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Generating Gowdy cosmological models*

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Using the analogy with stationary axisymmetric solutions, we present a method to generate new analytic cosmological solutions of Einstein's equation belonging to the class of T^3 Gowdy cosmological models. We show that the solutions can be generated from their data at the initial singularity and present the formal general solution for arbitrary initial data. We exemplify the method by constructing the Kantowski–Sachs cosmological model and a generalization of it that corresponds to an unpolarized T^3 Gowdy model. © 2004 American Institute of Physics.
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I. INTRODUCTION

Gowdy metrics,^{1,2} which are exact solutions of Einstein's vacuum field equations, represent cosmological models with various possible topologies ($S^3, S^1 \times S^2, T^3$) and are spatially compact inhomogeneous space–times admitting two commuting spacelike Killing vector fields. They are of special importance for the study of formation of singularities in general relativity. Many efforts^{3–6} have been made in order to derive and analyze these types of solutions, especially by using numerical methods. The physical structure of these metrics is sufficiently simple to expect that the singularities can be analyzed in detail, but their mathematical structure is sufficiently complicated so that the global dynamical behavior is still far from being completely understood. One of the most intriguing questions concerns the structure of the curvature singularity that is expected to appear at certain spacelike boundary of the associated space–time. Many studies have been devoted to the so-called “asymptotically velocity term dominated” (AVTD) behavior which states that near the singularity each point in space is characterized by a different spatially homogeneous cosmology.⁸ The idea of AVTD behavior was originally proposed in Ref. 9 more than 30 years ago, but is still being debated. Numerical analysis of the Gowdy models have shown that they do become AVTD near the singularity, except at a set of isolated points, where there are “spikes” in the behavior of the metric functions. The origin of these spikes is investigated in Ref. 6. Gowdy metrics have been also analyzed as toy models in quantum midisuperspace gravity.⁷

Numerical methods have been extensively used to investigate Gowdy models, but only recently it has been argued that solutions generating techniques can be applied in this case^{10–12} to generate new solutions and that even a “simple change of coordinates” can be applied to reinterpret certain stationary axisymmetric solutions as $S^1 \times S^2$ Gowdy cosmological models.¹³ The reason why these methods can be used also in this case is due to the well-known fact that the solution generating techniques are applicable to any space–time which admits two (commuting) Killing vector fields. In this work, we will concentrate on T^3 Gowdy cosmological models and will see that a complex coordinate transformation, together with a complex change of metric functions, allows us to apply in a straightforward manner the well-known solution generating

*We would like to dedicate this work to the memory of Professor Nail Sibgatullin who unexpectedly passed away on March 12, 2004.

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techniques that have been intensively used for stationary axisymmetric solutions.

This paper is organized as follows: In Sec. II we derive the “transformation” that relates stationary axisymmetric solutions with Gowdy T^3 models. We will show that due to this analogy, the AVTD behavior in Gowdy T^3 is mathematically equivalent to the behavior of stationary axisymmetric solutions near the axis of symmetry. In Sec. III we show that Sibgatullin’s method for constructing solutions can be applied in the case of Gowdy T^3 models and in Sec. IV we present several examples of exact solutions generated by using this method. Finally, Sec. V is devoted to the conclusions and some remarks about different possibilities of generalizing the results derived in this work.

II. STATIONARY AXISYMMETRIC SOLUTIONS AND GOWDY T^3 MODELS

Consider the line element for stationary axisymmetric spacetimes in the Lewis–Papapetrou form¹⁴

$$ds^2 = -e^{2\psi}(dT + \omega d\phi)^2 + e^{-2\psi}[\rho^2(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (1)$$

where ψ , ω , and γ are functions of the nonignorable coordinates ρ and z . The ignorable coordinates T and ϕ are associated with the two Killing vector fields $\eta_I = \partial/\partial T$ and $\eta_{II} = \partial/\partial \phi$. The field equations take the form

