

Lawrence Berkeley National Laboratory

Recent Work

Title

Z PHYSICS AND TESTS OF THE STANDARD MODEL.

Permalink

<https://escholarship.org/uc/item/5np0r23n>

Author

Cahn, Robert N.

Publication Date

1989



Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Physics Division

Presented at the Heavy Flavor Physics Symposium,
Beijing, PRC, August 11-20, 1988

LIBRARY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIF. 94720-1900

MAY 15 1989

LIBRARY AND
DOCUMENTS SECTION

Z Physics and Tests of the Standard Model

R.N. Cahn

January 1989



LBL-26433
c.2

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

Z Physics and Tests of the Standard Model *

Presented at the Heavy Flavor Physics Symposium

August 11 - 20, 1988

Peking University, Beijing

Robert N. Cahn

*Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720*

Abstract

Fundamental aspects of Z physics are reviewed with an emphasis on e^+e^- annihilation. The effects of radiative corrections, both from ordinary QED and from the electroweak interactions are considered from an elementary point of view, but in some detail. The possibility of mixing with an extra Z boson is discussed. The implications for experiments are stressed. Additional information that will be obtained from measurements of the W in collider experiments is considered.

*This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

1 Fundamental Properties of the Z

These lectures are intended to show how the standard model of electroweak interactions can be applied to some experimental situations, primarily the production of the Z by e^+e^- annihilation. The standard model makes precise predictions that depend on a very few physical parameters. The basic gauge theory has three parameters, g , g' , and v , two couplings and the vacuum expectation value of the Higgs field. These three parameters may be eliminated in favor of three physical quantities. Two of these are always chosen to be $\alpha \approx 1/137$ and $G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2}$. The third has typically been taken to be $\sin \theta_W$ as determined by neutral current data. However, once SLC and LEP are operating, the mass of the Z will be determined with great precision and it will be taken as the third physical parameter. Two other physical parameters are of immediate significance: the mass of the t -quark and the mass of the Higgs boson. These also affect predictions for quantities that can be measured at the Z peak.

In the first section of the lectures we present the standard model predictions for width of the Z , the forward-backward and left-right asymmetries, and the decays of the Z that yield a Higgs boson, all considered in the most orthodox model, which has a single Higgs doublet. Radiative corrections due to purely QED effects are discussed as well. In the second section we consider radiative corrections from the electroweak interactions. The treatment is not comprehensive but instead focuses on those corrections arising from vacuum polarization. These are the most important corrections and can be treated in a straightforward manner. The presentation is rather pedestrian, but many of the details are displayed in the belief that this is the best way to explain how variations in one parameter, like the t -quark mass, affect the predictions for other quantities, like the left-right asymmetry. The results are not new and the intent is purely pedagogical.

In the third section we consider how deviations from the standard model might manifest themselves in quantities that could be measured at the Z . Specifically we consider a class of models that contain an extra Z boson. The group theory that determines the couplings of such Z s if they arise from the breakdown of an E_6 gauge symmetry is explained. A convenient means is proposed

for comparing the data to such alternative models.

Data from hadron colliders will determine the $W - Z$ mass difference to rather good precision in the next few years. This will provide an important complement to the data from Z and is also discussed briefly in the third section.

Altogether, the aim of the lectures is to provide a practical guide to the analysis of data from the Z as a test of the standard model.

1.1 The Standard Model

The standard model is based on a spontaneously broken $SU(2) \times U(1)$ Yang-Mills theory coupled to chiral fermions and a single complex scalar Higgs isodoublet[1]. The $SU(2)$ gauge field is $\mathbf{W}$ and the $U(1)$ field is B . The gauge covariant derivative is

$$D_\mu = \partial_\mu - ig\mathbf{T} \cdot \mathbf{W}_\mu - ig'\frac{Y}{2}B_\mu \quad (1.1)$$

The field strengths are

$$\mathbf{W}_{\mu\nu} = \partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + g\mathbf{W}_\mu \times \mathbf{W}_\nu \quad (1.2)$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad (1.3)$$

The gauge sector Lagrangian is

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}\mathbf{W}_{\mu\nu} \cdot \mathbf{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \quad (1.4)$$

The fermions have $SU(2) \times U(1)$ quantum numbers satisfying the equivalent of the Gell-Mann Nishijima relation.

$$Q = T_3 + \frac{Y}{2} \quad (1.5)$$

Because the left-handed but not the right-handed fermions couple to $W^\pm$, the left-handed fermions are put in weak-isospin doublets while the right-handed fermions are singlets. The results are shown in Table 1.1.

The same quantum numbers apply analogously to μ, ν_μ, c, s , and τ, ν_τ, t, b . The three generations of quarks and leptons can be indicated

$$\begin{aligned} U &= (u, c, t) & D &= (d, s, b) \\ N &= (\nu_e, \nu_\mu, \nu_\tau) & L &= (e, \mu, \tau) \end{aligned} \quad (1.6)$$

	ϵ_L^-	ϵ_R^-	ν_L	u_L	u_R	d_L	d_R
Q	-1	-1	0	2/3	2/3	-1/3	-1/3
T_3	-1/2	0	+1/2	1/2	0	-1/2	0
$Y/2$	-1/2	0	-1/2	1/6	2/3	1/6	-1/3

Table 1.1: The weak quantum number assignments of the ϵ , ν_e , u , and d .

Of course we must treat the left-handed and right-handed parts separately. The familiar result is that the weak currents can be written in terms of U_L, U_R, D'_L , and D'_R where

$$D'_L = V D_L \quad (1.7)$$

and V is the Kobayashi-Maskawa 3×3 matrix. The gauge particles couple as

$$[\bar{U} \bar{D}'] \left(g \frac{\boldsymbol{\tau}}{2} \cdot \mathbf{W}_\mu + g' \frac{Y}{2} B_\mu \right) \begin{bmatrix} U \\ D' \end{bmatrix} \quad (1.8)$$

The τ matrix operates in the (u, d') , (c, s') , and (t, b') spaces.

The Higgs isodoublet couples as

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \quad (1.9)$$

where

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad (1.10)$$

$$\begin{aligned} \phi^- &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) & \phi^+ &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \\ \bar{\phi}^0 &= \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4) & \phi^0 &= \frac{1}{\sqrt{2}}(\phi_3 - i\phi_4) \end{aligned} \quad (1.11)$$

and the conjugate spinor is

$$\phi_c = \begin{pmatrix} \bar{\phi} \\ -\phi^- \end{pmatrix} \quad (1.12)$$

and

$$\phi^\dagger \phi = \frac{1}{2} (\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) \quad (1.13)$$

If $\lambda > 0$ and $\mu^2 < 0$ the potential in Eq. (1.9) has a minimum for

$$\langle \phi^\dagger \phi \rangle = \frac{-\mu^2}{2\lambda} \quad (1.14)$$

and we chose

$$\langle \phi_3 \rangle = \sqrt{\frac{-\mu^2}{\lambda}} \equiv v \quad (1.15)$$

Writing $\phi_3 = H + v$ we find that the constant term yields

$$D_\mu \phi \rightarrow \frac{-iv}{2\sqrt{2}} \left[g(W_1 - iW_2)_\mu \right] \quad (1.16)$$

Thus there is a quadratic piece in the Lagrangian

$$\frac{v^2}{8} \left[g^2(W_1^2 + W_2^2) + (gW_{3\mu} - g'B_\mu)^2 \right] \quad (1.17)$$

This gives for $W^\pm = (W_1 \mp iW_2)/\sqrt{2}$

$$m_W^2 = \frac{g^2 v^2}{4} \quad (1.18)$$

and for

$$Z = \frac{gW_3 - g'B}{\sqrt{g^2 + g'^2}}, \quad (1.19)$$

$$m_Z^2 = \frac{(g^2 + g'^2)}{4} v^2. \quad (1.20)$$

The field

$$A = \frac{g'W_3 + gB}{\sqrt{g^2 + g'^2}} \quad (1.21)$$

is the massless photon.

It is traditional to define

$$\tan \theta_W \equiv g'/g \quad (1.22)$$

so that

$$\begin{aligned} A &= \sin \theta W_3 + \cos \theta B \\ Z &= \cos \theta W_3 - \sin \theta B \end{aligned} \quad (1.23)$$

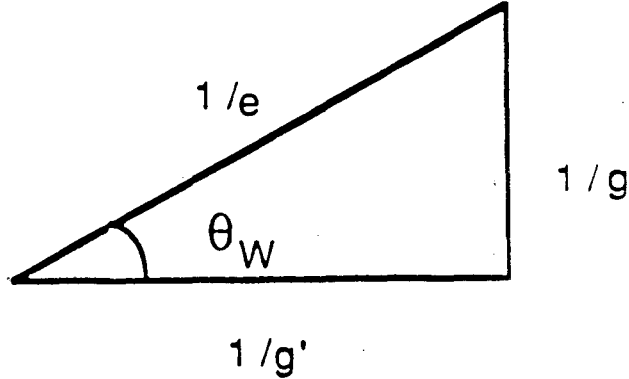


Figure 1.1: Mnemonic for the relation between g, g', e and θ_W

In terms of the physical fields the covariant derivative becomes

$$D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}}(T_+ W_\mu^+ + T_- W_\mu^-) - \frac{ig}{\cos \theta_W}(T_3 - Q \sin^2 \theta_W)Z_\mu - ieQ A_\mu \quad (1.24)$$

where $T_\pm = T_1 \pm iT_2$ and e is defined by

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2} \quad (1.25)$$

We have the mnemonic shown in Fig. 1.1

If the model is to correspond to V-A theory it must give the correct amplitude for μ decay. That amplitude is determined by charged W exchange. The effective interaction is determined by the couplings in Eq. (1.24)

$$\mathcal{L}_{eff} = \left(\frac{g}{\sqrt{2}}\right)^2 \frac{1}{m_W^2} \bar{\nu}_\mu \gamma^\lambda \frac{1}{2}(1 - \gamma_5) \mu \quad \bar{e} \gamma_\lambda \frac{1}{2}(1 - \gamma_5) \nu_e \quad (1.26)$$

This agrees with the usual form if

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{v^2}{2} \quad (1.27)$$

Now from Eqs. (1.22) and (1.25) or Fig. 1.1

$$g^2 = \frac{e^2}{\sin^2 \theta_W} = \frac{4\pi\alpha}{\sin^2 \theta_W} \quad (1.28)$$

so

$$m_W^2 = \frac{\pi\alpha}{\sqrt{2}\sin^2\theta_W G_F} \quad (1.29)$$

We shall see later that there are important radiative corrections to this relation with the consequence that $\alpha \simeq 1/137$ should be replaced by $\simeq 1/127.5$. This gives numerically

$$m_W^2 = \frac{(38.7 \text{ GeV})^2}{\sin^2\theta_W} \quad (1.30)$$

From Eqs. (1.18), (1.20), and (1.22)

$$m_Z^2 = \frac{m_W^2}{\cos^2\theta_W} \quad (1.31)$$

1.2 The Width of the Z .

The partial width of the Z to decay into a fermion-antifermion pair is best considered by looking at left-handed and right-handed fermions separately, e_L^-, e_R^- , etc. or by considering just left-handed fermions but both the particle and its charge-conjugate. If the coupling to the Z is

$$g_f Z_\mu \bar{f}_L \gamma^\mu f_L \quad (1.32)$$

then the partial width is

$$\Gamma(Z \rightarrow f_L \bar{f}_R) = \frac{g_f^2 m_Z}{24\pi} \quad (1.33)$$

and a similar formula applies for W decay, and for decays into right-handed fermions. In particular we have

$$\Gamma(Z \rightarrow \nu_L \bar{\nu}_R) = \frac{\sqrt{2} G_F m_Z^3}{24\pi} = 176 \text{ MeV} \left(\frac{m_Z}{93 \text{ GeV}} \right)^3 \quad (1.34)$$

$$\Gamma(Z \rightarrow f_L \bar{f}_R) = \frac{\sqrt{2} G_F m_Z^3}{6\pi} (T_3 - Q \sin^2\theta_W)^2 \quad (1.35)$$

$$\Gamma(Z \rightarrow f_R \bar{f}_L) = \frac{\sqrt{2} G_F m_Z^3}{6\pi} (Q \sin^2\theta_W)^2 \quad (1.36)$$

$$\Gamma(W \rightarrow e \bar{\nu}_e) = \frac{\sqrt{2} G_F m_W^3}{12\pi} = 241 \text{ MeV} \left(\frac{m_W}{82 \text{ GeV}} \right)^3 \quad (1.37)$$

	Γ before color	$\times QCD$	$\Gamma/\text{generation}$	generations	Γ	% each type
e_L	54	1	54	3	162	2.0
e_R	35	1	35	3	105	1.3
ν_L	176	1	176	3	528	6.6
u_L	87	1.04×3	272	2	544	10.2
u_R	15	1.04×3	48	2	96	1.8
d_L	128	1.04×3	398	3	1195	14.9
d_R	4	1.04×3	12	3	36	0.5

Table 1.2: Partial widths for the Z with mass 93 GeV. All widths are given in MeV. The total width without any t -quark contribution is 2666 MeV.

Of course each partial width into a quark and an antiquark must be multiplied by three for color. Furthermore, QCD corrections increase these partial widths by a factor $1 + \alpha_s(m_Z^2)/\pi \approx 1.04$. If we suppose that $Z \rightarrow t\bar{t}$ does not occur and that there are just three neutrino types, we can treat the final state fermions as massless. For a Z with mass 93 GeV (using $\sin^2 \theta = 0.222$) we find the results in Table 1.2.

The t -quark may appear in Z decays, but its phase space will be greatly suppressed. The UA-1 limit is $m_t > 41$ GeV. The partial width for $Z \rightarrow t\bar{t}$ including the effects of the quark mass is

$$\Gamma_t = \frac{\sqrt{2}\beta G_F m_Z^3}{\pi} \left[\frac{3 + \beta^2}{32} + \frac{2}{9}(3 - \beta^2) \sin^4 \theta_W - \frac{3 - \beta^2}{6} \sin^2 \theta_W \right] \quad (1.38)$$

where β is the t quark velocity

$$\beta = \sqrt{1 - \frac{4m_t^2}{m_Z^2}} \quad (1.39)$$

This partial width as a function of m_t is shown in Fig. 1.2.

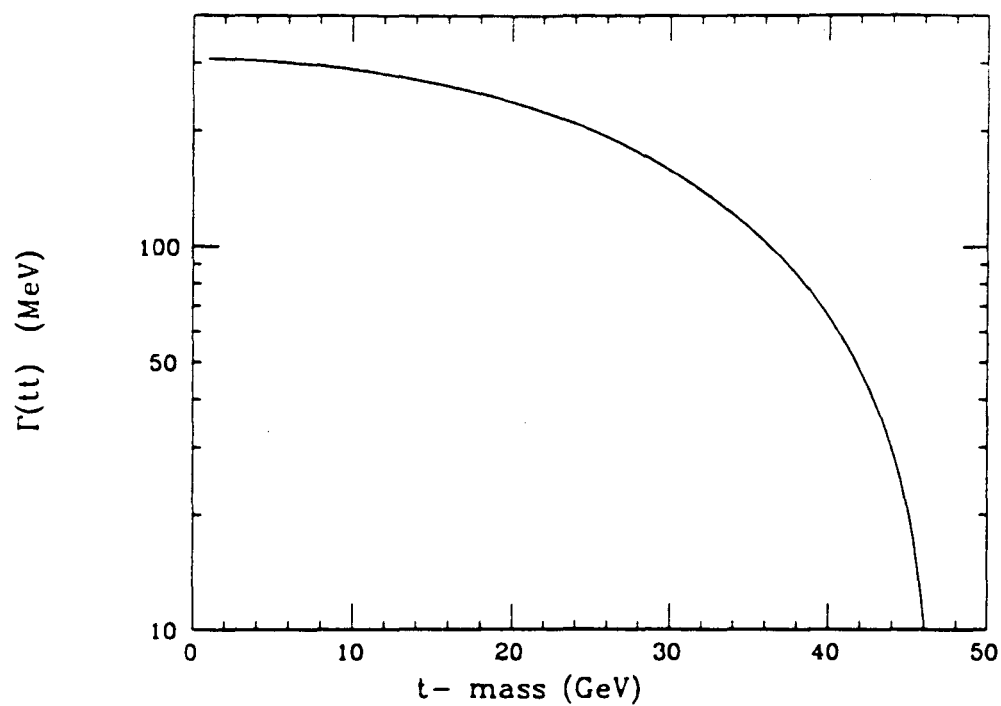


Figure 1.2: The partial width of the Z into $t\bar{t}$ as a function of the t-quark mass.

1.3 The Cross Section for $e^+e^- \rightarrow Z$

The Z will be a truly spectacular resonance in e^+e^- scattering. The Breit-Wigner formula gives

$$\sigma = \left[\frac{4\pi}{k^2} \frac{(2J+1)B(Z \rightarrow e^+e^-)}{(2S_1+1)(2S_2+1)} \right] \frac{\Gamma^2/4}{(W-M)^2 + \Gamma^2/4} \quad (1.40)$$

where $2J+1 = 3$ is the statistical weight for the final state, Z , $2S_1+1 = 2S_2+1 = 2$ is the statistical weight for the colliding particles, and k is the incident c.m. momentum. The factor in square bracket is 56 nb if $m_Z = 93$ GeV and $B(Z \rightarrow e^+e^-) = 0.033$.

The Breit-Wigner shape will be modified only slightly by the energy spread of the incident beams since that spread will be small compared to the width of the Z . On the other hand, corrections from initial state electromagnetic radiative corrections will be large[2].

These corrections are large because of the presence of a large logarithm, $\ln(m_Z^2/m_e^2)$. The strength of the corrections is determined by

$$t = \frac{2\alpha}{\pi} \left[\ln \frac{W^2}{m_e^2} - 1 \right] \quad (1.41)$$

which is 0.108 for $W^2 = m_Z^2$.

A convenient formalism is obtained from the work of Kuraev and Fadin[3]. The radiatively corrected cross section σ , is obtained from the uncorrected cross section σ_0 through

$$\sigma(W) = \left(1 + \frac{3t}{4}\right) t \int_0^E \frac{dk}{k} \left(\frac{k}{E}\right)^t \sigma_0(W-k) \quad (1.42)$$

Here $E = W/2$ is the beam energy.

From Eq. (1.40) the peak cross section in the absence of radiative corrections would be

$$\sigma_{\max} = \frac{12\pi}{m_Z^2} B(Z \rightarrow e^+e^-) \approx 56 \text{ nb} \quad (1.43)$$

Setting $k = \Gamma x/2$, $W - M = \lambda\Gamma/2$ we have

$$\frac{\sigma(W)}{\sigma_{\max}} \cong \left(1 + \frac{3t}{4}\right) t \left(\frac{\Gamma}{W}\right)^t \int_0^{2E/\Gamma} dx x^{t-1} [1 + (x - \lambda)^2]^{-1} \quad (1.44)$$

Since $E \gg \Gamma$, the upper limit may be set to ∞ . The integral can be done by contour integration. Consider

$$f(z) = (-z)^{t-1} [1 + (z - \lambda)^2]^{-1} \quad (1.45)$$

cut along the positive real axis from 0 to ∞ . Take a contour beginning at infinity below the cut, encircling the branch point, and continuing to positive infinity above the branch cut.

$$\int_C f(z) dz = +2i \sin \pi(1-t) \int_0^\infty dx x^{t-1} [1 + (x - \lambda)^2]^{-1} \quad (1.46)$$

On the other hand, wrapping the contour around the poles at $x = +\lambda \pm i$ gives

$$\int_C f(z) dz = 2\pi(1 + \lambda^2)^{\frac{t-1}{2}} \sin \left[(1-t) \cos^{-1} \frac{-\lambda}{\sqrt{1 + \lambda^2}} \right], \quad (1.47)$$

so

$$\begin{aligned} \Phi(\lambda) &\equiv t \int_0^\infty dx x^{t-1} [1 + (x - \lambda)^2]^{-1} \\ &= \frac{\pi t}{\sin \pi t} (1 + \lambda^2)^{\frac{t-1}{2}} \sin \left[(1-t) \cos^{-1} \left(\frac{-\lambda}{\sqrt{1 + \lambda^2}} \right) \right] \end{aligned} \quad (1.48)$$

In particular

$$\Phi(0) = \frac{\pi t/2}{\sin(\pi t/2)} \quad (1.49)$$

and thus at the old peak ($\lambda = 0$)

$$\begin{aligned} \frac{\sigma(m_Z)}{\sigma_{\max}} &= \left(1 + \frac{3t}{4}\right) \left(\frac{\Gamma}{W}\right)^t \frac{\pi t/2}{\sin \pi t/2} \\ &= 1.081 \times 0.682 \times 1.005 \\ &= 0.741 \end{aligned} \quad (1.50)$$

that is, a 26% reduction. If in the final result

$$\frac{\sigma(W)}{\sigma_{\max}} = \left(1 + \frac{3t}{4}\right) \left(\frac{\Gamma}{W}\right)^t \Phi(\lambda) \quad (1.51)$$

we expand in powers of t (that is, in powers of α) and keep only the first order corrections we get an inadequate approximation. A comparison of the full formula and the first order formula alone is shown in Fig. 1.3.

