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Generalizations of the Coinvariant Algebra

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by

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ABSTRACT OF THE DISSERTATION

Generalizations of the Coinvariant Algebra

by

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Doctor of Philosophy in Mathematics

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The classical coinvariant algebra is the quotient of the polynomial ring in n variables by the ideal generated by polynomials that are invariant under the action of variable permutation. The classical coinvariant algebra is a fundamental object of study in the theory of algebraic combinatorics and a variety of generalizations of it have been defined. In this dissertation we will explore a variety of generalizations and refinements of the coinvariant algebra.

Chapter 1

Introduction

Actions of the symmetric group $\mathfrak{S}_n$ are a fundamental object of study in the fields of algebra and combinatorics with applications throughout mathematics. For any finite group G, understanding the irreducible representations of G is key to understanding its general representation theory. The irreducible representations are equinumerous with (though in general not in canonical bijection with) the conjugacy classes of G. Thus for the symmetric group $\mathfrak{S}_n$, the irreducible representations can be indexed by partitions of n, and a specific way of doing is given by a construction of Specht of his now eponymous $\mathfrak{S}_n$-modules [17].

One tool to study the irreducible representations of G is to look at its (left/right) regular representation of G which is the representation obtained from the action of G on its group algebra by (left/right) multiplication. Every irreducible representation of G will appear in the regular representation of G with multiplicity equal to the dimension of that representation. Thus understanding of the regular representation of G can help lead to greater understanding of its more general representation theory.

The coinvariant algebra R_n of the symmetric group $\mathfrak{S}_n$ is the quotient of $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ by the ideal I_n generated by polynomials invariant under the action of $\mathfrak{S}_n$ by variable permutation. The ideal I_n is homogeneous and invariant under the action of $\mathfrak{S}_n$, so that R_n is a graded $\mathfrak{S}_n$-module. Further by a result of Chevalley [6] as an ungraded $\mathfrak{S}_n$-module R_n is isomorphic to the regular representation of $\mathfrak{S}_n$, and the multiplicities of irreducible representations appearing in it can be described in terms of the shapes of standard Young tableaux of size n.

Having R_n as a realization of the regular representation as a polynomial quotient is interesting in its own right, but as mentioned earlier R_n is also graded which provides additional structure since in general there is not a canonical way to impose a grading on the regular representation of a group. To generalize the result of Chevalley, the multiplicities of each irreducible representation in the component of degree d would ideally be given by the number of standard Young tableaux of size n with some statistic equal to d. Lusztig (unpublished) and Stanley [18] show that the major index of standard Young tableaux (which is the sum of all descents) fulfills this requirement.

This covers the basic results for the classical coinvariant algebra, but there have many directions in which the coinvariant algebra has been generalized. In this thesis we will review many of these generalizations and extend these generalization further. Frequently the extensions we give will generalize two existing extensions, thus we will in a sense complete a square of generalization.

This thesis will be organized as follows. In the next chapter we will start by giving a more detailed descirption of the requisite background material. We will then give an Adin, Brenti Roichmann [2] style refinement of the grading of $R_{n,k}$, a generalized coinvariant algebra defined by Haglund, Rhoades, and Shimizono [12] that has connections to the Delta conjecture from the field of Macdonald polynomials and to crystal theory. Then in chapter 3 we will extend the results of chapter 2 to an extension of $R_{n,k}$ due to Chan and Rhoades [5] that replaces $\mathfrak{S}_n$ with a more general class of reflection groups. In chapter 4 we will exmaine a generalized coinvariant algebra introduced by Rhoades[16] that is indexed by composition. In chapter 5 we will review Iwahori-Hecke algebras and give a general method for extending a certain class of polynomial quotients that are $\mathfrak{S}_n$ modules to Iwahori-Hecke algebra modules, and we will specifically examine the case of Tanisaki [21] quotients. In chapter 6 we will examine a 0-Hecke action whose combinatorics mirrors that of the Tanisaki quotient.

Chapter 2

Descent Representations of $R_{n,k}$

2.1 Definitions and Background

The classical coinvariant algebra R_n is constructed as follows: let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ by permutation of the variables $x_1, \ldots, x_n$. The polynomials that are invariant under this action are called symmetric polynomials, and we let I_n be the ideal generated by symmetric polynomials with vanishing constant term. Then R_n is defined as the algebra obtained by quotienting $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ by I_n, that is

$$R_n := \frac{\mathbb{Q}[x_1, x_2, \ldots, x_n]}{I_n}. \tag{2.1}$$

There are a number of sets of symmetric polynomials in $x_1, \ldots, x_n$ that algebraically generate all symmetric polynomials in the variables $x_1, \ldots, x_n$ with vanishing constant term. The set that is important for the generalization of R_n that we are considering is the elementary symmetric functions

$$e_d := \sum_{1 \leq i_1 < i_2 < \ldots < i_d \leq n} \prod_{j=1}^{d} x_{i_j}, \tag{2.2}$$

for $1 \leq d \leq n$. We then have

$$I_n = \langle e_1, e_2, \ldots, e_n \rangle. \tag{2.3}$$

Since I_n is homogeneous and invariant under the action of $\mathfrak{S}_n$, the coinvariant algebra is a graded $\mathfrak{S}_n$-module. Since the conjugacy classes of $\mathfrak{S}_n$ are indexed by partitions of n, the irreducible representations of $\mathfrak{S}_n$ are also indexed by partitions of n (an explicit construction of irreducibles is given by Specht modules). We let S^λ denote the irreducible representation corresponding to λ, and we let χ^λ_μ be the character of S^λ evaluated at an element of type μ.

The following relies on some definitions that we will cover in Section 2.1. Given a representation V of $\mathfrak{S}_n$, a natural question to ask is: "What is the multiplicity of S^λ in V for each partition of n?". All of this information can be contained in a single symmetric function called the **Frobenius image** of V, which is denoted $Frob(V)$. The Frobenius image has the following formula

$$Frob(V) = \sum_{\lambda \vdash n} c_\lambda s_\lambda, \tag{2.4}$$

where c_λ is the multiplicity of S^λ in V and s_λ is the Schur function associated to λ. We will take this formula as a definition. In the case of the classical coinvariant algebra this problem was solved by Chevalley [6] who showed that the multiplicity of S^λ in R_n is the number of standard Young tableaux of shape λ, that is that

$$Frob(R_n) = \sum_{T \in SYT(n)} s_{sh(T)}. \tag{2.5}$$

If V is a graded representation of $\mathfrak{S}_n$ with degree d component V_d, then we can also consider the Frobenius image of V_d for all d. This data can be combined into a single function called the **graded Frobenius image**, which is defined as follows:

$$grFrob(V;q) = \sum_{d=0}^{\infty} q^d Frob(V_d). \tag{2.6}$$

Lusztig (unpublished) and Stanley [18] showed that for the classical coinvariant algebra the multiplicity of S^λ in the degree d component of R_n is the number of standard Young tableaux with major index equal to d. Stated in terms of the graded Frobenius image,

$$grFrob(R_n;q) := \sum_{T \in SYT(n)} q^{maj(T)} s_{shape(T)}. \tag{2.7}$$

A further refinement of R_n is given as follows: define

$$P_{\trianglelefteq\mu} := span\{m \in \mathbb{Q}[x_1,\ldots,x_n] : \lambda(m) \trianglelefteq \mu\}, \tag{2.8}$$

and

$$P_{\triangleleft\mu} := span\{m \in \mathbb{Q}[x_1,\ldots,x_n] : \lambda(m) \triangleleft \mu\} \tag{2.9}$$

where m are monomials, $\lambda(m)$ is the exponent partition of m, and $\triangleleft$ is the dominance order on partitions. Then let $Q_{\trianglelefteq\mu}$ and $Q_{\triangleleft\mu}$ be the projections of $P_{\trianglelefteq\mu}$ and $P_{\triangleleft\mu}$ onto R_n respectively. Next define

$$R_{n,\mu} := Q_{\trianglelefteq\mu}/Q_{\triangleleft\mu}. \tag{2.10}$$

This is a refinement of the grading since the degree d component of R_n is equal to

$$\bigoplus_{\mu \vdash d} R_{n,\mu}. \tag{2.11}$$

Adin, Brenti, and Roichman [2] show that $R_{n,\mu}$ is zero unless μ is a partition with at most $n-1$ parts such that the differences between consecutive parts are at most 1. We call such partitions descent partitions. They also show that in the case that $R_{n,\mu}$ is not zero, the multiplicity of S^λ in $R_{n,\mu}$ is given by the number of standard Young tableaux of shape λ with descent set equal to the descent set of μ, where we define a descent of a partition μ as a value i such that $\mu_i > \mu_{i+1}$. For example if $n = 5$, and $\mu = (3,2,2,1)$, then the descents of μ are $1,3,4$, and the multiplicity of $S^{(2,2,1)}$ is 1 since the only standard Young tableau of shape $(2,2,1)$ with descent set $\{1,3,4\}$ is

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}. \tag{2.12}$$

Motivated by the Delta Conjecture in the theory of Macdonald polynomials, Haglund, Rhoades, and Shimozono [12] generalize this entire picture by defining the ideal

$$I_{n,k} := \langle x_1^k, x_2^k, \ldots x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle, \tag{2.13}$$

for a positive integer $k \leq n$. They then define a generalized coinvariant algebra as

$$R_{n,k} := \frac{\mathbb{Q}[x_1, \ldots, x_n]}{I_{n,k}}. \tag{2.14}$$

This is a generalization since in the case $n = k$, we recover the classical coinvariant algebra R_n, that is $R_{n,n} = R_n$. This is connected to the Delta Conjecture because Haglund, Rhoades, and Shimozono show that

$$(rev_q \circ \omega) grFrob(R_{n,k}; q) \tag{2.15}$$

is equal to $Rise_{n,k}(x;q,0), Rise_{n,k}(x;0,q)$, $Val_{n,k}(x;q,0)$, and $Val_{n,k}(x;0,q)$, where $Rise_{n,k}$ and $Val_{n,k}$ are combinatorially defined functions appearing in the Delta Conjecture, and ω is the standard involution on symmetric functions.

As in the classical case, $R_{n,k}$ is a graded $\mathfrak{S}_n$-module and we can refine the grading as follows.

Definition 2.1.1. *Let μ be a partition with at most n parts. Next define $S_{\trianglelefteq\mu}$ and $S_{\triangleleft\mu}$ to be the projections of $P_{\trianglelefteq\mu}$ and $P_{\triangleleft\mu}$ onto $R_{n,k}$. We then define*

$$R_{n,k,\mu} := S_{\trianglelefteq\mu} / S_{\triangleleft\mu}. \tag{2.16}$$

This is a refinement of the grading since the degree d component of $R_{n,k}$ is equal to

$$\bigoplus_{\mu \vdash d} R_{n,k,\mu}. \tag{2.17}$$

Our primary goal is to determine the multiplicities of S^λ in $R_{n,k,\mu}$ which we do in the following theorem, thus extending the results of Adin, Brenti, and Roichman on $R_{n,\mu}$ to $R_{n,k,\mu}$ and refining the results of Haglund, Rhoades and Shimozono.

Theorem 2.1.2. *The algebra $R_{n,k,\rho}$ is zero unless ρ fits in an $(n-1)\times k$ rectangle and $\rho_i - \rho_{i+1} \leq 1$ for $i > n-k$. In the case that $R_{n,k,\rho}$ is not zero, the multiplicity of S^λ in $R_{n,k,\rho}$ is given by*

$$|\{T \in SYT(\lambda) : Des_{n-k+1,n}(\rho) \subseteq Des(T) \subseteq Des(\rho)\}|. \tag{2.18}$$

A key component of the methods in [2] is the use of a basis for $\mathbb{Q}[x_1,\ldots,x_n]$ that arises from the theory of Cohen-Macaulay rings and the fact that $e_n, e_{n-1},\ldots,e_1$ form a regular sequence. We are not able to use these methods since the generators of $I_{n,k}$ do not form a regular sequence.

A different direction of generalization comes from considering the coinvariant algebra for general complex reflection groups $G(r,p,n)$, which reduce to $\mathfrak{S}_n$ in the case $r=p=1$. These algebras are studied by Bagno and Biagioli in [3]. Chan and Rhoades [5] give generalizations of these objects in the case $p=1$ for a parameter $k \leq n$. If we let $\mathbf{x}_n^r$ denote the set of variables $\{x_1^r, x_2^r, \ldots, x_n^r\}$ then the ideal we are interested in for this case is

$$J_{n,k} := \langle x_1^{kr}, x_2^{kr}, \ldots, x_n^{kr}, e_n(\mathbf{x}_n^r), e_{n-1}(\mathbf{x}_n^r), \ldots, e_{n-k+1}(\mathbf{x}_n^r)\rangle, \tag{2.19}$$

and the algebra is

$$S_{n,k} := \frac{\mathbb{C}[x_1,x_2,\ldots,x_n]}{J_{n,k}}. \tag{2.20}$$

This is a graded $G(r,1,n)$-module and we can again refine the grading by partitions of size d as follows.

Definition 2.1.3. *Let μ be a partition with at most n parts. Next define $S_{\trianglelefteq\mu}$ and $S_{\triangleleft\mu}$ to be the* *projections of* $P_{\trianglelefteq\mu}$ *and* $P_{\triangleleft\mu}$ *onto* $S_{n,k}$*. We then define*

$$S_{n,k,\mu} := S_{\trianglelefteq\mu}/S_{\triangleleft\mu}. \tag{2.21}$$

This refines the grading since the degree d component of $S_{n,k}$ is equal to

$$\bigoplus_{\mu\vdash d} S_{n,k,\mu}. \tag{2.22}$$

The following theorem gives the multiplicities of irreducible representations appearing in $S_{n,k,\mu}$.

Theorem 2.1.4. *The algebra* $S_{n,k,\rho}$ *is zero unless* ρ *fits in an* $n\times(kr)$ *rectangle,* $\rho_i - \rho_{i+1} \leq r$ *for* $i > n-k$*, and* $\rho_n < r$*. In the case that* $S_{n,k,\rho}$ *is not zero, the multiplicity of* $S^{\overline{\lambda}}$ *in* $S_{n,k,\rho}$ *is given by*

$$|\{T \in SYT(\overline{\lambda}) : Des^r_{n-k+1,n}(\rho) \subseteq Des(T) \subseteq Des^r(\rho), c_i(T) \equiv \rho_i (mod r)\}|. \tag{2.23}$$

2.2 Descents and Monomials

An important component of the results of [2] on R_n is the use of a certain monomial basis for R_n. We will recall this basis and the generalization of this basis given in [12] for $R_{n,k}$. This basis for R_n will be indexed by permutations, and will be defined in terms of the descents of the corresponding permutation.

Given a permutation $\sigma \in \mathfrak{S}_n$, i is a descent of σ if $\sigma(i) > \sigma(i+1)$. We denote by Des(σ) the set of descents of σ. We denote by $d_i(\sigma)$, the number of descents of σ that are at least as large as i, that is

$$d_i(\sigma) := |\{i, i+1, \ldots, n\} \cap Des(\sigma)|. \tag{2.24}$$

Finally for two integers i, j such that $1 \leq i \leq j \leq n$ we let $Des_{i,j}(\sigma)$ denote the set of descents of σ that are between i and j inclusively, that is

$$Des_{i,j}(\sigma) := Des(\sigma) \cap \{i, i+1, \dots, j-1, j\}. \tag{2.25}$$

For example if $\sigma = 31427865 \in \mathfrak{S}_8$, then $\mathrm{Des}(\sigma) = \{1,3,6,7\}$,
$(d_1(\sigma), \dots, d_8(\sigma)) = (4,3,3,2,2,2,1,0), and$
$Des_{2,6}(\sigma) = \{3,6\}$.

Descents are used to define a set of monomials which descend to a basis for R_n, see [7, 9].

Definition 2.2.1. *Given a permutation $\sigma \in \mathfrak{S}_n$, the* **Garsia-Stanton monomial** *or simply* **descent monomial** *associated to σ is*

$$gs_\sigma := \prod_{i=1}^{n} x_{\sigma(i)}^{d_i(\sigma)}. \tag{2.26}$$

These monomials descend to a basis for R_n.

For example, if $\sigma = 31427865 \in \mathfrak{S}_8$, then

$$gs_\sigma = x_3^4 x_1^3 x_4^3 x_2^2 x_7^2 x_8^2 x_6^1. \tag{2.27}$$

These monomials are generalized by Haglund, Rhoades, and Shimozono in [12] to (n,k)-descent monomials that are indexed by ordered set partitions of n with k blocks. Alternatively they can be indexed by pairs (π, I) consisting of a permutation $\pi \in \mathfrak{S}_n$ and a sequence $i_1, \dots, i_{n-k}$ such that

$$k - des(\pi) > i_1 \geq i_2 \geq \dots \geq i_{n-k} \geq 0. \tag{2.28}$$

This is done as follows:

Definition 2.2.2. *Given a permutation $\pi \in \mathfrak{S}_n$ and a sequence $I = (i_1, i_2, \dots, i_{n-k})$ such that*

$$k - des(\pi) > i_1 \geq i_2 \geq \dots \geq i_{n-k} \geq 0, \tag{2.29}$$

the (n,k)**-descent monomial** *associated to* (π, I) *is*

$$gs_{\pi,I} := gs_{\pi} x_{\pi(1)}^{i_1} x_{\pi(2)}^{i_2} \dots x_{\pi(n-k)}^{i_{n-k}}. \tag{2.30}$$

These monomials descend to a basis for $R_{n,k}$.

As an example if $\sigma = 31427865 \in \mathfrak{S}_8$, $k=6$, and $I=(1,0)$, then

$$gs_{\sigma,I} = gs_{\sigma} \cdot x_3^1 x_1^0 = x_3^5 x_1^3 x_4^3 x_2^2 x_7^2 x_8^2 x_6^1. \tag{2.31}$$

2.3 Permutations and Partitions

The way that Adin, Brenti, and Roichman [2] make use of the classical descent monomial basis is by using a basis for $\mathbb{Q}[x_1, \dots, x_n]$ given by Garsia in [7]. This basis is the set $\{gs_{\pi} e_{\mu}\}$, where π is an element of $\mathfrak{S}_n$, μ is a partition with parts of size at most n, and

$$e_{\mu} = e_{\mu_1} e_{\mu_2} \dots e_{\mu_{\ell(\mu)}}. \tag{2.32}$$

In making use of this basis it is necessary to associate certain permutations and partitions to monomials. Our results also use these, so we recall them here.

Definition 2.3.1. *The* **index permutation** *of a monomial* $m = \prod_{i=1}^{n} x_i^{p_i}$ *is the unique permutation* π*, such that the following hold:*

1. $p_{\pi(i)} \geq p_{\pi(i+1)}$
2. $p_{\pi(i)} = p_{\pi(i+1)} \pi(i) < \pi(i+1)$

We denote the index permutation of m as $\pi(m)$.

Definition 2.3.2. *The* **exponent partition** *of a monomial* $m = \prod_{i=1}^{n} x_i^{p_i}$ *is the partition* $(p_{\pi(1)}, p_{\pi(2)}, \dots, p_{\pi(n)})$*, where* $\pi = \pi(m)$*. We denote the exponent partition of m as* $\lambda(m)$.

We note that if λ is the exponent partition of a descent monomial, then $\lambda_n = 0$ and $\lambda_i - \lambda_{i+1} \leq 1$. We call a partition that satisfies these conditions a **descent partition**. If λ is the exponent partition of an (n,k)-descent monomial, then λ has less than n parts, and its parts are of size less than k. We call such partitions (n,k)**-partitions**.

Definition 2.3.3. *The* **complementary partition** *of a monomial m is the partition that is conjugate to $(\lambda_i - d_i(\pi))_{i=1}^n$, where $\pi = \pi(m)$ and $\lambda = \lambda(m)$. We denote the complementary partition of m as $\mu(m)$.*

To clarify these definitions we present an example.

Example 2.3.4. *Let $n = 8$, $k = 5$, $I = (2,2,1)$ and let*

$$m = x_1^6 x_2 x_3 x_4^2 x_6^4 x_7 x_8^2 = x_1^6 x_6^4 x_4^2 x_8^2 x_2 x_3 x_7, \tag{2.33}$$

then

$$\begin{gathered}
\pi(m) = 16482375, \\
\lambda(m) = (6,4,2,2,1,1,1,0), \\
Des(\pi(m)) = \{2,4,7\}, \\
gs_{\pi(m)} = x_1^3 x_6^3 x_4^2 x_8^2 x_2 x_3 x_7, \\
\mu(m)' = (3,1), \\
\mu(m) = (2,1,1), and \\
gs_{\pi(m),I} = x_1^5 x_6^5 x_4^3 x_8^2 x_2 x_3 x_7.
\end{gathered}$$

The final key component is a partial ordering on monomials of a given degree together with a result on how multiplying monomials by elementary symmetric functions interacts with this partial order. For a proof of Proposition 2.3.6 we refer the reader to [2].

