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Long-term capital growth: the good and bad properties of the Kelly and fractional Kelly capital growth criteria

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The main advantage of the Kelly criterion, which maximizes the expected value of the logarithm of wealth period by period, is that it maximizes the limiting exponential growth rate of wealth. The main disadvantage of the Kelly criterion is that its suggested wagers may be very large because the Arrow–Pratt risk aversion, the reciprocal of current wealth, is small compared with other commonly chosen utility functions. Hence, the Kelly criterion is relatively risky in the short term. And although most of the time Kelly bettors will have a lot of final wealth after a long sequence of favorable bets, it is possible, through bad scenarios, to lose most of one's wealth. Hence, care in the use of Kelly and fractional Kelly strategies is crucial.

In the one asset two valued payoff case, the optimal Kelly wager is the edge (expected return) divided by the odds. If there are multiple assets in a continuous time model with a lognormal distribution for returns, the

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Table 1. Average ratio of certainty equivalent loss for errors in means, variances and covariances. Source: Chopra and Ziemba (1993).

Risk tolerance*	Errors in means vs. covariances	Errors in means vs. variances	Errors in variances vs. covariances
25	5.38	3.22	1.67
50	22.50	10.98	2.05
75	56.84	21.42	2.68
	↓	↓	↓
	20	10	2
	Error mean	Error var	Error covar
	20	2	1

*Risk tolerance $= R_T(w) = 100/(1/2)R_A(w)$ where $R_A(w) = -u''(w)/u'(w)$.

optimal *Kelly strategy* also has a return relative to risk structure. The investments in assets are given by the product of the inverse of the covariance matrix for the idiosyncratic component and the vector of excess returns. When the covariance matrix is diagonal, an assumption that MacKinley and Pastor (2000) support with evidence, the investment fractions in assets are proportional to the Sharpe ratios. The total fraction of wealth invested depends on the investor's aversion to risk.

In practice, estimates for the mean vector and covariance matrix are used in the calculation of the Kelly investments in assets. Chopra and Ziemba (1993), following earlier studies by Kallberg and Ziemba (1981, 1984), showed for mean–variance asset allocation problems that the mean is much more important than the variances and covariances. For allocations based on average risk aversion, errors in means versus errors in variances, covariances are about 20:2:1 in importance as measured by the cash equivalent value of final wealth. However, for low risk aversion such as log utility with risk aversion equal to $1/w$, the impact of estimation error was about 100:3:1. Also, the bets may be exceedingly large and risky for favorable Kelly bets. The covariance can be used to yield improved Stein-type estimates of the means (MacLean *et al.* 2007.)

So the Elog maximizing bettor must be very careful not to over bet because of inaccurate mean estimates. Table 1 and figure 1 show this and illustrate that the relative importance of mis-estimated parameters depends on the bettor's degree of risk aversion. The lower is the Arrow–Pratt risk aversion,

$$R_A = -\frac{u''(w)}{u'(w)},$$

the greater are the relative errors from incorrectly estimated means. Chopra (1993) further shows that portfolio turnover is larger for errors in the means than for errors in variances and covariances, but the effects are much less than for the cash equivalent performance as shown in figure 2. The impact of estimation errors can be moderated with a fractional Kelly strategy, where a fraction α, with $0 \leqslant \alpha \leqslant 1$, of the total Kelly proportion is allocated to risky assets and the rest to cash. In the