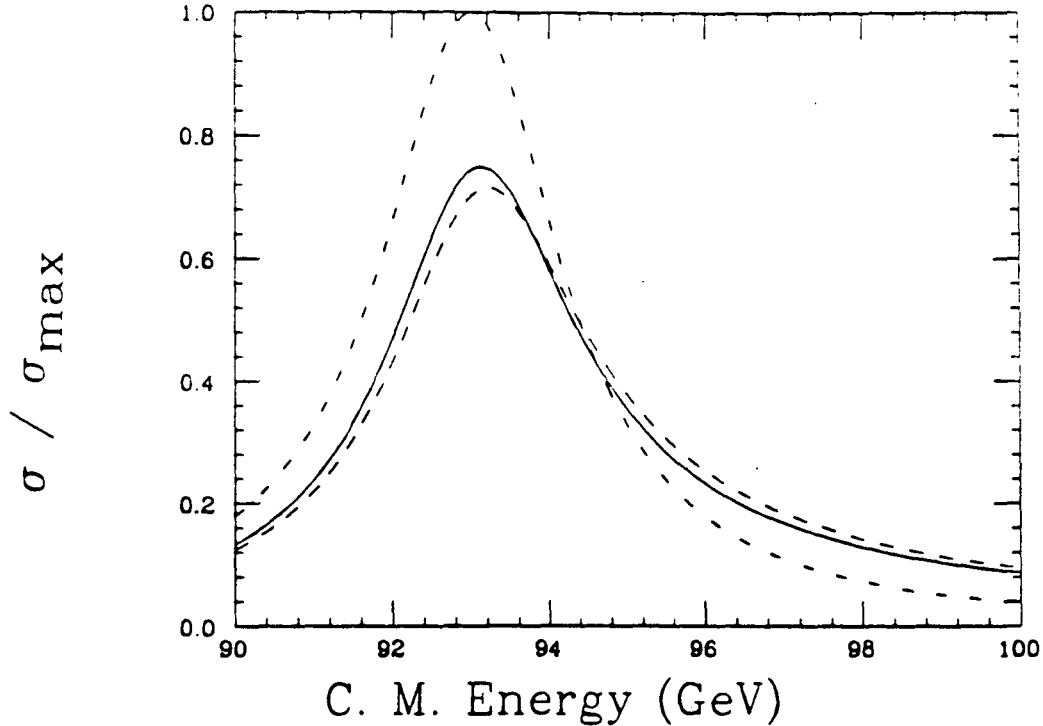


Figure 1.3: Comparison of the shape of the Z using just the first order QED corrections (dashed) and the full corrections (solid). The dotdash curve shows the shape without any QED correction.

Using Eq. (1.51) we can simulate the results that will be obtained at SLC and LEP. In Figure 1.4 we show a simulation with 10 nb^{-1} accumulated at each of the center of mass energies 91, 92, 93, 94, and 95 GeV. Statistical errors only have been included. A fit to the experimental points using the radiatively corrected line shape inferred a Z mass of 91.65 GeV while the actual mass for the simulation was 91.73 GeV. The inferred width was 2.954 GeV compared to an actual width of 2.694 GeV. This shows that 50 nb^{-1} will be enough to find the mass to considerable accuracy, but not enough to determine the width to less than the contribution of a single neutrino. In Figure 1.4 results are also shown for a simulation with 500 nb^{-1} . Here the error in the width was about 70 MeV, just less than half the contribution of a neutrino. About 500 nb^{-1} will thus be required to determine the number of neutrinos.

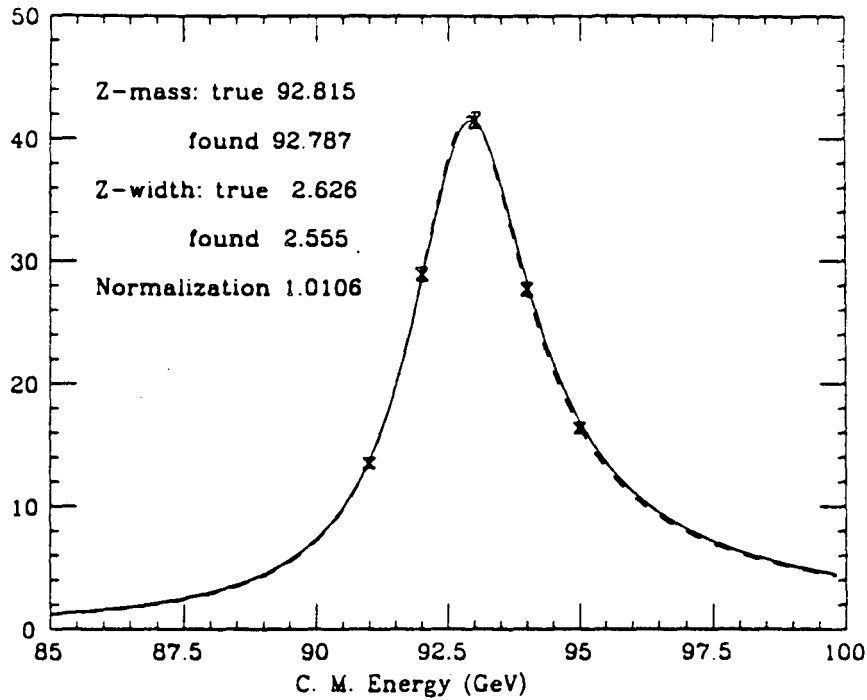
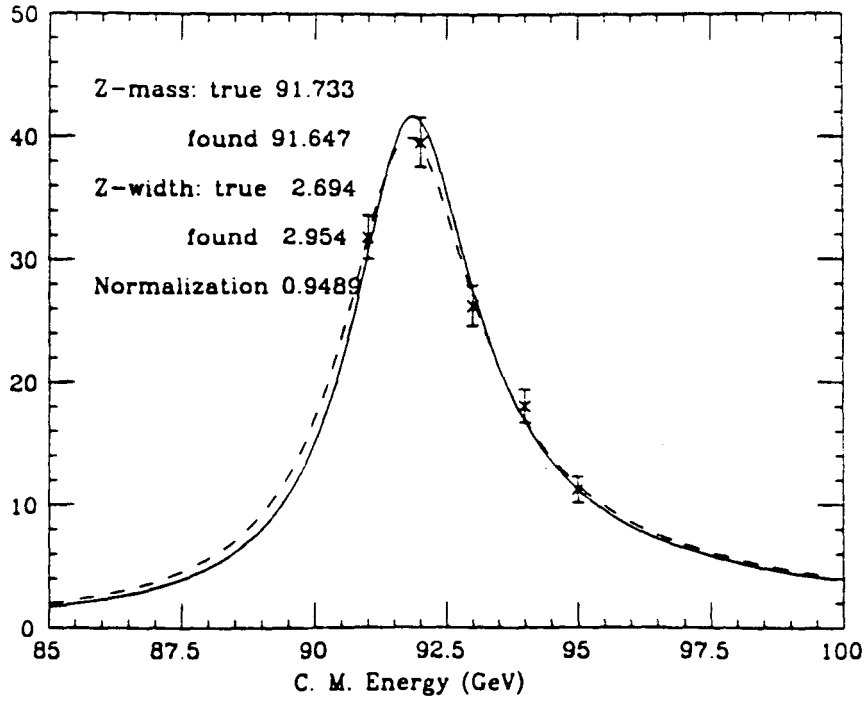


Figure 1.4: Results of a simulation of total cross section measurements near the Z , with the cross section given in nanobarns. Data were generated corresponding to 10 nb^{-1} at each of five energies. A fit (dashed) was made to the simulated data using the known radiatively corrected lineshape, but allowing the overall normalization to float. The second figure shows an analogous simulation with 100 nb^{-1} at each point. The solid curves show the true line shapes for the simulated masses and widths.

1.4 Asymmetries

Because the Z couples differently to left-handed and right-handed fermions, the angular distributions of the outgoing leptons and hadron jets are especially interesting. Here we consider effects in the absence of radiative corrections. We shall include, however, the important influence of $\gamma - Z$ interference.

Consider first $e_L^- e_R^+ \rightarrow f_L \bar{f}_R$. The spinor structure of the amplitude is the same for γ and Z exchanges (in the s-channel). In addition to the spinor structure there is a factor

$$\frac{Q_L^{\gamma e} Q_L^{\gamma f}}{s} + \frac{Q_L^{Z e} Q_L^{Z f}}{s - m_Z^2 + i\Gamma m_Z} \quad (1.52)$$

Here $Q_L^{\gamma e}$ is the photon coupling to $e_L^- e_R^+$ and $Q_L^{Z f}$ is the Z coupling to $f_L \bar{f}_R$ and so on. Of course $Q_L^{\gamma e} = Q_R^{\gamma e}$. The angular distribution for $e_L^- e_R^+ \rightarrow f_L \bar{f}_R$ is $(1 + \cos \theta)^2$, while for $e_L^- e_R^+ \rightarrow f_R \bar{f}_L$ we have $(1 - \cos \theta)^2$. This can be understood in terms of angular momentum conservation. If the left-handed electron enters moving in the negative z-axis it has $J_Z = +1/2$ while the right-handed positron also has $J_Z = +1/2$. The initial state thus has $J_Z = +1$. If the final state is $f_L \bar{f}_R$ it has 1 as its component of angular momentum along the direction of motion of $\bar{f}_R$. If the outgoing fermion (f_L) is moving in the same direction as the incoming fermion (the negative Z -direction) J_Z matches perfectly. However, if the outgoing fermion is going in the direction opposite the incoming fermion J_Z is not conserved. This is reflected in the vanishing of $(1 + \cos \theta)^2$ in the backward direction. For a similar reason $e_L^- e_R^+ \rightarrow f_R \bar{f}_L$ vanishes in the forward direction.

If we define

$$\xi = \left(1 - \frac{m_Z^2}{s} - i \frac{\Gamma m_Z}{s} \right)^{-1} \quad (1.53)$$

the amplitudes for the various helicity amplitudes are

$$\begin{aligned} |\mathcal{M}(e_L^- e_R^+ \rightarrow f_L \bar{f}_R)|^2 &\propto |Q_L^{\gamma e} Q_L^{\gamma f} + \xi Q_L^{Z e} Q_L^{Z f}|^2 (1 + \cos \theta)^2 \\ |\mathcal{M}(e_L^- e_R^+ \rightarrow f_R \bar{f}_L)|^2 &\propto |Q_L^{\gamma e} Q_R^{\gamma f} + \xi Q_L^{Z e} Q_R^{Z f}|^2 (1 - \cos \theta)^2 \\ |\mathcal{M}(e_R^- e_L^+ \rightarrow f_L \bar{f}_R)|^2 &\propto |Q_R^{\gamma e} Q_L^{\gamma f} + \xi Q_R^{Z e} Q_L^{Z f}|^2 (1 - \cos \theta)^2 \\ |\mathcal{M}(e_R^- e_L^+ \rightarrow f_R \bar{f}_L)|^2 &\propto |Q_R^{\gamma e} Q_R^{\gamma f} + \xi Q_R^{Z e} Q_R^{Z f}|^2 (1 + \cos \theta)^2 \end{aligned} \quad (1.54)$$

Here θ is the angle between the incident and final fermion directions (or equivalently, incident and final antifermion directions). At center of mass energies well below the Z , the energies of PEP, PETRA, and TRISTAN, $|\xi|$ is not large and it is the interference between the photon and the Z that is responsible for the deviations from the $1 + \cos^2 \theta$ distribution. Very near the Z , $|\xi| \gg 1$ and only the Z pole is important. There $d\sigma/d\Omega$, which is proportional to the sum of the four amplitudes squared in Eq. (1.54), has the form

$$\begin{aligned} \frac{d\sigma}{d\Omega} &\propto (Q_L^{Ze^2} + Q_R^{Ze^2}) (Q_L^{Zf^2} + Q_R^{Zf^2}) (1 + \cos^2 \theta) \\ &+ (Q_L^{Ze^2} - Q_R^{Ze^2}) (Q_L^{Zf^2} - Q_R^{Zf^2}) 2 \cos \theta. \end{aligned} \quad (1.55)$$

The axial and vector couplings are

$$\begin{aligned} Q_V &= Q_R^Z + Q_L^Z \\ Q_A &= Q_R^Z - Q_L^Z \end{aligned} \quad (1.56)$$

so

$$\frac{d\sigma}{d\Omega} \propto (Q_V^{e^2} + Q_A^{e^2}) (Q_V^{f^2} + Q_A^{f^2}) (1 + \cos^2 \theta) + 4Q_V^e Q_A^e Q_V^f Q_A^f \cdot 2 \cos \theta \quad (1.57)$$

The forward-backward asymmetry is defined by taking the cross section for $0 < \theta < \frac{\pi}{2}$ and subtracting the cross section for $\frac{\pi}{2} < \theta < \pi$, and dividing by the sum. Thus

$$\begin{aligned} A_{FB} &= \frac{3}{4} \cdot \frac{2Q_V^e Q_A^e}{Q_V^{e^2} + Q_A^{e^2}} \cdot \frac{2Q_V^f Q_A^f}{Q_V^{f^2} + Q_A^{f^2}} \\ &= \frac{3}{4} \cdot \frac{Q_L^{Ze^2} - Q_R^{Ze^2}}{Q_L^{Ze^2} + Q_R^{Ze^2}} \cdot \frac{Q_L^{Zf^2} - Q_R^{Zf^2}}{Q_L^{Zf^2} + Q_R^{Zf^2}} \end{aligned} \quad (1.58)$$

Of course A_{FB} can be measured only if the fermion (as opposed to antifermion) direction can be identified in the final state. This is possible for final states e, μ , and τ , though for the e^+e^- final state this analysis is incomplete because the non-resonant (Bhabha) scattering contributes. For the $e^+e^- \rightarrow \mu^+\mu^-$ events the uncertainty in A_{FB} will be largely limited by statistics so

$$\begin{aligned} \delta A_{FB} &= \frac{1}{\sqrt{N_\mu}} = \frac{1}{\sqrt{B(Z \rightarrow \mu^-\mu^+)N}} \\ &\approx \frac{5.5}{\sqrt{N}} \end{aligned} \quad (1.59)$$

where N is the total number of Z 's produced and N_μ is the number of $\mu^+\mu^-$ pairs produced. For $N = 10^6$, $\delta A_{FB} = 5.5 \times 10^{-3}$.

Let us assume that the Z couples to the e and the μ in the same way:

$$\begin{aligned} Q_L^{Ze} &= Q_L^{Z\mu} = \ell \\ Q_R^{Ze} &= Q_R^{Z\mu} = \tau \end{aligned} \quad (1.60)$$

Then

$$A_{FB} = \frac{3}{4} \left[\frac{1 - (r/\ell)^2}{1 + (r/\ell)^2} \right]^2 \quad (1.61)$$

and

$$\begin{aligned} \frac{\delta A_{FB}}{A_{FB}} &= 2 \cdot \frac{2}{1 - (r/\ell)^4} (-\delta(r/\ell)^2) \\ \frac{\delta A_{FB}}{0.034} &\approx -6.90 \delta(r/\ell)^2 \end{aligned} \quad (1.62)$$

where we have used our earlier estimate $r^2/\ell^2 = \Gamma((Z \rightarrow e_R^- e_L^+)/\Gamma(Z \rightarrow e_L^- e_R^+) = 35/54$.

Thus

$$\delta(r/\ell)^2 \approx 4.2 \times \delta A_{FB} \quad (1.63)$$

and for 10^6 Z 's,

$$\delta(r/\ell)^2 \approx 0.023 \quad (1.64)$$

Near the Z the asymmetry is energy dependent as a result of the interference between the Z and the photon. This is shown in Fig. 1.5.

An alternative way to measure $(r/\ell)^2$ is to use longitudinally polarized electron beams. If σ_L is the cross section with left-handed electrons incident we define

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} \quad (1.65)$$

From the helicity amplitudes squared, evaluated for $|\xi| \gg 1$,

$$A_{LR} = \frac{1 - (r/\ell)^2}{1 + (r/\ell)^2} \quad (1.66)$$

At SLC it is planned to measure A_{LR} using a polarized beam generated by directing circularly polarized laser light on the surface of a GaAs crystal[4].

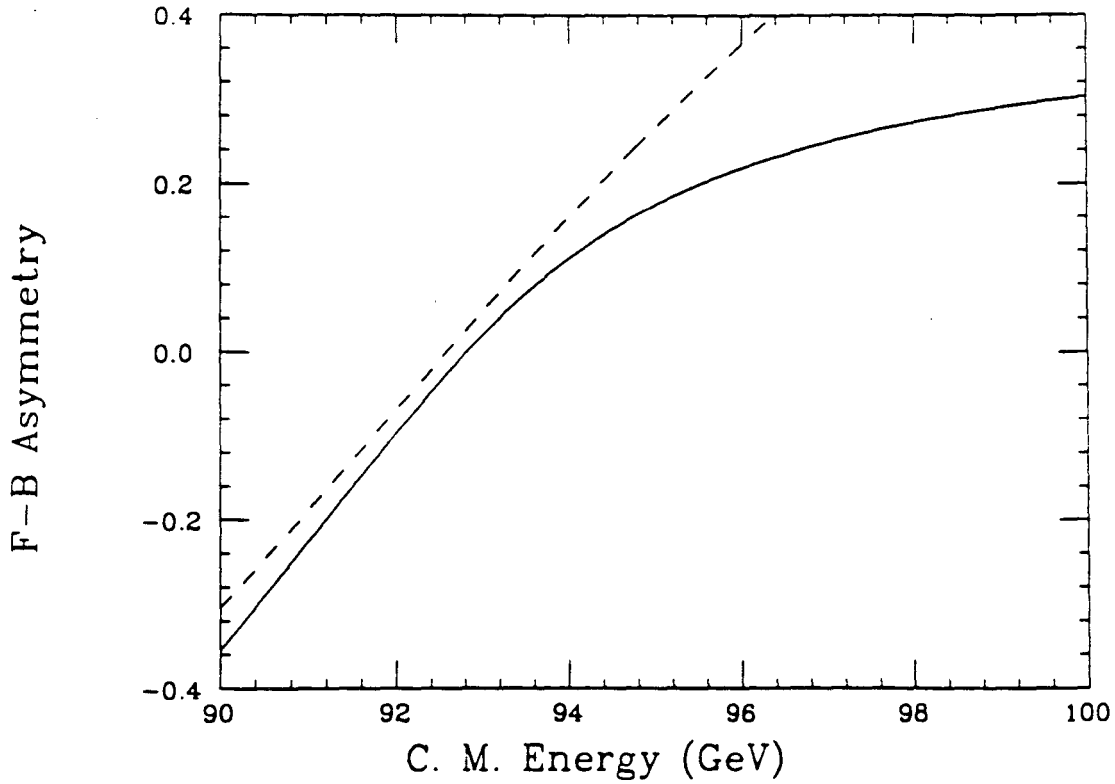


Figure 1.5: The forward-backward asymmetry for $e^+e^- \rightarrow \mu^+\mu^-$ as a function of the center of mass energy for $m_Z = 93$ GeV. The dashed curve shows the result without radiative corrections. The full curve is an approximation to the result with electromagnetic radiative corrections. A complete treatment of the radiative corrections for this quantity must include the non-collinearity of the outgoing muons when a photon is emitted.

