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NEMATIC LIQUID CRYSTAL SYSTEMS AND MAGNETO-HYDRODYNAMICS SYSTEM:THE PROPERTIES OF THEIR SOLUTIONS

A dissertation submitted in partial satisfaction of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

Mimi Dai

June 2012

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Abstract

Nematic Liquid Crystal Systems and Magneto-Hydrodynamics System:The Properties of Their Solutions

by

Mimi Dai

We study a simplified system for the flow of Nematic Liquid Crystals (LCD) in the cases of non-constant density and constant density with certain initial and boundary conditions. This is a coupled non-parabolic dissipative dynamic system.

For the system with non-constant density in the whole space $\mathbb{R}^3$, we establish the global existence of weak solutions.

For the system with non-constant density on a bounded domain $\Omega \in \mathbb{R}^n$, $n = 2, 3$, we establish the regularity and uniqueness of solutions. We obtain that, in $\mathbb{R}^2$, the global regularity with general data; in $\mathbb{R}^3$, the global regularity with small initial data and a local (short time) regularity with large data. In addition, with more smoothness assumption on initial data, we obtain the uniqueness both for dimension 2 and 3 cases. The main tools are higher order energy estimates (so called Ladyzhenskaya estimate, see [28]), Gagliardo-Nirenberg interpolation inequalities and the frozen coefficient method, see [28].

For the system with constant density in the whole space $\mathbb{R}^3$ with small initial data, we study the large time behavior of solutions and obtain algebraic decay rate for the solutions in energy space. The main ingredient to derive decay is Fourier splitting method which was originally introduced by M. Schonbek to study the large time behavior of solutions to Navier-Stokes equations.

We also study the magneto-hydrodynamics system (MHD). We demonstrate that the solutions to the Cauchy problem for the three dimensional incompressible MHD system can develop different types of norm inflations in $\dot{B}^{-1,\infty}_{\infty}$. Particularly the magnetic field can develop norm inflation in short time even when the velocity remains small and vice verse.

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Part I

Nematic Liquid Crystal Systems

Chapter 1

Introduction

The flows of nematic liquid crystals can be treated as slow moving particles where the fluid velocity and the alignment of the particles influence each other. The hydrodynamic theory of liquid crystals was established by Ericksen [12], [13] and Leslie [32], [33] in the 1960's. As Leslie points out in his 1968 paper : "liquid crystals are states of matter which are capable of flow, and in which the molecular arrangements give rise to a preferred direction". As a result, there is an anisotropy in the mechanical, electrical, magnetic, and optical properties. As the flow of liquid crystals is more like liquid, we use hydrodynamic equations of liquid to describe it.

Let ρ, u and d denote the density of fluid, the velocity of fluid and the mean molecular alignment at a point x in a given sample volumn Ω, respectively. The system satisfies the conservation laws,

- conservation of mass:
$$\frac{d}{dt}\int_{\Omega}\rho dx = 0 \tag{1.0.1}$$
- conservation of linear momentum:
$$\frac{d}{dt}\int_{\Omega}\rho u dx = \int_{\partial\Omega}\tau\cdot n d\sigma + \int_{\Omega}\rho F dx \tag{1.0.2}$$

and the dynamics of the gradient flow:

$$\dot{d} = -\nu\frac{\delta I}{\delta d} \tag{1.0.3}$$

with boundary condition $d|_{\partial\Omega} = d_0|_{\partial\Omega}$ and norm constraint $|d| = 1$.

In the above equations, the physical meaning of each term is given as

- τ denotes stress tensor
- F denotes external body force on per unit mass
- n denotes outward unit normal vector
- $\dot{d} = d_t + u \cdot \nabla d$, which is the so called material derivative
- ν is a constant which indicates how fast decay along the fastest decent direction
- I denotes free energy. In practice, we neglect the contribution from magnetic energy and gravitational energy, and assume that the only contribution is from the distortion energy which has the following simplified form from the interesting mathematical point view

$$I(d) = \int_\Omega W(d)dx = \int_\Omega \frac{1}{2}|\nabla d|^2 dx.$$

- $\frac{\delta I}{\delta d}$ denotes the variation of the energy $I(d)$ with respect to the vector field d.

Notice that the domain Ω is moving with the flow. Denote the domain at initial time by Ω_0, it follows that

$$\int_\Omega \rho(x,t)dx = \int_{\Omega_0} \rho(x(X,t),t)\mathrm{det}J dX$$

where $J = \frac{\partial x}{\partial X}$ is the Jacobi matrix, with $x \in \Omega$ and $X \in \Omega$, and $\dot{x} = u$. Then from (1.0.1), we have

$$\int_{\Omega_0} [\frac{\partial \rho}{\partial t} + \frac{dx}{dt} \cdot \frac{\partial \rho}{\partial x} + (\nabla \cdot u)\rho]\mathrm{det}J dX = 0$$

which gives the continuity equation

$$\rho_t + \mathrm{div}(\rho u) = 0, \tag{1.0.4}$$

due to the fact that $\mathrm{det}J > 0$.

In (1.0.2), using Gauss Theorem

$$\int_{\partial\Omega} \tau \cdot n d\sigma = \int_\Omega \nabla \cdot \tau dx.$$

By proceeding a similar approach, we arrive at the force balance equation (also called Navier-Stokes equation)

$$(\rho u)_t + \mathrm{div}(\rho u \otimes u) = \mathrm{div}\tau + \rho F. \tag{1.0.5}$$

By Leslie-Ericksen's theory (see [32]), we have the constitutive relations

$$\begin{aligned}\tau_{ij} &= W\delta_{ij} - p\delta_{ij} - \frac{\partial W}{\partial d_{k,j}} d_{k,i} + \tau_{ij}^D \\ &= \tau_{ij}^S + \tau_{ij}^D\end{aligned} \tag{1.0.6}$$

where τ^S denotes the static part and τ^D denotes the dissipative parts, p denotes the intrinsic pressure of the fluid, and with the notation $d_{i,j} = \partial d_i/\partial x_j$ for convenience. In this equation,

- W denotes the free energy density. As mentioned above, we assume

$$W = \frac{1}{2}|\nabla d|^2 \tag{1.0.7}$$

- δ_{ij} denotes Kronecker operator, which is defined as

$$\delta_{ij} = 1 \quad \text{if } i = j; \quad \text{otherwise } \delta_{ij} = 0;$$

- for simplicity, we assume

$$\tau^D = \frac{\mu}{2}(\nabla u + (\nabla u)^T) \tag{1.0.8}$$

  with the viscosity constant μ;

- Ericksen-stress $\frac{\partial W}{\partial d_{k,j}} d_{k,i}$ are the counter terms to the transport contributions in all the currents and quasi-currents. This term yields the coupling of the fluid with the direction field of the molecules.

Inserting the equations (1.0.7) and (1.0.8) into the constitutive relation equation (1.0.6), it follows that the equation (1.0.5) reads in components as

$$(\rho u_i)_t + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial}{\partial x_j}((\frac{1}{2}|\nabla d|^2 - p)\delta_{ij} - d_{k,j}d_{k,i} + \tau_{ij}^D)$$

which has a compact form by writing all the gradient terms into a new pressure term

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p = \mu\Delta u - \nabla \cdot (\nabla d \otimes \nabla d) + \rho F, \tag{1.0.9}$$

with the stress tensor $\nabla d \otimes \nabla d$ denoting the 3×3 matrix whose ij-th entry is given by "$\nabla_i d \cdot \nabla_j d$" for $1 \le i, j \le 3$.

To obtain the equation (1.0.3), we minimize the Dirichlet energy

$$\min_{d\in H^1(\Omega;S^2)} \int_\Omega W(d) = \min_{d\in H^1(\Omega;S^2)} \int_\Omega \frac{1}{2}|\nabla d|^2 dx \tag{1.0.10}$$

with Dirichlet boundary condition:

$$d|_{\partial\Omega} = d_0$$

Applying a calculus of variation method, we have

$$-\Delta d + \gamma(x)d = 0, \quad |d| = 1.$$

Multiplying the above equation by d gives that

$$\gamma(x) = -|\nabla d|^2.$$

Therefore, the gradient flow equation can be written as

$$d_t + u\cdot\nabla d = \nu(\Delta d + |\nabla d|^2 d). \tag{1.0.11}$$

Assume that there is no external force F acting on the system and that the fluid is incompressible, we obtain a simplified model for the flow of nematic liquid crystals:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0 \\ \rho u_t + \rho(u\cdot\nabla)u + \nabla p = \mu\Delta u - \lambda\nabla\cdot(\nabla d\otimes\nabla d) \\ d_t + (u\cdot\nabla)d = \nu(\Delta d + |\nabla d|^2 d) \\ \nabla\cdot u = 0 \end{cases} \tag{1.0.12}$$

In the system (1.0.12), the third equation describes the heat flow into sphere. The higher non-linear term $|\nabla d|^2 d$ makes the analysis very difficult. It is hard to get good enough estimate for d from the heat flow equation so that the coupling term $\nabla\cdot(\nabla d\otimes\nabla d)$ has a weak convergence.

In 2D, the system (1.0.12) with a constant density ($\rho = 1$) has been studied in a recent work of F. Lin, J. Lin and C. Wang [34]. With smallness assumption, the authors established the existence of global weak solutions on a bounded smooth domain which are smooth everywhere with possible exceptions of finite singular times. In 3D, the existence of weak solutions is still open.

To relax the non-linear constraint $|d| = 1$, one of the methods is to consider a form of penalization. We add the penalty term $F(d) = \frac{1}{4\sigma^2}(|d|^2 - 1)^2$ into the free energy density W in (1.0.7), and consider the new energy density

$$W(d) = \frac{1}{2}|\nabla d|^2 + \frac{1}{4\sigma^2}(|d|^2 - 1)^2$$

Minimizing the above energy

$$\min_{d \in H^1(\Omega;\mathbb{R}^3)} \int_\Omega \frac{1}{2}|\nabla d|^2 + \frac{1}{4\sigma^2}(|d|^2 - 1)^2 dx$$

yields that the gradient flow equation as

$$d_t + u \cdot \nabla d = \Delta d - \frac{1}{\sigma^2}(|d|^2 - 1)d. \tag{1.0.13}$$

For simplicity, we write

$$f(d) = \frac{1}{\sigma^2}(|d|^2 - 1)d. \tag{1.0.14}$$

Note that $f(d) = \nabla F(d)$. In fact $F(d)$ is called the penalty term of Ginzburg-Landau approximation of the original free energy of the director field with unit length.

Thus we obtain the following approximated nematic liquid crystal system

$$\begin{aligned} \rho_t + \nabla \cdot (\rho u) &= 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p &= \mu \Delta u - \nabla \cdot (\nabla d \otimes \nabla d) \\ d_t + u \cdot \nabla d &= \nu \Delta d - f(d), \\ \nabla \cdot u &= 0. \end{aligned} \tag{1.0.15}$$

In this thesis, we are going to work with the system (1.0.15) with $\mu = \nu = 1$ and σ being a fixed constant in $\Omega \times (0, T)$, where Ω is a domain in $\mathbb{R}^n$, with $n = 2, 3$, or $\Omega = \mathbb{R}^3$, with unknown functions ρ the density, p the pressure, u the velocity and d the director field

- $\rho : \Omega \times [0, T] \to \mathbb{R}$,
- $p : \Omega \times [0, T] \to \mathbb{R}$,
- $u : \Omega \times [0, T] \to \mathbb{R}^n$ and
- $d : \Omega \times [0, T] \to \mathbb{R}^n$.

There is a vast literature on the hydrodynamic of the liquid crystal system. For background we list a few names, with no intention to be complete: [14], [23], [24], [35], [36], [37], [5], [4], [5], [6], [7], [41], [21] and [55]. Particularly,

- on bounded domain in $\mathbb{R}^3$ with non-constant density, in [41] and [21], the global weak existence of solutions to the flow of nematic liquid crystals was obtained;
- on bounded domain, with constant density $\rho = 1$, in [35] the authors proved:
  existence of a global regular solution in 2D,
  existence of a global regular solution with large constant μ or short time regular solution in 3D,
  and a preliminary long time behavior of the regular solutions;
- on bounded domain, with constant density $\rho = 1$, in [55] the author established an algebraic decay for the global regular solutions by applying Łojasiewicz-Simon approach.

The work we have done will be summarized in the following.

In Chapter 2, with a slightly different setting on the initial density ρ_0, we extend the weak solutions on bounded domain into the whole space $\mathbb{R}^3$. The main method we use is similar as in [21]. However, we include this work in the thesis for completeness, since it is an independent work. In this work, we assume that $0 \leq \rho_0(x) \leq M$ for all $x \in \mathbb{R}^3$. When extending to the whole space, it is not reasonable to assume that the density is strictly positive. The fact that the density ρ could be zero (vacuum occurs) makes it difficult to obtain weak solutions on the whole space. We apply a renormalization method to modify the initial data introduced by P. L. Lions [38] and solve successfully this difficulty . We carry out some necessary details on this renormalization procedure in Subsection 2.3.1.

In Chapter 3, we focus on the existence of regular solutions to the system (1.0.15).In light of the regularity results to the pure fluid system established in [2] and [30] it is natural to ask if the regularity results in [35] can be extend to prove the regularity of the solutions for flows of nematic liquid crystals with non-constant fluid density. We give an answer in this chapter. Considering the system (1.0.15) in a bounded domain in $\mathbb{R}^2$ and $\mathbb{R}^3$ with strictly position initial density, we establish the existence of :

- Global classical solutions with large initial data on smooth bounded domains in $\mathbb{R}^2$;
- Global classical solutions with small initial data on smooth bounded domains in $\mathbb{R}^3$;
- Local classical solutions with large initial data on smooth bounded domain in $\mathbb{R}^3$.

The main tools used in our proofs are: a modified Ladyzhenskaya energy method (see [29]), the frozen coefficient method (see [28]) and a standard bootstrapping argument. To obtain the regularity, one key estimate is to show that $u \in L^\infty(0,T;L^p(\Omega))$ for $p > 2$ in 2D and $p > 3$ in 3D. It suffices to obtain the estimate $\nabla u \in L^\infty(0,T;L^2(\Omega))$. We use the higher order energy method (also called Ladyzhenskaya energy method), which consists in obtaining a uniform in time estimate for the following energy

$$\Phi(t)^2 = \|\nabla u(\cdot,t)\|^2_{L^2(\Omega)} + \|\Delta d(\cdot,t)\|_{L^2(\Omega)}$$

Specifically the Ladyzhenskaya energy method we used here is an interesting modification of the one used by F. Lin and C. Liu [35].

Since the density is not constant, we need to use a frozen coefficient method when applying Ladyzhenskaya energy method. The idea of frozen coefficient method is to divide the domain into a finite number of subdomains, in each of which the oscillations of the density function $\rho(x,t)$ are sufficiently small. In 3D, we established estimates that show the density function $\rho(x,t)$ oscillations on small balls are sufficiently small. This follows from using our estimates on the velocity $u \in L^2(0,T;H^2)$. In Appendix 3.6, we work out a simple example to show how the frozen coefficient method can be applied.

The other techniques used involve bootstrapping methods among the three equations in (1.0.15) combined with some standard L^p estimates for parabolic equations. This improves the regularity of solutions (ρ,u,d) and eventually yields a classical solution.

We point out that a self contained proof of existence of weak solutions is given in Appendix 3.5. There are two reasons we do not apply the existence

proof of weak solutions in Chapter 2 directly. The first reason is that, in Chapter 3, we assume the initial density $\rho_0 > 0$ and hence it is not necessary to apply the renormalization method as in Chapter 2. The second reason is that, in the approximation process as in Chapter 2, a viscosity term $\epsilon\Delta\rho$ is added to regularize the first equation of the system (1.0.15) and the term $\epsilon\nabla\rho\cdot\nabla u$ is added to the second equation of (1.0.15). The two terms seem to create difficulties when we establish the Ladyzhenskaya energy estimates for the approximating solutions derived by the Galerkin method. Thus in Appendix 3.5 we sketch a different proof of existence for the global weak solutions to the system (1.0.15) without the introduction of the viscosity term $\epsilon\Delta\rho$ for the density in the equation of the conservation of mass.

In Chapter 4, we establish an asymptotic behavior for the regular solutions to the LCD system with a constant density in the whole space $\mathbb{R}^3$. As mentioned above, on bounded domain, the asymptotic behavior of solutions to the flow of nematic liquid crystals with constant density was studied in [35, 55]. It was shown in [55] that, with suitable initial conditions, the velocity converges to zero and the direction field converges to the steady solution to the following equation

$$\begin{cases} -\Delta d + f(d) = 0, x \in \Omega \\ d(x) = d_0(x), x \in \partial\Omega. \end{cases} \tag{1.0.16}$$

In [55], the Łojasiewicz-Simon inequality in Lemma 2.1 is used to derive the convergence when Ω is a bounded domain. Lack of compactness considerations do not allow us to use similar arguments in the whole space $\mathbb{R}^3$.In this chapter we consider the asymptotic behavior of the solutions to (1.0.15) in the whole space $\mathbb{R}^3$. subject to the additional condition on the direction field which insures that the direction tends to a constant unit vector w_0, as the space variable tends to infinity:

$$\lim_{|x|\to\infty} d_0(x) = w_0. \tag{1.0.17}$$

This simplifies the situation and allows us to obtain the stability without needing the Liapunov reduction and Łojasiewicz-Simon inequality, since w_0 is a non-degenerate steady solution to (4.1.5).

We start from the basic energy estimates (4.2.1) and Ladyzhenskaya estimates (which will be obtained in the whole space in Appendix 4.4) for the system (1.0.15). We then establish the convergence of the direction field d to the constant

steady solution w_0 based on Gagliardo-Nirenberg interpolation techniques. More precisely, the convergence obtained is in $L^p(\mathbb{R}^3)$ for any $p > 1$, with an algebraic decay rate of $(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}$. We then focus on the conservation of momentum equation in (1.0.15). We apply the Fourier splitting technique [48, 49, 52] to obtain the decay of the velocity u with an algebraic decay rate of $(1+t)^{-\frac{1}{4}}$ in $L^2(\mathbb{R}^3)$ norm.

Again, for completeness, we extend the existence of regular solutions on bounded domain to the whole space in Appendix 4.4, based on the arguments in Chapter 3.

We point out that an optimal decay rate $(1+t)^{-\frac{3}{4}}$ for the velocity u in the energy space $L^2(\mathbb{R}^3)$ and the decay of solutions in higher order Sobolev spaces are obtained in a more recent work. To restrict the length of the thesis, this work will not be included here.

Chapter 2

Existence of Global Weak Solutions

2.1 The Settings and Main Result

In this chapter we are interested in the global weak solution to LCD (1.0.15) system in the whole space $\mathbb{R}^3 \times (0,\infty)$ with the following initial conditions:

$$\rho(x,0) = \rho_0(x), \quad 0 \le \rho_0(x) \le \bar{\rho}, \quad \rho_0 \in L^2(\mathbb{R}^3) \tag{2.1.1}$$

$$d(x,0) = d_0(x), \quad |d_0| \le 1 \tag{2.1.2}$$

$$m_0 \doteq \rho u|_{t=0} \in L^2(\mathbb{R}^3), \quad m_0 = 0 \ \text{ a.e. on } \{\rho_0 = 0\} \tag{2.1.3}$$

$$|m_0|^2/\rho_0 \in L^1(\mathbb{R}^3) \tag{2.1.4}$$

and the boundary conditions:

$$\lim_{|x|\to\infty} u(x,t) = 0, \quad \text{for} \ \ t \in (0,\infty) \tag{2.1.5}$$

$$\lim_{|x|\to\infty} [d(x,t) - d_0(x)] = 0, \quad \text{for} \ \ t \in (0,\infty) \tag{2.1.6}$$

The main existence result states as:

Theorem 2.1.1 *The system (1.0.15) with the initial boundary conditions (2.1.1)-(2.1.6) has a global weak solution (ρ,u,d) satisfying, for all $T \in (0,\infty)$, $R \in (0,\infty)$*

$$\rho \in L^\infty(0,T;L^2(B_R)) \tag{2.1.7}$$

$$u \in L^2(0,T;H^1(B_R)) \tag{2.1.8}$$

$$\rho|u|^2 \in L^\infty(0,T;L^1(\mathbb{R}^3)) \tag{2.1.9}$$

$$d \in L^2(0,T;H^2(B_R)) \tag{2.1.10}$$

$$d \in L^\infty(0,T;H^1(B_R)) \tag{2.1.11}$$

and the energy inequality

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} \frac{1}{2}\rho|u|^2 + \frac{1}{2}|\nabla d|^2 + F(d) \\ &+ \int_{\mathbb{R}^3} |\Delta d - f(d)|^2 + |\nabla u|^2 \\ &\leq C(\rho_0, m_0, d_0). \end{aligned} \tag{2.1.12}$$

The existence proof can be given in the following five steps:

- construct an approximated system on a bounded ball $B_R \times [0,T)$ by adding the viscosity term $\eta\triangle\rho$ to the first equation with modified initial data ρ_0^ϵ, u_0^ϵ, and d_0^ϵ;
- for each fixed η and ϵ, find a classical solution of the approximated equation using Faedo-Galerkin method;
- pass to limit by taking η to 0;
- pass to limit by taking ϵ to 0;
- pass to limit by taking R to infinity.

The rest of this chapter is organized as follows. In Section 2.2, we present the maximal principles and a formal energy law satisfied by the system (1.0.15). We also include a standard L^p estimate for the parabolic equation and a so-called Lions-Aubin compactness argument. In Section 2.3, we prove Theorem 2.1.1 following the five steps mentioned above.

2.2 Preliminary

2.2.1 Maximal Principles and Energy Law

We first prove two maximum principles related to the density ρ and the direction vector d respectively.

By the property of transport equations, we have

Lemma 2.2.1 *For all $t \in [0,T]$, $x \in \mathbb{R}^3$,*

$$\underline{\rho} \leq \rho(x,t) \leq \bar{\rho}.$$

Proof Formally, from the first equation in (1.0.15), we observe that if $\beta \in C^1([0,\infty);\mathbb{R})$ with bounded support, $\beta(\rho)$ satisfies

$$\frac{\partial \beta(\rho)}{\partial t} + \operatorname{div}(u\beta(\rho)) = \beta'(\rho)(\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho) = 0.$$

Then, integrating the left hand side over $\mathbb{R}^3$, we have that

$$\frac{d}{dt}\int_{\mathbb{R}^3} \beta(\rho) dx = 0.$$

Hence $\int_{\mathbb{R}^3} \beta(\rho) dx$ is independent of t.

In particular, for any fixed α, γ satisfying $0 \leq \alpha < \gamma < \infty$, we choose $\beta = g_n \in C^1([0,\infty);\mathbb{R})$ such that

$$\begin{cases} g_n(t) = 0 \quad \text{if } t \in [\alpha,\gamma]^C \\ g_n(t) = 1 \quad \text{if } t \in [\alpha + \frac{1}{n}, \gamma - \frac{1}{n}] (\text{take } n \geq \frac{2}{\gamma - \alpha}) \\ 0 \leq g_n \leq 1, \quad \text{otherwise.} \end{cases}$$

Letting $n \to \infty$ to have

$$meas\left\{x \in \mathbb{R}^3 | \alpha \leq \rho(x,t) \leq \gamma\right\} \in [0,\infty]$$

is independent of t. In particular, we have

$$0 \leq \rho(x,t) \leq \|\rho_0\|_{L^\infty}, \quad a.e..$$

□

To be applied to the approximated system (2.3.1), we shall also give a maximum principle in the following case.

Lemma 2.2.2 *If ρ satisfies the equation, for some fixed $\eta > 0$, on $B_R \times [0,T]$ for any fixed ball B_R*

$$\rho_t + u \cdot \nabla \rho = \eta \triangle \rho$$

and

$$\nabla \rho \cdot n|_{\partial B_R} = 0$$

with initial data ρ_0 satisfying

$$0 < \underline{\rho} \leq \rho_0(x) \leq \bar{\rho}, \quad \nabla \rho_0 \cdot n|_{\partial B_R} = 0,$$

then we have

$$\underline{\rho} \leq \rho(x,t) \leq \bar{\rho}$$

for all $t \in [0,T]$, $x \in B_R$.

Proof Suppose $(x_0, t_0) \in \partial B_R \times (0,T]$ is the maximum point of ρ on $B_R \times [0,T]$, then we observe that

$$\rho_t(x_0,t_0) = \eta \triangle \rho(x_0,t_0)$$

since $\nabla \rho = 0$ at x_0. As a function of t, we have $\rho_t \geq 0$ on $\partial B_R \times (0,T]$. On the other hand, on the boundary ∂B_R, we can decompose $\triangle \rho$ as

$$\triangle \rho = \partial_n \partial_n \rho + H \partial_n \rho + \triangle_T \rho,$$

where H is the Hessian, T is the tangential. At the point (x_0,t_0), we have that $\partial_n \partial_n \rho < 0$, and $\triangle_T \rho \leq 0$ since the interior can not contain the maximal point, while $\nabla \rho \cdot n|_{\partial B_R} = 0$ implies $\partial_n \rho|_{\partial B_R} = 0$. Thus, we get $\triangle \rho < 0$ at point (x_0,t_0), which is a contradiction. Therefore, the maximum can only be obtained at $t = 0$. It completes the proof. □

Next, we give a maximum principle for the direction vector d.

Lemma 2.2.3 *Suppose d satisfies the system (1.0.15)with initial d_0 and $|d_0| \leq 1$, then we have*

$$|d(x,t)| \leq 1$$

for all $t \in [0,\infty)$, $x \in \mathbb{R}^3$.

Proof Multiplying the third equation in (1.0.15) by d, we get

$$\frac{1}{2}\frac{d}{dt}|d|^2 + (u\cdot\nabla d)\cdot d = \triangle d\cdot d - \frac{1}{\eta^2}(|d|^2-1)|d|^2.$$

Since,

$$\triangle|d|^2 = \sum_{i=1}^{3}\partial_i\partial_i|d|^2 = 2\triangle d\cdot d + 2|\nabla d|^2,$$

we actually obtain that

$$\frac{1}{2}\frac{d}{dt}|d|^2 + (u\cdot\nabla d)\cdot d = \frac{1}{2}\triangle|d|^2 - |\nabla d|^2 - \frac{1}{\eta^2}(|d|^2-1)|d|^2 \tag{2.2.1}$$

Now we consider the function $|d|^2$. Assume there exists a point $(x_0,t_0)\in B_R\times(0,T]$ (the interior) such that $|d|^2$ obtains the maximum value at this point. Then we have

$$\frac{d}{dt}|d|^2|_{(x_0,t_0)} \geq 0.$$

And $\nabla|d|^2 = 0$ at x_0, which implies $\partial_i|d|^2 = 0$, for $i=1,2,3$. Thus, we can get

$$(u\cdot\nabla d)\cdot d = \sum_{i=1}^{3}\sum_{j=1}^{3}u^i(\partial_i d^j)d^j = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}u^i\partial_i(d^j)^2 = \frac{1}{2}\sum_{i=1}^{3}u^i\partial_i|d|^2 = 0.$$

So we conclude the left hand side of (2.2.1) is nonnegative at (x_0,t_0). While at this maximum point, $\triangle|d|^2\leq 0$. If $|d|>1$ at this maximum point, the right hand side of (2.2.1) is negative. This is a contradiction. Therefore, either we have that the maximum point of $|d|^2$ is at $t=0$ or $|d|<1$ at the maximum point. And both of them imply the desired result in the lemma. □

Next, we introduce the energy law which governs the system (1.0.15).

Lemma 2.2.4 *If (ρ,u,d) is a weak solution to (1.0.15) with the given data ρ_0, u_0 and d_0 in (2.1.1)-(2.1.4), there holds*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\rho|u|^2 + |\nabla d|^2 + 2F(d) + \int_{\mathbb{R}^3}|\triangle d - f(d)|^2 + |\nabla u|^2 \leq 0. \tag{2.2.2}$$

Proof Formally, if we work on the classical solution, we multiply the second equation in (1.0.15) by u and integrate by parts,

$$\int_{\mathbb{R}^3}\rho_t|u|^2 + \frac{1}{2}\rho\frac{d}{dt}|u|^2 - \int_{\mathbb{R}^3}\rho u^i u^j u_i^j = -\int_{\mathbb{R}^3}|\nabla u|^2 - \int_{\mathbb{R}^3}[(\triangle d - f(d))\cdot\nabla d]\cdot u.$$

Multiply the first equation in (1.0.15) by $|u|^2$ and integrate by parts,

$$\int_{\mathbb{R}^3} \rho_t |u|^2 - 2\int_{\mathbb{R}^3} \rho u^i u_i^j u^j = 0.$$

Combining the above two equations to get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \rho|u|^2 = -\int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} [(\triangle d - f(d)) \cdot \nabla d] \cdot u.$$

Then multiply the third equation in (1.0.15) by $-\Delta d + f(d)$ and integrate by parts,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla d|^2 + \int_{\mathbb{R}^3} d_t \cdot \nabla_d F(d) = \int_{\mathbb{R}^3} (u \cdot \nabla d)(\triangle d - f(d)) - \int_{\mathbb{R}^3} (\Delta d - f(d))^2.$$

So the energy identity in (2.2.2) is obtained by combining the last two equations. For weak solution, we have the energy inequality (2.2.2). □

2.2.2 L^p Estimate of Parabolic Equations

In this subsection, we introduce the L^p-estimate for parabolic equations. We start with some definitions.

Definition 2.2.5 *We say that a pair of Banach spaces (E_0, E_1) form a densely injected Banach couple if $E_1 \hookrightarrow E_0$ densely.*
Denote $\mathcal{L}(E_1, E_0)$ the set of linear operators from E_1 to E_0 with (E_0, E_1) being a densely injected Banach couple.

Definition 2.2.6 *We denote by*

$$\mathcal{H}(E_1, E_0)$$

the set of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as a linear operator in E_0 with domain E_1, is the infinitesimal generator of a strongly continuous analytic semigroup $\{e^{-tA}; t \geq 0\}$ on E_0, that is, in $\mathcal{L}(E_0)$.

Assumption 2.2.7 *Assume that $\mathcal{A} \subset \mathcal{H}(E_1, E_0)$ and there are constants $M > 0$ and $\vartheta \in (0, \pi/2)$ such that $\sum_\vartheta \subset \rho(-A)$ and*

$$\|A\|_{\mathcal{L}(E_1,E_0)} + (1 + |\lambda|)^{1-j} \|(\lambda + A)^{-1}\|_{\mathcal{L}(E_0,E_j)} \leq M$$

for $(\lambda, A) \in \sum_\vartheta \times \mathcal{A}$ and $j = 0, 1$.

In particular, Laplace $\triangle$ is in such $\mathcal{A}$ with $E_0 = L^q$, $E_1 = W^{2,q}$, or with $E_0 = W^{-1,q}$, $E_1 = W^{1,q}$, for any $q \in (1, \infty)$. For more detail, see Amann [1].

Definition 2.2.8 *The pair of Banach spaces (E_0, E_1) is said to be interpolation couple if there exists a locally convex space X such that $E_j \hookrightarrow X$, $j = 0, 1$.*

Definition 2.2.9 *Let (E_0, E_1) be an interpolation couple. Given $x \in E_0 + E_1$, define*

$$K(x,t) := K(x,t,E_0,E_1) := \inf\{\|x_0\|_{E_0} + t\|x_1\|_{E_1}; x = x_0 + x_1\}$$

for $t > 0$. Put

$$\|x\|_{\theta,q} := \|t^{-\theta}K(x,t)\|_{L^q(\dot{\mathbb{R}}^+, dt/t)}, \quad 0 < \theta < 1, \quad 1 \le q \le \infty,$$

and

$$(E_0, E_1)_{\theta,q} := (\{x \in E_0 + E_1; \|x\|_{\theta,q} < \infty\}, \|\dot{\|}_{\theta,q})$$

for $0 < \theta < 1$, $1 \le q \le \infty$. Denote $(E_0, E_1)_{\theta,q} = E_\theta$.

Lemma 2.2.10 *With the Assumption 2.2.7, given any $(g, A, f) \in E_{1-1/p} \times \mathcal{A} \times L^p(J; E_0)$ for $p \in (1, \infty)$, the Cauchy problem*

$$v_t + Av = f(x,t), \quad t \in \dot{J}, \quad v(0) = g$$

has a unique solution $v \in L^p(J; E_1) \cap W^{1,p}(J; E_0)$, and

$$\int_J \|v(t)\|_0^p dt + \int_J \|v_t(t)\|_0^p dt + \int_J \|Av(t)\|_0^p dt \le C(\int_J \|f(t)\|_0^p dt + \|g\|_{1-1/p}^p),$$

where C is independent of (g, A, f), $\|\cdot\|_0$ denotes the norm in E_0 and $\|\cdot\|_{1-1/p}$ denotes the norm in $E_{1-1/p}$.

2.2.3 Compactness Arguments

We introduce two compactness theorems which will be used often later when taking convergence of sequences.

Lemma 2.2.11 *[Lions-Aubin,[54]] Let X_0, X and X_1 be Banach spaces such that $X_0 \subset X \subset X_1$. Suppose that the injections $X_0 \to X \to X_1$ are continuous, and the injection from X_0 to X is compact. Let $Y = \{v | v \in L^{\alpha_0}(0,T; X_0), v_t \in L^{\alpha_1}(0,T; X_1)\}$ with norm $\|v\|_Y = \|v\|_{L^{\alpha_0}(0,T;X_0)} + \|v_t\|_{L^{\alpha_1}(0,T;X_1)}$, for some α_0, $\alpha_1 > 1$. Then, the injection from Y to $L^{\alpha_0}(0,T;X)$ is compact.*

Lemma 2.2.12 *[39] Let X and X_1 be Banach spaces and $X \subset X_1$ is compact. Let $Y = \{v | v \in L^\infty(0,T;X), v_t \in L^\alpha(0,T;X_1)\}$ for some $\alpha > 1$. Then, $Y \subset C(0,T;X_1)$ is compact.*

2.3 Existence Proof

In this section, we give the existence proof in five steps as pointed out in Section 2.1.

2.3.1 Construction of the approximated system on $B_R \times [0,T)$ with modified initial data

We first study the existence of solutions to the approximated system

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u) &= \eta \triangle \rho \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \eta \nabla \rho \cdot \nabla u + \nabla p &= \triangle u + \tau(d) \cdot \nabla d \\ d_t + u \cdot \nabla d &= \triangle d - f(d) \\ \nabla \cdot u &= 0 \end{aligned} \tag{2.3.1}$$

in $B_R \times [0,T)$, with boundary conditions

$$u|_{\partial B_R} = 0, \quad \nabla \rho \cdot n|_{\partial B_R} = 0, \quad d|_{\partial B_R} = d_0(x) \tag{2.3.2}$$

and modified initial conditions ρ_0^ϵ, u_0^ϵ, and d_0^ϵ defined in the following.