Definition 2.3.5. *For m_1, m_2 monomials of the same total degree, $m_1 \prec m_2$ if one of the following holds:*

1. $\lambda(m_1) \lhd \lambda(m_2)$

2. $\lambda(m_1) = \lambda(m_2)$ *and* $inv(\pi(m_1)) > inv(\pi(m_2))$,

where $\lhd$ *is the strict dominance order on partitions and inv is the inversion statistic on permutations.*

This partial order is useful because of how it interacts with multiplication of monomials and elementary symmetric functions. This interaction is encapsulated in the following proposition:

Proposition 2.3.6. *Let m be a monomial equal to* $x_1^{p_1} \ldots x_n^{p_n}$, *then among the monomials appearing in* $m \cdot e_\mu$, *the monomial*

$$\prod_{i=1}^{n} x_{\pi(i)}^{p_{(\pi(i))}+\mu_i'} \tag{2.34}$$

is the maximum with respect to $\prec$, *where* π *is the index permutation of m.*

Proof. We refer the reader to [2] for a proof of this theorem. □

2.4 Standard Young Tableaux

Our main results for this chapter come in the form of counting certain standard Young tableaux.

A **Ferrers diagram** is a collection of unit boxes which, since we are using English notation, are justified to the left and up. The lengths of the rows of a Ferrers diagram form a partition which we call the **shape** of the Ferrers diagram. A **semistandard Young tableau** of size n is a Ferrers diagram containing n boxes where each box is assigned a positive integer such that the integers increase weakly along rows and strictly down columns. A **standard Young tableau** is a semistandard Young tableau containing exactly the integers $1, 2, \ldots, n$. We denote the set of standard [semistandard] Young tableaux of size n by $SYT(n)[SSYT(n)]$. For a partition μ, we let $SYT(\mu)[SSYT(\mu)]$ denote the set of all standard [semistandard] Young tableaux of shape μ. The weight of a semistandard

Young tableau T is the vector $wt(T)$ where the ith entry of $wt(T)$ is the number of i's in T. The Schur functions are then defined as

$$s_\lambda := \sum_{T \in SSYT(\lambda)} \mathbf{x}^{wt(T)}, \tag{2.35}$$

where here $\mathbf{x}$ denotes a countably infinite set of variables. The Schur functions form a linear basis for symmetric functions, and there is a well known involution ω on the space of symmetric functions that sends s_λ to $s_{\lambda'}$ where λ' is the partition conjugate to λ.

An integer i is a **descent** of a standard Young tableaux T if the box containing $i+1$ is strictly below the box containing i. We denote by $Des(T)$ the set of all descents of T. Furthermore given two integers $1 \leq i \leq j \leq n$ we define $Des_{i,j}$ to be the set of descents of T that are between i and j inclusively, that is

$$Des_{i,j}(T) := Des(T) \cap \{i, i+1, \ldots j-1, j\}. \tag{2.36}$$

As examples, consider the following Young tableaux:

$$T_1 = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array} \quad T_2 = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & 6 & 8 \\ \hline\end{array} \quad T_3 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 7 & 8 \\ \hline 3 & 5 & 6 \\ \cline{1-3}\end{array}$$

$$S_1 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array} \quad S_2 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline\end{array} \quad S_3 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 3 & 4 \\ \cline{1-3}\end{array}$$

T_1, T_2, T_3 are standard Young tableaux, and S_1, S_2, S_3 are semistandard Young tableaux. The shape of T_1 and S_1 is $(4,3,1)$, the shape of T_2 and S_2 is $(4,4)$, and the shape of T_3 and S_3 is $(5,3)$. The descent sets of the standard Young tableaux are as follows:

$\mathrm{Des(T_1)} = \{1,2,4,7\}$,

$Des(T_2) = \{1,4,7\}$,

$Des(T_3) = \{2,4\}$.

Next, $Des_{5,7}(T_1) = Des_{5,7}(T_2) = \{7\}$, and $Des_{5,7}(T_3) = \emptyset$. The weight of the semistandard

Young tableaux are as follows:

$\mathrm{wt}(\mathrm{S}_1) = (2,2,3,1,0,0,\ldots)$,

$wt(S_2) = (4,4,0,0,0,0\ldots)$,

$wt(S_3) = (1,2,2,2,1,0,\ldots)$.

A **skew Young tableau** is a Young tableau that has had a Young tableau removed from its upper left corner. The definitions of both semistandard Young tableaux and Schur function extend to **semistandard skew Young tableaux** and **skew Schur functions**. A connected skew Young tableau that does not contain any 2×2 boxes is called a skew ribbon tableau. These two conditions make it so that the shape of a skew ribbon tableau is uniquely determined by the lengths of its rows, so that we can specify a skew-ribbon tableau shape by a sequence of positive integers. For example if we specify that a skew ribbon tableau has rows of lengths $(4,2,1,3)$, then the following are two examples of semistandard skew Young tableaux with the only possible shape:

$$\begin{array}{ccccccc} & & & 1 & 1 & 2 & 3 \\ & & 1 & 2 & & & \\ & & 3 & & & & \\ 1 & 2 & 4 & & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & 1 & 1 & 1 \\ & & 1 & 2 & & & \\ & & 2 & & & & \\ 1 & 1 & 3 & & & & \end{array}.$$

2.5 Main Results

In the case of the classical coinvariant algebra, Adin, Brenti, and Roichman determine the isomorphism type of $R_{n,\rho}$ by comparing the graded traces of the actions of $\mathfrak{S}_n$ on $\mathbb{Q}[x_1,\ldots,x_n]$ and on R_n. We will follow a similar path, but instead of considering the action of $\mathfrak{S}_n$ on $\mathbb{Q}[x_1,\ldots,x_n]$, we will consider its action on the space

$$P_{n,k} := span_{\mathbb{Q}}\{x_1^{p_1}x_2^{p_2}\ldots x_n^{p_n} : p_1,p_2,\ldots,p_n < k\}, \tag{2.37}$$

that is the space of rational polynomials in the variables $x_1,\ldots,x_n$ where the powers of each x_i are less than k.

We begin by giving a straightening lemma that is a similar to a lemma of Adin, Brenti, and

Roichman [2]. Our lemma differs from theirs in that we are considering monomials in $P_{n,k}$ instead of $\mathbb{Q}[x_1,\ldots,x_n]$, we use (n,k)-descent monomials instead of the classical descent monomials, and we consider elementary symmetric functions corresponding to partitions with parts of size at least $n-k+1$ instead of all elementary symmetric functions.

Lemma 2.5.1. *If $m = \prod_{i=1}^{n} x_i^{p_i}$ is a monomial in $P_{n,k}$ (that is $p_i < k$ for all i), then*

$$m = gs_{\pi,I}e_{\nu} + \sum, \tag{2.38}$$

where $\pi = \pi(m)$; $\sum$ is a sum of monomials $m' \prec m$; I is the length $n-k$ sequence defined by $i_\ell = \mu'_\ell - \mu'_{n-k+1}$, where μ is the complementary partition of m; and ν is the partition specified by:

1. $\nu'_\ell = \mu'_\ell$ *for* $\ell > n-k$
2. $\nu'_\ell = \mu'_{n-k+1}$ *for* $\ell \leq n-k$

Furthermore ν consists of parts of size at least $n-k+1$.

Proof. In order to show that $gs_{\pi,I}$ is well defined we need to check that $k - des(\pi) > i_1 \geq i_2 \geq \ldots \geq i_{n-k} \geq 0$. By definition $i_1 = \mu'_1 - \mu'_{n-k+1} \leq \mu'_1 = p_{\pi(1)} - d_1(\pi)$ and then by assumption $p_{\pi(1)} < k$, and $d_1(\pi) = des(\pi)$, thus

$$i_1 \leq p_{\pi(1)} - d_1(\pi) < k - des(\pi). \tag{2.39}$$

We also note that I is a non-negative weakly-decreasing sequence since it consists of the parts of a partition minus a constant that is at most as large as the smallest part of the partition. Thus I satisfies the condition so $gs_{\pi,I}$ is well defined.

Next we show that $gs_{\pi,I}$ and m have the same index permutation, that is that

$$\pi(gs_{\pi,I}) = \pi(m) = \pi. \tag{2.40}$$

To show this, we need to consider the sequence of the exponents of $x_{\pi(\ell)}$ in $gs_{\pi,I}$. This sequence is the sum of the sequences $d_\ell(\pi)$ and i_ℓ (where we take $i_\ell = 0$ for $\ell > n-k$). Since these are both weakly-decreasing sequences, their sum is also weakly-decreasing. Furthermore if $d_\ell(\pi) + i_\ell = d_{\ell+1}(\pi) + i_{\ell+1}$, then $d_\ell(\pi) = d_{\ell+1}(\pi)$, which by the definition of $d_\ell(\pi)$ implies that ℓ is not a descent of π, that is that $\pi(\ell) < \pi(\ell+1)$, thus π satisfies the two conditions of being the index permutation, and thus by uniqueness it is the index permutation.

Now by Proposition 2.3.6, the maximum monomial in $gs_{\pi,I}e_\nu$ will have the form $\prod_{\ell=1}^{n} x_{\pi(\ell)}^{q_\ell}$ where q_ℓ is given by:

1. $q_\ell = d_\ell(\pi) + i_\ell + \nu'_\ell$ for $\ell \leq n-k$
2. $q_\ell = d_\ell(\pi) + \nu'_\ell$ for $\ell > n-k$

By substitution, first using the definitions of i_ℓ and ν_ℓ and then the definition of the complementary partition, we get

$$q_\ell = d_\ell(\pi) + \mu'_\ell - \mu'_{n-k+1} + \mu'_{n-k+1} = d_\ell(\pi) + \mu'_\ell = p_{\pi(\ell)} \tag{2.41}$$

for $\ell \leq n-k$, and

$$q_\ell = d_\ell(\pi) - \mu'_\ell = p_{\pi(\ell)} \tag{2.42}$$

for $\ell > n-k$.

Finally, ν has parts of size at least $n-k+1$ because by definition, the first $n-k+1$ parts of ν' are all the same size. $\square$

This lemma gives rise to a basis for $P_{n,k}$ which will be key to relating how $\mathfrak{S}_n$ acts on $P_{n,k}$ to how it acts on $R_{n,k}$.

Proposition 2.5.2. *The set $D_{n,k}$ consisting of products $gs_{\pi,I}e_\nu$ where ν is a partition with parts of size at least $n-k+1$ and $(\lambda(gs_{\pi,I}) + \nu')_1 < k$ form a basis for $P_{n,k}$.*

Proof. The condition that $(\lambda(gs_{\pi,I})+\nu')_1 < k$ along with Lemma 2.5.1 guarantees that the maximum monomial in each element of $D_{n,k}$ is contained in $P_{n,k}$. Then since the partial order $\prec$ refines dominance order, all other monomials appearing in elements in $D_{n,k}$ are also contained in $P_{n,k}$. Therefore $D_{n,k}$ is contained in $P_{n,k}$.

Iteratively applying Lemma 2.5.1 lets us express any monomial in $P_{n,k}$ as a linear combination of elements in $D_{n,k}$, thus $D_{n,k}$ spans $P_{n,k}$. To show that this expansion is unique (up to rearrangement) it is sufficient to show that if the maximal monomials in $gs_{\pi,I}e_\nu$ and $gs_{\phi,J}e_\rho$ are the same, then $\pi = \phi$, $I = J$, and $\nu = \rho$. To see this, we note that as a corollary of the proof of Lemma 2.5.1, the index permutations of the maximal monomials are the same, and they are π and ϕ respectively, and thus $\pi = \phi$. Then, by Proposition 2.3.6, the power of $x_{\pi(\ell)}$ in each of these maximum monomials will be $d_\ell(\pi)+i_\ell+\nu'_\ell$ and $d_\ell(\pi)+j_\ell+\rho'_\ell$. This immediately gives that $\nu'_\ell = \rho'_\ell$ for $\ell > n-k$ since $i_\ell = j_\ell = 0$ for $\ell > n-k$. Then since the first $n-k+1$ parts of ν' are all equal and the first $n-k+1$ parts of ρ' are equal and since $\nu'_{n-k+1} = \rho'_{n-k+1}$, we have that $\nu' = \rho'$ which implies $\nu = \rho$. This then implies that $i_\ell = j_\ell$ for all ℓ, and therefore this expansion is unique. Therefore $D_{n,k}$ is linearly independent and is a basis. $\square$

Proposition 2.5.3. *Let p be the map projecting from $\mathbb{Q}[x_1,\ldots,x_n]$ to $R_{n,k}$ and let m be a monomial in $P_{n,k}$. Then*

$$p(m) = \sum_{\pi,I} \alpha_{\pi,I} gs_{\pi,I}, \tag{2.43}$$

where $\alpha_{\pi,I}$ are some constants, and the sum is over pairs (π,I) such that $\lambda(gs_{\pi,I}) \trianglelefteq \lambda(m)$.

Proof. Since $D_{n,k}$ is a basis, we can express $m = \sum_{\pi,I,\nu} \alpha_{\pi,I,\nu} gs_{\pi,I}e_\nu$ for some constants $\alpha_{\pi,I,\nu}$. By Lemma 2.5.1, $\alpha_{\pi,I,\nu}$ is zero if the leading monomial of $gs_{\pi,I}e_\nu$ is not weakly smaller than m under the partial order on monomials. But since the partial order on monomials refines the dominance order on exponent partitions, for each non-zero term the exponent partition of the leading monomial will be dominated by $\lambda(m)$, that is that

$$(\lambda(gs_{\pi,I}) + \nu') \trianglelefteq \lambda(m). \tag{2.44}$$

Then when we project down to $R_{n,k}$, each term with $\nu \neq \emptyset$ will vanish since e_ν is in $I_{n,k}$, so that,

$$p(m) = \sum_{\pi,I} \alpha_{\pi,I,\emptyset} gs_{\pi,I}, \tag{2.45}$$

where the sum is over (π, I) such that $\lambda(gs_{\pi,I}) \trianglelefteq \lambda(m)$. □

This proposition is the reason that we have that the degree d component of $R_{n,k}$ is isomorphic to

$$\bigoplus_{\mu \vdash d} R_{n,k,\mu}. \tag{2.46}$$

This proposition gives the following corollary:

Corollary 2.5.4. *$R_{n,k,\rho}$ is zero unless ρ is the exponent partition of an (n,k)-descent monomial, which occurs precisely when ρ is an (n,k)-partition such that the last k parts form a descent partition.*

This basis allows us to express the trace of the action of $\tau \in \mathfrak{S}_n$ on $P_{n,k}$ in terms of the trace of its action on $R_{n,k}$ in the basis of (n,k)-Garsia-Stanton monomials. To do this, let $gs_{\pi,I} e_\nu \in D_{n,k}$ and $\tau \in \mathfrak{S}_n$. As in the proof of Proposition 2.5.3 we will have that

$$\tau(gs_{\pi,I}) = \sum_{\phi,J,\mu} \alpha_{\phi,J,\mu} gs_{\phi,J} e_\mu, \tag{2.47}$$

for some constants $\alpha_{\phi,J,\mu}$, where $\alpha_{\phi,J,\mu} = 0$ unless $\lambda(gs_{\pi,I}) \trianglerighteq \lambda(gs_{\phi,J}) + \mu'$. Then

$$\tau(gs_{\pi,I} e_\nu) = \tau(gs_{\pi,I}) e_\nu = \sum_{\phi,J,\mu} \alpha_{\phi,J,\mu} gs_{\phi,J} e_\mu e_\nu. \tag{2.48}$$

The important thing for this equation is that for each $\alpha_{\phi,J,\mu}$ that is non-zero, $gs_{\phi,J} e_\mu e_\nu$ is an element in $D_{n,k}$ since $\lambda(gs_{\phi,J}) + \mu' + \nu' \trianglelefteq \lambda(gs_{\pi,I}) + \nu'$. Thus the coefficient of $gs_{\pi,I}e_\nu$ in $\tau(gs_{\pi,I}e_\nu)$ is $\alpha_{\pi,I,\emptyset}$. Next if we project $\tau(gs_{\pi,I})$ onto $R_{n,k}$, we get that in $R_{n,k}$

$$\tau(gs_{\pi,I}) = \sum_{\phi,J} \alpha_{\phi,J,\emptyset} gs_{\phi,J}, \tag{2.49}$$

since every term with $\mu \neq \emptyset$ vanishes when projected to $R_{n,k}$. Therefore the contribution of $gs_{\pi,I}$ to the trace of the action of τ on $R_{n,k}$ is also $\alpha_{\pi,I,\emptyset}$.

We now move to the lemmas that will allow us to prove our main result.

Lemma 2.5.5. *Given an (n,k)-partition μ and an (n,k)-descent partition ν there exists a (n,k)-partition ρ such that $\mu = \nu + \rho$ if and only if $Des(\nu) \subseteq Des(\mu)$. If it exists, ρ is unique.*

Proof. There is only one possible value for each part of ρ which is $\rho_i = \mu_i - \nu_i$, the only thing to check is whether this gives a partition, specifically we need to check whether $\rho_i - \rho_{i+1} = (\mu_i - \mu_{i+1}) - (\nu_i - \nu_{i+1}) \geq 0$. Since ν is a descent partition, $(\nu_i - \nu_{i+1})$ is 1 if i is a descent of ν and 0 if it is not. Similarly, $(\mu_i - \mu_{i+1})$ is at least 1 if i is a descent of μ and 0 otherwise. Thus in order for $(\mu_i - \mu_{i+1}) - (\nu_i - \nu_{i+1})$ to be non-negative, it is necessary and sufficient that if i is a descent of ν, then i is also a descent of μ. That is, ρ will be a partition if and only if $Des(\nu) \subset Des(\mu)$. $\square$

Example 2.5.6. *As an example of Lemma 2.5.5, let $n = 8, k = 6$ then let $\mu = (5,5,3,3,1,1,1,0)$,*
$\nu_1 = (2,2,1,1,0,0,0,0)$,
$\nu_2 = (3,3,2,2,1,1,0,0)$.

Then Des(μ) $= \{2,4,7\}$,
$Des(\nu_1) = \{2,4\}$,
$Des(\nu_2) = \{2,4,6,8\}$.

We then have that $Des(\nu_1) \subseteq Des(\mu)$, and that $\mu - \nu_1 = (3,3,2,2,1,1,1,0)$ is a partition. On the other hand, $Des(\nu_2) \nsubseteq Des(\mu)$, and $\mu - \nu_2 = (2,2,1,1,0,0,1,0)$ is not a partition.

Lemma 2.5.7. *Given an (n,k)-partition μ and a set $S \subseteq Des_{n-k+1,n}(\mu)$, there is a unique pair (ν,ρ) such that $\mu = \nu + \rho$ and ν is the exponent partition of an (n,k)-descent monomial with*

$Des_{n-k+1,n}(\nu) = S$, and ρ is an (n,k)-partition with $\rho_1 = \rho_2 = \ldots = \rho_{n-k+1}$, furthermore this means that $Des_{1,n-k}(\mu) = Des_{1,n-k}(\nu)$.

Proof. The last k values of the exponent partition of a descent monomial form a descent partition, so applying Lemma 2.5.5 to the partition determined by S determines the last k values of ρ. Then since we need that the first $n-k+1$ values of ρ are the same, this determines what ρ must be, and by subtraction what ν must be. We just need to check that ν is actually a partition, that is that $\nu_i - \nu_{i+1} \geq 0$ for $1 \leq i \leq n-k$. This is true since $\nu_i - \nu_{i+1} = \mu_i - \mu_{i+1} \geq 0$ because $\rho_i = \rho_{i+1}$ for $i \leq n-k$. The condition that $Des_{1,n-k}(\mu) = Des_{1,n-k}(\nu)$ follows from the fact that $\mu = \nu + \rho$ and that the first $n-k+1$ parts of ρ are the same. $\square$

We give an example of how Lemma 2.5.7 works.

Example 2.5.8. *Let $n = 8, k = 6$, and let*

$$\mu = (5,5,3,3,1,1,1,0), \tag{2.50}$$

and let $S = \{4\}$, then

$$\nu = (3,3,1,1,0,0,0,0), \tag{2.51}$$

and

$$\rho = (2,2,2,2,1,1,1,0) \tag{2.52}$$

We now give a proof of Theorem 2.1.2.

Proof of Theorem 2.1.2. The determination of when $R_{n,k,\rho}$ is zero is from Corollary 2.5.4.

Next we define an inner product on polynomials by $\langle m_1, m_2 \rangle = \delta_{m_1 m_2}$ for two monomials m_1, m_2, and then extending bilinearly. We then consider the graded trace of the action of $\tau \in \mathfrak{S}_n$ on $P_{n,k}$ defined for the monomial basis by

$$Tr_{P_{n,k}}(\tau) := \sum_m \langle \tau(m), m \rangle \cdot \bar{q}^{\lambda(m)} \tag{2.53}$$

where $\overline{q}^{\lambda} = \prod_{i=1}^{n} q_i^{\lambda_i}$ for any partition λ. Adin, Brenti, Roichman show that

$$Tr_{\mathbb{Q}[x_1,\ldots,x_n]}(\tau) = \sum_{\lambda \vdash n} \chi_{\mu}^{\lambda} \frac{\sum_{T \in SYT(\lambda)} \prod_{i=1}^{n} q_i^{d_i(T)}}{\prod_{i=1}^{n}(1 - q_1 q_2 \ldots q_i)} \tag{2.54}$$

(where μ is the cycle type of τ). From this we can recover $Tr_{P_{n,k}}(\tau)$ by restricting to powers of q_1 that are at most $k-1$. Doing this gives

$$\sum_{\lambda \vdash n} \chi_{\mu}^{\lambda} \sum_{T \in SYT(\lambda), \nu} \bar{q}^{\lambda_{Des(T)}} \bar{q}^{\nu}, \tag{2.55}$$

where the ν's are partitions such that $(\lambda_{Des(T)})_1 + \nu_1 < k$, and $\lambda_{Des(T)}$ is the descent partition with descent set T.