This technique was used in the classic experiment of C.Y. Prescott *et al.* that measured the left-right asymmetry in $e - d$ scattering[5]. The degree of polarization that can be achieved is 30-40%. The precision of the forward-backward measurement will be limited by statistics and by uncertainties in measuring the degree of polarization of the beam. Early running might achieve $\delta A_{LR} \approx 0.025$ while ultimately $\delta A_{LR} \approx 0.01$ might be attained, I estimate[4].

Now

$$\frac{\delta A_{LR}}{A_{LR}} = -\frac{2}{1 - (r/\ell)^4} \delta(r/\ell)^2 \quad (1.67)$$

$$\frac{\delta A_{LR}}{0.21} = -3.4 \left(\delta(r/\ell^2) \right)$$

$$\delta(r/\ell)^2 \approx 1.36 \delta A_{LR} \quad (1.68)$$

Even for the conservative $\delta A_{LR} = 0.025$ estimate, which is for $N = 10^4$ and a fractional uncertainty in the polarization $\delta P/P = 5\%$, $\delta(r/\ell)^2 = 0.033$, which is comparable to the precision obtained from forward-backward symmetry with $10^6 Z$ s. For the more optimistic $\delta A_{LR} = 0.01$ based on $10^5 Z$ s and $\delta P/P = 3\%$, the left-right asymmetry is a better probe than $10^6 Z$ s using the forward-backward asymmetry.

The quality of the tests furnished by the forward-backward and left-right asymmetries can be judged by interpreting the measurement of $(r/\ell)^2$ as a prediction of m_Z . Since

$$m_Z^2 = \frac{\text{const}}{\cos^2 \theta_W \sin^2 \theta_W} \quad (1.69)$$

we find that

$$\frac{2\delta m_Z}{m_Z} = \frac{\sin^2 \theta_W - \cos^2 \theta_W}{\sin^2 \theta_W \cos^2 \theta_W} \delta \sin^2 \theta_W \approx -3.2 \delta \sin^2 \theta_W \quad (1.70)$$

On the other hand,

$$\left(\frac{r}{\ell}\right)^2 = \left(\frac{\sin^2 \theta_W}{\frac{1}{2} - \sin^2 \theta_W}\right)^2 \quad (1.71)$$

and

$$\delta \left(\frac{r}{\ell}\right)^2 = -\frac{\sin^2 \theta_W}{(\frac{1}{2} - \sin^2 \theta_W)^3} \delta \sin^2 \theta_W \approx -10.3 \delta \sin^2 \theta_W \quad (1.72)$$

so that the uncertainty in the predicted Z mass is

$$\delta m_Z = 0.16 m_Z \delta \left(\frac{r}{\ell}\right)^2 = 14 \text{ GeV } \delta \left(\frac{r}{\ell}\right)^2 \quad (1.73)$$

The estimated uncertainty of $\delta A_{LR} = 0.025$ thus gives an prediction for the Z mass with an uncertainty of about 450 MeV, which can be compared with the uncertainty of perhaps 40 MeV that will come from the direct measurement.

Because of interference between the Z and the photon the left-right asymmetry will also show an energy dependence. Moreover, because the various final states couple differently to the photon, the energy dependence varies with the final state. AT the Z peak, the interference is unimportant and the left-right asymmetry is independent of the final fermion type. This is displayed in Figure 1.6.

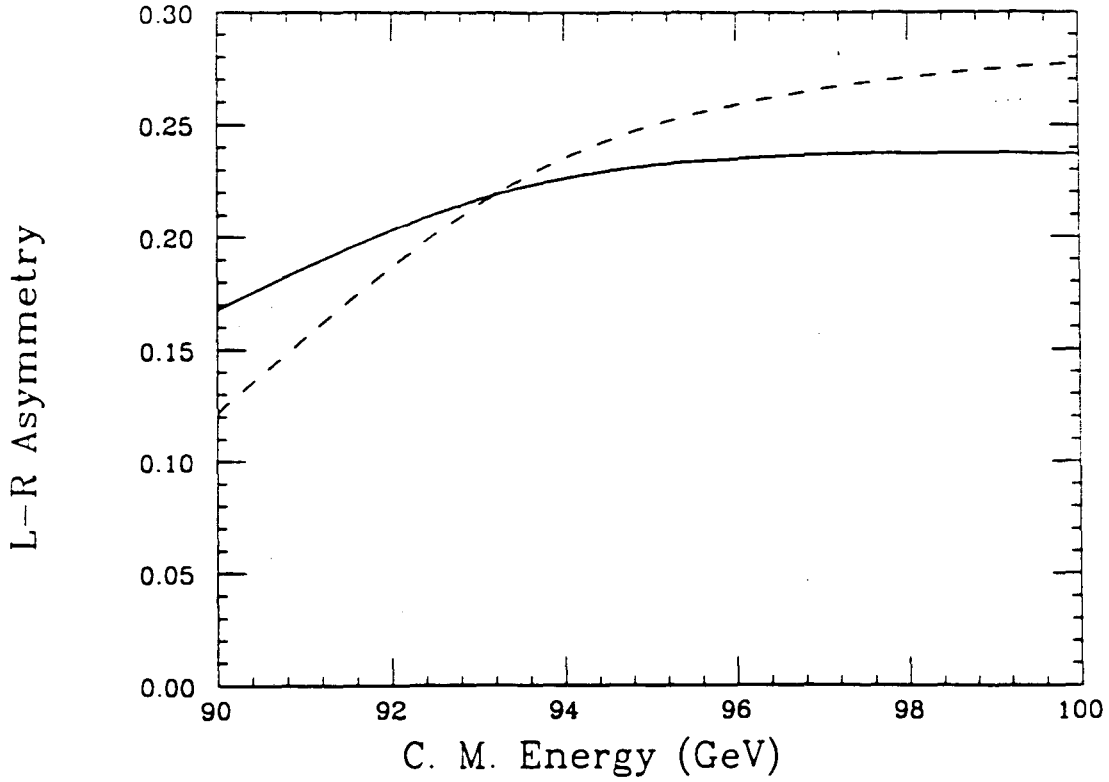


Figure 1.6: The left-right asymmetry as a function of the center of mass energy for $m_Z = 93$ GeV and $\Gamma = 2.8$ GeV. The solid curve corresponds to $e^+e^- \rightarrow \mu^+\mu^-$, the dashed to $e^+e^- \rightarrow u\bar{u}$, and the dotted to $e^+e^- \rightarrow d\bar{d}$. QED radiative corrections have been included by applying Eq. (1.51) to each helicity amplitude.

1.5 Z Decays to the Higgs Boson

In addition to the standard Z decays, there may be decays to other particles. Among the most interesting possibilities are those involving the Higgs boson. As pointed out by Ioffe and Khoze[6], the most important is $Z \rightarrow H e^+ e^-$ or $Z \rightarrow H \mu^+ \mu^-$. The charged lepton pair comes from the decay of a virtual Z , as shown in Fig. 1.7. Defining $x = 2E_H/m_Z$ with E_H the energy of the Higgs boson in the Z rest frame, the differential decay rate is [7]

$$\frac{1}{\Gamma(Z \rightarrow \mu^+ \mu^-)} \frac{d\Gamma}{dx} = \frac{\sqrt{2}G_F m_Z^2}{4\pi^2} \frac{\left(1 - x + \frac{x^2}{12} + \frac{2}{3} \frac{m_H^2}{m_Z^2}\right) \sqrt{x^2 - 4 \frac{m_H^2}{m_Z^2}}}{\left(x - \frac{m_H^2}{m_Z^2}\right)^2} \quad (1.74)$$

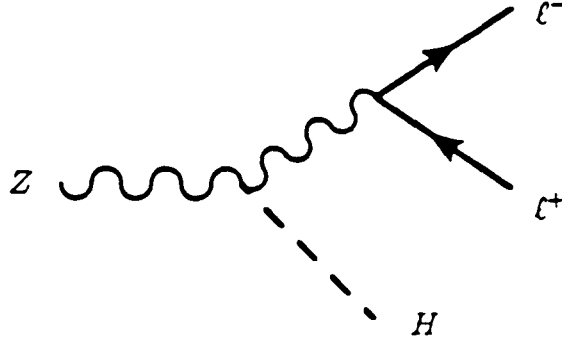


Figure 1.7: The decay $Z \rightarrow H\ell^+\ell^-$ or $Z \rightarrow H\mu^+\mu^-$.

The ratio $\Gamma(Z \rightarrow H\mu^+\mu^-)/\Gamma(Z \rightarrow \mu^+\mu^-)$ is shown in Fig. 1.8. Also shown is the relative rate for $Z \rightarrow H\gamma$, a process that proceeds through W -loops and fermion loops^{1.7}. The rate formula is [8]

$$\frac{\Gamma(Z \rightarrow H\gamma)}{\Gamma(Z \rightarrow \mu^+\mu^-)} = \frac{\alpha}{\pi^3} \sqrt{2} G_F m_W^2 \left(\frac{E_\gamma}{m_Z} \right)^3 \frac{A^2}{1 + (1 - 4 \sin^2 \theta_W)^2} \quad (1.75)$$

where

$$A \approx - \left[4.9 + 0.3 \frac{m_H^2}{m_W^2} \right] \quad (1.76)$$

and E_γ is the photon energy in the Z rest frame, $E_\gamma = m_Z^2 - m_H^2/(2m_Z)$. As can be seen the $Z \rightarrow H\mu^+\mu^-$ mode is more attractive for $m_H \leq 55$ GeV. With $10^7 Z$ s a good possibility at LEP, a Higgs boson with mass up to about 40 GeV ought to be discovered [9].

2 Introduction to Renormalization Effects in the Standard Model

The triumph of the Standard Model is two-fold. First, through the work of G.'t Hooft and others, spontaneously broken gauge theories were shown to be

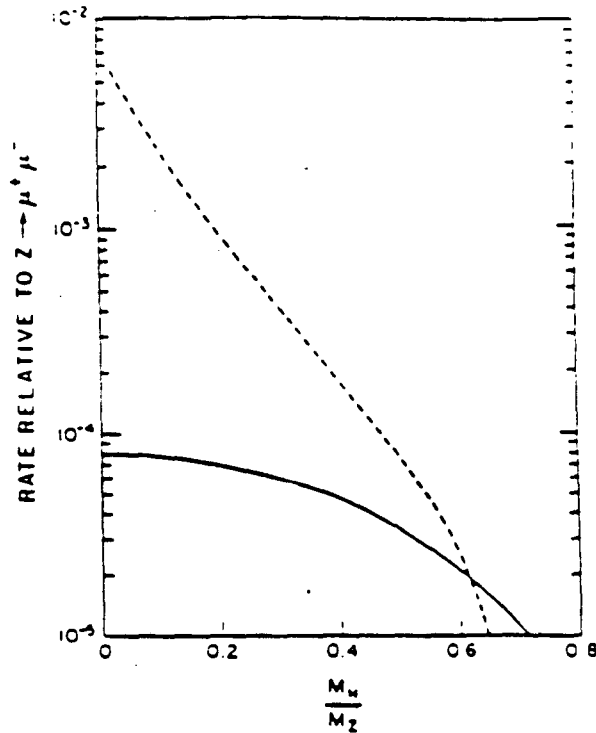


Figure 1.8: The rates for $Z \rightarrow H \mu^+ \mu^-$ and $Z \rightarrow H \gamma$ relative to $Z \rightarrow \mu^+ \mu^-$. [8]

renormalizable and thus the model of Glashow, Weinberg, and Salam became a real candidate for the theory of electroweak interactions. Second, the discovery and measurement of the neutral weak current and later the discovery of the W and Z verified the model. Detailed testing of the model is essential to confirm the prediction of the radiative corrections to the model or, even better, to find discrepancies between the theory and experiment.

The radiative corrections that occur are of several types. In the measurement of $e^+e^- \rightarrow Z$ the largest corrections are purely due to QED. These have been discussed above. The technically difficult part of the corrections involve the electroweak gauge sector of the model. An outstanding guide to such calculations is the classic paper of Fujikawa, Lee and Sanda [10].

It turns out that the most important radiative corrections, aside from the initial-state radiation, are a relatively simple class of vacuum polarization diagrams. It is these that we treat here. Our approach is that of Sirlin [11] which

we follow slavishly, dropping his discussion of the gauge sector, but providing explicit calculations of the various contributions. For a thorough treatment, see, for example, the work of Lynn, Peskin, and Stuart [12].

2.1 Physical Quantities

The Lagrangian of the gauge theory has three fundamental parameters, g (the $SU(2)$ coupling constant), g' (the $U(1)$ coupling constant), and v (the vacuum expectation value that breaks the symmetry). We know that in lowest order

$$\begin{aligned} m_Z^2 &= \frac{1}{4}(g^2 + g'^2)v^2 \\ m_W^2 &= \frac{1}{4}g^2v^2 \\ \frac{G_F}{\sqrt{2}} &= \frac{g^2}{8m_W^2} \\ \frac{1}{e^2} &= \frac{1}{4\pi\alpha} = \frac{1}{g^2} + \frac{1}{g'^2} \end{aligned} \tag{2.1}$$

Now we cannot require all four of these relations to remain true after radiative corrections since we have only three parameters. In the actual world α and G_F are very well known and m_Z soon will be. These can be taken to define the theory. Then m_W^2 and all other physical quantities can be predicted in terms of α , G_F and m_Z . At present m_Z is not known with good precision. Data on neutral current neutrino scattering are better and can be used instead of m_Z .

It is often useful, especially in dealing with the neutrino data to introduce

$$\tan \theta_W = g'/g \tag{2.2}$$

If we wish to consider radiative corrections it is necessary to be more precise. One alternative is to *define*

$$\cos^2 \theta_W = \frac{m_W^2}{m_Z^2} \tag{2.3}$$

but there are alternatives that differ by terms of order α .

Regardless of which physical parameters we use to define the theory, we can proceed by deriving the physical observables in terms of g , g' , and v . Then three

of the relations fix these three parameters and the remaining relations become predictions.

2.2 Lagrangian and Counterterms

After spontaneous symmetry breaking the gauge bosons of the electroweak theory have a mass matrix Lagrangian

$$\mathcal{L} = \frac{v_0^2}{2} \left[\frac{g_0^2}{2} W_\mu^+ W^{-\mu} + \frac{1}{4} (g'_0 B_\mu - g_0 W_\mu)^2 \right] \quad (2.4)$$

Following Sirlin we ignore wave function renormalization and generate counterterms by writing

$$\begin{aligned} g_0 &= g - \delta g \\ g'_0 &= g' - \delta g' \\ v_0^2 &= v^2 - \delta v^2 \end{aligned} \quad (2.5)$$

Working to first order in δs we have

$$\begin{aligned} \mathcal{L} &= \frac{m_Z^2}{2} Z_\mu Z^\mu + m_W^2 W_\mu^+ W^{-\mu} \\ &\quad - \left(\frac{\delta m_Z^2}{2} Z_\mu Z^\mu + \delta m_W^2 W_\mu^+ W^{-\mu} - \delta m_{ZA}^2 Z_\mu A^\mu \right) \end{aligned} \quad (2.6)$$

with

$$\begin{aligned} \delta m_W^2 &= \frac{1}{4} \delta (g^2 v^2) = \frac{1}{4} (v^2 \delta g^2 + g^2 \delta v^2) \\ \delta m_Z^2 &= \frac{1}{4} \delta [(g^2 + g'^2) v^2] \\ \delta m_{ZA}^2 &= \frac{m_Z^2}{\sqrt{g^2 + g'^2}} (c \delta g' - s \delta g) = \frac{v^2}{4} (-g' \delta g + g \delta g') \end{aligned} \quad (2.7)$$

where we have *defined*

$$\begin{aligned} \tan \theta_W &= g'/g \\ c &= \cos \theta_W, \quad s = \sin \theta_W \end{aligned} \quad (2.8)$$

The counterterms $\delta m_W^2, \delta m_Z^2, \delta m_{ZA}^2$ will be used to offset one-loop effects at specified points.

Consider the mass of the Z , as described by



We shall avoid considerations of gauges insofar as possible. Here let us imagine that the first diagram is given by $-ig_{\mu\nu}/(q^2 - m_Z^2)$ and the second by

$$\frac{-ig_{\mu\lambda}}{q^2 - m_Z^2} \left[+ig^{\lambda\sigma} A_{ZZ}(q^2) \right] \frac{-ig_{\sigma\nu}}{q^2 - m_Z^2} = \frac{-ig_{\mu\nu}}{q^2 - m_Z^2} \cdot \frac{A_{ZZ}(q^2)}{q^2 - m_Z^2} \quad (2.9)$$

In general the vacuum polarization bubble is

$$i \left(g^{\lambda\sigma} A + q^\lambda q^\sigma B \right). \quad (2.10)$$

but we shall not be concerned with B since we shall always couple somewhere to an electron line, for which not just the vector current, but also the axial current is conserved (in the $m_e = 0$ limit). The sum of all the diagrams with repetitions of the bubble is a geometric series with the sum

$$\frac{-ig_{\mu\nu}}{q^2 - m_Z^2 - A_{ZZ}(q^2)} \quad (2.11)$$

There is also the diagram generated by the counterterm



This gives

$$\frac{-ig_{\mu\nu}}{q^2 - m_Z^2} (-i\delta m_Z^2) \frac{-i}{q^2 - m_Z^2} \quad (2.12)$$

and for the full propagator

$$\frac{-ig_{\mu\nu}}{q^2 - m_Z^2 - A_{ZZ}(q^2) + \delta m_Z^2}. \quad (2.13)$$

Now we fix m_Z^2 to be the energy squared where the real part of the propagator vanishes by requiring

$$\delta m_Z^2 = \delta \left[\frac{v^2}{4} (g^2 + g'^2) \right] = \text{Re } A_{ZZ}(m_Z^2) \quad (2.14)$$

Similarly

$$\delta m_W^2 = \delta \left[\frac{v^2}{4} g^2 \right] = \text{Re } A_{WW}(m_W^2) \quad (2.15)$$

In all such expressions δ acts in the natural way, viz. $\delta(g^2 v^2) = v^2 \delta g^2 + g^2 \delta v^2 = 2gv^2 \delta g + g^2 \delta v^2$. We shall make this all explicit below by calculating A for the most important class of diagrams.