Definition of ρ_0^ϵ: let

$$\tilde{\rho_0} = \begin{cases} \rho_0, & \text{in } B_{R-\epsilon} \\ 1, & \text{in } B^c_{R-\epsilon} \end{cases}$$

and

$$\tilde{\rho_0}^\epsilon = \tilde{\rho_0} * \omega_{\epsilon/2}|_{B_R}$$

where ω_ϵ is defined as following

$$\omega(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases}$$

with $\int_{\mathbb{R}^3}\omega(x)dx = 1$ and let

$$\omega_\epsilon(x) = \frac{1}{\epsilon^3}\omega(\frac{x}{\epsilon}).$$

Then we define

$$\rho_0^\epsilon = \tilde{\rho_0}^\epsilon + \epsilon.$$

Thus following the steps of [38] we have

Lemma 2.3.1 *The modified initial density $\rho_0^\epsilon \in C_0^\infty(B_R)$ satisfies*

$$0 < \epsilon \le \rho_0^\epsilon \le C, \quad \nabla\rho_0^\epsilon \cdot n|_{B_R} = 0$$

where C is a constant independent of ϵ and n denotes the unit normal vector, and

$$\rho_0^\epsilon \to \rho_0 \quad in\ L^p(B_R)$$

for any $p \in [1, \infty)$.

Definition of m_0^ϵ and u_0^ϵ: let

$$\bar{m}_0^\epsilon = (m_0\rho_0^{-1/2})_\epsilon(\rho_0^{1/2})_\epsilon,$$

where

$$(\rho_0^{1/2})_\epsilon = \tilde{\rho}_0^{1/2} * \omega_{\epsilon/2}|_{B_R}$$
$$(m_0\rho_0^{-1/2})_\epsilon = (m_0\rho_0^{-1/2}1_{d>\epsilon}) * \omega_{\epsilon/2},$$

and

$$d = dist(x, \partial B_R).$$

Next, decompose $\bar{m}_0^\epsilon$ in the following way (Hodge decomposition)

Lemma 2.3.2 *If $\bar{m}_0^\epsilon \in L^2(B_R)$, then there is a unique solution $(\bar{u}_0^\epsilon, q_0^\epsilon)$ in $L^2(B_R)$ to the equation*

$$\begin{cases} \bar{m}_0^\epsilon = \rho_0^\epsilon\bar{u}_0^\epsilon + \nabla q_0^\epsilon, \quad \bar{u}_0^\epsilon, q_0^\epsilon \in C^\infty(\bar{B}_R) \\ \nabla\cdot\bar{u}_0^\epsilon = 0 \ \text{ in } \ B_R, \quad \bar{u}_0^\epsilon \cdot n|_{\partial B_R} = 0 \end{cases} \tag{2.3.3}$$

denoting by n the unit outward normal to ∂_{B_R}. Furthermore, if ρ_0^ϵ converges a.e. to ρ_0, and $\bar{m}_0^\epsilon$ converges to $\bar{m}_0$ in $L^2(B_R)$, then $\rho_0^\epsilon\bar{u}_0^\epsilon$, ∇q_0^ϵ also converge to p, q in $L^2(B_R)$ respectively, where $\bar{m}_0 = p + \nabla q$.

For proof, see Lemma 2.1 in [38].

Lemma 2.3.3 *We can find* $u_0^\epsilon \in C_0^\infty(B_R) \cap L^2(B_R)$ *such that*

$$\|u_0^\epsilon - \bar{u}_0^\epsilon\|_{L^2(B_R)} \leq \epsilon, \quad \nabla \cdot u_0^\epsilon = 0 \quad \text{in} \quad B_R. \tag{2.3.4}$$

Proof We first write $\bar{u}_0^\epsilon$ as

$$\bar{u}_0^\epsilon = (\bar{u}_0^\epsilon \cdot n)n + T$$

where T denotes tangential which is orthogonal to n. The the question is equivalent to solve the following equation,

$$\begin{cases} \nabla \cdot v = 0 \quad \text{in} \quad B_R \\ v = T \quad \text{on} \quad \partial B_R \end{cases} \tag{2.3.5}$$

where $v \in C^\infty(B_R) \cap L^2(B_R)$ and $\|v\|_{L^2(B_R)} \leq \epsilon$ for a fixed $\epsilon > 0$. Then we can take $u_0^\epsilon = \bar{u}_0^\epsilon - v$. So let us focus on finding a solution to (2.3.5).

In polar coordinate (r, θ), let

$$v = a(r, \theta)\partial r + \varphi(r, \theta)T$$

where $a|_{\partial B_R} = 0$,while φ is constructed as a smooth cut off function such that $\varphi(r, \theta) = 0$ for $r \leq R - \epsilon$ and $\|\varphi\|_{L^2(B_R)} < \epsilon$. Thus

$$\begin{aligned} \nabla \cdot v &= \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} v_i) \\ &= \frac{1}{r^{n-1}} \partial r (r^{n-1} a) + \frac{1}{r^{n-1}} \partial_i (r^{n-1} T_i \varphi) \end{aligned}$$

where g is the metric and n is the dimension. And $\nabla \cdot v = 0$ implies that,

$$\partial r (r^{n-1} a) = -r^{n-1} \mathrm{div}_T T\varphi.$$

So we have

$$a = -\frac{1}{r^{n-1}} \int_r^R s^{n-1} \varphi(s) ds \ \mathrm{div}_T T.$$

Since $a|_{\partial B_R} = 0$ and a is smooth, the proof is complete. □

Finally we set

$$m_0^\epsilon = \rho_0^\epsilon u_0^\epsilon + \nabla q_0^\epsilon. \tag{2.3.6}$$

Since $\rho^\epsilon > 0$ we may impose

$$u|_{t=0} = u_0^\epsilon \ \text{ in } \ B_R. \tag{2.3.7}$$

Lemma 2.3.4 *The modified momentum $\bar{m}_0^\epsilon \in C_0^\infty(B_R)$ satisfies*

$$\bar{m}_0^\epsilon \to m_0 \quad \textit{in } L^2(B_R) \tag{2.3.8}$$

$$\bar{m}_0^\epsilon (\rho_0^\epsilon)^{-1/2} \to m_0 \rho_0^{-1/2} \quad \textit{in } L^2(B_R), \tag{2.3.9}$$

and hence m_0^ϵ has the same convergences, namely

$$m_0^\epsilon \to m_0 \quad \textit{in } L^2(B_R) \tag{2.3.10}$$

$$m_0^\epsilon (\rho_0^\epsilon)^{-1/2} \to m_0 \rho_0^{-1/2} \quad \textit{in } L^2(B_R). \tag{2.3.11}$$

Proof We have

$$\begin{aligned}
\int_{B_R} |\bar{m}_0^\epsilon - m_0|^2 &\le \int_{B_R} |(m_0 \rho_0^{-1/2})_\epsilon (\rho_0^{1/2})_\epsilon - m_0 \rho_0^{-1/2} (\rho_0^{1/2})_\epsilon|^2 \\
&+ \int_{B_R} |m_0 \rho_0^{-1/2} (\rho_0^{1/2})_\epsilon - m_0 \rho_0^{-1/2} \rho_0^{1/2}|^2 \\
&\le \int_{B_R} |(m_0 \rho_0^{-1/2})_\epsilon - m_0 \rho_0^{-1/2}|^2 |(\rho_0^{1/2})_\epsilon|^2 \\
&+ \int_{B_R} |m_0 \rho_0^{-1/2}|^2 |(\rho_0^{1/2})_\epsilon - \rho_0^{1/2}|^2
\end{aligned}$$

It is clear that the first integral in the above inequality converges to 0 since $(\rho_0 1/2)_\epsilon$ has an uniform upper bound. To show the second integral converges to 0, we first need to show

$$|(\rho_0^{1/2})_\epsilon - \rho_0^{1/2}|^2 \to 0, \quad \text{a. e.}$$

By the definition of $(\rho_0^{1/2})_\epsilon$, $\int_{B_R} |(\rho_0^{1/2})_\epsilon - \rho_0^{1/2}|^2 \to 0$ and it implies the measure convergence. Riesz Theorem yields the needed pointwise convergence. The boundedness of $|m_0 \rho_0^{-1/2}|^2$ and $\rho_0^{1/2}$ imply that the second integral converges to 0 by the dominated convergence theorem.

Next we prove the convergence in (2.3.9),

$$\begin{aligned}
&\int_{B_R} |\bar{m}_0^\epsilon(\rho_0^\epsilon)^{-1/2} - m_0(\rho_0)^{-1/2}|^2 \\
&\leq \int_{B_R} |(m_0\rho_0^{-1/2})_\epsilon - m_0\rho_0^{-1/2}|^2 |(\rho_0^{1/2})_\epsilon(\rho_0^\epsilon)^{-1/2}|^2 \\
&+ \int_{B_R} |m_0\rho_0^{-1/2}|^2 |(\rho_0^{1/2})_\epsilon(\rho_0^\epsilon)^{-1/2} - 1|^2
\end{aligned}$$

Noticing that $(\rho_0^{1/2})_\epsilon \leq (\rho_0)_\epsilon^{1/2}$, by the definition of modification, the right hand side of last inequality converges to 0.

By the construction we have that

$$m_0^\epsilon = \bar{m}_0^\epsilon + \rho_0^\epsilon(u_0^\epsilon - \bar{u}_0^\epsilon)$$

$$m_0^\epsilon(\rho_0^\epsilon)^{-1/2} = \bar{m}_0^\epsilon(\rho_0^\epsilon)^{-1/2} + (\rho_0^\epsilon)^{1/2}(u_0^\epsilon - \bar{u}_0^\epsilon).$$

Thus, the convergence in (2.3.8) and (2.3.9) yields (2.3.10) and (2.3.11), respectively. □

Definition of d_0^ϵ: we now modify the initial data for the director field as,

$$d_0^\epsilon = d_0 * \omega_\epsilon|_{B_R}.$$

In the rest of the chapter we work with the modified initial data $(\rho_0^\epsilon, u_0^\epsilon, d_0^\epsilon)$.

2.3.2 Existence of approximated system on $B_R \times [0,T)$ for fixed η and ϵ

Our goal in this step is to establish the existence of approximating solution using a Faedo-Galerkin method. The main result is as follows,

Theorem 2.3.5 *Let ρ_0^ϵ, u_0^ϵ and d_0^ϵ be defined as in Subsection 2.3.1 , there exists a weak solution (ρ, u, d) to the system (2.3.1). For any $\phi \in C_c^\infty(B_R \times [0,T])$ with*

$div\phi = 0$, the solution satisfies

$$-\int_{B_R}\rho_0^\epsilon\phi(x,0)dx = \int_0^T\int_{B_R}\rho\phi_t - (u\cdot\nabla\rho)\phi + \eta\nabla\rho\cdot\nabla\phi dxdt,$$

$$-\int_{B_R} m_0^\epsilon\phi(x,0)dx - \int_0^T\int_{B_R}\rho u\phi_t dxdt$$

$$= \int_0^T\int_{B_R} -\nabla u\cdot\nabla\phi + (\rho u\otimes u)\cdot\nabla\phi - \eta(\nabla\rho\cdot\nabla u)\phi + \nabla d\cdot(-\triangle d + f(d))\phi dxdt,$$

$$-\int_{B_R} d_0^\epsilon\phi(x,0)dx = \int_0^T\int_{B_R} d\phi_t - (u\cdot\nabla\rho)\phi + (\triangle d - f(d))\phi dxdt,$$

and the energy inequality

$$\int_{B_R}\frac{1}{2}\rho|u|^2+\frac{1}{2}|\nabla d|^2+F(d)+\int_0^T\int_{B_R}|\nabla u|^2+|\triangle d-f(d)|^2 \le C(\rho_0^\epsilon, m_0^\epsilon, d_0^\epsilon) \quad (2.3.12)$$

Remark 2.3.6 *(i) The weak solution (ρ, u, d) is in fact $(\rho^{\eta,R}, u^{\eta,R}, d^{\eta,R})$ depending on η and R. But for convenience, we omit the index.*

(ii) Eventually, we will show that the constant $C(\rho_0^\epsilon, m_0^\epsilon, d_0^\epsilon)$ on the right hand side of (2.3.12) is bounded by a constant $C(\rho_0, m_0, d_0)$ only depending on ρ_0,m_0 and d_0.

The proof of Theorem 2.3.5 will be divided into four steps.

Step I We solve the first equation in (2.3.1) with boundary (2.3.2) and a smooth u.

Claim 2.3.7 *Assume $u \in C([0,T]; C^2(\bar{B}_R))$ with*

$$\nabla\cdot u = 0, \quad u|_{\partial B_R} = 0.$$

Then the first equation in (2.3.1) with boundary condition (2.3.2) and initial data ρ_0^ϵ has a unique classical solution ρ on $B_R\times[0,T]$ such that for any fixed $t\in[0,T]$

$$\rho(t) \in C^{2+\delta}(\bar{B}_R).$$

For a proof, see Theorem 5.1.21 in [44].

Claim 2.3.8 *By the argument of maximum principle Lemma 2.2.1, we have that*

$$\epsilon \le \rho(x,t) \le C$$

for all $t\in[0,T]$, $x\in B_R$, and C is a constant independent of ϵ.

Claim 2.3.9 *There exists a mapping $S = S(u)$,*

$$S : C([0,T]; C^2(\bar{B}_R)) \cap u|\nabla \cdot u = 0 \mapsto C([0,T]; C^{2+\delta}(\bar{B}_R))$$

satisfying the following properties:

1. *$\rho = S(u)$ is a unique classical solution of the first equation in (2.3.1) and (2.3.2);*

2. $$\epsilon \leq S(u)(x,t) \leq C, \forall (x,t) \in B_R \times [0,T];$$

3. $$\sup_{t\in[0,T]} \int_{B_R} |\nabla S(u)|^2 + \eta \int_0^T \int_{B_R} |\triangle S(u)|^2 \leq C(T, \eta, \rho_0^\epsilon, \|u\|_{L^\infty(B_R\times[0,T])});$$

4. $$\|S(u^1) - S(u^2)\|_{C([0,T];H^1(B_R))} \leq \sqrt{T} C \|u^1 - u^2\|_{C([0,T];L^\infty(B_R))}$$

   where $C = C(T, \eta, \rho_0^\epsilon, \|u^1\|_{L^\infty(B_R\times[0,T])}, \|u^2\|_{L^\infty(B_R\times[0,T])})$.

See [41] for a proof.

Step II In this step, we solve the third equation in (2.3.1) with initial data d_0^ϵ and boundary condition $d(x,t)|_{\partial_{B_R}} = d_0^\epsilon$. The existence of such d is guaranteed by the standard theory of parabolic equation, see [28].

Claim 2.3.10 *There exists a mapping $R = R(u)$,*

$$R : C([0,T]; C^2(\bar{B}_R)) \cap u|\nabla \cdot u = 0 \mapsto C([0,T]; C^{2+\delta}(\bar{B}_R))$$

satisfying the following properties:

1. *$d = R(u)$ is a C^2 solution of the third equation in (2.3.1) and (2.3.2);*

2. $$\sup_{t\in[0,T]} \int_{B_R} |\nabla R(u)|^2 + \int_0^T \int_{B_R} |\triangle R(u)|^2 \leq C(T, d_0^\epsilon, \|u\|_{L^\infty(B_R\times[0,T])});$$

3.

$$\|R(u^1) - R(u^2)\|_{C([0,T];H^1(B_R))} + \|R(u^1) - R(u^2)\|_{L^2(0,T;H^2(B_R))}$$
$$\leq \sqrt{T}C\|u^1 - u^2\|_{C([0,T];L^\infty(B_R))}$$

where $C = C(T, d_0^\epsilon, \|u^1\|_{L^\infty(B_R\times[0,T])}, \|u^2\|_{L^\infty(B_R\times[0,T])})$.

See [41] for a proof.

Step III We now search a solution u to the second equation in (2.3.1) with initial data u_0^ϵ and boundary (2.3.2) by Faedo-Galerkin method. First of all, we define some function spaces as,

$$U = C_0^\infty(B_R, \mathbb{R}^3) \cap u|\nabla \cdot u = 0$$

$$H = \text{closure of } U \text{ in } L^2(B_R, \mathbb{R}^3)$$

$$V = \text{closure of } U \text{ in } H_0^1(B_R, \mathbb{R}^3)$$

Let $\{\phi_i\}_{i=1}^\infty$ be an orthonormal basis of H satisfying

$$\triangle\phi_i + \nabla p_i = -\lambda_i\phi_i, \quad \phi_i|_{\partial_{B_R}} = 0.$$

Consider X_n =span$\{\phi_i\}_{i=1}^n$ for $n = 1, 2, 3, \ldots$. The approximating solutions $u_n \in C([0,T]; X_n)$ we search for are required to satisfy the integral equality,

$$\int_{B_R} \rho(t)u_n(t)\psi - \int_{B_R} m_o^\epsilon\psi \tag{2.3.13}$$
$$= \int_0^t \int_{B_R} [\triangle u_n - \text{div}(\rho u_n \otimes u_n) - \eta(\nabla\rho \cdot \nabla u_n) + \nabla d \cdot (-\triangle d + f(d))]\psi,$$

for all $t \in [0,T)$, and any $\psi \in X_n$. The above identity can be rephrased as,

$$u_n(t) = \mathcal{M}^{-1}[\rho]((m_0^\epsilon)^* + \int_0^t \mathcal{N}[\rho(s), d(s), u_n(s)]ds),$$

for more detail on operators $\mathcal{M}$ and $\mathcal{N}$ see [41]. Denote the image of $\mathcal{M}^{-1}$ by $\boldsymbol{\psi}$. It follows that

$$\|\boldsymbol{\psi}_n\|_{C([0,T];X_n)} \tag{2.3.14}$$
$$\leq \frac{1}{\epsilon}\sqrt{T}C(T, \eta, \rho_0, d_0, n, \|u_n\|_{L^\infty(B_R\times(0,T))})$$
$$+ \frac{1}{\epsilon}C_1(n)\|m_0^\epsilon\|_{L^1(B_R)}$$

and

$$\|\boldsymbol{\psi}_n^1 - \boldsymbol{\psi}_n^2\|_{C([0,T];X_n)} \leq \sqrt{T}C(T,\eta,\rho_0,d_0,n)\|u_n^1 - u_n^2\|_{L^\infty(B_R\times(0,T))}, \tag{2.3.15}$$

See [41] for detail.

Thus, by a fixed point argument, there exists a local solution (ρ_n, u_n, d_n) to the system (2.3.1) at least on a short time $[0, T_n]$, $T(n) \leq T$. In order to show that $T(n) = T$ for all n, it is enough to obtain an uniform estimate $\|\rho_n u_n\|_{L^1_{B_R}} < K$ on the whole $[0, T(n)]$, where K is a constant to be determined below and $\|u_n\|_{C([0,T];X_n)} \leq K$.

Differentiate (2.3.13) with respect to t, and let $\psi = u_n(t)$, we can obtain on $(0, T(n))$,

$$\frac{d}{dt}\int_{B_R}\frac{1}{2}\rho_n|u_n|^2 + \frac{1}{2}|\nabla d_n|^2 + F(d_n) + \int_{B_R}|\nabla u_n|^2 + |\triangle d_n - f(d_n)|^2 = 0 \tag{2.3.16}$$

Multiplying the first equation in (2.3.1) by ρ_n, and integrating by parts to obtain

$$\frac{d}{dt}\int_{B_R}|\rho_n|^2 + 2\eta\int_{B_R}|\nabla\rho_n|^2 = 0 \tag{2.3.17}$$

From (2.3.16), it follows,

$$\sup_{0\leq t\leq T(n)}\int_{B_R}\rho_n|u_n|^2 \leq \int_{B_R}F(d_0^\epsilon) + \rho_0^\epsilon|u_0^\epsilon|^2 + |\nabla d_0^\epsilon|^2 \leq M.$$

From (2.3.17), it follows

$$\sup_{0\leq t\leq T(n)}\int_{B_R}|\rho_n|^2 \leq \int_{B_R}|\rho_0^\epsilon|^2 \leq C(\rho_0)$$

where M is a constant independent of ϵ and n.

By Hölder inequality,

$$\int_{B_R}\rho_n|u_n| \leq |B_R|^{1/4}(\int_{B_R}\rho_n|u_n|^2)^{1/2}(\int_{B_R}|\rho_n|^2)^{1/4}.$$

Take $K = \frac{1}{\inf\rho_n}C_1(n)M^{1/2}(\int_{B_R}|\rho_n|^2)^{1/4} + 1$, hence by the inequality (2.3.14)

$$\sup_{0\leq t\leq T(n)}\frac{1}{\inf\rho_n}C_1(n)\|\rho_n u_n\|_{L^1(B_R)} \leq K.$$

Thus it is possible to repeat the fixed point argument to conclude, for any fixed n, that $T(n)$ can reach T after a finite number of steps. Therefore we proved that

Theorem 2.3.11 *For fixed η and ϵ, there exists a solution (ρ_n, u_n, d_n) to the system (2.3.1) with boundary (2.3.2) and the solution satisfies the energy identity (2.3.16) and (2.3.17), so consequently we have*

$$0 < \epsilon \leq \rho_n \leq C \tag{2.3.18}$$

$$\sup_{0\leq t\leq T} \int_{B_R} \rho_n |u_n|^2 \leq C(\rho_0, m_0, d_0) \tag{2.3.19}$$

$$\sup_{0\leq t\leq T} \int_{B_R} |\rho_n|^2 \leq C(\rho_0) \tag{2.3.20}$$

$$\sup_{0\leq t\leq T} \int_{B_R} |\nabla d_n|^2 \leq C(\rho_0, m_0, d_0) \tag{2.3.21}$$

$$\int_0^T \int_{B_R} |\nabla u_n|^2 \leq C(\rho_0, m_0, d_0) \tag{2.3.22}$$

$$\eta \int_0^T \int_{B_R} |\nabla \rho_n|^2 \leq C(\rho_0) \tag{2.3.23}$$

$$\int_0^T \int_{B_R} |\triangle d_n|^2 \leq C(\rho_0, m_0, d_0) \tag{2.3.24}$$

all the constants C are independent of n.

Step IV We finish the proof of Theorem (2.3.5) by passing the limit $n \to \infty$ using compactness Lemma 2.2.11 and 2.2.12.

Lemma 2.3.12 *By (2.3.22), we can find a subsequence such that*

$$u_n \rightharpoonup u \text{ in } L^2(0, T; V).$$

Lemma 2.3.13 *From the first equation in (2.3.1), we obtain*

$$\|\frac{\partial}{\partial t}\rho_n\|_{L^{4/3}(0,T;H^{-1}(B_R))} \leq C(T, \rho_0, m_0, d_0).$$

See page 19 of [41] for a proof.

Since $H^1 \hookrightarrow L^2 \subset H^{-1}$, by (2.3.23) and Lions-Aubin lemma 2.2.11, we have that

Lemma 2.3.14 *The sequence $\{\rho_n\}$ is precompact in $L^2(0,T;L^2(B_R))$. We can take subsequence such that*

$$\rho_n \rightharpoonup \rho \text{ in } L^2(0,T;H^1(B_R))$$

$$\rho_n \to \rho \text{ in } L^2(0,T;L^2(B_R))$$

with $0 < \epsilon \le \rho \le C$, C is independent of ϵ.

Lemma 2.3.15 *From the third equation in (2.3.1), we can obtain*

$$\|\frac{\partial}{\partial t} d_n\|_{L^{4/3}(0,T;L^2(B_R))} \le C(T, \rho_0, m_0, d_0).$$

See page 21 of [41] for a proof.

Since $H^2 \hookrightarrow H^1 \subset L^2$, by (2.3.24) and Lions-Aubin lemma 2.2.11, we have that

Lemma 2.3.16 *The sequence $\{d_n\}$ is precompact in $L^2(0,T;H^1(B_R))$. We can take subsequence such that*

$$d_n \rightharpoonup d \text{ in } L^2(0,T;H^2(B_R))$$

$$d_n \to d \text{ in } L^2(0,T;H^1(B_R)).$$

Lemma 2.3.17 *The sequence $\{\rho_n u_n\}$ converges weakly star in $L^\infty(0,T;L^{4/3}(B_R))$, namely,*

$$\rho_n u_n \rightharpoonup \rho u \text{ weakly star in } L^\infty(0,T;L^{4/3}(B_R)).$$

Proof Estimates (2.3.19)(2.3.20) imply that

$$\sup_{0\le t\le T} \int_{B_R} |\rho_n u_n|^{\frac{4}{3}} \le \sup_{0\le t\le T} \Big(\int_{B_R} \rho_n |u_n|^2\Big)^{\frac{2}{3}} \Big(\int_{B_R} |\rho_n|^2\Big)^{\frac{1}{3}} \le C(\rho_0, m_0, d_0).$$

So $\rho_n u_n \rightharpoonup m$ weakly star in $L^\infty(0,T;L^{4/3}(B_R))$ for some vector m. On the other hand, for any $\psi \in C_c^\infty(B_R \times (0,T))$, we have

$$\int_0^T \int_{B_R} (\rho_n u_n - \rho u)\psi = \int_0^T \int_{B_R} (\rho_n - \rho) u\psi + \int_0^T \int_{B_R} \rho_n (u_n - u)\psi$$

in which the first integral converges to 0 by lemma 2.3.14 and the second integral converges to 0 by lemma 2.3.14 and 2.3.12. Thus,

$$\rho u = m \text{ in } \mathcal{D}'(B_R \times (0,T)).$$

Notice that

$$\sup_{0\le t\le T}\int_{B_R}|\rho u|^{\frac{4}{3}}\le C|B_R|^{\frac{1}{3}}\sup_{0\le t\le T}(\int_{B_R}\rho|u|^2)^{\frac{2}{3}}\le C(\rho_0,m_0,d_0),$$

we conclude that

$$\rho u=m \text{ in } L^\infty(0,T;L^{4/3}(B_R)).$$

□

We are ready now to pass the limit $n\to\infty$ in the first equation in (2.3.1), namely,

Proposition 2.3.18 *The limit ρ satisfies the following equation for any $\phi\in C_c^\infty(B_R\times(0,T))$ with $div\phi=0$*

$$-\int_{B_R}\rho_0^\epsilon\phi(x,0)dx=\int_0^T\int_{B_R}\rho\phi_t-(u\cdot\nabla\rho)\phi+\eta\nabla\rho\cdot\nabla\phi dxdt.$$

Proof Since ρ_n satisfy

$$(\rho_n)_t+\mathrm{div}(\rho_n u_n)=\eta\triangle\rho_n,$$

integrate it to get

$$-\int_{B_R}\rho_0^\epsilon\phi(x,0)dx=\int_0^T\int_{B_R}\rho_n\phi_t+(\rho_n u_n)\nabla\phi+\eta\nabla\rho\cdot\nabla\phi dxdt.$$

The construction of ρ_0^ϵ guarantees the convergence of the integral at the left. The convergence of the first and third integral at the right are guaranteed by lemma 2.3.14 while the second one is guaranteed by lemma 2.3.17. It completes the proof. □

Lemma 2.3.19 *From the second equation in (2.3.1), we have that*

$$\|\frac{\partial}{\partial t}(\rho_n u_n)\|_{L^{7/6}(0,T;W^{-1,2}(B_R))}\le C(T,\rho_0,m_0,d_0).$$

Proof For any $\psi\in L^7(0,T;W_0^{1,2}(B_R))$, multiply the second equation in (2.3.1) by ψ and integrate to get

$$\begin{aligned}\int_{B_R}(\rho_n u_n)_t\psi=&\int_{B_R}\rho_n u_n\otimes u_n\nabla\psi-\eta\int_{B_R}(\nabla\rho_n\cdot\nabla u_n)\psi\\&-\int_{B_R}\nabla u_n\cdot\nabla\psi+\int_{B_R}\nabla d_n\cdot(-\triangle d_n+f(d_n))\psi\end{aligned}$$

By (2.3.18),

$$|\int_{B_R} \rho_n u_n \otimes u_n \nabla\psi| \le C\|\psi\|_{W^{1,2}(B_R)}(\int_{B_R} |u_n|^4)^{\frac{1}{2}}$$
$$\le C\|\psi\|_{W^{1,2}(B_R)}\|u_n\|^{\frac{1}{2}}_{L^2(B_R)}\|u_n\|^{\frac{3}{2}}_{L^6(B_R)}$$

Thus,

$$|\int_0^T \int_{B_R} \rho_n u_n \otimes u_n \nabla\psi|$$
$$\le C\|\psi\|_{L^7(0,T;W^{1,2}(B_R))} \sup_{0\le t\le T} \|u_n\|^{\frac{1}{2}}_{L^2(B_R)}(\int_0^T \|u_n\|^{\frac{7}{4}}_{L^6(B_R)})^{\frac{6}{7}}$$
$$\le C \tag{2.3.25}$$

due to (2.3.18), (2.3.19) and (2.3.22).

From (2.3.22), it follows

$$|\int_0^T \int_{B_R} \nabla u_n \cdot \nabla\psi| \le C.$$

From (2.3.21) and (2.3.24), it follows

$$|\int_0^T \int_{B_R} \nabla d_n \cdot (-\triangle d_n + f(d_n))\psi| \le C.$$

By (2.3.22) and (2.3.23), we obtain

$$|\eta \int_0^T \int_{B_R} (\nabla u_n \cdot \nabla\rho_n)\psi|$$
$$\le \eta \int_0^T \|\nabla u_n\|^{\frac{1}{2}}_{L^2(B_R)}\|\nabla\rho_n\|^{\frac{1}{3}}_{L^3(B_R)}\|\psi\|^{\frac{1}{6}}_{L^6(B_R)}$$
$$\le \eta\|\nabla u_n\|_{L^2(B_R\times(0,T))}\|\psi\|_{L^7(0,T;L^6(B_R))}(\int_0^T \|\nabla\rho_n\|^{\frac{14}{15}}_{L^3(B_R)})^{\frac{5}{14}}$$
$$\le C,$$

supposing that $\nabla\rho_n \in L^{\frac{14}{5}}(0,T;L^3(B_R))$. We shall use L^p theory to show this estimate in the following. Since $\rho_n u_n \in L^\infty(0,T;L^{\frac{4}{3}}(B_R)) \cap L^2(0,T;L^6(B_R))$, we can have $\rho_n u_n \in L^{\frac{14}{5}}(0,T;L^3(B_R))$ by interpolation. Then from the first equation in (2.3.1), it implies that $\nabla\rho_n$ is also in $L^{\frac{14}{5}}(0,T;L^3(B_R))$ by the so-called L^p-estimate, Lemma 2.2.10, where we take $A = -\triangle$, $E_0 = W^{-1,3}$, $E_1 = W^{1,3}$ and $p = \frac{14}{5}$. □

Lemma 2.3.20 *Using compactness of $L^{4/3}(B_R) \hookrightarrow W^{-1,2}(B_R)$, one deduce from Lemma 2.3.19 and Lemma 2.2.12,*

$$\rho_n u_n \to \rho u \text{ in } C([0,T]; W^{-1,2}(B_R)). \tag{2.3.26}$$

In order to continue, we have to show that the first equation in (2.3.1) holds, in fact, in the strong sense.

Lemma 2.3.21 *[16] There exists $r > 1$, $q > 2$ such that $\partial_t \rho_n$, $\triangle \rho_n$ are bounded in $L^r(B_R \times (0,T))$, $\nabla \rho_n$ is bounded in $L^q(0,T; L^2(B_R))$ independent of n. Consequently, the limit ρ belongs to the same class and satisfies the first equation in (2.3.1) a. e. on $B_R \times (0,T)$ and the boundary condition in the sense of traces.*

Then we obtain more estimate on ρ_n as,

Lemma 2.3.22

$$\int_{B_R} |\rho_n - \rho|^2(t) + 2\eta \int_0^t \int_{B_R} |\nabla \rho_n - \nabla \rho|^2 \to 0.$$

In particular,

$$\nabla \rho_n \to \nabla \rho \quad \text{in } L^2(B_R \times (0,T)).$$

Proof Multiplying the the first equation satisfied by the approximating solution ρ_n in (2.3.1) by ρ_n yields

$$\|\rho_n\|^2_{L^2(B_R)} + 2\eta \int_0^t \|\rho_n\|^2_{L^2(B_R)} ds = -\int_0^t \int_{B_R} \mathrm{div} u_n |\rho_n|^2 dx ds + \|\rho_0\|^2_{L^2(B_R)}. \tag{2.3.27}$$

Multiplying the the first equation in (2.3.1) by ρ (actually we multiply it by smooth mollifier of ρ and take a limit) yields

$$\|\rho\|^2_{L^2(B_R)} + 2\eta \int_0^t \|\rho\|^2_{L^2(B_R)} ds = -\int_0^t \int_{B_R} \mathrm{div} u |\rho|^2 dx ds + \|\rho_0\|^2_{L^2(B_R)}. \tag{2.3.28}$$

Using the fact that ρ_n and ρ are uniformly bounded (see Subsection 2.2.1), and applying Lemma 2.3.12 and Lemma 2.3.14, it follows

$$\lim_{n\to\infty} \int_0^t \int_{B_R} \mathrm{div} u_n |\rho_n|^2 dx ds = \int_0^t \int_{B_R} \mathrm{div} u |\rho|^2 dx ds.$$

Then by the equations (2.3.27) and (2.3.28), we have

$$\lim_{n\to\infty} \|\rho_n\|^2_{L^2(B_R)} = \|\rho\|^2_{L^2(B_R)}, \quad \lim_{n\to\infty} \int_0^t \|\rho_n\|^2_{L^2(B_R)} ds = \int_0^t \|\rho\|^2_{L^2(B_R)} ds.$$

The lemma is proved by combining the weak convergence and norm convergence. □

With all the lemmas above, we can pass limit in the second and third equation in (2.3.1). To point out, lemma 2.3.12 2.3.26 give the convergence of the term $\rho_n u_n \otimes u_n$, and lemma 2.3.12 2.3.22 imply the convergence of the term $\nabla \rho_n \cdot \nabla u_n$.

Thus we proved Theorem 2.3.5.

2.3.3 The vanishing viscosity limit by taking $\eta \to 0$

Denote the solution obtained in Subsection 2.3.2 as $(\rho^\eta, u^\eta, d^\eta)$. The main result in this subsection is that

Theorem 2.3.23 *There exists a weak solution (ρ, u, d) to the system*

$$\rho_t + div(\rho u) = 0 \tag{2.3.29}$$

$$(\rho u)_t + div(\rho u \otimes u) + \nabla p = \Delta u + \nabla d \cdot \tau(d) \tag{2.3.30}$$

$$d_t + u \cdot \nabla d = \Delta d - f(d) \tag{2.3.31}$$

$$\nabla \cdot u = 0 \tag{2.3.32}$$

on $B_R \times (0, T)$ with initial data ρ_0^ϵ, u_0^ϵ, d_0^ϵ and boundary condition (2.3.2) satisfying the energy inequality

$$\begin{aligned} &\int_{B_R} \frac{1}{2}\rho|u|^2 + \frac{1}{2}|\nabla d|^2 + F(d) + \int_0^T \int_{B_R} |\nabla u|^2 + |\Delta d - f(d)|^2 \\ &\leq C \int_{B_R} \frac{1}{2}\frac{m_0^2}{\rho_0} + \frac{1}{2}|\nabla d_0|^2 + F(d_0), \end{aligned} \tag{2.3.33}$$

and

$$\sup_{0 \leq t \leq T} \int_{B_R} |\rho|^2(t) \leq C(\rho_0). \tag{2.3.34}$$

First, we collect some compactness results.