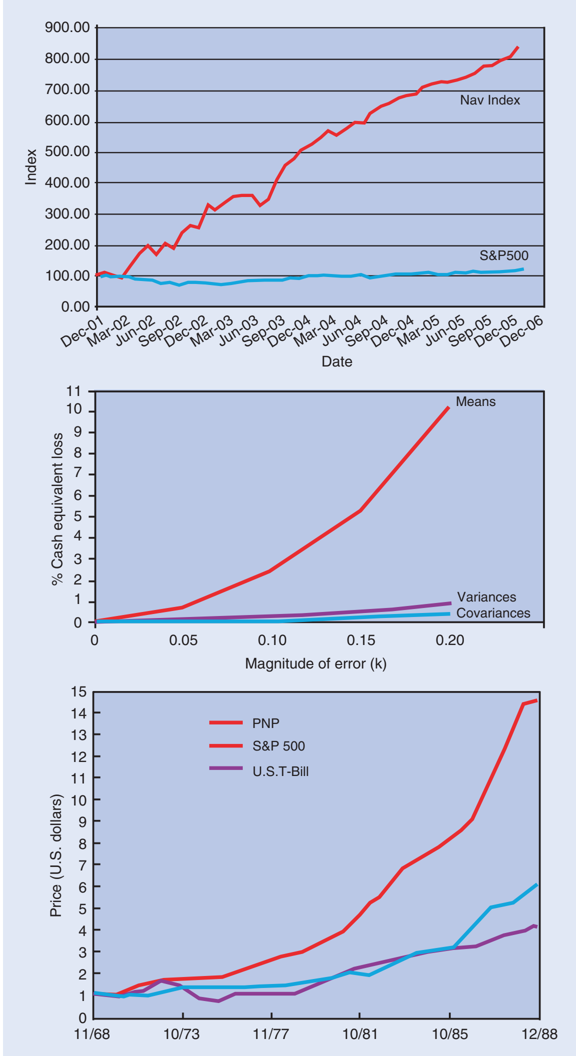

Figure 1. Mean percentage cash equivalent loss due to errors in inputs.

lognormal case the fraction is directly linked to the Arrow–Pratt absolute and relative risk aversion parameter $R_R = wR_A(w) = 1 - \delta$ of the negative power utility function

$$u(w) = \delta w^{\delta}, \quad \text{for } \delta < 0, \text{ namely } \alpha = \frac{1}{1-\delta} = \frac{1}{R_R}.$$

A variation on this, which cuts back on investment capital, w, is the *discretionary wealth approach* (Wilcox 2003), where a fraction of wealth, c, is set aside at each investment decision for consumption or savings. This is a special type of fractional Kelly strategy, where

$$f = \frac{w-c}{w}.$$

Since log has $R_A(w) = 1/w$, which is close to zero, the Kelly bets may be exceedingly large and risky for favorable Kelly bets. MacLean *et al.* (2010a) present simulations of medium term Kelly, fractional Kelly and proportional betting strategies. The results show that with favorable investment opportunities, Kelly bettors attain

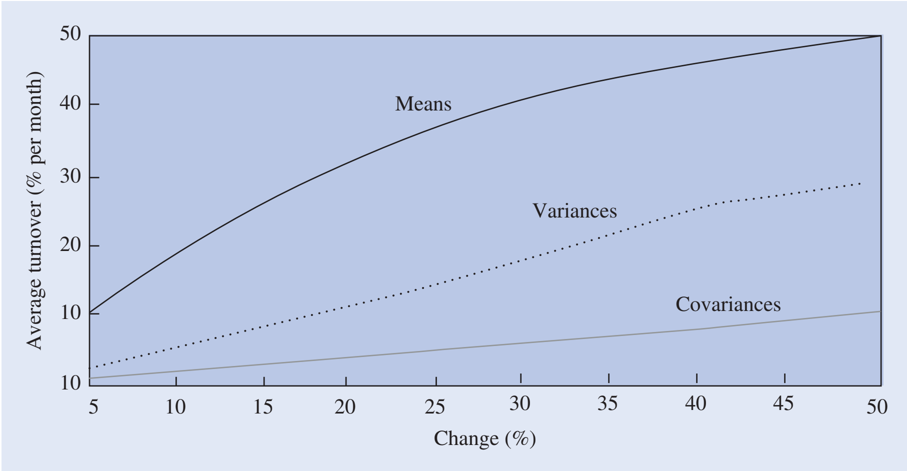

Figure 2. Average turnover for different percentage changes in means, variances and covariances. Source: Based on data from Chopra (1993).

large final wealth most of the time. But, because a long sequence of bad scenario outcomes is possible, any strategy can lose substantially, even if there are many independent investment opportunities and the chance of losing at each investment decision point is small. The Kelly and fractional Kelly rules, like all rules, are never a sure way of winning for every finite sequence.