Next consider the electric charge. The interactions of the gauge bosons with the fermions are governed by

$$\mathcal{L} = \bar{\psi} \left(g_0 \mathbf{W} \cdot \frac{\boldsymbol{\tau}}{2} + g'_0 \frac{Y}{2} B \right) \psi \quad (2.16)$$

which we again express in terms of g, g' , and counterterms using

$$\begin{aligned} Z &= \frac{-g' B + g W_3}{\sqrt{g^2 + g'^2}} = -s B + c W_3 \\ A &= \frac{g B + g' W_3}{\sqrt{g^2 + g'^2}} = c B + s W_3 \end{aligned} \quad (2.17)$$

The result is

$$\begin{aligned} \mathcal{L} &= \bar{\psi} \left[\frac{g}{\sqrt{2}} (T_+ W^+ + T_- W^-) + \frac{g}{c} (T_{3L} - Q s^2) Z + e Q A \right] \psi \\ &\quad - \bar{\psi} \left[\frac{\delta g}{\sqrt{2}} (T_+ W^+ + T_- W^-) + \left(\delta \sqrt{g^2 + g'^2} \right) (T_{3L} - Q s^2) Z \right. \\ &\quad \left. + \delta e Q A + g s^2 c \left(\frac{\delta g}{g} - \frac{\delta g'}{g'} \right) Q Z \right. \\ &\quad \left. + g s \left(\frac{\delta g}{g} - \frac{\delta g'}{g'} \right) (T_{3L} - Q s^2) A \right] \psi \end{aligned} \quad (2.18)$$

Here $T_{3L} = \frac{1}{2} \tau_3 \frac{1}{2} (1 - \gamma_5)$ and $Q = T_{3L} - Y/2$. The electric charge has been identified $e = (1/g^2 + 1/g'^2)^{-1/2}$ and $\delta e = \delta(1/g^2 + 1/g'^2)^{-1/2}$. We recognize the

electromagnetic and weak neutral currents:

$$\begin{aligned} J_\gamma^\mu &= \bar{\psi} \gamma^\mu Q \psi \\ J_Z^\mu &= \bar{\psi} \gamma^\mu (T_{3L} - s^2 Q) \psi \end{aligned} \quad (2.19)$$

In terms of these the Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{g}{\sqrt{2}} \bar{\psi} (T_+ \mathcal{W}^{++} + T_- \mathcal{W}^{--}) \psi + \frac{g}{c} J_Z^\mu Z_\mu + e J_\gamma^\mu A_\mu \\ &\quad - \frac{\delta g}{\sqrt{2}} \bar{\psi} (T_+ \mathcal{W}^{++} + T_- \mathcal{W}^{--}) \psi - A_\mu J_\gamma^\mu \delta c - Z_\mu J_Z^\mu \delta \sqrt{g^2 + g'^2} \\ &\quad - J_\gamma^\mu Z_\mu g s^2 c \left(\frac{\delta g}{g} - \frac{\delta g'}{g'} \right) - J_Z^\mu A_\mu c \left(\frac{\delta g}{g} - \frac{\delta g'}{g'} \right) \end{aligned} \quad (2.20)$$

The counterterms from δv^2 , δg , and $\delta g'$ are determined by three conditions, two of which have already been obtained by defining m_Z^2 and m_W^2 . The third condition is obtained from the definition of the charge e by considering the coupling of a soft photon to a charged particle.

The six contributions are shown in Fig.2.1.

Their sum is

$$ieJ_\gamma^\mu - i(s\delta g - c\delta g')J_Z^\mu + i\frac{g}{c}J_Z^\mu \frac{\delta m_{ZA}^2}{q^2 - m_Z^2} + ieJ_\gamma^\mu \frac{A_\gamma(q^2)}{2q^2} + i\frac{g}{c}J_Z^\mu \frac{A_Z(q^2)}{q^2 - m_Z^2} - iJ_\gamma^\mu \delta e \quad (2.21)$$

The $\frac{1}{2}$ appears because half of the diagram is absorbed in wave function renormalization. For this to make sense, the result must be proportional to J_γ^μ ; all the J_Z^μ pieces must cancel as $q^2 \rightarrow 0$. Now in fact from Eq. (2.7)

$$\frac{g}{c} \frac{\delta m_{ZA}^2}{m_Z^2} = (c\delta g' - s\delta g) \quad (2.22)$$

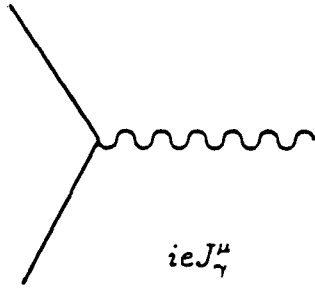
so the second and third pieces cancel. We shall show below that $A_Z(0) = 0$. Thus we need as well

$$\delta e = \lim_{q^2 \rightarrow 0} e \frac{A_\gamma(q^2)}{2q^2} \quad (2.23)$$

We shall see below that $A_\gamma(q^2) \propto q^2$ as $q^2 \rightarrow 0$. Following Sirlin's notation we define

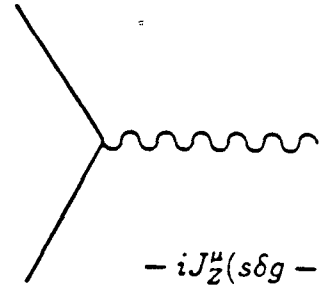
$$\Pi_\gamma(q^2) = -\frac{A_\gamma(q^2)}{q^2} \quad (2.24)$$

a.



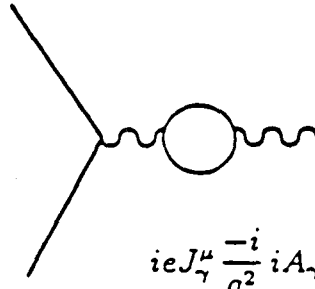
$$ieJ_\gamma^\mu$$

b.



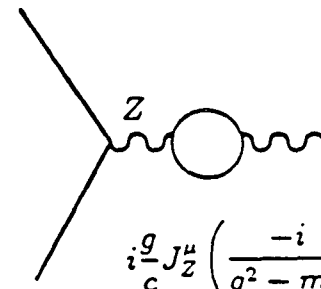
$$-iJ_Z^\mu(s\delta g - c\delta g')$$

c.



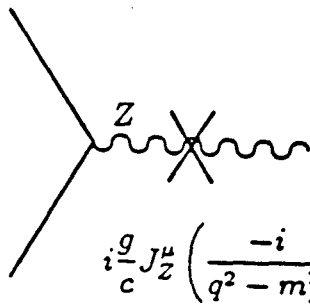
$$ieJ_\gamma^\mu \frac{-i}{q^2} iA_\gamma(q^2)$$

d.



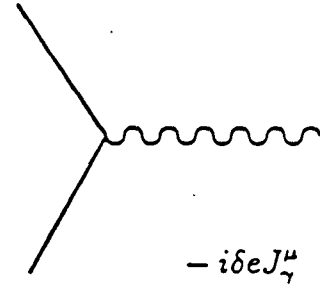
$$i\frac{g}{c}J_Z^\mu \left(\frac{-i}{q^2 - m_Z^2} \right) iA_Z(q^2)$$

e.



$$i\frac{g}{c}J_Z^\mu \left(\frac{-i}{q^2 - m_Z^2} \right) i\delta m_{ZA}^2$$

f.



$$-i\delta e J_\gamma^\mu$$

Figure 2.1: Diagrams contributing to the coupling of a soft photon to an electron. All the wiggly lines are photons excepted as noted. Diagrams b. and e. have vertices generated by the counterterms. Altogether these diagrams must give a contribution of ieJ_γ^μ to $(-i)$ times the Feynman amplitude in the limit $q^2 \rightarrow 0$.

so the requirement is

$$\delta e = -\frac{e\Pi_\gamma(0)}{2} \quad (2.25)$$

In summary, δg , $\delta g'$, and δv^2 are determined by

$$\begin{aligned} \delta \left[\frac{1}{4} v^2 (g^2 + g'^2) \right] &= \text{Re } A_{ZZ}(m_Z^2) \\ \delta \left[\frac{1}{4} v^2 g^2 \right] &= \text{Re } A_{WW}(m_W^2) \\ \frac{2}{e} \delta e &= -\Pi_\gamma(0) \end{aligned} \quad (2.26)$$

2.3 Vacuum Polarization

To clarify and make explicit the preceding we shall adopt a model in which the vacuum polarization is given simply by fermion loops with just leptons and quarks circulating. This is certainly a crude approximation for the hadronic contributions. Indeed a large uncertainty is introduced into the calculation through the choice of quark masses, especially for the light quarks. However, this model does allow us to see concretely what is really happening. Moreover, for the heavy quarks it is probably a reliable calculation.

We compute the diagram in Fig. 2.2 allowing the currents at the couplings to be different (e.g., J_γ^μ and J_Z^μ) and allowing different masses for the two portions of the fermion loop (if the current is charged).

We proceed by dimensional regularization with

$$\begin{aligned} \int \frac{d^n k}{(k^2 - m^2)^\alpha} &= \frac{i\pi^{n/2}}{(m^2)^{\alpha-n/2}} \frac{(-1)^\alpha \Gamma(\alpha - n/2)}{\Gamma(\alpha)} \\ \int \frac{d^n k k^2}{(k^2 - m^2)^\alpha} &= \frac{i\pi^{n/2}}{(m^2)^{\alpha-1-n/2}} \frac{(-1)^{\alpha-1} \Gamma(\alpha - 1 - n/2)}{\Gamma(\alpha)} \frac{n}{2} \\ \int \frac{d^n k k^\mu k^\nu}{(k^2 - m^2)^\alpha} &= \frac{i\pi^{n/2}}{(m^2)^{\alpha-1-n/2}} \frac{(-1)^{\alpha-1} \Gamma(\alpha - 1 - n/2)}{\Gamma(\alpha)} \frac{g^{\mu\nu}}{2} \end{aligned} \quad (2.27)$$

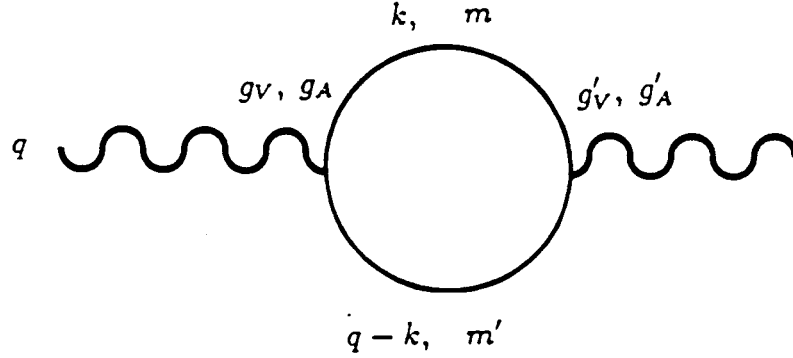


Figure 2.2: The vacuum polarization diagram for a fermion loop in a vector propagator.

Now we have

$$i\Pi_{\mu\nu}(q^2) = - \int \frac{d^n k}{(2\pi)^n} \text{Tr} \frac{i}{\not{k} - m} (-i)\gamma_\mu (g_V + g_A \gamma_5) \frac{i}{-(\not{q} - \not{k}) - m'} (-i)\gamma_\nu (g'_V + g'_A \gamma_5) \quad (2.28)$$

This is handled in the usual way by introducing a Feynman parameter and setting

$$\mu^2 = xm^2 + (1-x)m'^2 + (x^2 - x)q^2 \quad (2.29)$$

In this way we find

$$\begin{aligned} A(q^2) = & -\pi^{n/2} (2\pi)^{-n} n \Gamma(2 - \frac{n}{2}) \int_0^1 dx (\mu^2)^{\frac{n}{2}-2} \\ & \times \left\{ (g_V g'_V + g_A g'_A) \left[-\frac{1}{2}m^2 - \frac{1}{2}m'^2 - (\frac{1}{2} - x)(m^2 - m'^2) + 2x(1-x)q^2 \right] \right. \\ & \left. + (g_V g'_V - g_A g'_A) m m' \right\} \quad (2.30) \end{aligned}$$

The currents for Z and γ are diagonal so $m = m'$, $g_V = g'_V$, $g_A = g'_A$ and

$$\begin{aligned} A^{diag}(q^2) = & -\pi^{n/2} (2\pi)^{-n} n \Gamma(2 - \frac{n}{2}) \int_0^1 dx (\mu^2)^{\frac{n}{2}-2} \\ & \times \left\{ (g_V^2 + g_A^2) [-m^2 + 2x(1-x)q^2] + (g_V^2 - g_A^2) m^2 \right\} \end{aligned}$$

$$\begin{aligned}
&= -\pi^{n/2}(2\pi)^{-n}n\Gamma(2-\frac{n}{2})\int_0^1 dx(\mu^2)^{\frac{n}{2}-2} \\
&\quad \times \left\{ (g_A^2 + g_V^2)[+2x(1-x)q^2] - 2g_A^2 m^2 \right\}
\end{aligned} \tag{2.31}$$

For the charged weak current $g_V = g'_V = -g_A = -g'_A = g/2\sqrt{2}$

$$\begin{aligned}
A_{WW}(q^2) &= -\pi^{n/2}(2\pi)^{-n}n\Gamma(2-\frac{n}{2})\frac{g^2}{4}\int_0^1 dx(\mu^2)^{\frac{n}{2}-2} \\
&\quad \times \left[-\frac{1}{2}(m^2 + m'^2) - (\frac{1}{2} - x)(m^2 - m'^2) + 2x(1-x)q^2 \right]
\end{aligned} \tag{2.32}$$

In dimensional regularization we expand

$$\begin{aligned}
(\mu)^{\frac{n}{2}-2} &= \exp[(\frac{n}{2}-2)\ln \mu^2] \\
&= 1 + \left(\frac{n}{2}-2\right)\ln \mu^2 + \dots
\end{aligned} \tag{2.33}$$

Thus we write

$$\begin{aligned}
A_{WW}(q^2) &= -\pi^{n/2}(2\pi)^{-n}n\Gamma(2-\frac{n}{2})\frac{g^2}{4}\left[-\frac{1}{2}(m^2 + m'^2) + \frac{1}{3}q^2 \right] \\
&\quad + 4\pi^2(2\pi)^{-4}\frac{g^2}{4}\int_0^1 dx \ln \mu^2 \left[-\frac{1}{2}(m^2 + m'^2) \right. \\
&\quad \left. - (\frac{1}{2} - x)(m^2 - m'^2) + 2x(1-x)q^2 \right]
\end{aligned} \tag{2.34}$$

Here we have used

$$\left(\frac{n}{2}-2\right)\Gamma(2-\frac{n}{2}) = -\Gamma(3-\frac{n}{2}) \rightarrow -1 \tag{2.35}$$

We define three functions

$$\begin{aligned}
f(m^2, m'^2, q^2) &= \int_0^1 dx \ln \mu^2 \\
g(m^2, m'^2, q^2) &= \int_0^1 dx x(1-x) \ln \mu^2 \\
h(m^2, m'^2, q^2) &= \int_0^1 dx (\frac{1}{2} - x) \ln \mu^2
\end{aligned} \tag{2.36}$$

The last of these vanishes if $m^2 = m'^2$.

In terms of these Eq. (2.34) gives

$$\begin{aligned}
A_{WW}(q^2) = & -\pi^{n/2}(2\pi)^{-n}n\Gamma(2-\frac{n}{2})\frac{g^2}{4}\left[-\frac{1}{2}(m^2+m'^2)+\frac{1}{3}q^2\right] \\
& +\frac{g^2}{16\pi^2}\left[+2q^2g(m^2,m'^2,q^2)-\frac{1}{2}(m^2+m'^2)f(m^2,m'^2,q^2)\right. \\
& \left.-(m^2-m'^2)h(m^2,m'^2,q^2)\right]
\end{aligned} \tag{2.37}$$

The function $A_{\gamma\gamma}$ is similarly computed from Eq. (2.31) in terms of the fermion's charge, Q :

$$\begin{aligned}
A_{\gamma\gamma}(q^2) = & -\pi^{n/2}(2\pi)^{-n}n\Gamma(2-\frac{n}{2})\frac{1}{3}q^2e^2Q^2 \\
& +2q^2\frac{e^2Q^2}{4\pi^2}g(m^2,m'^2,q^2)
\end{aligned} \tag{2.38}$$

For the Z

$$\begin{aligned}
g_V &= \frac{g}{\cos\theta_W}\left(\frac{1}{2}T_{3L}-Qs^2\right) \\
g_A &= \frac{g}{\cos\theta_W}\left(-\frac{1}{2}T_{3L}\right)
\end{aligned} \tag{2.39}$$

and from Eq. (2.31)

$$\begin{aligned}
A_{ZZ}(q^2) = & -\pi^{n/2}(2\pi)^{-n}n\Gamma(2-\frac{n}{2})\frac{g^2}{c^2} \\
& \times\left[\frac{q^2}{3}\left(\frac{1}{2}T_{3L}^2-T_{3L}Qs^2+Q^2s^4\right)-\frac{1}{2}T_{3L}^2m^2\right] \\
& +\frac{1}{4\pi^2}\frac{g^2}{c^2}\left[2q^2\left(\frac{1}{2}T_{3L}^2-T_{3L}Qs^2+Q^2s^4\right)g(m^2,m'^2,q^2)\right. \\
& \left.-\frac{1}{2}T_{3L}^2m^2f(m^2,m'^2,q^2)\right]
\end{aligned} \tag{2.40}$$

Note that for $A_{\gamma Z}$ only the $g_V g_V$ term contributes and $A_{\gamma Z} \propto q^2$ so $A_{\gamma Z}(0) = 0$ as mentioned before.

The functions A_{ZZ} , A_{WW} , and $A_{\gamma\gamma}$ occur in the counterterms. If we calculate some physical amplitudes with our one-loop corrections we will need to include the effects of the counterterms. They should just cancel the divergences of the one-loop corrections and leave us with a finite answer.

The counterterm coefficients δg , $\delta g'$, and δv^2 are determined by Eq. (2.26)

$$\begin{aligned}\delta \ln\left[\frac{1}{4}v^2g^2\right] &= \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2} \\ \delta \ln\left[\frac{1}{4}v^2(g^2 + g'^2)\right] &= \frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} \\ \delta \ln\left(\frac{1}{g^2} + \frac{1}{g'^2}\right) &= \Pi_\gamma(0)\end{aligned}\quad (2.41)$$

which are linear in δg , $\delta g'$, and δv^2 . We find

$$\begin{aligned}\delta g &= -\frac{e}{2s}\Pi_\gamma(0) - \frac{ec^2}{2s^3}\left[\frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2}\right] \\ \delta g' &= -\frac{e}{2c}\Pi_\gamma(0) + \frac{e}{2c}\left[\frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2}\right]\end{aligned}\quad (2.42)$$

We can evaluate these explicitly. Using Eqs. (2.38) and (2.40), from each fermion there is a contribution

$$\begin{aligned}\Pi_\gamma(0) &= \pi^{n/2}(2\pi)^{-n}n\Gamma(2 - \frac{n}{2})\frac{1}{3}e^2Q^2 - \frac{e^2Q^2}{4\pi^2}2g(m^2, m^2, 0) \\ \frac{A_{ZZ}(m_Z^2)}{m_Z^2} &= \pi^{n/2}(2\pi)^{-n}n\Gamma(2 - \frac{n}{2})\frac{g^2}{c^2}\left[-\frac{1}{3}\left(\frac{1}{2}T_{3L}^2 - T_{3L}Qs^2 + Q^2s^4\right) + \frac{m^2}{2m_Z^2}T_{3L}^2\right] \\ &\quad + \frac{1}{4\pi^2}\frac{g^2}{c^2}\left[\left(\frac{1}{2}T_{3L}^2 - T_{3L}Qs^2 + Q^2s^4\right)2g(m^2, m^2, m_Z^2) \right. \\ &\quad \left. - \frac{1}{2}T_{3L}^2\frac{m^2}{m_Z^2}f(m^2, m^2, m_Z^2)\right]\end{aligned}\quad (2.43)$$

For a doublet, which could be leptons or quarks

$$\begin{aligned}\frac{A_{WW}(m_W^2)}{m_W^2} &= \pi^{n/2}(2\pi)^{-n}n\Gamma(2 - \frac{n}{2})\frac{g^2}{4}\left(-\frac{1}{3} + \frac{m^2 + m'^2}{2m_W^2}\right) \\ &\quad + \frac{g^2}{16\pi^2}\left[2g(m^2, m'^2, m_W^2) - \frac{1}{2}\frac{m^2 + m'^2}{m_W^2}f(m^2, m'^2, m_W^2) \right. \\ &\quad \left. - \frac{m^2 - m'^2}{m_W^2}h(m^2, m'^2, m_W^2)\right]\end{aligned}\quad (2.44)$$

From these we compute, with Q' the charge of the $T_3 = 1/2$ member of the doublet and $Q = Q' - 1$ the charge of the $T_3 = -1/2$ element,

$$\begin{aligned}
& \frac{A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{A_{WW}(m_W^2)}{m_W^2} \\
&= \pi^{n/2} (2\pi)^{-n} n \Gamma(2 - \frac{n}{2}) \frac{g^2}{c^2} \left[+\frac{1}{12} s^2 - \frac{1}{3} (Q^2 + Q'^2) s^4 \right] \\
&+ \frac{g^2}{4\pi^2 c^2} \left[\frac{1}{4} g(m'^2, m'^2, m_Z^2) + \frac{1}{4} g(m^2, m^2, m_Z^2) - \frac{1}{2} g(m^2, m'^2, m_W^2) \right. \\
&- \frac{1}{8} \frac{m'^2}{m_Z^2} f(m'^2, m'^2, m_Z^2) - \frac{1}{8} \frac{m^2}{m_Z^2} f(m^2, m^2, m_Z^2) + \frac{1}{8} \frac{m^2 + m'^2}{m_Z^2} f(m^2, m'^2, m_W^2) \\
&\quad \left. + \frac{m^2 - m'^2}{4m_Z^2} h(m^2, m'^2, m_W^2) \right] \\
&+ \frac{g^2}{4\pi^2 c^2} s^2 \left[Q g(m'^2, m'^2, m_Z^2) - Q' g(m^2, m^2, m_Z^2) + \frac{1}{2} g(m^2, m'^2, m_W^2) \right] \\
&+ \frac{g^2}{4\pi^2 c^2} s^4 \left[+2Q'^2 g(m'^2, m'^2, m_Z^2) + 2Q^2 g(m^2, m^2, m_Z^2) \right] \tag{2.45}
\end{aligned}$$

2.4 The Fermi Constant

We have not finished the renormalization program. Because $\alpha \cong 1/137.06$, $G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2}$, and m_Z will be the best known electroweak physical constants, we want to use them to fix g, g' , and v^2 . Now we have already dealt with $\delta g, \delta g'$, and δv^2 *implicitly* by defining them in terms of quantities that involve m_Z and m_W . In our scheme m_Z will be taken as fundamental and m_W will be predicted as a function of α, G_F , and m_Z .