Alternatively, we can calculate $Tr_{P_{n,k}}(\tau)$ by using the basis from Proposition 2.5.2, this gives
$\mathrm{Tr}_{P_{n,k}}(\tau) = \sum_{\sigma,I,\nu} \langle \tau(gs_{\sigma,I} e_{\nu}), gs_{\sigma,I} e_{\nu} \rangle \bar{q}^{\lambda(gs_{\sigma,I})} \bar{q}^{\nu'}$
$= \sum_{\sigma,I,\nu} \langle \tau(gs_{\sigma,I}), gs_{\sigma,I} \rangle \bar{q}^{\lambda(gs_{\sigma,I})} \bar{q}^{\nu'}$
$= \sum_{\lambda,\nu} Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda}) \bar{q}^{\lambda} \bar{q}^{\nu'}$ where the ν's are partitions with parts of size at least $n-k+1$ such that $(\lambda(gs_{\sigma,I}))_1 + (\nu')_1 < k$, and $Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda})$ is the coefficient of $\bar{q}^{\lambda}$ in the graded trace of the action of τ on $R_{n,k}$.

We now consider the coefficient of $\bar{q}^{\rho}$ for some partition ρ. Using the first calculation and Lemma 2.5.5, the inner sum can be reduced to T such that $Des(T) \subseteq Des(\rho)$, so that we get

$$\sum_{\lambda \vdash n} \chi_{\mu}^{\lambda} |\{T \in SYT(\lambda), Des(T) \subseteq Des(\rho)\}|. \tag{2.56}$$

Looking at the second calculation and using Lemma 2.5.7 gives

$$\sum_{S \subseteq Des_{n-k+1,n}(\rho)} Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda_S}), \tag{2.57}$$

where λ_S is the exponent partition of some (n,k)-descent monomial $gs_{\sigma,I}$ with $S = Des_{n-k+1,n}(\lambda(gs_{\sigma,I}))$, and $Des_{1,n-k}(\lambda(gs_{\sigma,I})) = Des_{1,n-k}(\rho)$. Together this gives that

$$\sum_{\lambda \vdash n} \chi^{\lambda}_{\mu} |\{T \in SYT(\lambda) : Des(T) \subseteq Des(\rho)\}| = \sum_{S \subseteq Des_{n-k+1,n}(\rho)} Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda_S}). \tag{2.58}$$

We want to further refine this result by showing that

$$\sum_{\lambda \vdash n} \chi^{\lambda}_{\mu} |\{T \in SYT(\lambda) : S' \subseteq Des(T) \subseteq Des(\lambda_{S'})\}| = Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda_{S'}}) \tag{2.59}$$

for any specific S'. What this refinement is saying is that out of all of the Standard Young Tableaux being counted by the left-hand side, the ones that correspond to a particular S' from the right-hand side are those that satisfy $Des_{n-k+1,n}(T) = S'$. We prove this refinement by induction on $|\lambda_{S'}|$. The base case of $\lambda_{S'} = \emptyset$ can be seen by taking $\rho = \emptyset$. In this case, both sides reduce to the desired expressions. If we take $\rho = \lambda_{S'}$, then $\lambda_{S'}$ will appear in the sum since we can take the ν from Lemma 2.5.7 to be 0, and all other λ_S's will be smaller since the corresponding ν's will be non-empty. Thus by the inductive hypothesis,

$$\sum_{\lambda \vdash n} \chi^{\lambda}_{\mu} |\{T \in SYT(\lambda) : S' \not\subseteq Des(T) \subseteq Des(\rho)\}| = \sum_{S \subsetneq S'} Tr_{R_{n,k}}(\tau; \bar{q}^{\lambda_S}). \tag{2.60}$$

In words, we are summing over all Standard Young Tableaux T that have a strict subset of S' as descents, thus if we subtract this from our result, the only terms remaining are those with all of S' as descents. This then proves the theorem since the exponent partition of any (n,k)-descent monomial $gs_{\sigma,I}$, there will be a $\lambda_{S'}$ that is equal to $gs_{\sigma,I}$ when we take $\rho = \lambda(gs_{\sigma,I})$. $\square$

Example 2.5.9. *Let* $n = 8$*,* $k = 6$*,* $\rho = (5,3,2,2,1,1,1)$, $\lambda = (4,3,1)$, *then* $Des_{1,2}(\rho) = \{1,2\}$, *and* $Des_{3,8} = \{4,7\}$.

The standard Young tableaux T *of shape* λ *with* $\{4,7\} \subseteq Des(T) \subseteq \{1,2,4,7\}$ *are as follows:*

$$\begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 6 & 8 & \\ \cline{1-3} 5 & & & \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & 6 & \\ \cline{1-3} 8 & & & \\ \cline{1-1} \end{array}$$

1	2	4	7
3	5	6	
8			

1	2	4	7
3	6	8	
5			

1	2	6	7
3	4	8	
5			

1	2	3	4
5	6	7	
8			

Therefore by Theorem 2.1.2, the coefficient of S^λ in $R_{n,k,\rho}$ is 7.

Using Theorem 2.1.2 we can also recover a result of Haglund, Rhoades, and Shimozono [12].

We will use the q-binomial coefficient which has the following formulation.

$$[n]_q := 1+q+\dots q^{n-1} \quad [n]_q! := [n]_q[n-1]_q\dots[1]_q \quad \binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \tag{2.61}$$

Additionally we will use the well known result that the coefficient of q^d in $\binom{n+m}{m}_q$ is the number of partitions of size d that fit in an $n \times m$ box.

Corollary 2.5.10. *Let $f_\lambda(q)$ be the generating function for the multiplicities of S^λ in the degree d component of $R_{n,k}$. Then*

$$f_\lambda(q) = \sum_{T \in SYT(\lambda)} q^{maj(T)} \binom{n-des(T)-1}{n-k}_q, \tag{2.62}$$

where the major index $maj(T)$ is the sum of the descents of T.

Proof. By Theorem 2.1.2, each standard Young tableau of shape λ contributes to $f_\lambda(q)$ once for each partition ρ such that ρ is the exponent partition of an (n,k)-descent monomial and $Des_{n-k+1,n}(\rho) \subseteq Des(T) \subseteq Des(\rho)$. All such ρ come from (n,k)-descent monomials $gs_{\pi,I}$ where π is a permutation with $Des(\pi) = Des(T)$ and I is a sequence such that $k - des(T) > i_1 \geq i_2 \geq \dots \geq i_{n-k} \geq 0$. This choice of I is the same as choosing a partition that fits in an $(n-k) \times (k-1-des(T))$ box. The generating function for the number of partitions of size d that fit in an $(n-k) \times (k-1-des(T))$ box is $\binom{(n-k)+(k-des(T)-1)}{n-k}_q = \binom{n-des(T)-1}{n-k}_q$. The factor of gs_π in the (n,k)-descent monomial then has degree $maj(T)$, so that each standard Young tableau T of shape λ will contribute $q^{maj(T)}\binom{n-des(T)-1}{n-k}_q$

to $f_\lambda(q)$. This completes the proof. $\square$

The proof of this result in [12] is fairly involved using a tricky recursive argument involving an auxiliary family of algebras. Our method gives a simpler proof for the result.

2.6 Relations to Crystal Theory

Theorem 2.1.2 is related to the a crystal structure that defined by Benkart, Colmenarejo, Harris, Orellana, Panova, Schilling, and Yip [4]. Like $R_{n,k}$, the crystal structure that they define is motivated by the Delta Conjecture, and its graded character is equal to

$$(rev_q \circ \omega) grFrob(R_{n,k}; q), \tag{2.63}$$

which, as we mentioned before, is equal to a special case of the combinatorial side of the Delta Conjecture. This crystal is built up from crystal structures on ordered multiset partitions in minimaj ordering with specificed descents sets, and the characters of these smaller crystals is given in terms of skew ribbon tableaux. Since $R_{n,k}$ is an algebra that corresponds to the entire crystal structure, it is natural to wonder if there are algebras that correspond to these smaller crystals. The algebras $R_{n,k,\rho}$ are these algebras.

In order to see this connection, we need to rewrite the Frobenius image of $R_{n,k,\rho}$ that we get from Theorem 2.1.2 to get an expression in terms of skew-ribbon tableaux. Using the combinatorial definition of s_λ and Theorem 2.1.2, we can write Frobenius image of $R_{n,k,\rho}$ as

$$Frob(R_{n,k,\rho}) = \sum_{(P,Q)} \mathbf{x}^{wt(P)} \tag{2.64}$$

where the sum is over pairs (P,Q) with the following conditions:

- P is a semistandard Young tableau of size n
- Q is a standard Young tableau of size n

- $sh(P) = sh(Q)$
- $Des_{n-k+1,n}(\rho) \subseteq Des(Q) \subseteq Des(\rho)$.

The Robinson–Schensted–Knuth(RSK) correspondence (see Chapter 7 of [19] for a review of the RSK correspondence) gives a weight-preserving bijection between pairs (P,Q) with the above conditions and words w of length n in the alphabet of positive integers with $Des_{n-k+1,n}(\rho) \subseteq Des(w) \subseteq Des(\rho)$. Therefore if we apply the reverse RSK correspondence to the Frobenius image it can be rewritten as

$$Frob(R_{n,k,\rho}) = \sum_{w} \mathbf{x}^{wt(w)} \tag{2.65}$$

where the sum is over words of length n with $Des_{n-k+1,n}(\rho) \subseteq Des(w) \subseteq Des(\rho)$.

Next let d_i be the difference between the ith and $(i-1)$th descents of ρ, taking d_1 to be the first descent. Then let p be the index of the largest descent smaller than $n-k+1$. With this notation, any word w as above can be split into subwords $w_1, w_2, \ldots, w_p$, and v such that $w = w_1 w_2 \ldots w_p v$ where each w_i has length d_i and has no descents, and v has descents at $d_{p+1}, d_{p+1}+d_{p+2}, \ldots, d_{p+1}+d_{p+2}+\ldots d_{des(\rho)}$. Any such collection of subwords gives an acceptable word w, thus $Frob(R_{n,k,\rho})$ can be written as product of terms of the form $\sum_{w_i} \mathbf{x}^{wt(w_i)}$ and $\sum_{v} \mathbf{x}^{wt(v)}$, where the sums are over words with the corresponding restrictions. These terms can be simplfied as follows. The term $\sum_{w_i} \mathbf{x}^{wt(w_i)}$ is equal to h_{d_i} and $\sum_{v} \mathbf{x}^{wt(v)}$ is equal to s_γ where γ is the skew ribbon shape with rows of lengths $(n-(d_1+d_2+\ldots+d_{des(\rho)}), d_{des(\rho)}, d_{des(\rho)-1}, \ldots, d_{p+1})$. This last part is because there is a bijection between fillings of γ and words with the conditions of v given by reading the fillings of γ row by row from bottom to top reading each row from left to right. Combining these together gives that the Frobenius image of $R_{n,k,\rho}$ is equal to

$$Frob(R_{n,k,\rho}) = s_\gamma \prod_{i=1}^{p} h_{d_i} \tag{2.66}$$

To clarify the above we will give an example. Let $n = 11, k = 8$, let $\rho = (7,7,5,3,3,3,3,2,1,1)$. Then $Des_{n-k+1,n}(\rho) = \{7,8,10\}$ and $Des(\rho) = \{2,3,7,8,10\}$, and the values of d_i written in a list are $2,1,4,1,2$ and $p=2$. The skew ribbon tableau that will appear will thus have row lengths $(1,2,1,4)$. A pair (P,Q) with the above conditions would then be

$$P=\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 5 & 5 \\ \cline{1-4} 3 & 4 \\ \cline{1-2}\end{array} \quad Q=\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 4 & 8 & 10 \\ \cline{1-4} 9 & 11 \\ \cline{1-2}\end{array}$$

Applying the reverse RSK correspondence to this pair gives the word

$$w = 34125553132. \tag{2.67}$$

This is then broken up into the words $w_1 = 34$, $w_2 = 1$, and $v = 25553132$ these are then put into semistandard Young (skew) tableaux as follows

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline\end{array}\ ,\ \begin{array}{ccccc} & & & & \boxed{2} \\ & & & \boxed{1} & \boxed{3} \\ & & & \boxed{3} & \\ \boxed{2} & \boxed{5} & \boxed{5} & \boxed{5} & \end{array}$$

If we apply ω to this product we get

$$\omega(Frob(R_{n,k,\rho})) = s_\gamma \prod_{i=1}^{p} e_{d_i}. \tag{2.68}$$

This expression (for the appropriately chosen values) is the character of the crystals that Benkart, Colmenarejo, Harris, Orellana, Panova, Schilling, and Yip [4] use to build up their main crystal strucuture. Therefore the algebras $R_{n,k,\rho}$ fill in a piece that was missing on the algebraic side of things.

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Chapter 3

Coinvariants for Wreath Products

This picture can be extended by looking at reflection groups other than $\mathfrak{S}_n$. Specifically we will look at the complex reflection group $G(r,1,n)$ which is equal to the wreath product of $\mathbb{Z}_r$ and $\mathfrak{S}_n$. This group acts on $\mathbb{C}[x_1,\ldots,x_n]$ by $\mathfrak{S}_n$ permuting the variables and by the ith copy of $\mathbb{Z}_r$ sending x_i to ξx_i where ξ is a primitive rth root of unity. Alternatively, we can view this group as the set of $n\times n$ matrices with exactly 1 non-zero entry in each row and column where the non-zero entries are rth roots of unity. The action of $G(r,1,n)$ on $\mathbb{C}[x_1,\ldots,x_n]$ in this case is matrix multiplication. A third way of thinking of this group is as permutation of n in which each number is assigned one out of r colors.

Throughout this section many of the objects we consider will depend on the positive integer r, but since we only ever consider a fixed r we will frequently suppresses the r in our notation in order to avoid cumbersome notation. To begin we will write G_n for the group $G(r,1,n)$.

As in the case of $\mathfrak{S}_n$ there is a coinvariant algebra S_n associated to this action of G_n that is defined as

$$S_n := \frac{\mathbb{C}[x_1,\ldots,x_n]}{J_n}, \tag{3.1}$$

where J_n is the ideal generated by all polynomials invariant under the action of G_n with zero constant term. Any polynomial that is invariant under the action of G_n must be a symmetric polynomials in the variables $x_1^r, x_2^r, \ldots, x_n^r$. We denote this set of variables as $\mathbf{x}_n^r$. Then $J_n = \langle e_n(\mathbf{x}_n^r), \ldots e_1(\mathbf{x}_n^r)\rangle$.

Our goal is to give the multiplicities of all irreducible representations of G_n in $S_{n,k,\rho}$. In order to do this, we will review some of the representation theory of G_n.

3.1 Background and Definitions

The elements of G_n can be viewed as r-colored permutations of length n which are defined as follows:

Definition 3.1.1. *An r-**colored permutation** of length n is a permutation $\pi = \pi_1 \dots \pi_n$ where each value π_i has been assigned a value c_i from the set $\{0, 1, \dots, r-1\}$. We can write this in the form $\pi_1^{c_1} \pi_2^{c_2} \dots \pi_n^{c_n}$.*

For example $3^0 5^2 2^1 1^2 4^0$ is a 3-colored permutation of length 5.

As before we define a statistic on r-colored permutation called descents.

Definition 3.1.2. *An index i is a **descent** of an r-colored permutation $g = \pi_1^{c_1} \pi_2^{c_2} \dots \pi_n^{c_n}$ if one of the following conditions hold:*

1. $c_i < c_{i+1}$
2. $c_i = c_{i+1}$ *and* $\pi_i > \pi_{i+1}$.

We denote the set of descents of an r-colored permutation by $Des(g)$. Furthermore we will denote $|Des(g)|$ as $des(g)$, and we will write $d_i(g)$ to be the number of descents of g that are at least as large as i, that is

$$d_i(g) := |Des(g) \cap \{i, i+1, \dots, n\}|. \tag{3.2}$$

For example if $g = 3^0 5^2 4^2 6^0 1^1 2^1$, then $Des(g) = \{1, 2, 4\}$ since 1 and 4 satisfy condition (1) and 2 satisfies condition (2).

Using these d_i values we follow Bagno and Biagioli[3] in defining flag descent values as

$$f_i(g) = r d_i(g) + c_i. \tag{3.3}$$

With these definitions we recall a set of monomials in $\mathbb{C}[\mathbf{x}_n]$ that descend to a vector-space basis for S_n provided by Bagno and Biagioli[3].

Definition 3.1.3. *Given a r-colored permutation $g = \pi_1^{c_1}\pi_2^{c_2}\dots\pi_n^{c_n}$, we define the* r**-descent monomial** b_g *as follows:*

$$b_g := \prod_{i=1}^{n} x_{\pi_i}^{f_i(g)} \tag{3.4}$$

The set of r-descent monomials descend to a basis for S_n.

We note that by the definition of f_i and descents of r-colored permutations that $f_i(g)$ is a weakly decreasing sequence such that $f_i(g) - f_{i+1}(g) \leq r$.

Chan and Rhoades[5] generalized these monomials to a set of monomials that descends to a basis for $S_{n,k}$.

Definition 3.1.4. *Given an r-colored permutation $g = \pi_1^{c_1}\dots\pi_n^{c_n}$ such that $des(g) < k$, and an integer sequence $I = (i_1, \dots i_{n-k})$ such that $k - des(g) > i_1 \geq i_2 \geq \dots \geq i_{n-k} \geq 0$, we define the* (n,k,r)**-descent monomial** *as*

$$b_{g,I} := b_g \cdot x_{\pi_1}^{ri_1} \dots x_{\pi_{n-k}}^{ri_{n-k}} \tag{3.5}$$

The set of (n,k,r)-descent monomials descend to a basis for $S_{n,k}$. We note that these monomials have individual powers strictly bounded by kr. These observations motivate the following definitions.

Definition 3.1.5. *We call a partition an* (n,k,r)**-partition** *if it has n parts (some of which might be zero), each of which is strictly less than rk.*

Definition 3.1.6. *Given a partition μ, we call an index i an* r**-descent** *of μ if*

$$\lfloor \frac{\mu_i}{r} \rfloor > \lfloor \frac{\mu_{i+1}}{r} \rfloor. \tag{3.6}$$

We will denote $Des^r(\mu)$ as the set of r-descents of μ.

Definition 3.1.7. *We call a partition an* r**-descent partition** *if the difference between consecutive parts is at most r, and the last part has size less than* r.

The exponent partitions of both r and (n,k,r)-descent monomials are (n,k,r)-partitions. Furthermore the exponent partition of an r-descent monomial is an r-descent partition with r-descents equal to $Des(g)$ for the corresponding r-colored permutation g. The last k parts of the exponent partition of an (n,k,r)-descent monomial is an r-descent partition with r-descents determined by $Des_{n-k+1,n}(g)$, and the first k parts have r-descents that are a superset of $Des_{1,n-k}(g)$. Furthermore it is straightforward to see that all such (n,k,r)-partitions arise as the exponent partition of some (n,k,r)-descent monomial.

In order to work with the basis of (n,k,r)-descent monomials we need to relate them to the partial order on monomials from the previous section. To do that we first give a way of associating an r-colored permutation to a monomial.

Definition 3.1.8. *Given a monomial* $x_1^{a_1}\dots x_n^{a_n}$ *we define its* **index** r**-colored permutation** $g(m) = \pi_1^{c_1}\pi_2^{c_2}\dots\pi_n^{c_n}$ *to be the unique* r*-colored permutation such that*

1. $a_{\pi_i} \geq a_{\pi_{i+1}}$ *for* $1 \leq i < n$
2. *if* $a_{\pi_i} = a_{\pi_{i+1}}$, *then* $\pi_i < \pi_{i+1}$
3. $a_i \equiv c_i (mod r)$

One last definition before we state some results is the following:

Definition 3.1.9. *Given a monomial* $m = x_1^{a_1}\dots x_n^{a_n}$ *the* r**-complementary** *partition* $\mu(m)$ *is the partition conjugate to*

$$\left(\frac{a_{\pi_1} - rd_1(g) - c_1(g)}{r}, \dots, \frac{a_{\pi_n} - rd_n(g) - c_n(g)}{r}\right) \tag{3.7}$$

where $g = g(m)$ *and* π *is the uncolored permutation of* g.

Implicit in these definitions is that they are well defined which is covered in [3].

We can now state the lemma that ties these objects together.

Lemma 3.1.10. *Let m be a monomial equal to $x_1^{p_1}\dots x_n^{p_n}$, then among the monomials appearing in $m\cdot e_\mu(\mathbf{x}_n^r)$, the monomial*

$$\prod_{i=1}^{n} x_{\pi(i)}^{p_{(\pi(i))}+r\mu_i'} \tag{3.8}$$

is the maximum with respect to $\prec$, where π is the index permutation of m.