Lemma 2.3.24 *Let ρ^η be the solution obtained in Subsection 2.3.2. Then*

$$\eta \triangle \rho^\eta \to 0 \text{ in } L^2(0, T; H^{-1}(B_R)).$$

Proof For any $\phi \in L^2(0,T;H^1(B_R))$,

$$\eta \int_0^T \int_{B_R} \nabla \rho^\eta \cdot \nabla \phi \leq \sqrt{\eta}(\eta \int_0^T \int_{B_R} |\nabla \rho^\eta|^2)^{\frac{1}{2}} (\int_0^T \int_{B_R} |\nabla \phi|^2)^{\frac{1}{2}} \to 0.$$

□

Lemma 2.3.25 *Let ρ^η, u^η be the solution obtained in Subsection 2.3.2. Then*

$$\eta \nabla \rho^\eta \cdot \nabla u^\eta \to 0 \text{ in } L^1(B_R \times (0,T)).$$

Proof For any $\phi \in L^2(0,T;H^1(B_R))$,

$$\eta \int_0^T \int_{B_R} \nabla \rho^\eta \cdot \nabla u^\eta \leq \sqrt{\eta}(\eta \int_0^T \int_{B_R} |\nabla \rho^\eta|^2)^{\frac{1}{2}} (\int_0^T \int_{B_R} |\nabla u^\eta|^2)^{\frac{1}{2}} \to 0.$$

□

Notice that the whole process of lemma 2.3.12 to lemma 2.3.16 is independent of η, thus we have

Lemma 2.3.26 *Extracting a subsequence if necessary, we have that*

$$\rho^\eta \rightharpoonup \rho \text{ in } L^2(B_R \times (0,T)) \text{ with } 0 < \epsilon \leq \rho \leq C$$

$$u^\eta \rightharpoonup u \text{ in } L^2(0,T;H_0^1(B_R))$$

$$d^\eta \rightharpoonup d \text{ in } L^2(0,T;H^2(B_R))$$

$$d^\eta \to d \text{ in } L^2(0,T;H^1(B_R))$$

And the limit functions ρ, d satisfy initial conditions

$$\rho(x,0) = \rho_0^\epsilon, \quad d(x,0) = d_0^\epsilon.$$

Using similarly argument as in Subsection 2.3.2 to produce Lemma 2.3.17, we also have

Lemma 2.3.27 *Let ρ^η and u^η be the solution obtained in Subsection 2.3.2. Then*

$$\rho^\eta u^\eta \to \rho u \text{ in } C([0,T];L^{6/5}(B_R)).$$

Now we are ready to show that

Proposition 2.3.28 *The limit functions (ρ, u, d) is a weak solution to the system (2.3.29)-(2.3.32), more precisely, for any $\phi \in C_c^\infty(B_R \times (0,T))$ with $div\phi = 0$,*

$$-\int_{B_R} \rho_0^\epsilon \phi(x,0)dx = \int_0^T \int_{B_R} \rho\phi_t - (u\cdot\nabla\rho)\phi dxdt,$$

$$\begin{aligned}
&-\int_{B_R} m_0^\epsilon \phi(x,0)dx - \int_0^T \int_{B_R} \rho u\phi_t dxdt \\
&= \int_0^T \int_{B_R} -\nabla u\cdot\nabla\phi + (\rho u\otimes u)\cdot\nabla\phi + \nabla d\cdot(-\triangle d + f(d))\phi dxdt,
\end{aligned}$$

$$-\int_{B_R} d_0^\epsilon \phi(x,0)dx = \int_0^T \int_{B_R} d\phi_t - (u\cdot\nabla\rho)\phi + (\triangle d - f(d))\phi dxdt.$$

The proof is straightforward due to the lemmas above. Thus, it also completes the proof of Theorem 2.3.23.

2.3.4 The modified initial data limit by taking $\epsilon \to 0$

Denote the solution obtained in Subsection 2.3.3 as $(\rho^\epsilon, u^\epsilon, d^\epsilon)$. Our main result in this subsection is that

Theorem 2.3.29 *There exists a weak solution (ρ, u, d) of the system (2.3.29)-(2.3.32) on $B_R \times (0,T)$ with initial data $\rho_0|_{B_R}$, $m_0|_{B_R}$, $d_0|_{B_R}$ and boundary condition*

$$u|_{\partial B_R} = 0, \quad d(x,t)|_{\partial B_R} = d_0(x)$$

satisfying (2.3.34) and the energy inequality

$$\begin{aligned}
&\int_{B_R} \frac{1}{2}\rho^\epsilon|u^\epsilon|^2 + \frac{1}{2}|\nabla d^\epsilon|^2 + F(d^\epsilon) + \int_0^t \int_{B_R} |\nabla u^\epsilon|^2 + |\triangle d^\epsilon - f(d^\epsilon)|^2 \qquad (2.3.35)\\
&\le \int_{B_R} \frac{m_0^2}{2\rho_0} + \frac{1}{2}|\nabla d_0|^2 + F(d_0).
\end{aligned}$$

To prove Theorem 2.3.29, we start with some compactness results.

Lemma 2.3.30 *Since all the estimates in Lemma 2.3.12 to Lemma 2.3.16 is independent of ϵ, thus we have by passing the limit (extracting a subsequence if necessary),*

$$\rho^\epsilon \rightharpoonup \rho \text{ in } L^2(B_R\times(0,T)) \text{ with } 0 \le \rho \le C$$

$$u^\epsilon \rightharpoonup u \ in \ L^2(0,T;H_0^1(B_R))$$

$$d^\epsilon \rightharpoonup d \ in \ L^2(0,T;H^2(B_R))$$

$$d^\epsilon \to d \ in \ L^2(0,T;H^1(B_R))$$

Since now $\rho \geq 0$, we use a different method to show that

Lemma 2.3.31

$$\rho^\epsilon u^\epsilon \rightharpoonup \rho u \ weak \ star \ in \ L^\infty(0,T;L^{4/3}(B_R)).$$

Proof For any $\phi \in C_0^\infty(B_R \times (0,T))$,

$$\begin{aligned}
&\sup_{0\leq t\leq T}\int_{B_R} |\rho^\epsilon u^\epsilon - \rho u|^{\frac{4}{3}}\phi \\
&\leq \sup_{0\leq t\leq T}\int_{B_R} |\rho^\epsilon|^{\frac{4}{3}}|u^\epsilon - u|^{\frac{4}{3}}\phi + \sup_{0\leq t\leq T}\int_{B_R} |u|^{\frac{4}{3}}|\rho^\epsilon - \rho|^{\frac{4}{3}}\phi
\end{aligned}$$

It is straight forward the first integral converges to 0. To see the second integral converges to 0, we use Riesz Theorem and Dominated Convergence Theorem which has similar process as to prove lemma 2.3.4. □

Then we use a similar argument as to produce Lemma 2.3.17 and get

Lemma 2.3.32 *Let ρ^ϵ and u^ϵ be the solution obtained in Subsection 2.3.3. Then*

$$\rho^\epsilon u^\epsilon \to \rho u \ in \ C([0,T];L^{6/5}(B_R)).$$

To prove Theorem 2.3.29, we need to show that, ρ, u and d satisfy the following equations, for any $\phi \in C_c^\infty(B_R \times [0,T))$ with $\mathrm{div}\phi = 0$,

$$\begin{cases}
\int_{B_R} \rho_0\phi(x,0) + \int_0^T \int_{B_R} \rho\phi_t + \rho u \cdot \nabla\phi = 0, \\
\int_{B_R} m_0\phi(x,0) + \int_0^T \int_{B_R} \rho u\phi_t + \rho u \otimes u \cdot \nabla\phi - \nabla u \cdot \nabla\phi + g\phi = 0, \\
\int_{B_R} d_0\phi(x,0) + \int_0^T \int_{B_R} d\phi_t + (u \cdot \nabla\ d)\phi + [\triangle d - F(d)]\phi = 0.
\end{cases} \tag{2.3.36}$$

We already know that weak solution $(\rho^\epsilon, u^\epsilon, d^\epsilon)$ satisfy,

$$\begin{cases}
\int_{B_R} \rho_0^\epsilon\phi(x,0) + \int_0^T \int_{B_R} \rho^\epsilon\phi_t + \rho^\epsilon u^\epsilon \cdot \nabla\phi = 0 \\
\int_{B_R} m_0^\epsilon\phi(x,0) + \int_0^T \int_{B_R} \rho^\epsilon u^\epsilon\phi_t + \rho^\epsilon u^\epsilon \otimes u^\epsilon \cdot \nabla\phi - \nabla u^\epsilon \cdot \nabla\phi = 0 \\
\int_{B_R} d_0^\epsilon\phi(x,0) + \int_0^T \int_{B_R} d^\epsilon\phi_t + (u^\epsilon \cdot \nabla\ d^\epsilon)\phi + [\triangle d^\epsilon - F(d^\epsilon)]\phi = 0.
\end{cases}$$

The convergence of the initial terms is guaranteed by Lemma 2.3.1 and Lemma 2.3.4. All the other terms are converged in a straightforward way based on the compact results we have obtained.

Next, let us specify the energy estimate (2.3.33). Since the solution $(\rho_0^\epsilon, u_0^\epsilon, d_0^\epsilon)$ we have now is weak solution, instead of the identity (2.3.16), we have inequality

$$\frac{d}{dt}\int_{B_R}\frac{1}{2}\rho^\epsilon|u^\epsilon|^2+\frac{1}{2}|\nabla d^\epsilon|^2+F(d^\epsilon)+\int_{B_R}|\nabla u^\epsilon|^2+|\triangle d^\epsilon-f(d^\epsilon)|^2\leq 0.$$

Thus integrating it to get

$$\begin{aligned}&\int_{B_R}\frac{1}{2}\rho^\epsilon|u^\epsilon|^2+\frac{1}{2}|\nabla d^\epsilon|^2+F(d^\epsilon)+\int_0^t\int_{B_R}|\nabla u^\epsilon|^2+|\triangle d^\epsilon-f(d^\epsilon)|^2\\&\leq\int_{B_R}\frac{1}{2}\rho_0^\epsilon|u_0^\epsilon|^2+\frac{1}{2}|\nabla d_0^\epsilon|^2+F(d_0^\epsilon)\end{aligned}$$

for all $t\in[0,T]$. Then, by the construction of initial data is subsection 3.1, we observe that

$$\begin{aligned}\int_{B_R}\rho_0^\epsilon|u_0^\epsilon|^2dx&=\int_{B_R}\frac{1}{\rho_0^\epsilon}|m_0^\epsilon-\nabla q_0^\epsilon|^2dx\\&=\int_{B_R}\frac{|m_0^\epsilon|^2}{\rho_0^\epsilon}+\frac{|\nabla q_0^\epsilon|^2}{\rho_0^\epsilon}-\frac{2}{\rho_0^\epsilon}(\rho_0^\epsilon u_0^\epsilon+\nabla q_0^\epsilon)\cdot\nabla q_0^\epsilon dx\\&=\int_{B_R}\frac{|m_0^\epsilon|^2}{\rho_0^\epsilon}-\frac{|\nabla q_0^\epsilon|^2}{\rho_0^\epsilon}dx\end{aligned}$$

due to the fact that $u_0^\epsilon=0$ on ∂B_R and $\mathrm{div}u_0^\epsilon=0$. Thus with lemma 2.3.4, we have that

$$\int_{B_R}\rho_0^\epsilon|u_0^\epsilon|^2dx\leq\int_{B_R}\frac{m_0^2}{\rho_0}dx.$$

Since $d_0^\epsilon\to d_0$ in $H^2(B_R)$, we can obtain energy inequality (2.3.35). It completes the proof of Theorem 2.3.29.

2.3.5 Extend the bounded ball B_R to the whole space by letting $R\to\infty$

Take a sequence of balls B_{R_n} with radius $R_n=n$. Denote the solutions obtained in Subsection 2.3.4 on ball $B_R\times[0,T]$ as $(\rho_R, m_R=(\rho u)_R, d_R)$. Since the estimates (2.3.34) and (2.3.35) are independent of radius R, with similar argument

as Lemma 2.3.12 to Lemma 2.3.16 and (2.3.26) we can find subsequences ρ_{n_1}, u_{n_1}, m_{n_1}, and d_{n_1} from ρ_R, u_R, m_R, d_R for $R \geq 1$ such that

$$\rho_{n_1} \rightharpoonup \rho^1 \text{ in } L^2(B_1 \times (0,T)) \text{ with } 0 \leq \rho^1 \leq C$$

$$u_{n_1} \rightharpoonup u^1 \text{ in } L^2(0,T; H_0^1(B_1)) \text{ on the set } \rho^1 \neq 0$$

$$d_{n_1} \rightharpoonup d^1 \text{ in } L^2(0,T; H^2(B_1))$$

$$d_{n_1} \to d^1 \text{ in } L^2(0,T; H^1(B_1))$$

$$m_{n_1} \to \rho^1 u^1 \text{ in } C([0,T]; L^{6/5}(B_1)).$$

and limit functions (ρ^1, u^1, d^1) satisfy theorem 2.3.29 for $R = 1$, i.e., on $B_1 \times [0,T]$.

Next, on $B_2 \times [0,T]$, we take subsequences ρ_{n_2}, u_{n_2}, and d_{n_2} from ρ_{n_1}, u_{n_1}, d_{n_1} such that they converge to ρ^2, u^2, and d^2 respectively. It is obviously that

$$\rho^2|_{B_1} = \rho^1, \quad u^2|_{B_1} = u^1, \ d^2|_{B_1} = d^1.$$

Repeatedly, on $B_k \times [0,T]$, we can take subsequences ρ_{n_k}, u_{n_k} and d_{n_k} from $\rho_{n_{k-1}}$, $u_{n_{k-1}}$, $d_{n_{k-1}}$ such that they converge to ρ^k, u^k and d^k respectively. And we have that

$$\rho^k|_{B_{k-1}} = \rho^{k-1}, \quad u^k|_{B_{k-1}} = u^{k-1}, \ d^k|_{B_{k-1}} = d^{k-1}.$$

Then we take the diagonal $(\rho_{kk}, u_{kk}, d_{kk})$ as a sequence and let $k \to \infty$. This sequence (if necessary, take a subsequence of it) converges to (ρ, u, d), in $\mathbb{R}^3 \times [0,T]$ respectively. At the end, we only need to check the boundary condition (2.1.6). From the above process, we have that, on any ball B_k,

$$d^k|_{\partial B_k} = d_0(x)|_{\partial B_k}.$$

Thus,

$$\lim_{k\to\infty} d^k|_{\partial B_k} = \lim_{k\to\infty} d_0(x)|_{\partial B_k}.$$

The left is $\lim_{|x|\to\infty} d(x,t)$ for each $x \in \mathbb{R}^3$, while the right is $\lim_{|x|\to\infty} d_0(x)$ for each $x \in \mathbb{R}^3$. It completes the proof of (2.1.6). Therefore, we find a weak solution (ρ, u, d) satisfying Theorem (2.1.1) which closes this chapter.

Chapter 3

Regularity of Solutions in $\mathbb{R}^2$ and $\mathbb{R}^3$

In this chapter we establish the regularity and uniqueness of solutions to LCD system (1.0.15) on bounded domain in $\mathbb{R}^2$ and $\mathbb{R}^3$. The results presented extend the regularity and uniqueness results for constant density liquid crystal systems, obtained by Fanghua Lin and Chun Liu, [35].

3.1 The Settings and Main Results

We focus on the regularity of solutions to the flow of nematic liquid crystals satisfying the initial conditions:

$$\rho(x,0) = \rho_0(x), \quad 0 < M_1 \leq \rho_0(x) \leq M_2, \tag{3.1.1}$$

$$u(x,0) = u_0(x), \quad \nabla \cdot u_0 = 0, \quad u_0|_{\partial\Omega} = 0, \tag{3.1.2}$$

$$d(x,0) = d_0(x), \quad |d_0(x)| = 1, \tag{3.1.3}$$

and the boundary conditions:

$$u(\cdot,t)|_{\partial\Omega} = 0, \quad d(\cdot,t)|_{\partial\Omega} = d_0|_{\partial\Omega}. \tag{3.1.4}$$

Notice that the initial density ρ_0 in (3.1.1) has a lower bound M_1, it is not necessary to apply the renormalization method in Chapter 2 to obtain the weak solutions. On the other hand, for further consideration, our proof of regularity uses

energy estimates introduced by Ladyzhenskaya on the approximate solutions. The added viscosity term $\eta\Delta\rho$ and $\eta\nabla\rho\cdot\nabla u$ in the approximating system would make the analysis to establish the Ladyzhenskaya energy estimates for the approximate solutions much more complicated. Thus in Appendix 3.5 we sketch a proof of existence for the global weak solutions to system (1.0.15) without the introduction of the viscosity term for the density in the equation of the conservation of mass.

We obtain interior regularity with a relatively weak conditions on the initial data. For more regular data, we are able to obtain solutions which are regular up to the boundary. In the rest of the introduction we briefly describe our main results.

Regularity in 2 dimensions:

Theorem 3.1.1 *Suppose that Ω is a smooth bounded domain in $\mathbb{R}^2$. Let ρ_0, u_0 and d_0 satisfy (3.1.1)-(4.1.4). Suppose that $\rho_0\in C^1(\Omega)$, $u_0\in H^1(\Omega)$ and $d_0\in H^2(\Omega)$. Then, the system (1.0.15) has a global classical solution (ρ,u,d), that is, for all $T>0$ and some $\alpha\in(0,1)$*

$$u\in C^{1+\alpha/2,2+\alpha}((0,T)\times\Omega)$$

$$\nabla p\in C^{\alpha/2,\alpha}((0,T)\times\Omega)$$

$$d\in C^{1+\alpha/2,2+\alpha}((0,T)\times\Omega)$$

$$\rho\in C^1((0,T)\times\Omega).$$

Corollary 3.1.2 *Suppose (ρ_0,u_0,d_0) satisfies the conditions in Theorem 3.1.1. In addition, $\rho_0\in C^1(\bar\Omega)$, $u_0\in C^{2+\alpha}(\bar\Omega)$ and $d_0\in C^{2+\alpha}(\bar\Omega)$. Then the solution (ρ,u,d) is regular up to boundary, that is, the conclusions in Theorem 3.1.1 hold in $[0,T]\times\bar\Omega$.*

Indeed, more regularity on the initial data d_0 implies more regularity on the boundary due to the second condition in (3.1.4). Thus, it yields the regularity up to boundary, see [26].

The regularity for the flow of the nematic liquid crystals in dimension 2 turns out to be not too difficult. We first establish the Ladyzhenskaya energy estimate (3.2.9) and (3.2.10), similar to that in [35]. Then we apply the regularity

result for transport equations in [2] to obtain the Hölder continuity of the fluid density. Therefore Theorem 3.1.1 follows from the L^p estimates and Hölder estimates in [28] and a more or less standard bootstrapping between the three equations in the system (1.0.15).

Provided we have sufficiently small data or we work with sufficiently short time in 3D, we also obtain interior regularity. Given more restriction on data, the regularity can be obtained up to the boundary.

Regularity in 3 dimension with small data or short time:

Theorem 3.1.3 *Suppose that Ω is a smooth bounded domain in $\mathbb{R}^3$. Let ρ_0, u_0 and d_0 satisfy (3.1.1)-(3.1.4). Assume that $\rho_0 \in C^1(\Omega)$, $u_0 \in H^1(\Omega)$ and $d_0 \in H^2(\Omega)$. Suppose that (ρ, u, d) is a weak solution to the system (1.0.15) in Theorem 3.5.1. Then*

1. There is a positive small number ϵ_0 such that if

$$\|u_0\|^2_{H^1(\Omega)} + \|\nabla d_0\|^2_{H^1(\Omega)} \leq \epsilon_0, \tag{3.1.5}$$

then the system (1.0.15) has a classical solution (ρ, u, d) in the time period $(0, T)$, for all $T > 0$. That is, for some $\alpha \in (0, 1)$

$$\begin{aligned} u &\in C^{1+\alpha/2, 2+\alpha}((0,T) \times \Omega) \\ \nabla p &\in C^{\alpha/2, \alpha}((0,T) \times \Omega) \\ d &\in C^{1+\alpha/2, 2+\alpha}((0,T) \times \Omega) \\ \rho &\in C^1((0,T) \times \Omega). \end{aligned} \tag{3.1.6}$$

2. For general data, there exists a positive number $\delta_0 = \delta_0(\rho_0, u_0, d_0)$ such that (3.1.6) holds in the interval $(0, T)$ for $T \leq \delta_0$.

Corollary 3.1.4 *Suppose in addition to the hypothesis in Theorem 3.1.3, that $\rho_0 \in C^1(\bar{\Omega})$, $u_0 \in C^{2+\alpha}(\bar{\Omega})$ and $d_0 \in C^{2+\alpha}(\bar{\Omega})$, then the solution is regular up to the boundary for data small in the sense (3.1.5) or for large data and sufficiently short time.*

The proof of the regularity of the solution to the system (1.0.15) in dimension 3 takes the same approach as in dimension 2 but is much more complicated.

First in contrast to the cases of dimension 2, we only get the Ladyzhenskaya energy estimates when either the initial data is small in the sense as described in (4.4.8) or T is small. Our calculations and estimates are based on those in [35], with interesting modifications. We use ideas of [35] making it work in a rather different way. We keep the potentially small terms $\|u\|_{L^2}$ and $\|\nabla d\|_{L^2}$ instead of throwing them away. This gives a more unified way to derive the Ladyzhenskaya energy estimates in the cases:

- of small data
- for short time.

After having the Ladyzhenskaya energy estimates, in contrast to the two dimensional case, we do not have the Hölder continuity for the fluid density. Instead we observe that we have small oscillations of the density over small balls in $\Omega \times [0, T]$ provided that either the initial data is small or for short time. This turns out to be enough to carry out the frozen coefficient method to improve the regularity of the fluid velocity. We refer the reader to [30] for a reference of the frozen coefficient method. We give the idea of the method in Appendix 6. Our key lemma on the oscillation of the fluid density is as follows:

Lemma 3.1.5 *Suppose that Ω is a smooth bounded domain in $\mathbb{R}^3$. Let ρ_0, u_0 and d_0 satisfy (3.1.1)-(3.1.4). Assume that $\rho_0 \in C^1(\bar{\Omega})$, $u_0 \in H^1(\Omega)$ and $d_0 \in H^2(\Omega)$. Suppose that (ρ, u, d) is a weak solution to the system (1.0.15) in Theorem 3.5.1. Let $t_1 \in (0, T)$ and $p \in \Omega$, define*

$$\mathcal{A}_{(p,t_1)} = (B_{r_0}(p) \cap \Omega) \times ([t_1 - r_0, t_1 + r_0] \cap [0, T]).$$

Then, for any $\epsilon > 0$, there exists $\epsilon_0 > 0$ and $r_0 > 0$ such that for $p \in \Omega$ and all $T > 0$,

$$\sup_{(q,t_2) \in \mathcal{A}_{(p,t_1)}} |\rho(q, t_2) - \rho(p, t_1)| \leq \epsilon, \tag{3.1.7}$$

provided that either

$$\|u_0\|^2_{H^1(\Omega)} + \|\nabla d_0\|^2_{H^1(\Omega)} \leq \epsilon_0 \quad \text{or } T \leq \delta_0.$$

Remark 3.1.6 *The interior regularity in Theorem 3.1.1 and Theorem 3.1.3 is obtained by bootstrapping argument.*

To close this section, we state the uniqueness of solution in the following sense,

Theorem 3.1.7 *Let (ρ, u, d) be the solution to system (1.0.15) and (3.1.1)-(3.1.4) obtained in Corollary 3.1.2 for two dimensions or in Corollary 3.1.4 for three dimensions. Let $(\bar{\rho}, \bar{u}, \bar{d})$ be a weak solution to system (1.0.15) with (3.1.1)-(3.1.4) satisfying the following energy inequalities:*

$$\int_\Omega |\bar{\rho}|^2 dx \le \int_\Omega |\rho_0|^2 dx, \tag{3.1.8}$$

$$\int_\Omega \frac{1}{2}\bar{\rho}|\bar{u}|^2 + \frac{1}{2}|\nabla \bar{d}|^2 + F(\bar{d})dx + \int_0^T \int_\Omega |\nabla \bar{u}|^2 + |\triangle \bar{d} - f(\bar{d})|^2 dxdt \tag{3.1.9}$$
$$\le \int_\Omega \frac{1}{2}\rho_0|u_0|^2 + \frac{1}{2}|\nabla d_0|^2 + F(d_0)dx.$$

Then $(\rho, u, d) \equiv (\bar{\rho}, \bar{u}, \bar{d})$.

Remark 3.1.8 *Note that the weak solutions established in [41] and [21] satisfy the inequalities recorded above. Hence these solutions coincide with solutions we obtained in Corollary 3.1.4 due to the uniqueness Theorem 3.1.7.*

3.2 Regular solutions in $\mathbb{R}^2$

In this section we use the Ladyzhenskaya energy method [29] to show that $u \in L^\infty([0,T], H^1(\Omega)) \bigcap L^2([0,T], H^2(\Omega))$ and $d \in L^\infty([0,T], H^2(\Omega)) \bigcap L^2([0,T], H^3(\Omega))$. Then we apply a result from [2] to get the Hölder continuity of the density ρ. With the continuity of the density ρ we apply the so-called frozen coefficient method to get higher regularities for ρ, p, u, d by bootstrapping. The proof of the Ladyzhenskaya energy inequality is similar to that in [35]. The key inequalities used often in this chapter are the following Gagliardo-Nirenberg inequality (cf. [15]) :

Lemma 3.2.1 *(Gagliardo-Nirenberg) If Ω is a smooth bounded domain in $\mathbb{R}^n$, then*

$$\|u\|^4_{L^4(\Omega)} \le C\|u\|^2_{L^2(\Omega)}(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}), \quad n = 2 \tag{3.2.1}$$

$$\|u\|^4_{L^4(\Omega)} \le C\|u\|_{L^2(\Omega)}(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})^{\frac{3}{2}}, \quad n = 3. \tag{3.2.2}$$

Moreover, if $u|_{\partial\Omega} = 0$, *then*

$$\|u\|^4_{L^4(\Omega)} \le C\|u\|^2_{L^2(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}, \tag{3.2.3}$$

when $n = 2$ *and*

$$\|u\|^4_{L^4(\Omega)} \le C\|u\|_{L^2(\Omega)}\|\nabla u\|^3_{L^2(\Omega)}, \tag{3.2.4}$$

when $n = 3$.

3.2.1 The $L^\infty(H^1)$ and $L^2(H^2)$ estimates of the velocity

Our strategy is similar as in [35]. We first establish the desired bounds for the Galerkin approximating solutions (ρ_m, u_m, d_m) in the sequence that one has from Galerkin method when proving the existence (cf. Appendix 3.5). By passing to the weak limit we then obtain the desired bounds for the weak solution (ρ, u, d). We first set

$$\Phi^2_m(t) = \|\nabla u^m\|^2_{L^2} + \|\triangle d^m\|^2_{L^2}. \tag{3.2.5}$$

Then to show that $u^m \in L^\infty(H^1)\bigcap L^2(H^2)$ and $d^m \in L^\infty(H^2)\bigcap L^2(H^3)$ we calculate $\frac{d}{dt}\Phi^2_m$. Namely,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Phi^2_m(t) &= \int_\Omega \nabla u^m \cdot \nabla u^m_t dx + \int_\Omega \triangle d^m \cdot \triangle d^m_t dx \\
&= -\int_\Omega \rho^m |u^m_t|^2 dx - \int_\Omega |\nabla\triangle d^m|^2 dx \\
&+ \int_\Omega \nabla\triangle d^m \cdot \nabla(u^m \cdot \nabla d^m) - \triangle d^m \cdot \triangle(f(d^m))dx \\
&+ \int_\Omega -\rho^m(u^m \cdot \nabla u^m)u^m_t - u^m_t \nabla d^m \triangle d^m dx \\
&= -\int_\Omega \rho^m |u^m_t|^2 dx - \int_\Omega |\nabla\triangle d^m|^2 dx \\
&+ I + II + II + IV,
\end{aligned}$$

here we used the equation (3.5.6). We find

$$\begin{aligned}
I &\le \varepsilon\|\nabla\triangle d^m\|^2_{L^2} + C\|\nabla u^m\|^2_{L^4}\|\nabla d^m\|^2_{L^4} + C\|u^m\|^2_{L^4}\|\triangle d^m\|^2_{L^4} + C\|u^m\|^2_{L^4} \\
&\le \varepsilon\|\triangle u^m\|^2_{L^2} + 2\varepsilon\|\nabla\Delta d^m\|^2_{L^2} + C\Phi^4_m + C\Phi^3_m + C\Phi^2_m + C\Phi_m,
\end{aligned}$$

by Gagliardo-Nirenberg inequality, the basic energy estimate (3.5.12) and the fact that

$$\|D^2 d^m\|_{L^4} \le C(\|\Delta d^m\|_{L^4} + \|d^m\|_{L^4}).$$

And similarly,

$$II \le C\Phi_m^2 + C,$$

$$III \le \varepsilon\|\triangle u^m\|_{L^2}^2 + \varepsilon\|u_t^m\|_{L^2}^2 + C\Phi_m^4 + C\Phi_m^3,$$

$$IV \le \varepsilon\|\nabla\triangle d^m\|_{L^2}^2 + \varepsilon\|u_t^m\|_{L^2}^2 + C\Phi_m^4 + C\Phi_m^2.$$

Moreover, from equation (3.5.6), we have

$$\|\triangle u^m\|_{L^2}^2 \le 2M_2\|u_t^m\|_{L^2}^2 + 2\|\nabla\Delta d^m\|_{L^2}^2 + C\Phi_m^4 + C\Phi_m^3 + C\Phi_m^2 \tag{3.2.6}$$

and

$$M_1\|u_t^m\|_{L^2}^2 \le 2\|\Delta u^m\|_{L^2}^2 + 2\|\nabla\Delta d^m\|_{L^2}^2 + C\Phi_m^4 + C\Phi_m^3 + C\Phi_m^2. \tag{3.2.7}$$

Therefore, we arrive at

$$\frac{d}{dt}\Phi_m^2(t) + \|\Delta u^m\|_{L^2}^2 + \|\nabla\Delta d^m\|_{L^2}^2 \le C\Phi_m^4(t) + C, \tag{3.2.8}$$

which implies, as in [35], again in the light of the basic energy inequality (3.5.12),

$$\Phi_m^2(t) \le (\Phi^2(0) + C)e^{C\int_0^T(\Phi_m^2(t)+1)dt} \le (\Phi(0)^2 + C)e^{CT+C}$$

where $\Phi^2(0) = \|\nabla u_0\|_{L^2}^2 + \|\triangle d_0\|_{L^2}^2$, and

$$\int_0^T \|\Delta u^m\|_{L^2}^2 dt + \int_0^T \|\nabla\Delta d^m\|_{L^2}^2 dt \le C.$$

Hence we have obtained

Theorem 3.2.2 *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and the initial data (ρ_0, u_0, d_0) satisfy the conditions of Theorem 3.1.1. Suppose that $\Phi(0)^2 = \|\nabla u_0\|_{L^2}^2 + \|\triangle d_0\|_{L^2}^2 < \infty$. Then there exists a constant C such that the approximating solutions (ρ^m, u^m, d^m) of the system (1.0.15) obtained in Theorem 3.5.1 satisfies*

$$\|\nabla u^m\|_{L^2}^2 + \|\Delta d^m\|_{L^2}^2 \le e^{CT+C}(\Phi^2(0) + C) \tag{3.2.9}$$

for all $t \in [0, T]$ and

$$\int_0^T \|\Delta u^m\|_{L^2}^2 dt + \int_0^T \|\nabla\Delta d^m\|_{L^2}^2 dt \le C. \tag{3.2.10}$$

From the last theorem it follows that

Corollary 3.2.3 *Under the same hypothesis of last theorem, there exists a solution* (ρ, u, d) *to (1.0.15) which satisfies the energy inequalities*

$$\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \leq e^{CT+C}(\Phi^2(0) + C) \tag{3.2.11}$$

for all $t \in [0, T]$ *and*

$$\int_0^T \|\Delta u\|_{L^2}^2 dt + \int_0^T \|\nabla \Delta d\|_{L^2}^2 dt \leq C. \tag{3.2.12}$$

Proof It follows by extracting a subsequence of the Galerkin approximations (ρ^m, u^m, d^m) and passing to the limit. □

3.2.2 Hölder continuity of the fluid density

Next we recall a regularity lemma for the transport equation from [2] to get Hölder continuity for the fluid density ρ.

Lemma 3.2.4 *([2]) Suppose that* $u \in L^\infty([0,T], H^1(\Omega)) \bigcap L^2([0,T], H^2(\Omega))$. *And suppose that*

$$\rho_t + u \cdot \nabla \rho = 0$$

in $\Omega \times (0, T)$. *Assume that* $\rho(0) \in C^1(\Omega)$ *and that* $\Omega \subset \mathbb{R}^2$ *is smooth and bounded. Then* $\rho \in C^\alpha(\Omega \times [0, T])$ *for some* $\alpha \in (0, 1)$ *which depends only on the initial data,* T *and* Ω.

3.2.3 Proof of Theorem 3.1.1

Based on [2], we sketch a proof here for completeness. Rewrite the third equation in (1.0.15) as

$$d_t - \triangle d = -u \cdot \nabla d - f(d). \tag{3.2.13}$$

It yields from the basic energy inequality (3.5.4) and estimate (3.2.9) that ,

$$u \cdot \nabla d \in L^\infty(0, T; L^r(\Omega)), \quad \text{for any } r \in [1, \infty).$$

Then due to standard estimates of solutions to parabolic equations (see [28] and [1]), from the equation (3.2.13) we have

$$d \in W^{1,r}(W^{2,s}), \quad \text{for any } r, s \in [1, \infty),$$

which implies that

$$d \in C^{\alpha/2, 1+\alpha}([0,T] \times \bar{\Omega}) \tag{3.2.14}$$

for any $\alpha \in (0,1)$. Now we go to the second equation in (1.0.15) and use the above estimates for d to improve the estimate on u via the frozen coefficient method [30], after we have the Hölder continuity for the fluid density ρ, as done in [2] to derive that

$$u \in W^{1,p}(W^{2,q}),$$

for some $q > 1$ and any $p > 1$. Therefore we have

$$u \in C^{\alpha/2, \alpha}([0,T] \times \bar{\Omega}).$$

Next going back to the third equation in (1.0.15), by the standard Hölder estimates for parabolic equations (see [28] Theorem 5.1 in Chapter VII) we have

$$d \in C^{1+\frac{\alpha}{2}, 2+\alpha}((0,T) \times \Omega),$$

for some $\alpha \in (0,1)$. Therefore we are able to go back to the the second equation in (1.0.15), again via frozen coefficient method [30], to derive that

$$u \in C^{1+\frac{\alpha}{2}, 2+\alpha}((0,T) \times \Omega).$$

Finally $\rho \in C^1((0,T) \times \Omega)$ follows from the regularity of u and the regularity of the pressure p follows easily from the regularity of (ρ, u, d) similarly as in [2].

Remark 3.2.5 *By bootstrapping argument, we can get higher regularity.*

Proof of Corollary 3.1.2 follows by Krylov's Theorem 10.3.3 in [26], which for convenience of reader, we recall here [26],

Theorem 3.2.6 *For any $h \in C^{\alpha/2,\alpha}((0,T)\times\Omega)$ and boundary $g \in C^{1+\alpha/2,2+\alpha}((0,T)\times\Omega)$ there exists a unique function $u \in C^{1+\alpha/2,2+\alpha}([0,T] \times \bar{\Omega})$ satisfying the equation $Lu - u_t = h$ in $(0,T) \times \Omega)$ and equal g on boundary of $(0,T) \times \Omega)$.*

We apply this to both the second and third equation in (1.0.15). Applying to the second equation, $h = u \cdot \nabla d + f(d)$, $g = d_0$ and $L = \triangle$. Applying to the third equation, $h = u \cdot \nabla u + \nabla p + \nabla \cdot (\nabla d \otimes \nabla d)$, $g = 0$ and $L = \triangle$. Proceeding as in the last theorem yields $h \in C^{\alpha/2,\alpha}((0,T) \times \Omega)$ for both equations. On the boundary of Ω, $u = 0$, $d = d_0$ and $d_0 \in C^{2+\alpha}(\Omega)$ by hypothesis. Hence the conclusion of the Corollary follows.