Now we have the exact relations

$$\begin{aligned}
\frac{1}{e^2} &= \frac{1}{g^2} + \frac{1}{g'^2} \\
m_Z^2 &= \frac{1}{4} (g^2 + g'^2) v^2 \tag{2.46}
\end{aligned}$$

We need one more relation and that comes from relating G_F to the μ decay rate.

In lowest order, the rate for $\mu \rightarrow e \nu \bar{\nu}$ is

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{2.47}$$

ignoring m_e/m_μ . The purely electromagnetic corrections give

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3} \left[1 + \frac{\alpha}{2\pi} \left(\frac{25}{4} - \pi^2 \right) + \dots \right] \quad (2.48)$$

Now the electroweak theory gives the decay amplitude in lowest order

$$\mathcal{M} = \frac{g^2}{8m_W^2} \bar{e} \gamma_\mu (1 - \gamma_5) \nu_e \bar{\nu}_\mu \gamma^\mu (1 - \gamma_5) \mu \quad (2.49)$$

and comparing with the usual V-A theory we find

$$\frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}} \quad (2.50)$$

The diagrams for the electroweak corrections generated by the vacuum polarization we consider in this section are shown in Fig. 2.3. These are distinct from the electromagnetic corrections that appear in Eq. (2.48).

Their sum is

$$\Delta \mathcal{M} = \mathcal{M} \left[-\frac{A_W(0)}{m_W^2} + \frac{\delta m_W^2}{m_W^2} - 2 \frac{\delta g}{g} \right] \quad (2.51)$$

Now since

$$\delta m_W^2 = \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2} \quad (2.52)$$

and

$$\frac{\delta g}{g} = -\Pi_\gamma(0) - \frac{c^2}{2s^2} \left[\frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2} \right] \quad (2.53)$$

we find

$$\begin{aligned} \Delta \mathcal{M} = \mathcal{M} & \left[\frac{\text{Re } A_{WW}(m_W^2) - A_{WW}(0)}{m_W^2} + \Pi_\gamma(0) \right. \\ & \left. + \frac{c^2}{s^2} \left(\frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2} \right) \right] \end{aligned} \quad (2.54)$$

Now of course this better turn out to be finite. The terms that diverge as $n \rightarrow 4$ are

$$\begin{aligned} \bar{A}_{WW}(q^2) &= -\pi^{n/2} (2\pi)^{-n} n \Gamma(2 - \frac{n}{2}) \frac{g^2}{4} \left[-\frac{1}{2}(m^2 + m'^2) + \frac{1}{3}q^2 \right] \\ \bar{A}_\gamma(q^2) &= -\pi^{n/2} (2\pi)^{-n} n \Gamma(2 - \frac{n}{2}) e^2 Q^2 \frac{1}{3} q^2 \\ \bar{A}_{ZZ}(q^2) &= -\pi^{n/2} (2\pi)^{-n} n \Gamma(2 - \frac{n}{2}) \\ &\quad \times \frac{g^2}{c^2} \left[\frac{1}{3} q^2 \left(\frac{1}{2} T_{3L}^2 - T_{3L} Q s^2 + Q^2 s^4 \right) - \frac{1}{2} T_{3L}^2 m^2 \right] \end{aligned} \quad (2.55)$$

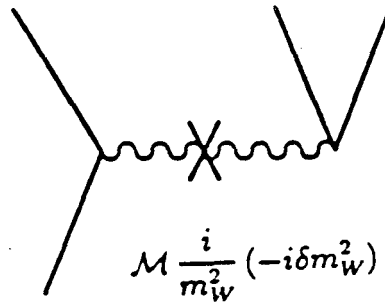
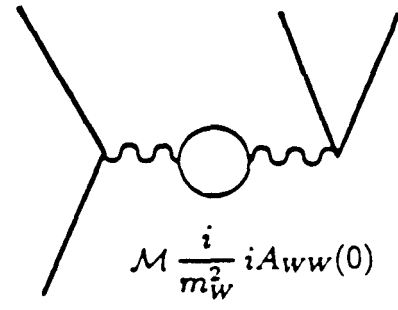
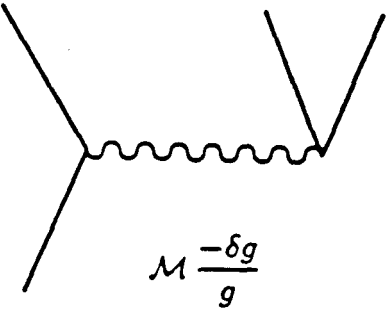
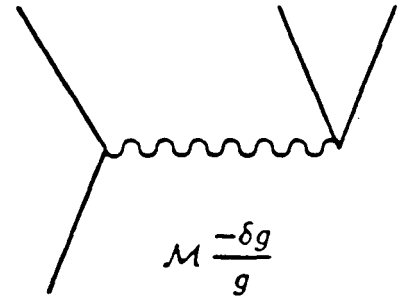
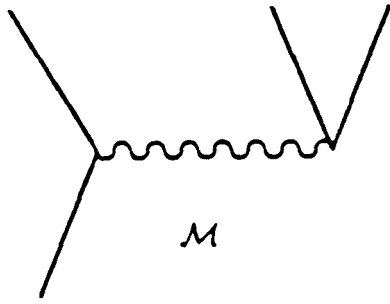


Figure 2.3: The diagrams contributing to muon decay. There are vertices generated by counterterms in the second and third diagrams.

From these we determine

$$\begin{aligned}
\frac{\text{Re } \bar{A}_{WW}(m_W^2) - \bar{A}_{WW}(0)}{m_W^2} &= -\pi^{n/2}(2\pi)^{-n}n\Gamma(2 - \frac{n}{2})\frac{g^2}{4} \cdot \frac{1}{3} \\
\bar{\Pi}_{\gamma\gamma}(0) &= \pi^{n/2}(2\pi)^{-n}n\Gamma(2 - \frac{n}{2})e^2(\frac{1}{2} + \frac{Y^2}{2})\frac{1}{3} \\
\frac{\text{Re } \bar{A}_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } \bar{A}_{WW}(m_W^2)}{m_W^2} &= -\pi^{n/2}(2\pi)^{-n}n\Gamma(2 - \frac{n}{2})\frac{g^2}{c^2} \\
&\quad \times \left[-\frac{1}{12}s^2 + \left(\frac{1}{6} + \frac{Y^2}{6} \right) s^4 \right]
\end{aligned} \tag{2.56}$$

where for the $\gamma\gamma$ and ZZ channels we summed over the two members of an $SU(2)$ doublet. The total contribution of these terms to $\Delta\mathcal{M}$ is

$$\begin{aligned}
\Delta\mathcal{M} &= \mathcal{M} \left[-\pi^{n/2}(2\pi)^{-n}n\Gamma(2 - n/2)g^2 \right] \\
&\quad \times \left\{ \frac{1}{12} - s^2 \left(\frac{1}{6} + \frac{Y^2}{6} \right) + \frac{1}{s^2} \left[-\frac{1}{12}s^2 + \left(\frac{1}{6} + \frac{Y^2}{6} \right) s^4 \right] \right\} \\
&= 0.
\end{aligned} \tag{2.57}$$

For the finite pieces that remain we find

$$\begin{aligned}
&\frac{\text{Re } A_{WW}(m_W^2)}{m_W^2} - \frac{A_{WW}(0)}{m_W^2} \\
&= \frac{g^2}{16\pi^2} \left\{ 2g(m^2, m'^2, m_W^2) - \frac{m^2 + m'^2}{2m_W^2} [f(m^2, m'^2, m_W^2) - f(m^2, m'^2, 0)] \right. \\
&\quad \left. - \frac{m^2 - m'^2}{m_W^2} [h(m^2, m'^2, m_W^2) - h(m^2, m'^2, 0)] \right\} \\
\Pi_{\gamma\gamma}(0) &= -\frac{e^2}{4\pi^2} [2Q^2g(m^2, m'^2, 0) + 2Q'^2g(m'^2, m'^2, 0)]
\end{aligned}$$

$$\frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2}$$

$$\begin{aligned}
&= \frac{g^2}{4\pi^2 c^2} \left[\frac{1}{4} g(m'^2, m'^2, m_Z^2) + \frac{1}{4} g(m^2, m^2, m_Z^2) - \frac{1}{2} g(m^2, m'^2, m_W^2) \right. \\
&\quad - \frac{1}{8} \frac{m'^2}{m_Z^2} f(m'^2, m'^2, m_Z^2) - \frac{1}{8} \frac{m^2}{m_Z^2} f(m^2, m^2, m_Z^2) + \frac{1}{8} \frac{m^2 + m'^2}{m_Z^2} f(m^2, m'^2, m_W^2) \\
&\quad \left. + \frac{m^2 - m'^2}{4m_Z^2} h(m^2, m'^2, m_W^2) \right] \\
&+ \frac{g^2}{4\pi^2 c^2} s^2 \left[Q g(m'^2, m'^2, m_Z^2) - Q' g(m^2, m^2, m_Z^2) + \frac{1}{2} g(m^2, m'^2, m_W^2) \right] \\
&+ \frac{g^2}{4\pi^2 c^2} s^4 \left[+2Q'^2 g(m'^2, m'^2, m_Z^2) + 2Q^2 g(m^2, m^2, m_Z^2) \right] \tag{2.58}
\end{aligned}$$

Altogether the change in the decay amplitude is $\Delta \mathcal{M} = \mathcal{M} \Delta r$ with

$$\begin{aligned}
\Delta r &= \frac{g^2}{4\pi^2 s^2} \left\{ \frac{1}{4} \left[g(m^2, m^2, m_Z^2) + g(m'^2, m'^2, m_Z^2) - 2g(m^2, m'^2, m_W^2) \right] \right. \\
&\quad - \frac{1}{8} \left[\frac{m^2}{m_Z^2} f(m^2, m^2, m_Z^2) + \frac{m'^2}{m_Z^2} f(m'^2, m'^2, m_Z^2) - \frac{m^2 + m'^2}{m_Z^2} f(m^2, m'^2, m_W^2) \right] \\
&\quad \left. + \frac{m^2 - m'^2}{4m_Z^2} h(m^2, m'^2, m_W^2) \right\} \\
&+ \frac{g^2}{4\pi^2} \left\{ Q g(m^2, m^2, m_Z^2) - Q' g(m'^2, m'^2, m_Z^2) + g(m^2, m'^2, m_W^2) \right. \\
&\quad - \frac{1}{8} \frac{m^2 + m'^2}{m_W^2} \left[f(m^2, m'^2, m_W^2) - f(m^2, m'^2, 0) \right] \\
&\quad \left. - \frac{1}{4} \frac{m^2 - m'^2}{m_W^2} \left[h(m^2, m'^2, m_W^2) - h(m^2, m'^2, 0) \right] \right\} \\
&+ \frac{e^2}{4\pi^2} \left\{ 2Q^2 \left[g(m^2, m^2, m_Z^2) - g(m^2, m^2, 0) \right] \right. \\
&\quad \left. + 2Q'^2 \left[g(m'^2, m'^2, m_Z^2) - g(m'^2, m'^2, 0) \right] \right\} \tag{2.59}
\end{aligned}$$

This is a formidable expression but we can make some sense out of it. First suppose that we regard m_W and m_Z as not too different and also suppose that m and m' , the masses of the $T_3 = -1/2$ and $T_3 = +1/2$ elements of the fermion isodoublet are similar and $m, m' \ll m_W, m_Z$. In this limit

$$\Delta r \approx \frac{e^2}{4\pi^2} \cdot 2 \left\{ \left[g(m^2, m^2, m_Z^2) - g(m^2, m^2, 0) \right] Q^2 \right.$$

$$+ \left[g(m'^2, m'^2, m_Z^2) - g(m'^2, m'^2, 0) \right] Q'^2 \} \quad (2.60)$$

We recall that

$$\begin{aligned} g(m^2, m'^2, q^2) &= \int_0^1 dx \, x(1-x) \ln \mu^2 \\ \mu^2 &= (1-x)m^2 + xm'^2 + (x^2-x)q^2 \end{aligned} \quad (2.61)$$

In fact, we really want only the real part of each function because only real functions appear in Δr

$$g(m^2, m^2, m_Z^2) - g(m^2, m^2, 0) = \int_0^1 dx \, x(1-x) \ln \left[1 - x(1-x) \frac{m_Z^2}{m^2} \right] \quad (2.62)$$

For $m_Z^2/m^2 < 4$ the integral is real. For $m_Z^2/m^2 > 4$ the integral becomes complex. For $a < 4$

$$\begin{aligned} &\int_0^1 dx \, x(1-x) \ln [1 - ax(1-x)] \\ &= \frac{1}{6} \left\{ -\frac{5}{3} - \frac{4}{a} + \left(\frac{4}{a} + 2 \right) \sqrt{\frac{4}{a} - 1} \sin^{-1} \sqrt{\frac{a}{4}} \right\} \end{aligned} \quad (2.63)$$

while for $a > 4$

$$\begin{aligned} &\int_0^1 dx \, x(1-x) \ln [1 - ax(1-x)] \\ &= \frac{1}{6} \left\{ -\frac{5}{3} - \frac{4}{a} + \left(\frac{4}{a} + 2 \right) \sqrt{1 - \frac{4}{a}} \left[\ln \left(\sqrt{\frac{a}{4}} + \sqrt{\frac{a}{4} - 1} \right) \pm \frac{i\pi}{2} \right] \right\} \end{aligned} \quad (2.64)$$

where the sign of $i\pi/2$ depends on whether a is above or below the cut. For $m_Z^2/m^2 \gg 1$

$$\text{Re} \left[g(m^2, m^2, m_Z^2) - g(m^2, m^2, 0) \right] \rightarrow \frac{1}{6} \left[-\frac{5}{3} + \ln \frac{m_Z^2}{m^2} \right] \quad (2.65)$$

Altogether

$$\Delta r \approx \frac{2\alpha}{\pi} \sum_f Q_f^2 \left(-\frac{5}{18} + \frac{1}{6} \ln \frac{m_Z^2}{m_f^2} \right) \quad (2.66)$$

	e	μ	τ	u	d	s	c	b	t
mass (GeV)	.0005	.106	1.8	0.15	0.15	0.35	1.5	5.0	50
$\Delta r \times 10^2$	1.7	0.9	0.5	1.2	0.3	0.2	0.7	0.1	0.0

Table 2.1: Table 2.1. Contributions to Δr from the simple formula using the fermion masses. The total here is about 0.055.

where the summation extends over all fermions, $e, \mu, \tau, u, d, s, c, b, t$ and of course a factor three must be supplied for the quarks since there are three colors. What values should be used for the quark masses? There is no precise answer. The vacuum polarization A_γ, A_W , depends on real physical states and our quark model is only a crude approximation. A careful treatment uses dispersion relations and data from $e^+e^- \rightarrow \text{hadrons}$, which are connected through

$$\text{Im } \Pi_\gamma^{\text{had}}(q^2) = \frac{q^2 \sigma_{e^+e^- \rightarrow \text{hadrons}}(q^2)}{4\pi} \quad (2.67)$$

and

$$\Pi_\gamma(q^2) - \Pi_\gamma(0) = \frac{\alpha}{3\pi} \int_0^\infty dq'^2 R(q'^2) \frac{q^2}{q'^2(q'^2 - q^2)} \quad (2.68)$$

Here $R(q'^2)$ is the ratio of the e^+e^- annihilation cross section into hadrons divided by that into $\mu^+\mu^-$:

$$R(q'^2) = \frac{3q'^2}{4\pi\alpha^2} \sigma_{e^+e^- \rightarrow \text{hadrons}}(q'^2) \quad (2.69)$$

It is convenient to use

$$\Pi_\gamma(0) = \Pi_\gamma(0) - \Pi_\gamma(q^2) + \Pi_\gamma(q^2) \quad (2.70)$$

and take q^2 spacelike.

Since our interest is pedagogical, we ignore these niceties and instead use a simple model with $m_u = m_d = 150 \text{ MeV}$, $m_s = 350 \text{ MeV}$, $m_c = 1.5 \text{ GeV}$, $m_b = 5 \text{ GeV}$, $m_t = 50 \text{ GeV}$. For the known quarks this gives thresholds at about the right place. The contributions we find this way are given in Table 2.1.

Because of the correction Δr , which we have only approximated (the more careful calculation gives about $\Delta r = 0.070$) we see that the Born level result

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{1}{2v^2} \quad (2.71)$$

should be replaced by

$$\frac{G_F}{\sqrt{2}} = \frac{1}{2v^2}(1 + \Delta r) \quad (2.72)$$

This may be added to the relations

$$\begin{aligned} m_Z^2 &= \frac{1}{4}(g^2 + g'^2)v^2 \\ \frac{1}{e^2} &= \frac{1}{g^2} + \frac{1}{g'^2} \end{aligned} \quad (2.73)$$

These then determine g, g' , and v^2 . In particular we find

$$m_W^2 = \frac{1}{2} \left(1 + \sqrt{1 - \frac{4\pi\alpha(1 + \Delta r)}{\sqrt{2}m_Z^2 G_F}} \right) m_Z^2 \quad (2.74)$$

With $m_Z = 93.0$ GeV, $\Delta r = 0.070$ we find $m_W = 82.1$ GeV. The sensitivity to Δr is given

$$\frac{\partial m_W}{\partial \Delta r} = -\frac{\sin^2 2\theta_W}{8 \cos 2\theta_W \cos \theta_W} \frac{m_Z}{\cos \theta_W} \approx -16.4 \text{ GeV} \quad (2.75)$$

A shift in Δr could come from heavy quarks or leptons as explored in the next section.