$$\psi_{\rho\rho} + \frac{1}{\rho}\psi_{\rho} + \psi_{zz} + \frac{e^{4\psi}}{2\rho^2}(\omega_{\rho}^2 + \omega_z^2) = 0, \quad (2)$$

$$\omega_{\rho\rho} - \frac{1}{\rho}\omega_{\rho} + \omega_{zz} + 4(\omega_{\rho}\psi_{\rho} + \omega_z\psi_z) = 0, \quad (3)$$

$$\gamma_{\rho} = \rho(\psi_{\rho}^2 - \psi_z^2) - \frac{e^{4\psi}}{4\rho^2}(\omega_{\rho}^2 - \omega_z^2), \quad (4)$$

$$\gamma_z = 2\rho\psi_{\rho}\psi_z - \frac{1}{2\rho}e^{4\psi}\omega_{\rho}\omega_z, \quad (5)$$

where the lower indices represent partial derivative with respect to the corresponding coordinate.

Consider now the following coordinate transformation $(\rho, t) \rightarrow (\tau, \sigma)$ and the complex change of coordinates $(\phi, z) \rightarrow (\delta, \chi)$ defined by

$$\rho = e^{-\tau}, \quad T = \sigma, \quad z = i\chi, \quad \phi = i\delta, \quad (6)$$

and introduce the functions P , Q , and λ by means of the relationships

$$\psi = \frac{1}{2}(P - \tau), \quad Q = i\omega, \quad \gamma = \frac{1}{2}\left(P - \frac{\lambda}{2} - \frac{\tau}{2}\right). \quad (7)$$

Introducing Eqs. (6) and (7) into the line element (1), we obtain

$$-ds^2 = e^{-\lambda/2}e^{\tau/2}(-e^{-2\tau}d\tau^2 + d\chi^2) + e^{-\tau}[e^P(d\sigma + Qd\delta)^2 + e^{-P}d\delta^2]. \quad (8)$$

Let us take $\tau \geq 0$ (what seems reasonable in virtue that the radial coordinate $\rho = e^{-\tau} \geq 0$) and “compactify” the new coordinates as $0 \leq \chi, \sigma, \delta \leq 2\pi$ (a less reasonable condition since in general $-\infty < z < +\infty$ and $T \geq 0$). The line element (8) with the coordinates τ, χ, σ , and δ in the range given above is known as the line element for Gowdy T^3 cosmological models.⁶ Furthermore, one can verify by direct calculation that the action of the transformations (6) and (7) on the field

equations (2)–(5) yields exactly the field equations for the Gowdy cosmological models which after some algebraic manipulations can be written as a set of two second order differential equations for P and Q ,

$$P_{\tau\tau} - e^{-2\tau} P_{\chi\chi} - e^{2P} (Q_\tau^2 - e^{-2\tau} Q_\chi^2) = 0, \quad (9)$$

$$Q_{\tau\tau} - e^{-2\tau} Q_{\chi\chi} + 2(P_\tau Q_\tau - e^{-2\tau} P_\chi Q_\chi) = 0, \quad (10)$$

and two first order differential equations for λ

$$\lambda_\tau = P_\tau^2 + e^{-2\tau} P_\chi^2 + e^{2P} (Q_\tau^2 + e^{-2\tau} Q_\chi^2), \quad (11)$$

$$\lambda_\chi = 2(P_\chi P_\tau + e^{2P} Q_\chi Q_\tau). \quad (12)$$

It should be emphasized that this method for “deriving” the Gowdy line element from the stationary axisymmetric one involves real as well as complex transformations at the level of coordinates and metric functions. It is, therefore, necessary to demand that the resulting metric functions P , Q , and λ be real. That means that in general it is not possible to take an axisymmetric stationary solution and apply the transformations to obtain a Gowdy cosmological model. If the resulting functions are not real, they cannot be physical reasonable solutions to the real equations (9)–(12). These transformations can be used only as a guide to get some insight into the form of the new solutions. In any case, the corresponding field equations have to be invoked in order to confirm the correctness of the solution.