3.3 Regular Solutions in $\mathbb{R}^3$

In this section we consider the solutions of (1.0.15) for $\Omega \subset \mathbb{R}^3$ a bounded domain. We establish regularity in two cases

- global regularity with small initial data
- short time regularity.

First we adopt the idea from [35] to derive the Ladyzhenskaya energy estimates. In contrast to the cases of dimension 2, we will only get the Ladyzhenskaya energy estimates for the above two cases as in [35]. Our calculations and estimates are similar to those in [35], with interesting modifications. After having the Ladyzhenskaya energy estimates, unlike the case of dimension 2, we will not have Hölder continuity for the fluid density right away. Instead, we observe that we can have the oscillation of the density over small balls in $\Omega \times [0,T]$ to be small, provided that either the initial data is small or we work for short time, which turns out to be enough to carry out the frozen coefficient method to improve the regularity of the fluid velocity.

3.3.1 Ladyzhenskaya Energy Estimates

Our derivation of the Ladyzhenskaya energy estimates in [35] in dimension 3 is rather an interesting modification of the original Ladyzhenskaya's argument for the pure fluid systems. In [35] for the argument to work, it needs either the viscosity for the fluid to be very large or the time to be very short. We use the same idea, but, instead we assume the initial data to be small or the time to be short while the viscosity of the fluid is a fixed constant. For the convenience of the arguments in

our context, without loss of generality we take the constant to be 1. Let us set as before:

$$\Phi_m^2(t) = \|\nabla u^m\|_{L^2}^2 + \|\triangle d^m\|_{L^2}^2 \tag{3.3.1}$$

Then, as in the previous section, we will first derive the Ladyzhenskaya energy estimates for the Galerkin approximate solutions (ρ^m, u^m, d^m) and pass to the weak limit to obtain the Ladyzgenskaya energy estimates for the weak solutions (ρ, u, d) to the system (1.0.15) as desired.

Again, Using $u^m|_{\partial\Omega} = d_t^m|_{\partial\Omega} = 0$, and $\triangle d^m|_{\partial\Omega} = 0$, by integration by parts, it still follows that,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Phi_m^2(t) &= \int_\Omega \nabla u^m \cdot \nabla u_t^m dx + \int_\Omega \triangle d^m \cdot \triangle d_t^m dx \\
&= -\int_\Omega \rho^m |u_t^m|^2 dx - \int_\Omega |\nabla \triangle d^m|^2 dx \\
&+ \int_\Omega \nabla \triangle d^m \cdot \nabla (u^m \cdot \nabla d^m) - \triangle d^m \cdot \triangle (f(d^m)) dx \\
&+ \int_\Omega -\rho^m (u^m \cdot \nabla u^m) u_t^m - u_t^m \nabla d^m \triangle d^m dx \\
&= -\int_\Omega \rho^m |u_t^m|^2 dx - \int_\Omega |\nabla \triangle d^m|^2 dx \\
&+ I + II + II + IV,
\end{aligned}$$

We will proceed the same way as we did in dimension 2 except the Gagliardo-Nirenberg inequality is different in dimension 3 from that in dimension 2. More importantly we will keep the terms $\|u^m\|_{L^2}$ and $\|\nabla d^m\|_{L^2}$ whenever necessary. We may derive

$$I \le \varepsilon \|\Delta u^m\|_{L^2}^2 + \varepsilon \|\nabla \Delta d^m\|_{L^2}^2 + (\|u^m\|_{L^2} + \|\nabla d^m\|_{L^2})(C\Phi_m^8 + C),$$

$$II \le C\Phi_m^2 + C\|\nabla d^m\|_{L^2},$$

$$III \le \varepsilon \|u_t^m\|_{L^2}^2 + \varepsilon \|\Delta u^m\|_{L^2}^2 + C\|u^m\|_{L^2}^2 \Phi_m^8 + C\Phi_m^2,$$

and

$$IV \le \varepsilon \|u_t^m\|_{L^2}^2 + \varepsilon \|\nabla \Delta d^m\|_{L^2}^2 + C\|\nabla d^m\|_{L^2}^2 \Phi_m^8 + C\|\nabla d^m\|_{L^2}^2.$$

To relate Δu^m back to u_t^m, from equation (3.5.6), we have that

$$\|\triangle u^m\|_{L^2}^2 \le 2M_2 \|u_t^m\|_{L^2}^2 + 2\|\nabla \Delta d^m\|_{L^2}^2 + (\|u^m\|_{L^2}^2 + \|\nabla d^m\|_{L^2}^2)(C\Phi_m^8 + C) + C\Phi_m^2$$

and

$$M_1\|u_t^m\|_{L^2}^2 \le 2\|\Delta u^m\|_{L^2}^2 + 2\|\nabla\Delta d^m\|_{L^2}^2 + (\|u^m\|_{L^2}^2 + \|\nabla d^m\|_{L^2}^2)(C\Phi_m^8 + C) + C\Phi_m^2.$$

Therefore we arrive at

$$\|\Delta u^m\|_{L^2}^2 + \|\nabla\Delta d^m\|_{L^2}^2 + \frac{d}{dt}\Phi_m^2 \le C\Phi_m^2 + (\|u^m\|_{L^2} + \|\nabla d^m\|_{L^2})(C\Phi_m^8 + C).$$

Set

$$\tilde{\Phi}_m^2 = \Phi_m^2 + \|u^m\|_{L^2} + \|\nabla d^m\|_{L^2}$$

and observe that

$$\frac{d}{dt}\tilde{\Phi}_m^2 \le (C + C(\|u^m\|_{L^2} + \|\nabla d^m\|_{L^2})\tilde{\Phi}_m^6)\tilde{\Phi}_m^2 \tag{3.3.2}$$

or set

$$\tilde{\Phi}_m^2 = \Phi_m^2 + \|u^m\|_{L^2} + \|\nabla d^m\|_{L^2} + 1$$

we have

$$\frac{d}{dt}\tilde{\Phi}_m^2 \le C\tilde{\Phi}_m^8. \tag{3.3.3}$$

It is clear from (3.3.3) one can prove the Ladyzhenskaya energy estimates when T is small. To get the Ladyzhenskaya energy estimates for small initial data we will use an idea similar to that in [35]. Suppose that

$$\|u_0\|_{H^1}^2 + \|d_0\|_{H^2}^2 = \theta_0.$$

Recall that by the basic energy estimate we have

$$\|u^m\|_{L^2} + \|\nabla d^m\|_{L^2} \le \|u_0\|_{H^1}^2 + \|d_0\|_{H^2}^2 \le \theta_0.$$

Then we claim that, if θ_0 is so small that

$$C\theta_0(4e^{C+1}\theta_0)^3 < 1, \tag{3.3.4}$$

then

$$\tilde{\Phi}_m^2 \le 4e^{C+1}\theta_0,$$

for all t. First we prove the claim for $t \in [0,1]$. Assume otherwise, there must be $t_0 \in (0,1)$ such that

$$\tilde{\Phi}_m^2(t_0) = 4e^{C+1}\theta_0 \tag{3.3.5}$$

and

$$\tilde{\Phi}_m^2(t) \le 4e^{C+1}\theta_0$$

for all $t \in (0, t_0]$. Therefore, from (3.3.2), by the choice of θ_0 in (3.3.4), we have

$$\frac{d}{dt}\tilde{\Phi}_m^2 \le (C+1)\tilde{\Phi}_m^2$$

for all $t \in (0, t_0)$ and $\tilde{\Phi}_m^2(0) \le 2\theta_0$, which implies that

$$\tilde{\Phi}_m^2(t_0) \le e^{C+1}\tilde{\Phi}_m^2(0) \le 2e^{C+1}\theta_0$$

and thus contradicts (3.3.5). For $t > 1$, we simply observe, as in [35], that the basic energy inequality (3.5.4)

$$\int_{t-1}^{t} \tilde{\Phi}_m^2(t)dt \le 2\theta_0$$

implies that there is $t_0 \in (t-1, t)$ such that

$$\tilde{\Phi}_m^2(t_0) \le 2\theta_0.$$

Then one may repeat the above argument to conclude that

$$\tilde{\Phi}_m^2(t) \le 4e^{C+1}\theta_0.$$

As in the case of 2 dimension, passing to the limit the existence of weak solutions will follow from the uniform estimates we have obtained. This weak solution satisfies

Theorem 3.3.1 *Suppose that Ω is a smooth bounded domain in $\mathbb{R}^3$. Let ρ_0, u_0 and d_0 satisfy (3.1.1)-(3.1.4). Assume that*

$$\|u_0\|_{H^1} + \|d_0\|_{H^2} < \infty.$$

Let (ρ, u, d) be a weak solution to the system (1.0.15) with data (ρ_0, u_0, d_0). There is $\epsilon_0 > 0$ such that if

$$\|u_0\|_{H^1} + \|d_0\|_{H^2} \le \epsilon_0,$$

then

$$\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \int_0^T (\|\Delta u\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2)dt \le C(\|u_0\|_{H^1} + \|d_0\|_{H^2}) \quad (3.3.6)$$

holds for any positive T. For data with no smallness condition there is a δ_0 depending on the data such that (3.3.6) holds for all $T \le \delta_0$.

3.3.2 Oscillation of the Fluid Density

In this subsection, we show that the oscillation of the density ρ over small balls in space time. This estimate follows by the estimate on u from the Ladyzhenskaya energy estimates obtained in the previous subsection. For convenience of the readers we restate Lemma 3.1.5.

Lemma 3.3.2 *Suppose that Ω is a smooth bounded domain in $\mathbb{R}^3$. Let ρ_0, u_0 and d_0 satisfy (3.1.1)-(3.1.4). Assume that $\rho_0 \in C^1(\bar{\Omega})$, $u_0 \in H^1(\Omega)$ and $d_0 \in H^2(\Omega)$. Suppose that (ρ, u, d) is a weak solution to the system (1.0.15) constructed in Theorem 3.5.1 (see Appendix). Let $t_1 \in (0,T)$ and $p \in \Omega$, define*

$$\mathcal{A}_{(p,t_1)} = (B_{r_0}(p) \cap \Omega) \times ([t_1 - r_0, t_1 + r_0] \cap [0,T]).$$

Then, for any $\epsilon > 0$, there exists $\epsilon_0 > 0$ and $r_0 > 0$ such that for $p \in \Omega$ and all $T > 0$,

$$\sup_{(q,t_2)\in\mathcal{A}_{(p,t_1)}} |\rho(q,t_2) - \rho(p,t_1)| \leq \epsilon,$$

provided that either

$$\|u_0\|^2_{H^1(\Omega)} + \|\nabla d_0\|^2_{H^2(\Omega)} \leq \epsilon_0 \quad \text{or} \quad T \leq \delta_0.$$

Proof The equation of the conservation of mass is

$$\rho_t + u \cdot \nabla \rho = 0. \tag{3.3.7}$$

For reasonably regular velocity we may solve (3.3.7) by the method of characteristics. Due to Osgood theorem [47], since $u \in H^2$ for almost all $t \in [0,T]$ and in light of the Ladyzhenskaya energy estimates in Theorem 3.3.1, there is a unique solution to the Cauchy problem corresponding to (3.3.7) by finding trajectories of the liquid particles

$$\frac{dy}{d\tau} = u(y,\tau), \quad y|_{\tau=t} = x; \quad x \in \Omega, \quad t \in (0,T)$$

and defining

$$\rho(x,t) = \rho_0(y(0,x,t)). \tag{3.3.8}$$

Step 1 Fix a time $t \in [0,T]$. Let x_1 and x_2 be two arbitrary points from Ω satisfying $|x_1 - x_2| \leq d < 1$. For any $\tau \in (0,t)$, assume

$$y_1 = y(\tau, x_1, t), \quad y_2 = y(\tau, x_2, t),$$

then the difference $z(\tau) = y_1 - y_2$ is the solution to the Cauchy problem

$$\frac{dz}{d\tau} = u(y_1, \tau) - u(y_2, \tau), \quad z|_{\tau=t} = x_1 - x_2.$$

For $0 \leq \alpha \leq 1/2$, by a standard Sobolev embedding theorem in [15], we have

$$\frac{d|z|}{d\tau} \leq C\|u(\tau)\|_{H^2(\Omega)}|z|^\alpha, \quad \tau \in (0, t).$$

Integrating form τ to t,

$$|z|^{1-\alpha} \leq |x_1 - x_2|^{1-\alpha} + CT^{\frac{1}{2}}\|u\|_{L^2(0,T;H^2(\Omega))}. \tag{3.3.9}$$

As a consequence of Theorem 3.3.1, we know that $|z|$ is as small as needed provided that $|x_1 - x_2|$ and $\|u_0\|_{H^1} + \|d_0\|_{H^2}$ are small or T is small. Therefore

$$|\rho(x_1, t) - \rho(x_2, t)| = |\rho_0(y(0, x_1, t)) - \rho_0(y(0, x_2, t))| \leq C|z|$$

is as small as one wants provided that $|x_1 - x_2|$ and $\|u_0\|_{H^1} + \|d_0\|_{H^2}$ are small or T is small.

Step 2 Fix a point $x \in \Omega$. Let $t_1, t_2 \in [0, T]$ arbitrary, let

$$y_1 = y(\tau, x, t_1), \quad y_2 = y(\tau, x, t_2).$$

Assume that $x' = y_2|_{\tau=t_1}$. Then due to uniqueness, the integral curve $y_2(\tau)$ can be considered as a solution to the Cauchy problem

$$\frac{dy_2}{d\tau} = u(y_2, \tau), \quad y_2|_{\tau=t_1} = x'$$

with initial data at $\tau = t_1$. Hence, the difference $z(\tau) = y_1 - y_2$ is the solution of the Cauchy problem

$$\frac{dz}{d\tau} = u(y_1, \tau) - u(y_2, \tau), \quad z|_{\tau=t_1} = x - x'.$$

By the definition of x', we have

$$x' = x - \int_{t_1}^{t_2} u(y_2(s, x, t_2), s)ds.$$

Therefore, again due to a standard Sobolev embedding Theorem,

$$|x - x'| \leq \int_{t_1}^{t_2} |u(y, s)|ds \leq C \int_{t_1}^{t_2} \|u\|_{H^2(\Omega)}ds \leq C\|u\|_{L^2(0,T;H^2(\Omega))}|t_1 - t_2|^{\frac{1}{2}}.$$

By Step 1, we conclude that

$$|\rho(x,t_1)-\rho(x,t_2)| = |\rho_0(y(0,x,t_1))-\rho_0(y(0,x,t_2))| = |\rho_0(y(0,x,t_1))-\rho_0(y(0,x',t_1))|$$

is as small as needed provided that $|t_1 - t_2|$ and $\|u_0\|_{H^1} + \|d_0\|_{H^2}$ are small or T is small. This completes the proof of the Lemma. □

3.3.3 Proof of Theorem 3.1.3

In this subsection, we finish the proof of Theorem 3.1.3. This is done by combining Theorem 3.3.1, Lemma 3.1.5 from the previous subsections, the frozen coefficient technique applied to $L^p(L^q)$ estimates, Hölder estimates and a bootstrapping argument between the three equations of the system (1.0.15).

In the appendix we briefly describe the frozen coefficient method. For simplicity, we show how the method works for the density dependent Navier-Stokes equation and obtain the estimate for L^q space with $q > \sqrt{3}$. We apply now this method to our approximating solutions (ρ^m, u^m, d^m). These solutions for each m satisfy the conclusion of Theorem 3.1.3. We now show that the estimates are uniformly in m applying the frozen coefficient method, in which case we let $q = 2$. That allows us to pass to the limit (ρ, u, d) and obtain (4.1.8) for weak solutions. We now show briefly the steps to yield the necessary uniform bound.

Here to simplify the notation we denote the approximating solutions by (ρ, u, d). We notice first that since

$$u \in L^\infty(0,T;L^6(\Omega)), \quad \nabla d \in L^\infty(0,T;L^6(\Omega)),$$

we have

$$u \cdot \nabla d \in L^\infty(0,T;L^3(\Omega)).$$

By standard parabolic estimates on the third equation in (1.0.15) (cf. [28] and [1]), we have

$$d \in W^{1,p}(W^{2,3}),$$

for all $1 < p < \infty$, which implies that $\nabla d \in L^\infty(0,T;L^q(\Omega))$ for any $q \in (1,\infty)$. Thus, we have

$$u \cdot \nabla d \in L^\infty(0,T;L^q(\Omega)), \quad \forall q \in (1,6).$$

Applying the same standard parabolic estimate on the third equation in (1.0.15) yields

$$d \in W^{1,p}(W^{2,q}), \forall p \in (1,\infty) \text{ and } q \in (1,6), \tag{3.3.10}$$

which implies that $d \in C^{\alpha/2,1+\alpha}([0,T] \times \bar{\Omega})$ for some $\alpha \in (0,1)$ and

$$\nabla d \Delta d \in L^{\infty}(0,T;L^{q}(\Omega)), \quad \forall q \in (1,6).$$

In the second equation of (1.0.15), the estimates for the conservation of momentum with constant density can be extended to the non-constant density cases when Lemma 3.1.5 is available. This is done via the frozen coefficient method.

We know that

$$u \cdot \nabla u \in L^{\infty}(0,T;L^{\frac{3}{2}}(\Omega)).$$

Now we apply the frozen coefficient method using the oscillation estimates for the density, (Lemma 3.1.5) to yield

$$u \in W^{1,p}(W^{2,3/2}), \quad \forall p \in (1,\infty)$$

Thus $u \in L^{\infty}(0,T;W^{1,3}(\Omega))$ and $u \cdot \nabla u \in L^{\infty}(0,T;L^{2}(\Omega))$. Repeating the above argument yields

$$u \in W^{1,p}(W^{2,2}), \quad \forall p \in (1,\infty), \tag{3.3.11}$$

from where it follows that $u \in C^{\alpha}(\Omega \times [0,T])$.

Back to the third equation in (1.0.15), we conclude that

$$d \in C^{1+\frac{\alpha}{2},2+\alpha}((0,T) \times \Omega)$$

for some $\alpha \in (0,1)$ via the standard Hölder estimates for parabolic equations (cf. [28], Chapter VII). From here the argument for the pure fluid systems works with no further significant modifications. This completes the proof of Theorem 3.1.3.

Proof of Corollary 3.1.4: Follows by Krylov's Theorem [27] just as in the case of two dimension.

3.4 Uniqueness of Solution

In this section we establish Theorem 3.1.7. For the LCD system with constant density, Lin and Liu [35] proved that the solution (u,d) is unique provided

$u \in L^\infty(0,T;H^1)$ and $d \in L^\infty(0,T;H^2)$. The idea is to calculate the energy law satisfied by the difference of two solutions and establish a Gronwall's inequality. In our case, to calculate the energy law of the difference of two solutions it has some extra terms involving the density. Hence the estimates are more involved requiring additional bounds on the strong solution (ρ, u, d) to yield a Gronwall's inequality. In 2D, we need

$$\nabla\rho, \nabla u \in L^\infty((0,T)\times\Omega), \quad u_t, u\cdot\nabla u \in L^\infty(0,T;L^q(\Omega)), q>2. \tag{3.4.1}$$

In 3D, we need

$$\nabla\rho, \nabla u \in L^\infty((0,T)\times\Omega), \quad u_t, u\cdot\nabla u \in L^\infty(0,T;L^3(\Omega)). \tag{3.4.2}$$

With the assumption on data, $\rho_0 \in C^1(\bar{\Omega})$, $u_0 \in C^{2+\alpha}(\bar{\Omega})$ and $d_0 \in C^{2+\alpha}(\bar{\Omega})$, the solution (ρ, u, d) from Corollary 3.1.2 or Corollary 3.1.4 satisfies (3.4.1) or (3.4.2), respectively .

Proof of Theorem 3.1.7: First recall that the regular solution (ρ, u, d) from Theorem 3.1.1 and Theorem 3.1.3 satisfies the energy equality:

$$\begin{aligned}&\int_\Omega \frac{1}{2}\rho|u|^2 + \frac{1}{2}|\nabla d|^2 + F(d)dx + \int_0^T\int_\Omega |\nabla u|^2 + |\triangle d - f(d)|^2 dxdt \\ &= \int_\Omega \frac{1}{2}\rho_0|u_0|^2 + \frac{1}{2}|\nabla d_0|^2 + F(d_0)dx.\end{aligned} \tag{3.4.3}$$

The density ρ is the strong solution of the transport equation, hence it satisfies that

$$\int_\Omega \rho^2 dx = \int_\Omega \rho_0^2 dx.$$

On the other hand side, $\bar{\rho}$ is a weak solution of the transport equation and $M_1 \le \bar{\rho} \le M_2$. We have by hypothesis that

$$\int_\Omega \bar{\rho}^2 dx \le \int_\Omega \rho_0^2 dx. \tag{3.4.4}$$

Thus,

$$\begin{aligned}\frac{1}{2}\int_\Omega |\rho-\bar{\rho}|^2 dx &= \frac{1}{2}\int_\Omega \rho^2 dx + \frac{1}{2}\int_\Omega \bar{\rho}^2 dx - \int_\Omega \rho\bar{\rho} dx \\ &\le \int_\Omega \rho_0^2 dx - \int_\Omega \rho\bar{\rho} dx.\end{aligned} \tag{3.4.5}$$

Since $\rho \in C^1([0,T]\times\bar{\Omega})$, we can take ρ as a test function. Thus, multiplying

$$\bar{\rho}_t + \bar{u}\cdot\nabla\bar{\rho} = 0$$

by ρ and integrating by parts yields

$$\begin{aligned}\int_\Omega \rho_0^2 dx - \int_\Omega \rho\bar{\rho}dx &= -\int_0^t\int_\Omega \bar{\rho}\rho_t dsdx - \int_0^t\int_\Omega (\bar{u}\cdot\nabla\rho)\bar{\rho}dsdx \\ &= \int_0^t\int_\Omega (u\cdot\nabla\rho)\bar{\rho}dsdx - \int_0^t\int_\Omega (\bar{u}\cdot\nabla\rho)\bar{\rho}dsdx.\end{aligned} \tag{3.4.6}$$

Here we used again that ρ is a classical solution of the transport equation. Substituting the equality (3.4.6) in (3.4.5) gives

$$\begin{aligned}\frac{1}{2}\int_\Omega |\rho-\bar{\rho}|^2 dx &\le \int_0^t\int_\Omega \bar{\rho}(u-\bar{u})\nabla\rho dsdx \\ &= \int_0^t\int_\Omega (\rho-\bar{\rho})(u-\bar{u})\nabla\rho dsdx.\end{aligned} \tag{3.4.7}$$

Next, calculate the following term

$$\begin{aligned}&\frac{1}{2}\int_\Omega \bar{\rho}|u-\bar{u}|^2dx + \frac{1}{2}\int_\Omega |\nabla d - \nabla\bar{d}|^2 dx \\ &= \frac{1}{2}\int_\Omega (\bar{\rho}-\rho)|u|^2dx + \frac{1}{2}\int_\Omega \rho|u|^2dx + \frac{1}{2}\int_\Omega \bar{\rho}|\bar{u}|^2dx - \int_\Omega \bar{\rho}u\cdot\bar{u}dx \\ &+ \frac{1}{2}\int_\Omega |\nabla d|^2dx + \frac{1}{2}\int_\Omega |\nabla\bar{d}|^2dx - \int_\Omega \nabla d\otimes\nabla\bar{d}dx,\end{aligned}$$

where $\nabla d\otimes\nabla\bar{d}$ denotes the 3×3 matrix whose ij-th entry is given by $\nabla_i d\cdot\nabla_j\bar{d}$ for $1\le i,j\le 3$.

Using the energy equality (3.4.3) for the regular solution (ρ,u,d) and inequality (3.1.8) for the weak solution $(\bar{\rho},\bar{u},\bar{d})$ combined with the last equality yields

$$\begin{aligned}&\frac{1}{2}\int_\Omega \bar{\rho}|u-\bar{u}|^2dx + \frac{1}{2}\int_\Omega |\nabla d-\nabla\bar{d}|^2dx \\ &\le -\int_0^t\int_\Omega |\nabla u-\nabla\bar{u}|^2dxdt - \int_0^t\int_\Omega |\triangle d-\triangle\bar{d}|^2dxdt \\ &+ \frac{1}{2}\int_\Omega (\bar{\rho}-\rho)|u|^2dx - \int_\Omega \bar{\rho}u\otimes\bar{u}-\rho_0|u_0|^2dx - \int_\Omega \nabla d\otimes\nabla\bar{d} - |\nabla d_0|^2dx \\ &- \int_\Omega F(d)dx - \int_\Omega F(\bar{d})dx + 2\int_\Omega F(d_0)dx - \int_0^t\int_\Omega |f(d)|^2dxdt \\ &- \int_0^t\int_\Omega |f(\bar{d})|^2dxdt - 2\int_0^t\int_\Omega \nabla u\cdot\nabla\bar{u}dxdt - 2\int_0^t\int_\Omega \triangle d\cdot\triangle\bar{d}dxdt \\ &+ 2\int_0^t\int_\Omega \triangle d\cdot f(d)dxdt + 2\int_0^t\int_\Omega \triangle\bar{d}\cdot f(\bar{d})dxdt.\end{aligned} \tag{3.4.8}$$

Since $u, d \in C^{1+\alpha/2,2+\alpha}([0,T]\times\bar{\Omega})$, we can take u, d as test functions for the weak solution $\bar{u}, \bar{d}$. Thus it follows that

$$\begin{aligned}\int_\Omega \bar{\rho}u\otimes\bar{u} - \rho_0|u_0|^2dx = &\int_0^t\int_\Omega \bar{\rho}\bar{u}u_t dxdt + \int_0^t\int_\Omega \bar{\rho}\bar{u}(\bar{u}\cdot\nabla u)dxdt \\ &- \int_0^t\int_\Omega \bar{\rho}\bar{u}(\bar{u}\cdot\nabla u)dxdt - \int_0^t\int_\Omega \bar{\rho}\bar{u}(\bar{u}\cdot\nabla u)dxdt,\end{aligned} \tag{3.4.9}$$

$$\begin{aligned}\int_\Omega \nabla d\otimes\nabla\bar{d} - |\nabla d_0|^2dx = &-\int_0^T\int_\Omega \triangle\bar{d}d_t dxdt + \int_0^T\int_\Omega \triangle d(\bar{u}\cdot\nabla\bar{d})dxdt \\ &- \int_0^T\int_\Omega \triangle\bar{d}\cdot\triangle d dxdt + \int_0^T\int_\Omega f(\bar{d})\triangle d dxdt.\end{aligned} \tag{3.4.10}$$

Indeed, formally, to get (3.4.9), we multiply equation

$$(\bar{\rho}\bar{u})_t + \operatorname{div}(\bar{\rho}\bar{u}\otimes\bar{u}) + \nabla\bar{p} = \triangle\bar{u} - \nabla\cdot(\nabla\bar{d}\otimes\nabla\bar{d})$$

by u and integrate by parts. To get (3.4.10), we multiply equation

$$\bar{d}_t + \bar{u}\cdot\nabla\bar{d} = \Delta\bar{d} - f(\bar{d})$$

by Δd and integrate by parts.

Substitute (3.4.9) and (3.4.10) in (3.4.8). Add (3.4.7) combined with the equalities in equations (1.0.15), yields

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\rho-\bar{\rho}|^2dx + \frac{1}{2}\int_\Omega \bar{\rho}|u-\bar{u}|^2dx + \frac{1}{2}\int_\Omega |\nabla d - \nabla\bar{d}|^2dx \\ &\le -\int_0^t\int_\Omega |\nabla u - \nabla\bar{u}|^2dxdt - \int_0^t\int_\Omega |\triangle d - \triangle\bar{d}|^2dxdt \\ &+ \int_0^t\int_\Omega (\bar{\rho}-\rho)(u-\bar{u})\nabla\rho dxdt - \int_0^t\int_\Omega \bar{\rho}\nabla u|u-\bar{u}|^2dx \\ &+ \int_0^t\int_\Omega (\bar{\rho}-\rho)(u-\bar{u})(u_t + u\cdot\nabla u)dxdt + \int_0^t\int_\Omega u(\nabla d - \nabla\bar{d})(\triangle d - \triangle\bar{d})dxdt \\ &- \int_0^t\int_\Omega (u-\bar{u})(\nabla d - \nabla\bar{d})\triangle d dxdt + \int_0^t\int_\Omega (\triangle d - \triangle\bar{d})(f(d)-f(\bar{d}))dxdt.\end{aligned} \tag{3.4.11}$$

Recall that the regular solution (ρ, u, d) satisfies (3.4.1) in 2D and (3.4.2) in 3D. By Hölder and Gagliardo-Nirenberg inequalities on the right hand side of (3.4.11), it follows that

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\rho(t)-\bar{\rho}(t)|^2 + \bar{\rho}(t)|u(t)-\bar{u}(t)|^2 + |\nabla d(t) - \nabla\bar{d}(t)|^2dx \\ &\le C\int_0^t\int_\Omega |\rho-\bar{\rho}|^2 + |u-\bar{u}|^2 + |\nabla d - \nabla\bar{d}|^2dxdt.\end{aligned} \tag{3.4.12}$$

To handle the last integral in (3.4.11), we used the fact that $|d|, |\bar{d}| \leq 1$ and hence $|f(d) - f(\bar{d})| \leq C|d - \bar{d}|$ by the definition of $f(d)$. Thus,

$$\int_0^t \int_\Omega |f(d) - f(\bar{d})|^2 dxdt \leq C \int_0^t \int_\Omega |d - \bar{d}|^2 dxdt \leq C(\Omega) \int_0^t \int_\Omega |\nabla d - \nabla \bar{d}|^2 dxdt$$

where the constant C depends on space domain Ω not on time T, and C depends on the dimension of the space.

Using the lower bound of $\bar{\rho} \geq M_1 > 0$, and Gronwall's inequality to (3.4.12) yields

$$\begin{aligned}
&\frac{1}{2} \int_\Omega |\rho(t) - \bar{\rho}(t)|^2 + \bar{\rho}|u(t) - \bar{u}(t)|^2 + |\nabla d(t) - \nabla \bar{d}(t)|^2 dx \\
&\leq \int_\Omega |\rho(0) - \bar{\rho}(0)|^2 + \bar{\rho}(0)|u(0) - \bar{u}(0)|^2 + |\nabla d(0) - \nabla \bar{d}(0)|^2 dx e^{Ct} \\
&= 0
\end{aligned}$$

for all $t > 0$ which implies

$$\bar{\rho} - \rho = \bar{u} - u = \bar{d} - d \equiv 0.$$

This completes the proof of Theorem 3.1.7.

3.5 Appendix: Existence of Weak Solutions

In this appendix we sketch an existence proof of weak solutions using Galerkin approximating method. As mentioned in Section 3.1 the existence of global weak solutions to the flow of nematic liquid crystals have been established in [41], [21] and Chapter 2 of this thesis for non constant fluid density. Unfortunately the Ladyzhenskaya energy estimates do not seem to work for the Galerkin approximating solutions constructed therein. The Galerkin approximate solutions constructed here will possess the Ladyzhenskaya energy estimates. When the initial fluid density has a positive lower bound, we are able to derive estimates on u_t^m and d_t^m so that we can employ the compactness lemma of Lions-Aubin. Since the Galerkin method has been widely used for fluid systems as well as on the system of liquid crystals the proof here will be rather brief (cf. [2], [35], [41], [21]).

In the following we construct a sequence of Galerkin approximating solutions that are uniformly bounded. These are the solutions used in Section 3.2 and

Section 3.3. The bounds obtained there through Ladyzhenskaya method yield a subsequence that will converge to the classical solution.
Let us first state the existence theorem for global weak solutions:

Theorem 3.5.1 *Assume that $u_0 \in L^2(\Omega)$ and $d_0 \in H^1(\Omega)$ with $d_0|_{\partial\Omega} \in H^{3/2}(\Omega)$. The system (1.0.15) with initial boundary conditions (3.1.1)-(3.1.4) has a weak solution (ρ^m, u^m, d^m) for each $m = 1, 2, 3, ...$ satisfying, for all $T \in (0, \infty)$,*

$$0 < M_1 \leq \rho^m \leq M_2 \tag{3.5.1}$$

$$u^m \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \tag{3.5.2}$$

$$d^m \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)), \quad |d^m| \leq 1, \tag{3.5.3}$$

and the energy inequality

$$\int_\Omega (|\triangle d^m - f(d^m)|^2 + |\nabla u^m|^2) dx + \frac{d}{dt} \int_\Omega (\frac{1}{2}\rho |u^m|^2 + \frac{1}{2}|\nabla d^m|^2 + F(d^m)) dx \leq 0. \tag{3.5.4}$$

Proof Let

$$\mathcal{H}(\Omega) = \text{ closure of } \{f \in C_0^\infty(\Omega, \mathbb{R}^3) : \nabla \cdot f = 0\} \text{ in } L^2(\Omega, \mathbb{R}^3)$$

and $\{\phi_i\}_{i=1}^\infty$ be an orthonormal basis of $\mathcal{H}$ and satisfying:

$$-\triangle \phi_i + \nabla P_i = \lambda_i \phi_i \text{ in } \Omega$$

$$\phi_i = 0 \text{ on } \partial\Omega$$

for $i = 1, 2, ...$ Here $0 < \lambda_1 \leq \lambda_2 ... \leq \lambda_n \leq ...$ with $\lambda_i \to \infty$. In other words, we choose an orthonormal basis of $\mathcal{H}$ which consists of the eigenfunctions of Stokes operator on Ω with vanishing Dirichlet boundary condition (see [54]). Let

$$P_m : \mathcal{H} \to \mathcal{H}_m = span\,\{\phi_1, ...\phi_m\}$$

be the orthonormal projection. We seek approximate solutions (ρ^m, u^m, d^m) with $u^m \in \mathcal{H}_m$, satisfy the following equations:

$$\rho_t^m + u^m \cdot \nabla \rho^m = 0, \quad \rho^m(x, 0) = \rho_0(x) \tag{3.5.5}$$

$$P_m(\rho^m \frac{\partial}{\partial t} u^m) = P_m(\triangle u^m - \rho^m u^m \cdot \nabla u^m - \nabla \cdot (\nabla d^m \otimes \nabla d^m)) \tag{3.5.6}$$

$$d_t^m + u^m \cdot \nabla d^m = \triangle d^m - f(d^m) \tag{3.5.7}$$

$$u^m(x,0) = P_m u_0(x), \quad d^m(x,0) = d_0(x), \quad d^m(x,t)|_{\partial\Omega} = d_0(x)|_{\partial\Omega}. \tag{3.5.8}$$

Let

$$u^m(x,t) = \sum_{l=1}^{m} g_i^m(t)\phi_i(x), \tag{3.5.9}$$

with $g_i^m(t) \in C^1[0,T]$. Hence (3.5.6) is equivalent to the following system of ordinary differential equations:

$$\sum_{i=1}^{m} \mathcal{A}_i^{mj}(t)\frac{d}{dt}g_i^m(t) = -\sum_{i,k}^{m} \mathcal{B}_{ik}^{mj}(t)g_i^m(t)g_k^m(t) - \sum_{i=1}^{m} \mathcal{C}_i^j g_i^m(t) + \mathcal{D}^{mj}(t), \tag{3.5.10}$$

for $j = 1, 2, ..., m$, where

$$\begin{cases} \mathcal{A}_i^{mj}(t) = \int_\Omega \rho^m(t)\phi_i(x)\phi_j(x)dx, \\ \mathcal{B}_{ik}^{mj}(t) = \int_\Omega \rho^m(t)(\phi_i(x)\cdot\nabla\phi_k(x))\phi_j(x)dx, \\ \mathcal{C}_i^j = \int_\Omega \nabla\phi_i(x)\cdot\nabla\phi_j(x)dx, \\ \mathcal{D}^{mj}(t) = \int_\Omega \sum_{k,l}(\frac{\partial}{\partial x_k}d^m \cdot \frac{\partial}{\partial x_l}d^m)\frac{\partial}{\partial x_l}\phi_j^k(x)dx. \end{cases} \tag{3.5.11}$$

Here $\phi_j^k(x)$ is the kth component of the vector $\phi_j(x)$. And

$$u^m(\cdot,0) = \sum_{i=1}^{m} g_i^m(0)\phi_i(x), \quad \text{where} \quad g_i^m(0) = \int_\Omega u_0(x)\phi_i(x)dx.$$

To finish the proof, we need the following lemma.

