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Degenerate Competing Three-Particle Systems

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We study systems of three interacting particles, in which drifts and variances are assigned by rank. These systems are degenerate: the variances corresponding to one or two ranks can vanish, so the corresponding ranked motions become ballistic rather than diffusive. Depending on which ranks are allowed to “go ballistic" the systems exhibit markedly different behavior, which we study here in some detail. Also studied are stability properties for the resulting planar process of gaps between successive ranks.

Keywords: Competing particle systems; local times; reflected planar Brownian motion; triple collisions; structure of filtrations

1. Introduction

Systems of three or more interacting particles that assign local characteristics to individual motions by rank, rather than by index (“name"), have received considerable attention in recent years under the rubric of “competing particle systems"; see for instance [35], [26], [25], [27], [39], [40], [7], [28], [5], [30], and the references cited there. A crucial common feature of these studies is that particles of all ranks are assigned some strictly positive local variance. This nondegeneracy smooths out the transition probabilities of particles.

We study here, and to the best of our knowledge for the first time, systems of such competing particles which are allowed to degenerate, meaning that the variances assigned to one or two ranks can vanish. This kind of degeneracy calls for an entirely new theory; we initiate such a theory in the context of systems consisting of three particles. Even with this simplification, the range of behavior these systems can exhibit is quite rich. We illustrate just how rich, by studying in detail the construction and properties of three such systems — respectively, in Sections 2, 3, and in an Appendix (Section A).

The systems of Sections 2 and A assign ballistic motions to the leader and laggard particles, and diffusive motion to the middle particle. The quadratic variations of both leader and laggard are zero, as are the cross-variations between any two particles; the resulting diffusion matrix is thus of rank one. The intensity of collisions between the middle particle, and the leader or the laggard, is measured by the growth of the local time for the respective gap in ranks. Skew-elastic colliding behavior between the middle and laggard particles is described by collision local times in Section A.

By contrast, the system of Section 3 assigns independent diffusive motions to the leader and laggard particles, and ballistic ranked motion to the middle particle. The quadratic variation of the middle particle is zero, and so are the cross-variations between any two particles; thus, the diffusion matrix is now of rank two. These different kinds of behavior are summarized in Table 1.

A salient feature emerging from the analysis, is that two purely ballistic ranked motions can never “pinch" a Brownian motion running between, and reflected off from, them (Proposition 2.1); whereas two Brownian ranked motions *can* pinch a purely ballistic one running in their midst and reflected off from them, resulting in a massive collision that involves all three particles (subsection 3.2). Using simple excursion-theoretic ideas, we show in Theorem 3.1 how such a system can extricate itself from

Section	Leader	Middle	Laggard	Double Collisions	Triple Collisions
2	Ballistic	Diffusive	Ballistic	Elastic	No triple collisions
3	Diffusive	Ballistic	Diffusive	Elastic	Triple collisions
A	Ballistic	Diffusive	Ballistic	Skewed	(open question)

Table 1. Different behavior for leader, middle and laggard particles in each of three particle systems.

such a triple collision; but also that the solution to the stochastic equations that describe its motion, which is demonstrably strong up until the first time a triple collision occurs, *ceases to be strong after that time*. This last question on the structure of filtrations had been open for several years. We also show that, even when triple collisions do occur in the systems studied here, they are “soft": the associated triple collision local time is identically equal to zero.

The analysis of these three-particle systems is connected to that of planar, semimartingale reflecting Brownian motions (SRBMs) and their local times on the faces of the nonnegative orthant. The survey paper [45] and the monograph [15] are excellent entry points to this subject and its applications, alongside the foundational papers [16], [43], [17], [18], [37]. In Sections 2 and A the planar process of gaps is a degenerate SRBM driven by a one-dimensional Brownian motion, while it is a non-degenerate SRBM driven by a two-dimensional Brownian motion in Section 3. The directions of reflection are the same in Sections 2 and 3, but different in Section A because of the skew-elastic collisions between the middle and laggard particles. Under appropriate conditions the planar process of gaps between ranked particles has an invariant distribution. We exhibit this distribution explicitly in one instance (subsection A.3) under the so-called “skew-symmetry" condition, and offer a conjecture for it in another. We show that the former is the product (A.20) of its exponential marginals, while the latter is determined by the distribution of the sum of its marginals and is *not* of product form (cf. Remarks 2.4, 2.6 in [23]).

The three-particle systems studied in this paper reveal some of the rich probabilistic structures that degenerate, three-dimensional continuous MARKOV processes can exhibit. It will be very interesting to extend the analysis of the present paper to $n-$particle systems with $n \geq 4$. We expect certain of the features studied here to hold also in higher dimensions, but leave such extensions to future work.

2. Diffusion in the Middle, with Ballistic Hedges

Given real numbers $\delta_1, \delta_2, \delta_3$ and $x_1 > x_2 > x_3$, we start by constructing a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ endowed with a right-continuous filtration $\mathbb{F} = \big\{\mathfrak{F}(t)\big\}_{0 \leq t < \infty}$, to which are adapted three independent Brownian motions $B_1(\cdot), B_2(\cdot), B_3(\cdot)$, and three continuous processes $X_1(\cdot), X_2(\cdot), X_3(\cdot)$ that satisfy

$$X_i(\cdot) = x_i + \sum_{k=1}^{3} \delta_k \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_k^X(t)\}} \, \mathrm{d}t + \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_2^X(t)\}} \, \mathrm{d}B_i(t)\,, \quad i = 1, 2, 3\,, \tag{2.1}$$

$$\int_0^{\infty} \mathbf{1}_{\{R_k^X(t) = R_\ell^X(t)\}} \, \mathrm{d}t = 0\,, \;\; \forall \;\, k < \ell; \qquad \Big\{ t \in (0, \infty) \, : \, R_1^X(t) = R_3^X(t) \Big\} = \emptyset \tag{2.2}$$

with probability one. Here we denote the descending order statistics by

$$\max_{j=1,2,3} X_j(t) =: R_1^X(t) \geq R_2^X(t) \geq R_3^X(t) := \min_{j=1,2,3} X_j(t)\,, \qquad t \in [0, \infty), \tag{2.3}$$

and adopt the convention of resolving ties always in favor of the lowest index i; for instance, we set

$$R_1^X(t) = X_1(t)\,,\;\; R_2^X(t) = X_3(t)\,,\;\; R_3^X(t) = X_2(t) \qquad \text{on} \;\; \big\{ X_1(t) = X_3(t) > X_2(t) \big\}\,.$$

The dynamics (2.1) mandate ballistic motions for the leader and laggard particles with drifts δ_1 and δ_3, respectively, which act here as "outer hedges"; and diffusive (Brownian) motion with drift δ_2, for the middle particle. The first condition in (2.2) posits that collisions of particles are *non-sticky,* in that the set of all collision times has zero LEBESGUE measure; while the second proscribes triple collisions.

As a canonical example, it is useful to keep in mind the symmetric configuration

$$\delta_3 = -\delta_1 = \gamma > 0 = \delta_2\,, \tag{2.4}$$

for which the system of equations (2.1) takes the appealing, symmetric form

$$X_i(\cdot) = x_i + \gamma \int_0^{\cdot} \Big(\mathbf{1}_{\{X_i(t) = R_3^X(t)\}} - \mathbf{1}_{\{X_i(t) = R_1^X(t)\}} \Big) \mathrm{d}t + \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_2^X(t)\}}\, \mathrm{d}B_i(t)\,. \tag{2.5}$$

This system was introduced and studied for the first time in the technical report [12]. Figure 1, which illustrates its path behavior, is taken from that report.

A very salient feature of the dynamics in (2.1) is that its dispersion structure is both degenerate and discontinuous. It should come then as no surprise, that the analysis of the system (2.1)–(2.2) might be not entirely trivial; in particular, it is not covered by the results in either [41] or [4]. The question then, is whether a process $X(\cdot)$ with (2.1) and (2.2) exists; and if so, whether its distribution — and, a bit more ambitiously, its sample path structure — is determined uniquely.

2.1. Analysis

Suppose that a solution to the system of equation (2.1) subject to the conditions of (2.2) has been constructed. Its descending order-statistics are given then as

$$R_1^X(t) \,=\, x_1 + \delta_1\, t + \frac{1}{2}\, \Lambda^{(1,2)}(t)\,, \qquad R_3^X(t) \,=\, x_3 + \delta_3\, t - \frac{1}{2}\, \Lambda^{(2,3)}(t) \tag{2.6}$$

$$R_2^X(t) \,=\, x_2 + \delta_2\, t + W(t) - \frac{1}{2}\, \Lambda^{(1,2)}(t) + \frac{1}{2}\, \Lambda^{(2,3)}(t) \tag{2.7}$$

for $0 \le t < \infty$, on the strength of the results in [3]. Here, the scalar process

$$W(\cdot) := \sum_{i=1}^{3} \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_2^X(t)\}}\, \mathrm{d}B_i(t) = \sum_{i=1}^{3} \left(X_i(\cdot) - x_i - \sum_{k=1}^{3} \delta_k \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_k^X(t)\}} \mathrm{d}t \right) \tag{2.8}$$

is standard Brownian motion by the P. LÉVY theorem; and we denote the local time accumulated at the origin by the continuous, nonnegative semimartingale $R_k^X(\cdot) - R_\ell^X(\cdot)$ over the time interval $[0,t]$, by

$$\Lambda^{(k,\ell)}(t) \,\equiv\, L^{R_k^X - R_\ell^X}(t)\,, \qquad k < \ell. \tag{2.9}$$

Throughout this paper we use the convention

$$L^{\Xi}(\cdot) \equiv L^{\Xi}(\cdot\,;0) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^{\cdot} \mathbf{1}_{\{\Xi(t) < \varepsilon\}} \mathrm{d}\langle M \rangle(t) = \int_0^{\cdot} \mathbf{1}_{\{\Xi(t) = 0\}} \mathrm{d}\Xi(t) = \int_0^{\cdot} \mathbf{1}_{\{\Xi(t) = 0\}} \mathrm{d}C(t) \tag{2.10}$$

for the “right" local time at the origin of a continuous, nonnegative semimartingale of the form $\Xi(\cdot)=\Xi(0)+M(\cdot)+C(\cdot)$, with $M(\cdot)$ a continuous local martingale and $C(\cdot)$ a process of finite first variation on compact intervals. The local time process $L^{\Xi}(\cdot)$ is continuous, adapted and nondecreasing, flat off the zero-set $\{t\geq 0: \Xi(t)=0\}$.

We denote now the sizes of the gaps between the leader and the middle particle, and between the middle particle and the laggard, by

$$G(\cdot) := R_1^X(\cdot)-R_2^X(\cdot), \qquad H(\cdot) := R_2^X(\cdot)-R_3^X(\cdot), \tag{2.11}$$

respectively, and obtain from (2.6)–(2.9) the semimartingale representations

$$G(t) = x_1-x_2-\big(\delta_2-\delta_1\big)\,t-W(t)-\frac{1}{2}\,L^H(t)+L^G(t), \qquad 0\leq t<\infty \tag{2.12}$$

$$H(t) = x_2-x_3-\big(\delta_3-\delta_2\big)\,t+W(t)-\frac{1}{2}\,L^G(t)+L^H(t), \qquad 0\leq t<\infty \tag{2.13}$$

where we recall the identifications $L^G(\cdot)\equiv\Lambda^{(1,2)}(\cdot)$, $L^H(\cdot)\equiv\Lambda^{(2,3)}(\cdot)$ from (2.11), (2.9). We introduce also the continuous semimartingales

$$U(t)=x_1-x_2-\big(\delta_2-\delta_1\big)\,t-W(t)-\frac{1}{2}\,L^H(t), \quad V(t)=x_2-x_3-\big(\delta_3-\delta_2\big)\,t+W(t)-\frac{1}{2}\,L^G(t),$$

and observe

$$G(\cdot) = U(\cdot)+L^G(\cdot)\geq 0, \qquad \int_0^\infty \mathbf{1}_{\{G(t)>0\}}\,\mathrm{d}L^G(t) = 0 \tag{2.14}$$

$$H(\cdot) = V(\cdot)+L^H(\cdot)\geq 0, \qquad \int_0^\infty \mathbf{1}_{\{H(t)>0\}}\,\mathrm{d}L^H(t) = 0. \tag{2.15}$$

In other words, the “gaps" $G(\cdot)$, $H(\cdot)$ are the SKOROKHOD reflections of the semimartingales $U(\cdot)$ and $V(\cdot)$, respectively. The theory of the SKOROKHOD reflection problem (e.g., Lemma VI.2.1 in [38]) provides now the relationships

$$L^G(t) = \max_{0\leq s\leq t}\big(-U(s)\big)^+ = \max_{0\leq s\leq t}\Big(-(x_1-x_2)+\big(\delta_2-\delta_1\big)\,s+W(s)+\frac{1}{2}\,L^H(s)\Big)^+ \tag{2.16}$$

$$L^H(t) = \max_{0\leq s\leq t}\big(-V(s)\big)^+ = \max_{0\leq s\leq t}\Big(-(x_2-x_3)+\big(\delta_3-\delta_2\big)\,s-W(s)+\frac{1}{2}\,L^G(s)\Big)^+ \tag{2.17}$$

between the two local time processes $L^G(\cdot)\equiv\Lambda^{(1,2)}(\cdot)$ and $L^H(\cdot)\equiv\Lambda^{(2,3)}(\cdot)$, once the scalar Brownian motion $W(\cdot)$ has been specified.