A very useful form for analyzing the field equations of space-times with two (commuting) Killing vector fields is the Ernst representation. In fact, this convenient form for the field equations was first proposed for axisymmetric stationary solutions,¹⁵ but since then it has been applied in many different configurations.^{16–18} Here we will present the Ernst representation of the main field equations (9) and (10) which is especially adapted to the coordinates used here. To this end, let us introduce a new coordinate t and a new function $R = R(t, \chi)$ by means of the equations^{10–13}

$$t = e^{-\tau}, \quad R_t = t e^{2P} Q_\chi, \quad R_\chi = t e^{2P} Q_t. \quad (13)$$

Then, the field equation (9) can be expressed as

$$t^2 \left(P_{tt} + \frac{1}{t} P_t - P_{\chi\chi} \right) + e^{-2P} (R_t^2 - R_\chi^2) = 0, \quad (14)$$

whereas Eq. (10) for the function Q turns out to be equivalent to the integrability condition $R_{t\chi} = R_{\chi t}$. However, an alternative and convenient equation is obtained by introducing Eq. (13) directly into Eq. (10). So we obtain

$$t e^P \left(R_{tt} + \frac{1}{t} R_t - R_{\chi\chi} \right) - 2[(t e^P)_t R_t - (t e^P)_\chi R_\chi] = 0, \quad (15)$$

an equation which of course becomes an identity if the integrability condition $R_{t\chi} = R_{\chi t}$ is satisfied. We can now introduce the complex Ernst potential E and the complex gradient operator D as

$$E = t e^P + i R, \quad \text{and} \quad D = \left(\frac{\partial}{\partial t}, i \frac{\partial}{\partial \chi} \right), \quad (16)$$

which allow us to write the main field equations in the *Ernst-type representation*

$$\text{Re}(E) \left(D^2 E + \frac{1}{t} D t D E \right) - (D E)^2 = 0. \quad (17)$$

It is easy to verify that the field equations (14) and (15) can be obtained as the real and imaginary part of the Ernst equation (17), respectively. For the sake of completeness, we rewrite the system of first order, partial, differential equations (11) and (12) in terms of the Ernst potential:

$$\lambda_t = -\frac{t}{2}(C_+ C_+^* + C_- C_-^*), \quad (18)$$

$$\lambda_\chi = -\frac{t}{2}(C_+ C_+^* - C_- C_-^*), \quad (19)$$

where

$$C_\pm = \frac{1}{\text{Re}(E)}(E_t \pm E_\chi) - \frac{1}{t}, \quad (20)$$

and the asterisk denotes complex conjugation.

If the Ernst potential E is known, then it is easy to recover the metric functions P , Q , and λ which enter the line element (8) of Gowdy T^3 cosmological models. In fact, from Eq. (16) one can algebraically construct the functions P and R . Then the function Q can be obtained by solving the system of two first order partial differential equations given in (13). Notice that the integrability condition of this last system is satisfied by virtue of Eq. (17). Finally, the system (18) and (19) for the function λ can be solved by quadratures since its integrability condition coincides with the Ernst equation (17). Consequently, all the information about any Gowdy T^3 cosmological model is contained in the corresponding Ernst potential.

One of the most important properties of the Ernst representation (17) is that it is very appropriate to investigate the symmetries of the field equations. In particular, the symmetries of the Ernst equation for stationary axisymmetric spacetimes have been used to develop the modern solution generating techniques,¹⁹ like the Bäcklund method, Belinsky–Zakharov inverse scattering method, the Hoenselaers–Kinnorsley–Xanthopoulos method, and others (for an introductory review and detailed references, see Ref. 20). In all these methods it is necessary to start from a given “seed” solution which has to be specified in the whole space–time (except, perhaps, in the regions where the metric possesses true curvature singularities). An alternative approach for exploring the symmetries inherent in the Ernst equation was explicitly developed by Sibgatullin²¹ and consists of constructing exact solutions to the Ernst equation from initial data specified only on certain hypersurface (submanifold) of the space–time. For instance, in the case of stationary axisymmetric space–times, Sibgatullin’s method allows one to construct exact solutions from their data on the axis of symmetry. In the following sections we will show that Sibgatullin’s method can be applied in the case of Gowdy cosmological models and will present several examples of its application.