Lemma 3.5.2 *There exists a weak solution (ρ^m, u^m, d^m) to the problem (3.5.5)-(3.5.8) in $Q_T = \Omega \times [0,T]$, for any $T \in (0,\infty)$, satisfying*

$$M_1 \le \rho^m \le M_2$$

$$u^m \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$$

$$d^m \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2), \quad |d^m| \le 1.$$

Moreover, (ρ^m, u^m, d^m) is smooth in the interior of Q_T and satisfies the energy equality,

$$\int_\Omega |\nabla u^m|^2 + |\triangle d^m - f(d^m)|^2 dx + \frac{d}{dt}\int_\Omega (\frac{1}{2}(\rho^m|u^m|^2 + |\nabla d^m|^2) + F(d^m))dx = 0. \tag{3.5.12}$$

The proof of this lemma is based on an application of the Leray-Schauder fixed point theorem. Let $v^m = \sum_{i=1}^m h_i^m \phi_i \in C^1(0,T;\mathcal{H}_m)$. For each m let ρ^m be a solution to

$$\rho + v^m \cdot \nabla \rho = 0 \tag{3.5.13}$$

with initial condition $\rho(\cdot,0) = \rho_0$. Let d^m be a solution to

$$d_t + v^m \cdot \nabla d = \Delta d - f(d)$$

with initial condition $d(\cdot,0) = d_0$ and boundary condition $d(x,t) = d_0(x)$. The reason the transport equation (3.5.13) is solvable for $v^m \in C^1(0,T;\mathcal{H}_m)$ is due to the regularity of the eigenfunctions of the Stokes operators (cf. [54] [29]). Let $u^m = \sum_{i=1}^m g_i^m \phi_i \in C^1(0,T;\mathcal{H}_m)$ be the solution of the system of linear equations

$$\sum_{i=1}^m \mathcal{A}_i^{mj}(t) \frac{d}{dt} g_i^m(t) = -\sum_{i,k}^m \mathcal{B}_{ik}^{mj}(t) h_i^m(t) g_k^m(t) - \sum_{i=1}^m \mathcal{C}_i^j g_i^m(t) + \mathcal{D}^{mj}(t).$$

This system of linear equations is solvable because the eigenvalues of matrix of the coefficients $\mathcal{A}_i^{mj}(t)$ are bounded from below, since

$$\mathcal{A}_i^{mj}\gamma_i\gamma_j = \int_\Omega \rho|\psi|^2 dx \geq M_1 \int_\Omega |\psi|^2 dx \quad \text{where } \psi = \sum_{i=1}^m \gamma_i \phi_i. \tag{3.5.14}$$

Thus we constructed a mapping $\mathcal{M}$ with $\mathcal{M}(v^m) = u^m$. The energy estimate (3.5.12) play the key rule to allow one to apply Leray-Schauder fixed point theorem for $\mathcal{M}$.

Remark 3.5.3 *We know that from the higher energy estimates obtained in Section 3.2 and Section 3.3, we can pass to the limit and conclude that, there exists* (ρ, u, d) *such that, taking subsequence if necessary,*

$$\rho^m \rightharpoonup \rho \quad \textit{in } L^p(\Omega \times [0,T]) \quad \textit{for any } p \in (1,\infty)$$

$$u^m \rightharpoonup u \quad \textit{weakly in } C(0,T;H^1) \quad \textit{and strongly in } C(0,T;L^2)$$

$$d^m \rightharpoonup d \quad \textit{weakly in } L^2(0,T;H^2) \quad \textit{and strongly in } L^2(0,T;H^1).$$

It follows easily from the above convergences that indeed (ρ, u, d) *is a weak solution to the system* (1.0.15).

□

3.6 Appendix: Frozen Coefficient Method

For completeness, in this section, we recall an application of the frozen coefficient method by considering the following problem (for more detail see [30])

$$\begin{aligned} \rho v_t - \Delta v + \rho v \cdot \nabla v + \nabla p = f, \\ \nabla \cdot v = 0, \quad v(0) = 0, \quad v|_{\partial\Omega} = 0, \end{aligned} \tag{3.6.1}$$

in the domain $\Omega \times (0,T)$. Here ρ is a given function in $\Omega \times (0,T)$ satisfying

$$0 < M_1 \le \rho \le M_2 \tag{3.6.2}$$

$$|\nabla \rho| \le M_3, \quad |\rho_t| \le M_4 \tag{3.6.3}$$

(the derivatives of ρ may be the generalized ones). And f is a given function in $L^q(\Omega \times (0,T))$ for $q > \sqrt{3}$.

The frozen coefficient method is now used to prove the following lemma,

Lemma 3.6.1 *If $v \in W^{1,q}(W^{2,q}(\Omega))$ and $\nabla p \in L^q(\Omega \times (0,T))$ is a solution of problem (3.6.1), then for an appropriate constant a depending on q*

$$\|v_t\|_{L^q} + \|\Delta v\|_{L^q} + \|\nabla p\|_{L^q} \le C(\|f\|_{L^q} + \|v\|^a_{L^q}), \tag{3.6.4}$$

where L^q denotes the norm in $L^q(\Omega \times (0,T))$, the constant C depends on q, Ω, M_1, M_2, M_3 and M_4.

Proof Let $\zeta_k^\lambda(x)$ be the set of nonnegative functions in $C^2(\mathbb{R}^n)$ depending on the parameter $\lambda \in (0,1)$, forming a partition of unity in Ω:

$$1 = \sum_{k=1}^{N_\lambda} \zeta_k^\lambda(x), \quad x \in \bar{\Omega}$$

with

$$supp \zeta_k^\lambda(x) \subset \Omega_k^\lambda$$

where the subdomains Ω_k^λ form a covering of Ω which satisfy the following properties

(i) the diameters of Ω_k^λ do not exceed λ;

(ii) there is a fixed number l such that the multiplicity of the covering of Ω by the subdomains Ω_k^λ is $\leq l$;

(iii) $\sum_{k=1}^{N_\lambda}(\zeta_k^\lambda(x))^q \geq \mu > 0$, $\forall x \in \Omega$;

(iv) $|D_x^\beta \zeta_k^\lambda(x)| \leq c\lambda^{-|\beta|}$, $|\beta| = 1, 2$, $\forall x \in \Omega$.

For simplicity, we consider the case when ρ is independent of t. Fix $x_k \in \Omega_k^\lambda$, for each $k = 1, 2, ..., N_\lambda$, the vector $v_k(x,t) = v(x,t)\zeta_k^\lambda(x)$ and the function $p_k(x,t) = p(x,t)\zeta_k^\lambda(x)$ are solutions in $\Omega \times (0,T)$ (in the same spaces as v and p) to the problem

$$\begin{aligned} &\rho(x_k)v_{kt} - \Delta v_k + \nabla p_k \\ &= f\zeta_k^\lambda(x) + [\rho(x_k) - \rho(x)]v_{kt} - \rho(x)v_k \cdot \nabla v_k + g_k(x,t), \\ &\nabla \cdot v_k = v \cdot \nabla \zeta_k^\lambda(x), \quad v_k(0) = 0, \quad v_k|_{\partial\Omega} = 0, \end{aligned} \tag{3.6.5}$$

where

$$g_k(x,t) = -2\sum_{i=1}^{n} v_{x_i}\zeta_{kx_i}^\lambda - v\Delta\zeta_k^\lambda + p\nabla\zeta_k^\lambda + \rho(x)[v\zeta_x^\lambda(v \cdot \nabla\zeta_k^\lambda) + (\zeta_k^\lambda - 1)\zeta_k^\lambda v \cdot \nabla v].$$

Assuming the right hand side of the first equation in (3.6.5) is the "free term", making use of the results of Lemma 2.2 in [30], we have the estimate for v_k, p_k

$$\begin{aligned} &\|v_{kt}\|_{L^q} + \|\Delta v_k\|_{L^q} + \|\nabla p_k\|_{L^q} \\ &\leq C_1(\|f\|_{L^q} + \max_{x\in\Omega_k^\lambda}|\rho(x_k) - \rho(x)|\|v_{kt}\|_{L^q} + \|v_k \cdot \nabla v_k\|_{L^q} + \|g_k\|_{L^q}), \end{aligned} \tag{3.6.6}$$

where constant C_1 depends on M_2. Since $\max_{x\in\Omega_k^\lambda}|\rho(x_k) - \rho(x)| \leq M_3\lambda$, and the properties of ζ_k^λ, we are able to derive estimate (3.6.4) from (3.6.6), by choosing appropriately $\lambda = (1 + M_3)^{-1}\min\left\{1, (2C_1)^{-1}\right\}$ and applying the general Gagliardo-Nirenberg inequality (see [2]). □

Remark 3.6.2 *As seen in this simple example once we have the bound on the small oscillations the desired estimate follows.*

Chapter 4

Asymptotic Behavior of Solutions in $\mathbb{R}^3$

In this chapter we study the large time behavior of solutions to the nematic liquid crystal system with constant density in the whole space $\mathbb{R}^3$. The fluid under consideration has constant density and small initial data.

4.1 The Settings and Main Results

We consider the LCD system with constant density

$$\begin{aligned} u_t + \nabla \cdot (u \otimes u) + \nabla p &= \mu \Delta u - \nabla \cdot (\nabla d \otimes \nabla d) \\ d_t + u \cdot \nabla d &= \nu \Delta d - f(d), \\ \nabla \cdot u &= 0. \end{aligned} \tag{4.1.1}$$

and with the following initial conditions:

$$u(x,0) = u_0(x), \quad \nabla \cdot u_0 = 0, \tag{4.1.2}$$

$$d(x,0) = d_0(x), \quad |d_0(x)| = 1, \tag{4.1.3}$$

and

$$u_0 \in H^1(\mathbb{R}^3), \quad d_0 - w_0 \in H^2(\mathbb{R}^3), \tag{4.1.4}$$

with a fixed vector $w_0 \in S^2$, i.e., $|w_0| = 1$.

The asymptotic behavior of solutions to the system (4.1.1) was studied for bounded domains in [35, 55]. It was shown in [55] that, with suitable initial conditions, the velocity converges to zero and the direction field converges to the steady solution to the following equation

$$\begin{cases} -\Delta d + f(d) = 0, x \in \Omega \\ d(x) = d_0(x), x \in \partial\Omega. \end{cases} \tag{4.1.5}$$

Wu's main approach was through the Łojasiewicz-Simon method [45], a method broadly used to study the uniqueness of asymptotic limits for nonlinear evolution equations. In 1983, Simon [45] obtained a breakthrough result for semi-linear parabolic equations with a nonlinearity $f(x, u)$, analytic in the unknown function u. He showed that bounded global solutions of such semi-linear equations converge to an equilibrium as $t \to \infty$. Simon's idea relies on a generalization of the Łojasiewicz inequality (see [42] and [43]) for analytic functions defined in the finite dimensional space $\mathbb{R}^m$. In [55] the author introduced a suitable Łojasiewicz-Simon type inequality for vector functions with non-homogeneous Dirichlet boundary condition (Lemma 2.1 in [55]) to be able to use Simon's ideas.

In this chapter we consider the asymptotic behavior of the solutions to (4.1.1) in the whole space $\mathbb{R}^3$. Due to the lack of compactness in the whole space, Łojasiewicz-Simon approach cannot be applied. Thus we consider a simple case subject to the additional condition on the direction field which insures that the direction tends to a constant unit vector w_0, as the space variable tends to infinity:

$$\lim_{|x|\to\infty} d_0(x) = w_0. \tag{4.1.6}$$

This simplifies the situation and allows us to obtain the stability without needing the Liapunov reduction and Łojasiewicz-Simon inequality, since w_0 is a non-degenerate steady solution to (4.1.5).

The existence of global regular solutions of (4.1.1) with small initial data in the whole space $\mathbb{R}^3$ has been established in Appendix 4.4, based on the arguments in Chapter 3. For convenience we state the existence theorem of global regular solutions here as follows:

Theorem 4.1.1 *Let u_0 and d_0 satisfy (4.1.2)-(4.1.4). Assume that $u_0 \in H^1(\mathbb{R}^3)$ and $d_0 - w_0 \in H^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ for a unit vector w_0. There is a positive small number ϵ_0 such that if*

$$\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|^2_{H^2(\mathbb{R}^3)} \leq \epsilon_0, \tag{4.1.7}$$

then the system (4.1.1) has a classical solution (u, p, d) in the time period $(0, T)$, for all $T > 0$. That is, for some $\alpha \in (0, 1)$

$$\begin{aligned} u &\in C^{1+\alpha/2, 2+\alpha}((0,T) \times \mathbb{R}^3) \\ \nabla p &\in C^{\alpha/2, \alpha}((0,T) \times \mathbb{R}^3) \\ d &\in C^{1+\alpha/2, 2+\alpha}((0,T) \times \mathbb{R}^3). \end{aligned} \tag{4.1.8}$$

And the solution (u, p, d) satisfies the following basic energy estimate and higher order energy estimate (also called Ladyzhenskaya energy estimate in [35])

$$\begin{aligned} &\int_{\mathbb{R}^3} |u|^2 + |\nabla d|^2 + 2F(d)dx + 2\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 + |\Delta d - f(d)|^2 dxdt \\ &\leq \|u_0\|^2_{L^2(\mathbb{R}^3)} + \|\nabla d_0\|^2_{L^2(\mathbb{R}^3)} \end{aligned} \tag{4.1.9}$$

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u|^2 + |\Delta d|^2 dx + \int_0^T \int_{\mathbb{R}^3} |\Delta u|^2 + |\nabla \Delta d|^2 dxdt \\ &\leq C(\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|^2_{H^2(\mathbb{R}^3)}). \end{aligned} \tag{4.1.10}$$

Furthermore, the solution d satisfies

$$\int_{\mathbb{R}^3} |d(t) - w_0| dx \leq (C_0 t + \int_{\mathbb{R}^3} |d_0 - w_0| dx) e^{Ct} \tag{4.1.11}$$

with the constants C_0 and C depending only on initial data and on η, respectively.

Remark 4.1.2 *The estimates in (4.1.8) are actually global. The higher order Hölder norms of solutions (u, d, p) on each bounded ball in (4.1.8) are essentially estimated in terms of the initial data $\|u_0\|_{H^1} + \|d_0 - w_0\|_{H^2}$, see Chapter 3 and [28](Theorem 5.1 in Chapter VII). And the assumption (4.1.4) on initial data is global.*

For the smooth solution obtained in Theorem 4.1.1, our main asymptotic result is:

Theorem 4.1.3 *Let (u, p, d) be a smooth solution obtained in Theorem 4.1.1. Assume additionally $u_0 \in L^1(\mathbb{R}^3)$ and $d_0 - w_0 \in L^p(\mathbb{R}^3)$, for any $p \geq 1$ and a unit vector w_0. There exists a small number $\epsilon_0 > 0$ such that if*

$$\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|^2_{H^2(\mathbb{R}^3)} \leq \epsilon_0, \tag{4.1.12}$$

then

$$\|d(\cdot, t) - w_0\|_{L^p(\mathbb{R}^3)} \leq C\|d_0 - w_0\|_{L^p(\mathbb{R}^3)}(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \tag{4.1.13}$$

$$\|\nabla(d(\cdot, t) - w_0)\|^2_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}}, \tag{4.1.14}$$

$$\|u(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{1}{2}}, \tag{4.1.15}$$

where the various constants C only depend on initial data.

The rest of the chapter is organized as follows: in Section 4.2 we establish the decay for the difference $d - w_0$, using the basic energy estimate (4.1.9) and the Ladyzhenskaya energy estimate (4.1.10). Combining the decay of $d - w_0$ and Fourier splitting technique [48], in Section 4.3 we obtain an algebraic decay for the velocity u in $L^2(\mathbb{R}^3)$. In Appendix 4.4, we sketch a proof for the existence theorem 4.1.1.

4.2 Convergence of the direction field

In this section we study the L^p decay of the direction field $d - w_0$ and the decay for the first derivative. The first step is to derive a uniform estimate of $d - w_0$ in $L^2(\mathbb{R}^3)$. This yields a uniform estimate for $d - w_0$ in $L^p(\mathbb{R}^3)$ for any $p \geq 1$. This L^p estimate, is the basis to establish the decay results. For ease of reading we state the basic energy estimate and Ladyzhenskaya energy estimate satisfied by the smooth solution (u, p, d) (see Appendix 4.4 for details),

$$\begin{aligned} &\|u\|^2_{L^2} + \|\nabla d\|^2_{L^2} + \|F(d)\|_{L^1} + 2\int_0^T \|\nabla u\|^2_{L^2} + \|\Delta d - f(d)\|^2_{L^2} dt \\ &\leq \|u_0\|^2_{L^2} + \|\nabla d_0\|^2_{L^2} \end{aligned} \tag{4.2.1}$$

$$\|\nabla u\|^2_{L^2} + \|\Delta d\|^2_{L^2} + \int_0^T \|\Delta u\|^2_{L^2} + \|\nabla \Delta d\|^2_{L^2} dt \leq C(\|u_0\|^2_{H^1} + \|d_0 - w_0\|^2_{H^2}). \tag{4.2.2}$$

In the sequel we need to use a general Gagliardo-Nirenberg interpolation inequality in whole space $\mathbb{R}^n$. For completeness we recall from [17] the inequality here

Proposition 4.2.1 *[17] Let $w \in W^{m,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, for $1 \le p \le \infty$ and $1 \le q \le \infty$. Then*

$$\|D^k w\|_{L^r(\mathbb{R}^n)} \le C\|D^m w\|^a_{L^p(\mathbb{R}^n)}\|w\|^{1-a}_{L^q(\mathbb{R}^n)} \tag{4.2.3}$$

for any integer $k \in [0, m-1]$, where

$$\frac{1}{r} = \frac{k}{n} + a(\frac{1}{p} - \frac{m}{n}) + (1-a)\frac{1}{q} \tag{4.2.4}$$

with $a \in [\frac{k}{m}, 1]$, either if $p = 1$ or $p > 1$ and $m - k - \frac{n}{p} \notin \mathcal{N} \cup \{0\}$, while $a \in [\frac{k}{m}, 1)$, if $p > 1$ and $m - k - \frac{n}{p} \in \mathcal{N} \cup \{0\}$.

4.2.1 Uniform estimate of $d - w_0$ in $L^2(\mathbb{R}^3)$

In this part, we show that the integrals

$$\int_{\mathbb{R}^3} |d(x,t) - w_0|^2 dx \text{ and } \int_0^T \int_{\mathbb{R}^3} |\nabla(d(x,t) - w_0)|^2 dxdt$$

are uniformly bounded by the initial data, applying estimates for the solutions obtained in Theorem 4.1.1. We have the following lemma,

Lemma 4.2.2 *Let d be the solution obtained in Theorem 4.1.1. There exists ϵ_0 sufficiently small so that if $\|u_0\|_{L^2(\mathbb{R}^3)} + \|\nabla d_0\|_{L^2(\mathbb{R}^3)} \le \epsilon_0$. Then*

$$\int_{\mathbb{R}^3} |d(x,t) - w_0|^2 dx + \int_0^T \int_{\mathbb{R}^3} |\nabla(d(x,t) - w_0)|^2 dxdt \le C, \tag{4.2.5}$$

where the constant C depends on the initial data and the norm $\|d_0 - w_0\|_{L^2(\mathbb{R}^3)}$.

Proof Since $|w_0| = 1$ and $f(w_0) = \frac{1}{\eta^2}(|w_0|^2 - 1)w_0 = 0$, the second equation in (1.0.15) can be expressed as

$$(d - w_0)_t + u \cdot \nabla(d - w_0) = \Delta(d - w_0) - f(d) + f(w_0). \tag{4.2.6}$$

Applying the mean value theorem for vector valued functions, we have

$$f(d) - f(w_0) = \left(\int_0^1 Df(w_0 + s(d - w_0))ds\right) \cdot (d - w_0), \tag{4.2.7}$$

where Df denotes the Jacobian matrix of f. Multiplying (4.2.6) by $d - w_0$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |d-w_0|^2 dx \qquad (4.2.8)\\
&= -\int_{\mathbb{R}^3} [u\cdot\nabla(d-w_0)](d-w_0)dx\\
&+ \int_{\mathbb{R}^3} \Delta(d-w_0)\cdot(d-w_0)dx\\
&- \int_{\mathbb{R}^3} (d-w_0)^T \left(\int_0^1 Df(w_0+s(d-w_0))ds\right)(d-w_0)dx,\\
&\equiv -I_4 + I_5 + I_6.
\end{aligned}$$

The three terms I_4, I_5 and I_6 are estimated as follows:

$$\begin{aligned}
|I_4| &= |\lim_{R\to\infty}\int_{B_R} [u\cdot\nabla(d-w_0)](d-w_0)dx| \qquad (4.2.9)\\
&= |\lim_{R\to\infty}\frac{1}{2}\int_{\partial B_R} |d-w_0|^2 u\cdot n d\sigma|\\
&\le C\lim_{R\to\infty}\left(\int_{\partial B_R}|d-w_0|^4 d\sigma\right)^{1/2}\left(\int_{\partial B_R}|u|^2 d\sigma\right)^{1/2}\\
&\le C\lim_{R\to\infty}\left(\int_{\partial B_R}|d-w_0| d\sigma\right)^{1/2}\left(\int_{\partial B_R}|u|^2 d\sigma\right)^{1/2}.
\end{aligned}$$

Denote the set $A = \left\{R : \int_{\partial B_R}|d-w_0|d\sigma \ge M\right\}$, for a certain constant $M > 0$. A is closed and its complement A^c is open. Write

$$\int_{\mathbb{R}^3}|d(t)-w_0|dx = \int_0^\infty\int_{\partial B_R}|d(t)-w_0|d\sigma dR.$$

Recall that by Theorem (4.1.1) the last expression is bounded by the initial data, for any fixed time $t > 0$. Thus by Chebyshev's inequality we have

$$\mu\{A\} \le \frac{C}{M} \qquad (4.2.10)$$

where μ denotes the measure of a set, and C denotes a constant depending only on the initial data. Since the energy estimate (4.2.1) implies that

$$\int_{\mathbb{R}^3}|u|^2dx = \int_0^\infty\int_{\partial B_R}|u|^2 d\sigma dR$$

is bounded by initial data, there exists a sequence $\{R_i\}_{i=1}^\infty \subset A^c$ with $R_i \to \infty$ such that

$$\int_{\partial B_{R_i}}|u|^2 d\sigma \to 0 \qquad (4.2.11)$$

Combining the inequalities (4.2.9), (4.2.10) and (4.2.11), yields for all $t > 0$,

$$I_4 = 0. \tag{4.2.12}$$

For I_5, we have

$$I_5 = \lim_{R\to\infty}[\int_{\partial B_R} \frac{\partial(d-w_0)}{\partial n}\cdot(d-w_0)d\sigma - \int_{B_R}|\nabla(d-w_0)|^2dx] \tag{4.2.13}$$

The boundary term is estimated as follows:

$$\begin{aligned}
&\int_{\partial B_R} \frac{\partial(d-w_0)}{\partial n}\cdot(d-w_0)d\sigma \qquad (4.2.14)\\
&\leq \left(\int_{\partial B_R}|\nabla(d-w_0)|^2d\sigma\right)^{1/2}\left(\int_{\partial B_R}|d-w_0|^2d\sigma\right)^{1/2}\\
&\leq C\left(\int_{\partial B_R}|\nabla(d-w_0)|^2d\sigma\right)^{1/2}\left(\int_{\partial B_R}|d-w_0|d\sigma\right)^{1/2}
\end{aligned}$$

From Theorem 4.1.1, we have that $\|\nabla(d-w_0)\|_{L^2(\mathbb{R}^3)}$ is uniformly bounded for any fixed $t > 0$, $\|d(t)-w_0\|_{L^1(\mathbb{R}^3)}$ has a time dependent bound. For the inequality (4.2.14), we apply a similar argument used to derive (4.2.12) and obtain the existence of a sequence R_i approaching infinity satisfying

$$\lim_{R_i\to\infty}\int_{\partial B_{R_i}} \frac{\partial(d-w_0)}{\partial n}\cdot(d-w_0)d\sigma \to 0$$

It follows then that

$$I_5 = -\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx. \tag{4.2.15}$$

With respect to I_6, let $\tilde{d} = w_0 + s(d-w_0)$ and $z = d - w_0$. Using the definition of $f(d) = \frac{1}{\eta^2}(|d|^2-1)d$, a straightforward calculation yields

$$\begin{aligned}
&(d-w_0)^T\left(\int_0^1 Df(w_0+s(d-w_0))ds\right)(d-w_0) = z^T\int_0^1 Df(\tilde{d})dsz\\
&= \int_0^1[(2\tilde{d}_1^2+|\tilde{d}|^2-1)z_1^2+(2\tilde{d}_2^2+|\tilde{d}|^2-1)z_2^2+(2\tilde{d}_3^2+|\tilde{d}|^2-1)z_3^2\\
&+4\tilde{d}_1\tilde{d}_2z_1z_2+4\tilde{d}_1\tilde{d}_3z_1z_3+4\tilde{d}_2\tilde{d}_3z_2z_3]ds\\
&= \int_0^1[2\tilde{d}_1^2z_1^2+2\tilde{d}_2^2z_2^2+2\tilde{d}_3^2z_3^2+4\tilde{d}_1\tilde{d}_2z_1z_2+4\tilde{d}_1\tilde{d}_3z_1z_3\\
&+4\tilde{d}_2\tilde{d}_3z_2z_3]+(|\tilde{d}|^2-1)(z_1^2+z_2^2+z_3^2)ds\\
&= \int_0^1 2[\tilde{d}_1z_1+\tilde{d}_2z_2+\tilde{d}_3z_3]^2+(|\tilde{d}|^2-1)|z|^2ds\\
&= \int_0^1 2[\tilde{d}\cdot z]^2+(|\tilde{d}|^2-1)|z|^2ds.
\end{aligned}$$

In the above equation, the third and forth equality comes from regrouping terms and completing a perfect square. Thus, I_6 can be written as

$$\begin{aligned} I_6 &= -\frac{1}{\eta^2}\int_{\mathbb{R}^3}\int_0^1 2[(w_0+s(d-w_0))\cdot(d-w_0)]^2 \\ &\quad + (|w_0+s(d-w_0)|^2-1)|d-w_0|^2 ds dx \\ &= -\frac{1}{\eta^2}\int_{\mathbb{R}^3} 2[w_0\cdot(d-w_0)]^2 + 3w_0\cdot(d-w_0)|d-w_0|^2 + |d-w_0|^4 dx \\ &= -\frac{1}{\eta^2}\int_{\mathbb{R}^3} 2[w_0\cdot(d-w_0)+\frac{3}{4}|d-w_0|^2]^2 - \frac{1}{8}|d-w_0|^4 dx \\ &\leq \frac{1}{8\eta^2}\int_{\mathbb{R}^3}|d-w_0|^4 dx. \end{aligned} \tag{4.2.16}$$

Combining (4.2.8) and the inequalities (4.2.12), (4.2.15), and (4.2.16) gives

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|d-w_0|^2dx + \int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx \leq \frac{1}{8\eta^2}\int_{\mathbb{R}^3}|d-w_0|^4dx. \tag{4.2.17}$$

The right hand side of the inequality (4.2.17) can be estimated as

$$\begin{aligned} \int_{\mathbb{R}^3}|d-w_0|^4dx &\leq (\int_{\mathbb{R}^3}|d-w_0|^2dx)^{1/2}(\int_{\mathbb{R}^3}|d-w_0|^6dx)^{1/2} \\ &\leq C(\int_{\mathbb{R}^3}|d-w_0|^2dx)^{1/2}(\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx)^{3/2} \\ &\leq C\int_{\mathbb{R}^3}|d-w_0|^2dx\left(\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx. \end{aligned}$$

Gagliardo-Nirenberg interpolation inequality (Proposition 4.2.1) yields

$$\|\nabla(d-w_0)\|^2_{L^2(\mathbb{R}^3)} \leq C\|d-w_0\|_{L^2(\mathbb{R}^3)}\|\Delta(d-w_0)\|_{L^2(\mathbb{R}^3)}.$$

Combining the last two inequalities with (4.2.17) gives

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3}|d-w_0|^2dx + \int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx \\ &\leq C\left(\int_{\mathbb{R}^3}|d-w_0|^2dx\right)^2\int_{\mathbb{R}^3}|\Delta(d-w_0)|^2dx. \end{aligned} \tag{4.2.18}$$

Denote $\phi(t) = \int_{\mathbb{R}^3}|d(t)-w_0|^2dx$. Then

$$\frac{d\phi}{\phi^2} \leq C\int_{\mathbb{R}^3}|\Delta(d-w_0)|^2dxdt.$$

Integrating the last inequality over $[0,t]$ yields

$$-\frac{1}{\phi(t)} + \frac{1}{\phi(0)} \leq C\int_0^t\int_{\mathbb{R}^3}|\Delta(d-w_0)|^2dxdt.$$

Thus,

$$\phi(t) \leq \frac{\phi(0)}{1 - C\phi(0)\int_0^t \int_{\mathbb{R}^3} |\Delta(d - w_0)|^2 dxdt}.$$

Since $|f(d)|^2 = \frac{1}{4\eta^2}|d|^2 F(d) \leq CF(d)$ for fixed constant η and $\int_{\mathbb{R}^3} F(d)dx$ is uniformly bounded by initial data from basic energy estimate (4.2.1), it then follows from (4.2.1) and the smallness hypothesis that

$$\int_0^t \int_{\mathbb{R}^3} |\Delta(d - w_0)|^2 dxdt \leq C(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2) \leq \epsilon_0.$$

Assume that ϵ_0 is so small that $C\epsilon_0\phi(0) < 1/2$, then

$$C\phi(0)\int_0^t \int_{\mathbb{R}^3} |\Delta(d - w_0)|^2 dxdt < \frac{1}{2}.$$

Hence for any $t > 0$,

$$\phi(t) \leq 2\phi(0),$$

that is,

$$\int_{\mathbb{R}^3} |d(t) - w_0|^2 dx \leq 2\int_{\mathbb{R}^3} |d_0 - w_0|^2 dx. \tag{4.2.19}$$

Due to the estimates (4.2.19) and (4.2.18), we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} |d - w_0|^2 dx + \int_{\mathbb{R}^3} |\nabla(d - w_0)|^2 dx \leq C\int_{\mathbb{R}^3} |\Delta(d - w_0)|^2 dx,$$

where the constant C only depends on the initial data. Integrating over $[0, t]$, by the basic energy inequality (4.2.1) it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |d(t) - w_0|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla(d - w_0)|^2 dxdt \\
&\leq \int_{\mathbb{R}^3} |d_0 - w_0|^2 dx + C\int_0^t \int_{\mathbb{R}^3} |\Delta(d - w_0)|^2 dxdt \\
&\leq C.
\end{aligned}$$

Thus,

$$\int_0^t \int_{\mathbb{R}^3} |\nabla(d - w_0)|^2 dxdt \leq C, \tag{4.2.20}$$

where the constant C only depends on initial data. This completes the proof of the lemma. □

The following auxiliary estimate shows that, provided the initial data is small enough, the norm $\|d(\cdot, t) - w_0\|_{L^\infty(\mathbb{R}^3)}$ will be as small as necessary. This smallness yields that $|d|$ will be close to 1, for all time.

Lemma 4.2.3 *Let d be the solution obtained in Theorem 4.1.1. Then*

$$\|d(\cdot,t)-w_0\|_{L^\infty(\mathbb{R}^3)} \le C\|\nabla d(t)\|^{1/2}_{L^2(\mathbb{R}^3)}\|\Delta d(t)\|^{1/2}_{L^2(\mathbb{R}^3)}, \tag{4.2.21}$$

where C is an absolute constant.

Proof Let $\mathcal{F}$ denote the Fourier transform. By Lemma 4.2.2, we can take the Fourier transform of $d-w_0$

$$\begin{aligned}
&\|d(\cdot,t)-w_0\|_{L^\infty(\mathbb{R}^3)} \le \int_{\mathbb{R}^3} |\mathcal{F}(d-w_0)|d\xi \qquad (4.2.22)\\
&= \int_{|\xi|\le\lambda} |F(d-w_0)|d\xi + \int_{|\xi|\ge\lambda} |F(d-w_0)|d\xi\\
&= \int_{|\xi|\le\lambda} \frac{1}{|\xi|}\cdot|\xi||F(d-w_0)|d\xi + \int_{|\xi|\ge\lambda} \frac{1}{|\xi|^2}\cdot|\xi^2||\mathcal{F}(d-w_0)|d\xi\\
&\le \left(\int_{|\xi|\le\lambda}\frac{1}{|\xi|^2}d\xi\right)^{\frac12}\left(\int_{|\xi|\le\lambda}|\xi||\mathcal{F}(d-w_0)|^2d\xi\right)^{\frac12}\\
&+ \left(\int_{|\xi|\ge\lambda}\frac{1}{|\xi|^4}d\xi\right)^{\frac12}\left(\int_{|\xi|\ge\lambda}|\xi^2\mathcal{F}(d-w_0)|^2d\xi\right)^{\frac12}\\
&\le C\left(\int_{r\le\lambda}\frac{r^2}{r^2}dr\right)^{\frac12}\left(\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx\right)^{\frac12}\\
&+ C\left(\int_{r\ge\lambda}\frac{r^2}{r^4}dr\right)^{\frac12}\left(\int_{\mathbb{R}^3}|\Delta(d-w_0)|^2dx\right)^{\frac12}\\
&\le C\lambda^{\frac12}\|\nabla d(t)\|_{L^2(\mathbb{R}^3)} + C\lambda^{-\frac12}\|\Delta d(t)\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

To find a λ that minimizes the right hand side of the last inequality, take the derivative in λ and set the right hand side equal to zero:

$$\lambda^{\frac12}\|\nabla d(t)\|_{L^2(\mathbb{R}^3)} = \lambda^{-\frac12}\|\Delta d(t)\|_{L^2(\mathbb{R}^3)},$$

yielding

$$\lambda = \|\Delta d(t)\|_{L^2(\mathbb{R}^3)}/\|\nabla d(t)\|_{L^2(\mathbb{R}^3)}.$$

Thus using this λ in (4.2.22) gives inequality (4.2.21), and the proof of the Lemma is complete. □

Corollary 4.2.4 *Suppose the initial data $\|u_0\|_{H^1}+\|d_0-w_0\|_{H^2}$ are small enough. Then, $|d(x,t)|\ge\frac12$.*

Proof It follows combining (4.2.1), (4.2.2), (4.2.21) since $|w_0| = 1$. □

4.2.2 Uniform estimate of $d - w_0$ in L^p with any $p \geq 1$

Here we show provided the data is small enough, all the L^p norms of $d - w_0$ are bounded.