• Finally, we note that the equations of (2.12)-(2.13) can be cast in the form

$$\begin{pmatrix} G(t) \\ H(t)\end{pmatrix} =: \mathfrak{G}(t) = \mathfrak{g}+\mathfrak{Z}(t)+\mathcal{R}\,\mathfrak{L}(t), \qquad 0\leq t<\infty, \tag{2.18}$$

of [16], where

$$\mathcal{R} = \mathcal{I}-\mathcal{Q}, \qquad \mathcal{Q} := \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0\end{pmatrix}, \qquad \mathfrak{g}=\mathfrak{G}(0), \qquad \mathfrak{L}(t) = \begin{pmatrix} L^G(t) \\ L^H(t)\end{pmatrix},$$

$$\mathfrak{Z}(t) = \begin{pmatrix} (\delta_1 - \delta_2)t - W(t) \\ (\delta_2 - \delta_3)t + W(t) \end{pmatrix}, \qquad 0 \le t < \infty . \tag{2.19}$$

One reflects off the faces of the nonnegative quadrant, in other words, the degenerate, two-dimensional Brownian motion $\mathfrak{Z}(\cdot)$ with drift vector and covariance matrix given respectively by

$$\boldsymbol{m} = \big(\delta_1 - \delta_2 \, , \, \delta_2 - \delta_3\big)', \qquad \mathcal{C} := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.20}$$

The directions of reflection are the row vectors of the *reflection matrix* $\mathcal{R}$, and the matrix $\mathcal{Q} = \mathcal{I} - \mathcal{R}$ has spectral radius strictly less than 1, as postulated by [16]. The process $\mathfrak{Z}(\cdot)$ is allowed in [16] to be degenerate, i.e., its covariation matrix can be only nonnegative-definite.

2.2. Synthesis

We trace now the steps of subsection 2.1 in reverse: start with given real numbers $\delta_1, \delta_2, \delta_3$, and $x_1 > x_2 > x_3$, and construct a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbb{F} = \big\{\mathfrak{F}(t)\big\}_{0 \le t < \infty}$ rich enough to support a scalar, standard Brownian motion $W(\cdot)$. In fact, we select the filtration $\mathbb{F}$ to be $\mathbb{F}^W = \big\{\mathfrak{F}^W(t)\big\}_{0 \le t < \infty}$, the smallest right-continuous filtration to which $W(\cdot)$ is adapted.

Informed by the analysis of the previous section we consider, by analogy with (2.16)-(2.17), the two-dimensional SKOROKHOD reflection system

$$A(t) = \max_{0 \le s \le t} \Big(-(x_1 - x_2) + \big(\delta_2 - \delta_1\big)\, s + W(s) + \frac{1}{2}\, \Gamma(s) \Big)^+, \qquad 0 \le t < \infty \tag{2.21}$$

$$\Gamma(t) = \max_{0 \le s \le t} \Big(-(x_2 - x_3) + \big(\delta_3 - \delta_2\big)\, s - W(s) + \frac{1}{2}\, A(s) \Big)^+, \qquad 0 \le t < \infty \tag{2.22}$$

for two continuous, nondecreasing and $\mathbb{F}^W$–adapted processes $A(\cdot)$ and $\Gamma(\cdot)$ with $A(0) = \Gamma(0) = 0$. This system of equations is of the type studied in [16]. From Theorem 1 of that paper, we know that it possesses a unique, $\mathbb{F}^W$–adapted solution.

Once the solution $\big(A(\cdot), \Gamma(\cdot)\big)$ to this system has been constructed, we define the processes

$$U(t) := x_1 - x_2 - \big(\delta_2 - \delta_1\big)\, t - W(t) - \frac{1}{2}\, \Gamma(t), \quad V(t) := x_2 - x_3 - \big(\delta_3 - \delta_2\big)\, t + W(t) - \frac{1}{2}\, A(t) \tag{2.23}$$

and then "fold" them to obtain their SKOROKHOD reflections; that is, the continuous semimartingales

$$G(t) := U(t) + \max_{0 \le s \le t} \big(-U(s)\big)^+ = x_1 - x_2 - \big(\delta_2 - \delta_1\big)\, t - W(t) - \frac{1}{2}\, \Gamma(t) + A(t) \ge 0 \tag{2.24}$$

$$H(t) := V(t) + \max_{0 \le s \le t} \big(-V(s)\big)^+ = x_2 - x_3 - \big(\delta_3 - \delta_2\big)\, t + W(t) - \frac{1}{2}\, A(t) + \Gamma(t) \ge 0 \tag{2.25}$$

for $t \in [0, \infty)$, in accordance with (2.14)–(2.17). From the theory of the SKOROKHOD reflection problem once again, we deduce the a.e. properties

$$\int_0^\infty \mathbf{1}_{\{G(t) > 0\}}\, \mathrm{d}A(t) = 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t) > 0\}}\, \mathrm{d}\Gamma(t) = 0\,; \tag{2.26}$$

and the theory of semimartingale local time ([38], Chapter VI), gives

$$\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\,\mathrm{d}t = \int_0^\infty \mathbf{1}_{\{G(t)=0\}}\,\mathrm{d}\langle G\rangle(t) = 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t)=0\}}\,\mathrm{d}t = \int_0^\infty \mathbf{1}_{\{H(t)=0\}}\,\mathrm{d}\langle H\rangle(t) = 0\,. \tag{2.27}$$

2.2.1. Constructing the Ranks

We introduce now, by analogy with (2.6)-(2.7), the processes

$$R_1(t) := x_1 + \delta_1 t + \frac{1}{2}A(t)\,, \qquad R_3(t) := x_3 + \delta_3 t - \frac{1}{2}\Gamma(t)\,, \tag{2.28}$$

$$R_2(t) := x_2 + \delta_2 t + W(t) - \frac{1}{2}A(t) + \frac{1}{2}\Gamma(t) \tag{2.29}$$

for $0 \le t < \infty$ and note the relations $R_1(\cdot) - R_2(\cdot) = G(\cdot) \ge 0$, $R_2(\cdot) - R_3(\cdot) = H(\cdot) \ge 0$ in conjunction with (2.24) and (2.25). In other words, we have the a.e. comparisons, or "descending rankings", $R_1(\cdot) \ge R_2(\cdot) \ge R_3(\cdot)$. It is clear from the discussion following (2.21), (2.22), that these processes are adapted to the filtration generated by the driving Brownian motion $W(\cdot)$, whence the inclusion $\mathbb{F}^{(R_1,R_2,R_3)} \subseteq \mathbb{F}^W$.

Let us show that these rankings never collapse. To put things a bit colloquially: *"Two ballistic motions cannot squeeze a diffusive (Brownian) motion"*. We are indebted to Drs. Robert FERNHOLZ (cf. [11]) and Johannes RUF for the argument that follows.

Proposition 2.1. *With probability one, we have:* $R_1(\cdot) - R_3(\cdot) = G(\cdot) + H(\cdot) > 0$.

Proof: We shall show that there cannot possibly exist numbers $T \in (0,\infty)$ and $r \in \mathbb{R}$, such that $R_1(T) = R_2(T) = R_3(T) = r$.

We argue by contradiction: If such a configuration were possible for some $\omega \in \Omega$ and some $T = T(\omega) \in (0,\infty)$, $r = r(\omega) \in \mathbb{R}$, we would have

$$r - \delta_3(T-t) \le R_3(t,\omega) \le R_2(t,\omega) \le R_1(t,\omega) \le r - \delta_1(T-t)\,, \qquad 0 \le t < T\,.$$

This is already impossible if $\delta_1 > \delta_3$, so let us assume $\delta_1 \le \delta_3$ and try to arrive at a contradiction in this case as well. The above quadruple inequality implies, a fortiori,

$$r - \delta_3(T-t) \le \overline{R}(t,\omega) := \frac{1}{3}\big(R_3(t,\omega) + R_2(t,\omega) + R_1(t,\omega)\big) \le r - \delta_1(T-t)\,, \qquad 0 \le t < T\,.$$

But we have $r - \overline{R}(t,\omega) = \overline{R}(T,\omega) - \overline{R}(t,\omega) = \overline{\delta}(T-t) + \big(W(T,\omega) - W(t,\omega)\big)/3$, where $\overline{\delta} := (\delta_1 + \delta_2 + \delta_3)/3$, and back into the above inequality this gives

$$3\left(\delta_1 - \overline{\delta}\right) \le \frac{W(T,\omega) - W(t,\omega)}{T-t} \le 3\left(\delta_3 - \overline{\delta}\right), \qquad 0 \le t < T\,.$$

However, from the PAYLEY-WIENER-ZYGMUND theorem for the Brownian motion $W(\cdot)$, this double inequality would force ω into a $\mathbb{P}-$null set. $\Box$

On this case the sequence $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (0,1,0)$ of local covariances-by-rank is concave, so the lack of triple collisions just established is in formal accordance with known results; although not obviously expected, let alone deduced, from them, because the local variance structure here is degenerate.

2.2.2. Identifying the Increasing Processes $A(\cdot)$, $\Gamma(\cdot)$ as Local Times

We claim that, in addition to (2.26) and (2.27), the properties

$$\int_0^\infty \mathbf{1}_{\{H(t)=0\}} \, \mathrm{d}A(t) = 0 \,, \qquad \int_0^\infty \mathbf{1}_{\{G(t)=0\}} \, \mathrm{d}\Gamma(t) = 0 \tag{2.30}$$

are also valid a.e. Indeed, we know from (2.26) that $A(\cdot)$ is flat off the set $\{t \geq 0 : G(t) = 0\}$, so we have $\int_0^\infty \mathbf{1}_{\{H(t)=0\}} \, \mathrm{d}A(t) = \int_0^\infty \mathbf{1}_{\{H(t)=G(t)=0\}} \, \mathrm{d}A(t)$; but this last expression is a.e. equal to zero because, as we have shown, $\{t \geq 0 : G(t) = H(t) = 0\} = \emptyset$ holds mod. $\mathbb{P}$. This proves the first equality in (2.30); the second is argued similarly.

But now, the local time at the origin of the continuous, nonnegative semimartingale $G(\cdot)$ is given as

$$L^G(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}} \, \mathrm{d}G(t) = \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}} \, \mathrm{d}A(t) - \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}} \, \frac{\mathrm{d}\Gamma(t)}{2} + \big(\delta_1 - \delta_2\big) \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}} \, \mathrm{d}t$$

from (2.10) and (2.24). From (2.27) and (2.30) the last two integrals vanish, so (2.26) leads to

$$L^G(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}} \, \mathrm{d}A(t) = A(\cdot) \,; \qquad \text{and} \qquad L^H(\cdot) = \Gamma(\cdot). \tag{2.31}$$

***Remark* 2.1.** *The Structure of Filtrations:* We have identified the components of the pair $(A(\cdot), \Gamma(\cdot))$, solution of the system (2.21)-(2.22), as the local times at the origin of the continuous semimartingales $R_1(\cdot) - R_2(\cdot) = G(\cdot) \geq 0$ and $R_2(\cdot) - R_3(\cdot) = H(\cdot) \geq 0$. In particular, this implies the filtration inclusions $\mathbb{F}^{(A,\Gamma)} = \mathbb{F}^{(R_1,R_3)} \subseteq \mathbb{F}^{(G,H)} \subseteq \mathbb{F}^{(R_1,R_2,R_3)}$; and back in (2.29), it gives $\mathbb{F}^W \subseteq \mathbb{F}^{(R_1,R_2,R_3)}$. But we have already noted the reverse of this inclusion, so we conclude that the process of ranks generates exactly the same filtration as the driving scalar Brownian motion: $\mathbb{F}^{(R_1,R_2,R_3)} = \mathbb{F}^W$.

2.2.3. Constructing the Individual Motions ("Names")

Once the "ranks" $R_1(\cdot) \geq R_2(\cdot) \geq R_3(\cdot)$ have been constructed in §2.2.1 on the filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbb{F} = \{\mathfrak{F}(t)\}_{0 \leq t < \infty}$, with $\mathbb{F}$ selected as the smallest right-continuous filtration $\mathbb{F}^W = \{\mathfrak{F}^W(t)\}_{0 \leq t < \infty}$ to which the scalar Brownian motion $W(\cdot)$ is adapted, we can construct as in the proof of Theorem 5 in [28] the "names" that generate these ranks — that is, processes $X_1(\cdot), X_2(\cdot), X_3(\cdot)$, as well as a three-dimensional Brownian motion $(B_1(\cdot), B_2(\cdot), B_3(\cdot))$ defined on this same space, such that the equation (2.1) is satisfied and $R_k^X(\cdot) \equiv R_k(\cdot)$, $k = 1, 2, 3$.

It is also clear from our construction that the conditions (2.2) are also satisfied: the first thanks to the properties of (2.27), the second because of Proposition 2.1.

• Alternatively, the construction of a pathwise unique, strong solution for the system (2.1) can be carried out along the lines of Proposition 8 in [25]. We start at time $\tau_0 \equiv 0$ and follow the paths of the top particle and of the pair consisting of the bottom two particles *separately*, until the top particle collides with the leader of the bottom pair (at time ϱ_0). Then we follow the paths of the bottom particle and of the pair consisting of the top two particles *separately*, until the bottom particle collides with the laggard of the top pair (at time τ_1). We repeat the procedure until the first time we see a triple collision, obtain two interlaced sequences of stopping times $\{\tau_k\}_{k \in \mathbb{N}_0}$ and $\{\varrho_k\}_{k \in \mathbb{N}_0}$ with

$$0 = \tau_0 \leq \varrho_0 \leq \tau_1 \leq \varrho_1 \leq \cdots \leq \tau_k \leq \varrho_k \leq \cdots \,, \tag{2.32}$$

and denote by the first time a triple collision occurs

$$\mathcal{S} := \inf\big\{t \in (0,\infty) : X_1(t) = X_2(t) = X_3(t)\big\} = \lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \varrho_k\,. \tag{2.33}$$

During each interval of the form $[\tau_k, \varrho_k)$ or $[\varrho_k, \tau_{k+1})$, a pathwise unique, strong solution of the corresponding two-particle system is constructed as in Theorem 4.1 in [14].

We end up in this manner with a three-dimensional Brownian motion $(B_1(\cdot), B_2(\cdot), B_3(\cdot))$, and with three processes $X_1(\cdot), X_2(\cdot), X_3(\cdot)$ that satisfy (2.1) and the first requirement (2.2), once again thanks to results in [14]. For this system, the ranked processes $R_1^X(\cdot) \geq R_2^X(\cdot) \geq R_3^X(\cdot)$ as in (2.3), satisfy the equations we studied in § 2.2.1, and generate the same filtration $\mathbb{F}^W = \{\mathfrak{F}^W(t)\}_{0\leq t<\infty}$ as the scalar Brownian motion $W(\cdot)$ above (Remark 2.1). We have seen in Proposition 2.1 that for such a system there are no triple collisions: $\mathcal{S} = \infty$. Thus the second condition in (2.2) is satisfied as well, all inequalities in (2.32) are strict, and we have proved the following result.

