu_1 \cdots f^{\epsilon_{n-2}}(\underbrace{\mu_{n-2}f\hat{\mu}_{n-1}f^{-1}}_{\in M})ff.$$

it follows by Case 1 that $(\hat{\omega}_n A)$ shrinks. Thus, by (5.10), $(\omega_{n+1}(A \cup B_{\epsilon_{n+1}}))_{n\in\mathbf{P}}$ shrinks. $\qquad\square$

5.2. The boundary map.
We construct a Γ-equivariant map from the Gromov boundary of $\Gamma = M\star_\phi$ to the flag manifold $\mathrm{Flag}(\tau_{\mathrm{mod}})$:

$$\xi : \partial_\infty\Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}}). \tag{5.11}$$

Recall that $\partial_\infty\Gamma$ decomposes into $\partial_I\Gamma \sqcup \partial_{II}\Gamma$, where $\partial_I\Gamma = \Gamma \cdot (\partial_\infty M)$. As in the case of amalgamated free products (§4.2), we define ξ separately on $\partial_I\Gamma$ and $\partial_{II}\Gamma$; see §5.2.1 for the definition of $\xi|_{\partial_I\Gamma}$ and §5.2.2 for the definition of $\xi|_{\partial_{II}\Gamma}$.

5.2.1. *Definition of the boundary map for type I points.*
For every point $\varepsilon \in \partial_I\Gamma$, we may pick some element $\gamma \in \Gamma$ such that $\gamma^{-1}\varepsilon \in \partial_\infty M$. Since M is τ_{mod}-Anosov, we have a M-equivariant boundary embedding $\xi : \partial_\infty M \to \Lambda_{\tau_{\mathrm{mod}}}(M) \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$. We define

$$\xi(\varepsilon) := \gamma\xi(\gamma^{-1}\varepsilon). \tag{5.12}$$

We check that $\xi(\varepsilon)$ is well-defined, i.e., does not depend on the choice of $\gamma \in \Gamma$ in (5.12): If $\gamma_1 \in \Gamma$ is any other element such that $\gamma_1^{-1}\varepsilon \in \partial_\infty M$, then the intersection $(\gamma^{-1}\gamma_1)\partial_\infty M \cap \partial_\infty M$ is nonempty since $\gamma^{-1}\varepsilon$ is a common point. Thus, in the Bass-Serre tree T, M and $(\gamma^{-1}\gamma_1)M$

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[9]This follows by the observation above that $(\hat{\omega}_{n-1})_{n\in\mathbf{P}}$ is τ_{mod}-regular.

are equal or adjacent vertices, showing that $\mathrm{rl}(\gamma^{-1}\gamma_1) \leq 1$ (see Lemma 2.21). If $\mathrm{rl}(\gamma^{-1}\gamma_1) = 0$, then $\gamma_1 \in \gamma\mathrm{M}$, and, in this case, the well-definedness of (5.12) follows by M-equivariance of $\xi : \partial_\infty\mathrm{M} \to \mathrm{Flag}(\tau_{\mathrm{mod}})$. So, let us assume that $\mathrm{rl}(\gamma^{-1}\gamma_1) = 1$. In this case, $\gamma^{-1}\gamma_1 = \mu_0 f^\epsilon \mu_1$, where $\mu_0, \mu_1 \in \mathrm{M}$ and $|\epsilon| = 1$. Let us also assume that $\epsilon = 1$, since the case $\epsilon = -1$ is similar. So, $\gamma_1 = \gamma\mu_0 f\mu_1$ and $\gamma^{-1}\varepsilon \in \mu_0 f\mu_1(\partial_\infty\mathrm{M}) \cap \partial_\infty\mathrm{M} = \mu_0(\partial_\infty\mathrm{H}_+)$. Hence,

$$\mu_0^{-1}\gamma^{-1}\varepsilon \in \partial_\infty\mathrm{H}_+. \tag{5.13}$$

Since $f \in G$ conjugates H_- and H_+, i.e., $f\mathrm{H}_- f^{-1} = \mathrm{H}_+$, for all $\varepsilon' \in \partial_\infty\mathrm{H}_+$,

$$\xi|_{\partial_\infty\mathrm{H}_-}(f\varepsilon') = f\xi|_{\partial_\infty\mathrm{H}_+}(\varepsilon'). \tag{5.14}$$