III. CONSTRUCTING SOLUTIONS FROM AVTD DATA

As we have mentioned above, an important property of Gowdy cosmological models is its AVTD behavior near the initial singularity. In the case of T^3 models it can be shown that the singularity is approached in the limit $\tau \rightarrow \infty$. The AVTD behavior implies that at the singularity all spatial derivatives of the field equations can be neglected and only the temporal behavior is relevant. On the other hand, the transformation (6) indicates that the limit $\tau \rightarrow \infty$ is equivalent to the limit $\rho \rightarrow 0$; however, this is true only at the level of coordinates and a more detailed analysis is necessary to make sure that this analogy is also valid at the level of explicit solutions. To this end, let us consider the system of partial differential equations for ψ and ω given in Eqs. (2) and (3). If we neglect the spatial dependence on z , which according to the transformation (6) is equivalent to the spatial dependence on χ in Gowdy models, then we obtain a system of differential equations which can be solved by quadratures and yields

$$\psi = \frac{1}{2} \ln[a(\rho^{1+c} + b^2 \rho^{1-c})], \quad \omega = \frac{ib}{a(\rho^{1+c} + b^2 \rho^{1-c})} + id, \quad (21)$$

where a , b , c , and d are arbitrary real functions of z . Clearly, this solution is meaningless when considered as a stationary axisymmetric spacetime. However, if we now follow the prescription given in Eqs. (6) and (7) for obtaining Gowdy models, we will find that solution (21) “corresponds” to the Gowdy model

$$P = \ln[a(e^{-c\tau} + b^2 e^{c\tau})], \quad Q = \frac{b}{a(e^{-2c\tau} + b^2)} + d, \quad (22)$$

where now a , b , c , and d are to be considered as arbitrary real functions of the coordinate χ . It is straightforward to verify that the expressions given in Eq. (22) satisfy the Gowdy field equations (9) and (10) in its “truncated” form, i.e., when the spatial derivatives are neglected. The solution (22) is known in the literature as the AVTD solution for Gowdy T^3 models⁶ and dictates the behavior of these models near the singularity $\tau \rightarrow \infty$. Thus, we have “derived” the AVTD solution starting from its stationary axisymmetric counterpart. This is a further indication that the behavior of Gowdy models at the initial singularity is mathematically equivalent to the behavior of stationary axisymmetric solutions at the axis. For the sake of completeness we also quote here the value of the function λ corresponding to the AVTD solution (22) that can be obtained by integrating Eq. (11):

$$\lambda = \lambda_0 - c^2 \ln t, \quad (23)$$

where λ_0 is an additive constant. Furthermore, the corresponding AVTD Ernst potential can be obtained by introducing Eq. (22) into Eqs. (16) and (17). Then,

$$E = a[e^{-(1+c)\tau} + b^2 e^{-(1-c)\tau}] + iR^{\text{avtd}} \quad \text{with} \quad R_{\chi}^{\text{avtd}} = -2abc. \quad (24)$$

If we define

$$E(\tau \rightarrow \infty, \chi) = e(\chi) \quad (25)$$

as the Ernst potential at the singularity, we see from Eq. (24) that for $c \in (-1, 1)$ only the imaginary part remains, $e(\chi) = iR^{\text{avtd}}$. This means that the real part of $e(\chi)$ is arbitrary and since R^{avtd} is given in terms of the real part it is also arbitrary. If $c \notin (-1, 1)$, the Ernst potential diverges at the singularity for arbitrary values of the functions a and b . In the limiting case $c = \pm 1$, the Ernst potential at the singularity is regular, but again no conditions appear for the behavior of the functions a and b . Consequently, the AVTD behavior does not impose any conditions on the function $e(\chi)$. We will now see that it is possible to use this function to construct the corresponding Ernst potential $E(\tau, \chi)$.