Proof. The proof of this is similar to the proof of Prop 2.3.6. □

For the representation side of things we will give only a cursory overview of pertinent details, a more thorough treatment can be found in [20]. The analog of partitions which index the irreducible representations of $\mathfrak{S}_n = G(1,1,n)$ are r-partitions.

Definition 3.1.11. *An **r-partition** of n is an r-tuple of partitions $(\mu^0,\mu^1,\dots,\mu^{r-1})$ such that $\sum_{i=0}^{r-1}|\mu_i| = n$. We will use Greek letters with a bar to denote r-partitions, and will write $\overline{\mu}\vdash_r n$ to denote that $\overline{\mu}$ is an r-partition of n.*

The conjugacy classes of G_n, and thus the irreducible representations of G_n, are indexed by r-partitions of n. Given an r-partition $\overline{\lambda}$, we denote the irreducible representation of G_n corresponding to $\overline{\lambda}$ as $S^{\overline{\lambda}}$. The analog of standard Young tableaux in G_n are standard Young r-tableaux.

Definition 3.1.12. *A **standard Young r-tableau** of shape $(\mu^0,\mu^1,\dots,\mu^{r-1}) = \overline{\mu}$ is a way of assigning the integers $1,2,\dots,n$ to the boxes of r Ferrers diagrams of shapes $\mu^0,\mu^1,\dots\mu^{r-1}$ such that in each of the Ferrers diagrams the integers increase down columns and along rows. We denote the set of all standard Young r-tableaux of shape $\overline{\mu}$ as $SYT(\overline{\mu})$*

As with standard Young tableaux we have the notion of descents.

Definition 3.1.13. *An index i is a **descent** of a standard Young r-tableaux T if one of the following holds:*

1. *$i+1$ is in a component with a higher index than i*

2. *$i+1$ and i are in the same component and $i+1$ is strictly below i.*

Similar to other descents, we will denote $Des(T)$ as the set of all descents of T, $des(T)$ will be the number of descents of T, $d_i(T)$ will be the number of descents of T that are i or bigger. One last statistic related to descents is

$$f_i(T) = r \cdot d_i(T) + c_i(T) \tag{3.9}$$

where $c_i(T)$ is the index of the component of T that contains i.

The result connecting standard Young r-tableaux to our problem is the following:

Proposition 3.1.14. *The graded trace of the action of $\overline{\tau}$ on S_n has the following formula*

$$Tr_{\mathbb{C}[\mathbf{x}_n]}(\tau) = \frac{1}{\prod_{i=1}^{n}(1-q_1^r q_2^r \dots q_i^r)} \sum_{\overline{\lambda} \vdash_r n} \chi_{\overline{\tau}}^{\overline{\lambda}} \sum_{T \in SYT(\overline{\lambda})} \prod_{i=1}^{n} q_i^{f_i(T)}, \tag{3.10}$$

where $\chi_{\overline{\tau}}^{\overline{\lambda}}$ is the character $S^{\overline{\lambda}}$ evaluated at an element of type $\overline{\tau}$.

The proposition is proved in [3], though it is in a more general form since the formula that they give is for the entire family of groups $G(r,p,n)$. The formula that we state here is how it simplifies in the case $p = 1$.

3.2 Results

In order to calculate the multiplicities of $S^{\overline{\lambda}}$ in $S_{n,k,\mu}$ we will calculate the graded trace of the action on an element of type $\overline{\tau}$ on the space of polynomials in $\mathbb{C}[\mathbf{x}_n]$ where the individual exponents of each variable are less than kr. We denote this space $\mathbb{C}_{kr}[\mathbf{x}_n]$. First, we will calculate this trace using Proposition 3.1.14. Then we will use a basis for $\mathbb{C}_{kr}[\mathbf{x}_n]$ created from the descent basis for $S_{n,k}$

.

Lemma 3.2.1. *If $m = \prod_{i=1}^{n} x_i^{a_i}$ is a monomial in $\mathbb{C}_{rk}[\mathbf{x}_n]$ (that is $a_i < kr$ for all i), then*

$$m = b_{g,I} e_{\nu}(\mathbf{x}_n^r) + \sum, \tag{3.11}$$

where $g = g(m)$; $\sum$ is a sum of monomials $m' \prec m$; I is a sequence defined by $i_\ell = \mu'_\ell - \mu'_{n-k+1}$ where $\mu = \mu(g)$; and ν is the partition specified by:

1. $\nu'_\ell = \mu'_\ell$ *for* $\ell > n-k$

2. $\nu'_\ell = \mu'_{n-k+1}$ *for* $\ell \leq n-k$

Furthermore ν consists of parts of size at least $n-k+1$.

Proof. In order for $b_{g,I}$ to be well defined, we need that $k - des(g) > i_1 \geq \ldots \geq i_{n-k} \geq 0$. Since I is defined by taking a weakly decreasing, non-negative sequence and subtracting a constant which is smaller than the smallest part, I satisfies $i_1 \geq i_2 \geq \ldots \geq i_{n-k} \geq 0$. Letting π be the uncolored permutation of g and using definitions we get

$$i_1 = \mu'_1 - \mu'_{n-k+1} \leq \mu'_1 = \frac{a_{\pi(1)} - rd_1(g) - c_1(g)}{r} \tag{3.12}$$

and then by assumption $a_{\pi(1)} < rk$, and by definition $d_1(g) = des(g)$, thus

$$i_1 < \frac{rk - rdes(g)}{r} = k - des(g). \tag{3.13}$$

Therefore the use of $b_{g,I}$ is well defined.

We now show that $b_{g,I}$ and m have the same index r-colored permutation, specifically that

$$g(b_{g,I}) = g(m) = g. \tag{3.14}$$

We look at the sequence of the exponents of $x_{\pi(\ell)}$ in $b_{g,I}$. This is the sum of $rd_\ell(g) + c_\ell(g)$ and ri_ℓ (where we take $i_\ell = 0$ for $\ell > n-k$). Both of these sequences are weakly-decreasing, and therefore their sum is also weakly-decreasing. Additionally if the ℓth and $(\ell+1)$th entries are the same, then $rd_\ell(g)+c_\ell(g) = rd_{\ell+1}(g)+c_{\ell+1}(g)$, and thus $d_\ell(g) = d_{\ell+1}(g)$ and $c_\ell(g) = c_{\ell+1}(g)$. By the definition of $d_\ell(g)$, this implies that ℓ is not a descent of g. Since $c_\ell(g) = c_{\ell+1}(g)$ this means that $\pi(\ell) < \pi(\ell+1)$, thus g satisfies the first two conditions of being the index r-colored permutation, and the 3rd condition follows by the definition.

Now by Lemma 3.1.10, the maximum monomial in $b_{g,I}e_\nu(\mathbf{x}_n^r)$ will have the form $\prod_{\ell=1}^{n} x_{\pi(\ell)}^{q_\ell}$ where q_ℓ is given by:

1. $q_\ell = rd_\ell(g)+c_\ell(g)+ri_\ell+r\nu'_\ell$ for $\ell \leq n-k$
2. $q_\ell = d_\ell(g)+c_\ell(g)+r\nu'_\ell$ for $\ell > n-k$

Substituting using the definitions of i_ℓ and ν_ℓ and then the definition of the r-complementary partition, we have that

$$q_\ell = d_\ell(g)+c_\ell(g)+r\mu'_\ell - r\mu'_{n-k+1} + r\mu'_{n-k+1} = d_\ell(g)+c_\ell(g)+r\mu'_\ell = a_{\pi(\ell)} \tag{3.15}$$

for $\ell \leq n-k$, and

$$q_\ell = d_\ell(g)+c_\ell(g) - r\mu'_\ell = a_{\pi(\ell)} \tag{3.16}$$

for $\ell > n-k$

Finally ν has parts of size at least $n-k+1$ because by definition, the first $n-k+1$ parts of ν' are all the same size. $\square$

Proposition 3.2.2. *The set $B_{n,k}$ which consists of products $b_{g,I}e_\nu(\mathbf{x}_n^r)$ for ν a partition with parts of size at least $n-k+1$ and $(\lambda(b_{g,I})+r\nu')_1 < rk$ form a basis for $\mathbb{C}_{rk}[\mathbf{x}_n]$.*

Proof. The condition that $(\lambda(b_{g,I})+r\nu')_1 < rk$ along with Lemma 3.2.1 guarantees that each of the elements of $B_{n,k}$ are in $\mathbb{C}_{nk}[\mathbf{x}_n]$.

Applying Lemma 3.2.1 iteratively lets us express any monomial in $\mathbb{C}_{kr}[\mathbf{x}_n]$ as a linear combination elements of $B_{n,k}$, which means that $B_{n,k}$ spans $\mathbb{C}_{rk}[\mathbf{x}_n]$. To see that (up to rearrangement) this expansion is unique it is sufficient to show that if the maximal monomials in $b_{g,I}e_{\nu}(\mathbf{x}_n^r)$ and $b_{h,J}e_{\rho}(\mathbf{x}_n^r)$ are the same, then $g = h$, $I = J$ and $\nu = \rho$. As a corollary of the proof of Lemma 3.2.1, the index r-colored permutations of the maximal monomials are the same, and they are both g and h, and thus $g = h$. By Lemma 3.1.10, the power of $x_{\pi(\ell)}$ in each of the maximum monomials will be $rd_\ell(g) + c_\ell(g) + ri_\ell + r\nu'_\ell$ and $rd_\ell(h) + c_\ell(h) + rj_\ell + r\rho'_\ell$. This means that $\nu'_\ell = \rho'_\ell$ for $\ell > n-k$ since $i_\ell = j_\ell = 0$ for $\ell > n-k$. Next since the first $n-k+1$ parts of ν' are all equal and the first $n-k+1$ parts of ρ' are equal and $\nu'_{n-k+1} = \rho'_{n-k+1}$, we have that $\nu' = \rho'$ which implies $\nu = \rho$. This then implies that $i_\ell = j_\ell$ for all ℓ, and therefore this expansion is unique. Therefore $B_{n,k}$ is linearly independent and is a basis. $\square$

Proposition 3.2.3. *Let p be the map projecting from $\mathbb{C}[\mathbf{x}_n]$ to $S_{n,k}$ and let m be a monomial in $\mathbb{C}_{rk}[\mathbf{x}_n]$. Then*

$$p(m) = \sum_{g,I} \alpha_{g,I} b_{g,I} \tag{3.17}$$

where $\alpha_{g,I}$ are some constants, and the sum is over pairs g, I such that $\lambda(b_{g,I}) \trianglelefteq \lambda(m)$.

Proof. Since $B_{n,k}$ is a basis we can express $m = \sum_{g,I,\nu} \alpha_{g,I,\nu} b_{g,I} e_{\nu}(\mathbf{x}_n^r)$. By Lemma 3.2.1 $\alpha_{g,I,\nu}$ is zero if the leading monomial of $b_{g,I}e_{\nu}$ is not weakly smaller than m under the partial order on monomials. But since the partial order on monomials refines the dominance order on exponent partitions, for each non-zero term the exponent partition of the leading monomial will be dominated by $\lambda(m)$ that is that

$$(\lambda(b_{g,I}) + r\nu') \trianglelefteq \lambda(m). \tag{3.18}$$

Then when we project down to $S_{n,k}$, each term with $\nu \neq \emptyset$ will vanish since $e_{\nu}(\mathbf{x}_n^r)$ is in $J_{n,k}$, so that

$$p(m) = \sum_{g,I} \alpha_{g,I,\emptyset} b_{g,I} \tag{3.19}$$

where the sum is over (g, I) such that $\lambda(b_{g,I}) \trianglelefteq \lambda(m)$. $\square$

Proposition 3.2.3 gives the following corollary:

Corollary 3.2.4. *$S_{n,k,\rho}$ is zero unless ρ is the exponent partition of an (n,k,r)-descent monomial, which occurs precisely when ρ is an (n,k,r)-partition such that the last k parts form an r-descent partition.*

Similarly to the case for the symmetric group, this basis allows us to calculate the trace the action of $\tau \in G_n$ on $\mathbb{C}_{rk}[\mathbf{x}_n]$ in terms of the trace of its action on $S_{n,k}$ with the basis of (n,k,r)-descent monomials. Specifically the contribution to the trace of the element $b_{g,I}$ in $S_{n,k}$ will be equal to the contribution of $b_{g,I}e_\nu(\mathbf{x}_n^r)$ in $\mathbb{C}_{rk}[\mathbf{x}_n]$.

Lemma 3.2.5. *Given an (n,k,r)-partition μ and an (n,k,r) r-descent partition ν there exists a unique (n,k,r)-partition ρ such that $\mu = \nu + r\rho$ if and only if $Des^r(\nu) \subseteq Des^r(\mu)$ and $\mu_i \equiv \nu_i (mod r)$ for all i.*

Proof. The only possible value for each part of ρ is $\rho_i = \frac{\mu_i - \nu_i}{r}$. The mod r condition is necessary and sufficient for these values to be integers. In order for this to be a partition we need

$$\rho_i - \rho_{i+1} = \frac{1}{r}[(\mu_i - \mu_{i+1}) - (\nu_i - \nu_{i+1})] \geq 0. \tag{3.20}$$

Let c_i be the common remainder of μ_i and ν_i mod r. Since ν is an r-descent partition, $\frac{1}{r}((\nu_i - c_i) - (\nu_{i+1} - c_{i+1}))$ is 1 if i is an r-descent of ν and 0 if it is not. Similarly, $\frac{1}{r}((\mu_i - c_i) - (\mu_{i+1} - c_{i+1}))$ is at least 1 if i is an r-descent of μ and 0 otherwise. Thus in order for

$$\frac{1}{r}[(\mu_i - c_i) - (\mu_{i+1} - c_{i+1})] - \frac{1}{r}[(\nu_i - c_i) - (\nu_{i+1} - c_{i+1})] = \frac{1}{r}[(\mu_i - \mu_{i+1}) - (\nu_i - \nu_{i+1})] \tag{3.21}$$

to be non-negative, it is necessary and sufficient that if i is an r-descent of ν, then i is also an r-descent of μ. That is, ρ will be a partition if and only if $Des^r(\nu) \subseteq Des^r(\mu)$. □

Lemma 3.2.6. *Given an (n,k,r)-partition μ and a set $S \subseteq Des^r_{n-k+1,n}(\mu)$, there is a unique pair (ν, ρ) such that $\mu = \nu + r\rho$ and ν is the exponent partition of an (n,k,r)-descent monomial with $Des^r_{n-k+1,n}(\nu) = S$, and ρ is an (n,k)-partition with $\rho_1 = \rho_2 = \ldots = \rho_{n-k+1}$.*

Proof. The last k values of the exponent partition of an (n,k,r)-descent monomial form an r-descent partition, so by Lemma 3.2.5 applied to the partition determined by S and the values of $\mu_i (mod r)$ the last k values of ρ are determined. Since the first $n-k+1$ values of ρ need to be the same, this determines what ρ must be, and by subtraction what ν must be. We only need to check that ν is actually a partition, that is that $\nu_i - \nu_{i+1} \geq 0$ for $1 \leq i \leq n-k$. This is true since $\nu_i - \nu_{i+1} = (\mu_i - r\rho_i) - (\mu_{i+1} - r\rho_{i+1}) = \mu_i - \mu_{i+1} \geq 0$ since $\rho_i = \rho_{i+1}$. □

We now give the proof of Theorem 2.1.4.

Proof of Theorem 2.1.4. The condition on when $S_{n,k,\rho}$ is zero is covered by Corollary 3.2.4.

We consider the graded trace of the action of $\tau \in G_n$ on $\mathbb{C}_{rk}[\mathbf{x}_n]$ defined by

$$Tr_{\mathbb{C}_{rk}[\mathbf{x}_n]}(\overline{\tau}) := \sum_m \langle \overline{\tau}(m), m \rangle \cdot \bar{q}^{\lambda(m)}. \tag{3.22}$$

From 3.1.14 we have that

$$Tr_{\mathbb{C}[\mathbf{x}_n]}(\tau) = \frac{1}{\prod_{i=1}^n (1 - q_1^r q_2^r \dots q_i^r)} \sum_{\bar{\lambda}} \chi_{\bar{\mu}}^{\bar{\lambda}} \sum_{T \in SYT(\bar{\lambda})} \prod_{i=1}^n q_i^{f_i(T)} \tag{3.23}$$

(where $\bar{\mu}$ is the cycle type of τ). From this we can recover $Tr_{\mathbb{C}_{rk}[\mathbf{x}_n]}(\tau)$ by restricting to powers of q_1 that are at most $rk-1$. Doing this gives

$$\sum_{\bar{\lambda} \vdash_r n} \chi_{\bar{\mu}}^{\bar{\lambda}} \sum_{T \in SYT(\bar{\lambda}), \nu} \bar{q}^{\lambda_{F(T)}} \bar{q}^{r\nu}, \tag{3.24}$$

where $F(T) = (f_1(T), \dots, f_n(T))$ and the ν's are partitions such that $(\lambda_{F(T)})_1 + r\nu_1 < rk$.

Alternatively, we can calculate $Tr_{\mathbb{C}_{rk}[\mathbf{x}_n]}(\tau)$ by using the basis from Proposition 3.2.2, this gives $\mathrm{Tr}_{\mathbb{C}_{rk}[\mathbf{x}_n]}(\tau) = \sum_{g,I,\nu} \langle \tau(b_{g,I} e_\nu(\mathbf{x}_n^r)), b_{g,I} e_\nu(\mathbf{x}_n^r) \rangle \bar{q}^{\lambda(b_{g,I})} \bar{q}^{r\nu'}$
$= \sum_{g,I,\nu} \langle \tau(b_{g,I}), b_{g,I} \rangle \bar{q}^{\lambda(b_{g,I})} \bar{q}^{r\nu'}$
$= \sum_{\phi,\nu} Tr_{S_{n,k}}(\tau; \bar{q}^\phi) \bar{q}^\phi \bar{q}^{r\nu'}$ where the ν's are partitions with parts of size at least $n-k+1$ such that $(\phi)_1 + (r\nu')_1 < rk$, the ϕ's are the exponent partitions of (n,k,r)-descent monomials, and $Tr_{S_{n,k}}(\tau; \bar{q}^\phi)$

is the coefficient of $\bar{q}^{\phi}$ in the graded trace of the action of τ on $S_{n,k}$.

We now consider the coefficient of $\bar{q}^{\rho}$ for some (n,k,r)-partition ρ. Using the first calculation and Lemma 3.2.5, the inner sum can be reduced to T such that $Des(T) \subseteq Des^r(\rho)$, and such that $c_i(T) \equiv \rho_i (mod r)$ for all i, so that we get

$$\sum_{\bar{\lambda} \vdash_r n} \chi^{\bar{\lambda}}_{\bar{\mu}} |\{T \in SYT(\lambda) : Des(T) \subseteq Des^r(\rho), and c_i(T) \equiv \rho_i (mod r)\}|. \tag{3.25}$$

Looking at the second calculation and using Lemma 3.2.6 gives

$$\sum_{\phi} Tr_{S_{n,k}}(\tau; \bar{q}^{\phi}), \tag{3.26}$$

where the sum is over the set consisting of the all exponent partitions ϕ of (n,k,r)-descent monomials that satisfy the following two conditions:

1. $Des^r_{n-k+1,n}(\phi)$ is a subset of $Des^r_{n-k+1,n}(\rho)$.

2. $\phi_i \equiv \rho_i$ $(mod r)$ for all i.

Together this gives that

$$\sum_{\bar{\lambda} \vdash_r n} \chi^{\bar{\lambda}}_{\bar{\mu}} |\{T \in SYT(\bar{\lambda}) : Des(T) \subseteq Des^r(\rho), c_i(T) \equiv \rho_i (mod r)\}| = \sum_{\phi} Tr_{S_{n,k}}(\tau; \bar{q}^{\phi}). \tag{3.27}$$

We can further refine this result by showing that

$$\sum_{\bar{\lambda} \vdash_r n} \chi^{\bar{\lambda}}_{\bar{\mu}} |\{T \in SYT(\bar{\lambda}) : Des_{n-k+1,n}(\phi) \subseteq Des(T) \subseteq Des(\phi), c_i(T) \equiv \rho_i (mod r)\}| \tag{3.28}$$

$$= Tr_{S_{n,k}}(\tau; \bar{q}^{\phi}) \tag{3.29}$$

for any specific ϕ'. We do this by induction on $|\phi'|$. The base case of ϕ' being empty can be easily seen by taking $\rho = \emptyset$. If we take $\rho = \phi'$, then ϕ' will appear in the sum, and all other ϕ's will be smaller, so by the inductive hypothesis,

$$\sum_{\overline{\lambda}\vdash_r n} \chi_{\overline{\mu}}^{\overline{\lambda}} |\{T \in SYT(\overline{\lambda}) : Des_{n-k+1,n}(\phi) \not\subseteq Des(T) \subseteq Des(\phi), c_i(T) \equiv \rho_i (mod r)\}| \tag{3.30}$$

$$= \sum_{\phi \neq \phi'} Tr_{S_{n,k}}(\tau; \bar{q}^{\phi}) \tag{3.31}$$

Subtracting this from our result gives the desired refinement. This then proves the theorem since the exponent partition of any (n,k,r)-descent monomials $b_{g,I}$ appears when we take $\rho = \lambda(b_{g,I})$. $\square$

As an example of Theorem 2.1.4, let $n = 7$, $k = 5$ and let $r = 2$. Then consider letting $\rho = (9,5,5,4,3,2,0)$. The standard Young r-tableaux T that we must consider will have $4,6,7$ in the 0-component, and $1,2,3,5$ in the 1-component. Furthermore they will have $\{4,6\} \subseteq Des(T) \subseteq \{1,4,6\}$. The possibilities for the 0-component and the 1-component are independent. The possibilities for the 0-component are

$$\begin{array}{|c|c|}\hline 4 & 6 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|}\hline 4 \\ \hline 6 \\ \hline 7 \\ \hline \end{array}.$$

The possibilities for the 1-component are

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Therefore the multiplicites of $S^{(2,1),(4)}$, $S^{(2,1),(2,2)}$, $S^{(2,1),(2,1,1)}$, $S^{(1,1,1),(4)}$, $S^{(1,1,1),(2,2)}$, and $S^{(1,1,1),(2,1,1)}$ in $S_{n,k,\rho}$ are 1, and the multiplicities of $S^{(2,1),(3,1)}$ and $S^{(1,1,1),(3,1)}$ are 2. All other multiplicities are zero.