Thus,

$$\begin{aligned}
\gamma_1\xi(\gamma_1^{-1}\varepsilon) &= \gamma_1\xi(\mu_1^{-1}f^{-1}\mu_0^{-1}\gamma^{-1}\varepsilon) \\
&= \gamma_1\mu_1^{-1}\xi(f^{-1}\mu_0^{-1}\gamma^{-1}\varepsilon) \\
&= \gamma_1\mu_1^{-1}f^{-1}\xi(\mu_0^{-1}\gamma^{-1}\varepsilon) \\
&= \gamma_1\mu_1^{-1}f^{-1}\mu_0^{-1}\xi(\gamma^{-1}\varepsilon) = \gamma\xi(\gamma^{-1}\varepsilon),
\end{aligned}$$

where the second equality is valid because $(f^{-1}\mu_0^{-1}\gamma^{-1}\varepsilon) \in \partial_\infty\mathrm{M}$, the third equality is verified by (5.13) and (5.14), and the fourth equality is valid because $\gamma^{-1}\varepsilon \in \partial_\infty\mathrm{M}$.

The following result is immediate from the definition of ξ above:

Lemma 5.13. *The map $\xi : \partial_\mathrm{I}\Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ defined by (5.12) is Γ-equivariant.*

5.2.2. *Definition of the boundary map for type II points.* For $\varepsilon \in \partial_{\mathrm{II}}\Gamma$, consider an alternating sequence (see Definition 2.2) $\omega_n \xrightarrow{\mathrm{Cay}} \varepsilon$ given by Proposition 2.13. Define

$$\xi(\varepsilon) := \bigcap_{n \in \mathbf{N}} \omega_n(A \cup B_{\epsilon_n}), \tag{5.15}$$

see Proposition 5.7. The following result shows that $\xi(\varepsilon)$ is well-defined.

Lemma 5.14. *For $\varepsilon \in \partial_{\mathrm{II}}\Gamma$ and for any sequence (γ_n) in Γ, if $\gamma_n \xrightarrow{\mathrm{Cay}} \varepsilon$, then $\gamma_n \xrightarrow{\mathrm{flag}} \xi(\varepsilon)$.*

Proof. The proof is similar to Lemma 4.14. We omit the details. $\qquad\square$

By Corollary 5.9, we have:

Corollary 5.15. *The map $\xi : \partial_{\mathrm{II}}\Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ defined by (5.15) is Γ-equivariant.*

5.2.3. *The boundary map preserves convergence dynamics.* The following result is reminiscent of Lemma 5.14.

Lemma 5.16. *Let $\varepsilon \in \partial_\mathrm{I}\Gamma$ and let (γ_n) be a sequence in Γ. If $\gamma_n \xrightarrow{\mathrm{Cay}} \varepsilon$, then $\gamma_n \xrightarrow{\mathrm{flag}} \xi(\varepsilon)$.*

Proof. Using the action of Γ on $\partial_\mathrm{I}\Gamma$, it will be enough to prove the proposition for $\varepsilon \in \partial_\infty\mathrm{M}$. We argue by contradiction:

If the result is false, then there exists a sequence (γ_n) in Γ and $\varepsilon \in \partial_\infty\mathrm{M}$ such that

$$\gamma_n \xrightarrow{\mathrm{Cay}} \varepsilon, \text{ but } \xi(\varepsilon) \text{ is not a } \tau_{\mathrm{mod}}\text{-accumulation point of } (\gamma_n). \tag{5.16}$$

Equip each γ_n with a D-normal form (see Lemma 2.18),

$$\gamma_n = \mu_{n,0} f^{\epsilon_{n,1}} \mu_{n,1} \cdots f^{\epsilon_{n,l_n}} \mu_{n,l_n}. \tag{5.17}$$