Sibgatullin's method²¹ has been developed to construct exact stationary axisymmetric solutions starting from their data on the axis of symmetry. It is based upon the fact that the Ernst equation possesses symmetry properties associated with an infinite-dimensional Lie group which transforms one solution of this equation into another solution of the same equation. This implies remarkable analyticity properties that make it possible to reduce the Ernst equation to a system of linear integral equations which can be integrated explicitly if initial data is known, for instance, on the axis of symmetry. It is clear that the Ernst-type representation (17) possesses similar symmetry properties. On the other hand, we have shown that the behavior of stationary axisymmetric solutions near the axis is mathematically equivalent to the behavior of Gowdy T^3 cosmological models near the singularity. Thus, it should be possible to construct Gowdy cosmological models starting from the value of the corresponding Ernst potential at the singularity. It turns out that Sibgatullin's method can be generalized in a straightforward manner to include the case of Gowdy models. A detailed explanation of the procedure necessary to obtain the system of linear integral equations

associated with the Ernst equation is given in Ref. 21. Here we will only quote the main steps of the construction. Assume that the value of the Ernst potential is known at the initial singularity, i.e., $e(\chi)$ is given. Then, the Ernst potential can be generated by means of the integral equation

$$E(t, \chi) = \frac{1}{\pi} \int_{-1}^1 \frac{e(\xi) \mu(\xi)}{\sqrt{1-s^2}} ds, \quad (26)$$

where the unknown function $\mu(\xi)$ has to be found from the singular integral equation

$$\int_{-1}^1 \frac{\mu(\xi) [e^*(\eta) + e(\xi)]}{(s-\kappa)\sqrt{1-s^2}} ds = 0, \quad (27)$$

with the normalization condition

$$\int_{-1}^1 \frac{\mu(\xi)}{\sqrt{1-s^2}} ds = \pi, \quad (28)$$

where $\xi = \chi + ts$, $\eta = \chi + t\kappa$, with $s, \kappa \in [-1, 1]$.

Notice that for this method no condition is imposed on the behavior of $e(\chi)$. This is in accordance with the result obtained above about the AVTD behavior of the Ernst potential near the singularity. Once $e(\chi)$ is given in any desired form, one only has to calculate the integral (26) to find the Ernst potential. However, to calculate this integral one first has to find the function $\mu(\xi)$ by means of the singular equation (27) and the normalization condition (28). In practice, for a given $e(\xi)$ one has to make a reasonable ansatz for $\mu(\xi)$ such that it allows the existence of solutions for the integral singular equation (27).

IV. EXAMPLES OF GOWDY T^3 MODELS

The cases where the Ernst potential at the initial singularity behaves as a rational function are relatively easy to analyze. In this section we will present two such examples. Let us consider the following simple example of an Ernst potential at the singularity

$$e(\chi) = \frac{\chi_0 - \chi}{\chi_0 + \chi}, \quad (29)$$

where χ_0 is a real constant. The first step of the construction is to find the unknown function μ according to Eqs. (27) and (28). A reasonable ansatz is again a rational function²¹

$$\mu = A_0 + \frac{A_1}{\xi - \xi_1}, \quad (30)$$

where ξ_1 is the root of the equation $e(\xi) + \bar{e}(\xi) = 0$ (in this case $\xi_1 = \chi_0$) and A_0, A_1 , are functions of t and χ . To handle the integrals which follow from the singular integral equation we use the following standard formulas

$$\int_{-1}^1 \frac{ds}{\sqrt{1-s^2}} = \pi, \quad (31)$$

$$\int_{-1}^1 \frac{ds}{(a+isb)\sqrt{1-s^2}} = \frac{\pi}{\sqrt{a^2+b^2}}, \quad (32)$$

$$\int_{-1}^1 \frac{ds}{(s-\kappa)(s-\gamma)\sqrt{1-s^2}} = \frac{\pi}{(\kappa-\gamma)\sqrt{\gamma^2-1}}, \quad (33)$$

where a , b , and γ are arbitrary constants.