Theorem 2.1.4 also allows us to recover the following corollary which is equivalent (up to a change of indexing) to a result of Chan and Rhoades [5].

Corollary 3.2.7. *Let $f_{\overline{\lambda}}(q)$ be the generating function for the multiplicities of $S^{\overline{\lambda}}$ in the degree d*

component of $S_{n,k}$. Then

$$f_{\bar{\lambda}}(q) = \sum_{T \in SYT(\bar{\lambda})} q^{maj(T)} \binom{n - des(T) - 1}{n-k}_{q^r}, \tag{3.32}$$

where the major index $maj(T)$ is equal to $\sum_{i=1}^{n} rd_i(T) + c_i(T)$

Proof. By Theorem 2.1.4, each standard Young r-tableau of shape $\bar{\lambda}$ contributes to $f_{\bar{\lambda}}(q)$ once for each partition ρ such that ρ is the exponent partition of an (n,k,r)-descent monomial and $Des^r_{n-k+1,n}(\rho) \subseteq Des(T) \subseteq Des^r(\rho)$. All such ρ come from (n,k,r)-descent monomials $b_{g,I}$ where g is an r-colored permutation with $Des^r(g) = Des(T)$, $c_i(g) = c_i(T)$ for all i, and I is a sequence such that $k - des(T) > i_1 \geq i_2 \geq \ldots \geq i_{n-k} \geq 0$. This choice of I is the same as choosing a partition that fits in an $(n-k) \times (k-1-des(T))$ box. The generating function for the number of partitions of size d that fit in an $(n-k) \times (k-1-des(T))$ box is $\binom{(n-k)+(k-des(T)-1)}{n-k}_q = \binom{n-des(T)-1}{n-k}_q$. But in $b_{g,I}$ we are multiplying the values in I by r, so we need to plug q^r into this q-binomial coefficient to get $\binom{n-des(T)-1}{n-k}_{q^r}$. The factor of b_g in the (n,k,r)-descent monomial then has degree $maj(T)$, so that each standard Young tableau T of shape λ will contribute $q^{maj(T)}\binom{n-des(T)-1}{n-k}_{q^r}$ to $f_{\bar{\lambda}}(q)$. This completes the proof. □

The proof of this result in [5] is fairly involved using a tricky recursive argument involving an auxiliary family of algebras. We manage to give a simpler proof for this result.

Overlapping notations slightly, Chan and Rhoades [5] also defined the ideal

$$I_{n,k} := \langle x_1^{kr+1}, x_2^{kr+1}, \ldots, x_n^{kr+1}, e_n(\mathbf{x}_n^r), e_{n-1}(\mathbf{x}_n^r), \ldots, e_{n-k+1}(\mathbf{x}_n^r) \rangle, \tag{3.33}$$

and the algebra

$$R_{n,k} := \frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{I_{n,k}}. \tag{3.34}$$

As before we can refine the grading on this algebra to define $R_{n,k,\rho}$, and can ask what the graded isomorphism type of this G_n module is. By slightly modifying the results of this section (looking at partitions with largest part kr instead of $kr-1$, and using the extended descent monomials from [5] instead of the descent monomials) we can obtain a result that is analogous to Theorem 2.1.4.

3.3 A Method for Defining Ideals

One path to take from here would be to try to extend the (n,k)-coinvariant algebras introduced by Chan and Rhoades [5] for $G(r,1,n)$ to all complex reflection groups. It seems that the simplest groups to consider are $G(2,2,n)$ which are equal to the real reflection groups of Coxeter-Dynkin type D_n. In the case that $G(r,1,n)$ is a real reflection group, the structure of the corresponding (n,k)-coinvariant algebra is governed by the combinatorics of the k-dimensional faces of the associated Coxeter complex. We can define a candidate graded algebras for D_n that will satisfy this property by using a more general technique of Garsia and Procesi [8] which we recall here .

We start by taking a finite set of points $X \subset \mathbb{C}^n$. We then consider the set of polynomials in $\mathbb{C}[x_1,x_2,\dots,x_n]$ that vanish on X, that is

$$\{f \in \mathbb{C}[x_1,\dots,x_n] : f(\bar{x}) = 0 \; forall \; \bar{x} \in X\}. \tag{3.35}$$

This set is an ideal in $\mathbb{C}[x_1,\dots,x_n]$, and we will denote it by $\mathbb{I}(X)$. Next, we consider the quotient $\frac{\mathbb{C}[x_1,\dots,x_n]}{\mathbb{I}(X)}$.

The elements of this quotient can be viewed as $\mathbb{C}$-valued function on X. We do this by taking a representative polynomial in $\mathbb{C}[x_1,\dots,x_n]$, viewing it as a function from $\mathbb{C}^n$ to $\mathbb{C}$, and then restricting its domain to X. Two polynomials will give rise to the same $\mathbb{C}$-valued function if and only if their difference vanishes on X, which occur precisely if the difference is in $\mathbb{I}(X)$. Therefore this is well defined. Furthermore for every element $\bar{x} \in X$, we can construct an indicator function for $\bar{x}$ as follows. For each element $\bar{y} \in X \backslash \bar{x}$ choose an index $i_{\bar{y}}$ at which $\bar{x}$ and $\bar{y}$ differ, then

$$\prod_{\bar{y} \in X \backslash \bar{x}} \frac{x_{i_{\bar{y}}} - \bar{y}_{i_{\bar{y}}}}{\bar{x}_{i_{\bar{y}}} - \bar{y}_{i_{\bar{y}}}} \tag{3.36}$$

is an indicator function for $\bar{x}$. Therefore $\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{I}(X)}$ is isomorphic as a vector space to $\mathbb{C}[X]$ where $\mathbb{C}[X]$ is the coordinate ring of X.

Any subgroup W of $GL(\mathbb{C}^n)$ acts on $\mathbb{C}[x_1,\ldots,x_n]$ by linear substitution. If X is invariant under W, then $\mathbb{I}(X)$ is invariant under W, and thus both $\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{I}(X)}$ and $\mathbb{C}[X]$ are W-modules. Furthermore in addition to being isomorphic as vector spaces, these two objects are isomorphic as W-modules. Unfortunately, $\mathbb{I}(X)$ will not generally be homogeneous, and thus we will not have that $\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{I}(X)}$ is graded. In order to fix this we introduce a function τ that sends a non-zero polynomial to its top degree component. For example

$$\tau(x_1^2+x_2^2+x_2x_3-x_1-x_2-x_3+3)=x_1^2+x_2^2+x_2x_3 \tag{3.37}$$

and

$$\tau(x_1^4+x_1x_2x_3x_4+x_3^4-x_2^3-x_2^2+3)=x_1^4+x_1x_2x_3x_4+x_3^4. \tag{3.38}$$

We then consider the ideal $\mathbb{T}(X)$ generated by the top degrees of polynomials that vanish on X, that is

$$\mathbb{T}(X):=\langle\{\tau(f):f\in\mathbb{I}(X)-\{0\}\}\rangle. \tag{3.39}$$

This ideal is homogeneous and invariant under W, therefore $\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{T}(X)}$ is a graded W-module. Furthermore it can be shown (see [8] for details) that

$$\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{T}(X)}\cong_W\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{I}(X)}\cong_W\mathbb{C}[X]. \tag{3.40}$$

Then if we take W to be D_n and take X to be a set of points containing exactly one point in each of the k-dimensional faces of the Coxeter complex of D_n such that X is invariant under D_n, our candidate algebra will then be $\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{T}(X)}$.

There are two difficulties that we run into at this point. The first is the question of how we choose X. Different choices of X lead to isomorphic ungraded D_n-modules, but the graded structure in general depends on X, and it is not clear what the "correct" choice is. The second difficulty is getting a nice generating set for $\mathbb{T}(X)$. We do have a general method to get a (potentially ugly) description of $\mathbb{T}(X)$ from X which is the following.

The idea behind our method is that if we find a set $P \subset \mathbb{T}(X)$ such that

$$dim\left(\frac{\mathbb{C}[x_1,\ldots,x_n]}{\langle P\rangle}\right) = dim\left(\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{T}(X)}\right) = |X|, \tag{3.41}$$

then P generates $\mathbb{T}(X)$. For a given $P \subset \mathbb{T}(X)$, let $st(P)$ be the standard monomial basis for $\frac{\mathbb{C}[x_1,\ldots,x_n]}{\langle P\rangle}$ with respect to some graded monomial ordering (see [1] for more details). We will have found a P that works when we have that $|st(P)| = |X|$.

All monomials of degree $|X|$ will appear in $\mathbb{T}(X)$. In order to avoid cumbersome notation we will show this with an example. If $n = 3$ and X consists of the points $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$, then have $\mathrm{x}_1^3 = \tau((x_1-\alpha_1)(x_1-\beta_1)(x_1-\gamma_1))$
$x_1^2x_2 = \tau((x_1-\alpha_1)(x_1-\beta_1)(x_2-\gamma_2))$
$x_1^2x_3 = \tau((x_1-\alpha_1)(x_1-\beta_1)(x_3-\gamma_3))$, and so on. This idea generalizes to show that all degree $|X|$ monomials will appear in $\mathbb{T}(X)$. We will thus start with P consisting of all monomials of degree d. Then $st(P)$ will consist of monomials of degree less than d, which is a finite set.

We now describe a method for adding an element to P that will reduce the size of $st(P)$. Let $m_1, m_2, \ldots m_s$ be the elements of $st(P)$, and let $p_1, p_2, \ldots p_t$ be the elements of X. We then create a $t \times s$ matrix M by setting $M_{ij} = m_j(p_i)$. If $t < s$, then the null space of M is non-zero, so we can take a non-zero vector $v = (v_1, v_2, \ldots, v_s)$ in the null space. We then consider the polynomial $f = \sum_{j=1}^{s} v_j m_j$. Evaluating this polynomial at p_i gives $\sum_{j=1}^{s} v_j m_j(p_i) = (Mv)_i = 0$. Thus f vanishes on X which means that $\tau(f)$ is in $\mathbb{T}(X)$. The leading monomial of $\tau(f)$ is an element of $st(P)$, and adding $\tau(f)$ to P will at least eliminate this leading monomial from $st(P)$. We then iterate this process until $|st(P)| = |X|$.

This method also gives us the standard monomial basis for $\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{T}(X)}$. This allows us to give examples of when different choices of X lead to different graded structures. If we let X be the orbits of $(1,1,2)$, $(-1,1,2)$, and $(1,2,2)$ under D_3, then the Hilbert series of $\frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{T}(X)}$ is

$5q^5+11q^4+10q^3+6q^2+3+1$. If instead we take the orbits of $(1,1,2)$, $(-1,1,2)$, and $(1,\sqrt{\frac{5}{2}},\sqrt{\frac{5}{2}})$, then we get Hilbert series $11q^5+9q^4+7q^3+5q^2+3q+1$, and if we take the orbits of $(1,1,2)$, $(-1,1,2)$, and $(0,\sqrt{3},\sqrt{3})$ we get Hilbert series $4q^6+8q^5+8q^4+7q^3+5q^2+3q+1$. From experimental data it does appear that there is a generic isomorphism type, but even for X that give rise to isomorphic graded D_n-modules, the ideals $\mathbb{T}(X)$ can be different.

In later chapters we will modify this method to work in the context of actions of the Iwahori-Hecke algebra.

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Chapter 4

Generalized Coinvariant Algebras for Compositions

4.1 Background

In [16] Rhoades studies the space of spanning configurations associated to a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ and an integer k which consist of tuples $(V_1, V_2, \ldots V_r)$ where each V_i is a linear subspace of $\mathbb{R}^k$ of dimension α_i such that $span(V_1, V_2, \ldots, V_r) = \mathbb{R}^k$. In studying this space he defines a quotient

$$R_{\alpha,k} := \mathbb{Q}[x_1, \ldots x_n]/I_{\alpha,k}, \tag{4.1}$$

where n is the size of α, and $I_{\alpha,k}$ is generated by two types of elements. The first type are the elementary symmetric functions

$$e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n), \tag{4.2}$$

where $\mathbf{x}_n$ denotes the set of variables $\{x_1, x_2, \ldots, x_n\}$. For the second type, let $\mathbf{x}_i^\alpha$ denote the set of α_i x variables indexed from $1 + \sum_{j=1}^{i-1} \alpha_j$ to $\sum_{j=1}^{i} \alpha_j$. We will refer to the sets of variables $\mathbf{x}_i^\alpha$ as the ith batch of variables. Similarly we will refer to the set of indices of $\mathbf{x}_i^\alpha$ as the ith batch of values. Then the second type of generators are then the homogeneous symmetric functions

$$h_k(\mathbf{x}_i^\alpha), h_{k-1}(\mathbf{x}_i^\alpha), \ldots, h_{k-\alpha_i+1}(\mathbf{x}_i^\alpha) \tag{4.3}$$

for all $1 \leq i \leq r$.

The quotient $R_{\alpha,k}$ reduces to the quotient $R_{n,k}$ of Haglund, Rhoades, and Shimizono [12] when $\alpha = 1^n$. This quotient in turn is a generalization of the classical coinvariant algebra R_n. Both R_n and $R_{n,k}$ are graded $\mathfrak{S}_n$-modules since the ideals that define them are homogeneous quotients that carry an action of $\mathfrak{S}_n$. On the other hand, the ideal $I_{\alpha,k}$ while graded, is not invariant under $\mathfrak{S}_n$ acting by variable permutation. However it is invariant under the action of the Young subgroup $\mathfrak{S}_\alpha \subset \mathfrak{S}_n$ that permutes each batch of variables within themselves, that is

$$\mathfrak{S}_\alpha \cong \mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \ldots \times \mathfrak{S}_{\alpha_r}. \tag{4.4}$$

This with the fact that the generators of $I_{\alpha,k}$ are homogeneous means that $R_{\alpha,k}$ is a graded $\mathfrak{S}_\alpha$ module.

The ungraded isomorphism types of R_n and $R_{n,k}$ as $\mathfrak{S}_n$-modules are the same as the $\mathfrak{S}_n$ modules given by the actions of $\mathfrak{S}_n$ on permutations of length n and ordered set partitions of $[n]$ into k blocks respectively. The analogous objects for $R_{\alpha,k}$ are (α,k)-ordered set partitions which are closely related to the more studied (α,k)-ordered multiset partitions. We start by defining these objects.

Definition 4.1.1. (α,k)-*Ordered Multiset Partitions*

Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$, and a non-negative integer k, a (α,k)-ordered multiset partition σ is a collection of k ordered blocks $B_1, B_2, \ldots, B_k$ with the following properties:

- *For each $j \in [k]$, B_j is a non-empty subset of $[r]$.*
- $\cup_{i=1}^k B_i = [r]$.
- *For each $i \in [r]$ there are exactly α_i blocks that contain i.*

We will denote the set of (α, k)-ordered multiset partitions by $\mathcal{OMP}_{\alpha,k}$, and we will denote its cardinality by $Stir(\alpha, k)$. Further we will write (α, k)-ordered multiset partitions in the form $(B_1|B_2|\ldots|B_k)$.

We will call a composition α, k-compatible if $\mathcal{OMP}_{\alpha,k}$ is non empty. This is equivalent to the condition that $\alpha_i \leq k$ for all i, and that $\sum_{i=1}^{r} \alpha_i \geq k$.

Example 4.1.2. *Let $\alpha = (2,4,3)$ and let $k = 5$, then*

$$\sigma = (13|123|2|23|2), \tag{4.5}$$

is an element of $\mathcal{OMP}_{\alpha,k}$.

Definition 4.1.3. (α, k)*-Ordered Set Partitions*

Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ of size n, and an non-negative integer k, an (α, k)-ordered set partition is an ordered set partition of n with k blocks such that the integers from $1 + \sum_{i=1}^{j-1} \alpha_i$ to $\sum_{i=1}^{j} \alpha_i$ all appear in distinct blocks. In other words the integers 1 to n are split into r batches of consecutive integers where the jth batch is of size α_j, and each part of the partition contains at most one integer from each batch. We will denote the set of (α, k)-ordered set partitions by $\mathcal{OSP}_{\alpha,k}$.

Example 4.1.4. *For example if $\alpha = (2,4,3)$, and $k = 5$ then*

$$\pi = (27|149|3|68|5), \tag{4.6}$$

is an element of $\mathcal{OSP}_{\alpha,k}$

The symmetric group $\mathfrak{S}_n$ acts on ordered set partitions of n by permuting the values $[n]$, but in general $\mathcal{OSP}_{\alpha,k}$ will not be preserved by this action. It will however be preserved by the Young subgroup $\mathfrak{S}_\alpha$. Furthermore the action of $\mathfrak{S}_\alpha$ on $\mathcal{OSP}_{\alpha,k}$ is free since the elements of each batch are all in different blocks.

We can define a map $\rho : \mathcal{OSP}_{\alpha,k} \to \mathcal{OMP}_{\alpha,k}$ by having $\rho(\sigma)$ be the ordered multiset partition formed by replacing every element in the ith batch of values in $\sigma \in \mathcal{OSP}_{\alpha,k}$ by i for all $i \in [r]$. For example if we take σ and π as in Examples 4.1.2 and 4.1.4, then $\rho(\pi) = \sigma$. Since the action of $\mathfrak{S}_\alpha$ does not change which blocks contain elements from each batch of values, ρ is constant on the orbits of the action of $\mathfrak{S}_\alpha$. Conversely, if two elements of $\mathcal{OSP}_{\alpha,k}$ are equal under ρ, then the are in the same orbit. Therefore we have that

$$|\mathcal{OSP}_{\alpha,k}| = |\mathfrak{S}_\alpha| \cdot |\mathcal{OMP}_{\alpha,k}|. \tag{4.7}$$

We can define a function ϕ to be a one-sided inverse of ρ by having $\phi(\sigma)$ be defined by replacing the i's in σ by the ith batch of variables increasing from left to right. We can then define a bijective map

$$\tau : \mathfrak{S}_\alpha \times \mathcal{OMP}_{\alpha,k} \to \mathcal{OSP}_{\alpha,k}, \tag{4.8}$$

by

$$\tau(\pi,\sigma) := \pi \circ (\phi(\sigma)). \tag{4.9}$$

Later it will be useful to break down the inverse of τ in the following way. Given $\sigma \in \mathcal{OSP}_{\alpha,k}$, define $t_i(\sigma)$ to be the component of $\mathfrak{S}_{\alpha_i}$ in $\tau^{-1}(\sigma)$. With these definitions we have that $\tau^{-1}(\sigma) = ((t_1(\sigma), t_2(\sigma), \ldots, t_r(\sigma)), \rho(\sigma))$.

We can generalize the recurrance relation that the Striling numbers of the second kind satisfies as follows.

Proposition 4.1.5. *The quantity $Stir(\alpha,k)$ satisfy the following relation*

$$\alpha_i Stir(\alpha,k) = (k-\alpha_i+1)Stir(\hat{\alpha},k) + kStir(\hat{\alpha},k-1) \tag{4.10}$$

where $i \in [r]$, and $\hat{\alpha}$ is the composition obtained by decreasing the ith entry of α by 1. This along with the boundary conditions that $Stir((),0) = 1$, and that $Stir(\alpha,k) = 0$ if α is not k-compatible gives us a algorithm for calculating any value of $Stir(\alpha,k)$.

Proof. We will prove this bijectively. The lefthand side of the equation is the cardinality of

$$[\alpha_i] \times \mathcal{OMP}_{\alpha,k}, \tag{4.11}$$

and the righthand side is the cardinality of

$$([k-\alpha_i+1] \times \mathcal{OMP}_{\hat{\alpha},k}) \sqcup ([k] \times \mathcal{OMP}_{\hat{\alpha},k-1}). \tag{4.12}$$

For any element of $\mathcal{OMP}_{\alpha,k}$ define some way of labelling the α_i i's with the elements of $[\alpha_i]$. Next for any element of $\mathcal{OMP}_{\hat{\alpha},k}$ define a way of labelling the $k-\alpha_i+1$ blocks that do not contain an i by the elements of $[k-\alpha_i+1]$. Finally for any element of $\mathcal{OMP}_{\hat{\alpha},k-1}$ define some way of labelling the k spots between its blocks (including before the first block and after the last block) with the elements of $[k]$.