Passing to a subsequence, we may assume that $\epsilon_{n,1}$ are the same, ϵ, for all n. In this situation, $(\mu_{n,0}^{-1})$ cannot diverge away from H_ϵ: Otherwise, after extraction, $\mu_{n,0} \xrightarrow{\text{Cay}} \varepsilon$ and

$$\gamma_n B_{\epsilon_{n,l_n}} \subset \mu_{n,0} B_\epsilon \to \xi(\varepsilon),$$

showing that $\gamma_n \xrightarrow{\text{flag}} \xi(\varepsilon)$. However, since $(\mu_{n,0})$ fellow-travels γ_n, $\gamma_n \xrightarrow{\text{Cay}} \varepsilon$. Thus, $\varepsilon = \varepsilon$ and, therefore, the assumption (5.16) is violated.

So, after extraction, $\mu_{n,0}$ are all the same, μ_0. We may now repeat the same procedure to the sequence $(f^{-1} \mu_0^{-1} \gamma_n)_n$ to show that after another extraction $\mu_{n,1}$ are all the same. We can continue doing this procedure an arbitrary number of times: Thus, for all $l \in \mathbf{N}$, there exists a subsequence (γ_n') of (γ_n) such that suitable normal forms of the elements of (γ_n') share at least l common leftmost letters. For $l = 3$, Proposition 2.14 shows that (γ_n') has no accumulation points in the boundary of M. This is a contradiction with the assumption that $\gamma_n \xrightarrow{\text{Cay}} \varepsilon \in \partial_\infty M$. $\qquad\square$

Combining Lemmas 5.14 and 5.16, we obtain:

Corollary 5.17. *The map $\xi : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ preserves convergence dynamics.*

In particular, Γ is τ_{mod}-regular; see Remark 3.11.

5.2.4. *The boundary map is antipodal.*

Proposition 5.18. *The map $\xi : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ in (5.11) (obtained by combining (5.12) and (5.15)) is* antipodal*: That is, for every pair of distinct points $\varepsilon_\pm \in \partial_\infty \Gamma$, the points $\tau_\pm := \xi(\varepsilon_\pm) \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ are antipodal to each other.*

Proof. We consider the following cases. Recall that ξ is Γ-equivariant (by Corollary 5.15 and Lemma 5.13).

Case 1. *Suppose that both $\varepsilon_\pm$ are type I points.* Using the Γ-equivariance, we may assume that $\varepsilon_- \in \partial_\infty M$. If ε_+ is also in $\partial_\infty M$, then $\xi(\varepsilon_\pm)$ are antipodal since $\xi : \partial_\infty M \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ is an antipodal map. Else, $\varepsilon_+ \notin \partial_\infty M$ but there exists $\gamma \in \Gamma$ such that $\mathrm{rl}(\gamma) \geq 1$ and $\varepsilon := \gamma^{-1} \varepsilon_+ \in \partial_\infty M$. Let $\gamma = \mu_0 f^{\epsilon_1} \mu_1 \cdots f^{\epsilon_n} \mu_n$ be a normal form. So,

$$\mu_0^{-1} \xi(\varepsilon_+) \in \mu_0^{-1} \gamma(\Lambda_{\tau_{\mathrm{mod}}}(M)) = \mu_0^{-1} f^{\epsilon_1} \mu_1 \cdots f^{\epsilon_n}(\Lambda_{\tau_{\mathrm{mod}}}(M)).$$

If $\mathrm{rl}(\gamma) = 1$, then $\mu_0^{-1} \xi(\varepsilon_+) \in f^{\epsilon_1}(\Lambda_{\tau_{\mathrm{mod}}}(M))$. Moreover, $\mu_0^{-1} \varepsilon_+ \notin f^{\epsilon_1} \partial_\infty H_{-\epsilon_1}$, since we have assumed that $\varepsilon_+ \notin \partial_\infty M$. Thus, it follows that $f^{-\epsilon_1} \mu_0^{-1} \xi(\varepsilon_+) \in \Lambda_{\tau_{\mathrm{mod}}}(M) \setminus \Lambda_{\tau_{\mathrm{mod}}}(H_{-\epsilon_1})$. Since $B_{-\epsilon_1}$ is antipodal to $\Lambda_{\tau_{\mathrm{mod}}}(M) \setminus \Lambda_{\tau_{\mathrm{mod}}}(H_{-\epsilon_1})$, $f^{-\epsilon_1} \mu_0^{-1} \xi(\varepsilon_-)$, which is an element of $B_{-\epsilon_1}$, is antipodal to $\Lambda_{\tau_{\mathrm{mod}}}(M) \setminus \Lambda_{\tau_{\mathrm{mod}}}(H_{-\epsilon_1})$. Thus, $\xi(\varepsilon_+)$ is antipodal to $\xi(\varepsilon_-)$.