2.5 Heavy t-quark

Let us suppose the t-quark is very heavy, so heavy that we can ignore m_W or m_Z by comparison. Then in the expression for Δr it is the terms accompanied by m^2/m_W^2 or m^2/m_Z^2 that dominate. We approximate

$$\begin{aligned} f(m_t^2, m_t^2, m_Z^2) &\approx f(m_t^2, m_t^2, 0) \\ f(m_b^2, m_t^2, m_W^2) &\approx f(m_b^2, m_t^2, 0) \\ h(m_b^2, m_t^2, m_W^2) &= h(m_b^2, m_t^2, 0) \end{aligned} \quad (2.76)$$

The contribution to Δr is

$$\begin{aligned} \Delta r^t &= \frac{3g^2}{4\pi^2 s^2} \left\{ -\frac{1}{8} \left[\frac{m_b^2}{m_Z^2} f(m_b^2, m_b^2, 0) + \frac{m_t^2}{m_Z^2} f(m_t^2, m_t^2, 0) - \frac{m_b^2 + m_t^2}{m_Z^2} f(m_b^2, m_t^2, 0) \right] \right. \\ &\quad \left. + \frac{(m_b^2 - m_t^2)}{4m_Z^2} h(m_b^2, m_t^2, 0) \right\} \end{aligned} \quad (2.77)$$

From the definition of f and h

$$\begin{aligned}
f(m^2, m'^2, 0) &= \ln m^2 \\
f(m^2, m'^2, 0) &= \int_0^1 dx \ln[xm'^2 + (1-x)m^2] \\
&= \frac{m'^2 \ln m'^2 - m^2 \ln m^2}{m'^2 - m^2} - 1 \\
h(m^2, m'^2, 0) &= \int_0^1 dx \left(\frac{1}{2} - x\right) \ln[xm'^2 + (1-x)m^2] \\
&= \frac{m^2 m'^2}{2(m'^2 - m^2)^2} \ln \frac{m'^2}{m^2} - \frac{1}{4} \frac{m^2 + m'^2}{m'^2 - m^2} \quad (2.78)
\end{aligned}$$

Substituting these values we find for the contribution of a heavy t-quark,

$$\begin{aligned}
\Delta r &= -\frac{3G_F c^2}{8\pi^2 \sqrt{2} s^2} \left\{ m_t^2 + m_b^2 - \frac{2m_t^2 m_b^2}{m_t^2 - m_b^2} \ln \frac{m_t^2}{m_b^2} \right\} \\
&\approx -1.1 \times 10^{-6} [m_t(\text{ GeV})]^2 \quad (2.79)
\end{aligned}$$

Using Eq. (2.75) we find [12]

$$\Delta m_W = \frac{3}{16\pi^2 \sqrt{2}} G_F m_t^2 \frac{\cos^2 \theta_W}{\cos 2\theta_W} \approx 180 \text{ MeV} \left(\frac{m_t}{100 \text{ GeV}} \right)^2 \quad (2.80)$$

where of course this is really only reliable for m_t large compared to m_W .

2.6 Corrections to $e^+e^- \rightarrow \mu^+\mu^-$

The left-right asymmetry, which can be measured with longitudinally polarized Z s, is also affected by radiative corrections: the presence of heavy quarks or new particles in the vacuum polarization loops might be detected if very high precision measurements are made. To derive relations for the change in the left-right asymmetry due to radiative corrections we consider explicitly $e^+e^- \rightarrow \mu^+\mu^-$, though the observable A_{LR} does not depend on the choice of the final state fermion. Indeed, because all the final states have the same left-right asymmetry on the Z peak this measurement is especially attractive.

The Feynman diagrams required are shown in Fig.2.4. In the figure all the wiggly lines are Z s except where γ is indicated. Only diagrams with at least one resonant (Z) propagator need be considered.

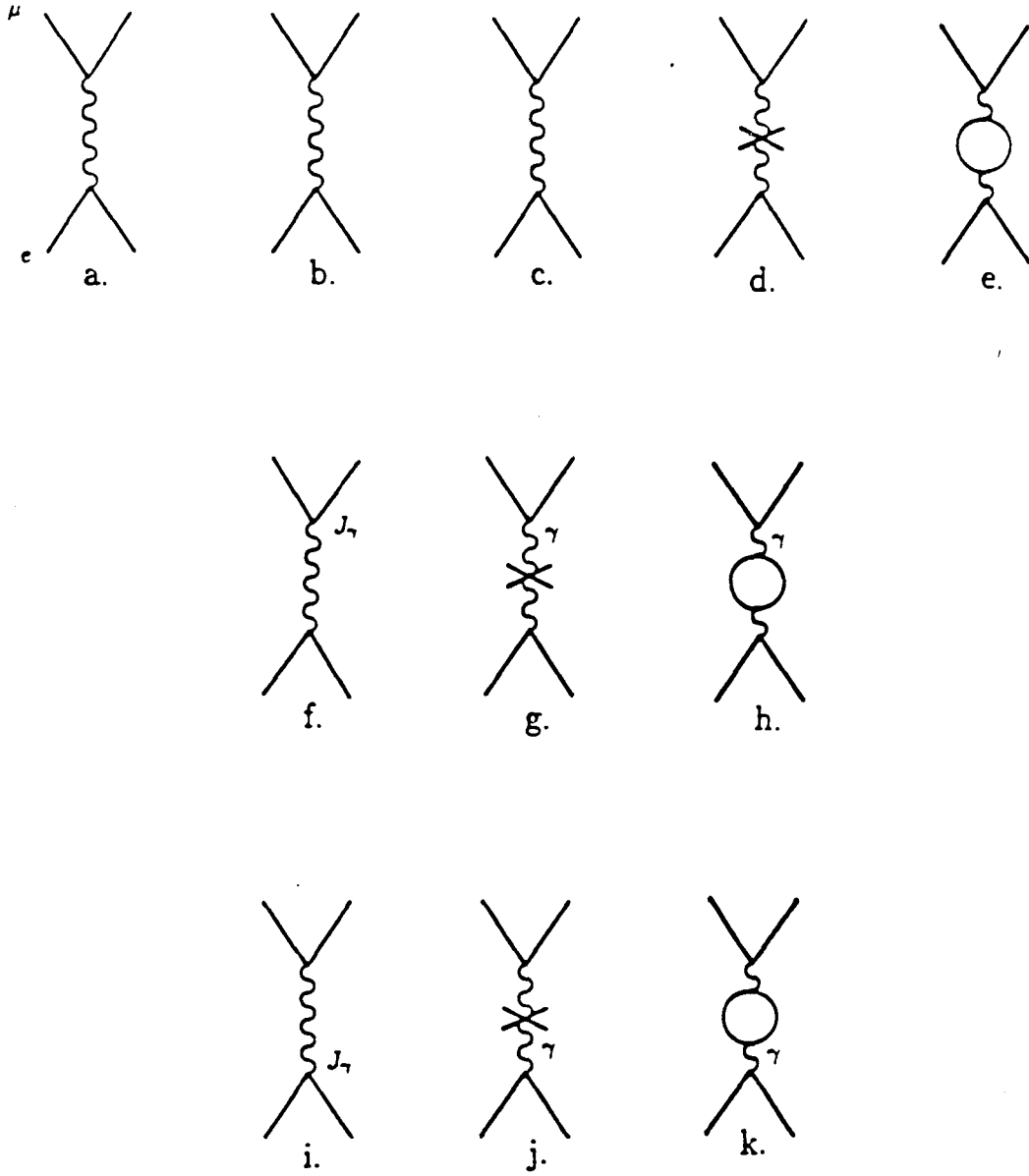


Figure 2.4: Diagrams contributing to $e^+e^- \rightarrow \mu^+\mu^-$. All wiggly lines are Z except where noted. In diagrams b. and c. the Z couples at one end or the other through a counterterm induced coupling.

Consider first diagrams a-e in which the couplings are all through J_Z . Each diagram contains a factor $J_Z^\epsilon J_Z^\mu$. If we indicate the cm energy squared by q^2 and use

$$\begin{aligned} c &= \cos \theta_W \\ s &= \sin \theta_W \end{aligned} \quad (2.81)$$

then for the Born term we have

$$-i\mathcal{M} = \frac{ig^2}{c^2} \frac{J_Z^\epsilon J_Z^\mu}{q^2 - m_Z^2} \quad (2.82)$$

Diagrams b-e give

$$-i\mathcal{M} = \frac{ig^2}{c^2} \frac{J_Z^\epsilon J_Z^\mu}{q^2 - m_Z^2} \left[-\frac{2c}{g}(c\delta g + s\delta g') - \frac{\delta m_Z^2}{q^2 - m_Z^2} + \frac{A_{ZZ}(q^2)}{q^2 - m_Z^2} \right] \quad (2.83)$$

Now

$$\delta m_Z^2 = \text{Re} A_{ZZ}(m_Z^2) \quad (2.84)$$

so we write

$$A_{ZZ}(q^2) = A_{ZZ}(m_Z^2) + (q^2 - m_Z^2)A'_{ZZ}(m_Z^2) + \mathcal{O}((q^2 - m_Z^2)^2) \quad (2.85)$$

Thus the first five diagrams give

$$-i\mathcal{M} = \frac{ig^2}{c^2} \frac{J_Z^\epsilon J_Z^\mu}{q^2 - m_Z^2} \left[1 - \frac{2c}{g}(c\delta g + s\delta g') + \frac{i \text{Im} A_{ZZ}(m_Z^2)}{q^2 - m_Z^2} + A'_{ZZ}(m_Z^2) \right] \quad (2.86)$$

plus nonresonant terms. The $\text{Im} A_{ZZ}(m_Z^2)$ is reabsorbed by writing

$$\begin{aligned} & \frac{1}{q^2 - m_Z^2} \left(1 + \frac{i \text{Im} A_{ZZ}(m_Z^2)}{q^2 m_Z^2} \right) \\ & \rightarrow \frac{1}{q^2 - m_Z^2 + i \text{Im} A_{ZZ}(m_Z^2)} \end{aligned} \quad (2.87)$$

From the relations for δg and $\delta g'$

$$\begin{aligned} \delta g &= \frac{-e\Pi_\gamma(0)}{2s} - \frac{ec^2}{2s^3} \left[\frac{\text{Re} A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re} A_{WW}(m_W^2)}{m_W^2} \right] \\ \delta g' &= \frac{-e\Pi_\gamma(0)}{2c} + \frac{e}{2c} \left[\frac{\text{Re} A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re} A_{WW}(m_W^2)}{m_W^2} \right] \end{aligned} \quad (2.88)$$

it follows that

$$c\delta g + s\delta g' = \frac{-e\Pi_\gamma(0)}{2sc} + \frac{e}{2} \left(\frac{s^2 - c^2}{cs^3} \right) \left[\frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2} \right] \quad (2.89)$$

Altogether for diagrams a through e

$$\begin{aligned} -i\mathcal{M} = & \frac{ig^2}{c^2} \frac{J_Z^\epsilon J_Z^\mu}{g^2 - m_Z^2 + i \text{Im } A_{ZZ}(m_Z^2)} \left[1 + A'_{ZZ}(m_Z^2) \right. \\ & \left. + \Pi_\gamma(0) - \frac{s^2 - c^2}{s^2} \left(\frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2} \right) \right] \quad (2.90) \end{aligned}$$

We need this to be finite. The divergent pieces are (summed over the two elements of a doublet)

$$\bar{A}'_{ZZ}(m_Z^2) = -\pi^{\frac{n}{2}} (2\pi)^{-n} n \Gamma(2 - \frac{n}{2}) \frac{1}{3} \frac{g^2}{c^2} \left[\frac{1}{4} - \frac{1}{2} s^2 + (Q^2 + Q'^2) s^4 \right]$$

$$\bar{\Pi}_\gamma(0) = \pi^{n/2} (2\pi)^{-n} n \Gamma(2 - \frac{n}{2}) \frac{1}{3} e^2 [Q^2 + Q'^2]$$

$$\frac{\bar{A}_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\bar{A}_{WW}(m_W^2)}{m_W^2} = \pi^{n/2} (2\pi)^{-n} \Gamma(2 - \frac{n}{2}) \frac{g^2}{c^2} \left[\frac{s^2}{12} - \frac{1}{3} s^4 (Q^2 + Q'^2) \right] \quad (2.91)$$

Indeed it follows that the sum of diagrams a-e is finite.

Diagrams f, g, and h give contributions proportional to $J_Z^\epsilon J_\gamma^\mu$. Diagrams i, j, and k give analogous contributions proportional to $J_\gamma^\epsilon J_Z^\mu$. The former are

$$\begin{aligned} -i\mathcal{M}_f &= J_Z^\epsilon J_\gamma^\mu i s c (s\delta g - c\delta g') \frac{-i}{q^2 - m_Z^2} \frac{-ig}{c} \\ -i\mathcal{M}_g &= J_Z^\epsilon J_\gamma^\mu (-ie) \frac{-i}{q^2} (i\delta m_{ZA}^2) \frac{-i}{q^2 - m_Z^2} \frac{-ig}{c} \\ -i\mathcal{M}_h &= J_Z^\epsilon J_\gamma^\mu (-ie) \frac{-i}{q^2} [iA_{ZA}(q^2)] \frac{-i}{q^2 - m_Z^2} \frac{-ig}{c} \quad (2.92) \end{aligned}$$

Since we are retaining only the resonant pieces, it is good enough to replace q^2 by m_Z^2 except in the expression $q^2 - m_Z^2$. Using

$$\begin{aligned}\delta m_{ZA}^2 &= \frac{m_Z^2}{\sqrt{g^2 + g'^2}}(c\delta g' - s\delta g) \\ -i\mathcal{M}_{f+g+h} &= J_Z^\epsilon J_\gamma^\mu \frac{-i}{q^2 - m_Z^2} \frac{-ig}{c} i \left\{ 2sc(s\delta g - c\delta g') - \frac{eA_{ZA}(m_Z^2)}{m_Z^2} \right\}\end{aligned}\quad (2.93)$$

Summed over the two elements of a doublet,

$$\begin{aligned}\frac{A_{ZA}(q^2)}{q^2} &= \frac{eg}{c} \left[\frac{1}{4} - s^2(Q^2 + Q'^2) \right] \left[-\frac{1}{3}\pi^{n/2}(2\pi)^{-n}n\Gamma(2 - \frac{n}{2}) \right] \\ &\quad + \frac{eg}{2\pi^2 c} Q' \left(\frac{1}{4} - Q's^2 \right) g(m^2, m'^2, q^2) \\ &\quad + \frac{eg}{2\pi^2 c} Q \left(-\frac{1}{4} - Qs^2 \right) g(m^2, m'^2, q^2)\end{aligned}\quad (2.94)$$

While

$$2sc(s\delta g - c\delta g') = -\frac{ec}{s} \left(\frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2} \right) \quad (2.95)$$

whose divergent part is

$$-\frac{eg^2 s}{c} \pi^{n/2} (2\pi)^{-n} n \Gamma(2 - \frac{n}{2}) \left[\frac{1}{12} - \frac{1}{3} s^2 (Q^2 + Q'^2) \right] \quad (2.96)$$

Thus the term proportional to $J_Z^\epsilon J_\gamma^\mu$ is finite.

Let us isolate the leading contribution for $m_t \gg m_W$. These come entirely from

$$\frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2} \quad (2.97)$$

and in fact solely from the $f(m^2, m'^2, q^2)$ and $h(m^2, m'^2, q^2)$ pieces in Eq. (2.58).

$$\begin{aligned}\frac{\text{Re } A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\text{Re } A_{WW}(m_W^2)}{m_W^2} &\longrightarrow \\ &\frac{g^2}{16\pi^2 m_W^2} \left[-\frac{1}{2} m^2 f(m^2, m^2, m_Z^2) - \frac{1}{2} m'^2 f(m'^2, m'^2, m_Z^2) \right]\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}(m^2 + m'^2)f(m^2, m'^2, m_W^2) + (m^2 - m'^2)h(m^2, m'^2, m_W^2)] \\
& \longrightarrow \frac{g^2}{16\pi^2 m_W^2} \left[-\frac{1}{4}(m^2 + m'^2) + \frac{m^2 m'^2 \ln m'^2/m^2}{2(m'^2 - m^2)} \right] \quad (2.98)
\end{aligned}$$

Thus the t-quark contribution is, including a factor of three for the color sum,

$$-i\mathcal{M}_{f+g+h} \approx J_Z^c j_\gamma^\mu \frac{-i}{q^2 - m_Z^2} \left(\frac{-ig}{c} \right)^2 \left(\frac{-3c^2}{8\pi^2} \right) \frac{G_F m_t^2}{\sqrt{2}} \quad (2.99)$$

with an analogous contribution for $J_\gamma^c J_Z^\mu$. The full amplitude then is proportional to

$$J_Z^c J_Z^\mu [1 - \mathcal{O}(\alpha)] + (J_Z^c J_\gamma^\mu + J_\gamma^c J_Z^\mu) \left(\frac{-3c^2}{8\pi^2} \frac{G_F m_t^2}{\sqrt{2}} \right) \quad (2.100)$$

so the effect is to replace

$$J_Z \rightarrow J_Z - \frac{3c^2}{8\pi^2} \frac{G_F m_t^2}{\sqrt{2}} J_\gamma \quad (2.101)$$

Now initially

$$J_Z \propto T_{3L} - QX \quad (2.102)$$

where X is just the usual

$$x_W = s^2 = \frac{g'^2}{g^2 + g'^2} \quad (2.103)$$

The left-right asymmetry is determined by the relative strengths of the T_{3L} and Q pieces in the current. Thus the t-quark contribution induces a change in X

$$\delta X = \frac{3c^2}{8\pi^2} \frac{G_F m_t^2}{\sqrt{2}} \quad (2.104)$$

There is, however, an additional contribution that comes from the shift in m_W^2 , Eq. (2.75). Since the t-quark produces a shift

$$\frac{\delta m_W^2}{m_Z^2} = \frac{3G_F m_t^2}{8\pi^2 \sqrt{2}} \frac{\cos^4 \theta_W}{\cos 2\theta_W} \quad (2.105)$$

This, itself gives a shift in $x_W = 1 - m_W^2/m_Z^2$

$$\delta x_W = -\frac{3G_F m_t^2}{8\pi^2 \sqrt{2}} \frac{\cos^4 \theta_W}{\cos 2\theta_W} \quad (2.106)$$

Together they yield

$$\delta X = -\frac{3G_F m_t^2}{8\pi^2 \sqrt{2}} \frac{s^2 c^2}{c^2 - s^2} = \frac{-3G_F m_t^2}{8\pi^2 \sqrt{2}} \frac{x_W(1 - x_W)}{1 - 2x_W} \quad (2.107)$$

Now

$$A_{LR} = \frac{\frac{1}{4} - X}{\frac{1}{4} - X + 2X^2} \quad (2.108)$$

So we find [12]

$$\begin{aligned} \delta A_{LR} &= \frac{2X^2 - X}{\left[2X^2 - X + \frac{1}{4}\right]^2} \delta X \\ &= \frac{3G_F m_t^2}{8\pi^2 \sqrt{2}} \frac{X^2(1 - X)}{\left[2X^2 - X + \frac{1}{4}\right]^2} \\ &= 3 \frac{4\alpha s^2}{\pi \left[1 + 16(s^2 - \frac{1}{4})^2\right]^2} \frac{m_t^2}{m_Z^2} \end{aligned} \quad (2.109)$$

For example, with $s^2 = 0.222$, $m_Z = 93$ GeV

$$\delta A_{LR} = 0.028(m_t/200 \text{ GeV})^2 \quad (2.110)$$

Such effects might actually be observed.

3 Extra Z s

In the previous sections we have detailed the predictions of the standard model for the Z , regarding as part of the standard model the yet-to-be-found t -quark and the Higgs boson. If deviations from the standard model of electroweak interactions are to be found in e^+e^- annihilation at the Z , they are likely to arise from mixing of the standard Z with an additional neutral gauge boson. Such mixing would shift the mass of the lighter Z downward and would modify its couplings to the fermions. These changes would produce deviations from the predictions of the standard model, predictions based on the measured mass of the Z . In practice, the measured Z mass would be used to infer the value of the weak mixing angle, or equivalently $x_W \equiv \sin^2 \theta_W$. Because of the mass shift, the inferred value of x_W would be greater than the true value, which could be

determined if a very precise measurement of the W mass were available. The deviations between the values of the observables at the Z and the standard model predictions depend then on the mixing between the Z s, characterized by an angle λ , and the difference δx_W between the true value of x_W and the value $\bar{x}_W$ inferred from the measured mass of the Z .

At the Z several quantities will be measured with precision. These include the total width, the forward-backward asymmetry for the $\mu^+\mu^-$ final state, the left-right asymmetry for incident polarized electrons and the ratio of charged leptonic to hadronic final final states. The deviations caused in these quantities by the existence of an additional Z depend on the couplings of that Z . The class of possible Z s is too large to be studied in general, but it is possible to study three particular choices motivated by E_6 , superstring-inspired models. To test the standard model requires comparison with an alternative and these models provide convenient examples.