Theorem 2.2. *The system of equations* (2.1)*,* (2.2) *admits a pathwise unique strong solution.*

Figure 1, reproduced here from [11], illustrates trajectories of $X_1(\cdot)$, $X_2(\cdot)$, $X_3(\cdot)$ for the canonical example (2.4). It is clear from this picture and from the construction in §2.2.1, that the middle particle $R_2(\cdot)$ undergoes Brownian motion $W(\cdot)$ with reflection at the upper and lower boundaries, respectively $R_1(\cdot)$ and $R_3(\cdot)$, of a time-dependent domain.

In contrast to the "double SKOROKHOD map" of [31], where the upper and lower reflecting boundaries are given constants, these boundaries $R_1(\cdot)$ and $R_3(\cdot)$ are here random continuous functions of time, of finite first variation on compact intervals. They are "sculpted" by the Brownian motion $W(\cdot)$ itself via the local times $L^G(\cdot) \equiv A(\cdot)$ and $L^H(\cdot) \equiv \Gamma(\cdot)$, in the manner of the system (2.21), (2.22). The upper (respectively, lower) boundary decreases (resp., increases) by linear segments at a 45^o-angle, and increases (resp., decreases) by a singularly continuous CANTOR-like random function, governed by the local time $L^G(\cdot) \equiv A(\cdot)$ (resp., $L^H(\cdot) \equiv \Gamma(\cdot)$).

2.3. Positive Recurrence, Ergodicity, Laws of Large Numbers

We present now a criterion for the process $(G(\cdot), H(\cdot))$ in (2.12)-(2.13) to reach an arbitrary open neighborhood of the origin in finite expected time. We carry out this analysis along the lines of [20]. The system studied there, is a *non-degenerate* reflected Brownian motion $(\boldsymbol{X}_\cdot, \boldsymbol{Y}_\cdot)$ in the first orthant driven by a planar Brownian motion $(\boldsymbol{B}_\cdot, \boldsymbol{W}_\cdot)$, namely

$$\boldsymbol{X}_t = \boldsymbol{x} + \boldsymbol{B}_t + \boldsymbol{\mu}\, t + \boldsymbol{L}_t^{\boldsymbol{X}} + \boldsymbol{\alpha} \boldsymbol{L}_t^{\boldsymbol{Y}}\,, \quad \boldsymbol{Y}_t = \boldsymbol{y} + \boldsymbol{W}_t + \boldsymbol{\nu}\, t + \boldsymbol{\beta} \boldsymbol{L}_t^{\boldsymbol{X}} + \boldsymbol{L}_t^{\boldsymbol{Y}}\,, \quad 0 \leq t < \infty\,. \tag{2.34}$$

Here $(\boldsymbol{x}, \boldsymbol{y})$ is the initial state in the nonnegative quadrant, and $\boldsymbol{\mu}$, $\boldsymbol{\nu}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are real constants. A necessary and sufficient condition for the positive recurrence of $(\boldsymbol{X}_\cdot, \boldsymbol{Y}_\cdot)$ in (2.34) is

$$\boldsymbol{\mu} + \boldsymbol{\alpha} \boldsymbol{\nu}^- < 0\,, \qquad \boldsymbol{\nu} + \boldsymbol{\beta} \boldsymbol{\mu}^- < 0\,; \tag{2.35}$$

and $x^- = \max(-x, 0)$ is the negative part of $x \in \mathbb{R}$ (see Proposition 2.3 of [20]). By contrast, our system (2.12)-(2.13) is driven by the single Brownian motion $W(\cdot)$, thus *degenerate,* and has the form

$$\boldsymbol{X}_t = \boldsymbol{x} - \boldsymbol{W}_t + \boldsymbol{\mu} t + \boldsymbol{L}_t^{\boldsymbol{X}} + \boldsymbol{\alpha} \boldsymbol{L}_t^{\boldsymbol{Y}}\,, \quad \boldsymbol{Y}_t = \boldsymbol{y} + \boldsymbol{W}_t + \boldsymbol{\nu} t + \boldsymbol{\beta} \boldsymbol{L}_t^{\boldsymbol{X}} + \boldsymbol{L}_t^{\boldsymbol{Y}}\,, \quad 0 \leq t < \infty \tag{2.36}$$

that one obtains by replacing formally the planar Brownian motion $(\boldsymbol{B}_\cdot, \boldsymbol{W}_\cdot)$ in (2.34) by $(-\boldsymbol{W}_\cdot, \boldsymbol{W}_\cdot)$. The system (2.12)-(2.13) can be cast in the form (2.36), if we replace formally the triple $(\boldsymbol{X}_\cdot, \boldsymbol{Y}_\cdot, \boldsymbol{W}_\cdot)$ by the triple $(G(\cdot), H(\cdot), W(\cdot))$ and substitute $\boldsymbol{\mu} = -(\delta_2 - \delta_1)$, $\boldsymbol{\nu} = -(\delta_3 - \delta_2)$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = -1/2$.

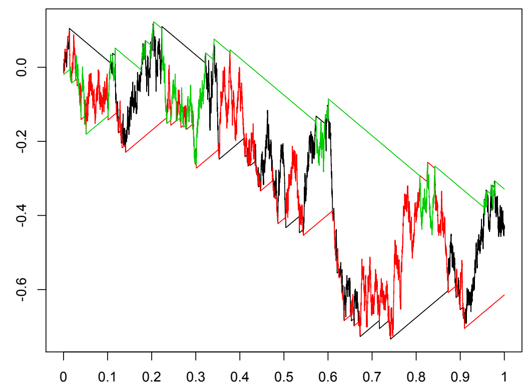

Figure 1. Simulated processes for the system in (2.5) with $\gamma = 1$: Black $= X_1(\cdot)$, Red $= X_2(\cdot)$, Green $= X_3(\cdot)$. The 3-D process $(X_1(\cdot), X_2(\cdot), X_3(\cdot))$ carries the same information content as a scalar Brownian Motion.

2.3.1. Existence, Uniqueness and Ergodicity of an Invariant Distribution

Theorem 2.3. *Under the conditions*

$$2(\delta_3 - \delta_2) + \big(\delta_1 - \delta_2\big)^- > 0\,, \qquad 2(\delta_2 - \delta_1) + \big(\delta_2 - \delta_3\big)^- > 0\,, \tag{2.37}$$

the process $\big(G(\cdot), H(\cdot)\big)$ *in (2.12)–(2.13) is positive-recurrent, and has a unique invariant measure* $\boldsymbol{\pi}$ *with* $\boldsymbol{\pi}((0,\infty)^2) = 1$, *to which its time-marginal distributions converge as* $t \to \infty$.

Let us note that the condition (2.37) is a simple recasting of (2.35). It is satisfied in the special case of (2.4); and more generally, under the strict ordering

$$\delta_1 \,<\, \delta_2 \,<\, \delta_3\,. \tag{2.38}$$

This condition is strictly stronger than (2.37); e.g., the choices $\delta_1 = 1/3,\ \delta_2 = 0,\ \delta_3 = 1$ satisfy (2.37) but not (2.38). We note that (2.37) implies $\delta_1 < \delta_3$, and at least one of $\delta_2 > \delta_1$, $\delta_3 > \delta_2$; that is, (2.37) excludes the possibility $\delta_3 \le \delta_2 \le \delta_1$. These claims are discussed in detail in Remark 2.2.

Proof: We start with a remark on the positivity of the transition probability

$$p^t(y, A) := \mathbb{P}\big((G(t), H(t)) \in A \,\big|\, (G(0), H(0)) = y\big)\,, \qquad y = (y_1, y_2) \in (0,\infty)^2, \ \ A \in \mathcal{B}\big((0,\infty)^2\big).$$

We recall that $G(\cdot) + H(\cdot)$ is of finite first variation, and decreases monotonically until $(G(\cdot), H(\cdot))$ hits one of the edges, i.e., $\mathrm{d}(G(t) + H(t)) = -(\delta_3 - \delta_1)\mathrm{d}t + (1/2)(\mathrm{d}L^G(t) + \mathrm{d}L^H(t))$, $t \ge 0$.

Consider now a trapezoid $\mathcal{T}_{a,b} := \{(x,y) \in [0,\infty)^2 : -x + a \le y \le -x + b\}$ whose intersection with A has positive LEBESGUE measure $\mathrm{Leb}(\mathcal{T}_{a,b} \cap A) > 0$, for some $a, b > 0$. Two cases arise:
(i) If $y_1 + y_2 \ge b$, then the point y is located in the north-east of $\mathcal{T}_{a,b}$, and we see that

$$p^t(y, A) \ge p^t(y, A \cap \mathcal{T}_{a,b}) > 0 \tag{2.39}$$

is valid for every $t \ge (y_1 + y_2 - a)\,/\,(\delta_3 - \delta_1)$, by considering the paths which do not touch the edges.

(ii) If $y_1 + y_2 < b$, then considering the paths for which either $L^G(\cdot)$ or $L^H(\cdot)$ exhibit an increase and invoking the MARKOV property, we see that (2.39) holds for every $t > 0$.

In either case, we deduce that for every $y \in (0,\infty)^2$, $A \in \mathcal{B}((0,\infty)^2)$ with positive LEBESGUE measure $\mathrm{Leb}(A) > 0$, there exists a positive real number t^* such that $p^t(y, A) > 0$ holds for every $t \ge t^*$. This shows that the skeleton MARKOV chain of the process $(G(\cdot), H(\cdot))$ is irreducible.

• Let us define now, inductively, two sequences of stopping times $\tau := \tau_1 = \inf\{s \ge 0 : G(s) = 0\}$, $\sigma := \sigma_1 = \inf\{s \ge \tau : H(s) = 0\}$, $\tau_n := \inf\{s \ge \sigma_{n-1} : G(s) = 0\}$, $\sigma_n := \inf\{s \ge \tau_n : H(s) = 0\}$ for $n = 2, 3, \ldots$. Also let us define $\mathbf{T}_0 := \inf\{s \ge 0 : G(s)H(s) = 0\}$ and

$$\mathbf{T}_\dagger := \inf\{s \ge 0 : G(s) \le x_0, H(s) = 0\}\,, \qquad \mathbf{T}_r := \inf\{s \ge 0 : (G(s), H(s)) \in \mathrm{B}_0(r)\}\,,$$

where $\mathrm{B}_0(r)$ is the ball of radius $r > 0$ centered at the origin.

Most of the arguments in [20] carry over smoothly to the degenerate system (2.12)-(2.13). In fact, we can replace $B(\cdot)$ by $-W(\cdot)$ in the proof of Propositions 2.1-2.2 of [20], and deduce that, under (2.37), there exists a large enough $x_0 > 0$ such that for $x_1 - x_2 \ge x_0$ we have

$$\mathbb{E}^{(x_1 - x_2, 0)}[G(\sigma_1)] \le (x_1 - x_2)/2\,, \qquad \mathbb{E}^{(x_1 - x_2, 0)}[\sigma_1] \le 2C(x_1 - x_2)\,,$$

where C is some positive constant. Moreover, again replacing $B(\cdot)$ by $-W(\cdot)$ in the first part of the proof of Proposition 2.3 of [20], we deduce $\mathbb{E}[\mathbf{T}_\dagger] \le C(1 + \sqrt{(x_1 - x_2)^2 + (x_2 - x_3)^2}\,)$. We claim that there exists a constant $\boldsymbol{\delta} > 0$ such that a uniform estimate

$$\inf_{0 < y \le x_0} \mathbb{P}^{(y,0)}\Big(\mathbf{T}_\varepsilon \le \mathbf{T}_{2x_0} \wedge 1\Big) \ge \boldsymbol{\delta} > 0 \tag{2.40}$$

holds; once (2.40) has been established, positive recurrence under the condition (2.37) will follow.

Instead of showing (2.40), we shall argue under the condition (2.37) that for every $0 < \varepsilon < x_0$, there exists a positive constant $\boldsymbol{\delta} > 0$ such that

$$\inf_{\varepsilon < y \le x_0} \mathbb{P}^{(y,0)}\Big(\widetilde{\mathbf{T}}_\varepsilon \le \widetilde{\mathbf{T}}_{2x_0} \wedge \mathbf{t}_0(y)\Big) \;\ge\; \boldsymbol{\delta} > 0\,, \tag{2.41}$$

where $\mathbf{t}_0(y) := (y - (5/6)\varepsilon)\,/\,(\delta_3 - \delta_1 - (1/2)(\delta_3 - \delta_2)^+) > 0$, $\varepsilon < y \le x_0$ and

$$\widetilde{\mathbf{T}}_r := \inf\{s \ge 0 : G(s) + H(s) = r\}\,, \quad r \ge 0\,.$$

In fact, we shall evaluate the smaller probability $\inf_{\varepsilon < y \le x_0} \mathbb{P}^{(y,0)}\big(\widetilde{\mathbf{T}}_\varepsilon \le \widetilde{\mathbf{T}}_{2x_0} \wedge \mathbf{t}_0(y)\,, \widetilde{\mathbf{T}}_\varepsilon < \tau\big)$, where we recall $\tau := \inf\{s > 0 : G(s) = 0\}$; that is, the probability that the process $(G(\cdot), H(\cdot))$, starting from the point $(G(0), H(0)) = (y, 0)$ on the axis, with $y \in (0, x_0]$, reaches the neighborhood of the origin before going away from the origin and before attaining the other axis.