If $\mathrm{rl}(\gamma) \geq 2$, and $\mu_1 \notin H_{\epsilon_2}$, then

$$\mu_0^{-1} \xi(\varepsilon_+) \in \mu_0^{-1} \gamma A \subset f^{\epsilon_1} \mu_1 B_{\epsilon_2} \subset B_{\epsilon_1}^\circ.$$

If $\mu_1 \in H_{\epsilon_2}$, then $\epsilon_1 = \epsilon_2$ and, hence,

$$\mu_0^{-1} \xi(\varepsilon_+) \in \mu_0^{-1} \gamma A \subset f^{\epsilon_1} \mu_1 f^{\epsilon_2}(A \cup B_{\epsilon_2}) = f^{\epsilon_1} B_{\epsilon_1} \subset B_{\epsilon_1}^\circ,$$

In both cases, $\mu_0^{-1}\xi(\varepsilon_+)$ is antipodal to A and in particular, to $\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{M})$. Thus, $\xi(\varepsilon_+)$ is antipodal to $\Lambda_{\tau_{\mathrm{mod}}}(\mathrm{M})$ and, in particular, to $\xi(\varepsilon_-)$.

Case 2. *Suppose that both $\varepsilon_\pm$ are type II points.* Let $(\omega_n^\pm)$ be alternating sequences such that $\omega_n^\pm \xrightarrow{\mathrm{Cay}} \varepsilon_\pm$ (see Lemma 2.16).

Lemma 5.19. *There exists $n \in \mathbf{N}$ such that $\omega_n^+ \notin \omega_n^-\mathrm{M}$.*

Proof. The proof is similar to the one of Lemma 4.20. $\square$

Let n_0 be the smallest natural number such that $\omega_{n_0}^+ \notin \omega_{n_0}^-\mathrm{M}$. Using an argument similar to the Case 2 in the proof of Proposition 4.18, one can show that $\xi((\omega_{n_0}^+)^{-1}\varepsilon_\pm)$ are antipodal, which is equivalent to $\xi(\varepsilon_\pm)$ being antipodal.

Case 3. *Suppose that ε_- is a type I point, but ε_+ is a type II point.* By Γ-equivariance, we may assume that $\varepsilon_- \in \partial_\infty\mathrm{M}$. Let (ω_n) alternating sequence,

$$\omega_n = \mu_0 f^{\epsilon_1}\mu_1 f^{\epsilon_2}\mu_2\cdots f^{\epsilon_{n-1}}\mu_{n-1}f^{\epsilon_n},$$

such that $\omega_n \xrightarrow{\mathrm{Cay}} \varepsilon_+$. Then, $\xi(\varepsilon_+) = \lim_{n\to\infty}\omega_n A$, see (5.15).

If $\mu_1 \in \mathrm{H}_{\epsilon_2}$, then $\epsilon_1 = \epsilon_2$, and it follows that

$$\mu_0^{-1}\omega_n A \subset f^{\epsilon_1}B_{\epsilon_1} \subset B_{\epsilon_1}^\circ.$$

Thus, $\mu_0^{-1}\xi(\varepsilon_+) \in B_{\epsilon_1}^\circ$. Else, if $\mu_1 \notin \mathrm{H}_{\epsilon_2}$, then $(\mu_0 f^{\epsilon_1}\mu_1)^{-1}\omega_n A \subset B_{\epsilon_2}$, also showing that $\mu_0^{-1}\xi(\varepsilon_+) \in f^{\epsilon_1}\mu_1 B_{\epsilon_2} \subset B_{\epsilon_1}^\circ$. However, $\mu_0^{-1}\xi(\varepsilon_-) \in A$. It follows that $\xi(\varepsilon_-)$ and $\xi(\varepsilon_+)$ are antipodal. $\square$