Introducing Eq. (30) into the normalization condition (28), we obtain

$$A_0 + \frac{A_1}{r_-} = 1, \quad (34)$$

where $r_- = \sqrt{(\chi - \chi_0)^2 - t^2}$, whereas the integral singular equation (27) leads to

$$A_0 - \frac{A_1(r_+ + r_-)}{2\chi_0 r_-} = 0. \quad (35)$$

The last two equations can be used to find the explicit values of A_0 and A_1 which then can be replaced in the result of the integration of Eq. (26) and yield

$$E(t, \chi) = -A_0 - \frac{A_1 - 2\chi_0 A_0}{r_+} = \frac{2\chi_0 - r_+ - r_-}{2\chi_0 + r_+ + r_-}, \quad (36)$$

where $r_+ = \sqrt{(\chi + \chi_0)^2 - t^2}$. It is easy to check that indeed this is a solution to the Ernst equation (17). Since the resulting Ernst potential is real, the solution corresponds to a polarized ($Q=0$) Gowdy model. The expression for the metric function P can easily be obtained from the definition (16) and Eq. (36), and the remaining function λ can be calculated (up to an additive constant) by quadratures from Eqs. (18) and (19):

$$\lambda = \ln \left[\frac{1}{t} \frac{(r_+ r_-)^2}{(r_+ + r_- + 2\chi_0)^4} \right]. \quad (37)$$

The physical significance of this solution becomes plausible in a different system of coordinates which we introduce in two steps. Let us first introduce in the (τ, χ) -sector of the line element (8) coordinates x and y by means of the relationships

$$e^{-2\tau} = t^2 = \chi_0^2 (1-x^2)(1-y^2), \quad \chi = \chi_0 xy, \quad (38)$$

or the inverse transformation law

$$x = \frac{r_+ + r_-}{2\chi_0}, \quad y = \frac{r_+ - r_-}{2\chi_0}, \quad (39)$$

so that the metric functions become

$$P = \ln \left[\frac{1-x}{\chi_0 \sqrt{(1-x^2)(1-y^2)}(1+x)} \right], \quad \lambda = \ln \left[\frac{(x^2 - y^2)^2}{\chi_0 \sqrt{(1-x^2)(1-y^2)}(1+x)^4} \right]. \quad (40)$$

The second transformation affects now all the sectors of the line element (8) and is defined by

$$x = \frac{T}{\chi_0} - 1, \quad y = \cos \theta, \quad \sigma = r, \quad \delta = \phi. \quad (41)$$

Then, after some algebraic manipulations, the metric can be written as

$$-ds^2 = - \left(\frac{2\chi_0}{T} - 1 \right)^{-1} dT^2 + \left(\frac{2\chi_0}{T} - 1 \right) dr^2 + T^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (42)$$

an expression that can immediately be recognized as the Kantowski–Sachs cosmological model.^{23,24} Thus, we have shown that the Kantowski–Sachs metric can be constructed from the value of its Ernst potential at the singularity (29).