Many great investors have used Kelly or fractional Kelly strategies. Three of them, Keynes, Buffett, and Soros, were long-term investors whose wealth paths were quite rocky but with good long-term outcomes. Our analyses suggest that Buffett and Soros seem to act similar to full Kelly bettors and Keynes like an 80% Kelly bettor with the negative power utility function $-w^{-0.25}$ (Ziemba 2003).

Graphs such as figure 3 show that growth is traded off for security with the use of fractional Kelly strategies and negative power utility functions. Log maximizes the long-run growth rate. Utility functions such as positive power that bet more than Kelly have more risk and lower growth, so they are growth-security dominated. Betting in this area invariably leads to disaster. One of the properties shown below that is illustrated in the graph is that for processes which are well approximated by continuous time, the growth rate becomes zero plus the risk free rate when one bets exactly twice the Kelly wager.

As you exceed the Kelly bets more and more, risk increases and long-term growth falls, eventually becoming more and more negative. Long Term Capital Management, Amarath and the Victor Niederhoffer funds are three of many real world instances in which overbetting led to disaster. See Ziemba and Ziemba (2007) for additional examples.

Thus long-term growth maximizing investors should bet Kelly (the expected log maximizing strategy period by period) or less. We call betting less than Kelly 'fractional Kelly', which is simply a blend of Kelly and cash. Consider the negative power utility function $\delta\omega^{\delta}$ for $\delta<0$. This utility function is concave and when $\delta\to 0$ it converges to log utility. As δ becomes more negative, the investor is less aggressive since his absolute

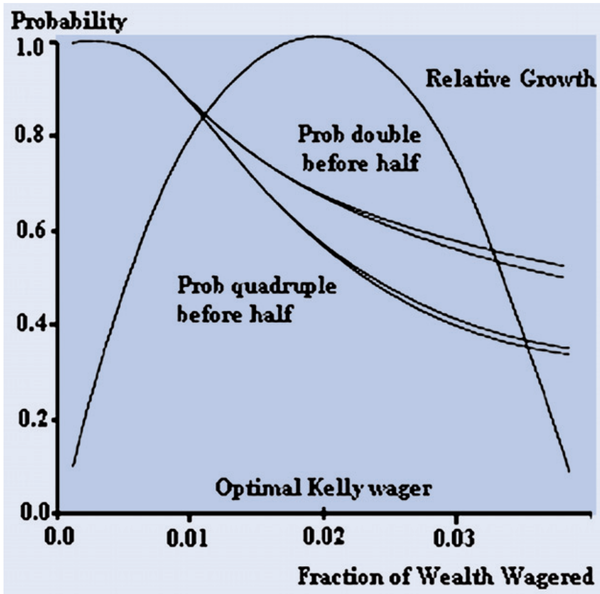

Figure 3. Continuous time approximation to the probability of doubling and quadrupling before halving and relative growth rates versus fraction of wealth wagered for Blackjack (2% advantage, $p=0.51$ and $q=0.49$). Source: MacLean and Ziemba (1999).

Arrow–Pratt risk aversion index is also higher. For the case of a stationary lognormal process and a given δ for utility function δw^{δ} with $\alpha=1/(1-\delta)$ between 0 and 1, the optimal portfolio has α invested in the Kelly portfolio and $1-\alpha$ invested in cash.

This handy formula relating the coefficient of the negative power utility function to the Kelly fraction is correct for lognormally distributed investment assets (MacLean *et al.* 2005). For example, half Kelly is $\delta=-1$ and quarter Kelly is $\delta=-3$. So if you want a less aggressive path than Kelly pick an appropriate δ. This formula does not apply more generally. For example, for coin tossing, where $Pr(X=1)=p$, $Pr(X-1)=q$, $p+q=1$,

$$f_{\delta}^{*}=\frac{p^{1/(1-\delta)}-q^{1/(1-\delta)}}{p^{1/(1-\delta)}+q^{1/(1-\delta)}}=\frac{p^{\alpha}-q^{\alpha}}{p^{\alpha}+q^{\alpha}},$$

which is not αf^*, where $f^* = p - q \geqslant 0$ is the Kelly bet. Thorp (2010) shows that, for non-lognormally distributed assets, this handy formula is not accurate.