With these labellings we will now define a function f from $[\alpha_i] \times \mathcal{OMP}_{\alpha,k}$ to $([k-\alpha_i+1] \times \mathcal{OMP}_{\hat{\alpha},k}) \sqcup ([k] \times \mathcal{OMP}_{\hat{\alpha},k-1})$. Consider $(j,\sigma) \in [\alpha_i] \times \mathcal{OMP}_{\alpha,k}$ and let i_j be the index of the block containing the i labelled by j in σ. If the i_jth block is not a singleton block, we define $f(j,\sigma) := (j',\sigma')$ where σ' is obtained from σ by removing the i in the i_jth block, and j' is the label of the block from which we removed an i. If the i_jth block is a singleton, then we define $f(j,\sigma) := (j',\sigma')$ where σ' is formed from σ by removing the i_jth block, and j' is the label of the spot that that block used to be.

The inverse of f is then defined by reinserting an i into a block or a singleton block $\{i\}$ into the spot indicated by the label, and then including the label of that i. By consturction these function are inverses of each other, and thus the result is shown.

$\square$

We will introduce two statistics on $\mathcal{OMP}_{\alpha,k}$ that are generalizations of the major index and inversion number, and we will show that these two statistics are equidistributed. For describing the major index, we will use an alternate description of (α,k)-ordered multiset partitions as descent starred words.

Definition 4.1.6. *Descent Starred Words*

An (α,k)-descent starred word is a word in the alphabet $[r]$ with content α in which $n-k$ descents are marked with a star to the right of the descent. The set of (α,k)-descent starred words are in bijection with $\mathcal{OMP}_{\alpha,k}$ by writing each block of an (α,k)-ordered multiset partition in decreasing order and putting a star in between each element in a block, or inversely by putting a bar in between any two entries in a descent starred word that do not have a star between them.

Definition 4.1.7. *Inversions*

Let $\sigma = (B_1|B_2|\dots|B_k)$ be an ordered multiset partition. Then for two indices $i < j$, the number of inversions between B_i and B_j, denoted $inv_{i,j}(\sigma)$, is the number of entries in B_i that are larger than the minimal entry of B_j, that is

$$inv_{i,j}(\sigma) := |B_i| - |B_i \cap [min(B_j)]|. \tag{4.13}$$

The inversion statistic of σ, denoted $inv(\sigma)$ is then

$$inv(\sigma) := \sum_{i<j\leq k} inv_{i,j}(\sigma). \tag{4.14}$$

Definition 4.1.8. *Major Index Given an element of $\mathcal{OMP}_{\alpha,k}$ written as a descent starred word $w = w_1 \dots w_n$, define the quantity d_i to be the number of stars weakly to the left of w_i in w, then the major index of w, denoted $maj(w)$ is*

$$maj(w) := \sum_{i\in Des(w)} (i - d_i), \tag{4.15}$$

where $Des(w)$ is the set of descents in w.

An alternative way of calculating $maj(w)$ is to sum all descents of w, and then subtract the *number of descents weakly to the right of each starred index, that is*

$$maj(w) = (\sum_{i \in Des(w)} i) - (\sum_{i \in Star(w)} r_i), \tag{4.16}$$

where $Star(w)$ is the starred indices of w, and r_i is the number of descents weakly to the right of i.

We can also consider separating starred and unstarred descents in the term $\sum_{i \in Star(w)} r_i$. The starred descents will always be $\sum_{i=1}^{n-k} i = \binom{n-k+1}{2}$. So for a third way of calculating $maj(w)$, we have

$$maj(w) = (\sum_{i \in Des(w)} i) - (\sum_{i \in Star(w)} u_i) - \binom{n-k+1}{2}, \tag{4.17}$$

where u_i is the number of unstarred descents weakly to the right of an index i in w.

In [22] Wilson shows that Inv and Maj and a third statistic of diagonal inversions are equidistributed on $\mathcal{OMP}_{\alpha,k}$. Rhoades [15] shows that a fourth statistic minimaj is also equidistributed with the other three. We will show that Inv, and thus the other three statistic as well, satisfy a certain set of recurrence relations and initial conditions. We will denote their shared generating function by $Stir'_q(\alpha,k)$.

Proposition 4.1.9. *If we let*

$$D^{inv}_{\alpha,k}(q) := \sum_{w \in \mathcal{OMP}_{\alpha,k}} q^{inv(w)}, \tag{4.18}$$

then $D^{inv}_{\alpha,k}(q)$ satisfy the following recurrsion relation:

$$[\alpha_i]_q D^{inv}_{\alpha,k}(q) = q^{\alpha_i - 1}[k - \alpha_i + 1]_q D^{inv}_{\hat{\alpha},k}(q) + [k]_q D^{inv}_{\hat{\alpha},k-1}(q), \tag{4.19}$$

with the initial conditions that that $D^{inv}_{(),0}(q) = 1$, and that $D^{inv}_{\alpha,k}(q) = 0$ if any part of α is greater than k, or if the size of α is less than k.

Proof. This result is a q-analogue of Proposition 4.1.5, and we will prove it using a similar, graded bijection. The idea is that the labels that we will choose will correspond to the terms in $[\alpha_i]_q, [k - \alpha_i +$

$1]_q$, and $[k]_q$, and then we just need to actually define these labellings and check that the total weights of all the terms match. For ease of notation we will have our labelling sets start at 0, corresponding to q-numbers having lowest term q^0.

For simplicity we will prove this result specifically for the case that $i = r$, at a later point we will show that these functions are invariant under permuting the entries of α, so that it is true for all i.

We will now give a description of the three labellings. For labelling the r's in an element of $\mathcal{OMP}_{\alpha,k}$, starting at zero, label the r's in singleton blocks from right to left, and then label the rest of the r's from left to right. Then, the weight of an element $(j,\sigma) \in \{0,1,\ldots,\alpha_r - 1\} \times \mathcal{OMP}_{\alpha,k}$ is $j + inv(\sigma)$.

For example:

$$(123^2|3^1|12|23^3|3^0|13^4) \tag{4.20}$$

For an element of $\mathcal{OMP}_{\hat{\alpha},k}$ label the blocks that do not contain r's from right to left starting at 0. Then, the weight of an element $(j,\sigma) \in \{0,1,\ldots,k-\alpha_r\} \times \mathcal{OMP}_{\hat{\alpha},k}$ is given by $\alpha_r - 1 + j + inv(\sigma)$

For an element of $\mathcal{OMP}_{\hat{\alpha},k-1}$ label the spaces between blocks (including before the first and after the last) from right to left starting at 0. Then the weight of an element $(j,\sigma) \in \{0,1,\ldots,k-1\} \times \mathcal{OMP}_{\hat{\alpha},k-1}$ is $j + inv(\sigma)$.

We then need to show that with these labellings and weights, the bijection from Proposition 4.1.5 is weight preserving. Let $(j,\sigma) \in \{0,1,\ldots,\alpha_r - 1\} \times \mathcal{OMP}_{\alpha,k}$, and let $f(j,\sigma) = (j',\sigma')$. Further let i_j be the index of the block containing the r labelled j. We start by consider how removing the r in block i_j affects the inversion statistic, in symbols this will be the quantity $inv(\sigma) - inv(\sigma')$. If the i_jth block is not a singleton, then removing the r in it will reduce the inversion statistic by the number of blocks to the right of the i_jth block that are not equal to $\{r\}$ (and thus have a minimum that is strictly less than r). Next by how we defined our labellings, j is $\alpha_r - 1$ minus the number of non-singleton blocks to the right of the i_jth block that contain an r. Thus $inv(\sigma) - inv(\sigma') + j - (\alpha_r - 1)$ is equal to the number of blocks to the right of the i_jth block that do not contain an r. But by how we have defined our labels, that is exactly the quantity j'. Thus we have that

$$inv(\sigma) - inv(\sigma') + j - (\alpha_r - 1) = j', \tag{4.21}$$

which can be rewritten as

$$in(\sigma) + j = inv(\sigma') + j' + (\alpha_r - 1), \tag{4.22}$$

or in other words the weight of (j, σ) and $f(j, \sigma)$ are the same.

If the i_jth block is not equal to $\{r\}$, then removing that block will decrease the inversion statistic by 1 for each block to the right of the i_jth block with minimum less than r, that is all blocks not equal to $\{r\}$. The label j is the number of blocks to the right of the i_j block that are equal to $\{r\}$, and thus $inv(\sigma) - inv(\sigma') + j$, is the number of blocks to the right of the i_jth block. But this is the label j', so that

$$inv(\sigma) + j = inv(\sigma') + j', \tag{4.23}$$

and again the weights of (j, σ) and $f(j, \sigma)$ are the same. Thus in all cases, this function is weight preserving, and the result follows from that.

□

4.2 Complementary Statistics

For this paper, the statistic in which we are most interested is not maj, but rather the complementary statistic $comaj$ defined as

$$comaj(\sigma) = max\{maj(\rho) : \rho \in \mathcal{OMP}_{\alpha,k}\} - maj(\sigma). \tag{4.24}$$

In order to study $comaj$, we give a formula for the maximal value of maj on $\mathcal{OMP}_{\alpha,k}$.

Lemma 4.2.1. *For any composition α of size n and length r let $m_{\alpha,k}$ denote the maximal value of*

maj on $\mathcal{OMP}_{\alpha,k}$ if $\mathcal{OMP}_{\alpha,k}$ is non-empty and zero otherwise. Then if α is k-compatible,

$$m_{\alpha,k} = kn - \binom{k}{2} - \sum_{i=1}^{r} \binom{\alpha_i + 1}{2}. \tag{4.25}$$

Proof. We will prove this by induction on n using the recursive relation from Proposition 4.1.9. The base cases when n is zero are trivial. For the inductive step let $k > 0$, and let α be a k-compatible composition. Then by considering the degree of each term in the recurrance relation we get,

$$deg(D^{maj}_{\alpha,k}) + \alpha_r - 1 = max\{m_{\hat{\alpha},k} + k - 1, m_{\hat{\alpha},k-1} + k - 1\}. \tag{4.26}$$

(Note that this equation holds even if one of $m_{\hat{\alpha},k}$ or $m_{\hat{\alpha},k-1}$ is zero, since then the other term will be weakly larger.)

We then note that if $\hat{\alpha}$ is k and $(k-1)$-compatible, then $m_{\hat{\alpha},k} \geq m_{\hat{\alpha},k-1}$ since

$$m_{\hat{\alpha},k} - m_{\hat{\alpha},k-1} = kn - (k-1)n - (\binom{k}{2} - \binom{k-1}{2}) = n - k + 1 \geq 0, \tag{4.27}$$

since $n \geq k - 1$. Thus in this case, or in the case that $\hat{\alpha}$ is not $k-1$ compatible, the first term will be the maximum. Thus

$$deg(D^{maj}_{\alpha,k}) = m_{\hat{\alpha},k} + k - \alpha_r = k(n-1) - \binom{k}{2} - (\sum_{i<r} \binom{\alpha_i + 1}{2}) - \binom{\alpha_r}{2} + k - \alpha_r = \tag{4.28}$$

$$kn - \binom{k}{2} - \sum_{i} \binom{\alpha_i + 1}{2}, \tag{4.29}$$

which is the desired result. The remaining possible case is that $\hat{\alpha}$ is not k-compatible in which case $m_{\hat{\alpha},k-1} \geq m_{\hat{\alpha},k}$. This will only happen when $n = k$, and thus we have that

$$deg(D^{maj}_{\alpha,k}) = m_{\hat{\alpha},k-1} + k - \alpha_r = (k-1)^2 - \binom{k-1}{2} - (\sum_{i=1}^{r-1} \binom{\alpha_i + 1}{2}) - \binom{\alpha_r}{2} + k - \alpha_r = \tag{4.30}$$

$$k^2 - \binom{k}{2} - \sum_{i=1}^{r} \binom{\alpha_i + 1}{2} = nk - \binom{k}{2} - \sum_{i=1}^{r} \binom{\alpha_i + 1}{2}, \tag{4.31}$$

which is again the desired result, thus by induction the result is shown.

□

Using Lemma 4.2.1 we are able find a recurisive function that the generating function for $comaj$ satisfies.

Proposition 4.2.2. *Letting*

$$D^{comaj}_{\alpha,k}(q) := \sum_{\sigma \in \mathcal{OMP}(\alpha,k)} q^{comaj(\sigma)}, \tag{4.32}$$

then $D^{comaj}_{\alpha,k}(q)$ satisfies the following recurrance relation:

$$[\alpha_i]_q D^{comaj}_{\alpha,k}(q) = [k - \alpha_i + 1]_q D^{comaj}_{\hat{\alpha},k}(q) + q^{n-k}[k]_q D^{comaj}_{\hat{\alpha},k-1}(q). \tag{4.33}$$

Proof. We will start by q-reversing the relation we have for $D^{maj}(q)$, and then multiply by $q^{m_{\alpha,k}+\alpha_i-1}$. Then using the fact that

$$D^{comaj}_{\alpha,k}(q) = q^{m_{\alpha,k}} D^{maj}_{\alpha,k}(q^{-1}), \tag{4.34}$$

and simplifying some algebra gives the result.

□

We will denote the polynomial $D^{comaj}_{\alpha,k}(q)$ by $Stir_q(\alpha,k)$.

Theorem 4.2.3. *The Hilbert series of $R_{\alpha,k}$ is the following*

$$Hilb(R_{\alpha,k};q) = [\alpha]_q! Stir_q(\alpha,k) \tag{4.35}$$

.

Conjecture 4.2.4. *The graded S_α Frobenius image of $R_{\alpha,k}$ is given by*

$$Stir_q(\alpha,k) \prod_{i=1}^{r} grFrob(R_{\alpha_i}, \mathbf{x}_i; q) \tag{4.36}$$

4.3 Extending statistics to $\mathcal{OSP}_{\alpha,k}$

We noted previously that $\mathcal{OSP}_{\alpha,k}$ can be related to $\mathcal{OMP}_{\alpha,k}$ by the map ρ. We can use ρ to define statistics on $\mathcal{OSP}_{\alpha,k}$ that generalize the statistics we have on $\mathcal{OMP}_{\alpha,k}$. The other thing we will need to do is to look at the subwords consisting only of letters from each batch of values. Let s_i be the function on $\mathcal{OSP}_{\alpha,k}$, that outputs the subword consisting of the ith batch of variables, where the α will be implicit. We note that these subwords do not depend on if we wrote the elements of $\mathcal{OSP}_{\alpha,k}$ as ascent or descent starred permutations.

Definition 4.3.1. *Major index for $\mathcal{OSP}_{\alpha,k}$*

Let σ be an element of $\mathcal{OSP}_{\alpha,k}$, define the major index of σ as follows

$$maj(\sigma) = maj(\rho(\sigma)) + \sum_{i=1}^{r} maj(s_i(\sigma)) \tag{4.37}$$

We can think of this definition as saying that there are two types of descents that we are considering. Descents between different batches, counted by the first term, and descents within each set that are counted by the rest of the terms. By grouping terms based on the value of $\rho(\sigma)$ and $s_i(\sigma)$ we can see that the generating function for maj on $\mathcal{OSP}_{\alpha,k}$ will be the product of the generating function for maj on $\mathcal{OMP}_{\alpha,k}$ times the product of the generating function for maj on $\mathfrak{S}_{\alpha_i}$ for all $1 \leq i \leq r$, that is

$$[\alpha]_q! Stir'_q(\alpha, k). \tag{4.38}$$

We are also interested in the complementary statistic $comaj$ which is defined as

$$comaj(\sigma) = max\{maj(\rho) : \rho \in \mathcal{OSP}_{\alpha,k}\} - maj(\sigma). \tag{4.39}$$

In order to understand $comaj$, it is useful to have a formula for the maximum value of maj on $\mathcal{OSP}_{\alpha,k}$. By the previous paragraph the maximum will be the maximum value of maj on $\mathcal{OMP}_{\alpha,k}$ plus the degree of $[\alpha]_q!$, which is $\sum_{i=1}^{r} \binom{\alpha_i}{2}$. Summing these two quantities together gives that the maximum of maj on $\mathcal{OSP}_{\alpha,k}$ is equal to

$$n(k-1)-\binom{k}{2}. \tag{4.40}$$

Knowing this lets us give the following characterization of $comaj$.

Proposition 4.3.2. *Given an element σ of $\mathcal{OSP}_{\alpha,k}$ written as a descent starred permutation, the comajor index can be calculated as follows:*

$$comaj(w)=\sum_{i\in Asc(w)} i-\sum_{i=1}^{r} maj(s_i(\sigma))+\sum_{i\in Star(w)} u_i, \tag{4.41}$$

where $Asc(w)$ is the set of indices $i\in[n-1]$ that are not descents of $\rho(\sigma)$, in other words $Asc(w)=[n-1]\backslash Des(\rho(w))$, and u_i, as in Definition 4.1.8 is the number of unstarred descents to the right of i in w.

Proof. In order to prove this it is sufficient to show that the sum of this claimed expression for $comaj(w)$ and $maj(w)$ is equal to $n(k-1)-\binom{k}{2}$, the maximal value of maj on $\mathcal{OSP}_{\alpha,k}$. Taking this sum with the third expression for $maj(\rho(\sigma))$ from Definition 4.1.8, the second and third sums of our claimed expression for $comaj(w)$ directly cancel leaving

$$\sum_{i\in Des(\rho(w))} i+\sum_{i\in Asc(w)} i-\binom{n-k+1}{2}=\sum_{i=1}^{n-1} i-\binom{n-k+1}{2}= \tag{4.42}$$

$$\binom{n}{2}-\binom{n-k+1}{2}=\frac{1}{2}(n^2-n-(n-k+1)(n-k))=n(k-1)-\binom{k}{2}, \tag{4.43}$$

where the first equality follows from the fact that $Des(\rho(w))\cup Asc(w)=[n-1]$.

$\square$

4.4 Hilbert Series of $R_{\alpha,k}$

The primary reason that we are interested in the statistic $comaj$ on $\mathcal{OSP}_{\alpha,k}$ is because it gives a description of the Hilbert series of $R_{\alpha,k}$.

Theorem 4.4.1. *The Hilbert series of $R_{n,k}$ is the same as the generating function for $comaj$ on $\mathcal{OSP}_{\alpha,k}$ that is*

$$Hilb(R_{\alpha,k};q) = \sum_{\sigma \in \mathcal{OSP}_{\alpha,k}} q^{comaj(\sigma)} = [\alpha]_q Stir_q(\alpha,k) \tag{4.44}$$

Proof. We will prove this result using a basis for $R_{\alpha,k}$ given by Rhoades in [16]. The basis is defined in two steps. We begin by recursively defining a set $M_{\alpha,k} \subset \mathcal{OSP}_{\alpha,k}$ that contains a representative from each orbit of the $\mathfrak{S}_\alpha$ action, which can be indexed by $\mathcal{OMP}_{\alpha,k}$. For an element $\sigma \in \mathcal{OMP}_{\alpha,k}$, the constuction starts by defining $\sigma^{(0)}$ as the ordered set partition with k empty blocks, we then define $\sigma^{(i)}$ as follows:

- Label the empty blocks of $\sigma^{(i-1)}$ from right to left with increasing integers starting at 0.
- Label the non-empty blocks of $\sigma^{(i-1)}$ from left to right with increasing integers ending at $k-1$.
- Insert the ith batch of values into $\sigma^{(i-1)}$ in the blocks that contain i's in σ, such that increasing values have decreasing labels.

If we were to remove an r from σ, only the last step of this process would change, and the resulting ordered set partition will be the same except that the that value in the rth batch corresponding to the removed r is removed, and the values larger than it are shifted down (thus the relative order of the remaining values are the same).

Let g be the function from $\mathcal{OMP}_{\alpha,k}$ to $M_{\alpha,k}$ that applies this process.

An element $\sigma = (B_1|B_2|\ldots|B_k) \in M_{\alpha,k}$ is then associated to the monomial with exponent sequence $(c_1, c_2 \ldots, c_n)$ defined as follows:

$$c_i := |\{j' > j : \min(B_{j'}) > i\}|, \tag{4.45}$$

if $i = \min(B_j)$ and

$$c_i := |\{j' > j : \min(B_{j'}) > i\}| + (j-1) \tag{4.46}$$

if $i \in (B_j)$ and $i > \min(B_j)$.

Let h be the function from $S_{\alpha,k}$ to these associated monomials. Further let $S_{\alpha,k}$ be the image of h.

Next we define a non-standard action of $\mathfrak{S}_n$ on $\mathbb{Q}[\mathbf{x}_n]$. Let s_i be the adjacent transposition that swaps i and $i+1$, and let m be any monomial that is not divisible by x_i or x_{i+1}, then we define an action of s_i as follows:

$$s_i \circ (x_i^a x_{i+1}^b m) := x_i^b x_{i+1}^{a-1} m \text{ if } a > b \tag{4.47}$$

and

$$x_i^{b+1} x_{i+1}^a m \text{ if } a \leq b \tag{4.48}$$

The set of all monomials obtained by applying an element of $\mathfrak{S}_\alpha$ to an element of $S_{\alpha,k}$ under this non-standard action is the monomial basis for $R_{\alpha,k}$.