We argue as follows: The process $(G(\cdot), H(\cdot))$ does not accumulate any local time $A(\cdot)$ before τ:

$$0 < G(t) \,=\, y - (\delta_2 - \delta_1)t - W(t) - (1/2)\Gamma(t)\,, \qquad 0 \le H(t) \,=\, -(\delta_3 - \delta_2)t + W(t) + \Gamma(t)\,,$$

and consequently $G(t) + H(t) \,=\, y - (\delta_3 - \delta_1)t + (1/2)\Gamma(t)$, for $0 \le t \le \tau$. From the SKOHOKHOD construction, we obtain the upper bound

$$\Gamma(t) \,=\, \max_{0 \le s \le t}\big(-W(s) + (\delta_3 - \delta_2)s\big)^+ \le \max_{0 \le s \le t}(-W(s))^+ + (\delta_3 - \delta_2)^+ t\,, \quad 0 \le t \le \tau$$

for the local time $\Gamma(\cdot)$. Thus we obtain

$$G(t) \geq y - \Big(\delta_2 - \delta_1 + \frac{1}{2}(\delta_3 - \delta_2)^+\Big)t - W(t) - \frac{1}{2}\max_{0 \leq s \leq t}\big(-W(s)\big)^+, \tag{2.42}$$

$$G(t) + H(t) \leq y - \Big(\delta_3 - \delta_1 - \frac{1}{2}(\delta_3 - \delta_2)^+\Big)t + \frac{1}{2}\max_{0 \leq s \leq t}\big(-W(s)\big)^+; \quad 0 \leq t \leq \tau. \tag{2.43}$$

Now let us consider the event

$$A(y) := \Big\{\omega \in \Omega : \max_{0 \leq s \leq \mathbf{t}_0(y)} |W(s,\omega)| \leq \varepsilon/3\Big\}, \qquad \varepsilon < y \leq x_0.$$

Since $\delta_3 - \delta_1 - (1/2)(\delta_3 - \delta_2)^+ \leq \delta_2 - \delta_1 + (1/2)(\delta_3 - \delta_2)^+$, for every $\omega \in A(y)$ we obtain

$$\min_{0 \leq t \leq \mathbf{t}_0(y)} \Big[y - \Big(\delta_2 - \delta_1 + \frac{1}{2}(\delta_3 - \delta_2)^+\Big)t - W(t,\omega) - \frac{1}{2}\max_{0 \leq s \leq t}\big(-W(s,\omega)\big)^+\Big] \geq \frac{\varepsilon}{3} > 0,$$

hence, combining with (2.42), we obtain $A(y) \subset \{\mathbf{t}_0(y) < \tau\}$. Moreover, for every $\omega \in A(y)$ we have

$$\min_{0 \leq t \leq \mathbf{t}_0(y)} \big(G(t,\omega) + H(t,\omega)\big) \leq \min_{0 \leq t \leq \mathbf{t}_0(y)} \Big[y - \Big(\delta_3 - \delta_1 - \frac{1}{2}(\delta_3 - \delta_2)^+\Big)t + \frac{1}{2}\max_{0 \leq s \leq t}\big(-W(s,\omega)\big)^+\Big] \leq \varepsilon,$$

thus also $\max_{0 \leq t \leq \mathbf{t}_0(y)} \big(G(t,\omega) + H(t,\omega)\big) \leq x_0 + \varepsilon < 2x_0$. We deduce the set-inclusion $A(y) \subset \{\widetilde{\mathbf{T}}_\varepsilon \leq \widetilde{\mathbf{T}}_{2x_0} \wedge \mathbf{t}_0(y), \widetilde{\mathbf{T}}_\varepsilon < \tau\}$, so the reflection principle for Brownian motion gives

$$\inf_{\varepsilon < y \leq x_0} \mathbb{P}^{(y,0)}(\widetilde{\mathbf{T}}_\varepsilon \leq \widetilde{\mathbf{T}}_{2x_0} \wedge \mathbf{t}_0(y)) \geq \inf_{\varepsilon < y \leq x_0} \mathbb{P}^{(y,0)}(\widetilde{\mathbf{T}}_\varepsilon \leq \widetilde{\mathbf{T}}_{2x_0} \wedge \mathbf{t}_0(y), \widetilde{\mathbf{T}}_\varepsilon < \tau)$$

$$\geq \inf_{\varepsilon < y \leq x_0} \mathbb{P}^{(y,0)}(A(y)) \geq 1 - \Big(\frac{\mathbf{t}_0(x_0)}{2\pi}\Big)^{1/2} \cdot \frac{4}{\varepsilon/3} \cdot \exp\Big(-\frac{(\varepsilon/3)^2}{2\mathbf{t}_0(x_0)}\Big).$$

Selecting $\varepsilon \in (0,1)$ small enough so that this right-most expression is positive, and denoting it by $\boldsymbol{\delta} > 0$, we obtain (2.41). We appeal now to the second half of the proof of Proposition 2.3 in [20], page 393, and conclude that the system (2.12)-(2.13) is positive-recurrent for neighborhoods, under (2.37).

For the remaining claims of the Theorem, let us recall the equations of (2.12)-(2.13) written in the HARRISON & REIMAN [16] form (2.18)-(2.20), and note that the process $\mathfrak{Z}(\cdot)$ of (2.19) has independent, stationary increments with $\mathfrak{Z}(t) = 0$ and $\mathbb{E}\big|\mathfrak{Z}(1)\big| < \infty$. Now, as is relatively easy to verify (and shown in Remark 2.2), the conditions of (2.37) imply that the components of the vector

$$-\mathcal{R}^{-1}\,\mathbb{E}\big(\mathfrak{Z}(1)\big) = \frac{2}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \delta_2 - \delta_1 \\ \delta_3 - \delta_2 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} \delta_2 + \delta_3 - 2\delta_1 \\ 2\delta_3 - \delta_1 - \delta_2 \end{pmatrix} =: \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \boldsymbol{\lambda} \tag{2.44}$$

are both strictly positive; cf. (2.49) below. Then Corollary 2.1 in [29] implies that the planar process $\mathfrak{G}(\cdot) = \big(G(\cdot), H(\cdot)\big)$ is positive recurrent, has a unique invariant probability measure $\boldsymbol{\pi}$, and converges to this measure in distribution as $t \to \infty$. The claim $\boldsymbol{\pi}\big((0,\infty)^2\big) = 1$ follows now from (2.27) and the strong law of large numbers (for bounded, measurable $f : [0,\infty)^2 \to \mathbb{R}$):

$$\lim_{T \to \infty} \frac{1}{T}\int_0^T f\big(G(t), H(t)\big)\,\mathrm{d}t = \int_{[0,\infty)^2} f(g,h)\,\boldsymbol{\pi}(\mathrm{d}g, \mathrm{d}h), \quad \text{a.e.} \qquad \square$$

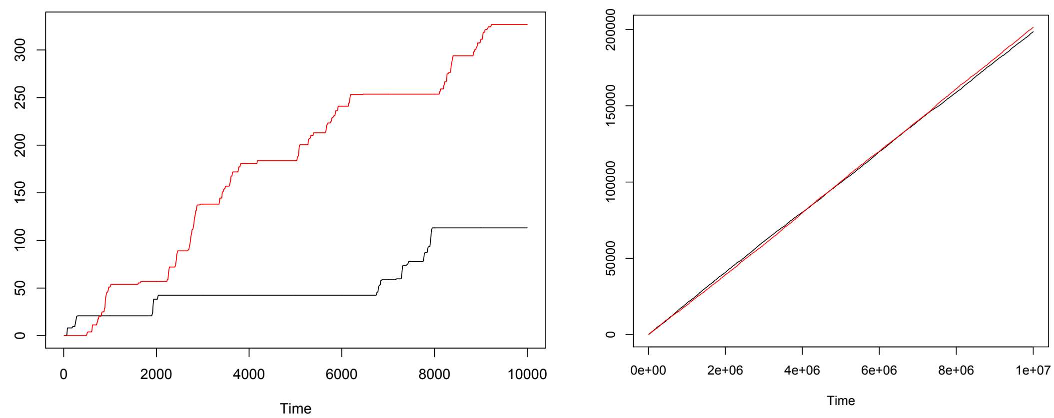

Figure 2. Simulated local times $L^G(\cdot)$ (black) and $L^H(\cdot)$ (red) for a short time (left panel) and for a long time (right panel) with $\delta_1 = 0.01$, $\delta_2 = 0.02$, $\delta_3 = 0.03$, thus $\lambda_1 = \lambda_2 = 0.02$. The long-term growth rates converge in the manner of (2.45), as the time-horizon increases; whereas over short time horizons, the CANTOR-function-like nature of local time becomes quite evident.

Proposition 2.4. *Under the conditions of (2.37), the local times accumulated at the origin by the "gap" processes $G(\cdot)$ and $H(\cdot)$ satisfy in the notation of (2.44) the strong laws of large numbers*

$$\lim_{t\to\infty} \frac{L^G(t)}{t} = \lambda_1 , \qquad \lim_{t\to\infty} \frac{L^H(t)}{t} = \lambda_2 , \qquad a.e. \tag{2.45}$$

Proof: The two-dimensional process $\big(G(\cdot), H(\cdot)\big)$ of gaps has under (2.37) a unique invariant probability measure $\boldsymbol{\pi}$ on $\mathcal{B}\big((0,\infty)^2\big)$, to which it converges in distribution. This implies, *a fortiori*, that

$$\lim_{t\to\infty} \frac{G(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \frac{H(t)}{t} = 0 \tag{2.46}$$

hold in distribution, thus also in probability. Back into (2.24), (2.25) and in conjunction with the law of large numbers for the Brownian motion $W(\cdot)$, these observations give that

$$\lim_{t\to\infty} \frac{2\,L^G(t) - L^H(t)}{2\,t} = \delta_2 - \delta_1 , \qquad \lim_{t\to\infty} \frac{2\,L^H(t) - L^G(t)}{2\,t} = \delta_3 - \delta_2 \tag{2.47}$$

hold in probability, and thus the same is true of (2.45). There exist then sequences $\{t_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ and $\{\tau_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ which increase strictly to infinity, and along which we have, a.e.,

$$\lim_{k\to\infty} \frac{L^G(\tau_k)}{\tau_k} = \frac{2}{3}\big(\delta_2 + \delta_3 - 2\,\delta_1\big) , \qquad \lim_{k\to\infty} \frac{L^H(t_k)}{t_k} = \frac{2}{3}\big(2\,\delta_3 - \delta_1 - \delta_2\big).$$

Theorem II.2 in [2] gives that $\lim_{t\to\infty}\big(L^G(t)/t\big)$ and $\lim_{t\to\infty}\big(L^H(t)/t\big)$ do exist almost everywhere (see also [17], sections 7, 8). It follows that the limits in (2.45) are valid not just in probability, but also almost everywhere; the same is true then for those of (2.46). □

The long-term growth rates of the local times $(L^G(\cdot), L^H(\cdot))$ in (2.45) are consistent with simulated local times based on the SKOROKHOD map in [16]. The simulations, reported in Figure 2, demonstrate the long-term linear growth of these local times with the rates of (2.45).

***Remark* 2.2** (Discussion of Condition (2.37), and a Sanity Check). We note that if $\delta_1 \ge \delta_2 \ge \delta_3$, the conditions of (2.37) cannot hold; this is because we have then

$$2(\delta_2 - \delta_1) + \big(\delta_2 - \delta_3\big)^- = 2(\delta_2 - \delta_1) \le 0\,, \qquad 2(\delta_3 - \delta_2) + \big(\delta_1 - \delta_2\big)^- = 2(\delta_3 - \delta_2) \le 0\,.$$

Thus, under (2.37), we have either $\delta_2 > \delta_1$ or $\delta_3 > \delta_2$. Only three cases are compatible with (2.37):

$$\text{(i) } \delta_1 < \delta_2 < \delta_3\,, \qquad \text{(ii) } \delta_2 > \delta_1 \text{ and } \delta_2 \ge \delta_3\,, \qquad \text{(iii) } \delta_3 > \delta_2 \text{ and } \delta_1 \ge \delta_2\,.$$

It can be shown that, in all these cases, the conditions of (2.37) imply

$$\delta_3 > \delta_1\,, \tag{2.48}$$

$$2\delta_3 - \delta_1 - \delta_2 > 0\,, \qquad \delta_2 + \delta_3 - 2\delta_1 > 0\,; \tag{2.49}$$

then, the a.e. limits in (2.45) are positive.

The inequalities of (2.49) imply both (2.48) and (2.37). As observed by the referee, the condition (2.49) has the interpretation that if any partition of $\{1,2,3\}$ into two subsets of consecutive indices, the leftmost group of particles has a larger average drift than the rightmost group; cf. [35], page 2187.

2.3.2. Exponential Convergence

Proposition 2.5. *Under the assumptions of Theorem 2.3, the function*

$$V(g,h) := \exp\Big\{\sqrt{g^2 + gh + h^2}\,\Big\}\,, \qquad (g,h) \in [0,\infty)^2 \setminus (0,0) \tag{2.50}$$

is LYAPUNOV *for the process $\mathfrak{G}(\cdot)$ in (2.12)–(2.13); i.e., there exist constants $a, b, \kappa > 0$ such that*

$$\mathcal{Z}(\cdot) := V\big(\mathfrak{G}(\cdot)\big) - V\big(\mathfrak{G}(0)\big) + \int_0^{\cdot} \Big(\kappa \cdot V\big(\mathfrak{G}(t)\big) - b \cdot \mathbf{1}_{\mathcal{T}_a}\big(\mathfrak{G}(t)\big)\Big)\mathrm{d}t \tag{2.51}$$

is a supermartingale, for $\mathcal{T}_a := \big\{(g,h) \in [0,\infty)^2 : g + h \le a\big\}$. In particular, the time-marginal distributions of the positive-recurrent process $\mathfrak{G}(\cdot)$ of gaps between ranks, converge exponentially fast in total variation to the unique invariant probability measure $\boldsymbol{\pi}$ of the process.