Lemma 4.2.5 *Let d be the solution obtained in Theorem 4.1.1. There exist λ_p, depending on p, so that if $\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|^2_{H^2(\mathbb{R}^3)} \leq \lambda_p$, then for $p > 1$*

$$\begin{aligned} &\frac{1}{p}\int_{\mathbb{R}^3} |d(x,t) - w_0|^p dx + \frac{2(p-1)}{p^2}\int_0^T \int_{\mathbb{R}^3} |\nabla |d(x,t) - w_0|^{p/2}|^2 dxdt \\ &\leq C_p \int_{\mathbb{R}^3} |d_0 - w_0|^p dx, \end{aligned} \tag{4.2.23}$$

where the constant C_p depends on p and λ_p. And for $p = 1$ we have

$$\begin{aligned} \int_{\mathbb{R}^3} |d - w_0| dx &\leq \int_{\mathbb{R}^3} |d_0 - w_0| dx + \int_0^T \int_{\mathbb{R}^3} |\nabla(d - w_0)|^2 + |\Delta(d - w_0)|^2 dxdt \\ &\leq C. \end{aligned}$$

Proof Recall that since $f(w_0) = 0$, we have that the direction equation can be rewritten as

$$(d - w_0)_t + u \cdot \nabla(d - w_0) = \Delta(d - w_0) - f(d) + f(w_0).$$

Multiplying the last equation by $(d - w_0)|d - w_0|^{p-2}$, for any $p \geq 2$ (or alternatively by $(d - w_0)/(|d - w_0| + \epsilon)^{2-p}$, when $p \in [1, 2)$, and letting $\epsilon \to 0$) yields

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3} |d - w_0|^p dx \\ &= -\int_{\mathbb{R}^3} [u \cdot \nabla(d - w_0)](d - w_0)|d - w_0|^{p-2} dx \\ &+ \int_{\mathbb{R}^3} \Delta(d - w_0)(d - w_0)|d - w_0|^{p-2} dx \\ &- \int_{\mathbb{R}^3} 2[w_0 \cdot (d - w_0) + \frac{3}{4}|d - w_0|^2]^2 |d - w_0|^{p-2} - \frac{1}{8}|d - w_0|^{p+2} dx \\ &\equiv I_7 + I_8 + I_9, \end{aligned} \tag{4.2.24}$$

where the I_9 was obtained similarly as in the previous calculation for I_6. We estimate I_7, I_8 and I_9 as follows:

Integrating by parts over ball B_R gives

$$\begin{aligned} I_7 = &\lim_{R\to\infty}\left(-\int_{\partial B_R}|d-w_0|^p u\cdot n d\sigma\right)+\\ &\lim_{R\to\infty}\left(\int_{B_R}[u\cdot\nabla(d-w_0)](d-w_0)|d-w_0|^{p-2}dx\right)\\ &+\lim_{R\to\infty}\left(\int_{B_R}[u\otimes(d-w_0)](d-w_0)\nabla|d-w_0|^{p-2}dx\right), \end{aligned}$$

while in the last integral we have that

$$\nabla|d-w_0|^{p-2}=(p-2)|d-w_0|^{p-3}\frac{(d-w_0)\cdot\nabla(d-w_0)}{|d-w_0|}$$

and applying the components calculation to the third integral yields that

$$\int_{B_R}[u\otimes(d-w_0)](d-w_0)\nabla|d-w_0|^{p-2}dx=(p-2)\int_{B_R}[u\cdot\nabla(d-w_0)](d-w_0)|d-w_0|^{p-2}dx.$$

It then follows that

$$I_7=\lim_{R\to\infty}\left(-\int_{\partial B_R}|d-w_0|^p u\cdot n d\sigma\right)-(p-1)I_7.$$

Therefore, we have that

$$\begin{aligned} p|I_7| &= |\lim_{R\to\infty}\int_{\partial B_R}|d-w_0|^p u\cdot n d\sigma| \qquad (4.2.25)\\ &\le \lim_{R\to\infty}\left(\int_{\partial B_R}|d-w_0|^2 d\sigma\right)^{1/2}\left(\int_{\partial B_R}|u|^2 d\sigma\right)^{1/2} \end{aligned}$$

for any $p\ge 1$, where we used that $|d-w_0|\le C$.

By Lemma 4.2.2 we know that $\int_{\mathbb{R}^3}|d-w_0|^2dx$ is bounded, and $\int_{\mathbb{R}^3}|u|^2dx$ is bounded from the energy estimate (4.2.1). Thus, for the inequality (4.2.25), using arguments similar to the ones applied to derive the convergence (4.2.12), will yield

$$\int_{\partial B_{R_i}}|d-w_0|^p u\cdot n d\sigma\to 0$$

for an appropriate sequence $R_i\to\infty$, for any $p\ge 1$. Thus,

$$I_7=0. \qquad (4.2.26)$$

For I_8, integrating by parts over B_R yields

$$\begin{aligned} I_8 = \lim_{R\to\infty}[&\int_{\partial B_R} \frac{\partial(d-w_0)}{\partial n}\cdot(d-w_0)|d-w_0|^{p-2}d\sigma \\ &- \int_{B_R}|\nabla(d-w_0)|^2|d-w_0|^{p-2}dx \\ &- (p-2)\int_{B_R}|\nabla(d-w_0)\cdot(d-w_0)|^2|d-w_0|^{p-4}dx] \\ &\equiv \lim_{R\to\infty}(K_1-K_2-K_3). \end{aligned} \tag{4.2.27}$$

The boundary term K_1 is estimated as

$$\begin{aligned} K_1 &= \int_{\partial B_R} \frac{\partial(d-w_0)}{\partial n}\cdot(d-w_0)|d-w_0|^{p-2}d\sigma \\ &\le \int_{\partial B_R}|\nabla(d-w_0)||d-w_0|^{p-1}d\sigma \\ &\le \int_{\partial B_R}|\nabla(d-w_0)|d\sigma \end{aligned} \tag{4.2.28}$$

where we used that $|d-w_0|^{p-1}\le C$ for any $p\ge 1$.

In Proposition (4.2.1), let $k=1$, $m=2$, $r=1$, $p=2$ and $q=1$. For $a=\frac{2}{7}$ the inequality (4.2.3) yields

$$\begin{aligned} &\int_{\mathbb{R}^3}|\nabla(d(t)-w_0)|dx \\ &\le C\left(\int_{\mathbb{R}^3}|\Delta(d(t)-w_0)|dx\right)^{1/7}\left(\int_{\mathbb{R}^3}|d(t)-w_0|dx\right)^{5/7} \\ &\le C_0(C(t))^{5/7}, \end{aligned}$$

where we used Ladyzhenskaya estimate (4.1.10) and the estimate (4.1.11), C_0 is a constant depending on initial data and $C(t)$ is the time dependent function in (4.1.11). Thus, for any fixed $t>0$,

$$\int_{\partial B_{R_i}}|\nabla(d-w_0)|d\sigma\to 0$$

for a sequence $R_i\to\infty$. Hence, from (4.2.28), we have

$$\lim_{R\to\infty}K_1=0. \tag{4.2.29}$$

Since

$$-|\nabla(d-w_0)|^2|d-w_0|^2\le -|\nabla(d-w_0)\cdot(d-w_0)|^2,$$

we have

$$-|\nabla(d-w_0)|^2|d-w_0|^{p-2} \le -|\nabla(d-w_0)\cdot(d-w_0)|^2|d-w_0|^{p-4},$$

which implies from (4.2.27) that

$$\begin{aligned} -K_2 - K_3 &\le (p-1)\int_{B_R} -|\nabla(d-w_0)\cdot(d-w_0)|^2|d-w_0|^{p-4}dx \\ &= -\frac{4(p-1)}{p^2}\int_{B_R}|\nabla|d-w_0|^{\frac{p}{2}}|^2dx. \end{aligned} \tag{4.2.30}$$

Combining (4.2.27), (4.2.29) and (4.2.30) yields

$$I_8 \le -\frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla|d-w_0|^{\frac{p}{2}}|^2dx \tag{4.2.31}$$

for any $p \ge 1$. Slightly modifying the process to estimate I_6 gives

$$I_9 \le \frac{1}{8\eta^2}\int_{\mathbb{R}^3}|d-w_0|^{p+2}dx. \tag{4.2.32}$$

Combining (4.2.24) with inequalities (4.2.26), (4.2.31), and (4.2.32) yields

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|d-w_0|^p dx + (p-1)\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2|d-w_0|^{p-2}dx \\ &\le C\int_{\mathbb{R}^3}|d-w_0|^{p+2}dx. \end{aligned} \tag{4.2.33}$$

Denote $v = |d-w_0|^{p/2}$. Then (4.2.33) can be rewritten as

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|v|^2dx + \frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla v|^2dx \le C\int_{\mathbb{R}^3}|v|^2|d-w_0|^2dx. \tag{4.2.34}$$

Applying Hölder inequality and Sobolev inequality to the right hand side yields

$$\begin{aligned} \int_{\mathbb{R}^3}|v|^2|d-w_0|^2dx &\le \left(\int_{\mathbb{R}^3}|v|^6dx\right)^{1/3}\left(\int_{\mathbb{R}^3}|d-w_0|^3dx\right)^{2/3} \\ &\le C\int_{\mathbb{R}^3}|\nabla v|^2dx\left(\int_{\mathbb{R}^3}|d-w_0|^2dx\right)^{2/3}\|d-w_0\|^{2/3}_{L^\infty(\mathbb{R}^3)} \end{aligned}$$

From Lemma 4.2.2 and Lemma 4.2.3, when $p > 1$, we can choose initial data small enough so that

$$C\left(\int_{\mathbb{R}^3}|d-w_0|^2dx\right)^{2/3}\|d-w_0\|^{2/3}_{L^\infty(\mathbb{R}^3)} \le \frac{2(p-1)}{p^2}. \tag{4.2.35}$$

It follows from (4.2.34) that

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|v|^2dx + \frac{2(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla v|^2dx \le 0. \tag{4.2.36}$$

The inequality (4.2.23), for $p > 1$, can be obtained integrating (4.2.36) over $[0, T]$. When $p = 1$, we have from (4.2.33)

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3}|d-w_0|dx &\leq C\int_{\mathbb{R}^3}|d-w_0|^3dx\\ &\leq C\int_{\mathbb{R}^3}|d-w_0|^2dx\|d-w_0\|_{L^\infty(\mathbb{R}^3)}\\ &\leq C\int_{\mathbb{R}^3}|\nabla d|^2+|\Delta d|^2dx\end{aligned}$$

where we used Lemma 4.2.2 and Lemma 4.2.3. Integrating over time $[0, T]$ yields

$$\begin{aligned}&\int_{\mathbb{R}^3}|d-w_0|dx\\ &\leq \int_{\mathbb{R}^3}|d_0-w_0|dx + C\int_0^T\int_{\mathbb{R}^3}|\nabla d|^2+|\Delta d|^2dxdt \leq C\end{aligned}$$

where we used the energy estimate (4.2.1). This completes the proof of the lemma. □

4.2.3 Decay of $d - w_0$ and $\nabla(d - w_0)$

We first establish that $d - w_0$ decays in $L^p(\mathbb{R}^3)$, for $p > 1$ at the rate $(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}$.

Theorem 4.2.6 *Let d be the solution obtained in Theorem 4.1.1. Assume $d - w_0 \in L^p(\mathbb{R}^3)$, $p \geq 1$. Assume the initial data satisfies the conditions of Lemma 4.2.5. Then for any $1 < p < \infty$, $t > 0$*

$$\|d(\cdot,t)-w_0\|_{L^p(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \tag{4.2.37}$$

where the constant C depends on λ_p as defined in Lemma 4.2.5.

Proof Note that as $p \to \infty$ the constants λ_p in Lemma 4.2.5 will tend to zero, hence we cannot pass to the limit as $p \to \infty$. Therefore this result does not give the decay for the L^∞ norm.

We proceed by induction for k with $p = 2^k$. The other powers p follow by interpolation. When $k = 0$ the theorem follows by Lemma 4.2.5. Suppose it holds for $s = k$, then we have

$$\|d(\cdot,t)-w_0\|_{L^{2^k}(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{2^k})}, \tag{4.2.38}$$

Let $v = |d(\cdot,t) - w_0|^{2^k}$.

Recall the inequality (4.2.36) (which holds provided the data satisfies (4.2.35))

$$\frac{d}{dt}\int_{\mathbb{R}^3} |v|^2 dx + C_p \frac{2(p-1)}{p}\int_{\mathbb{R}^3} |\nabla v|^2 dx \leq 0, \tag{4.2.39}$$

By Gagliardo-Nirenberg inequality we have

$$\int_{\mathbb{R}^3} |v|^2 dx \leq C\left(\int |\nabla v|^2 dx\right)^{3/5}\left(\int |v| dx\right)^{4/5}$$

Hence using the inductive hypothesis on the last integral on the right hand side we have

$$-\int |\nabla v|^2 dx \leq -C\left(\int_{\mathbb{R}^3} |v|^2 dx\right)^{5/3}(1+t)^{2(2^k-1)}$$

Combining the last inequality with (4.2.39) yields

$$\frac{d}{dt}\frac{\int_{\mathbb{R}^3}|v|^2 dx}{\left(\int_{\mathbb{R}^3}|v|^2 dx\right)^{5/3}} \leq -(1+t)^{2(2^k-1)}$$

Integrating and reordering terms yields

$$\int_{\mathbb{R}^3} |v|^2 dx \leq \left[\frac{v_0}{(1+(1+t))^{2(2^k-1)+1}}\right]^{3/2}$$

Since $\frac{3}{2}(2(2^k-1)+1) = \frac{3}{2}(2^{k+1}-1)$ the induction step is obtained, establishing the conclusion of the theorem.

□

As a consequence of the last theorem, we derive the decay of $\nabla(d - w_0)$.

Corollary 4.2.7 *Let d be the solution to system (4.1.1) obtained in Theorem 4.1.1. Then*

$$\|\nabla(d-w_0)\|_{L^2}^2 \leq C(1+t)^{-\frac{3}{4}},$$

where C depends on initial data.

Proof Take $p = 2$ in Theorem 4.2.6,

$$\|d(\cdot,t) - w_0\|_{L^2(\mathbb{R}^3)} \leq C\|d_0 - w_0\|_{L^2(\mathbb{R}^3)}(1+t)^{-\frac{3}{4}}. \tag{4.2.40}$$

Gagliardo-Nirenberg inequality (4.2.3) yields

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx &\leq C\left(\int_{\mathbb{R}^3}|d-w_0|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|\Delta(d-w_0)|^2dx\right)^{\frac{1}{2}}\\ &\leq C\left(\int_{\mathbb{R}^3}|d-w_0|^2dx\right)^{\frac{1}{2}}\\ &\leq C(1+t)^{-\frac{3}{4}},\end{aligned}$$

where in the last two steps we used Ladyzhenskaya energy estimate (4.2.2) and (4.2.40), respectively. The constant C depends on initial data. It completes the proof. □

4.3 Decay of Velocity

An application of the Fourier Splitting method [48] is used to establish L^2 decay of velocity u.

Theorem 4.3.1 *Let u be the solution obtained in Theorem 4.1.1. If additionally $u_0 \in L^1(\mathbb{R}^3)$, then*

$$\|u(\cdot,t)\|_{L^2}^2 \leq C(1+t)^{-\frac{1}{2}},$$

where C depends on initial data, the L^1 and L^2 norm of u_0.

Proof Multiplying the Navier-Stokes equation in system (4.1.1) by u and integrating by parts yields

$$\frac{d}{dt}\int_{\mathbb{R}^3}|u|^2+2\int_{\mathbb{R}^3}|\nabla u|^2dx = 2\int_{\mathbb{R}^3}\nabla u(\nabla d\otimes\nabla d)dx. \tag{4.3.1}$$

Hölder and Cauchy Schwartz inequalities yield

$$2\int_{\mathbb{R}^3}\nabla u(\nabla d\otimes\nabla d)dx \leq \int_{\mathbb{R}^3}|\nabla u|^2dx + C\int_{\mathbb{R}^3}|\nabla d\otimes\nabla d|^2dx.$$

Thus, we derive from (4.3.1)

$$\frac{d}{dt}\int_{\mathbb{R}^3}|u|^2+\int_{\mathbb{R}^3}|\nabla u|^2dx \leq C\int_{\mathbb{R}^3}|\nabla d\otimes\nabla d|^2dx.$$

The right hand side of above inequality can be estimated as

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla d \otimes \nabla d|^2 dx &= \int_{\mathbb{R}^3} (\nabla d \otimes \nabla d)(\nabla d \otimes \nabla d) dx \\ &= -3 \int_{\mathbb{R}^3} (d - w_0)\, \Delta d\, \nabla d \otimes \nabla d \\ &\leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla d \otimes \nabla d|^2 dx + C \int_{\mathbb{R}^3} |d - w_0|^2 |\Delta d|^2 dx,\end{aligned}$$

from which it follows

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla d \otimes \nabla d|^2 dx &\leq C \int_{\mathbb{R}^3} |d - w_0|^2 |\Delta d|^2 dx \\ &\leq C \left(\int_{\mathbb{R}^3} |d - w_0|^p dx \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^3} |\Delta d|^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}} \\ &= C \|d - w_0\|^2_{L^p(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3} |\Delta d|^{2 + \frac{4}{p-2}} dx \right)^{\frac{p-2}{p}} \\ &\leq C \|d - w_0\|^2_{L^p(\mathbb{R}^3)},\end{aligned}$$

for $p \geq 2$, where the last step followed from Ladyzhenskaya estimate (4.2.2) and the fact that $\|\Delta d\|_{L^\infty(\mathbb{R}^3 \times [0,T])}$ is bounded since d is regular in the sense stated in Theorem 4.1.1. Thus, it follows from Theorem 4.2.6 that

$$\int_{\mathbb{R}^3} |\nabla d \otimes \nabla d|^2 dx \leq C(1+t)^{-3(1-\frac{1}{p})},$$

for any $p \geq 2$. Therefore,

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx \leq C(1+t)^{-3(1-\frac{1}{p})}. \tag{4.3.2}$$

Applying Plancherel's theorem to (4.3.2) gives

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi + \int_{\mathbb{R}^3} |\xi|^2 |\hat{u}|^2 d\xi \leq C(1+t)^{-3(1-\frac{1}{p})}. \tag{4.3.3}$$

The idea is to decompose the frequency domain $\mathbb{R}^3$ in integral $\int_{\mathbb{R}^3} |\xi|^2 |\hat{u}|^2 d\xi$ into two time-dependent subdomains. The time dependent subdomains are a 3-dimensional sphere, $S(t)$, centered at the origin with an appropriate time dependent radius and its complement. For this we rewrite (4.3.3) as

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi \leq - \int_{S(t)^c} |\xi|^2 |\hat{u}|^2 d\xi - \int_{S(t)} |\xi|^2 |\hat{u}|^2 d\xi + C(1+t)^{-3(1-\frac{1}{p})},$$

where $S(t)$ is the ball

$$S(t) = \{\xi \in \mathbb{R}^3 : |\xi| \le r(t) = (\frac{k}{1+t})^{1/2}\}$$

for a certain k, which will be determined below. Hence

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} |\hat{u}|^2 d\xi &\le -\frac{k}{1+t}\int_{S(t)^c} |\hat{u}|^2 d\xi - \int_{S(t)} |\xi|^2 |\hat{u}|^2 d\xi + C(1+t)^{-3(1-\frac{1}{p})} \\ &= -\frac{k}{1+t}\int_{\mathbb{R}^3} |\hat{u}|^2 d\xi + \int_{S(t)} (\frac{k}{1+t} - |\xi|^2)|\hat{u}|^2 d\xi + C(1+t)^{-3(1-\frac{1}{p})}\end{aligned}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\hat{u}|^2 d\xi + \frac{k}{1+t}\int_{\mathbb{R}^3} |\hat{u}|^2 d\xi \le \frac{k}{1+t}\int_{S(t)} |\hat{u}|^2 d\xi + C(1+t)^{-3(1-\frac{1}{p})}. \tag{4.3.4}$$

The following estimate, which will be established later, is needed

$$|\hat{u}(\xi, t)| \le C|\xi|^{-1} \tag{4.3.5}$$

for $\xi \in S(t)$, where C is a constant only depending on the initial data. Combining the inequalities (4.3.4) and (4.3.5) yields

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\hat{u}|^2 d\xi + \frac{k}{1+t}\int_{\mathbb{R}^3} |\hat{u}|^2 d\xi \le \frac{C}{1+t}\int_{S(t)} |\xi|^{-2} d\xi + C(1+t)^{-3(1-\frac{1}{p})}.$$

Multiplying by the integrating factor $(1+t)^k$ yields

$$\frac{d}{dt}\left[(1+t)^k \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi\right] \le C(1+t)^{k-1}\int_{S(t)} |\xi|^{-2} d\xi + C(1+t)^{k-3(1-\frac{1}{p})}.$$

Since $p \ge 2$ and

$$\int_{S(t)} |\xi|^{-2} d\xi \le C\int_0^{r(t)} r^2 r^{-2} dr \le C(1+t)^{-1/2},$$

it follows that

$$\frac{d}{dt}\left[(1+t)^k \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi\right] \le C(1+t)^{k-\frac{3}{2}}.$$

Integrating in time yields

$$(1+t)^k \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi \le \int_{\mathbb{R}^3} |\hat{u}(\xi, 0)|^2 d\xi + C[(1+t)^{k-\frac{1}{2}} - 1].$$

Thus,

$$\int_{\mathbb{R}^3} |\hat{u}|^2 d\xi \le (1+t)^{-k}\int_{\mathbb{R}^3} |\hat{u}(\xi, 0)|^2 d\xi + C[(1+t)^{-\frac{1}{2}} - (1+t)^{-k}].$$

Since $u_0 \in L^2$, it follows that $\hat{u}(0) \in L^2$ from Plancherel's theorem. Hence

$$\int_{\mathbb{R}^3} |\hat{u}|^2 d\xi \leq C(1+t)^{-\frac{1}{2}},$$

Hence

$$\int_{\mathbb{R}^3} |u|^2 dx \leq C(1+t)^{-\frac{1}{2}}.$$

To complete the proof we need to establish the inequality (4.3.5). Taking the Fourier transform of Navier-Stokes equation in system (4.1.1) yields

$$\hat{u}_t + |\xi|^2 \hat{u} = G(\xi, t) \tag{4.3.6}$$

where

$$G(\xi, t) = -\mathcal{F}(u \cdot \nabla u) - \mathcal{F}(\nabla p) - \mathcal{F}(\nabla \cdot (\nabla d \otimes \nabla d)),$$

and $\mathcal{F}$ indicates the Fourier transform. Multiplying (4.3.6) by the integrating factor $e^{|\xi|^2 t}$ yields

$$\frac{d}{dt}[e^{|\xi|^2 t}\hat{u}] = e^{|\xi|^2 t} G(\xi, t).$$

Integrating in time gives

$$\hat{u}(\xi, t) = e^{-|\xi|^2 t}\hat{u}_0 + \int_0^t e^{-|\xi|^2 (t-s)} G(\xi, s) ds. \tag{4.3.7}$$

We assume for the moment the following auxiliary estimate, which we will prove below,

$$|G(\xi, t)| \leq C|\xi|. \tag{4.3.8}$$

Combining (4.3.7) and (4.3.8) yields

$$|\hat{u}(\xi, t)| \leq e^{-|\xi|^2 t}|\hat{u}_0| + \int_0^t e^{-|\xi|^2 (t-s)} |\xi| ds. \tag{4.3.9}$$

Since $u_0 \in L^1$, we have $|\hat{u}_0| \leq C$ for all ξ and some constant C.Performing integration in (4.3.9) gives

$$|\hat{u}(\xi, t)| \leq Ce^{-|\xi|^2 t} + \frac{C}{|\xi|}(1 - e^{-|\xi|^2 t}) \leq C|\xi|^{-1}$$

for $\xi \in S(t)$. To finish the proof we need to establish (4.3.8). For this purpose we analyze each term in $G(\xi, t)$ separately. We have

$$|\mathcal{F}(u \cdot \nabla u)| = |\mathcal{F}(\nabla \cdot (u \otimes u))| \leq \sum_{i,j} \int_{\mathbb{R}^3} |u^i u^j| |\xi_j| dx.$$

Since $u \in L^\infty(L^2)$ by the basic energy estimate, we have

$$|\mathcal{F}(u \cdot \nabla u)| \leq C|\xi|.$$

By the basic energy inequalities (4.2.1, 4.2.2) we have $\nabla d \in L^\infty(L^2)$ proceeding similarly as for the last inequality we have

$$|\mathcal{F}(\nabla \cdot (\nabla d \otimes \nabla d))| \leq C|\xi|.$$

Taking divergence of Navier-Stokes equation in system (1.0.15) gives that

$$\Delta p = -\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(u^i u^j) - \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(\nabla d^i \nabla d^j).$$

Taking the Fourier transform then yields

$$|\xi|^2 \mathcal{F}(p) = -\sum_{i,j} \xi_i \xi_j \mathcal{F}(u^i u^j) - \sum_{i,j} \xi_i \xi_j \mathcal{F}(\nabla d^i \nabla d^j).$$

Since $\mathcal{F}(u^i u^j) \in L^\infty$ and $\mathcal{F}(\nabla d^i \nabla d^j) \in L^\infty$, it follows that

$$\mathcal{F}(p) \leq C,$$

and thus $\mathcal{F}(\nabla p) \leq C|\xi|$. It completes the proof of (4.3.8) and hence completes the proof of theorem. □

Combining Theorem 4.2.6, Corollary 4.2.7 and Theorem 4.3.1 yields the proof of the main Theorem 4.1.3.

Remark 4.3.2 *The decay rate for the velocity u in L^2 obtained in [55], for the bounded domain case, is $(1+t)^{-\frac{\theta}{1-2\theta}}$ where $\theta \in (0, \frac{1}{2})$. When θ is close to 0, then $-\frac{\theta}{1-2\theta}$ would be very small, meaning the decay is very slow. In this paper, we obtained the decay rate for velocity u in L^2 with $(1+t)^{-\frac{1}{4}}$, a fixed constant algebraic rate. The advantage comes from the fact that we work on the whole space $\mathbb{R}^3$ where we can apply the Fourier splitting method.*

Remark 4.3.3 *It was pointed out in Section 4.1 that there is an essential difficulty to apply Łojasiewicz-Simon approach in whole space $\mathbb{R}^3$. However, in weighted Sobolev spaces of $\mathbb{R}^3$, the compactness is recovered. Thus, we expect there is hope to*

construct certain Lojasiewicz-Simon type inequality in weighted Sobolev spaces and proceed with the method in [55] to derive the decay of solutions to the LCD system in weighted Sobolev spaces.

4.4 Appendix: Existence of Classical Solutions in $\mathbb{R}^3$

In this section we sketch a brief proof of the existence Theorem 4.1.1 in Section 4.1. As mentioned in there, for bounded domains in $\mathbb{R}^3$, the existence of global regular solutions to the flow of nematic liquid crystals with constant density has been established in [35] provided the viscosity is large enough. The existence of global regular solutions to the flow of nematic liquid crystals with non-constant density has been established in Chapter 3 provided the initial data is small enough. In both of the above contexts, a Ladyzhenskaya energy estimate (higher order derivative estimate) was derived and hence a relatively standard bootstrapping argument yielded a regular solution.

The proof of Theorem 4.1.1 will be given through four steps. In the first step, on a sequence of balls B_{R_n} with radius R_n, centered at the origin, we obtain the existence of a Galerkin approximated solution $(u^{n,m}, d^{n,m})$ for the system (4.1.1) with modified initial data, for each $m = 1, 2, 3, ...$ In the second step, we establish an estimate of $d^{n,m} - w_0$ in $L^1(B_{R_n})$ for any fixed time $t > 0$. In the third step, we take the limit $m \to \infty$. In the forth step, we take the limit $R_n \to \infty$. In fact, we are able to show that all the estimates in Ladyzhenskaya energy method in step one are independent of the domain size. Thus we can take a subsequence of solutions on balls B_{R_n} which converge to a limit in $\mathbb{R}^3$ when R_n goes to infinity.

Lemma 4.4.1 *Assume $u_0 \in H^1(\mathbb{R}^3)$ and $\tilde{d}_0 \equiv d_0(x) - w_0 \in H^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ with $|w_0| = 1$. There exists a sequence of functions $\{(u_0^n, d_0^n)\}_{n=1}^{\infty}$ and a sequence of real numbers $\{R_n\}_{n=1}^{\infty}$ with $R_n \to \infty$ as $n \to \infty$ such that,*

$$u_0^n \in H_0^1(B_{R_n}), \text{ with } u_0^n \to u_0 \text{ in } H^1(\mathbb{R}^3) \text{ as } n \to \infty \tag{4.4.1}$$

$$\tilde{d}_0^n \in H_0^2(B_{R_n}), \text{ with } \tilde{d}_0^n \to \tilde{d}_0 \text{ in } H^2(\mathbb{R}^3) \text{ as } n \to \infty \tag{4.4.2}$$

where $\tilde{d}_0^n = d_0^n - w_0$. *Moreover,*

$$\|\tilde{d}_0^n\|_{L^1(B_{R_n})} \leq C\|\tilde{d}_0\|_{L^1(\mathbb{R}^3)}, \tag{4.4.3}$$

and $|d_0^n| \leq 1$.

Proof Such a sequence of functions can be constructed easily as follows. Let ζ_n be a sequence of smooth functions such that

$$\zeta_n(x) = \begin{cases} 1, & \text{in } B_{R_n} \\ 0, & \text{in } B_{2R_n} \end{cases}$$

and $|\zeta_n(x)| \leq 1$ for all $x \in \mathbb{R}^3$.
Define $u_0^n = \zeta_n u_0$ and $\tilde{d}_0^n = \zeta_n \tilde{d}_0$. Let $d_0^n = \tilde{d}_0^n + w_0$. These ζ_n can be chosen such that (4.4.1), (4.4.2) and (4.4.3) are satisfied. In addition, we have

$$|d_0^n| = |\zeta_n d_0 + (1-\zeta_n)w_0| \leq \zeta_n + (1-\zeta_n) = 1.$$

□

For the sequel we assume as initial conditions

$$\begin{cases} u(x,0) = u_0^n(x), \\ d(x,0) = d_0^n(x), \end{cases}, \text{ in } B_{R_n} \times [0,T), \tag{4.4.4}$$

where u_0^n and d_0^n are as obtained in Lemma 4.4.1, and the boundary conditions

$$\begin{cases} u(x,t) = 0, \\ d(x,t) - w_0 = 0, \end{cases}, \text{ on } \partial B_{R_n} \times [0,T). \tag{4.4.5}$$

We have the following existence of solutions to system (4.1.1) with (4.4.4) and (4.4.5) on each ball B_{R_n}.

Theorem 4.4.2 *Let* B_{R_n} *be the ball centered at origin with radius* R_n *in* $\mathbb{R}^3$. *Assume that* $u_0 \in H^1(\mathbb{R}^3)$ *and* $d_0 - w_0 \in H^2(\mathbb{R}^3)$. *The system (4.1.1) with initial and boundary conditions (4.4.4) and (4.4.5) has a smooth solution* $(u^{n,m}, p^{n,m}, d^{n,m})$ *for each* $m = 1, 2, 3, \ldots$ *satisfying, for any* $T > 0$

$$u^{n,m} \in L^2(0,T; H_0^1(B_{R_n})) \cap L^\infty(0,T; L^2(B_{R_n}))$$

$$\tilde{d}^{n,m} \in L^2(0,T;H_0^2(B_{R_n})) \cap L^\infty(0,T;H_0^1(B_{R_n}))$$

with $\tilde{d}^{n,m} = d^{n,m} - w_0$ *and* $|d^{n,m}| \le 1$. *Moreover, it satisfies the energy inequality*

$$\begin{aligned} &\frac{d}{dt}\int_{B_{R_n}} \frac{1}{2}|u^{n,m}|^2 + \frac{1}{2}|\nabla d^{n,m}|^2 + F(d^{n,m})dx \\ &+ \int_{B_{R_n}} |\nabla u^{n,m}|^2 + |\Delta d^{n,m} - f(d^{n,m})|^2 dx \le 0. \end{aligned} \tag{4.4.6}$$

Proof The existence proof is obtained through the standard Galerkin approximation method, see [35] and Chapter 3 of this thesis. We only need to give a brief explanation on the claim $|d^{n,m}| \le 1$ by applying a maximum principle argument. Notice that the approximated initial data d_0^n satisfies $|d_0^n(x)| \le 1$ for all $x \in \mathbb{R}^3$. Suppose there exists a point (x_0, t_0) in the interior of the domain $B_{R_n} \times [0,T)$, such that $|d^{n,m}|^2$ attains a maximum value at this point. Multiplying the equation

$$d_t^{n,m} + u^{n,m} \cdot \nabla d^{n,m} = \Delta d^{n,m} - \frac{1}{\eta^2}(|d^{n,m}|^2 - 1)d^{n,m}$$

by $d^{n,m}$ yields

$$\begin{aligned} &\frac{d}{dt}|d^{n,m}|^2 + u^{n,m} \cdot \nabla |d^{n,m}|^2 \cdot d^{n,m} \\ &= \triangle |d^{n,m}|^2 - 2|\nabla d^{n,m}|^2 - \frac{2}{\eta^2}(|d^{n,m}|^2 - 1)|d^{n,m}|^2. \end{aligned} \tag{4.4.7}$$

At the maximum point (x_0, t_0), we have $\frac{d}{dt}|d^{n,m}|^2 = \nabla |d^{n,m}|^2 = 0$ and $\triangle |d^{n,m}|^2 \le 0$. Thus, it follows from the equation (4.4.7) that, at the point (x_0, t_0)

$$(|d^{n,m}|^2 - 1)|d^{n,m}|^2 \le 0.$$

This insures that $|d^{n,m}| \le 1$ at any interior maximum point (x_0, t_0). Therefore, $|d^{n,m}| \le 1$ for all points in $B_{R_n} \times [0,T)$. □

For the solution $(u^{n,m}, d^{n,m})$ obtained in the above theorem on $B_{R_n} \times [0,T)$, we define energy quantity

$$\Phi_{n,m}^2(t) = \|\nabla u^{n,m}\|_{L^2(B_{R_n})}^2 + \|\triangle (d^{n,m} - w_0)\|_{L^2(B_{R_n})}^2.$$

With a slight modification of the proof of Theorem 3.1 in Chapter 3, we are able to show that

Theorem 4.4.3 *Assume that $u_0 \in H^1(B_{R_n})$ and $d_0 \in H^2(B_{R_n})$, and $\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|\tilde{d}_0\|^2_{H^2(\mathbb{R}^3)} < \infty$. Let $(u^{n,m}, d^{n,m})$ be solutions obtained in Theorem 4.4.2. There is a positive small number ϵ_0 such that if*

$$\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|^2_{H^2(\mathbb{R}^3)} \le \epsilon_0, \tag{4.4.8}$$

then

$$\begin{aligned} &\int_{B_{R_n}} |\nabla u^{n,m}|^2 + |\Delta(d^{n,m} - w_0)|^2 dx \\ &+ \int_0^T \int_{B_{R_n}} |\Delta u^{n,m}|^2 + |\nabla\Delta(d^{n,m} - w_0)|^2 dx \\ &\le C(\|u_0\|^2_{H^1(B_{R_n})} + \|d_0 - w_0\|^2_{H^2(B_{R_n})}), \end{aligned} \tag{4.4.9}$$

for any $T > 0$, where the constant C is independent of domain size R_n and m.