Since the action of s_i on a monomial m switches whether the exponents of x_i and x_{i+1} are strictly decreasing or not and simultaneously increases or decreases the total degree of m by 1, for any permutation $\sigma \in \mathfrak{S}_n$ if the exponenet sequence of m is strictly decreasing, then

$$deg(\sigma \circ m) = deg(m) - inv(\sigma). \tag{4.49}$$

If $\sigma \in \mathfrak{S}_\alpha$, then we only need the exponent sequences of each batch of variables to be strictly decreasing, which Rhoades shows is true for all monomials in $S_{\alpha,k}$. Thus if we denote the quantity

$$\sum_{m \in S_{\alpha,k}} q^{deg(m)} \tag{4.50}$$

by $D^S_{\alpha,k}$, and the quantity $deg([\alpha]_q!) = \sum_{i=1}^{r} \binom{\alpha_i}{2}$ by d_α , the Hilbert series of $R_{\alpha,k}$ will be

$$[\alpha]_{q^{-1}}!D^S_{\alpha,k} = q^{d_{\alpha,k}}[\alpha]_q!D^S_{\alpha,k}, \tag{4.51}$$

since $[\alpha]_q!$ is the generating function for *inv* on $\mathfrak{S}_\alpha$.

We want to show that this quantity is equal to

$$[\alpha]_q!Stir_q(\alpha,k). \tag{4.52}$$

This will follow from showing that

$$D^S_{\alpha,k} = Stir_q(\alpha,k)q^{d_{\alpha,k}}. \tag{4.53}$$

This in turn we will prove by showing that $D^S_{\alpha,k}$ satisfies a modified version of the recurrance realtion for $Stir_q(\alpha,k)$. Combining the above relation between $D^S_{\alpha,k}$ and $Stir_q(\alpha,k)$, the recurrance for $Stir_q(\alpha,k)$, and the fact that $d_\alpha - d_{\hat{\alpha}} = \alpha_r - 1$ we get that the recurrance for $D_{\alpha,k}$ that we need is

$$[\alpha_r]_q D_{\alpha,k} = [k-\alpha_r+1]_q q^{\alpha_r-1} D_{\hat{\alpha},k} + q^{n-k+\alpha_r-1}[k]_q D_{\hat{\alpha},k-1}. \tag{4.54}$$

The proof of this will be similar to our proof for the recurrance for $D^{maj}_{\alpha,k}$. We will start by giving labeling for the bijection f from Proposition 4.1.5. Label the r's in an (α,k)-ordered multiset partition σ, from 0 to $\alpha_r - 1$ as follows:

- Starting with 0, label the r's in non-singleton blocks from right to left.
- Then label r's in singleton blocks from left to right ending at $\alpha_r - 1$.

Then the weight of an element $(j,\sigma) \in \{0,1,\ldots,\alpha_r - 1\} \times \mathcal{OMP}_{\alpha,k}$ is $j + deg(h(g(\sigma))$.

For elements of $\mathcal{OMP}_{\hat{\alpha},k}$ label the blocks that do not contain an r from left to right starting at 0. The weight of an element $(j,\sigma) \in \{0,1,\ldots,k-\alpha_r\} \times \mathcal{OMP}_{\hat{\alpha},k}$ is given by $j + deg(h(g(\sigma)) + \alpha_r - 1$.

For elements of $\mathcal{OMP}_{\hat{\alpha},k-1}$ label the spaces between blocks (including before the first after the last) from left to right starting at zero. The weight of an element $(j,\sigma) \in \{0,1,\ldots,k-1\} \times \mathcal{OMP}_{\hat{\alpha},k-1}$ is $j + deg(h(g(\sigma)) + (n-k+\alpha_r-1)$.

These choices of weights make it so that if we show that the bijection with this choice of labellings is weight preserving, then the equation is true.

Let $(j,\sigma) \in \{0,1,\ldots,\alpha_r - 1\} \times \mathcal{OMP}_{\alpha,k}$, and let $f(j,\sigma) = (j',\sigma')$. Further let ℓ be the value in the rth batch that corresponds to the r labelled j, and let the block containing that r be the i_jth block. Next consider the case that the i_jth block is not a singleton. Since the only blocks in $g(\sigma)$ to the right of the i_jth block with minimums greater than ℓ, will be those that correspond to blocks equal to $\{r\}$. Thus removing the r labelled j from σ will decrease the degree of the associated monomial by $i_j - 1$ plus the number of blocks to the right of the i_jth block that is equal to $\{r\}$. The label j is the number of non-singleton blocks to the right of the i_jth block that contain an r, thus $deg(h(g(\sigma)) - deg(h(g(\sigma')) + j$ is i_j plus the number of blocks to the right of the i_jth block that contain an r. Since this expression counts all blocks (other than the i_jth) that contain r's this can be reexpressed as $\alpha_r - 1$ plus the number of blocks to the left of the i_j block that do not contain r. That last term is exactly j', so that $deg(h(g(\sigma)) - deg(h(g(\sigma')) + j = \alpha_r - 1 + j'$ which is eqivalent to

$$j + deg(h(g(\sigma)) = j' + deg(h(g(\sigma')) + \alpha_r - 1, \tag{4.55}$$

which show that this bijection is weight preserving in this case.

Next consider the case that the i_jth block of σ is equal to $\{r\}$. Then removing that block redues the index of each index of each block to the right of the i_jth block by 1, which reduces every non-minimal exponent of those blocks by 1. Further ℓ is larger than every element in every block to its left in $g(\sigma)$, so removing it decreases all of those exponents by 1. Finally the exponent of x_ℓ, which is being removed, is the number of blocks equal to $\{r\}$ in σ to the right of the i_jth block. Adding these terms together give $n-1$ plus the number of blocks to the right of the i_jth block that are not equal to $\{r\}$. The value of j is then $\alpha_r - 1$ minus the number of blocks to the right of the i_jth block that are equal to $\{r\}$. Thus

$$deg(h(g(\sigma)) - deg(h(g(\sigma')) + j = n - 1 + \alpha_r - 1 - (k - i_j). \tag{4.56}$$

The index j' on the other hand is equal to $i_j - 1$, thus

$$n - k + \alpha_r - 1 + j' = deg(h(g(\sigma)) - deg(h(g(\sigma')) + j \tag{4.57}$$

which implies that

$$j + deg(h(g(\sigma)) = j' + deg(h(g(\sigma')) + (n - k + \alpha_r - 1), \tag{4.58}$$

so that in this case as well the bijection is weight preserving. Thus the bijection is weight preserving, and thus the result is shown.

□

From the definition of $R_{\alpha,k}$ it is clear that the Hilbert series will be invariant under permuting the entries of α, and our earlier claim that $Stir_q(\alpha,k)$ and $Stir'_q(\alpha,k)$ are invariant under reordering the entries of α is justified.

4.5 Descent Monomial Basis

We will now give a (conjectural) descent-type monomial basis that generalizes Garsia-Stanton descent monomial basis of R_n and the (n,k)-descent monomial basis of $R_{n,k}$. First define a partial order $\prec_\alpha$ on $[n]$ by $i \prec_\alpha j$ if $i < j$ and x_i and x_j are in different batches of variables for $R_{\alpha,k}$. Then for a permutation $\pi \in \mathfrak{S}_n$, written in one-line notation as $\pi_1\pi_2\ldots\pi_n$, define an α-descent of π as an index $i < n$ such that $\pi_i \prec_\alpha \pi_{i+1}$, and define an α-ascent of π as any index $i < n$ that is not an α-descent. Let $Des_\alpha(\pi)$ denote the set of α-descents of π and let $Asc_\alpha(\pi)$ denote the set of all α-ascents of π.

Given a permutation $\pi \in \mathfrak{S}_n$, define the monomial $gs_\alpha(\pi)$ as follows:

$$gs_\alpha(\pi) := \prod_{i \in Asc_\alpha(\pi)} \prod_{j=1}^{i} x_{\pi_j} \tag{4.59}$$

Next, for the ith batch of variables let $\pi^i = \pi^i_1\pi^i_2\ldots\pi^i_{\alpha_i}$ be the subword of $\pi_1\ldots\pi_n$ containing only indices corresponding to the i batch of variables. Then define

$$\overline{gs_\alpha(\pi)} := gs_\alpha(\pi) \cdot (\prod_{i=1}^{r} \prod_{j \in Des(\pi^i)} \prod_{\ell=1}^{j} x_{\pi^i_\ell})^{-1}. \tag{4.60}$$

The set of all such monomials $\overline{gs_\alpha(\pi)}$ is then the conjectural basis.

It is easy to show that the set of all monomial $\overline{gs_\alpha(\pi)}$ have the correct degrees in order to be a basis, and by computation for small values they do seem to form a basis.

Chapter 5

Iwahori-Hecke Quotients

Iwahori-Hecke algebras are generalizations of the group algebras of Coxeter groups. One way of constructing them is as follows. Let (W,S) be a Coxeter system, that is W is a Coxeter group and $S = \{s_1, s_2, \dots, s_\ell\}$ are generators with the relations $(s_i s_j)^{m_{ij}}$ for some symmetric matrix $M = \{m_{ij}\}$ of positive integers. Next let $\mathbb{F}(q)$ be the field of rational function in an indeterminate q over a field $\mathbb{F}$ of characteristic 0. Then the (one-parameter) Iwahori-Hecke algebra $I_{(W,S)}(\mathbb{F}(q))$ can be defined as the unital, commutative $\mathbb{F}(q)$-algebra generated by elements T_i, for $i \in [\ell]$ subject to the following relations where we use 1 to denote the identity element of the algebra:

- $T_i T_j T_i \dots = T_j T_i T_j \dots$ for $1 \leq i \neq j \leq \ell$ where each side has m_{ij} terms
- $(T_i + 1)(T_i - q) = 0$ for $i \in [\ell]$

Sometimes the second relation here is replaced by the non-standard relation $T_i^2 = (1-q)T_i + q$, which is equivalent to setting T_i to $-T_i$.

This is a q-deformation of the group algebra of the Coxeter group W since when q goes to 1, the first of the above relations reduces to the usual braid relations, and the second reduces to the relation $T_i^2 = 1$ which is the usual involution condition for generators of Coxeter groups. For the specific case that the Coxeter group is the symmetric group $\mathfrak{S}_n$, which will mean that $\ell = n-1$, and $m_{ij} = 3$ for $|i-j| \leq 1$, and $m_{ij} = 2$ otherwise, there are a number of actions of this Iwahori-Hecke algebra. We will denote this Iwahori-Hecke algebra by $H_n(q)$. We wll give two such actions. Both of them will be defined using the divided difference operator ∂_i. The operator ∂_i acts on a polynomial f by

$$\partial_i \circ f := \frac{f - s_i \circ (f)}{x_i - x_{i+1}} \tag{5.1}$$

where s_i is the permutation that swaps i and $i+1$ and is acting on f by swapping x_i and x_{i+1}. Next an isobaric version of this operator π_i can be defined by

$$\pi_i \circ f := \partial_i \circ (x_i f). \tag{5.2}$$

A modified version of this action can be given by

$$\bar{\pi}_i \circ f := \pi_i \circ f - f. \tag{5.3}$$

Finally we can define an action of T_i by

$$T_i \circ f := q s_i \circ f + (1-q)\bar{\pi}_i \circ f. \tag{5.4}$$

These operators satisfy the above relations, so they define an action of $H_n(q)$. Another action of $H_n(q)$ (taken with the non-standard second relation) can be defined by

$$T_i \circ f = \partial_i \circ (x_i f) - q x_i \partial_i \circ f. \tag{5.5}$$

Again since these operators satisfy the (non-standard) relations they form an action of $H_n(q)$ (and if we wanted we could multiply it by -1 to make it satisfy the standard relations).

Since these Iwahori-Hecke algebra actions generalizes the action of the symmetric group algebra on $\mathbb{Q}[\mathbf{x}_n]$, it is natural to ask how we can generalize the $\mathfrak{S}_n$-modules that are defined as quotients of this polynomial ring (such as the coinvariant algebra or the Tanisaki quotients) to $H_n(q)$ modules that are defined as quotients of $\mathbb{Q}(q)[\mathbf{x}_n]$. Most of the steps in the method that Garsia and

Procesi used to define $\mathfrak{S}_n$ ideals still works in this setting. Given a finite set of points X in $\mathbb{Q}(q)^n$, we can consider the ideal of polynomials in $\mathbb{Q}(q)[\mathbf{x}_n$ that vanishes on X. Additionally we can take the ideal of top degree of polynomials that vanish on X to create a homogeneous ideal that gives rise to a graded quotient. The point at which it falls apart is that these actions of $H_n(q)$ on $\mathbb{Q}(q)[\mathbf{x}_n]$ do not arise from an action of $H_n(q)$ on $\mathbb{Q}(q)^n$, so there is no meaningful way in which the pointset X can be invariant under the action of $H_n(q)$. The way in which we can address this is by taking a point-set X in $\mathbb{Q}^n$, that is invariant under the action of $\mathfrak{S}_n$, and performing a q-deformation of X to create a pointset $X^{(q)}$ that is in $\mathbb{Q}(q)^n$. Generalizing a deformation used for a specific pointset used by Huang, Rhoades, and Scrimshaw [14], we define this deformation as follows:

Definition 5.0.1. *Given a pointset $X \subset \mathbb{Q}^n$, let $t = (t_1, t_2 \ldots, t_n)$ be a point in X. Then define the point $t^{(q)}$ in $\mathbb{Q}(q)^n$, to be $(q^{\alpha_1}t_1, q^{\alpha_2}t_2, \ldots q^{\alpha_n}t_n)$, where α_i is equal to $|\{j < i : t_j = t_i\}|$. In words α_i is the number of entries of t that come before t_i and are equal to t_i. The pointset $X^{(q)}$ is then defined to be the set of all points $t^{(q)}$ where t is a point of X, in symbols*

$$X^{(q)} := \{t^{(q)} : t \in X\}. \tag{5.6}$$

Since the individual entries of points $t^{(q)}$ differ from t by factors of powers of q, setting $q = 1$ recovers X, so that $X^{(q)}$ is a q-deformation of X. The points $t^{(q)}$ have been defined to have two particular properties that we will use and which we give in the following lemma.

Lemma 5.0.2. *If $t_i \neq t_{i+1}$, then $(s_i \circ t)^{(q)} = s_i \circ (t^{(q)})$, where s_i is the permutation that is swapping i and $i+1$. If $t_i = t_{i+1}$, then $t^{(q)}_{i+1} = qt^{(q)}_i$.*

Proof. This is because the number of entires to the left of and equal to t_i or t_{i+1} will not change by swapping them since they are unequal, and all other entries that were to their left will remain to their left. This follows from the fact that we will have that $\alpha_{i+1} = 1 + \alpha_i$, since the number of entries to the left and equal to t_{i+1} will exactly t_i, and all of the entires to the left of and equal to t_i. These two properties are what allow us to prove the following result

$\square$

Proposition 5.0.3. *Let X be a finite pointset in $\mathbb{Q}^n$ that is invariant under the action of $\mathfrak{S}_n$, then the ideal $\mathbb{T}(X^{(q)})$ consisting of the top degrees of polynomial in $\mathbb{Q}(q)[\mathbf{x}_n]$ that vanish on $X^{(q)}$ is invarinat under the two actions of $H_n(q)$ that we have given above.*

Proof. To show this we will show that the ideal $I(X^{(q)})$ consisting of polynomials in $\mathbb{Q}(q)[\mathbf{x}_n]$ that vanish on $X^{(q)}$ are invariant under our two actions. Then since these actions act in a homogeneous manner this will prove the result for the top degree ideal. The way in which we will show that $I(X^{(q)})$ is invariant is by showing that for any polynomial $f \in I(X^{(q)})$, any index $i \in [n-1]$, and any point $t^{(q)} \in X^{(q)}$, that $(T_i \circ f)(t^{(q)}) = 0$. This will imply that $T_i \circ f \in I(X^{(q)})$ for any T_i, and thus since the set of T_i generate $H_n(q)$ this will prove the result.

We break this into three cases based on the values of t_i and t_{i+1} in the point $t \in X$. If $t_i \neq t_{i+1}$, then applying s_i will commute with the process of our q-deformation, that is

$$(s_i \circ t)^{(q)} = s_i \circ (t^{(q)}). \tag{5.7}$$

This is because moving two different values past each other will not affect the power of q by which we are multiplying. Then since X is invariant under the action of $\mathfrak{S}_n$, $s_i \circ t \in X$, so that $(s_i \circ t)^{(q)} = s_i \circ (t^{(q)}) \in X^{(q)}$. Therefore $f(t^{(q)}) = f(s_i \circ t^{(q)}) = 0$ since these two points are in $X^{(q)}$ and f vanishes on that pointset. The first action is then

$$(T_i \circ f)(t^{(q)}) = [q s_i \circ f + (1-q)\bar{\pi}_i(f)](t^{(q)}) = \tag{5.8}$$

$$q f(s_i \circ t^{(q)}) + (1-q)[\frac{q^{\alpha_i} t_i \cdot f(t^{(q)}) - q^{\alpha_{i+1}} t_{i+1} \cdot f(s_i \circ (t^{(q)}))}{q^{\alpha_i} t_i - q^{\alpha_{i+1}} t_{i+1}} - f(t^{(q)})] = \tag{5.9}$$

$$q \cdot 0 + (1-q)[\frac{q^{\alpha_i} t_i \cdot 0 - q^{\alpha_{i+1}} t_{i+1} \cdot 0}{q^{\alpha_i} t_i - q^{\alpha_{i+1}} t_{i+1}} - 0] = 0 \tag{5.10}$$

where the denominator is non-zero since $t_i \neq t_{i+1}$ (which is important since we would not be able to evaluate the polynomial in this manner if the denominator was zero). Similarly for the second action

$$(T_i \circ f)(t^{(q)}) = [\partial_i \circ (q^{\alpha_i} t_i f) - q(q^{\alpha_i} t_i)(\partial_i \circ f)](t^{(q)}) = \tag{5.11}$$

$$\frac{q^{\alpha_i} t_i \cdot f(t^{(q)}) - q^{\alpha_{i+1}} t_{i+1} \cdot f(s_i \circ (t^{(q)}))}{q^{\alpha_i} t_i - q^{\alpha_{i+1}} t_{i+1}} - q^{\alpha_i+1} t_i \frac{f(t^{(q)}) - f(s_i \circ t^{(q)})}{q^{\alpha_i} t_i - q^{\alpha_{i+1}} t_{i+1}} = \tag{5.12}$$

$$\frac{q^{\alpha_i} t_i \cdot 0 - q^{\alpha_{i+1}} t_{i+1} \cdot 0}{q^{\alpha_i} t_i - q^{\alpha_{i+1}} t_{i+1}} - q^{\alpha_i+1} t_i \frac{0 - 0}{q^{\alpha_i} t_i - q^{\alpha_{i+1}} t_{i+1}} = 0, \tag{5.13}$$

where again the denominators are non-zero. This completes the first case.

The second case is that $t_i = t_{i+1} \neq 0$. In this case we will not have that $(s_i \circ t)^{(q)} = s_i \circ (t^{(q)})$, so we will only know that $f(t^{(q)}) = 0$ and we will not know anything about $f(s_i \circ t^{(q)})$. Instead we will have that have that the i and $(i+1)$th entries of $t^{(q)}$ differ by exactly a factor of q, in other words that $\alpha_{i+1} = \alpha_i + 1$. Using this we get that the first action of $H_n(q)$ will act as follows:

$$(T_i \circ f)(t^{(q)}) = [q s_i \circ f + (1-q)\bar{\pi}_i(f)](t^{(q)}) = \tag{5.14}$$

$$q f(s_i \circ t^{(q)}) + (1-q)[\frac{q^{\alpha_i} t_i \cdot f(t^{(q)}) - q^{\alpha_i+1} t_i \cdot f(s_i \circ (t^{(q)}))}{q^{\alpha_i} t_i - q^{\alpha_i+1} t_i} - f(t^{(q)})] = \tag{5.15}$$

$$q f(s_i \circ t^{(q)}) + (1-q)[\frac{q^{\alpha_i} t_i (0 - q f(s_i \circ (t^{(q)}))}{q^{\alpha_i} t_i (1-q)} - 0)] = \tag{5.16}$$

$$q f(s_i \circ t^{(q)}) - q f(s_i \circ t^{(q)}) = 0. \tag{5.17}$$

Similarly for the second action we have in this case that

$$(T_i \circ f)(t^{(q)}) = [\partial_i \circ (q^{\alpha_i} t_i f) - q(q^{\alpha_i} t_i)(\partial_i \circ f)](t^{(q)}) = \tag{5.18}$$

$$\frac{q^{\alpha_i} t_i \cdot f(t^{(q)}) - q^{\alpha_i+1} t_i \cdot f(s_i \circ (t^{(q)}))}{q^{\alpha_i} t_i - q^{\alpha_i+1} t_i} - q^{\alpha_i+1} t_i \frac{f(t^{(q)}) - f(s_i \circ t^{(q)})}{q^{\alpha_i} t_i - q^{\alpha_i+1} t_i} = \tag{5.19}$$

$$\frac{q^{\alpha_i} t_i (0 - q f(s_i \circ (t^{(q)}))}{q^{\alpha_i} t_i (1-q)} - q^{\alpha_i+1} t_i \frac{0 - f(s_i \circ t^{(q)})}{q^{\alpha_i} t_i (1-q)} = \tag{5.20}$$

$$\frac{-qf(s_i \circ t^{(q)})}{1-q} + q\frac{f(s_i \circ t^{(q)})}{1-q} = 0. \tag{5.21}$$

This completes the second case.