We now list various important good and bad Kelly criterion properties.

The good properties

Good Maximizing expected utility, $E\log X$, asymptotically maximizes the rate of asset growth (Breiman 1961, Algoet and Cover 1988).

Good The expected time to reach a preassigned goal A is asymptotically least as A increases without limit with a strategy maximizing $E\log X_N$ (Breiman 1961, Algoet and Cover 1988, Browne 1997).

Good Under fairly general conditions, maximizing $E\log X$ also asymptotically maximizes median $\log X$ (Ethier 1987, 2004, 2010).

Good The $E\log X$ bettor never risks ruin (Hakansson and Miller 1975).

Good The absolute amount bet is monotone increasing in wealth.

Good The $E\log X$ bettor has an optimal myopic policy. He does not have to consider prior nor subsequent investment opportunities. This is a crucially important result for practical use. Hakansson (1971) proved that the myopic policy obtains for dependent investments with the log utility function. For independent investments and any power utility a myopic policy is optimal (Mossin 1968). In fact, past outcomes can be taken into account by maximizing the conditional expected logarithm given the past (Algoet and Cover 1988).

Good Simulation studies show that the $E\log X$ bettor's fortune pulls ahead of other 'essentially different' strategies' wealth for most reasonable-sized samples. Essentially different has a limited meaning. For example $g^* \geqslant g$ but $g^* - g = \epsilon$ will not lead to rapid separation if ϵ is small enough The key again is risk (Bicksler and Thorp 1973, Ziemba and Hausch 1986, MacLean *et al.* 2010a). General formulas are given by Aucamp (1993).

Good If you wish to have higher security by trading it off for lower growth, then use a negative power utility function, δw^δ, or fractional Kelly strategy. See MacLean *et al.* (2004), who show how to compute the coefficient to stay above a growth path at discrete points in time with given probability or to be above a given drawdown with a certain confidence limit. MacLean *et al.* (2009) add the feature that path violations are penalized with a convex cost function. See also Stutzer (2010) for a related but different model of such security.

Good Competitive optimality. Kelly gambling yields wealth X^* such that $E(X/X^*) \leqslant 1$, for all other strategies X. This follows from the Kuhn Tucker conditions. Thus by Markov's inequality, $Pr[X \geqslant tX^*] \leqslant 1/t$, for $t \geqslant 1$, and for all other induced wealth x. Thus an opponent cannot outperform X^* by a factor t with probability greater than $1/t$. This inequality can be improved when $t = 1$ by allowing fair randomization U. Let U be drawn according to a uniform distribution over the interval $[0, 2]$, and let U be independent of X^*. Then the result improves to $Pr[X \geqslant UX^*] \leqslant 1/2$ for all portfolios X. Thus fairly randomizing one's initial wealth and then investing it according to the Kelly criterion, one obtains a wealth UX^* that can only be beaten half the time. Since a competing investor can use the same strategy, probability 1/2 is the best competitive performance one can expect. Kelly gambling is the heart of the solution of this two-person zero sum game of who ends up with the most money. So X^* (actually UX^*) is competitively optimal in a single investment period (Bell and Cover 1980, 1988).

Good If X^* is the wealth induced by the log optimal (Kelly) portfolio, then the expected wealth ratio is no greater than one, i.e. $E(X/X^*) \leqslant 1$, for the wealth X induced by any other portfolio (Bell and Cover, 1980, 1988).

Good Super St Petersburg. Any cost c for the St Petersburg random variable X, $Pr[X = 2^k] = 2^{-k}$, is acceptable. But the larger the cost c, the less wealth one should invest. The growth rate G^* of wealth resulting from repeating such investments is

$$G^* = \max_{0 \leqslant f \leqslant 1} E\ln\left(1 - f + \frac{f}{c}X\right),$$

where f is the fraction of wealth invested. The maximizing f^* is the Kelly proportion (Bell and Cover 1980). The Kelly fraction f^* can be computed even for a super St Petersburg random variable $Pr[Y = 2^{2^k}] = 2^{-k}$, $k = 1, 2, \ldots$, where $E\ln Y = \infty$, by maximizing the relative growth rate