Consider now the more general case

$$e(\chi) = \frac{\chi_0 - \chi - i\chi_1}{\chi_0 + \chi + i\chi_1}, \quad (43)$$

where χ_0 and χ_1 are real constants. The unknown function $\mu(\xi)$ can be sought in the form

$$\mu = A_0 + \frac{A_1}{\xi - \xi_1} + \frac{A_2}{\xi - \xi_2}, \quad (44)$$

where $\xi_{1,2} = \pm \alpha = \pm \sqrt{\chi_0^2 - \chi_1^2}$ are the roots of the equation $e(\xi) + e^*(\xi) = 0$. Substituting Eq. (44) in the integral equation (27) we obtain the system

$$-A_0 + \frac{A_1}{\chi_0 + i\chi_1 + \alpha} + \frac{A_2}{\chi_0 + i\chi_1 - \alpha} = 0, \quad (45)$$

$$-\frac{A_1(\alpha + i\chi_1)}{r_-(\chi_0 + i\chi_1 + \alpha)} + \frac{A_2(\alpha - i\chi_1)}{r_+(\chi_0 + i\chi_1 - \alpha)} = 0, \quad (46)$$

where $r_{\pm} = \sqrt{(\chi \pm \alpha)^2 - t^2}$. On the other hand, the normalization condition (28) yields

$$A_0 + \frac{A_1}{r_-} + \frac{A_2}{r_+} = 1, \quad (47)$$

an equation which together with Eqs. (45) and (46) form a closed algebraic system that determines the coefficients of the function μ :

$$\begin{aligned} A_0 &= \frac{\alpha(r_+ + r_-) + i\chi_1(r_+ - r_-)}{\alpha(r_+ + r_-) + i\chi_1(r_+ - r_-) + 2\alpha\chi_0}, \\ A_1 &= \frac{r_-(\alpha - i\chi_1)(\chi_0 + i\chi_1 + \alpha)}{\alpha(r_+ + r_-) + i\chi_1(r_+ - r_-) + 2\alpha\chi_0}, \\ A_2 &= \frac{r_+(\alpha + i\chi_1)(\chi_0 + i\chi_1 - \alpha)}{\alpha(r_+ + r_-) + i\chi_1(r_+ - r_-) + 2\alpha\chi_0}. \end{aligned} \quad (48)$$

Finally, we calculate the Ernst potential according to Eq. (26) and obtain

$$E(t, \chi) = -A_0 + \frac{A_1(\chi_0 - i\chi_1 - \alpha)}{r_-(\chi_0 + i\chi_1 + \alpha)} + \frac{A_2(\chi_0 - i\chi_1 + \alpha)}{r_+(\chi_0 + i\chi_1 - \alpha)} = \frac{2\alpha\chi_0 - \alpha(r_+ + r_-) - i\chi_1(r_+ - r_-)}{2\alpha\chi_0 + \alpha(r_+ + r_-) + i\chi_1(r_+ - r_-)}. \quad (49)$$

The calculation of the corresponding metric functions can be carried out as described in the last section. When integrating the systems of first order differential equations (13) for Q and (18) and (19) for λ , constants of integration appear which we choose such that a simpler representation is obtained in terms of the coordinates used. To write down the final form of the metric functions it is convenient to use the coordinates (x, y) as defined in Eq. (39) with χ_0 replaced by α . Then

$$P = \ln \frac{\chi_0^2 - \alpha^2 x^2 - \chi_1^2 y^2}{\alpha \sqrt{(1-x^2)(1-y^2)} [(\chi_0 + \alpha x)^2 + \chi_1^2 y^2]}, \quad (50)$$

$$Q = \frac{2\chi_1\chi_0(1-y^2)(\chi_0+\alpha x)}{\chi_0^2-\alpha^2x^2-\chi_1^2y^2}, \quad (51)$$

$$\lambda = \ln \frac{\chi_0^3(x^2-y^2)^2}{\sqrt{(1-x^2)(1-y^2)}[(\chi_0+\alpha x)^2+\chi_1^2y^2]^2}. \quad (52)$$

This metric corresponds to an unpolarized generalization of the Kantowski–Sachs cosmological model which is obtained in the limiting case $\chi_1=0$.