3.1 E_6 generated Z s

The electroweak symmetry $SU(2) \times U(1)$ may well be part of a much larger symmetry encompassing $SU(3)_{color}$ as well. The most economical possibility is the $SU(5)$ of Georgi and Glashow. A more recent approach derives from superstrings. There a gauge group E_6 breaks down ultimately to $SU(3) \times SU(2) \times U(1)$. One way this may occur is through the chain $E_5 \rightarrow SO(10) \times U(1), SO(10) \rightarrow SU(5) \times U(1)$. Each $U(1)$ has a corresponding neutral gauge boson, a Z .

How can we determine the couplings of these extra Z s? This is just an exercise in group theory. We sketch here the techniques based on Dynkin diagrams that provide a simple and powerful means for analyzing such problems[13].

We shall proceed by referring to the well-known group $SU(3)$. The eight standard generators of $SU(3)$ are $T_3, Y, T_\pm, U_\pm, V_\pm$. The first two have the special property that the other six are eigenvectors of the action of commutation with the first two:

$$[aT_3 + bY, T_\pm] = \pm aT_\pm$$

$$\begin{aligned}
[aT_3 + bY, U_{\pm}] &= \pm(-\frac{1}{2}a + b)U_{\pm} \\
[aT_3 + bY, V_{\pm}] &= \pm(\frac{1}{2}a + b)V_{\pm}
\end{aligned} \tag{3.1}$$

while T_3 and Y commute with each other. The space spanned by linear combinations of T_3 and Y is called the *Cartan subalgebra*. The raising and lowering operators are eigenvectors of the operation of commutation with an element of the Cartan subalgebra. The eigenvalue depends linearly on the element of the Cartan subalgebra chosen. Put more abstractly, each raising or lowering operator has associated with it a linear function, α , that acts on elements H of the Cartan subalgebra to give a real number $\alpha(H)$ (namely, the eigenvalue: $[H, E_{\alpha}] = \alpha(H)E_{\alpha}$). Thus the α associated with T_+ has the properties $\alpha_{T_+}(T_3) = 1$ and $\alpha_{T_+}(Y) = 0$.

The α s are really vectors (in the dual space of the Cartan subalgebra) and it is possible to define a scalar product on the space they span. The vectors α are called *roots*. It is possible to select a particular basis $\{H_{\alpha}\}$ for the Cartan subalgebra labeled by the roots α . In this basis

$$[H_{\beta}, E_{\alpha}] = \beta \cdot \alpha E_{\alpha} \tag{3.2}$$

It is easy to see that

$$[E_{\alpha}, E_{\beta}] \propto E_{\alpha+\beta} \tag{3.3}$$

since

$$\begin{aligned}
[H_{\gamma}, [E_{\alpha}, E_{\beta}]] &= -[E_{\alpha}, [E_{\beta}, H_{\gamma}]] - [E_{\beta}, [H_{\gamma}, E_{\alpha}]] \\
&= \beta \cdot \gamma [E_{\alpha}, E_{\beta}] + \alpha \cdot \gamma [E_{\alpha}, E_{\beta}]
\end{aligned} \tag{3.4}$$

which follows from the Jacobi identity, $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$. The vectors α are independent of the normalization chosen for the E_{α} .

The dimension of the Cartan subalgebra is called the *rank* of the group. The roots are vectors with this same dimension. For $SU(3)$ the rank is two so we can conveniently represent these vectors. The roots form a familiar hexagon as shown in Fig. 3.1.

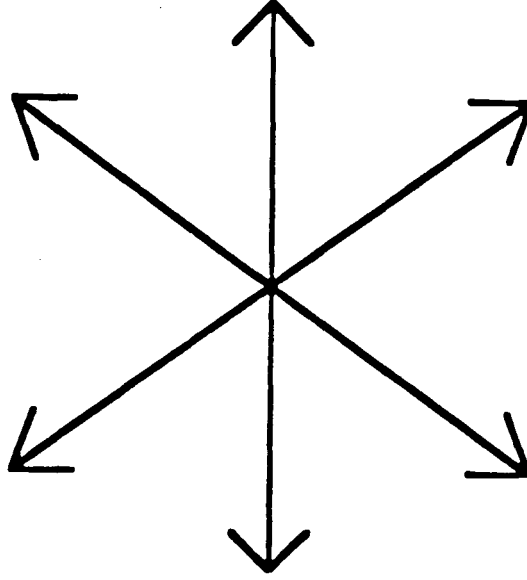


Figure 3.1: The root diagram for $SU(3)$.

Since the root space for $SU(3)$ is two dimensional we can choose two independent roots as a basis. Suppose we define an ordering of vectors in this space as follows. Let α_1 and α_2 be the distinguished basis and let any vector be expressed as

$$\gamma = C_1\alpha_1 + C_2\alpha_2 \quad (3.5)$$

Now if

$$\gamma' = C'_1\alpha_1 + C'_2\alpha_2 \quad (3.6)$$

we say $\gamma' > \gamma$ if $C'_2 > C_2$. If $C'_1 = C_1$ and $C'_2 > C_2$ we say $\gamma' > \gamma$. This defines an ordering of all the vectors in this space. In particular α_1 and α_2 are positive: $\alpha_1 > 0, \alpha_2 > 0$ Now the simple roots are those that are positive and that cannot be written as the sum of two positive roots. It is always possible to take a basis of simple roots. For $SU(3)$ this means taking α_1 and α_2 120° apart. *Note that the basis of simple roots is generally not an orthogonal basis.*

The Cartan matrix is given by

$$A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} \quad (3.7)$$

where α_i, α_j are simple roots.

For the groups we consider all the roots have the same magnitude so A_{ij} is symmetric. This is not true in general. It is easy to see that for $SU(3)$ the Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (3.8)$$

because $\cos 120^\circ = -1/2$.

In addition to the group itself we must consider representations of the group. A representation is a set of matrices corresponding to the generators of the group such that the commutation relations of the matrices are the same as those of the corresponding generators. The matrices act on a vector space whose elements we indicate by ϕ . It is possible to diagonalize the matrices corresponding to the Cartan subalgebra (because all its elements commute) and we write

$$H_\beta \phi_M = \beta \cdot M \phi_M \quad (3.9)$$

The *weight* M is a vector in the space spanned by the roots. It is easy to show that

$$E_\alpha \phi_M \propto \phi_{M+\alpha} \quad (3.10)$$

unless it is zero. Actually for a fixed weight M there may be more than one independent weight. This cannot happen for a root. There is only one independent generator E_α such that $[H_\beta, E_\alpha] = \beta \cdot \alpha E_\alpha$.

For a given irreducible representation there is a highest weight Λ , that is $\Lambda > M$ for every other weight M . Every weight can be indicated by a set of numbers

$$M_i = \frac{2M \cdot \alpha_i}{\alpha_i \cdot \alpha_i} \quad (3.11)$$

that turn out always to be integers. For the highest weight, the M_i are always non-negative. All the weights less than Λ are obtained by subtracting roots successively. An algorithm enables us to do such calculations efficiently.

Consider for example the 8-dimensional representation of $SU(3)$. The highest weight is $\Lambda = \alpha_1 + \alpha_2$ and $\Lambda_1 = 2\Lambda \cdot \alpha_1 / \alpha_1 \cdot \alpha_1 = 2(1 - 1/2)/1 = \Lambda_2 = 1$. We write the vector (1,1) beneath the Cartan matrix as shown in Figure 3.2.

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\boxed{1 \quad 1}$$

$$\boxed{-1 \quad 2} \quad \boxed{2 \quad -1}$$

$$\boxed{0 \quad 0}$$

$$\boxed{-2 \quad 1} \quad \boxed{1 \quad -2}$$

$$\boxed{-1 \quad -1}$$

Figure 3.2: The algorithm for finding the weights of a representation from the components of the highest weight. Each box corresponds to a weight M and the elements are the values of $M_i = 2M \cdot \alpha_i / \alpha_i \cdot \alpha_i$.

Now the rule is if there is a positive element in the first component subtract the first row of the Cartan matrix, i.e., $(1, 1) - (2, -1) = (-1, 2)$. This gives us the components of the weight $\Lambda - \alpha_1$. Similarly for the second component. Now in the next row we find for the first weight a component $+2$ in the second place so we subtract the second row twice, giving $(0, 0)$ and $(1, -2)$. The process terminates at $(-1, -1)$. All representations can be constructed in this fashion.

Now we know $SU(3)$ can break to the subgroup $SU(2) \times U(1)$. The hypercharge that generates the $U(1)$ points perpendicular to the $SU(2)$ direction. (Generally, if we have a product $G_1 \times G_2$ with α_1 a root for G_1 and α_2 a root for G_2 then $\alpha_1 \cdot \alpha_2 = 0$. This is so because if H_{α_1} is an element of the Cartan subalgebra of G_1 and E_{α_2} is a generator of G_2 they must commute, but $[H_{\alpha_1}, E_{\alpha_2}] = \alpha_1 \cdot \alpha_2 E_{\alpha_2} = 0$.) If α_1 is associated with the $SU(2)$ direction and β with the Y direction, $\beta \cdot \alpha_1 = 0$. Thus from the Cartan matrix we see

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\boxed{1 \quad 1} (3)$$

$$\boxed{-1 \quad 2} (3) \quad \boxed{2 \quad -1} (0)$$

$$\boxed{0 \quad 0} (0)$$

$$\boxed{-2 \quad 1} (0) \quad \boxed{1 \quad -2} (-3)$$

$$\boxed{-1 \quad -1} (-3)$$

Figure 3.3: The weights for the octet representation of $SU(3)$ together with their hypercharge quantum numbers. The hypercharge (with some normalization) is $M_1 + 2M_2$, i.e. the sum of the first element in the box and twice the second.

that $\beta \propto \alpha_1 + 2\alpha_2$. To calculate the values of Y for the weights of the octet representation we calculate $\beta \cdot \mathbf{M}$ for each weight $\mathbf{M}$. Now

$$\begin{aligned}
2\beta \cdot \mathbf{M} &= 2(\alpha_1 + 2\alpha_2) \cdot \mathbf{M} \\
&= M_1 + 2M_2
\end{aligned} \tag{3.12}$$

if $\alpha_1 \cdot \alpha_1 = \alpha_2 \cdot \alpha_2 = 1$. In this way we find for $2\beta \cdot \mathbf{M}$ the array shown in Figure 3.3.

There are actually two states at the center (like π^0 and η). Thus we find an isodoublet with $2\beta \cdot \mathbf{M} = 3$, another with $2\beta \cdot \mathbf{M} = -3$ and an isotriplet plus an isosinglet with $2\beta \cdot \mathbf{M} = 0$.

These same techniques can be used with larger Lie algebras, including the exceptional algebras. Each Lie algebra has a Cartan matrix. The Cartan matrix can be represented by a Dynkin diagram. If the rank of the algebra is n , there

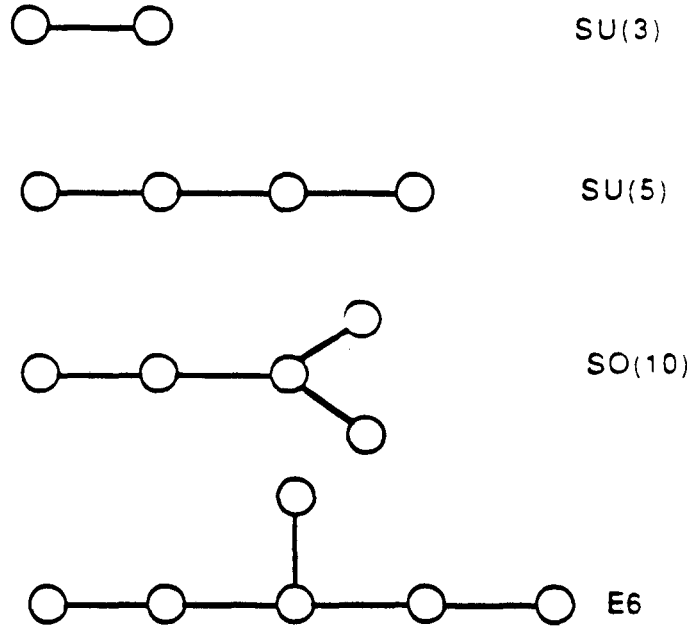


Figure 3.4: Dynkin diagrams for $SU(3)$, $SU(5)$, $SO(10)$, and E_6 .

are n dots. The i th and j th dots are joined by $A_{ij}A_{ji}$ lines. Thus for $SU(3)$ we have Figure 3.4.

Now we can see that $SO(10)$ can be obtained from E_6 simply by dropping a circle from either end of the long chain and $SU(5)$ can be obtained from $SO(10)$ by dropping either of the circles just after the branch.

Now we confront the question of interest. When $E_6 \rightarrow SO(10) \times U(1)_\psi$ and $SO(10) \rightarrow SU(5) \times U(1)_\chi$ what are the couplings of the Z_ψ and Z_χ to the fundamental fermions? This question is easy to answer for the breakdown of E_6 because all the standard fermions belong to the same irreducible representation of $SO(10)$: Z_ψ couples with equal strength to $e_L^-, e_L^+, u_L, \bar{u}_L \dots$ For $SO(10) \rightarrow SU(5) \times U(1)_\chi$ the calculation is more difficult.

We need to find a β for $SO(10)$ that is perpendicular to all the α_i 's in $SU(5)$. this is just like the $SU(3) \rightarrow SU(2) \times U(1)$ case. Now examining the

Cartan matrix for $SO(10)$

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad (3.13)$$

we see that the combination $2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4 + 5\alpha_5 \propto (0, 0, 0, 0, 1)$. This means that if

$$\beta \propto 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4 + 5\alpha_5 \quad (3.14)$$

then

$$\beta \cdot \alpha_i = 0 \quad (3.15)$$

for $i = 1, \dots, 4$. Now the $\alpha_i, i = 1, \dots, 4$ generate $SU(5)$ so this is just what we need. Now to determine the couplings of the Z_χ we must evaluate the χ charge for each fermion in the 16 dimensional representation of $SO(10)$. This charge is proportional to $2\beta \cdot M = 2M \cdot (2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4 + 5\alpha_5)$.

The representation is indicated in Figure 3.5.

We find 10 states with a value +1, five with the value -3 and one with the value +5. This corresponds to the decomposition of the 16 of $SO(10)$ into a 10, a $\bar{5}$, and a 1 of $SU(5)$. Now the generator T_3 is normalized so that $\text{Tr } T_3^2 = \frac{1}{4}(1 + 1 + 3 + 3) = 2$ where the e_L^- and ν_L^- contribute 1/4 each and u_L and d_L contribute 3/4 each. We want to normalize β similarly. As it stands, $\text{Tr } \beta^2 = 25 + 10 \cdot 1 + 5 \cdot 9 = 80$ so we need to take $Q_\chi = \beta/\sqrt{40}$. Thus the fermions in the 10 of $SU(5)$ have a χ charge of $1/(2\sqrt{10})$, while those in the $\bar{5}$ have a charge $-3/(2\sqrt{10})$. Of course these results are correct at the scale at which $SU(5)$ is the correct symmetry. They will be modified by renormalization group evolution to the low energy scale[14].

This brief sketch indicates how the charges of and extra Z might be related to an underlying theory[15]. In addition to Z_ψ and Z_χ , we shall also consider a particular combination of interest to string-based E_6 theories, Z_η , whose couplings are related to the other two by

$$Q_\eta = \sqrt{5/8}Q_\psi + \sqrt{3/8}Q_\chi \quad (3.16)$$

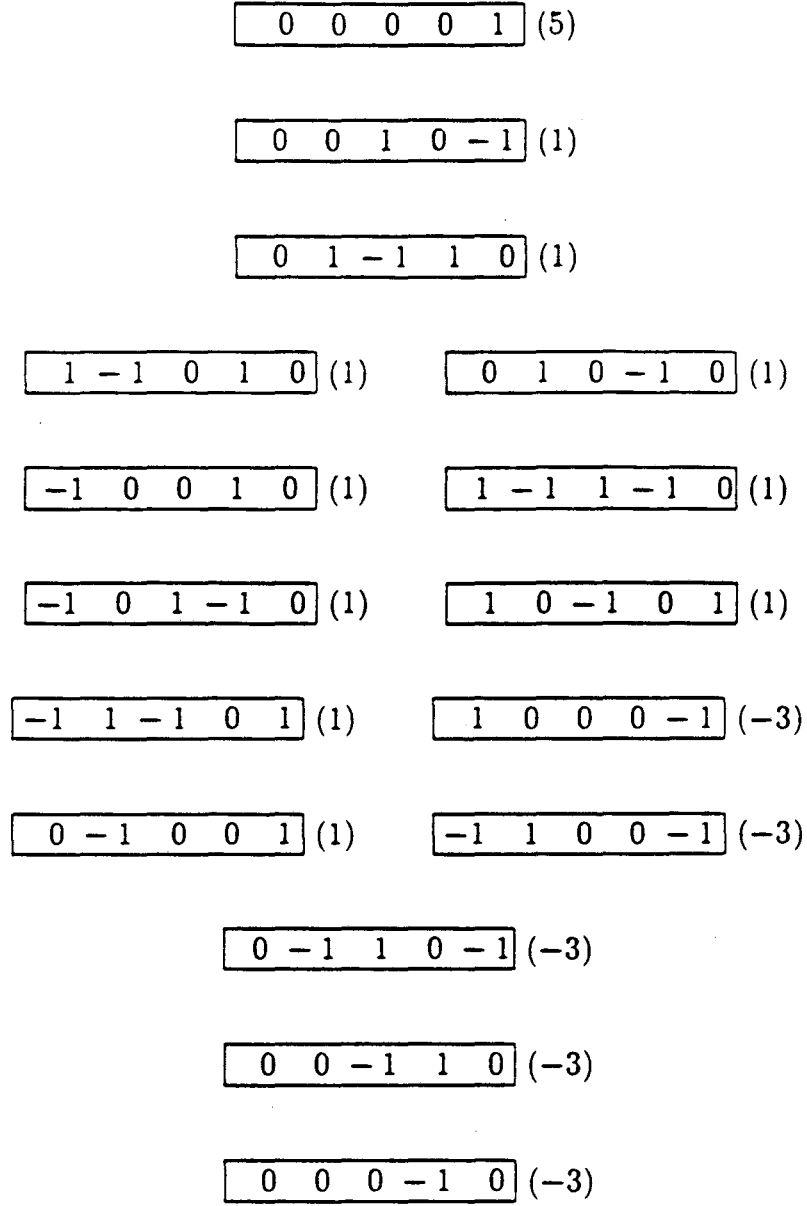


Figure 3.5: The 16 dimensional representation of $SO(10)$ and the quantum number associated with β . Each box represents a weight. The $U(1)$ quantum number in parentheses is obtained by multiplying the five elements inside a box by 2, 4, 6, 3, and 5 respectively, and taking the sum.