There is no need to prove the theorem except that we need a brief explanation on the last claim that constant C is independent of R_n. In the proof of Ladyzhenskaya energy estimate in Chapter 3 we only use the Gagliardo-Nirenberg interpolation inequalities and standard elliptic inequalities. That is we use

$$\|u^{n,m}\|^4_{L^4} \le C\|u^{n,m}\|_{L^2}\|\nabla u^{n,m}\|^3_{L^2}$$

$$\|\nabla(d^{n,m} - w_0)\|^4_{L^4} \le C\|\nabla(d^{n,m} - w_0)\|_{L^2}\|\Delta(d^{n,m} - w_0)\|^3_{L^2}$$

and the elliptic estimate

$$\|D^2 u^{n,m}\|_{L^2} \le C\|\Delta u^{n,m}\|_{L^2}$$

for $u^{n,m}$ and $d^{n,m} - w_0$ vanishing on the boundary. In the above inequalities, the various constants C are independent of the size of the domain. Thus the constant C in (4.4.9) is independent of R_n.

For the second step, we derive a time dependent estimate of $d^{n,m}(t) - w_0$ in $L^1(B_{R_n})$.

Lemma 4.4.4 *Let $d^{n,m}$ be the solution obtained in Theorem 4.4.2. In addition, assume $d_0 - w_0 \in L^1(\mathbb{R}^3)$. Then*

$$\int_{B_{R_n}} |d^{n,m}(t) - w_0| dx \le (C_0 t + \int_{\mathbb{R}^3} |d_0 - w_0| dx) e^{Ct} \tag{4.4.10}$$

where the constant C_0 only depends on initial data and constant C only depends on η.

Proof By the second equation in (4.1.1) we have,

$$\begin{aligned}
&\frac{d}{dt}\int_{B_{R_n}} |d^{n,m}(t)-w_0|dx \qquad (4.4.11)\\
&=\int_{B_{R_n}} \frac{(d^{n,m}(t)-w_0)\cdot d_t^{n,m}}{|d^{n,m}(t)-w_0|}dx\\
&=\int_{B_{R_n}} \frac{(d^{n,m}(t)-w_0)\cdot \Delta d^{n,m}(t)}{|d^{n,m}(t)-w_0|}dx-\int_{B_{R_n}} \frac{(d^{n,m}(t)-w_0)\cdot [u^{n,m}\cdot\nabla d^{n,m}(t)]}{|d^{n,m}(t)-w_0|}dx\\
&-\int_{B_{R_n}} \frac{(d^{n,m}(t)-w_0)\cdot f(d^{n,m}(t))}{|d^{n,m}(t)-w_0|}dx\\
&\equiv I_1+I_2+I_3.
\end{aligned}$$

There is no need to worry about the singular points of $(d^{n,m}-w_0)^{-1}$ in the above equation, since each term on the right hand side contains $\frac{d^{n,m}-w_0}{|d^{n,m}-w_0|}$. We deal with the three terms I_1, I_2 and I_3 in the following way.

Since $d^{n,m}(t)-w_0=0$ on the boundary ∂B_{R_n}. Replacing d_t by its value in the direction equation and integration by parts yields

$$\begin{aligned}
I_1=&-\int_{B_{R_n}} \frac{|\nabla(d^{n,m}(t)-w_0)|^2}{|d^{n,m}(t)-w_0|}dx \qquad (4.4.12)\\
&+\int_{B_{R_n}} |(d^{n,m}(t)-w_0)\cdot\nabla(d^{n,m}(t)-w_0)|^2|d^{n,m}(t)-w_0|^{-3}dx\\
&\le 0,
\end{aligned}$$

since $|(d^{n,m}(t)-w_0)\cdot\nabla(d^{n,m}(t)-w_0)|^2\le|(d^{n,m}(t)-w_0)|^2|\nabla(d^{n,m}(t)-w_0)|^2$.

By Hö lder inequality we have

$$\begin{aligned}
|I_2|&\le\int_{B_{R_n}}|u^{n,m}||\nabla(d^{n,m}-w_0)|dx \qquad (4.4.13)\\
&\le C\left(\int_{B_{R_n}}|u^{n,m}|^2dx\right)^{1/2}\left(\int_{B_{R_n}}|\nabla d^{n,m}|^2dx\right)^{1/2}\le C_0,
\end{aligned}$$

where we used the energy estimate (4.4.6), and the constant C_0 only depends on the initial data.

Recall that by definition $f(d^{n,m})=\frac{1}{\eta^2}(|d^{n,m}|^2-1)d^{n,m}$, and $|d^{n,m}|\le 1$ from Theorem 4.4.2 and $|w_0|=1$, hence

$$\begin{aligned}
|I_3|&\le\frac{1}{\eta^2}\int_{B_{R_n}}|f(d^{n,m})|dx \qquad (4.4.14)\\
&\le C\int_{B_{R_n}}|d^{n,m}-w_0||d^{n,m}+w_0||d^{n,m}|dx\le C\int_{B_{R_n}}|d^{n,m}-w_0|dx,
\end{aligned}$$

where the constant C depends on η. Combining the inequalities (4.4.11), (4.4.12), (4.4.13) and (4.4.14) yields

$$\frac{d}{dt}\int_{B_{R_n}} |d^{n,m}(t) - w_0|dx \leq C\int_{B_{R_n}} |d^{n,m}(t) - w_0|dx + C_0.$$

Integrating over $[0,t]$, and Gronwall's inequality (see [15]) gives,

$$\int_{B_{R_n}} |d^{n,m}(t) - w_0|dx \leq (C_0 t + \int_{B_{R_n}} |d_0^n - w_0|dx)e^{Ct},$$

for any $t > 0$. Inequality (4.4.10) now follows from the last estimate in Lemma 4.4.1. This concludes the second step.

In the third step, we take the limit of the Galerkin approximating solutions $(u^{n,m}, d^{n,m})$ as $m \to \infty$. By the estimates (4.4.6) and (4.4.9), there exists (u^n, d^n) for each $n = 1, 2, 3, ...$ such that, taking subsequence if necessary,

$$u^{n,m} \rightharpoonup u^n \quad \text{weakly in } L^2(0,T; H_0^1(B_{R_n})),$$

$$u^{n,m} \to u^n \quad \text{strongly in } L^\infty(0,T; L^2(B_{R_n})),$$

$$d^{n,m} \rightharpoonup d^n \quad \text{weakly in } L^2(0,T; H^2(B_{R_n})),$$

$$d^{n,m} \to d^n \quad \text{strongly in } L^2(0,T; H^1(B_{R_n})) \text{ with } d^n = w_0 \text{ on } \partial B_{R_n}.$$

It follows easily from the above convergence that (u^n, d^n) is a weak solution to the system (4.1.1) with initial condition (4.4.4) and boundary condition (4.4.5) on $B_{R_n} \times [0,T)$. Moreover, the solutions (u^n, d^n) satisfy the basic energy inequality

$$\begin{aligned} &\frac{d}{dt}\int_{B_{R_n}} \frac{1}{2}|u^n|^2 + \frac{1}{2}|\nabla d^n|^2 + F(d^n)dx \\ &+ \int_{B_{R_n}} |\nabla u^n|^2 + |\Delta d^n - f(d^n)|^2 dx \leq 0, \end{aligned} \tag{4.4.15}$$

and the higher order energy inequality

$$\begin{aligned} &\int_{B_{R_n}} |\nabla u^n|^2 + |\Delta(d^n - w_0)|^2 dx \\ &+ \int_0^t \int_{B_{R_n}} |\Delta u^n|^2 + |\nabla\Delta(d^n - w_0)|^2 dx \\ &\leq C(\|u_0\|^2_{H^1(B_{R_n})} + \|d_0 - w_0\|^2_{H^2(B_{R_n})}), \end{aligned} \tag{4.4.16}$$

for any $t > 0$, where the constant C is independent of domain size R_n. In addition, from Lemma 4.4.4 it follows that d^n satisfies the estimate

$$\int_{B_{R_n}} |d^n(t) - w_0| dx \leq (C_0 t + \int_{\mathbb{R}^3} |d_0 - w_0| dx) e^{Ct} \tag{4.4.17}$$

where the constant C_0 only depends on initial data and constant C only depends on η.

In the forth step, we extend the solutions (u^n, d^n) on B_{R_n} to the whole space $\mathbb{R}^3$ by taking limit $R_n \to \infty$. With the estimates (4.4.15) and (4.4.16) we can extract a subsequence $\left\{(u^{1k}, d^{1k})\right\}_{k=1}^{\infty}$ from (u^n, d^n) for $n \geq 1$ such that

$$u^{1k} \rightharpoonup u_{(1)} \text{ in } L^2(0, T; H^1(B_{R_1}))$$

$$u^{1k} \to u_{(1)} \text{ in } L^\infty(0, T; L^2(B_{R_1}))$$

$$d^{1k} \rightharpoonup d_{(1)} \text{ in } L^2(0, T; H^2(B_{R_1}))$$

$$d^{1k} \to d_{(1)} \text{ in } L^2(0, T; H^1(B_{R_1}))$$

and the limit $(u_{(1)}, d_{(1)})$ satisfies system (4.1.1) in distribution sense and the estimates (4.4.15) (4.4.16) on $B_{R_1} \times [0, T)$.

On $B_{R_2} \times [0, T)$, we take subsequence $\left\{(u^{2k}, d^{2k})\right\}_{k=1}^{\infty}$ from $\left\{(u^{1k}, d^{1k})\right\}_{k=1}^{\infty}$ such that $\left\{u^{2k}\right\}_{k=1}^{\infty}$, $\left\{p^{2k}\right\}_{k=1}^{\infty}$ and $\left\{d^{2k}\right\}_{k=1}^{\infty}$ converge to $u_{(2)}$ and $d_{(2)}$ respectively in the same convergence sense as above. And we have that

$$u_{(2)}|_{B_1} = u_{(1)}, \quad d_{(2)}|_{B_1} = d_{(1)}.$$

Repeating the process on each $B_{R_n} \times [0, T)$, we can take subsequence $\left\{(u^{nk}, d^{nk})\right\}_{k=1}^{\infty}$ from the sequence $\left\{(u^{(n-1)k}, d^{(n-1)k})\right\}_{k=1}^{\infty}$, such that $\left\{u^{nk}\right\}_{k=1}^{\infty}$ and $\left\{d^{nk}\right\}_{k=1}^{\infty}$ converge to $u_{(n)}$ and $d_{(n)}$ respectively. And we have that

$$u_{(n)}|_{B_{n-1}} = u_{(n-1)}, \quad d_{(n)}|_{B_{n-1}} = d_{(n-1)}.$$

Then we take the diagonal sequence $\left\{(u^{kk}, d^{kk})\right\}_{k=1}^{\infty}$ and let $k \to \infty$. This sequence (if necessary, take a subsequence of it) converges to (u, d), in $\mathbb{R}^3 \times [0, T)$. The limit (u, d) satisfies the system (4.1.1) in the sense of distributions and satisfies the energy estimates

$$\begin{aligned} &\int_{\mathbb{R}^3} |u|^2 + |\nabla d|^2 + 2F(d) dx + 2 \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 + |\Delta d - f(d)|^2 dx dt \\ &\leq \int_{\mathbb{R}^3} |u_0|^2 + |\nabla d_0|^2 dx \end{aligned} \tag{4.4.18}$$

$$\int_{\mathbb{R}^3} |\nabla u|^2 + |\Delta d|^2 dx + \int_0^T \int_{\mathbb{R}^3} |\Delta u|^2 + |\nabla \Delta d|^2 dx dt \tag{4.4.19}$$
$$\leq C(\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|^2_{H^2(\mathbb{R}^3)}).$$

In addition, the solution d satisfies the estimate

$$\int_{\mathbb{R}^3} |d(t) - w_0| dx \leq (C_0 t + \int_{\mathbb{R}^3} |d_0 - w_0| dx) e^{Ct}$$

with constant C_0 only depending on initial data and constant C only depending on η.

Estimates (4.4.18) and (4.4.19) allow us to apply the "bootstrapping argument" as used in Chapter 3 and [35], and prove that the limit (u, p, d) is a classical solution to system (4.1.1) satisfying the desired estimates in Theorem 4.1.1.

We mention that the estimates obtained in (4.1.8) are in fact global. The higher order Hölder norms of solutions (u, d, p) on each bounded ball in (4.1.8) are essentially estimated in terms of initial data $\|u_0\|_{H^1} + \|d_0 - w_0\|_{H^2}$ on a larger ball. For instance, $|\Delta d|_{C^\alpha(B_R)}$ is mainly estimated in terms of $|\nabla d|_{C^0(B_{R+\delta})}$ and $|u|_{C^0(B_{R+\delta})}$for some constant $\delta > 0$ (see [28] Theorem 5.1 in Chapter VII and Subsection 3.2.3 . The norms $|\nabla d|_{C^0(B_{R+\delta})}$ and $|u|_{C^0(B_{R+\delta})}$ can be estimated through Sobolev inequalities by higher order energy estimates $\|\nabla u\|_{L^\infty} + \|\Delta d\|_{L^\infty}$ which is estimated by initial data $\|u_0\|_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|_{H^2(\mathbb{R}^3)}$ from (4.4.19). Since the assumption (4.1.4) on initial data is global, it follows that the higher order Hölder estimates in (4.1.8) are global.

This completes the proof of Theorem 4.1.1. □

Part II

Magneto-Hydrodynamics System

Chapter 5

Introduction

In this part of the thesis we consider the three dimensional incompressible magneto-hydro -dynamics (MHD) system:

$$\begin{aligned} u_t - \triangle u + u \cdot \nabla u - b \cdot \nabla b + \nabla p = 0, \\ b_t - \triangle b + u \cdot \nabla b - b \cdot \nabla u = 0, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{aligned} \tag{5.0.1}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \tag{5.0.2}$$

and

$$\nabla \cdot u_0 = 0, \quad \nabla \cdot b_0 = 0, \tag{5.0.3}$$

where $x \in \mathbb{R}^3$, $t \geq 0$, u is the fluid velocity, b is the magnetic field. When the magnetic field $b(x,t)$ vanishes, incompressible MHD system is just incompressible Navier-Stokes equations. The solutions to MHD system also share the same scaling property of solutions to Navier-Stokes equations, that is,

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t),\ b_\lambda(x,t) = \lambda b(\lambda x, \lambda^2 t),\ p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

solve the MHD system (5.0.1) with initial data

$$u_{0\lambda} = \lambda u_0(\lambda x),\ b_{0\lambda} = \lambda b_0(\lambda x),$$

if $u(x,t)$ and $b(x,t)$ solve the MHD system (5.0.1) with the initial data $u_0(x)$ and $b_0(x)$. The spaces that are invariant under the above scaling are called the critical spaces. Examples of critical spaces in three dimension are

$$\dot{H}^{\frac{1}{2}} \hookrightarrow L^3 \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}^{-1,\infty}_{\infty}.$$

(see [8], for example, for the discussions of the embeddings).

The study of Navier-Stokes equations as well as of MHD system in critical spaces has been one of the focus of the research activities since the initial work of Kato [22]. In case of Navier-Stokes equations, on one hand, in 2001, Koch and Tataru [25] were able to establish global well-posedness of Navier-Stokes equations with small initial data in the space BMO^{-1}, on the other hand, recently, Bourgain and Pavlović [3] showed the ill-posedness for Navier-Stokes equations in $\dot{B}^{-1,\infty}_{\infty}$. More precisely, Bourgain and Pavlović constructed some arbitrarily small initial data in $\dot{B}^{-1,\infty}_{\infty}$ and produced the so-called norm inflation in the sense that the solution becomes arbitrarily large in $\dot{B}^{-1,\infty}_{\infty}$ after an arbitrarily short time.

In a recent work [46], Miao, Yuan and Zhang proved the existence of a global mild solution in BMO^{-1} for small initial data and uniqueness of such solution in $C([0,\infty); BMO^{-1})$. It is then an interesting problem to study the solutions to MHD system with initial data in the space $\dot{B}^{-1,\infty}_{\infty}$. In this paper, we discuss different cases of norm inflation phenomena for the MHD system in $\dot{B}^{-1,\infty}_{\infty}$. We construct arbitrarily small initial data (u_0, b_0) in $\dot{B}^{-1,\infty}_{\infty} \times \dot{B}^{-1,\infty}_{\infty}$. This data when evolved in time through the MHD system give raise to "norm inflation" in $\dot{B}^{-1,\infty}_{\infty}$ for the corresponding solutions (u, b). One particularly interesting scenario is that the magnetic field b shows norm inflation while the velocity u remains small. Namely, we show that

Theorem 5.0.5 *For any $\delta > 0$ there exists a solution (u, b, p) to the MHD system (5.0.1) with data $u_0 \in \mathcal{S}$ and $b_0 \in \mathcal{S}$ which satisfy*

$$\|u(0)\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \delta, \quad \|b(0)\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \delta,$$

such that for some $0 < T < \delta$

$$\|b(T)\|_{\dot{B}^{-1,\infty}_{\infty}} \gtrsim \frac{1}{\delta}$$

but for any $0 < t < T < \delta$

$$\|u(t)\|_{B_\infty^{-1,\infty}} \lesssim \delta.$$

Remark 5.0.6 *We refer the reader to the beginning of section of Preliminaries for the definition of the symbol* $\lesssim$.

Our proof follows the methods introduced by Bourgain and Pavlović in [3]. Efforts are made to give a very expository development of the inginious ideas of Bourgain and Pavlović in [3].

We now recall some auxiliary concepts related to plane waves, which are necessary in the sequel:

- The "diffusion" of a plane wave $v\sin(k\cdot x)$ in R^3 is given by

  $$e^{\Delta t}v\sin(k\cdot x) = e^{-|k|^2 t}v\sin(k\cdot x)$$

  Thus the magnitude of the diffusion of a plane wave dies down in time in the scale that is measured by the square of the magnitude of the wave vector k.

- It is easy to see that $u = b = e^{-|k|^2 t}v\sin(k\cdot x)$ solve MHD system when the wave vector k is orthogonal to the amplitude vector v.

- The nonlinear interaction of two such diffusions in MHD system can be captured, and it only produces a slower diffusion if the two wave vectors are close.

We note that these observations, are the basis of the original argument of Bourgain and Pavlović in [3]. We will use them to construct a combination of such "diffusions" with least nonlinear interactions yet producing enough slower "diffusions" to cause the norm inflation in short time.

Remark 5.0.7 *It is interesting to observe that even though the initial velocity is zero the velocity can be triggered to develop norm inflation while the magnetic field stays under control. More precisely we can show that, for any* $\delta > 0$ *there exists a solution* (u, b, p) *to the MHD system (5.0.1) with vanishing initial velocity and some* $b_0 \in \mathcal{S}$ *which satisfies that*

$$\|b(0)\|_{\dot{B}_\infty^{-1,\infty}} \lesssim \delta$$

and that for some $0 < T < \delta$

$$\|u(T)\|_{\dot{B}^{-1,\infty}_{\infty}} \gtrsim 1/\delta,$$

while for all $0 < t < T < \delta$

$$\|b(t)\|_{B^{-1,\infty}_{\infty}} \lesssim \delta.$$

Remark 5.0.8 *We also note that due to the interaction between the velocity and the magnetic field, if initially they are the same, they may restrain each other from norm inflations.*

In our work, we present our results in $\mathbb{T}^3$. But, as pointed out in [3], the proof can be modified to $\mathbb{R}^3$.

Chapter 6

Proofs of the Main Results

6.1 Preliminaries

6.1.1 Notation

We denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some constant C, and by $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with some constants C_1, C_2. For completeness we recall the defining norms for the Besov space $\dot{B}^{-1,\infty}_{\infty}$ and the BMO^{-1} space

$$\|f\|_{\dot{B}^{-1,\infty}_{\infty}} = \sup_{t>0} t^{1/2} \|e^{t\triangle} f\|_{L^\infty}. \tag{6.1.1}$$

$$\|f\|_{BMO^{-1}} = \sup_{x_0 \in R^3, R>0} \left(\frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0,\sqrt{R})} |e^{t\triangle} f(y)|^2 \, dy \, dt \right)^{\frac{1}{2}}. \tag{6.1.2}$$

We will also work with the so-called inhomogeneous Besov space $B^{-1,\infty}_{\infty}$ with the norm

$$\|f\|_{B^{-1,\infty}_{\infty}} = \sup_{0<t<1} \sqrt{t} \|e^{t\Delta} f\|_{L^\infty}. \tag{6.1.3}$$

Clearly

$$\|f\|_{B^{-1,\infty}_{\infty}} \leq \|f\|_{\dot{B}^{-1,\infty}_{\infty}}.$$

and,

$$\|f\|_{B^{-1,\infty}_{\infty}} \leq \|f\|_{L^\infty},$$

since $\|e^{t\Delta} f\|_{L^\infty} \leq \|f\|_{L^\infty}$.

6.1.2 The well-posedness result of the incompressible MHD system in BMO^{-1}

We recall the well-posedness result of C. Miao, B. Yuan and B. Zhang in BMO^{-1} in [46]. For this we introduce the spaces X_T and the corresponding norm.

Definition 6.1.1 *Let $u(x,t)$ be a measurable function on $\mathbb{R}^3 \times [0,T)$ for $T>0$ and let*

$$\|u(\cdot,\cdot)\|_{X_T} = \sup_{0<t<T} t^{1/2}\|u(\cdot,t)\|_{L^\infty} \tag{6.1.4}$$
$$+\sup_{x_0\in R^3, 0<R<T} \left(\frac{1}{|B(x_0,\sqrt{R})|} \int_0^R \int_{B(x_0,\sqrt{R})} |u(y,t)|^2\, dy\, dt \right)^{\frac{1}{2}}.$$

Then the space-time space X_T is defined by

$$X_T = \{f(x,t) \in L^2(0,T;L^2(\mathbb{R}^3)) : \|f\|_{X_T} < \infty\}$$

It is worth to mention that, for each $t \in (0,T]$,

$$\|f(\cdot,t)\|_{L^\infty} \le \frac{1}{\sqrt{t}}\|f\|_{X_T}.$$

In [46], Miao, Yuan and Zhang proved the following existence theorem:

Theorem 6.1.2 *(Miao, Yuan and Zhang) Let $(u_0(x), b_0(x)) \in BMO^{-1} \times BMO^{-1}$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then, there exists a positive constant ε such that if $\|(u_0,b_0)\|_{BMO^{-1}} < \varepsilon$ then the MHD system has a unique global mild solution $(u(x,t), b(x,t)) \in X_T \times X_T$ satisfying that $\|(u(x,t),b(x,t))\|_{X_T\times X_T} \le 2\varepsilon$ for all $T>0$.*

6.1.3 Bilinear operators

Let $\mathbb{P}$ denote the projection on divergence-free vector fields, which acts on a function ϕ as

$$\mathbb{P}(\phi) = \phi + \nabla \cdot (-\triangle)^{-1} div\phi.$$

As shown in [25, 46] the bilinear operator

$$\mathcal{B}(u,v) = \int_0^t e^{(t-\tau)\triangle}\mathbb{P}(u\cdot\nabla v)\, d\tau,$$

maps $X_T \times X_T$ into X_T continuously. More precisely we have,

$$\|\mathcal{B}(u,v)\|_{X_T} \lesssim \|u\|_{X_T}\|v\|_{X_T}. \tag{6.1.5}$$

6.1.4 Rewriting the MHD system

Following ideas from [3] we rewrite the MHD system (5.0.1) introducing the expression

$$u = e^{t\triangle}u_0 - u_1 + y \tag{6.1.6}$$

$$b = e^{t\triangle}b_0 - b_1 + z \tag{6.1.7}$$

where

$$u_1(x,t) = \mathcal{B}(e^{t\triangle}u_0(x), e^{t\triangle}u_0(x)) - \mathcal{B}(e^{t\triangle}b_0(x), e^{t\triangle}b_0(x)), \tag{6.1.8}$$

$$b_1(x,t) = \mathcal{B}(e^{t\triangle}u_0(x), e^{t\triangle}b_0(x)) - \mathcal{B}(e^{t\triangle}b_0(x), e^{t\triangle}u_0(x)), \tag{6.1.9}$$

An easy calculation shows that

$$\begin{aligned} y_t - \triangle y + G_0 + G_1 + G_2 = 0, \\ z_t - \triangle z + K_0 + K_1 + K_2 = 0, \end{aligned} \tag{6.1.10}$$

where

$$\begin{aligned} G_0 = \mathbb{P}[(e^{t\triangle}u_0 \cdot \nabla)u_1 + (u_1 \cdot \nabla)e^{t\triangle}u_0 + (u_1 \cdot \nabla)u_1] \\ -\mathbb{P}[(e^{t\triangle}b_0 \cdot \nabla)b_1 + (b_1 \cdot \nabla)e^{t\triangle}b_0 + (b_1 \cdot \nabla)b_1] \end{aligned}$$

$$\begin{aligned} G_1 = \mathbb{P}[(e^{t\triangle}u_0 \cdot \nabla)y + (u_1 \cdot \nabla)y + (y \cdot \nabla)e^{t\triangle}u_0 + (y \cdot \nabla)u_1] \\ -\mathbb{P}[(e^{t\triangle}b_0 \cdot \nabla)z + (b_1 \cdot \nabla)z + (z \cdot \nabla)e^{t\triangle}b_0 + (z \cdot \nabla)b_1] \end{aligned}$$

$$G_2 = \mathbb{P}[(y \cdot \nabla)y] - \mathbb{P}[(z \cdot \nabla)z]$$

and

$$\begin{aligned} K_0 = \mathbb{P}[(e^{t\triangle}u_0 \cdot \nabla)b_1 + (u_1 \cdot \nabla)e^{t\triangle}b_0 + (u_1 \cdot \nabla)b_1] \\ -\mathbb{P}[(e^{t\triangle}b_0 \cdot \nabla)u_1 + (b_1 \cdot \nabla)e^{t\triangle}u_0 + (b_1 \cdot \nabla)u_1] \end{aligned}$$

$$\begin{aligned} K_1 = \mathbb{P}[(e^{t\triangle}u_0 \cdot \nabla)z + (u_1 \cdot \nabla)z + (y \cdot \nabla)e^{t\triangle}b_0 + (y \cdot \nabla)b_1] \\ -\mathbb{P}[(e^{t\triangle}b_0 \cdot \nabla)y + (b_1 \cdot \nabla)y + (z \cdot \nabla)e^{t\triangle}u_0 + (z \cdot \nabla)u_1] \end{aligned}$$

$$K_2 = \mathbb{P}[(y \cdot \nabla)z] - \mathbb{P}[(z \cdot \nabla)y].$$

Here G_0 and K_0 are constants, G_1 and K_1 are linear, while G_2 and K_2 are quadratic in terms of y and z.

Remark 6.1.3 *Note that although the second equation in the MHD system (5.0.1) has no pressure, since u and b are both divergence free, the term $u \cdot \nabla b - b \cdot \nabla u$ is automatically divergence free. Hence the projector $\mathbb{P}$ acting on this term does not change the second equation and hence we can write b_1 and K_i's as described above.*

6.2 Interactions of plane waves

In this section we show how the diffusions of plane waves interact in MHD system. These interactions are the basis for the constructions of initial data which will evolve into the different cases of velocity and magnetic field norm inflations .

6.2.1 Diffusion of a plane wave

As a first step, we consider the same initial data for velocity and magnetic using one single plane wave. Suppose $k \in \mathbb{R}^3$, $v \in \mathbb{S}^2$ and $k \cdot v = 0$. Let

$$u_0 = b_0 = v \cos(k \cdot x).$$

Then $\nabla \cdot u_0 = 0$, and $\nabla \cdot b_0 = 0$. and

$$e^{t\triangle} v \cos(k \cdot x) = e^{-|k|^2 t} v \cos(k \cdot x). \tag{6.2.1}$$

In fact the "diffusions" $(e^{t\Delta} v \cos(k \cdot x), e^{t\Delta} v \cos(k \cdot x))$ of a plane wave solve the MHD system with vanishing pressure. It is important to notice that

- $\|v \cos(k \cdot x)\|_{\dot{B}^{-1,\infty}_{\infty}} \sim \frac{1}{|k|}$,
- $\|e^{t\Delta} v \cos(k \cdot x)\|_{X_T} \lesssim \frac{1}{|k|}$.

which says that the size of a plane wave in the space $\dot{B}^{-1,\infty}_{\infty}$ is reciprocal to the magnitude of its wave vector, and in X_T it is bounded by this same reciprocal.

6.2.2 Interaction of plane waves

Now we consider different plane wave initial data for the velocity and magnetic. Suppose $k_i \in \mathbb{R}^3$, $v_i \in \mathbb{S}^2$ and $k_i \cdot v_i = 0$, for $i = 1, 2$. Let

$$u_0 = \cos(k_1 \cdot x)v_1,$$

$$b_0 = \cos(k_2 \cdot x)v_2.$$

Using the decomposition given in Subsection 6.1.4,

$$u = e^{t\Delta}u_0, \quad b = e^{t\Delta}b_0 - b_1 + z$$

solve the MHD system with vanishing pressure. To simplify our calculations we assume that

$$k_2 \cdot v_1 = 0, \quad \text{and} \quad k_1 \cdot v_2 = \frac{1}{2},$$

which eliminates the term $e^{t\Delta}u_0 \cdot \nabla(e^{t\Delta}b_0)$ and gives

$$\begin{aligned} e^{t\Delta}b_0 \cdot \nabla(e^{t\Delta}u_0) &= -e^{-(|k_1|^2+|k_2|^2)t}v_1 \sin(k_1 \cdot x)\cos(k_2 \cdot x)(k_1 \cdot v_2) \\ &= -\frac{1}{4}e^{-(|k_1|^2+|k_2|^2)t}v_1(\sin((k_1 - k_2) \cdot x) + \sin((k_1 + k_2) \cdot x)). \end{aligned}$$

Hence

$$\begin{aligned} b_1 &= \frac{1}{4}v_1 \sin((k_1 - k_2) \cdot x) \int_0^t e^{-(|k_1|^2+|k_2|^2)\tau}e^{-|k_1-k_2|^2(t-\tau)}d\tau \\ &\quad + \frac{1}{4}v_1 \sin((k_1 + k_2) \cdot x) \int_0^t e^{-(|k_1|^2+|k_2|^2)\tau}e^{-|k_1+k_2|^2(t-\tau)}d\tau \\ &= b_{1,0} + b_{1,1}, \end{aligned}$$

where

$$b_{1,0} = \frac{1}{4}v_1 \sin((k_1 - k_2) \cdot x)\frac{-e^{-(|k_1|^2+|k_2|^2)t} + e^{-|k_1-k_2|^2t}}{2k_1 \cdot k_2}$$

and

$$b_{1,1} = \frac{1}{4}v_1 \sin((k_1 + k_2) \cdot x)\frac{e^{-(|k_1|^2+|k_2|^2)t} - e^{-|k_1+k_2|^2t}}{2k_1 \cdot k_2}.$$

Therefore, if we can manage to control z in the light of the continuity of the bilinear operator $\mathcal{B}$ in X_T, then the interaction of two plane waves are small in $\dot{B}^{-1,\infty}_\infty$ if neither the sum nor the difference of their wave vectors is small in magnitude. In the mean time, the interaction is sizable in $\dot{B}^{-1,\infty}_\infty$ if either the sum or the difference of their wave vectors is small in magnitude.

6.3 Proof of Theorem 5.0.5

In this section we will follow the idea from [3] to construct initial data to produce norm inflation for solutions to MHD system. From the discussions in the previous sections we know that the interaction of two plane waves is not enough to show the norm inflation. We need to build interactions of more plane waves. The construction in [3] depends on the rather sophisticated choices of plane waves. We will use a similar scheme.

6.3.1 Construction of initial data for the MHD system

For a fixed small number $\delta > 0$ we will specify later following initial data :

$$u_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} |k_s| v_s \cos(k_s \cdot x) \tag{6.3.1}$$

and

$$b_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} |k'_s| v'_s \cos(k'_s \cdot x). \tag{6.3.2}$$

We expect for each s, the interaction of the two plane waves $v_s \cos(k_s \cdot x)$ and $v'_s \cos(k'_s \cdot x)$ is sizable in $\dot{B}^{-1,\infty}_{\infty}$; while the interactions of plane waves of different s is small, as demonstrated in Subsection 6.2.2. Hence

- Wave vectors: The wave vectors $k_s \in \mathbb{R}^3$ are parallel to a given vector $k_0 \in \mathbb{R}^3$. The modulo $|k_0|$ will be taken to be large, depending on Q. The magnitude of k_s is defined by,

$$|k_s| = 2^s |k_0||k_{s-1}|, \qquad s = 1, 2, 3, ..., r. \tag{6.3.3}$$

  The wave vectors $k'_s \in \mathbb{R}^3$ is defined by

$$k_s - k'_s = \eta \tag{6.3.4}$$

  for a given vector $\eta \in \mathbb{S}^2$.

- Amplitude vectors: The amplitude vectors $v_s, v'_s \in \mathbb{S}^2$ satisfy

$$k_s \cdot v_s = k'_s \cdot v'_s = 0 \tag{6.3.5}$$

  to ensure the initial data are divergence free.

- Auxiliary assumptions: We also require that

$$\eta \cdot v_s = 0, \quad \eta \cdot v_s' = \frac{1}{2} \tag{6.3.6}$$

to simplify our calculations. In fact we will choose $v_s = v$ a fixed vector.

We first point out the following simple facts to further motivate the choices of the magnitudes of k_s.

Lemma 6.3.1

$$\sum_{l<s} |k_l| \sim |k_{s-1}| \quad \textit{and} \quad \sum_{l<s} |k_l'| \sim |k_{s-1}'| \tag{6.3.7}$$

$$\sum_{s=1}^{r} |k_s| e^{-|k_s|^2 t} \lesssim \frac{1}{\sqrt{t}} \quad \textit{and} \quad \sum_{s=1}^{r} |k_s'| e^{-|k_s'|^2 t} \lesssim \frac{1}{\sqrt{t}} \tag{6.3.8}$$

$$v_i \cdot k_j = v_i \cdot k_j' = v_i \cdot \eta = 0, \quad \forall \quad i, j = 1, 2, \ldots, r. \tag{6.3.9}$$

$$\sum_{i=1}^{r} |k_i| e^{-\frac{|k_i|^2}{|k_0|^2}} \lesssim 1, \quad \textit{and} \quad \sum_{i=1}^{r} |k_i'| e^{-\frac{|k_i'|^2}{|k_0|^2}} \lesssim 1. \tag{6.3.10}$$

Proof By the definition (6.3.3), it is clear that $|k_{l-1}| < \frac{1}{2}|k_l|$, which easily implies the first statement. For second statement, again due to the definition (6.3.3), we know that $|k_s| \sim |k_s| - |k_{s-1}|$. Thus,

$$\sum_{s=1}^{r} |k_s| e^{-|k_s|^2 t} \sim \sum_{s=1}^{r} (|k_s| - |k_{s-1}|) e^{-|k_s|^2 t},$$

while the later one can be considered as a finite Riemman summation of the function $e^{-x^2 t}$. Therefore

$$\sum_{s=1}^{r} |k_s| e^{-|k_s|^2 t} \lesssim \int_0^\infty e^{-x^2 t} dx = \frac{1}{\sqrt{t}} \int_0^\infty e^{-x^2 t} d(x\sqrt{t}) = \frac{\sqrt{\pi}}{2\sqrt{t}}.$$

For the third statement, we note that, k_s are parallel to the given vector k_0 for all s and $v_s = v$ is a fixed vector by the above choice. Hence, by (6.3.5)

$$v_i \cdot k_j = 0, \quad \text{for all } i, j = 1, 2, ..., r.$$

On the other hand side, from (6.3.4) and (6.3.6), we have

$$v_i \cdot k_j' = v_i \cdot (k_j - \eta) = v_i \cdot k_j - v_i \cdot \eta = 0.$$

The forth statement in the Lemma follows from the second one provided an appropriate choice of k_0. It completes the proof of Lemma. □

Next we calculate the norm of our initial data.