Proof: Applying ITÔ's formula to $V(\mathfrak{G}(\cdot))$, in conjunction with the easy consequence

$$\mathrm{d}\big(G^2(t) + G(t)H(t) + H^2(t)\big) = \Big[1 - \frac{3}{2}\big(\lambda_1 G(t) + \lambda_2 H(t)\big)\Big]\mathrm{d}t + \big(H(t) - G(t)\big)\,\mathrm{d}W(t) \tag{2.52}$$

of (2.24), (2.25) we obtain the decomposition $V(\mathfrak{G}(t)) = V(\mathfrak{G}(0)) + M^V(t) + A^V(t)$, where

$$M^V(t) := \int_0^t \frac{V(\mathfrak{G}(s)) \cdot (G(s) - H(s))}{2\sqrt{G^2(s) + G(s)H(s) + H^2(s)}}\,\mathrm{d}W(s), \qquad A^V(t) := \int_0^t [\mathcal{A}V](\mathfrak{G}(s))\,\mathrm{d}s,$$

$$[\mathcal{A}V](g,h) := \frac{V(g,h)}{2\sqrt{g^2 + gh + h^2}}\Big[1 - \frac{3}{2}(\lambda_1 g + \lambda_2 h)\Big] + \frac{V(g,h)(g-h)^2}{8(g^2 + gh + h^2)} - \frac{V(g,h)(g-h)^2}{8(g^2 + gh + h^2)^{3/2}}.$$

For arbitrary small $\varepsilon > 0$ and sufficiently large $a > 0$, the drift function $[\mathcal{A}V](g,h)$ satisfies

$$[\mathcal{A}V](g,h) \le -\kappa \cdot V(g,h) + b \cdot \mathbf{1}_{\mathcal{T}_a}(g,h)\,; \quad (g,h) \in \mathcal{T}_{\varepsilon,\infty}$$

for some $\kappa := (3/4)\min(\lambda_1, \lambda_2) > 0$, $b := \sup\{V(g,h)((1/8) + 1/(2\sqrt{g^2+gh+h^2})) : (g,h) \in \mathcal{T}_{\varepsilon,a}\}$, with the trapezoids $\mathcal{T}_{\varepsilon,a} := \{(x,y) \in [0,\infty)^2 : -x+\varepsilon \le y \le -x+a\}$ and $\mathcal{T}_{\varepsilon,\infty} := \{(x,y) \in [0,\infty)^2 : -x+\varepsilon \le y\}$. Then $\mathcal{Z}(\cdot)$ in (2.51) is a local supermartingale satisfying $\mathcal{Z}(t) \ge -V(\mathfrak{G}(0)) - bt$ for $t \ge 0$, hence a supermartingale by FATOU's lemma. The function V of (2.50) and its derivatives are not defined at the origin; but by Proposition 2.1, the process $\mathfrak{G}(\cdot)$ does not attain the origin when started away from it. Proposition 3.1 in [8] (and its references, as well as Definitions 5, 6 of [40] in the context of SRBM), shows that V is a LYAPUNOV function. As in the proof of Theorem 2.3, $\mathfrak{G}(\cdot)$ has an irreducible skeleton chain and hence, by Theorem 6.1 of [34], is aperiodic. We appeal now to the results in [40] (cf. [19], [34], [9], [8])These show that $\mathfrak{G}(\cdot)$ is positive recurrent, has a unique invariant distribution, and is $V-$uniformly ergodic. □

3. Ballistic Middle Motion, Diffusive Hedges

We take up in this section the "obverse" of the three-particle system in (2.1), (2.2), by which we mean replacing the equations in (2.1) by

$$X_i(\cdot) = x_i + \sum_{k=1}^{3} \delta_k \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_k^X(t)\}}\, \mathrm{d}t + \int_0^{\cdot} \Big(\mathbf{1}_{\{X_i(t)=R_1^X(t)\}} + \mathbf{1}_{\{X_i(t)=R_3^X(t)\}} \Big)\, \mathrm{d}B_i(t) \tag{3.1}$$

for $i = 1,2,3$, and replacing in the notation of (2.3) and (2.10) the conditions of (2.2) by

$$\int_0^{\infty} \mathbf{1}_{\{R_k^X(t)=R_\ell^X(t)\}}\, \mathrm{d}t = 0\,, \quad \forall\ k<\ell\,; \qquad L^{R_1^X - R_3^X}(\cdot) \equiv 0. \tag{3.2}$$

The processes $B_1(\cdot), B_2(\cdot), B_3(\cdot)$, are again independent scalar Brownian motions. It is now the leading and laggard particles that undergo diffusion, and the middle particle that "goes ballistic". The dynamics (3.1) involve again dispersion functions that are both discontinuous and degenerate.

In contrast to Proposition 2.1, however, we shall see here that *"the two Brownian motions can eventually squeeze the ballistic motion in the middle",* and thus triple points can occur; in fact, *with probability one in the case* $\delta_1 = \delta_2 = \delta_3$. Yet also, that the resulting triple collisions are "soft", in that the local time $L^{R_1^X - R_3^X}(\cdot)$ associated with them is identically equal to zero, as postulated in the second requirement of (3.2). The first requirement there, mandates that all collisions are non-sticky.

3.1. Analysis

Let us assume that a weak solution to this system of (3.1), (3.2) has been constructed on an appropriate filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbb{F} = \{\mathfrak{F}(t)\}_{0 \le t < \infty}$. Reasoning as before, we have the analogues

$$R_1^X(t) = x_1 + \delta_1 t + W_1(t) + \frac{1}{2}\Lambda^{(1,2)}(t)\,, \quad R_2^X(t) = x_2 + \delta_2 t - \frac{1}{2}\Lambda^{(1,2)}(t) + \frac{1}{2}\Lambda^{(2,3)}(t)\,,$$
$$R_3^X(t) = x_3 + \delta_3 t + W_3(t) - \frac{1}{2}\Lambda^{(2,3)}(t)\,; \quad t \ge 0 \tag{3.3}$$

of (2.6)-(2.7) in the notation of (2.9). As in (2.8), the processes

$$W_k(\cdot) := \sum_{i=1}^{3} \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_k^X(t)\}} \,\mathrm{d}B_i(t)\,, \qquad k=1,3 \tag{3.4}$$

are independent Brownian motions by the P. LÉVY theorem. It is fairly clear that the center of gravity of this system evolves as Brownian motion with drift, since $\sum_{i=1}^{3} X_i(t) = x + \delta\, t + \sqrt{2}\, Q(t)$ for $x = x_1 + x_2 + x_3$, $\delta = \delta_1 + \delta_2 + \delta_3$, and $Q(\cdot) = (W_1(\cdot) + W_3(\cdot))/\sqrt{2}$ is standard Brownian motion.

Now, the gaps $G(\cdot) := R_1^X(\cdot) - R_2^X(\cdot)$ and $H(\cdot) := R_2^X(\cdot) - R_3^X(\cdot)$ are given as

$$G(t) = U(t) + L^G(t)\,, \qquad H(t) = V(t) + L^H(t)\,, \qquad 0 \le t < \infty$$

in the manner of (2.12), (2.13), where again $L^G(\cdot) \equiv \Lambda^{(1,2)}(\cdot)$, $L^H(\cdot) \equiv \Lambda^{(2,3)}(\cdot)$, and

$$U(t) := x_1 - x_2 - \big(\delta_2 - \delta_1\big)\, t + W_1(t) - \frac{1}{2}\, L^H(t)\,, \quad V(t) := x_2 - x_3 - \big(\delta_3 - \delta_2\big)\, t - W_3(t) - \frac{1}{2}\, L^G(t)\,.$$

The theory of the SKOROKHOD reflection problem provides the system of equations linking the two local time processes $L^G(\cdot)$, $L^H(\cdot)$, an analogue of the system (2.16), (2.17):

$$L^G(t) = \max_{0 \le s \le t} \big(-U(s)\big)^+ = \max_{0 \le s \le t} \Big(-(x_1 - x_2) + \big(\delta_2 - \delta_1\big)\, s - W_1(s) + \frac{1}{2}\, L^H(s)\Big)^+ \tag{3.5}$$

$$L^H(t) = \max_{0 \le s \le t} \big(-V(s)\big)^+ = \max_{0 \le s \le t} \Big(-(x_2 - x_3) + \big(\delta_3 - \delta_2\big)\, s + W_3(s) + \frac{1}{2}\, L^G(s)\Big)^+. \tag{3.6}$$

3.2. Synthesis

Starting again with given real numbers δ_1, δ_2, δ_3 and $x_1 > x_2 > x_3$, we construct a filtered probability space $(\Omega, \widetilde{\mathfrak{F}}, \mathbb{P})$, $\widetilde{\mathbb{F}} = \big\{\widetilde{\mathfrak{F}}(t)\big\}_{0 \le t < \infty}$ which supports three independent, standard Brownian motions $W_k(\cdot)$, $k = 1, 2, 3$. We consider the analogue

$$A(t) = \max_{0 \le s \le t} \Big(-(x_1 - x_2) + \big(\delta_2 - \delta_1\big)\, s - W_1(s) + \frac{1}{2}\, \Gamma(s)\Big)^+, \quad 0 \le t < \infty \tag{3.7}$$

$$\Gamma(t) = \max_{0 \le s \le t} \Big(-(x_2 - x_3) + \big(\delta_3 - \delta_2\big)\, s + W_3(s) + \frac{1}{2}\, A(s)\Big)^+, \quad 0 \le t < \infty \tag{3.8}$$

of the system of equations (3.5) and (3.6) for two continuous, nondecreasing and adapted processes $A(\cdot)$ and $\Gamma(\cdot)$ with $A(0) = \Gamma(0) = 0$. Once again, the theory of [16] guarantees the existence of a unique continuous solution $\big(A(\cdot), \Gamma(\cdot)\big)$ for the system (3.7), (3.8), adapted to the filtration $\mathbb{F}^{(W_1, W_3)}$ generated by the 2-D Brownian motion $(W_1(\cdot), W_3(\cdot))$:

$$\mathfrak{F}^{(A,\Gamma)}(t) \subseteq \mathfrak{F}^{(W_1, W_3)}(t)\,, \qquad 0 \le t < \infty\,. \tag{3.9}$$

With the processes $A(\cdot)$, $\Gamma(\cdot)$ thus in place, we consider the continuous semimartingales

$$U(t) := x_1 - x_2 - \big(\delta_2 - \delta_1\big)\, t + W_1(t) - \frac{1}{2}\, \Gamma(t)\,, \quad V(t) := x_2 - x_3 - \big(\delta_3 - \delta_2\big)\, t - W_3(t) - \frac{1}{2}\, A(t)$$

and then “fold" them to obtain their SKOROKHOD reflections

$$G(t) := U(t) + \max_{0\le s\le t} \big(- U(s)\big)^+ = x_1 - x_2 - \big(\delta_2 - \delta_1\big)\, t + W_1(t) - \frac{1}{2}\,\Gamma(t) + A(t) \ge 0 \quad (3.10)$$

$$H(t) := V(t) + \max_{0\le s\le t} \big(- V(s)\big)^+ = x_2 - x_3 - \big(\delta_3 - \delta_2\big)\, t - W_3(t) - \frac{1}{2}\, A(t) + \Gamma(t) \ge 0 \quad (3.11)$$

for $t \in [0,\infty)$. This system of equations (3.10), (3.11) can be cast in the HARRISON-REIMAN form

$$\begin{pmatrix} G(t) \\ H(t) \end{pmatrix} = \begin{pmatrix} G(0) \\ H(0) \end{pmatrix} + \mathfrak{Z}(t) + \mathcal{R}\,\mathfrak{L}(t)\,, \qquad 0 \le t < \infty$$

of (2.18), now with covariance matrix

$$\mathcal{C} := \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}, \qquad \text{reflection matrix} \qquad \mathcal{R} = \mathcal{I} - \mathcal{Q}\,, \quad \mathcal{Q} := \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \text{ and}$$

$$\mathfrak{L}(t) = \begin{pmatrix} L^G(t) \\ L^H(t) \end{pmatrix}, \qquad \mathfrak{Z}(t) = \begin{pmatrix} (\delta_1 - \delta_2)t + W_1(t) \\ (\delta_2 - \delta_3)t - W_3(t) \end{pmatrix}, \qquad 0 \le t < \infty\,.$$

We obtain easily the analogues

$$\int_0^\infty \mathbf{1}_{\{G(t)>0\}}\, \mathrm{d}A(t) = 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t)>0\}}\, \mathrm{d}\Gamma(t) = 0\,, \quad (3.12)$$

$$\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}t = 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}t = 0 \quad (3.13)$$

of the properties in (2.26), (2.27) using, respectively, the theories of the SKOROKHOD reflection problem and of semimartingale local time. We claim that we also have here the analogues

$$\int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}A(t) = 0\,, \qquad \int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}\Gamma(t) = 0 \quad (3.14)$$

of the properties in (2.30), though now for a different reason.

Let us elaborate: The system of (3.10), (3.11) characterizes a non-degenerate, two-dimensional Brownian motion $(G(\cdot),\, H(\cdot))$ with drift $(\delta_1 - \delta_2,\, \delta_2 - \delta_3)$, reflected off the faces of the nonnegative quadrant. But now, in contrast to the situation prevalent in Section 2, it becomes perfectly possible for this planar motion to hit the corner of the nonnegative orthant with positive probability. In fact, according to Theorem 2.2 of [43] (see also [39] [21]; this theory is not directly applicable to the setting of Section 2, or to that of Section A, because there the driving Brownian motions are one-dimensional), *when* $\delta_1 = \delta_2 = \delta_3$ *this process will hit the corner of the quadrant with probability one:* $\mathbb{P}(G(t) = H(t) = 0, \text{ for some } t > 0) = 1$.

Yet, we have always with probability one:

$$\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}\Gamma(t) = \int_0^\infty \mathbf{1}_{\{G(t)=H(t)=0\}}\, \mathrm{d}\Gamma(t) = 0\,, \quad (3.15)$$

$$\int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}A(t) = \int_0^\infty \mathbf{1}_{\{G(t)=H(t)=0\}}\, \mathrm{d}A(t) = 0\,. \quad (3.16)$$

Here the first two equalities come from those in (3.12), and the second two equalities from Theorem 1 in [37]. The claims in (3.14) are thus established. Armed with the properties (3.12)-(3.14), we obtain here again the identifications $L^G(\cdot) \equiv A(\cdot)$, $L^H(\cdot) \equiv \Gamma(\cdot)$ of the processes $A(\cdot), \Gamma(\cdot)$ in (3.7), (3.8) as local times. Details are omitted, as they are very similar to what was done before.