It is easy to see that near the singularity ($\tau \rightarrow \infty$) the general Ernst potential (49) approaches the corresponding AVTD potential as given in Eq. (43). This could be interpreted as an indirect proof of the AVTD behavior of the solution obtained here. A more direct proof can be given by analyzing the hyperbolic velocity $v = \sqrt{P_\tau^2 + e^{2P} Q_\tau^2}$ (see Ref. 4). In terms of the coordinates x and y , the hyperbolic velocity of the solutions (50) and (51) is given as a rather cumbersome expression. Nevertheless, it is possible to perform an analysis of its behavior by considering the different domains of the coordinates according to the definition equation (39). One can show that in general $0 \leq v < 1$ which according to Ref. 4 implies that the solution is AVTD.

An important property of the method presented here is that it allows us to calculate an arbitrary Gowdy T^3 cosmological solution with any degree of accuracy. To this end, let us expand in Eq. (26) the value of the Ernst potential $e(\xi)$ in Fourier series,

$$e(\xi) = \sum_{k=0}^{\infty} e_k(t, \chi) \cos(k\varphi), \quad (53)$$

where $e_0 = e(\chi)$ and we have represented the parameter s as $s = \cos \varphi$. Then the unknown function $\mu(\xi)$ can be expanded in a similar way,

$$\mu(\xi) = \sum_{k=0}^{\infty} \mu_k(t, \chi) \cos(k\varphi). \quad (54)$$

The normalization condition (28) can easily be calculated and implies that $\mu_0 = 1$. Furthermore, the general solution of the integral equation (26) can be written as²¹

$$E(t, \chi) = e(\chi) + \frac{1}{2} \sum_{k=1}^{\infty} e_k \mu_k. \quad (55)$$

According to Eq. (27), the coefficients μ_k have to satisfy the following system of algebraic equations:

$$\sum_{l=1}^{\infty} \mu_k (e_{k+l} - e_{k+l}^* + e_{|k-l|} - e_{|k-l|}^*) = -2e_k. \quad (56)$$

Thus, once the value of the Ernst potential is given at the initial singularity [$e(\chi) = E(\chi, \tau \rightarrow \infty)$] the general solution of the Ernst equation reduces to an infinite series with coefficients satisfying a set of pure algebraic equations.

V. CONCLUSIONS

We have shown that it is possible to generate Gowdy T^3 cosmological models starting from their data near the initial singularity. To this end, we first show that the Gowdy T^3 line element can be obtained from the line element of stationary axisymmetric solutions by means of complex transformation that involves the metric functions and the coordinates. The behavior of stationary axisymmetric solutions at the axis of symmetry is shown to be mathematically equivalent to the

behavior of Gowdy T^3 models near the singularity. In particular, we have derived the AVTD solution from its stationary axisymmetric counterpart. We then use the Ernst representation of the field equations and apply Sibgatullin's method to the Ernst potential which can be given at the singularity as any arbitrary function of the angle coordinate χ . In particular, we have shown that the Kantowski–Sachs cosmological model can be derived in this manner by starting from a specific form of the Ernst potential in terms of a rational function. We then have found an unpolarized generalization of the Kantowski–Sachs cosmological model. This generalization has been obtained in the same way as the Kerr metric is obtained from its value at the axis of symmetry by using Sibgatullin's method.²¹ It is possible to consider more general examples of Ernst potentials at the axis in terms of rational functions. It turns out that the system of integral equations (27) and (28) forms a closed algebraic system from which the value of the function $\mu(\xi)$ can be found and the expression for the Ernst potential can be calculated.²² This method could also be applied in the case of Gowdy cosmological models considered here.

By expanding the value of the Ernst potential at the singularity in terms of a Fourier series, it is possible to write explicitly the general solution for this type of models (including the unpolarized case) by using only a recurrence algebraic formula. This is a result that could find some application in numerical investigations since it allows us to “control” the accuracy of the analysis by truncating the series at any desired level.

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