$$\max_{0 \leqslant f \leqslant 1} E\ln\frac{1 - f + (f/c)Y}{(1/2) + (1/2c)Y}.$$

This is bounded for all f in [0, 1]. Now, although the exponential growth rate of wealth is infinite for all proportions f and it seems that all $f \in [0, 1]$ are equally good, the maximizing f^* in the previous equation guarantees that the f^* portfolio will asymptotically exponentially outperform any other portfolio $f \in [0, 1]$. Both investors' wealth have super exponential growth, but the f^* investor will exponentially outperform any other essentially different investor.

Some bad properties

Bad The bets may be a large fraction of current wealth when the wager is favorable and the risk of loss is very small. For one such example, see Ziemba and

Hausch (1986, pp. 159–160). There, in the inaugural 1984 Breeders' Cup Classic \$3 million race, the optimal fractional wager, using the Dr Z place and show system using the win odds as the probability of winning, on the 3–5 shot Slew of Gold was 64%. (See also the 74% futures bet on the January effect in MacLean *et al.* 1992.) Thorp and Ziemba actually made this place and show bet and won with a low fractional Kelly wager. Slew finished third but the second place horse Gate Dancer was disqualified and placed third. Wild Again won this race; the first great victory by the masterful jockey Pat Day.

Bad For coin tossing, any fixed fraction strategy has the property that if the number of wins equals the number of losses then the bettor is behind. For n wins and n losses and initial wealth W_0 we have $W_{2n} = W_0(1 - f^2)^n$.

Bad The unweighted average rate of return converges to half the arithmetic rate of return. Thus you may regularly win less than you expect. This is a consequence of weighting equally rather than by size of the wager (Ethier and Tavaré 1983, Griffin 1985).

Some observations

(1) For an i.i.d. process and a myopic policy, which results from maximizing expected utility in case the utility function is log or a negative power, the result is fixed fraction betting, hence fractional Kelly includes all these policies.

(2) A betting strategy is 'essentially different' from Kelly if $S_n \equiv \sum_{i=1}^{n} \mathrm{E}\log(1 + f_i^* X_i) - \sum_{i=1}^{n} \mathrm{E}\log(1 + f_i X_i)$ tends to infinity as n increases. The sequence $\{f_i^*\}$ denotes the Kelly betting fractions and the sequence $\{f_i\}$ denotes the corresponding betting fractions for the essentially different strategy.

(3) The Kelly portfolio does not necessarily lie on the efficient frontier in a mean–variance model (Thorp 1971). Markowitz (1976) argues that the Kelly Elog maximizing strategy is a limiting mean–variance portfolio. Roll (1973) tests the differences between Kelly and mean–variance portfolios and, for his assumptions and data, finds them closely related.

(4) Despite its superior long-run growth properties, Kelly, like any other strategy, can have a very poor return outcome. For example, making 700 wagers all of which have a 14% advantage, the least of which has a 19% chance of winning, can turn \$1000 into \$18. But with full Kelly 16.6% of the time \$1000 turns into at least \$100,000 (Ziemba and Hausch 1986). Half Kelly does not help much as \$1000 can become \$145 and the growth is much lower with only over \$100,000 final wealth 0.1% of the time. For more such calculations, see Bicksler and Thorp (1973) and MacLean *et al.* (2010a).

(5) Fallacy: If maximizing $\mathrm{E}\log X_N$ almost certainly leads to a better outcome, then the expected utility of its outcome exceeds that of any other rule provided N is sufficiently large. *Counterexample:* $u(x) = x$, $1/2 < p < 1$, Bernoulli trials $f = 1$ maximizes $Eu(x)$ but $f = 2p - 1 < 1$ maximizes $\mathrm{E}\log X_N$ (Samuelson 1971, Thorp 1971, 2006).