- *Construction of the Ranked Motions:* We introduce now, by analogy with (2.28)-(2.29), the processes

$$\begin{aligned} R_1(t) &:= x_1 + \delta_1 t + W_1(t) + \frac{1}{2} A(t), \qquad R_2(t) := x_2 + \delta_2 t - \frac{1}{2} A(t) + \frac{1}{2} \Gamma(t), \\ R_3(t) &:= x_3 + \delta_3 t + W_3(t) - \frac{1}{2} \Gamma(t) \end{aligned} \tag{3.17}$$

for $0 \le t < \infty$, and note again the relations $R_1(\cdot) - R_2(\cdot) = G(\cdot) \ge 0$, $R_2(\cdot) - R_3(\cdot) = H(\cdot) \ge 0$ and the comparisons $R_1(\cdot) \ge R_2(\cdot) \ge R_3(\cdot)$. The range

$$R_1(t) - R_3(t) = G(t) + H(t) = x_1 - x_3 + \big(\delta_1 - \delta_3\big)\, t + W_1(t) - W_3(t) + \frac{1}{2}\Big(A(t) + \Gamma(t)\Big), \quad 0 \le t < \infty$$

is a nonnegative semimartingale with $\langle R_1 - R_3 \rangle(t) = 2\,t$ and local time at the origin

$$L^{R_1 - R_3}(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{G(t)+H(t)=0\}} \Big[\big(\delta_1 - \delta_3\big)\, \mathrm{d}t + \frac{1}{2}\Big(\mathrm{d}A(t) + \mathrm{d}\Gamma(t)\Big) \Big] = 0 \tag{3.18}$$

by virtue of (2.10) and (3.13), (3.14). This is in accordance with the second property posited in (3.2).

Whereas, we argued already that, at least when $\delta_1 = \delta_2 = \delta_3$, the first time of a triple collision is a.e. finite: i.e., $\mathbb{P}(\mathcal{S} < \infty) = 1$ for

$$\mathcal{S} := \inf\big\{ t \ge 0 : R_1(t) = R_3(t) \big\} = \inf\{ t \ge 0 : G(t) = H(t) = 0 \}. \tag{3.19}$$

***Remark* 3.1** (Structure of Filtrations)**.** It follows from (3.17), (3.9) that the so-constructed triple $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ is adapted to the filtration $\mathbb{F}^{(W_1, W_3)}$ of the planar Brownian motion $(W_1(\cdot), W_3(\cdot))$:

$$\mathfrak{F}^{(R_1, R_2, R_3)}(t) \subseteq \mathfrak{F}^{(W_1, W_3)}(t), \qquad 0 \le t < \infty. \tag{3.20}$$

On the other hand, the identifications $A(\cdot) = L^G(\cdot) = L^{R_1 - R_2}(\cdot)$, $\Gamma(\cdot) = L^H(\cdot) = L^{R_2 - R_3}(\cdot)$ show that $(A(\cdot), \Gamma(\cdot))$ is adapted to the filtration $\mathbb{F}^{(R_1, R_2, R_3)}$ generated by the triple $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$; on account of (3.17), it follows that the same is true of the 2-D Brownian motion $(W_1(\cdot), W_3(\cdot))$.

In other words, the reverse inclusion of (3.20) is also valid, and we conclude that the triple $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ and the pair $(W_1(\cdot), W_3(\cdot))$ generate exactly the same filtration:

$$\mathfrak{F}^{(R_1, R_2, R_3)}(t) = \mathfrak{F}^{(W_1, W_3)}(t), \qquad 0 \le t < \infty. \tag{3.21}$$

- *Construction of the Individual Motions Up Until a Triple Collision:* The methodologies deployed in §2.2.3, show here as well how to construct a *strong* solution to the system (3.1) subject to the requirements of (3.2), up until the first time $\mathcal{S}$ of (3.19) when a triple collision occurs. The difference here, of course, is that this can happen now in finite time, with positive probability; in fact, with probability one, i.e., $\mathbb{P}(\mathcal{S} < \infty) = 1$, when $\delta_1 = \delta_2 = \delta_3$ as we have seen.

Thus, we need to find another way to construct a solution *beyond* this time, that is, on the event $\{S < \infty\}$. For concreteness, and in order to simplify terminology and notation, we shall assume for the remainder of the present subsection that this event has full $\mathbb{P}-$measure.

• *Construction of the Individual Motions After a Triple Collision:* In order to construct the processes that satisfy (3.1) after the first triple collision time $\mathcal{S}$, we consider the excursions of the rank-gap process $(G(\cdot), H(\cdot))$ and unfold them, by permuting randomly the names of the individual components.

More precisely, for the semimartingales $G(\cdot)$ and $H(\cdot)$ let us define the first passage time

$$\sigma_0 := \inf\big\{t \geq 0 : G(t) \wedge H(t) = 0\big\},$$

the zero sets

$$\mathfrak{Z}^G := \{t \geq 0 : G(t) = 0\}, \qquad \mathfrak{Z}^H := \{t \geq 0 : H(t) = 0\},$$

and the corresponding countably-many excursion intervals $\{\mathcal{C}^G_\ell, \ell \in \mathbb{N}\}$, $\{\mathcal{C}^H_m, m \in \mathbb{N}\}$ away from the origin in a measurable manner, i.e.,

$$\mathbb{R}_+ \setminus \mathfrak{Z}^G = \bigcup_{\ell \in \mathbb{N}} \mathcal{C}^G_\ell, \qquad \mathbb{R}_+ \setminus \mathfrak{Z}^H = \bigcup_{m \in \mathbb{N}} \mathcal{C}^H_m.$$

We need to permute the indices in a proper and consistent way, so we define the permutation matrices

$$\mathfrak{P}_{1,2} := \begin{pmatrix} 0\,1\,0 \\ 1\,0\,0 \\ 0\,0\,1 \end{pmatrix}, \qquad \mathfrak{P}_{2,3} := \begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix}. \tag{3.22}$$

Here $\mathfrak{P}_{1,2}$ permutes the first and second elements, and $\mathfrak{P}_{2,3}$ permutes the second and third elements.

We enlarge the probability space with I.I.D. random (permutation) matrices $\{\Xi^G_{\ell,m},\, \ell \in \mathbb{N},\, m \in \mathbb{N}\}$ and $\{\Xi^H_{\ell,m},\, \ell \in \mathbb{N},\, m \in \mathbb{N}\}$, independent of each other and of the filtration $\mathbb{F}^R(\cdot)$ generated by the rank process $(R_1(\cdot), R_2(\cdot), R_3(\cdot))'$. Here, for each (ℓ, m), the random matrix $\Xi^G_{\ell,m}$ takes each of the values in $\{\mathcal{I}, \mathfrak{P}_{1,2}\}$ with probability $1/2$; whereas $\Xi^H_{\ell,m}$ takes each of the values in $\{\mathcal{I}, \mathfrak{P}_{2,3}\}$ with probability $1/2$. With these ingredients we introduce the simple, matrix-valued process

$$\begin{aligned}\boldsymbol{\eta}(\cdot) := \sum_{\ell \in \mathbb{N}} \sum_{m \in \mathbb{N}} \mathbf{1}_{\mathcal{C}^G_\ell \cap \mathcal{C}^H_m \cap [\sigma_0, \infty)}(\cdot) \Big(\big(\Xi^G_{\ell,m} - \mathcal{I}\big)\, \mathbf{1}_{\{\inf \mathcal{C}^G_\ell > \inf \mathcal{C}^H_m\}} \\ + \big(\Xi^H_{\ell,m} - \mathcal{I}\big)\, \mathbf{1}_{\{\inf \mathcal{C}^G_\ell < \inf \mathcal{C}^H_m\}} \Big)\end{aligned}, \tag{3.23}$$

then define the matrix-valued process $Z(\cdot)$ as the solution to the stochastic integral equation

$$Z(\cdot) = \mathcal{I} + \int_0^{\cdot} Z(t)\, \mathrm{d}\boldsymbol{\eta}(t). \tag{3.24}$$

To construct this solution, we proceed via an approximating scheme as in (3.28)-(3.29) below.

The definition of the process $\boldsymbol{\eta}(\cdot)$ in (3.23), after the time σ_0, is understood as follows:

(i) On the interval $\mathcal{C}^G_\ell \cap \mathcal{C}^H_m$ of the excursion which starts from a point in $\mathfrak{Z}^G$ (i.e., $\inf \mathcal{C}^G_\ell > \inf \mathcal{C}^H_m$), the simple process $\boldsymbol{\eta}(\cdot)$ assigns to this excursion the $\mathbf{0}$ matrix with probability $1/2$, or with probability $1/2$, the non-zero matrix

$$\mathfrak{P}_{1,2} - \mathcal{I} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.25}$$

(ii) On the interval $\mathcal{C}_\ell^G \cap \mathcal{C}_m^H$ of the excursion which starts from a point in $\mathfrak{Z}^H$ (i.e., $\inf \mathcal{C}_\ell^G < \inf \mathcal{C}_m^H$), the simple process $\boldsymbol{\eta}(\cdot)$ assigns to this excursion the $\mathbf{0}$ matrix with probability $1/2$, or with probability $1/2$, the non-zero matrix

$$\mathfrak{P}_{2,3} - \mathcal{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}. \tag{3.26}$$

(iii) When the excursion starts from the corner $\{t \ge 0 : G(t) = H(t) = 0\}$ (that is, $\inf \mathcal{C}_\ell^G = \inf \mathcal{C}_m^H$ for some ℓ and m), then the process $\boldsymbol{\eta}(\cdot)$ assigns the $\mathbf{0}$ matrix to this excursion.

The value $Z(t)$ of the matrix-valued process defined in (3.24) represents the product of (countably many, random) permutations listed in (3.22), until time $t \ge 0$. Since products of permutations are also permutations, the process $Z(\cdot)$ takes values in the collection of permutation matrices.

Finally, with $R(\cdot) = (R_1(\cdot), R_2(\cdot), R_3(\cdot))'$ constructed as in (3.17), we define the vector process

$$X(\cdot) \equiv \big(X_1(\cdot), X_2(\cdot), X_3(\cdot)\big)' := Z(\cdot)R(\cdot). \tag{3.27}$$

• We introduce at this point the enlarged filtration $\mathbb{F} := \{\mathfrak{F}(t),\, t \ge 0\}$ via $\mathfrak{F}(t) := \widetilde{\mathfrak{F}}(t) \vee \mathfrak{F}^Z(t)$. Since the sequences of I.I.D. random matrices $\{\Xi^G_{\ell,m};\, \ell \in \mathbb{N},\, m \in \mathbb{N}\}$ and $\{\Xi^H_{\ell,m};\, \ell \in \mathbb{N},\, m \in \mathbb{N}\}$ are independent of $\mathbb{F}^R$, it can be shown as in [36] that both triples $(W_1(\cdot), W_2(\cdot), W_3(\cdot))$ and $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ are semimartingales of this enlarged filtration $\mathbb{F}$.

We can state now and prove the following result.

Theorem 3.1. *On the filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F} = \{\mathfrak{F}(t)\}_{t \ge 0}$ *just constructed, and with the process* $X(\cdot)$ *as in (3.27), there exists a three-dimensional Brownian motion* $B(\cdot) = (B_1(\cdot), B_2(\cdot), B_3(\cdot))'$ *such that* $(\Omega, \mathcal{F}, \mathbb{P}),\, \mathbb{F} = \{\mathfrak{F}(t)\}_{t \ge 0},\, (X(\cdot), B(\cdot))$ *is a weak solution for the system (3.1), (3.2).*

This solution is unique in the sense of the probability distribution; thus, $X(\cdot)$ *has the strong* MARKOV *property. It is also pathwise unique and strong, up until the first time* $\mathcal{S}$ *a triple collision occurs; however, both pathwise uniqueness and strength fail after time* $\mathcal{S}$.

Proof. We split the argument in three distinct parts.

(i) Existence: We show that, on a suitable filtered probability space with independent Brownian motions $B_1(\cdot), B_2(\cdot), B_3(\cdot)$, the process $X(\cdot)$ defined by (3.27), with $Z(\cdot)$ in (3.24) and $\boldsymbol{\eta}(\cdot)$ in (3.23), satisfies the dynamics (3.1) and the requirement (3.2). The proof is based on the technique of unfolding semimartingales, in the manner of [24] for WALSH semimartingales.

We start by defining recursively the sequence $\{\tau_\ell^\varepsilon, \ell \in \mathbb{N}_0\}$ of stopping times as $\tau_0^\varepsilon := 0$,

$$\tau_{2\ell+1}^\varepsilon := \inf\{t > \tau_{2\ell}^\varepsilon : G(t) \wedge H(t) \ge \varepsilon\}, \quad \tau_{2\ell+2}^\varepsilon := \inf\{t > \tau_{2\ell+1}^\varepsilon : G(t) \wedge H(t) = 0\}, \tag{3.28}$$

along with the approximating processes $X^\varepsilon(\cdot) := Z^\varepsilon(\cdot)R(\cdot)$, where

$$Z^\varepsilon(\cdot) = \mathcal{I} + \int_0^\cdot Z^\varepsilon(t)\mathrm{d}\boldsymbol{\eta}^\varepsilon(t), \qquad \boldsymbol{\eta}^\varepsilon(\cdot) := \sum_{\ell \in \mathbb{N}} \boldsymbol{\eta}(\cdot)\, \mathbf{1}_{[\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)}(\cdot) \tag{3.29}$$

for every $\varepsilon \in (0,1)$. For these approximating processes, the product rule gives

$$X^\varepsilon(\cdot) = \int_0^\cdot \mathrm{d}\big(Z^\varepsilon(t)R(t)\big) = \int_0^\cdot Z^\varepsilon(t)\,\mathrm{d}R(t) + \int_0^\cdot \mathrm{d}Z^\varepsilon(t)\,R(t). \tag{3.30}$$

Now, as $\varepsilon \downarrow 0$, the process $X^\varepsilon(\cdot)$ converges to $X(\cdot) = Z(\cdot)R(\cdot)$ in (3.27), and the first term on the right-hand side converges in probability to the stochastic integral $\int_0^\cdot Z(t)\,\mathrm{d}R(t)$.