5.3. **Proof of Theorem B.** We show that the subgroup $\Gamma = \langle\mathrm{M}, f\rangle < G$ in the conclusion of Theorem B, which is naturally isomorphic to $\mathrm{M}\star_\phi$ (by Proposition 3.16), is a τ_{mod}-Anosov subgroup; this is equivalent to showing that Γ is a τ_{mod}-asymptotically embedded subgroup (see Definition 3.10) of G:

  (i) That Γ is hyperbolic is the content of Corollary 5.5.
  (ii) Finally, the boundary map $\xi : \partial_\infty\Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ in Equation (5.11) is Γ-equivariant (by Lemma 5.13 and Corollary 5.15), antipodal (by Proposition 5.18), and preserves convergence dynamics (by Corollary 5.17).

This concludes the proof of the Theorem B. $\square$

6. An application to Θ-positive representations of surface groups

In this section, we discuss a rich class of Anosov representations of surface groups in certain simple Lie groups G, called Θ-*positive* representations.

Guichard-Wienhard [12, 13], generalizing Lustzig's classical notion of total positivity, introduced the notion of Θ-*positivity* for certain flag varieties: Let G be a simple Lie group of noncompact type and with a finite center, let P_Θ be a parabolic subgroup conjugate to its opposite, and let P_Θ^{opp} be an opposite parabolic subgroup. A *positive structure* on the pair (G, P_Θ) is a sharp open sub-semigroup $U_\Theta^>$ of the unipotent radical U_Θ of P_Θ, invariant under the conjugation action by the identity component of $P_\Theta \cap P_\Theta^{\mathrm{opp}}$. Here, sharp means that if $u, v \in \mathrm{cl}(U_\Theta^>)$ and $uv = 1_G$, then $u = v = 1_G$. The subset $D := U_\Theta^> \cdot [P_\Theta^{\mathrm{opp}}] \subset G/P_\Theta$ is called

a *diamond* and $D^\vee := (U_\Theta^>)^{-1} \cdot [P_\Theta^{\mathrm{opp}}] \subset G/P_\Theta$ is called the *opposite diamond*. Crucial to the discussion of Θ-positivity is the fact that the subsets D and $D^\vee$ are connected components of $C([P_\Theta]) \cap C([P_\Theta^{\mathrm{opp}}])$, [12, Theorem 4.7]. The sharp semigroup property of $U_\Theta^>$ implies that D and $D^\vee$ are antipodal to each other. See [13, §10] for a detailed discussion of diamonds.

An n-tuple $(\sigma_1, \ldots, \sigma_n)$ of distinct points in G/P_Θ is said to be *positive* if there exists $g \in \mathrm{Aut}(G)$, $u_1, \ldots, u_{n-2} \in U_\Theta^>$ such that

$$g \cdot (\sigma_1, \ldots, \sigma_n) = ([P_\Theta], \underbrace{u_{n-2} \cdots u_1 [P_\Theta^{\mathrm{opp}}], \ldots, u_1 [P_\Theta^{\mathrm{opp}}]}_{\in D}, [P_\Theta^{\mathrm{opp}}]).$$

Definition 6.1. Fix a cyclic order in the circle S^1. A map $\xi : S^1 \to G/P_\Theta$ is said to be *positive* if ξ sends every cyclically ordered n-tuple of distinct points in S^1 to a *positive n-tuple*.

Note that positive maps are not assumed to be continuous.