(6) It can take a long time for any strategy, including Kelly, to dominate an essentially different strategy. For instance, in continuous time with a geometric Wiener process, suppose $\mu_a = 20\%$, $\mu_b = 10\%$, $\sigma_a = \sigma_b = 10\%$. Then in five years, A is ahead of B with 95% confidence. But if $\sigma_a = 20\%$, $\sigma_b = 10\%$ with the same means, it takes 157 years for A to beat B with 95% confidence. As another example, in coin tossing suppose game A has an edge of 1.0% and game B 1.1%. It takes two million trials to have an 84% chance that game B dominates game A (Thorp 2006).

(7) Fractional Kelly, with fraction k, where $0 < k \leqslant 1$, applied to capital w_0, is equivalent to full Kelly applied to kw_0, with($1 - k)w_0$ in cash. Question: For what other utility functions does such a decomposition hold? Lemma: It is true if the utility function $u(w)$ has the property $u(\lambda w) = f(\lambda)g(w)$. Proof: the solutions to $0 = u(\lambda x) = f(\lambda)g(w)$ are the solutions to $g(w) = 0$, which does not depend on λ. An example is $u(w) = -w^{-a}$. It is not known if the property in the lemma is also necessary.

The theory and practical application of the Kelly criterion is straightforward when the underlying probability distributions are fairly accurately known.† However, in investment applications this is usually not the case. Realized future equity returns may be very different from what one would expect using estimates based on historical returns. Consequently, practitioners who wish to protect capital above all, sharply reduce risk as their drawdown increases.

Prospective users of the Kelly criterion can check our list of good properties, bad properties and observations to test whether Kelly is well suited to their intended application. Given the extreme sensitivity of Elog calculations to errors in mean estimates, these estimates must be accurate and, to be on the safe side, the size of the wagers should be reduced.

For long-term compounders, the good properties dominate the bad properties of the Kelly criterion. But the bad properties may dampen the enthusiasm of naive prospective users of the Kelly criterion. The Kelly and fractional Kelly strategies are very useful if applied carefully with good data input and proper financial engineering risk control.

†With transactions and price pressures, one solves a simple stochastic program or, in the case of discrete probability distributions, a nonlinear program.

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Appendix A:

In continuous time, with a geometric Wiener process, betting exactly double the Kelly criterion amount leads to a growth rate equal to the risk-free rate. This result is due to Stutzer (2010), Janacek (1998) and Thorp (2006), and possibly others. The following simple proof, under the further assumption of the Capital Asset Pricing Model, is due to Harry Markowitz.

In continuous time

$$g_p = E_p - \frac{1}{2} V_p,$$

where E_p, V_p and g_p are the portfolio expected return, variance and expected log, respectively. In the CAPM,

$$\begin{aligned} E_p &= r_0 + (E_M - r_0)X, \\ V_p &= \sigma_M^2 X^2, \end{aligned}$$

where X is the portfolio weight, E_M is the expected market return, and r_0 is the risk-free rate. Collecting terms and setting the derivative of g_p with respect to X equal to zero yields

$$X = \frac{(E_M - r_0)}{\sigma_M^2},$$

which is the optimal Kelly bet with optimal growth rate

$$\begin{aligned} g^* &= r_0 + (E_M - r_0)^2 - \frac{1}{2}\left[\frac{(E_M - r_0)}{\sigma_M^2}\right]^2 \sigma_M^2 \\ &= r_0 + \frac{(E_M - r_0)^2}{\sigma_M^2} - \frac{1}{2}\left[\frac{(E_M - r_0)^2}{\sigma_M^2}\right] \\ &= r_0 + \frac{1}{2}\left[\frac{(E_M - r_0)}{\sigma_M}\right]^2. \end{aligned}$$

Substituting double Kelly, namely $Y = 2X$ for X above, into

$$g_p = r_0 + (E_M - r_0)Y - \frac{1}{2}\sigma_M^2 Y^2,$$

and simplifying yields

$$g_0 - r_0 = 2\left[\frac{(E_M - r_0)^2}{\sigma_M^2}\right] - \frac{4}{2}\left[\frac{(E_M - r_0)^2}{\sigma_M^2}\right] = 0.$$

Hence $g_0 = r_0$ when $Y = 2X$.

The CAPM assumption is not needed. For a more general proof and illustration, see Thorp (2006).