Let us analyze the semimartingale dynamics of this last integral. Since $Z(\cdot)$ is a permutation-matrix-valued process, the absolutely continuous finite-variation ("drift") components of $\int_0^\cdot Z(t)\,\mathrm{d}R(t)$ are

$$\int_0^\cdot Z(t)\begin{pmatrix}\delta_1\\ \delta_2\\ \delta_3\end{pmatrix}\mathrm{d}t = \int_0^\cdot \sum_{k=1}^3 \begin{pmatrix}\delta_k \mathbf{1}_{\{X_1(t)=R_k(t)\}}\\ \delta_k \mathbf{1}_{\{X_2(t)=R_k(t)\}}\\ \delta_k \mathbf{1}_{\{X_3(t)=R_k(t)\}}\end{pmatrix}\mathrm{d}t\,.$$

Similarly, the martingale ("noise") components of $\int_0^\cdot Z(t)\,\mathrm{d}R(t)$ are given by

$$\begin{aligned}\int_0^\cdot Z(t)\begin{pmatrix}\mathrm{d}W_1(t)\\ 0\\ \mathrm{d}W_3(t)\end{pmatrix} &= \int_0^\cdot \begin{pmatrix}\mathbf{1}_{\{X_1(t)=R_1(t)\}}\mathrm{d}W_1(t) + \mathbf{1}_{\{X_1(t)=R_3(t)\}}\mathrm{d}W_3(t)\\ \mathbf{1}_{\{X_2(t)=R_1(t)\}}\mathrm{d}W_1(t) + \mathbf{1}_{\{X_2(t)=R_3(t)\}}\mathrm{d}W_3(t)\\ \mathbf{1}_{\{X_3(t)=R_1(t)\}}\mathrm{d}W_1(t) + \mathbf{1}_{\{X_3(t)=R_3(t)\}}\mathrm{d}W_3(t)\end{pmatrix}\\ &= \int_0^\cdot \begin{pmatrix}\big(\mathbf{1}_{\{X_1(t)=R_1(t)\}} + \mathbf{1}_{\{X_1(t)=R_3(t)\}}\big)\,\mathrm{d}B_1(t)\\ \big(\mathbf{1}_{\{X_2(t)=R_1(t)\}} + \mathbf{1}_{\{X_2(t)=R_3(t)\}}\big)\,\mathrm{d}B_2(t)\\ \big(\mathbf{1}_{\{X_3(t)=R_1(t)\}} + \mathbf{1}_{\{X_3(t)=R_3(t)\}}\big)\,\mathrm{d}B_3(t)\end{pmatrix};\end{aligned} \tag{3.31}$$

here, on account of the P. Lévy theorem, the processes

$$B_i(\cdot) := \sum_{k=1}^3 \int_0^\cdot \mathbf{1}_{\{X_i(t)=R_k(t)\}}\mathrm{d}W_k(t)\,, \qquad i = 1,2,3 \tag{3.32}$$

are independent $\mathbb{F}-$Brownian motions (recall that $W_1(\cdot), W_2(\cdot), W_3(\cdot)$ are independent $\mathbb{F}-$Brownian motions). Finally, the local time components contributed by the term $\int_0^\cdot Z(t)\mathrm{d}R(t)$ are

$$\int_0^\cdot Z(t)\begin{pmatrix}(1/2)\,\mathrm{d}L^G(t)\\ -(1/2)\,\mathrm{d}L^G(t) + (1/2)\,\mathrm{d}L^H(t)\\ -(1/2)\,\mathrm{d}L^H(t)\end{pmatrix} = \frac{1}{2}\int_0^\cdot Z(t)\left(\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}\mathrm{d}L^G(t) + \begin{pmatrix}0\\ 1\\ -1\end{pmatrix}\mathrm{d}L^H(t)\right). \tag{3.33}$$

On the other hand, in the limit as $\varepsilon \downarrow 0$ of the term $\int_0^\cdot \mathrm{d}Z^\varepsilon(t)R(t)$ in (3.30), local time components appear and cancel those in (3.33). More precisely, by (3.24), we have

$$\int_0^T \mathrm{d}Z^\varepsilon(t)R(t) = \int_0^T Z^\varepsilon(t)\big(\mathrm{d}\boldsymbol{\eta}^\varepsilon(t)\big)R(t)\,, \qquad \int_0^T \mathrm{d}\boldsymbol{\eta}^\varepsilon(t)R(t) = \sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}\le T\}} \boldsymbol{\eta}^\varepsilon(\tau^\varepsilon_{2\ell+1})R(\tau^\varepsilon_{2\ell+1}). \tag{3.34}$$

The random vector $\boldsymbol{\eta}^\varepsilon(\tau^\varepsilon_{2\ell+1})\,R(\tau^\varepsilon_{2\ell+1})$ can take values

$$\big(\mathfrak{P}_{1,2} - \mathcal{I}\big)R(\tau^\varepsilon_{2\ell+1}) = \begin{pmatrix}-R_1(\tau^\varepsilon_{2\ell+1}) + R_2(\tau^\varepsilon_{2\ell+1})\\ R_1(\tau^\varepsilon_{2\ell+1}) - R_2(\tau^\varepsilon_{2\ell+1})\\ 0\end{pmatrix} = \varepsilon\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}$$

or 0, each with equal probability $1/2$, if it corresponds to the excursion from $\mathfrak{Z}^G$ for sufficiently small $\varepsilon > 0$; and it can take values

$$\big(\mathfrak{P}_{2,3}-\mathcal{I}\big)R(\tau^{\varepsilon}_{2\ell+1}) = \begin{pmatrix} 0 \\ -R_2(\tau^{\varepsilon}_{2\ell+1})+R_3(\tau^{\varepsilon}_{2\ell+1}) \\ R_2(\tau^{\varepsilon}_{2\ell+1})-R_3(\tau^{\varepsilon}_{2\ell+1}) \end{pmatrix} = \varepsilon \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

or 0, each with probability $1/2$, if it corresponds to the excursion from $\mathfrak{Z}^H$, for sufficiently small $\varepsilon > 0$. We are deploying here (3.23), (3.25)-(3.26), and the continuity of the sample paths of $R(\cdot)$.

We recall now the excursion-theoretic characterization of semimartingale local time (Theorem VI.1.10 of [38]): for a continuous scalar semimartingale, the rescaled number of its "downcrossings" approximates in a weak-law-of-large-numbers fashion the local time accumulated at the origin (see [36], [22], and the proof of Theorem 2.1 in [24], for crucial applications of this result). Applying this approximation to each set of excursions from $\mathfrak{Z}^G$ and $\mathfrak{Z}^H$ in the summation of (3.34), for $G(\cdot)$ and $H(\cdot)$, respectively, leads to the limiting behavior

$$\int_0^T \mathrm{d}\boldsymbol{\eta}^{\varepsilon}(t)R(t) \underset{\varepsilon\downarrow 0}{\longrightarrow} \int_0^T \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\mathrm{d}L^G(t) + \int_0^T \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\mathrm{d}L^H(t)$$

in probability. Combining this limit with (3.34), we obtain the convergence in probability

$$\int_0^T \mathrm{d}Z^{\varepsilon}(t)R(t) \underset{\varepsilon\downarrow 0}{\longrightarrow} \frac{1}{2}\int_0^T Z(t)\left[\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\mathrm{d}L^G(t) + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\mathrm{d}L^H(t)\right] \tag{3.35}$$

and observe that the local time components in (3.33) are cancelled by the limit (3.35) of $\int_0^T \mathrm{d}Z^{\varepsilon}(t)R(t)$.

We conclude that the process $X(\cdot)$ in (3.27) satisfies the requirements of (3.1)-(3.2), and yields a weak solution $(\Omega,\mathcal{F},\mathbb{P})$, $\mathbb{F}=\{\mathfrak{F}(t)\}_{t\geq 0}$, $(X(\cdot),B(\cdot))$ for this system as described above.

(ii) Uniqueness in Distribution: Suppose that there are two probability measures $\mathbb{P}_1$, $\mathbb{P}_2$ under which $X(\cdot)$ in (3.27) satisfies (3.1)-(3.2), and $B(\cdot)$ is three-dimensional Brownian motion. For $j=1,2$ we have $\mathbb{P}_j(\mathcal{S}<\infty)=1$. By analogy with the discussion in § 2.2.3, up to the first triple collision time $\mathcal{S}$ in (2.33) this solution is pathwise unique, thus also *strong;* that is, adapted to the filtration $\mathbb{F}^{(B_1,B_2,B_3)}$ generated by the 3-D Brownian motion $(B_1(\cdot),B_2(\cdot),B_3(\cdot))$. Hence, its probability distribution is uniquely determined over the interval $[0,\mathcal{S})$; in other words, $\mathbb{P}_1\equiv\mathbb{P}_2$ on $\mathcal{F}(\mathcal{S}-)$.

At time $t=\mathcal{S}$ we have $X_1(\mathcal{S})=X_2(\mathcal{S})=X_3(\mathcal{S})$, $\mathbb{P}_j$-a.e., for $j=1,2$, and ties are resolved in favor of the lowest index. For $t>\mathcal{S}$, every given "name" appears in each rank equally likely, since the system (3.1)-(3.2) is invariant under permutations; in particular, for every $t>0$, we have

$$\mathbb{P}_j\big(X_i(t)=R^X_k(t)\,|\,t>\mathcal{S}\big)=1/3\,; \qquad (i,k)\in\{1,2,3\}\,,\quad j=1,2\,. \tag{3.36}$$

Here the distribution of the rank process $R^X_k(\cdot)$, $k=1,2,3$ in (3.3) is uniquely determined through (3.17) by the distribution of the reflected Brownian motion $(G(\cdot),H(\cdot))$ in subsection 3.2. Since the distribution of $X(t)$, $t>\mathcal{S}$ is determined by the process $R^X(\cdot)$ of ranks and the name-rank correspondence, it is uniquely determined. Arguments based on the MARKOV property, allow us now to extend these considerations to the finite-dimensional distributions: $\mathbb{P}_1(\cdot\cap\{t\geq\mathcal{S}\})\equiv\mathbb{P}_2(\cdot\cap\{t\geq\mathcal{S}\})$ for every $t>0$. Combining these considerations with the uniqueness in distribution before time $\mathcal{S}$, we deduce that the weak solution we constructed is unique in distribution, that is, $\mathbb{P}_1\equiv\mathbb{P}_2$ on $\mathcal{F}(\infty)$.

(iii) Failure of Pathwise Uniqueness, and of Strength: In the construction of the matrix-valued processes $\boldsymbol{\eta}(\cdot)$ in (3.23) and $Z(\cdot)$ in (3.24), the excursion starting from the corner $\{t\geq 0: G(t)=$

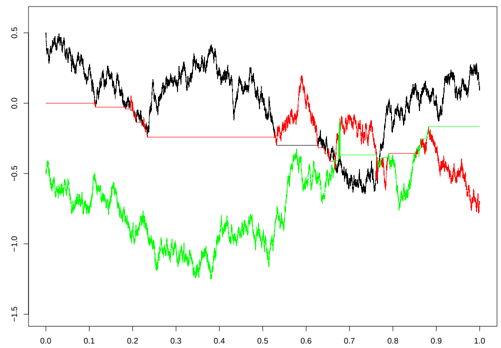

Figure 3. Simulated processes; Black $=X_1(\cdot)$, Red $=X_2(\cdot)$, Green $=X_3(\cdot)$. Here we have taken $\delta_1=-0.5$, $\delta_2=0$ and $\delta_3=0.5$ in (3.1).

$H(t)=0\}$ *does not* appear explicitly, because the triple collision local time $L^{G+H}(\cdot) = L^{R_1^X-R_3^X}(\cdot)$ is identically equal to zero as in (3.18). The corresponding construction of $X(\cdot)$ does not change the name-rank correspondence immediately before or after the triple collision. Since the triple collision local time $L^{G+H}(\cdot)$ does not grow, one may perturb in the above construction the weak solution, by randomly permuting the names of particles immediately after the triple collision time $\mathcal{S}$ — and still obtain the same stochastic dynamics (3.1)-(3.2), hence the same probability distribution for $X(\cdot)$.

Then the resulting sample path of $X(\cdot)$ is different from the original sample path, so pathwise uniqueness fails. But here we have uniqueness in distribution, so the solution of (3.1)-(3.2) cannot be strong after the first triple collision $\mathcal{S}$; this is because uniqueness in distribution, coupled with strong existence, implies pathwise uniqueness (the "dual YAMADA-WATANABE theorem" of [10], [6]). □

***Remark* 3.2** (Some Open Questions). To the best of our knowledge, the result of Theorem 3.1 — to the effect that the solution ceases to be strong after the first triple collision — is the first of its kind for competing particle systems. We conjecture that this feature holds in general such systems with $n\geq 3$ particles. Figure 3 shows simulated paths of particles in (3.1) with $\delta_1=-0.5$, $\delta_2=0$ and $\delta_3=0.5$, based on the construction discussed in Theorem 3.1.

The approach to (3.1)-(3.2) is akin to the construction of the WALSH Brownian motion, and to the splitting stochastic flow of the TANAKA equation. It would be interesting to examine the solvability of (3.1)-(3.2) via the spectral measures of classical/non-classical noises, and via the theory of stochastic flows in [42], [44], and [32], [33] (see also [1] and its references). It would also be quite interesting to determine whether the filtration of the post$-\mathcal{S}$ process $X(\cdot)$ might fail, in the spirit of [42], to be generated by *any* Brownian motion of *any* dimension. We leave these issues to further research.

Appendix A: Middle Diffusion, Ballistic Hedges, Skew-Elastic Collisions

Double collisions were completely "elastic" in the systems of Sections 2 and 3: when two particles there collided, they split their collision local times evenly. We study here briefly a variant of the system (2.1) — with the same purely ballistic motions for the leader and laggard particles, and the same diffusive motion for the middle particle — but now with “skew-elastic" collisions, as in [13], between the second- and third-ranked particles.