Suppose that $\xi : S^1 \to G/P_\Theta$ is a positive map. Given a cyclically ordered triple $(\varepsilon_+, \varepsilon, \varepsilon_-)$ in S^1, let $D_\varepsilon(\varepsilon_+, \varepsilon_-)$ denote the connected component of $C(\xi(\varepsilon_+)) \cap C(\xi(\varepsilon_-))$ that contains $\xi(\varepsilon)$. Then, for any other point $\varepsilon' \in S^1$ such that $(\varepsilon_+, \varepsilon', \varepsilon_-)$ is cyclically ordered, we have $D_\varepsilon(\varepsilon_+, \varepsilon_-) = D_{\varepsilon'}(\varepsilon_+, \varepsilon_-)$. Moreover, if $(\varepsilon_1, \ldots, \varepsilon_4)$ is any cyclically ordered 4-tuple in S^1 and $g \in \mathrm{Aut}(G)$ is any element such that $g \cdot D_{\varepsilon_2}(\varepsilon_1, \varepsilon_3) = D$, then $g \cdot D_{\varepsilon_4}(\varepsilon_3, \varepsilon_1) = D^\vee$; in particular,

$$D_{\varepsilon_2}(\varepsilon_1, \varepsilon_3) \text{ is antipodal to } D_{\varepsilon_4}(\varepsilon_3, \varepsilon_1). \tag{6.1}$$

See [14, Proposition 2.5(3)].

An interesting characteristic property of positive maps $\xi : S^1 \to G/P_\Theta$ is the following nesting property ([14, Corollary 3.9]), which reflects the fact that to $U_\Theta^>$ is a sharp semigroup: If $(\varepsilon_1, \ldots, \varepsilon_5)$ is any cyclically ordered 5-tuple in S^1, then

$$D_{\varepsilon_3}(\varepsilon_1, \varepsilon_5) \supset \mathrm{cl}\,(D_{\varepsilon_3}(\varepsilon_2, \varepsilon_4)). \tag{6.2}$$

Definition 6.2. Let Γ be a surface group. A representation $\rho : \Gamma \to G$ is called Θ-*positive* if there exists a ρ-equivariant positive map $\xi : \partial_\infty \Gamma \to G/P_\Theta$.

Using the Combination Theorems,[10] (6.1), and (6.2), we give a proof of the following:

Theorem 6.3 (Guichard-Labourie-Wienhard, [14, Theorem B])**.** *Let S be a closed orientable surface of genus ≥ 2. If $\rho : \pi_1(S) \to G$ is a Θ-positive representation, then ρ is a Θ-Anosov representation.*

Proof. We will identify $\Gamma := \pi_1(S)$ with a Fuchsian subgroup of $\mathrm{PSL}(2, \mathbf{R})$ and, thus, we obtain a hyperbolic structure on S. Let c be a simple closed separating[11] geodesic in S, whose free homotopy class is represented by an element $\eta \in \Gamma$. Then, Γ can be written as an amalgamated free product (cf. §1.2)

$$\Gamma = \Gamma_A \star_H \Gamma_B,$$

---

[10]We remark that the Combination Theorems mainly circumvent the discussion of *tripod metrics* in the original proof of this result in [14].

[11]Alternatively, one could also choose a non-separating one and work with HNN extensions.

where $H := \langle \eta \rangle \cong \mathbf{Z}$. The subgroup Γ_A (and, similarly, Γ_B) of $\mathrm{PSL}(2, \mathbf{R})$ is *Schottky*, i.e., there exist generators $\alpha_1, \ldots, \alpha_k$ of Γ_A and pairwise disjoint closed intervals in $\partial_\infty \mathbb{H}^2$,

$$I_1^+, I_1^-, \ldots, I_k^+, I_k^-,$$

such that for all $j \in \{1, \ldots, k\}$, α_j maps the exterior of I_j^- onto the interior of I_j^+.

Claim. $\rho|_{\Gamma_A}$ *(and, similarly, $\rho|_{\Gamma_B}$) is a Θ-Anosov representation.*

Cf. [4, Theorem 1.3].

Proof of claim. We first show that for all $j \in \{1, \ldots, k\}$, $H_j := \langle \rho(\alpha_j) \rangle$ is Θ-Anosov. Let $\varepsilon_j^\pm$ be the attractive/repulsive fixed point of α_j in $\partial_\infty \mathbb{H}^2$, which lies in the interior of $I_j^\pm$, respectively. Let $x_j^\pm, y_j^\pm$ denote the endpoints of $I_j^\pm$; choose these names in a way such that each triple $(x_j^\pm, \varepsilon_j^\pm, y_j^\pm)$ is cyclically ordered. If follows that the 5-tuples $(x_j^\pm, \alpha_j^{\pm 1} x_j^\pm, \varepsilon_j^\pm, \alpha_j^{\pm 1} y_j^\pm, y_j^\pm)$ are also cyclically ordered.