Lemma 6.3.2 *For u_0 and b_0 given in (6.3.1) (6.3.2) we have*

$$\|u_0\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \frac{Q}{\sqrt{r}}, \quad \|b_0\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \frac{Q}{\sqrt{r}}. \tag{6.3.11}$$

Proof For the given initial data u_0, we have that, due to(6.2.1)

$$e^{\tau\triangle}u_0 = \frac{Q}{\sqrt{r}}\sum_{s=1}^{r}|k_s|v_s\cos(k_s\cdot x)e^{-|k_s|^2\tau}. \tag{6.3.12}$$

Hence by Lemma 6.3.1

$$\begin{aligned}\|u_0\|_{\dot{B}^{-1,\infty}_{\infty}} &\sim \frac{Q}{\sqrt{r}}\sup_{t>0}\sqrt{t}\sum_{s=1}^{r}|k_s|e^{-|k_s|^2t}\\ &\lesssim \frac{Q}{\sqrt{r}},\end{aligned} \tag{6.3.13}$$

The bound for $\|b_0\|_{\dot{B}^{-1,\infty}_{\infty}}$ follows in the similar way. □

Lemma 6.3.3 *For u_0 and b_0 given in (6.3.1) (6.3.2) we have*

$$\|e^{t\Delta}u_0\|_{X_T} \lesssim Q, \quad \|e^{t\Delta}b_0\|_{X_T} \lesssim Q. \tag{6.3.14}$$

Proof We only need to verify one of the two.

$$\|e^{t\Delta}u_0\|_{X_T} \lesssim \frac{Q}{\sqrt{r}}\left(1+\sup_{t\in[0,T]}\left(\int_0^t\left(\sum_{i=1}^{r}|k_i|e^{-|k_i|^2\tau}\right)^2d\tau\right)^{\frac{1}{2}}\right),$$

where

$$\left(\sum_{i=1}^{r}|k_i|e^{-|k_i|^2\tau}\right)^2 \lesssim \sum_{i=1}^{r}|k_i|^2e^{-2|k_i|^2\tau} + 2\sum_{i=1}^{r}|k_i|e^{-|k_i|^2\tau}\sum_{j<i}|k_j| \lesssim \sum_{i=1}^{r}|k_i|^2e^{-|k_i|^2\tau}$$

Hence

$$\int_0^t\left(\sum_{i=1}^{r}|k_i|^2e^{-|k_i|^2\tau}\right)^2d\tau) \lesssim \sum_{i=1}^{r}(1-e^{-|k_i|^2t}) \lesssim r,$$

which implies

$$\|e^{t\Delta}u_0\|_{X_T} \lesssim \frac{Q}{\sqrt{r}} + Q.$$

This completes the proof of the Lemma. □

Finally we make a note that

Lemma 6.3.4 *For $t\in[0,+\infty)$,*

$$\|e^{t\Delta}u_0\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \frac{Q}{\sqrt{r}}e^{-|k_0|^2t}, \quad \|e^{t\Delta}b_0\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \frac{Q}{\sqrt{r}}e^{-|k_0|^2t}. \tag{6.3.15}$$

6.3.2 Analysis of u_1

As demonstrated in Subsection 6.2.2 we consider the decomposition

$$u = e^{t\Delta}u_0 - u_1 + y$$
$$b = e^{t\Delta}b_0 - b_1 + z$$

We want to handle the u_1 first. Recall the definition (6.1.8)

$$u_1 = \mathcal{B}(e^{t\Delta}u_0, e^{t\Delta}u_0) - \mathcal{B}(e^{t\Delta}b_0, e^{t\Delta}b_0)$$

By our discussions in Section 3.2 the interactions should be small. By the fact that $v_i \cdot k_j = 0$ it is immediately seen that

$$e^{t\triangle}u_0 \cdot \nabla e^{t\triangle}u_0 = 0.$$

Then a straight calculation shows

$$\begin{aligned} e^{t\triangle}b_0 \cdot \nabla e^{t\triangle}b_0 &= -\frac{Q^2}{r}\sum_{i,j=1}^{r} |k_i'||k_j'|e^{-(|k_i'|^2+|k_j'|^2)t}(v_i' \cdot k_j')v_j' \cos(k_i' \cdot x)\sin(k_j' \cdot x) \\ &= -\frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j'|e^{-(|k_i'|^2+|k_j'|^2)t}(v_i' \cdot k_j')v_j' \sin(k_i' + k_j') \cdot x \\ &\quad - \frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j'|e^{-(|k_i'|^2+|k_j'|^2)t}(v_i' \cdot k_j')v_j' \sin(k_j' - k_i') \cdot x \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}e^{t\triangle}b_0 \cdot \nabla e^{t\triangle}b_0 &= -\frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j'|e^{-(|k_i'|^2+|k_j'|^2)t}(v_i' \cdot k_j')u_j \sin(k_j' + k_i') \cdot x \\ &\quad - \frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j'|e^{-(|k_i'|^2+|k_j'|^2)t}(v_i' \cdot k_j')w_j \sin(k_j' - k_i') \cdot x \\ &= E_1 + E_2, \end{aligned}$$

where u_j is the projection of v_j' to the orthogonal to $k_j' + k_i'$ and w_j is the projection of v_j' to the orthogonal to $k_j' - k_i'$. Hence

$$\mathcal{B}((e^{t\Delta}b_0, e^{t\Delta}b_0) = \int_0^t e^{(t-\tau)\Delta}E_1 d\tau + \int_0^t e^{(t-\tau)\Delta}E_2 d\tau = F_1 + F_2.$$

We work for F_1 in the following. F_2 will be handled similarly and so the detail is omitted.

$$F_1 = \frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j'|(v_i' \cdot k_j')u_j \sin(k_j' + k_i') \cdot x \frac{e^{-(|k_i'|^2+|k_j'|^2)t} - e^{-|k_i'+k_j'|^2 t}}{|k_i' + k_j'|^2 - (|k_i'|^2 + |k_j'|^2)},$$

By the fact that $k_i' \cdot v_i' = 0$ and since the function $\frac{1-e^{-x}}{x}$ is bounded for $x > 0$, we have

$$|e^{\tau\Delta} F_1| \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} \sum_{i<j} |k_i'||k_j'|^2 t e^{-(|k_i'|^2+|k_j'|^2)t} e^{-|k_i'+k_j'|^2\tau}$$

and

$$|F_1| \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} \sum_{i<j} |k_i'||k_j'|^2 t e^{-(|k_i'|^2+|k_j'|^2)t}.$$

Hence

$$|F_1| \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} \sum_{i<j} |k_j'|^2 t e^{-\frac{1}{4}|k_j'|^2 t} |k_i'| e^{-\frac{1}{4}|k_j'|^2 t} \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} |k_{j-1}'| e^{-\frac{1}{4}|k_j'|^2 t}.$$

and

$$|e^{\tau\Delta} F_1| \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} |k_{j-1}'| e^{-\frac{1}{4}|k_j'|^2 t} e^{-\frac{1}{2}|k_j'|^2 \tau},$$

where we use the fact that xe^{-x} is bounded for $x > 0$ and Lemma 6.3.1.

$$\begin{aligned} \|F_1\|_{X_T} \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} |k_{j-1}'| \sup_{t\in[0,T]} \sqrt{t} e^{-\frac{1}{4}|k_j'|^2 t} \\ + \sup_{t\in[0,T]} \Big(\int_0^t \Big(\sum_{j=1}^{r} |k_{j-1}'| e^{-\frac{1}{4}|k_j'|^2 \tau}\Big)^2 d\tau\Big)^{\frac{1}{2}} \end{aligned}$$

where

$$\sum_{j=1}^{r} |k_{j-1}'| \sup_{t\in[0,T]} \sqrt{t} e^{-\frac{1}{4}|k_j'|^2 t} \lesssim \sum_{j=1}^{r} \frac{|k_{j-1}'|}{|k_j'|} \lesssim 1$$

and

$$\sup_{t\in[0,T]} \Big(\int_0^t \Big(\sum_{j=1}^{r} |k_{j-1}'| e^{-\frac{1}{4}|k_j'|^2 \tau}\Big)^2 d\tau\Big)^{\frac{1}{2}} \lesssim \Big(\sum_{j=1}^{r} \frac{|k_{j-1}'|}{|k_j'|}\Big)^{\frac{1}{2}} \lesssim 1$$

by the same argument as used in the proof of Lemma 6.3.3. And similarly

$$\|F_1\|_{\dot{B}_\infty^{-1,\infty}} = \sup_{\tau>0} \sqrt{\tau} \|e^{\tau\Delta} F_1\|_{L^\infty} \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} \frac{|k_{j-1}'|}{|k_j'|} \lesssim \frac{Q^2}{r}.$$

Therefore we conclude that

Lemma 6.3.5

$$\|u_1\|_{\dot{B}_\infty^{-1,\infty}} \lesssim \frac{Q^2}{r}, \quad \|u_1\|_{X_T} \lesssim \frac{Q^2}{r}. \tag{6.3.16}$$

Proof F_2 can be handled just like what we did with F_1. □

6.3.3 Analysis of b_1

First we recall that from (6.1.9)

$$b_1(x,t) = \mathcal{B}(e^{t\triangle}u_0(x), e^{t\triangle}b_0(x)) - \mathcal{B}(e^{t\triangle}b_0(x), e^{t\triangle}u_0(x)).$$

Similar to the calculations in the previous section, first, due to the fact that $v_i \cdot k_j' = 0$, we have

$$e^{\tau\triangle}u_0 \cdot \nabla e^{\tau\triangle}b_0 = 0.$$

And

$$\begin{aligned} e^{\tau\triangle}b_0 \cdot \nabla e^{\tau\triangle}u_0 = &-\frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j| e^{-(|k_i'|^2+|k_j|^2)t}(v_i' \cdot k_j)v_j \sin(k_j + k_i') \cdot x \\ &- \frac{Q^2}{2r}\sum_{i\neq j}^{r} |k_i'||k_j| e^{-(|k_i'|^2+|k_j|^2)t}(v_i' \cdot k_j)v_j \sin(k_j - k_i') \cdot x \\ &- \frac{Q^2}{2r}\sum_{i=1}^{r} |k_i||k_i'| e^{-(|k_i|^2+|k_i'|^2)t}(v_i' \cdot k_i)v_i \sin(k_i - k_i') \cdot x. \end{aligned}$$

We will write

$$e^{\tau\triangle}u_0 \cdot \nabla e^{\tau\triangle}b_0 - e^{\tau\triangle}b_0 \cdot \nabla e^{\tau\triangle}u_0 = \tilde{b}_{1,0} + \tilde{b}_{1,1},$$

where

$$\begin{aligned} \tilde{b}_{1,0} &= \frac{Q^2}{2r}\sum_{i=1}^{r} |k_i||k_i'| e^{-(|k_i|^2+|k_i'|^2)t} \sin(k_i - k_i') \cdot x (v_i' \cdot k_i)v_i \\ &= \frac{Q^2}{4r}\sin(\eta \cdot x)\sum_{i=1}^{r} |k_i||k_i'| e^{-(|k_i|^2+|k_i'|^2)t} v_i, \end{aligned}$$

due to our choices of wave vectors and amplitude vectors in Subsection 6.3.1. We then set

$$\tilde{b}_{1,1} = \tilde{d}_{1,1} + \tilde{e}_{1,1},$$

where

$$\begin{aligned} \tilde{d}_{1,1} &= \frac{Q^2}{2r}\sum_{i=1}^{r} |k_i||k_i'| e^{-(|k_i|^2+|k_i'|^2)t} \sin(k_i + k_i') \cdot x (v_i' \cdot k_i)v_i \\ &= \frac{Q^2}{4r}\sum_{i=1}^{r} |k_i||k_i'| e^{-(|k_i|^2+|k_i'|^2)t} v_i \sin(k_i + k_i') \cdot x. \end{aligned}$$

By the choices of k_i', which behaves more or less like k_i for each i when $|k_0|$ is very large, we conclude that $\tilde{e}_{1,1}$ can be handled just like what we did for E_1 in the

previous section. We then have

$$\begin{aligned} d_{1,1} &= \int_0^t e^{\Delta(t-\tau)}\mathbb{P}\tilde{d}_{1,1}(\tau)d\tau \\ &= \frac{Q^2}{4r}\sum_{i=1}^{r}|k_i||k_i'|v_i\sin(k_i+k_i')\cdot x\frac{e^{-(|k_i|^2+|k_i'|^2)t}-e^{-|k_i+k_i'|^2t}}{|k_i+k_i'|^2-(|k_i|^2+|k_i'|^2)} \end{aligned}$$

which gives

$$|d_{1,1}| \lesssim \frac{Q^2}{r}\sum_{i=1}^{r}|k_i|^2te^{-\frac{1}{2}|k_i|^2t} \lesssim \frac{Q^2}{r}\sum_{i=1}^{r}e^{-\frac{1}{4}|k_i|^2t}$$

and

$$|e^{\tau\Delta}d_{1,1}| \lesssim \frac{Q^2}{r}\sum_{i=1}^{r}e^{-\frac{1}{4}|k_i|^2t}e^{-\frac{1}{2}|k_i|^2\tau}.$$

Hence it is even easier to handle $d_{1,1}$ than to handle F_1 in the previous section. Now, if denote

$$b_{1,1} = \int_0^t e^{\Delta(t-\tau)}\mathbb{P}\tilde{b}_{1,1}(\cdot,\tau)d\tau,$$

we may conclude that

Lemma 6.3.6

$$\|b_{1,1}\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \frac{Q^2}{r}, \quad \|b_{1,1}\|_{X_T} \lesssim \frac{Q^2}{r}. \tag{6.3.17}$$

The focus is now on

$$\begin{aligned} b_{1,0} &= \int_0^t e^{\Delta(t-\tau)}\mathbb{P}\tilde{b}_{1,0}(\cdot,\tau)d\tau \\ &= \frac{Q^2}{4r}e^{-t}\sin(\eta\cdot x)v\sum_{i=1}^{r}|k_i||k_i'|\int_0^t e^{(1-(|k_i|^2+|k_i'|^2)\tau}d\tau, \\ &= \frac{Q^2}{4r}\sin(\eta\cdot x)v\sum_{i=1}^{r}|k_i||k_i'|\frac{e^{-t}-e^{-(|k_i|^2+|k_i'|^2)t}}{|k_i|^2+|k_i'|^2-1} \end{aligned} \tag{6.3.18}$$

since $v_i = v$ is fixed. Therefore we have

Lemma 6.3.7 *Suppose* $\frac{1}{|k_1|^2} \ll T \ll 1$. *Then*

$$\|b_{1,0}(\cdot,T)\|_{B^{-1,\infty}_{\infty}} = \sup_{\tau\in(0,1)}\sqrt{\tau}\|e^{\tau\Delta}b_{1,0}\|_{L^\infty} \sim Q^2, \quad \|b_{1,0}\|_{X_T} \lesssim \sqrt{T}Q^2. \tag{6.3.19}$$

Proof By (6.3.18), it follows that

$$b_{1,0} \sim Q^2\sin(\eta\cdot x)v, \tag{6.3.20}$$

for $\frac{1}{|k_1|^2} \ll T \ll 1$. Indeed, $T \ll 1$ insures $e^{-t} \sim 1$, for $t \leq T$; $\frac{1}{|k_1|^2} \ll T$ insures $e^{-(|k_s|^2+|k_s'|^2)t} \sim 0$. And, since $|k_1|$ is very large, $|k_s| \sim |k_s'|$ for every s by (6.3.4). Thus,

$$\begin{aligned} \|b_{1,0}\|_{B_\infty^{-1,\infty}} &\sim Q^2 \sup_{0<t<1} \sqrt{t} \|e^{t\Delta} \sin(\eta \cdot x)\|_{L^\infty} \\ &\sim Q^2 \sup_{0<t<1} \sqrt{t} e^{-|\eta|^2 t} \\ &\sim Q^2. \end{aligned} \tag{6.3.21}$$

And

$$\begin{aligned} \|b_{1,0}\|_{X_T} &\sim Q^2 \sup_{0<t<T} \sqrt{t} \| \sin(\eta \cdot x)\|_{L^\infty} \\ &+ Q^2 \sup_{x_0, 0<R<T} \left(\frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0,\sqrt{R})} |\sin(\eta \cdot x)|^2 dx dt \right)^{\frac{1}{2}} \\ &\lesssim \sqrt{T} Q^2. \end{aligned} \tag{6.3.22}$$

□

6.3.4 Analysis of y and z

In this section we analyze the parts y and z of the solution The idea is to control y and z using the boundedness of the bilinear operator $\mathcal{B}$ in the space X_T. Naively one would hope that nonlinear terms turn out to be even smaller. But the trouble is at the linear term G_1 and K_1 since

$$\|e^{t\Delta} u_0\|_{X_T} \lesssim Q, \quad \text{and} \quad \|e^{t\Delta} b_0\|_{X_T} \lesssim Q,$$

by Lemma 6.3.3, which is the best we can have. The problem is the plane waves should not be lumped together in one single time scale. Plane waves that have much bigger wave vectors diffuse much quicker. Therefore it makes sense to analyze how y and z evolve in different time scales and see how different plane waves contribute. In [3], Bourgain and Pavlović very skillfully designed time steps to group appropriately the plane waves. We will use the same idea. We now introduce the time step division as used in [3]. Let

$$0 < T_1 < T_2 < \cdots < T_\beta,$$

where $\beta = Q^3$ and

$$T_\alpha = |k_{r_\alpha}|^{-2}, \quad r_\alpha = r - \alpha Q^{-3} r, \quad \alpha = 1, 2, \dots.$$

In particular, $r_\beta = 0$ and $T_\beta = |k_0|^{-2}$. The following are the key estimates for the time step design in [3].

Lemma 6.3.8 *Suppose that r is sufficiently large for a fixed large number Q. Then*

$$\|(e^{t\triangle}u_0)\chi_{[T_\alpha,T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} \lesssim Q^{-1/2}, \quad \|(e^{t\triangle}b_0)\chi_{[T_\alpha,T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} \lesssim Q^{-1/2}.$$

Proof The proof is the same as in [3]. For the convenience of the reader we outline the proof showing how the design of the time steps. First use the decomposition

$$(e^{t\triangle}u_0)\chi_{[T_\alpha,T_{\alpha+1}]}(t) \approx L_1 + L_2 + L_3,$$

where

$$\begin{aligned}
L_1 &= \frac{Q}{\sqrt{r}} \sum_{s<r_{\alpha+1}} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2 t} \chi_{[T_\alpha,T_{\alpha+1}]}(t) \\
L_2 &= \frac{Q}{\sqrt{r}} \sum_{s=r_{\alpha+1}}^{r_\alpha} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2 t} \chi_{[T_\alpha,T_{\alpha+1}]}(t) \\
L_3 &= \frac{Q}{\sqrt{r}} \sum_{r_\alpha<s\le r} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2 t} \chi_{[T_\alpha,T_{\alpha+1}]}(t).
\end{aligned}$$

The first group are those plane waves whose sizes are small. Provided $\frac{Q}{\sqrt{r}} \lesssim Q^{-\frac{1}{2}}$ we have

$$\|L_1\|_{X_{T_{\alpha+1}}} \lesssim \frac{Q}{\sqrt{r}} \sqrt{T_{\alpha+1}} |k_{r_{\alpha+1}-1}| + \frac{Q}{\sqrt{r}} (T_{\alpha+1} |k_{r_{\alpha+1}-1}|^2)^{\frac{1}{2}} \lesssim \frac{Q}{\sqrt{r}} \lesssim Q^{-\frac{1}{2}},$$

The second group is the group of active plane waves in the time scale $[T_\alpha, T_{\alpha+1}]$. But, by the design, the number of plane waves in this group is small.

$$\begin{aligned}
\|L_2\|_{X_{T_{\alpha+1}}} &\lesssim \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (r_\alpha - r_{\alpha+1})^{\frac{1}{2}} \\
&= \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (Q^{-3} r)^{\frac{1}{2}} \\
&\lesssim Q^{-1/2},
\end{aligned}$$

if $\frac{Q}{\sqrt{r}} \lesssim Q^{-\frac{1}{2}}$. The last group of plane waves are those that have diffused too much and become small in size.

$$\|L_3\|_{X_{T_{\alpha+1}}} \lesssim \frac{Q}{\sqrt{r}} \sup_{t<T_{\alpha+1}} \sum_{s=r_\alpha+1}^{r} |k_s|\sqrt{t}e^{-|k_s|^2 t} + \frac{Q}{\sqrt{r}} \sup_{t<T_{\alpha+1}} \left(\int_0^t | \sum_{s=r_\alpha+1}^{r} |k_s|^2 e^{-|k_s|^2 t} \chi_{[T_\alpha, T_{\alpha+1}]}(\tau)| d\tau \right)^{\frac{1}{2}}$$

The first supremum of the last line is controlled by the integral

$$\begin{aligned} \int_{|k_{r_\alpha+1}|}^{|k_r|} \sqrt{t}e^{-x^2 t} dx &= \int_{|k_{r_\alpha+1}|\sqrt{t}}^{|k_r|\sqrt{t}} e^{-y^2} dy \\ &\leq \int_{|k_{r_\alpha+1}|\sqrt{t}}^{|k_r|\sqrt{t}} e^{-y} dy \\ &\leq e^{-|k_{r_\alpha+1}|\sqrt{t}} \leq e^{-|k_{r_\alpha+1}|/|k_{r_\alpha}|} \ll 1, \end{aligned}$$

where we used the fact that $|k_{r_\alpha+1}|\sqrt{t} > 1$ for $t \in [T_\alpha, T_{\alpha+1}]$ to the second step and the last step follows from (6.3.3).

The second supremum is controlled by

$$\begin{aligned} \left(\sum_{s=r_\alpha+1}^{r} e^{-|k_s|^2 T_{\alpha+1}} \right)^{1/2} &\leq (r - r_\alpha) e^{-|k_{r_\alpha+1}|^2/|k_{r_\alpha+1}|^2} \\ &\lesssim (\alpha Q^{-3} r) e^{-4^{r-\alpha Q^{-3}r}} \ll 1. \end{aligned}$$

In the same way one can show the same estimate for b_0. The proof of Lemma is complete. □

Lemma 6.3.9 *For $T > T_\beta$,*

$$\|(e^{t\triangle} u_0)\chi_{[T_\beta, T]}(t)\|_{X_T} \lesssim \frac{Q}{\sqrt{r}}, \tag{6.3.23}$$

$$\|(e^{t\triangle} b_0)\chi_{[T_\beta, T]}(t)\|_{X_T} \lesssim \frac{Q}{\sqrt{r}}. \tag{6.3.24}$$

Proof From Lemma 6.3.1 and (6.3.12), we see that

$$\begin{aligned} \|(e^{t\triangle} u_0)\chi_{[T_\beta, T]}(t)\|_{X_T} &\lesssim \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} |k_s| e^{-\frac{|k_s|^2}{|k_0|^2}} |(T - T_\beta)^{\frac{1}{2}} \\ &\lesssim \frac{Q}{\sqrt{r}}. \end{aligned}$$

The second one follows in the same way. □

Recall the equations for y and z from Section 2.4 that

$$y_t - \triangle y + G_0 + G_1 + G_2 = 0,$$
$$z_t - \triangle z + K_0 + K_1 + K_2 = 0,$$

Note that, $y(0) = z(0) = 0$. Hence, $t \in [T_\alpha, T_{\alpha+1}]$,

$$y(t) = -\int_0^t e^{(t-\tau)\triangle} G(\tau) d\tau \tag{6.3.25}$$
$$= -\int_0^t e^{(t-\tau)\triangle} G(\tau)\chi_{[0,T_\alpha]}(\tau) d\tau - \int_0^t e^{(t-\tau)\triangle} G(\tau)\chi_{[T_\alpha,T_{\alpha+1}]}(\tau) d\tau,$$

where $G = G_0 + G_1 + G_2$. So we can write

$$\|y\|_{X_{T_{\alpha+1}}} \leq I_1 + I_2 \tag{6.3.26}$$

to see how y develop in the time step $[T_\alpha, T_{\alpha+1}]$.

Similarly for z, we have

$$\|z\|_{X_{T_{\alpha+1}}} \tag{6.3.27}$$
$$\leq \| \int_0^t e^{(t-\tau)\triangle} K(\tau)\chi_{[0,T_\alpha]} d\tau \|_{X_{T_{\alpha+1}}} + \| \int_0^t e^{(t-\tau)\triangle} K(\tau)\chi_{[T_\alpha,T_{\alpha+1}]} d\tau \|_{X_{T_{\alpha+1}}}$$
$$= J_1 + J_2,$$

where $K = K_0 + K_1 + K_2$. Now we are ready to estimate the increments of y and z during the time scale $[T_\alpha, T_{\alpha+1}]$.

Lemma 6.3.10 *With appropriate choice of r and T, we have*

$$\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}} \lesssim Q^3(\frac{1}{r} + \sqrt{T_\beta}) + Q(\|y\|_{X_{T_\alpha}} + \|z\|_{X_{T_\alpha}}). \tag{6.3.28}$$

Proof Applying bilinear estimate (6.1.5), estimates in space X_{T_α} from Lemma 6.3.3, (6.3.19), (6.3.17) and (6.3.16), we have

$$\begin{aligned} I_1 \lesssim & (Q+\frac{Q^2}{r}+\|y\|_{X_{T_\alpha}})\|y\|_{X_{T_\alpha}} \\ & +(Q+Q^2\sqrt{T_\alpha}+\frac{Q^2}{r}+\|z\|_{X_{T_\alpha}})\|z\|_{X_{T_\alpha}} \\ & +(Q+\frac{Q^2}{r})\frac{Q^2}{r} \\ & +(Q+Q^2\sqrt{T_\alpha}+\frac{Q^2}{r})(Q^2\sqrt{T_\alpha}+\frac{Q^2}{r}). \end{aligned} \tag{6.3.29}$$

We next apply the Lemma 6.3.8 and estimate

$$\begin{aligned} I_2 \lesssim & (Q^{-1/2}+\frac{Q^2}{r}+\|y\|_{X_{T_{\alpha+1}}})\|y\|_{X_{T_{\alpha+1}}} \\ & +(Q^{-1/2}+Q^2\sqrt{T_{\alpha+1}}+\frac{Q^2}{r}+\|z\|_{X_{T_{\alpha+1}}})\|z\|_{X_{T_{\alpha+1}}} \\ & +(Q^{-1/2}+\frac{Q^2}{r})\frac{Q^2}{r} \\ & +(Q^{-1/2}+Q^2\sqrt{T_{\alpha+1}}+\frac{Q^2}{r})(Q^2\sqrt{T_{\alpha+1}}+\frac{Q^2}{r}). \end{aligned} \tag{6.3.30}$$

We choose r sufficiently large and T appropriately small such that

$$\frac{Q^2}{r} < Q^{-1/2}, \quad Q^2\sqrt{T} < Q^{-1/2}. \tag{6.3.31}$$

Hence we combine (6.3.26), (6.3.29) and (6.3.30) to arrive at

$$\begin{aligned} \|y\|_{X_{T_{\alpha+1}}} \lesssim & Q^3(\frac{1}{r}+\sqrt{T_\beta})+Q(\|y\|_{X_{T_\alpha}}+\|z\|_{X_{T_\alpha}}) \\ & +Q^{-1/2}(\|y\|_{X_{T_{\alpha+1}}}+\|z\|_{X_{T_{\alpha+1}}})+\|y\|^2_{X_{T_{\alpha+1}}}+\|z\|^2_{X_{T_{\alpha+1}}}. \end{aligned} \tag{6.3.32}$$

Similarly we can obtain

$$\begin{aligned} \|z\|_{X_{T_{\alpha+1}}} \lesssim & Q^3(\frac{1}{r}+\sqrt{T_\beta})+Q(\|y\|_{X_{T_\alpha}}+\|z\|_{X_{T_\alpha}}) \\ & +Q^{-1/2}(\|y\|_{X_{T_{\alpha+1}}}+\|z\|_{X_{T_{\alpha+1}}})+\|y\|_{X_{T_{\alpha+1}}}\|z\|_{X_{T_{\alpha+1}}}. \end{aligned} \tag{6.3.33}$$

Therefore, adding (6.3.32) and (6.3.33), we have

$$\begin{aligned} \|y\|_{X_{T_{\alpha+1}}}+\|z\|_{X_{T_{\alpha+1}}} \lesssim & Q^3(\frac{1}{r}+\sqrt{T_\beta})+Q(\|y\|_{X_{T_\alpha}}+\|z\|_{X_{T_\alpha}}) \\ & +(\|y\|_{X_{T_{\alpha+1}}}+\|z\|_{X_{T_{\alpha+1}}})^2. \end{aligned}$$

So, for much larger r and $|k_0|$, we have $\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}}$ small and

$$\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}} \lesssim Q^3(\frac{1}{r} + \sqrt{T_\beta}) + Q(\|y\|_{X_{T_\alpha}} + \|z\|_{X_{T_\alpha}}).$$

□

By iterating (6.3.28) it then follows easily that

Lemma 6.3.11

$$\|y\|_{X_{T_\beta}} + \|z\|_{X_{T_\beta}} \lesssim Q^{\beta+2}(\frac{1}{r} + \sqrt{T_\beta}) \tag{6.3.34}$$

Next for $T > T_\beta$, in the light of Lemma 6.3.9, one may repeat the argument in the proof of (6.3.28) and obtain

Lemma 6.3.12 *For appropriate choice of r and T,*

$$\|y\|_{X_T} + \|z\|_{X_T} \lesssim 4Q^4T. \tag{6.3.35}$$

This implies that,

$$\|y(\cdot,T)\|_{B_\infty^{-1,\infty}} \lesssim \|y(\cdot,T)\|_{L^\infty} \lesssim T^{-1/2}\|y\|_{X_T} \lesssim 4Q^4\sqrt{T} \tag{6.3.36}$$

and

$$\|z(\cdot,T)\|_{B_\infty^{-1,\infty}} \lesssim \|z(\cdot,T)\|_{L^\infty} \lesssim T^{-1/2}\|z\|_{X_T} \lesssim 4Q^4\sqrt{T}. \tag{6.3.37}$$

6.3.5 Finishing the Proof

Now we are ready to complete the proof of Theorem 5.0.5. Since (6.3.17) implies

$$\|b_{1,1}(\cdot,T)\|_{B_\infty^{-1,\infty}} \lesssim \|b_{1,1}(\cdot,T)\|_{L^\infty} \lesssim T^{-\frac{1}{2}}\|b_{1,1}\|_{X_T} \lesssim \frac{Q^2}{r\sqrt{T}}, \tag{6.3.38}$$

from (6.1.7) we combine (6.3.19), (6.3.38) and (6.3.37) to obtain that

$$\begin{aligned}
\|b(\cdot,T) - e^{T\triangle}b_0\|_{\dot{B}_\infty^{-1,\infty}} &\geq \|b(\cdot,T) - e^{T\triangle}b_0\|_{B_\infty^{-1,\infty}} \\
&\geq \|b_{1,0}(\cdot,T)\|_{B_\infty^{-1,\infty}} - \|b_{1,1}(\cdot,T)\|_{B_\infty^{-1,\infty}} - \|z(\cdot,T)\|_{B_\infty^{-1,\infty}} \\
&\gtrsim Q^2 - \|b_{1,1}\|_{L^\infty} - \|z\|_{L^\infty} \\
&\gtrsim Q^2(1 - \frac{1}{r\sqrt{T}} - 4Q^2\sqrt{T}).
\end{aligned}$$

Therefore, in the light of (6.3.15),

$$\|b(T)\|_{\dot{B}^{-1,\infty}_{\infty}} \gtrsim Q^2.$$

On the other hand, from (6.1.6), we combine (6.3.15), (6.3.16), and (6.3.36) and have, for any $t \in [0, T]$,

$$\|u(\cdot, t)\|_{B^{-1,\infty}_{\infty}} \lesssim \frac{Q}{\sqrt{r}} + \frac{Q^2}{r} + Q^4\sqrt{T} \tag{6.3.39}$$

and remains small in $B^{-1,\infty}_{\infty}$. Thus we proved theorem 5.0.5.

Remark 6.3.13 *We would like to note the following simple chart to indicate how the choices of the several parameters are made.*

$$\delta \longrightarrow Q \longrightarrow T \longrightarrow |k_0| \longrightarrow |k_s|, \quad Q \longrightarrow r.$$

6.4 Other scenarios of norm inflations

In this section, we consider other interesting norm inflation phenomena for MHD system. The essence of the construction introduced by Bourgain and Pavlović in [3] and used in this paper is that the collisions of plane waves with similar but large wave vectors can cause norm inflations in NSE as well as in MHD system. More precisely we see that the collisions in the quadratic terms trigger the norm inflations. Hence we can arrange the initial data to have collisions of plane waves with similar wave vectors either in u_1 or in b_1 to produce various norm inflation modes for MHD system. For example, even the initial velocity is zero, if the initial magnetic field contains enough plane waves to collide, we can produce the scenario where the velocity develops norm inflation in $\dot{B}^{-1,\infty}_{\infty}$ while the magnetic field remains small in the space $B^{-1,\infty}_{\infty}$. Namely,

Theorem 6.4.1 *Let* $u_0 \equiv 0$ *and*

$$b_0 = \frac{Q}{\sqrt{r}} \sum_{i=1}^{r} (|k_i| v_i \cos(k_i \cdot x) + |k_i'| v_i' \cos(k_i' \cdot x)).$$

Then, for any $\delta > 0$, there exists a solution (u, b, p) to the MHD system (5.0.1) with initial data u_0 and b_0 as in the above satisfying

$$\|b(0)\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \delta,$$

such that for some $0 < T < \delta$

$$\|u(T)\|_{\dot{B}^{-1,\infty}_{\infty}} \gtrsim 1/\delta,$$

while for any $0 < t < T < \delta$

$$\|b(t)\|_{B^{-1,\infty}_{\infty}} \lesssim \delta.$$

The proof will be more or less the same as the proof in [3] in the light of our discussions in the previous section. Another interesting case mentioned in Remark 5.0.8 is that when the initial velocity and the initial magnetic field are the same, although they both include many plane waves that are to collide, the collisions cancel each other in the evolution of the MHD system and produce no norm inflations.

Remark 6.4.2 *Finally we would like to mention that, in [9], Cheskidov and Shvydkoy introduced a different construction of initial data to prove the ill-posedness of NSE in certain Besov spaces. There are two more works about the ill-posedness results for Navier-Stokes equations by Germain [18] and Yoneda [56].*

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