More precisely, we consider in the notation of (2.3), (2.10), and with δ_1, δ_2, δ_3, $x_1 > x_2 > x_3$ given real numbers, the system of equations, first introduced and studied in [12] :

$$X_i(\cdot) = x_i + \sum_{k=1}^{3} \delta_k \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_k^X(t)\}}\, \mathrm{d}t + \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_2^X(t)\}}\, \mathrm{d}B_i(t) \tag{A.1}$$

$$+ \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_2^X(t)\}}\, \mathrm{d}L^{R_2^X - R_3^X}(t) + \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=R_3^X(t)\}}\, \mathrm{d}L^{R_2^X - R_3^X}(t)$$

for $i = 1, 2, 3$. We shall try to find a weak solution to this system; in other words, construct a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbb{F} = \{\mathfrak{F}(t)\}_{0 \le t < \infty}$ rich enough to accommodate independent Brownian motions $B_1(\cdot)$, $B_2(\cdot)$, $B_3(\cdot)$ and continuous semimartingales $X_1(\cdot)$, $X_2(\cdot)$, $X_3(\cdot)$ so that, with probability one, the equations of (A.1) are satisfied, along with the requirements

$$\int_0^{\infty} \mathbf{1}_{\{R_k^X(t)=R_\ell^X(t)\}}\, \mathrm{d}t = 0\,, \quad \forall\ k < \ell\,; \qquad L^{R_1^X - R_3^X}(\cdot) \equiv 0\,. \tag{A.2}$$

A.1. Analysis

Assuming that such a weak solution to the system of (A.1), (A.2) has been constructed, the ranked processes $R_k^X(\cdot)$ as in (2.3) are continuous semimartingales with decompositions

$$R_1^X(t) = x_1 + \delta_1\, t + \frac{1}{2} \Lambda^{(1,2)}(t)\,, \qquad R_3^X(t) = x_3 + \delta_3\, t + \frac{1}{2} \Lambda^{(2,3)}(t) \tag{A.3}$$

$$R_2^X(t) = x_2 + \delta_2\, t + W(t) - \frac{1}{2} \Lambda^{(1,2)}(t) + \frac{3}{2} \Lambda^{(2,3)}(t) \tag{A.4}$$

by analogy with (2.6)-(2.7); though also with the clear difference, that the collision local time $\Lambda^{(2,3)}(\cdot)$ is not split now evenly between the second- and third-ranked particles, but rather in a 1:3 proportion. We are using here the exact same notation for the standard Brownian motion $W(\cdot)$ as in (2.8), and for the collision local times $\Lambda^{(k,\ell)}(\cdot)$ as in (2.9). For the gaps $G(\cdot) = R_1^X(\cdot) - R_2^X(\cdot)$, $H(\cdot) = R_2^X(\cdot) - R_3^X(\cdot)$ we have the SKOROKHOD-type representations of the form (2.14)-(2.15), now with

$$U(t) = x_1 - x_2 + \big(\delta_1 - \delta_2\big)\, t - W(t) - \frac{3}{2}\, L^H(t)\,, \quad V(t) = x_2 - x_3 + \big(\delta_2 - \delta_3\big)\, t + W(t) - \frac{1}{2}\, L^G(t)\,.$$

Whereas, from the theory of the SKOROKHOD reflection problem we obtain now the relationships linking the two local time processes $L^G(\cdot)$ and $L^H(\cdot)$, namely

$$L^G(t) = \max_{0 \le s \le t} \big(- U(s) \big)^+ = \max_{0 \le s \le t} \Big(x_2 - x_1 + \big(\delta_2 - \delta_1\big)\, s + W(s) + \frac{3}{2}\, L^H(s) \Big)^+\,, \tag{A.5}$$

$$L^H(t) = \max_{0 \le s \le t} \big(- V(s) \big)^+ = \max_{0 \le s \le t} \Big(x_3 - x_2 + \big(\delta_3 - \delta_2\big)\, s - W(s) + \frac{1}{2}\, L^G(s) \Big)^+ \tag{A.6}$$

- The resulting system for the two nonnegative gap processes

$$G(t) \,=\, x_1 - x_2 + \big(\delta_1 - \delta_2\big)\, t - W(t) - \frac{3}{2}\, L^H(t) + L^G(t)\,, \qquad 0 \le t < \infty \tag{A.7}$$

$$H(t) \,=\, x_2 - x_3 + \big(\delta_2 - \delta_3\big)\, t + W(t) - \frac{1}{2}\, L^G(t) + L^H(t)\,, \qquad 0 \le t < \infty \tag{A.8}$$

is again of the HARRISON & REIMAN [16] type (2.18). It amounts to reflecting off the faces of the nonnegative orthant the degenerate, two-dimensional Brownian motion $\mathfrak{Z}(\cdot)$ as in (2.19), with drift vector $\boldsymbol{m} = \big(\delta_1 - \delta_2,\, \delta_2 - \delta_3\big)'$, covariance matrix $\mathcal{C}$ as in (2.20), but now with reflection matrix

$$\mathcal{R} := \mathcal{I} - \mathcal{Q}\,, \qquad \mathcal{Q} = \begin{pmatrix} 0 & 3/2 \\ 1/2 & 0 \end{pmatrix}, \qquad \text{thus} \qquad \mathcal{R}^{-1}\boldsymbol{m} \,=\, 2 \begin{pmatrix} 2\,\delta_1 + \delta_2 - 3\,\delta_3 \\ \delta_1 + \delta_2 - 2\,\delta_3 \end{pmatrix}.$$

Here the matrix $\mathcal{Q}$ has spectral radius strictly less than 1, and the *skew-symmetry condition* $\mathcal{R} + \mathcal{R}' = 2\,\mathcal{C}$ of [17] is satisfied by these covariance and reflection matrices.

A.2. Synthesis

Let us start now with given real numbers $\delta_1,\, \delta_2,\, \delta_3$, and $x_1 > x_2 > x_3$, and construct a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $\mathbb{F} = \big\{\mathfrak{F}(t)\big\}_{0 \le t < \infty}$ rich enough to support a standard Brownian motion $W(\cdot)$. By analogy with (A.5)-(A.6), we consider the following system of equations for two continuous, nondecreasing and adapted processes $A(\cdot)$ and $\Gamma(\cdot)$ with $A(0) = \Gamma(0) = 0$:

$$A(t) \,=\, \max_{0 \le s \le t} \Big(x_2 - x_1 + \big(\delta_2 - \delta_1\big)\, s + W(s) + \frac{3}{2}\, \Gamma(s) \Big)^+, \qquad 0 \le t < \infty \tag{A.9}$$

$$\Gamma(t) \,=\, \max_{0 \le s \le t} \Big(x_3 - x_2 + \big(\delta_3 - \delta_2\big)\, s - W(s) + \frac{1}{2}\, A(s) \Big)^+, \qquad 0 \le t < \infty. \tag{A.10}$$

Theorem 1 of [16] guarantees that this system has a unique continuous solution $(A(\cdot), \Gamma(\cdot))$, adapted to the smallest right-continuous filtration $\mathbb{F}^W$ to which the driving Brownian motion $W(\cdot)$ is itself adapted. With this solution in place, we construct the continuous semimartigales

$$U(t) := x_1 - x_2 + \big(\delta_1 - \delta_2\big)\, t - W(t) - \frac{3}{2}\, \Gamma(t)\,, \quad V(t) := x_2 - x_3 + \big(\delta_2 - \delta_3\big)\, t + W(t) - \frac{1}{2}\, A(t)\,, \tag{A.11}$$

and then “fold" them, to obtain their SKOROKHOD reflections

$$G(t) \,:=\, U(t) + \max_{0 \le s \le t} \big(-U(s) \big)^+ \,=\, x_1 - x_2 + \big(\delta_1 - \delta_2\big)\, t - W(t) - \frac{3}{2}\, \Gamma(t) + A(t) \,\ge\, 0 \tag{A.12}$$

$$H(t) \,:=\, V(t) + \max_{0 \le s \le t} \big(-V(s) \big)^+ \,=\, x_2 - x_3 + \big(\delta_2 - \delta_3\big)\, t + W(t) - \frac{1}{2}\, A(t) + \Gamma(t) \,\ge\, 0 \tag{A.13}$$

for $t \in [0, \infty)$. As before, for these two continuous, nonnegative semimartingales the theories of the SKOROKHOD reflection problem and of semimartingale local time give, respectively,

$$\int_0^\infty \mathbf{1}_{\{G(t) > 0\}}\, \mathrm{d}A(t) \,=\, 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t) > 0\}}\, \mathrm{d}\Gamma(t) \,=\, 0\,, \tag{A.14}$$

$$\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}t \,=\, 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}t \,=\, 0\,. \tag{A.15}$$

The following additional properties are checked easily:

$$\int_0^\infty \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}\Gamma(t) \,=\, 0\,, \qquad \int_0^\infty \mathbf{1}_{\{H(t)=0\}}\, \mathrm{d}A(t) \,=\, 0\,. \tag{A.16}$$

- We need to identify the regulating processes $A(\cdot)\,,\ \Gamma(\cdot)$ as local times. We start by observing

$$L^G(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}G(t) = \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}} \Big[\mathrm{d}A(t) - \frac{3}{2}\mathrm{d}\Gamma(t) - \mathrm{d}W(t) + \big(\delta_1 - \delta_2\big)\, \mathrm{d}t \Big]$$

from (2.10) and (A.12). The last (LEBESGUE) and next-to-last (ITÔ) integrals in this expression vanish on the strength of (A.15), whereas the third-to-last integral vanishes on account of (A.16); so we deduce the identification $L^G(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{G(t)=0\}}\, \mathrm{d}A(t) \equiv A(\cdot)$, where the second equality comes on the heels of (A.14). We identify similarly $L^H(\cdot) \equiv \Gamma(\cdot)$.

- By analogy with (A.3)-(A.4), we construct now the $\mathbb{F}^W-$adapted *processes of ranks*

$$R_1(t) \,:=\, x_1 + \delta_1\, t + \frac{1}{2}\, A(t)\,, \qquad R_3(t) \,:=\, x_3 + \delta_3\, t + \frac{1}{2}\, \Gamma(t)\,, \tag{A.17}$$

$$R_2(t) \,:=\, x_2 + \delta_2\, t + W(t) - \frac{1}{2}\, A(t) + \frac{3}{2}\, \Gamma(t) \tag{A.18}$$

and note $R_1(\cdot) - R_2(\cdot) = G(\cdot) \geq 0\,,\ R_2(\cdot) - R_3(\cdot) = H(\cdot) \geq 0$, thus $R_1(\cdot) \,\geq\, R_2(\cdot) \,\geq\, R_3(\cdot)$. The continuous process $R_1(\cdot) - R_3(\cdot) = G(\cdot) + H(\cdot) \geq 0$ is of finite first variation on compact intervals, so its local time at the origin vanishes, as posited in (A.2): $L^{R_1 - R_3}(\cdot) \equiv 0$. The other properties posited there are direct consequences of (A.15). Finally, the identifications $A(\cdot) \equiv L^G(\cdot) \equiv L^{R_1 - R_2}(\cdot), \Gamma(\cdot) \equiv L^H(\cdot) \equiv L^{R_2 - R_3}(\cdot)$ show, in conjunction with (A.18), that the rank vector process $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ and the scalar, standard Brownian motion $W(\cdot)$ generate the same filtration.

- We can construct now on a suitable filtered probability space independent Brownian motions $B_1(\cdot)$, $B_2(\cdot)$, $B_3(\cdot)$ and continuous, adapted processes $X_1(\cdot)$, $X_2(\cdot)$, $X_3(\cdot)$ so that, with probability one, the equations of (A.1) are satisfied, along with those of (A.2), up until the first time of a triple collision, as well as $R^X_k(t) = R_k(t)\,,\ 0 \leq t < \mathcal{S}$ for $k = 1, 2, 3$. Just as before, this is done by considering the particles two-by-two in the manner of [25], and applying the results in [13], [14].

A simulation of the paths of the resulting process $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ with $\delta_1 = -1,\ \delta_2 = -2$ and $\delta_3 = -1$ is depicted in Figure 4, reproduced here from [12]. *We believe, but have not been able to show, that* $\mathbb{P}(\mathcal{S} = \infty) = 1$ *holds in this case.*

A.3. Invariant Distribution

Assuming that both components of the vector

$$\boldsymbol{\lambda} \equiv \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} := -\,\mathcal{R}^{-1}\boldsymbol{m} \,=\, 2 \begin{pmatrix} 3\,\delta_3 - 2\,\delta_1 - \delta_2 \\ 2\,\delta_3 - \delta_1 - \delta_2 \end{pmatrix} \tag{A.19}$$

are positive numbers, and following the reasoning of subsection 2.3, we deduce that here again the two-dimensional, degenerate process $\big(G(\cdot), H(\cdot)\big)$ of gaps is positive recurrent, has a unique invariant

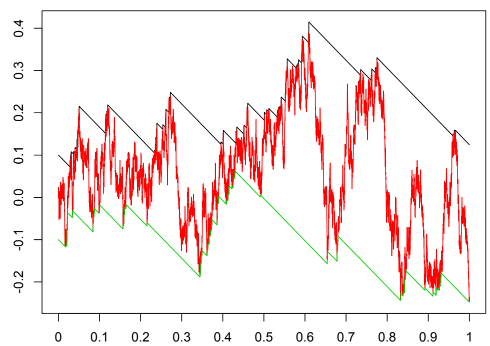

Figure 4. Simulated processes; Black $=R_1(\cdot)$, Red $=R_2(\cdot)$, Green $=R_3(\cdot)$. Here we have taken $\delta_1=-1$, $\delta_2=-2$ and $\delta_3=-1$ in (A.1). We are indebted to Dr. E.R. FERNHOLZ for this picture.

measure π with $\pi((0,\infty)^2)=1$, and converges to this probability measure in distribution as $t\to\infty$. The fact that the covariance matrix $\mathcal{C}$ and the reflection matrix $\mathcal{R}$ satisfy the skew-symmetry condition $\mathcal{R}+\mathcal{R}'=2\mathcal{C}$ implies that *the invariant probability measure should be the product of exponentials*

$$\pi\big(\mathrm{d}g,\mathrm{d}h\big) = 4\,\lambda_1\,\lambda_2\,e^{-2\,\lambda_1\,g-2\,\lambda_2\,h}\,\mathrm{d}g\,\mathrm{d}h\,, \qquad (g,h)\in(0,\infty)^2\,. \tag{A.20}$$

We send the reader to [23] for the proof of this result.

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