Let $A_j^\pm := \mathrm{cl}(D_{\varepsilon_j^\pm}(x_j^\pm, y_j^\pm))$. Then, (6.2) implies that

$$(A_j^\pm)^\circ \supset \mathrm{cl}\left(D_{\varepsilon_j^\pm}(\alpha_j^{\pm 1} x_j^\pm, \alpha_j^{\pm 1} y_j^\pm) \right) = \rho(\alpha_j^{\pm 1}) A_j^\pm,$$

whereas (6.1) and (6.2) imply that A_j^+ is antipodal to A_j^-. Then, by the same proof as in Lemma 5.2(i), we see that $H_j = \langle \rho(\alpha_j) \rangle$ is Θ-Anosov and $\Lambda_{\tau_{\mathrm{mod}}}(H_j) = \{\xi(\varepsilon_j^+), \xi(\varepsilon_j^-)\}$.

The subsets $A_j := A_j^+ \cup A_j^-$, for $j = 1, \ldots, k$ are also pairwise antipodal. Then,

$$\alpha_j^n \left(\bigcup_{i \neq j} (I_i^+ \cup I_i^-) \right) \subset (I_j^+ \cup I_j^-)^\circ,$$

for all nonzero $n \in \mathbf{Z}$, simply translates to (using (6.2)) the fact that

$$\rho(\alpha_j^n) \left(\bigcup_{i \neq j} A_i \right) \subset A_j^\circ.$$

Since we already observed in the preceding paragraph that $H_1, \ldots, H_k$ are Θ-Anosov subgroups of G, using the above, the combination theorem [5, Corollary 6.3] from our previous work yields that $\rho : \Gamma_A \to \langle H_1, \ldots, H_k \rangle$ is Θ-Anosov. $\square$

Let I_A and I_B denote the closures of the connected components of the complement of the 2-point limit set $\{\varepsilon_+, \varepsilon_-\}$ of H in $\partial_\infty \mathbb{H}^2$. Pick any interior points $\varepsilon_A \in I_A$ and $\varepsilon_B \in I_B$. We chose these names so that $\Lambda(\Gamma_A) \subset I_A$, $\Lambda(\Gamma_B) \subset I_B$, and $(\varepsilon_A, \varepsilon_+, \varepsilon_B, \varepsilon_-)$ is cyclically ordered. Let

$$A := \mathrm{cl}(D_{\varepsilon_A}(\varepsilon_-, \varepsilon_+)) \quad \text{and} \quad B := \mathrm{cl}(D_{\varepsilon_B}(\varepsilon_+, \varepsilon_-));$$

by (6.1), A° and B° are antipodal to each other. Moreover, since, for all $\alpha \in \Gamma_A \setminus H$ and $\beta \in \Gamma_B \setminus H$,

$$\alpha I_B \subset I_A^\circ \quad \text{and} \quad \beta I_A \subset I_B^\circ,$$

it follows that

$$\rho(\alpha)B \subset A^\circ \quad \text{and} \quad \rho(\beta)A \subset B^\circ.$$

Furthermore, $\rho(\mathrm{H})$ preserves $\xi(I_A)$ and $\xi(I_B)$, so that $\rho(\mathrm{H})A = A$ and $\rho(\mathrm{H})B = B$. Thus, (A, B) is an interactive pair for $(\rho(\Gamma_A), \rho(\Gamma_B); \rho(\mathrm{H}))$. Finally, using the antipodality property of the Θ-limit sets of $\rho(\Gamma_A)$ and $\rho(\Gamma_B)$ and Lemmas 4.1 and 4.3, one may directly verify the hypothesis (ii) in the statement of Theorem A; see also Remark 1.1. Thus, Theorem A implies that ρ is Θ-Anosov; cf. the examples in §1.2. $\qquad\square$

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