In these last two calculations we are canceling factors of t_i from our fractions, which is why we needed to have $t_i = t_{i+1} = 0$ as a separate case. In this case the only monomials in f that could contribute a non-zero term to $f(t^{(q)})$ would be monomials with no powers of x_i or x_{i+1}. For both actions of $H_n(q)$, T_i act by multiplication by a constant on such monomials (multiplication by q for the first action and 1 for the second). Further for both of these actions T_i acts homogeneously in terms of the total degree of x_i and x_{i+1} which mean that the image of monomials that do contain some power of x_i or x_{i+1} will be a sum of monomials that all contain some power of x_i or x_{i+1}. Therefore the non-zero terms in $(T_i \circ f)(t^{(q)})$ will be the same (up to a constant) as those in $f(t^{(q)})$, so that $f(t^{(q)}) = 0$, which shows the third case. Therefore since our cases cover all possible values for t_i and t_{i+1} we are done.

□

This result allows us to define an Iwahori-Hecke version of any $\mathfrak{S}_n$-module that can be defined by the method of Garsia and Procesi. Specifically we can do this for Tanisaki quotients which are what Garsia and Procesi were originally looking at [8]. The Tanisaki quotient can be defined as the quotient of $\mathbb{Q}[\mathbf{x}_n]$ by the ideal $\mathbb{T}(X_\lambda)$ where λ is a partition of n, and X_λ is the $\mathfrak{S}_n$ orbit of a point of the form $(a_1^{\lambda_1}, a_2^{\lambda_2}, \ldots, a_k^{\lambda_k})$ where k is the number of parts of λ, a_i are all distinct elements of $\mathbb{Q}$, and the exponents are denoting multiplicities. From the above result the quotient of $\mathbb{Q}(q)[\mathbf{x}_n]$ by $\mathbb{T}(X_\lambda^{(q)})$ under either action will be a Iwahori-Hecke version of the Tanisaki quotient.

While this approach gives us a description of our ideal with basically no effort, the description that it gives is not particularly nice, and the ideal will thus not be particularly easy to work with. The classical Tanisaki ideals are generated by elementary symmetric polynomials in partial sets of variables. We will prove that an appropriate q-deformation of these partial elementary symmetric functions form a generating set for $\mathbb{T}(X_\lambda)$.

Definition 5.0.4. *Given a set of variables $S \subset \mathbf{x}_n$, let $S = \{x_{i_1}, x_{i_2}, \ldots x_{i_k}\}$ where $i_1 < i_2 < \ldots i_k$. Then* *define $S^{(q)}$ as*

$$S^{(q)} := \{q^{\alpha_1} x_{i_1}, q^{\alpha_2} x_{i_2}, \dots q^{\alpha_k} x_{i_k}\}, \tag{5.22}$$

where α_i is defined recusively by

$$\alpha_1 = 0 \tag{5.23}$$

and

$$\alpha_i = \alpha_i - (\alpha_{i+1} - \alpha_i - 1) = 2\alpha_i - \alpha_{i+1} + 1 \tag{5.24}$$

for $i > 1$. The idea of this definition is that we are multiplying the variables in our set by powers of q, where we start with q^0, and then decrease the power by how many variables we are skipping at each step.

Example 5.0.5. *If we take*

$$S = \{x_1, x_3, x_6, x_7\} \tag{5.25}$$

we would have that

$$S^{(q)} = \{x_1, q^{-1} x_3, q^{-3} x_6, q^{-3} x_7\} \tag{5.26}$$

Theorem 5.0.6. *Let*

$$d_k(\lambda) := \sum_{i=n-k+1}^{n} \lambda'_i \tag{5.27}$$

where λ' is the partition conjugate to λ and we take λ'_i to be zero if i is larger than the number of parts of λ'. Then the ideal $\mathbb{T}(X_\lambda^{(q)})$ is generated by elementary symmetric functions $e_r(S^{(q)})$ where $S \subset \mathbf{x}_n$, $|S| = k$, and $k \geq r > k - d_k(\lambda)$.

Proof. We first show that this set of partial elementary symmetric functions are indeed in $\mathbb{T}(X_\lambda^{(q)})$. We start with the case that $S = \{x_1, x_2, \dots, x_k\}$. The fact that $e_r(S^{(q)})$ is in $\mathbb{T}(X^{(q)})$ by directly following the proof of this statement for the classical case by Garsia and Procesi in [8].

Then we consider the first action of T_i on $e_r(S^{(q)})$ for an S such that x_i is the variable with the largest index in S. We then claim that

$$T_i \circ (e_r(S^{(q)})) = qe_r((s_i \circ S)^{(q)}). \tag{5.28}$$

In other words acting by T_i gives us the desired partial elementary symmetric function (up to a factor of q) where we have replaced x_i with x_{i+1}. To see this we note that monomials m in $e_r(S^{(q)})$ fall into two types, either m contains a single x_i or it contains no x_i. If m contains a single power of x_i then $T_i \circ m = s_i \circ m$. If m does not contain x_{i+1}, then

$$T_i \circ (m) = qm = qs_i \circ m. \tag{5.29}$$

In other words the monomials without an x_i get an extra factor of q, and those with an x_i do not.

By the definition of our q deformation of variables the coefficient of x_i in $S^{(q)}$, will be larger than the coefficient of x_{i+1} in $(s_i \circ S)^{(q)}$ by a factor of q. Thus if we compare the coefficient of a monomial m appearing in $e_r(S^{(q)})$ to the coefficient of the monomial $s_i \circ m$ in $e_r((s_i \circ S)^{(q)})$ we can see that coefficient will be the same if x_i is not in m, and the power of q will be 1 larger in m if x_i is in m. Thus in going from the coefficient of $s_i \circ m$ in $e_r((s_i \circ S)^{(q)})$ to the coefficient of m in $e_r(S)^{(q)}$ and then to the coefficient of $s_i \circ (m)$ in $T_i \circ e_r(S^{(q)})$, we multiply by q in either the first of second step (depending on if the monomial has a x_i or not) so that we get

$$T_i \circ (e_r(S^{(q)})) = qe_r((s_i \circ S)^{(q)}). \tag{5.30}$$

Then since $\mathbb{T}(X^{(q)})$ is invariant under the action of T_i (and under multiplication by q) and we have that $e_r(\{x_1, x_2, \ldots, x_k\})$ is in $\mathbb{T}(X^{(q)})$, by repeatedly applying the above claim we can get that $e_r(S^{(q)})$ (for r and S as specificed in the theorem) is in $\mathbb{T}(X^{(q)})$. Therefore we have that the claimed generating set is in the ideal.

Since taking q to 1 recovers the generating set forthe ideal I_λ, by results in [10] the dimension of the quotient by this set of partial elementary symmetric functions will be equal to the dimension of R_λ, and thus they will generate $\mathbb{T}(X_\lambda^{(q)})$, and the result is shown.

$\square$

Chapter 6

0-Hecke Tanisaki Harmonics

In the previous chapter we considered the Iwahori-Hecke algebras which were defined in terms of an indeterminate q. As we noted if q is set to 1 then the Iwahori-Hecke algebra reduces to the group algebra of the underlying Coxeter group. If q is set to a different generic constant, we get a deformation of the group algebra of the Coxeter group, but the representation theory is essentially the same as that of the Coxeter group, and is thus not particularly interesting. However if we set $q = 0$, then the algebra changes drastically. Looking specifically at the case where we take $\mathfrak{S}_n$ as our Coxeter group the generating relations reduce to

1. $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ for $1 \leq i < n-1l$
2. $T_iT_j = T_jT_j$ for $|i-j| \geq 2$
3. $T_i^2 = -T_i$ for $i \in [n-1]$

We will denote this algebra by $H_n(0)$, and refer to it as the 0-Hecke algebra.

As an example of an action of $H_n(0)$, this algebra acts on $\mathfrak{S}_n$ by "bubble sorting," by which T_i acts on a permutation σ by switching σ_i and σ_{i+1} if $\sigma_i > \sigma_{i+1}$, and does nothing if $\sigma_i < \sigma_{i+1}$. Unlike the representation theory for $H_n(q)$ with generic q, the representations of $H_n(0)$ are no longer semi-simple. The irreducible representations of $H_n(0)$ are all 1-dimensional. By the relation $T_i^2 = -T_i$ it is easy to see that for a 1-dimensional representation T_i will either act by multiplication by 0 or -1. Further it can be shown that these choices of how each T_i acts completely determines the representation, so that there are 2^{n-1} irreducible representations that have a natural way of being indexed by subsets of $[n-1]$. Using the fact that bases for quasisymmetric functions are also indexed by subsets of $[n-1]$ a generalization of the Frobenius image of a representation of $H_n(0)$ can be defined by having the irreducble representation indexed by a subset be mapped to the corresponding Gessel fundamental quasisymmetric function F_S [11] which is defined as

$$F_S(\mathbf{x}) := \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_n} \prod_{j=1}^{n} x_{i_j} \tag{6.1}$$

where the inequality $i_j \leq i_{j+1}$ is strict if $j \in S$.

For actions of $H_n(0)$ on $\mathbb{Q}[\mathbf{x}_n]$ we could take the actions of $H_n(q)$ from the previous chapter and set q to zero, but the method of deforming pointsets to define ideals falls apart. This is because setting q equal to zero in our pointset deformation causes the process to no longer be injective, and thus the resulting ideals would not have the correct dimensions. However we can consider a slight modification of a "transfer action" of $H_n(0)$ defined by Huang [13] which acts as follows:

$$T_i \circ (m x_i^{\alpha} x_{i+1}^{\beta}) := \begin{cases} -m x_i^{\alpha} x_{i+1}^{\beta} & \alpha > \beta \\ 0 & \alpha = \beta \\ m x_i^{\beta} x_{i+1}^{\alpha} & \alpha < \beta \end{cases}$$

(6.2)

where m is any monomial that is not divisible by x_i or by x_{i+1}.

We will now define a 0-Hecke module that mirrors the combinatorics of the Tanisaki quotient R_λ. Instead of a quotient, our module will be defined as the span of a certain set of monomials, so it will have more of the flavor of harmonics rather than coinvariants. We will start be defining a set of sequences based on a partition λ of size n.

Definition 6.0.1. *For any partition λ consider the set of sequences*

$$(\lambda_1'-1,\lambda_1'-2\ldots,1,0),(\lambda_2'-1,\lambda_2'-2,\ldots,1,0),\ldots(\lambda_\ell'-1,\lambda_\ell'-2,\ldots,1,0) \tag{6.3}$$

where λ' is the conjugate partition to λ and $\ell=\lambda_1$. We call a shuffle of these sequences a λ-sequence where a shuffle of a set of sequence S is a single sequence of length n such that the set of sequences S all appear as disjoint subsequences.

With this definition we can now define our $H_n(0)$ module.

Definition 6.0.2. *Let M_λ be the $\mathbb{Q}$ vector space spanned by the set of all monomials m such that the exponent sequence of m is weakly below some λ sequence.*

We will first prove that M_λ is a $H_n(0)$-module under the transfer action.

Theorem 6.0.3. *The vector space M_λ is invariant under the transfer action of $H_n(0)$, and is thus a $H_n(0)$-module.*

Proof. We will show this by showing that any for any monomial $m\in M_\lambda$, and any generator T_i, that $T_i\circ(m)$ is in M_λ. Let α,β be the exponenets of x_i and x_{i+1} in m respectively. If $\alpha\geq\beta$, then $T_i\circ(m)$ is equal to m up to multiplication by a constant (0 or -1), so that $T_i\circ(m)\in M_\lambda$. If $\alpha<\beta$, then since these are integers we have that $\alpha\leq\beta-1$. Next let c_i and c_{i+1} be the ith and $(i+1)$th elements of a λ sequence that the exponent sequence of m is weakly below. Thus we will have that $\alpha\leq c_i$ and $\beta\leq c_{i+1}$. Then if c_i and c_{i+1} are from the same subsequence, then $c_{i+1}=c_i-1$ so that

$$\alpha\leq\beta-1\leq\beta\leq c_{i+1}, \tag{6.4}$$

and

$$\beta\leq c_{i+1}=c_i-1\leq c_i, \tag{6.5}$$

so that the exponent sequence of the monomial $T_i\circ(m)=s_i\circ(m)$ is weakly below the same sequence. If c_i and c_{i+1} are from different sequences, then the sequence formed by swapping c_i and c_{i+1} will also be a λ sequence so that

$$T_i \circ (m) = s_i \circ (m) \in M_\lambda. \tag{6.6}$$

Thus in either case $T_i \circ (m)$ is in M_λ, so that by linearity M_λ is a $H_n(0)$-module.

□

We next show that dimensionally M_λ the same as R_λ.

Theorem 6.0.4. *The Hilbert series of R_λ and M_λ are equal.*

Proof. We will start by giving a way of associating a λ-sequence $c = (c_1, c_2, \ldots, c_n)$ to any monomial $m \in M_\lambda$ with exponent sequence $d = (d_1, d_2, \ldots, d_n)$ such that c is component-wise weakly above d. We will proceed in a recursive manner. By the definition of a λ-sequence, the first value of any λ-sequence will be equal to $\lambda'_i - 1$ for some choice of i. We claim that we can set c_1 to be the smallest value $\lambda'_i - 1$ that is weakly larger than d_1. To see this let $e = (e_1, e_2, \ldots, e_n)$ be some λ-sequence that is weakly above d. If e_1 is greater than c_1, then we can modify e as follows. Consider a partition of e into the subsequences from the defintion of λ-sequences. Next let s be the subsequence that starts with index 1, and let i be the first index that starts a subsequence such that $e_i = c_1$. We then alter e by removing $e_i = c_1$ and then using the empty space to shift all elements of s that are to the left of i to the right, and then inserting c_1 into the now empty first spot. In this modified sequence every changed entry other than entry 1 has become larger, and by how we chose c_1, $c_1 \geq d_1$, so that the modified sequence starts with c_1 and is weakly above d, so the claim is proven.

Next if we define $\lambda^{(i)}$ to be the partition in which λ_{i+1} is decreased by 1 (or equivantly that is conjugate to λ' with an $i+1$ replaced with an i), we can note that if $(a_1, a_2, \ldots, a_n)$ is a λ-sequence then $(a_2, a_3, \ldots, a_n)$ is a $\lambda^{(a_1)}$ sequence. Thus we can inductively define the remaining values of c by looking at the momonial m with the powers of x_1 removed. Thus we will have the relation that

$$Hilb(M_\lambda; q) = \sum_{i=0}^{n} q^i Hilb(M_{\lambda^{(i)}}; q). \tag{6.7}$$

This (up to slightly different notation) is the same as a recurrence given by Garsia and Procesi[8] for

R_λ, thus we have that

$$Hilb(R_\lambda) = Hilb(M_\lambda). \tag{6.8}$$

$\square$

A partial order can be put on the monomials in M_λ by $m_1 \prec m_2$, if we can get m_2 by applying an element of $H_n(0)$ to m_1. Under this partial order comparable elements will have exponent sequences that are permutations of each other, and maximal elements will have exponent sequences that are weakly decreasing. This will be a partial order since the action of $H_n(0)$ is essentially a sorting operation, which does not give any way of unsorting. Using this partial order we can calculate the multiplicities of irreducible representations in R_λ just by looking at $\langle T_i \circ (m), m\rangle$ for each monomial m, that is by calculating the contribution of m to the trace of the action. For any monomials m the span of monomials strictly or weakly above m will be invariant under this action, so thus by comparing these two spaces we get the claimed result. We will use this in the following theorem to calculate the Frobenius image of M_λ.

Theorem 6.0.5. *The Frobenius image of M_λ is equal to h_λ where h_λ is the complete homogeneous symmetric function.*

Proof. For any monomial $m \in M_\lambda$ the monomial will contribute -1 to the trace of the action of T_i if the exponent of x_i in m is strictly larger than the exponent of x_{i+1} in m, or in other words if i is a descent in the exponent partition of m. Otherwise the contribution will be 0. Thus for each m the contribution to the Frobenius image will be $F_{Des(m)}$ where we let $Des(m)$ denote the descent set of the exponent partition of m. Thus we can express the Frobenius image of M_λ by summing over all monomial $m \in M_\lambda$, that is

$$Frob(M_\lambda) = \sum_{m \in M_\lambda} F_{Des(m)}, \tag{6.9}$$

where we abuse notation slightly to have M_λ denote the set of monomials in M_λ.

We next consider how h_λ expands into the Gessel fundamental basis, which we do by first expressing it in terms of the Schur basis. By the Pieri rule

$$h_\lambda = \sum_T s_{\mathrm{sh}(T)} \tag{6.10}$$

where the sum is over semistandard Young tableaux T with content λ. Then from [11] expanding s_λ into the Gessel Fundamental basis gives

$$s_\lambda = \sum_{T \in SYT(\lambda)} F_{Des(T)}. \tag{6.11}$$

Chaining these two expansions we can get that

$$h_\lambda = \sum_{T,S} F_{Des(S)} \tag{6.12}$$

where the sum is over pairs (T,S) where T is semistandard Young tableau of content λ and S is a standard Young tableau of shape $\mathrm{sh}(T)$. We can perform the reverse RSK algorithm on such pairs of tableaux to get a bijection to W_λ, the set of words of content λ. Further this bijection preserves descent sets, so that if (T,S) is sent to w, then $Des(w) = Des(S)$. Thus we have that

$$h_\lambda = \sum_{w \in W_\lambda} F_{Des(w)}. \tag{6.13}$$

At this point what we need to do is creat a bijection between words with content λ and monomials in M_λ that preserves descent sets. We will define such a function

$$f_\lambda : M_\lambda \to W_\lambda \tag{6.14}$$

by induction on n (where we are abusing notation slightly to have M_λ be the set of monomials in M_λ. For the base case of $n = 1$, the only function that will works will be the one that sends 1 (as a monomial) to 1 (as a word). Then for some λ of size n and some monomial m we consider

$f_{\lambda^{(i)}}(\bar{m}) =: w'$ where i is first entry of the λ sequence for m from the previous theorem, and $\bar{m}$ is the monomial formed by removing x_1 from m (and shifting indices). Let α be the exponent of x_1 in m, then we want to define $f_\lambda(m)$ to be the word formed by a prepending $\alpha+1$ to the beginning of w' (doing this will make inverting this function easy), but in general $\alpha+1$ is not the value by which λ and $\lambda^{(i)}$ differ. Instead they differ by 1 in the $(i+1)$th part, where $\alpha \le i$. We can remedy this as follows.

We first claim that there is a descent set preserving bijection between W_c and $W_{s_k \circ c}$ for any composition c where s_i is acting by swapping the kth and $(k+1)$th parts of the composiiton. This can be shown by just taking the case that $c = (c_1, c_2)$, and $k = 1$ (the general case will follow by applying the map to the subword containing i's and $(i+1)$'s). We define such a function for a word $w \in C_{(c_1,c_2)}$ by taking each consecutive subword $1^{a_1}2^{a_2}$ that is preceded by (or at the start of word) and followed by (or at the end or the word) 21 and replacing it with $1^{a_2}2^{a_1}$. For example

$$2(21)1(21)11122(21)12(21) \mapsto 1(21)2(21)11222(21)12(21), \tag{6.15}$$

where we have put the consecutive substrings of 21 in parenthesis for clarity. This map is defined to leave descents alone, swap the number of 1's and 2's, and is its own inverse so the claim is shown. The way in which we use this map is to apply them to w' for $k = i, i-1, \ldots \alpha - 1$ and then append α to the result. This process is clearly bijective (given a word in W_λ remove the first letter α taking $x_1^{\alpha-1}$, and then apply the above bijections for $k = \alpha-1, \ldots i-1, i$, and then recursively continue with the resulting word), so we just need to verify that it preserves descent sets. To see that we note that the first letter in w' is the exponent of x_2, denote this by β. The letter swapping we do interchanges values from α to i all of which are weakly larger than α, so that there wil be a descent at 1 if and only if $\alpha > \beta$. Therefore the result is shown.

$\square$

Since the Frobenius image of R_λ is also h_λ this immediately gives the following corollary.

Corollary 6.0.6. *The Frobenius images of R_λ and M_λ are equal.*

The natural extension of this result would be to show that the graded Frobenius images of R_λ and M_λ are the same. This is a fairly reasonable result to expect since we have shown that the Hilbert series and ungraded Frobenius images are both equal, and by calculation for small values of λ it holds. Unfortunately, while one might expect that the above bijection could be refined to a graded bijection, this is not particularly easy, mainly since doing so would have to involve the cocharge statistic which does not have a particularly nice description.

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