3.2 Z Mixing and the Mass Shift

Let us suppose that at low energies the electroweak interactions come not from a gauge theory based on $SU(2) \times U(1)$ but from one based on $SU(2) \times U(1) \times U(1)$ so there is an extra Z of the sort just discussed. Let the two Z s prior to mixing be denoted Z_{10} and Z_{20} and let their masses be M_{10} and M_{20} . The couplings to the left-handed fermion f are denoted by Q'_{10} and Q'_{20} . The convention for the mixing angle λ and the physical states Z_1 and Z_2 are defined by

$$\begin{aligned} Z_1 &= \cos \lambda Z_{10} + \sin \lambda Z_{20} \\ Z_2 &= -\sin \lambda Z_{10} + \cos \lambda Z_{20} \end{aligned} \quad (3.17)$$

The charges these states couple to are

$$\begin{aligned} Q'_1 &= \cos \lambda Q'_{10} - \sin \lambda Q'_{20} \\ Q'_2 &= \sin \lambda Q'_{10} + \cos \lambda Q'_{20} \end{aligned} \quad (3.18)$$

Here Q'_{10} is the usual electroweak neutral current coupling:

$$Q'_{10} = \frac{e}{\sin \theta_W \cos \theta_W} (T'_{3L} - \sin^2 \theta_W Q_{em}) \quad (3.19)$$

As a result of the mixing, the mass of the lower Z is decreased to its physical value, M_1

$$\begin{aligned} M_1^2 - M_{10}^2 &\approx -\lambda^2 (M_{20}^2 - M_{10}^2) \\ &\approx -\lambda^2 (M_2^2 - M_1^2) \end{aligned} \quad (3.20)$$

The standard model relations for the masses of the W and Z are

$$\begin{aligned} m_W^2 &= \frac{1}{2} \left(1 + \sqrt{1 - \frac{4\pi\alpha(1 + \Delta r)}{\sqrt{2}m_Z^2 G_F}} \right) m_Z^2 \\ \cos^2 \theta_W &= m_W^2 / m_Z^2 \end{aligned} \quad (3.21)$$

where Δr is the radiative correction discussed in Section 2 with the value 0.0713 ± 0.0013 if the t-quark mass is 45 GeV and the Higgs mass is 100 GeV [16]. Once the Z mass is determined to high precision, it will be used to infer the weak mixing angle, or equivalently, x_W . A precise measurement of the W would indeed lead to the correct value of x_W , but in the presence of Z mixing, the value inferred from the Z mass will be

$$\tilde{x}_W = \frac{1}{2} \left(1 + \sqrt{1 - \frac{4\pi\alpha(1 + \Delta r)}{\sqrt{2}M_1^2 G_F}} \right) \quad (3.22)$$

while the correct value is obtained from

$$\tilde{x}_W = \frac{1}{2} \left(1 + \sqrt{1 - \frac{4\pi\alpha(1 + \Delta r)}{\sqrt{2}M_{10}^2 G_F}} \right) \quad (3.23)$$

though of course M_{10} is cannot be directly measured. The mistake in the inferred value of x_W is thus

$$\begin{aligned} \delta x_W &= x_W - \tilde{x}_W \\ &\approx -\lambda^2 \cdot \frac{x_W(1 - x_W)}{1 - 2x_W} \cdot \frac{M_2^2 - M_1^2}{M_1^2} \end{aligned} \quad (3.24)$$

which is always negative.

Taking δx_W and λ as the two parameters for the mixing has some advantages over the standard choice of M_2^2 and λ . The deviations in the observable quantities in e^+e^- annihilation at the Z are linear in δx_W and λ , so each measurement yields a linear band as an allowed region in the $\delta x_W - \lambda$ plane. In addition, the allowed range will turn out to be a finite portion of the plane. The curves of fixed M_2^2 are parabolas and it is not hard to read the value of M_2 from the plot.

3.3 Partial Widths

All of the predictions of the standard model at the Z can be expressed in terms of the partial widths for $Z \rightarrow f_L \bar{f}_R$, where $f_L = e_L^-, e_L^+, u_L, \bar{u}_L$ etc. We assume universality among the three generations. The partial widths are simply

$$\Gamma_f \equiv \Gamma(Z \rightarrow f_L \bar{f}_R) = \frac{m_Z}{24\pi} Q_1^{f2} \quad (3.25)$$

assuming there is no phase space suppression. We shall assume that $Z \rightarrow t\bar{t}$ is kinematically forbidden. In contradistinction to Eq. 3.25, the partial width predicted from the measured Z mass would be

$$\tilde{\Gamma}_f = \frac{m_Z}{24\pi} Q_{10}^{f2} \quad (3.26)$$

with Q_{10}^f evaluated using $\tilde{x}_W$ in place of x_W . To first order in δx_W and λ ,

$$\begin{aligned} \frac{\delta\Gamma_f}{\Gamma_f} = \frac{\Gamma_f - \tilde{\Gamma}_f}{\Gamma_f} &= \lambda \frac{\partial \ln Q_1^{f2}}{\partial \lambda} + \delta x_W \frac{\partial \ln Q_{10}^{f2}}{\partial x_W} \\ &\equiv -2\lambda q^f + \delta x_W r^f \end{aligned} \quad (3.27)$$

where

$$q^f = \frac{Q_{20}^f}{Q_{10}^f} \quad (3.28)$$

is the ratio of the “new” Z charge of the fermion to the “old” Z charge, and

$$r^f = \frac{-1 + 2x_W}{x_W(1 - x_W)} - \frac{2Q_{em}^f}{T_{3L}^f - x_W Q_{em}^f} \quad (3.29)$$

Unlike q^f , r^f is independent of the couplings of the heavier Z . Values for r^f for the various fundamental fermions are given in Table 3.1. Values for q^f are given in Table 3.2 for three frequently considered models of the heavy Z : Z_χ , Z_ϕ , and Z_η .

From Eq. 3.27 it is clear that for each decay mode of the Z there is some straight line in the δx_W - λ plane along which the deviation, $\delta\Gamma$, vanishes. The same is true of the various observables like the total width and the asymmetries, which can be expressed in terms of the partial widths.

Table 3.1: Values of the neutral weak charge Q_{10}^f , the branching ratio b^f for the Z into $f\bar{f}$, the coefficient r^f , and its product with the branching ratio for the various fundamental fermions, f .

	Q_{10}^f	B^f	r^f	$r^f B^f$
e^-	-0.210	0.021	-10.413	-0.218
e^+	-0.168	0.013	5.790	0.077
ν	0.378	0.068	-3.219	-0.218
u	0.266	0.101	-7.007	-0.706
$\bar{u}$	0.112	0.018	5.790	0.103
d	-0.322	0.148	-4.784	-0.706
$\bar{d}$	-0.056	0.005	5.790	0.026

Table 3.2: Values of the ratio $q^f = Q_{20}^f/Q_{10}^f$ of the neutral weak charges for Z_χ , Z_ψ , and Z_η , and the product of q^f with the branching ratio, B^f of the Z into $f\bar{f}$.

	q_χ^f	q_ψ^f	q_η^f	$q_\chi^f B^f$	$q_\psi^f B^f$	$q_\eta^f B^f$
e^-	-1.038	-0.447	-0.283	-0.022	-0.009	-0.006
e^+	0.433	-0.559	0.707	0.006	-0.008	0.010
ν	0.577	0.248	0.157	0.039	0.017	0.011
u	-0.273	0.353	-0.446	-0.028	0.036	-0.045
$\bar{u}$	-0.650	0.839	-1.061	-0.012	0.016	-0.019
d	0.226	-0.291	0.369	0.033	-0.043	0.054
$\bar{d}$	-3.900	-1.678	-1.061	-0.017	-0.008	-0.005

3.4 Total Width

The total width is given by

$$\Gamma = 3(\Gamma_{e^-} + \Gamma_{e^+} + \Gamma_\nu + \Gamma_d + \Gamma_{\bar{d}}) + 2(\Gamma_u + \Gamma_{\bar{u}}) \quad (3.30)$$

It is simple to see that the deviation in the total width, that is, the difference between the standard model expectation based on the measured value of the Z mass, and the correct value is given by

$$\begin{aligned} \frac{\delta\Gamma}{\Gamma} = \frac{\Gamma - \tilde{\Gamma}}{\Gamma} = & -2\lambda \left[3(q^{e^-} B^{e^-} + q^{e^+} B^{e^+} + q^\nu B^\nu + q^d B^d + q^{\bar{d}} B^{\bar{d}}) \right. \\ & \left. + 2(q^u B^u + q^{\bar{u}} B^{\bar{u}}) \right] \\ & + \delta x_W \left[3(r^{e^-} B^{e^-} + r^{e^+} B^{e^+} + r^\nu B^\nu + r^d B^d + r^{\bar{d}} B^{\bar{d}}) \right. \\ & \left. + 2(r^u B^u + r^{\bar{u}} B^{\bar{u}}) \right] \end{aligned} \quad (3.31)$$

From Tables 3.1 and 3.2 we find for the three cases of Z_χ , Z_ϕ , and Z_η respectively,

$$\frac{\delta\Gamma}{\Gamma} = -2\lambda \begin{Bmatrix} 0.039 \\ -0.051 \\ 0.064 \end{Bmatrix} + \delta x_W(-4.11) \quad (3.32)$$

the total width of the Z will be measured at SLC and LEP to a precision of about ± 40 MeV, or $\pm 1.4\%$. Thus a measurement in perfect accord with the prediction of the standard model would provide a limit

$$\left| -2\lambda \begin{Bmatrix} 0.039 \\ -0.051 \\ 0.064 \end{Bmatrix} - 4.11\delta x_W \right| < 0.014 \quad (3.33)$$

This region is indicated on Figures 3.6, 3.7, and 3.8 for the cases of Z_χ , Z_ψ , and Z_η .

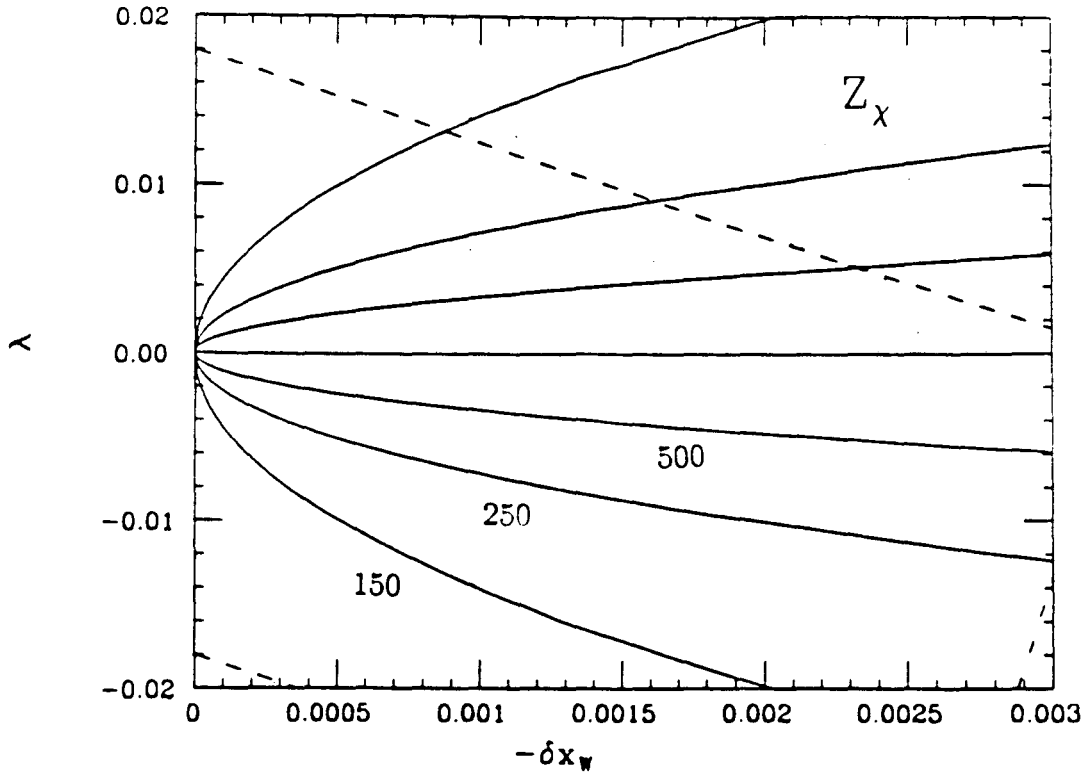


Figure 3.6: The $\delta x_W - \lambda$ plane showing the region that would be allowed by a measurement of R_μ to 1.7% (below the dotted line) and by a measurement of A_{LR} to ± 0.025 (between the dashed lines) for the Z_χ . The measurement of the total width provides no effective constraint, just cutting off the lower right corner. From outside to inside, the parabolas indicate $M_2 = 150$ GeV, 250 GeV, and 500 GeV.

3.5 Left-Right Asymmetry

A longitudinally polarized electron beam will make it possible to measure the left-right asymmetry in the production of the Z by e^+e^- annihilation. The asymmetry is defined in terms of the cross section to produce a Z with a completely polarized beam:

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} \quad (3.34)$$

This asymmetry depends only on the couplings of the Z to the electron and clearly is given by

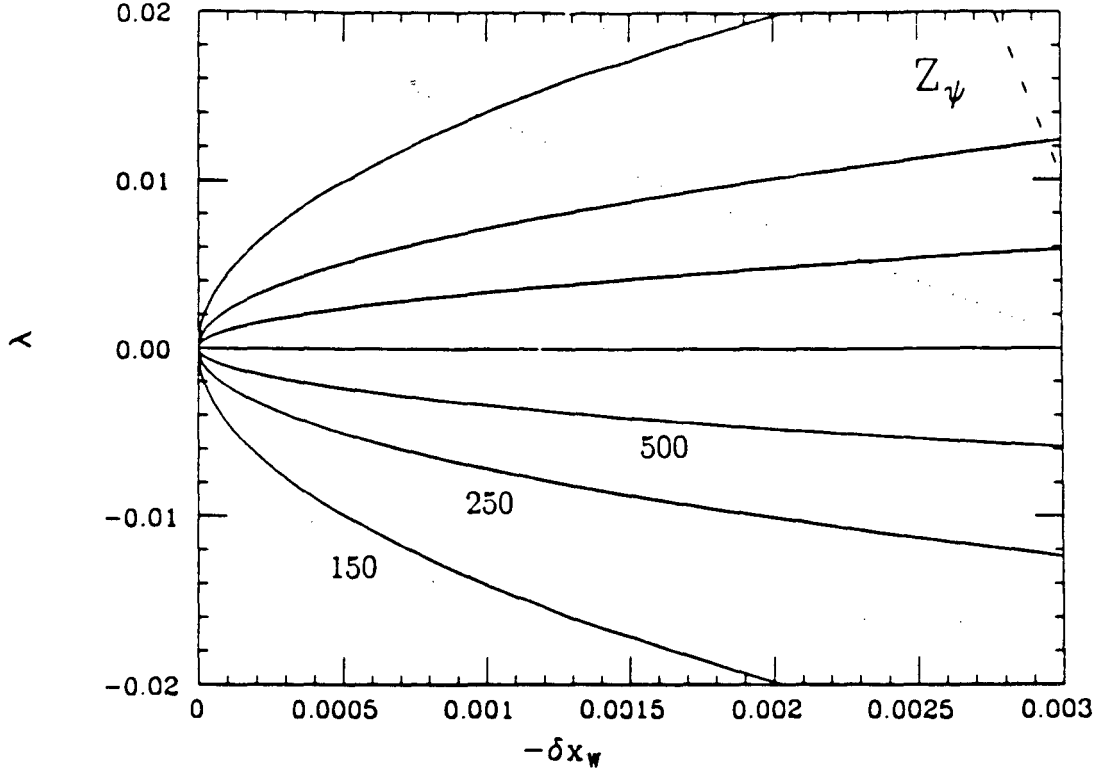


Figure 3.7: The $\delta x_W - \lambda$ plane showing the region that would be allowed by a measurement of R_μ to 1.7% (below the dotted line) and by a measurement of the total width to 40 MeV (all but the corner cut off by the dotdash line) for the Z_ψ . The measurement of A_{LR} provides no effective constraint. From outside to inside, the parabolas indicate $M_2 = 150$ GeV, 250 GeV, and 500 GeV.

$$A_{LR} = \frac{\Gamma_{e^-} - \Gamma_{e^+}}{\Gamma_{e^-} + \Gamma_{e^+}} \quad (3.35)$$

It follows that the deviation of the asymmetry is given by

$$\begin{aligned} \frac{\delta A_{LR}}{A_{LR}} &= \frac{A_{LR} - \tilde{A}_{LR}}{A_{LR}} \\ &= -2 \frac{\Gamma_{e^-} - \Gamma_{e^+}}{\Gamma_{e^-}^2 - \Gamma_{e^+}^2} \left(\frac{\delta \Gamma_{e^+}}{\Gamma_{e^+}} - \frac{\delta \Gamma_{e^-}}{\Gamma_{e^-}} \right) \\ &= -2 \frac{B^{e^-} B^{e^+}}{B^{e^-2} - B^{e^+2}} \left[-2\lambda(q^{e^+} - q^{e^-}) + \delta x_W(r^{e^+} - r^{e^-}) \right] \end{aligned} \quad (3.36)$$

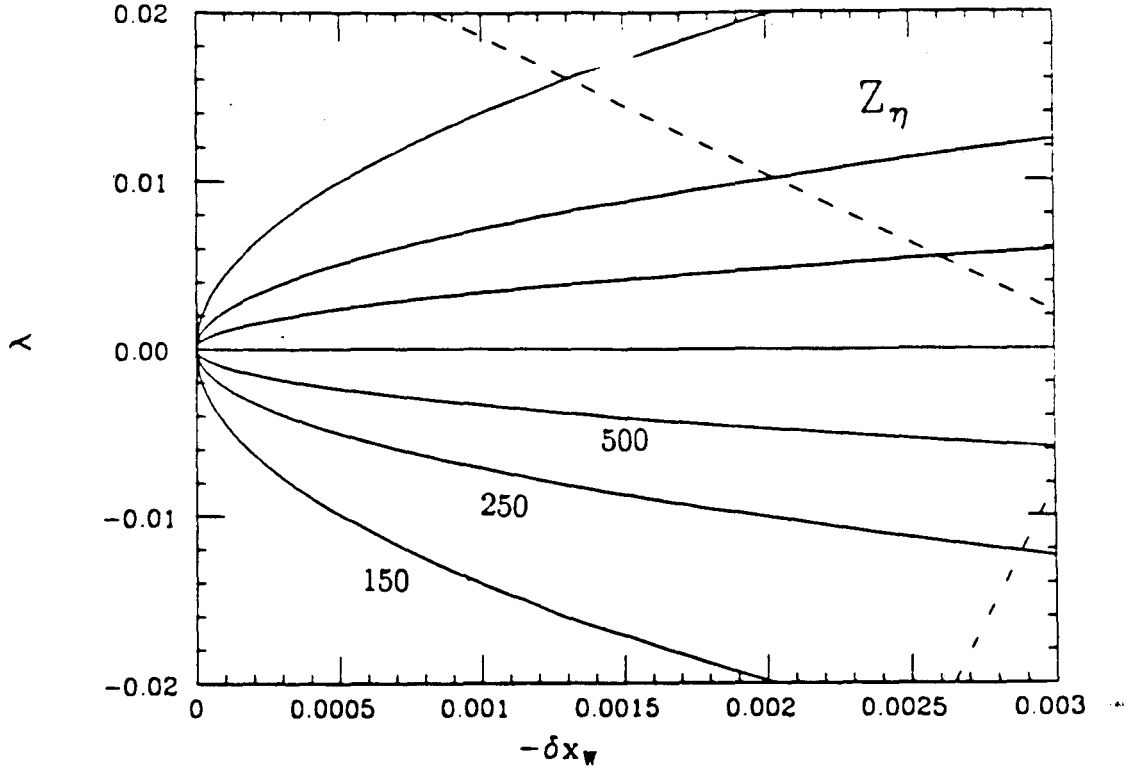


Figure 3.8: The $\delta x_W - \lambda$ plane showing the region that would be allowed by a measurement of R_μ to 1.7% (left of the dotted line), by a measurement of A_{LR} to ± 0.025 (below the dashed line), and the total width to 40 MeV (above the dotdash line) for the Z_η . From outside to inside, the parabolas indicate $M_2 = 150$ GeV, 250 GeV, and 500 GeV.

Using Tables 3.1 and 3.2 for Z_χ , Z_ψ , and Z_η ,

$$\frac{\delta A_{LR}}{A_{LR}} = -2.15 \left[-2\lambda \begin{Bmatrix} 1.47 \\ -0.11 \\ 0.99 \end{Bmatrix} + \delta x_W (16.2) \right] \quad (3.37)$$

The anticipated value of A_{LR} is about 0.21. The proposal to measure the left-right asymmetry at the SLC cites as an attainable uncertainty, $\Delta A_{LR} = 0.025$ with 10^4 Z s and a measurement of the polarization of the beam to $\Delta P/P = 5\%$. With 10^5 Z s and $\Delta P/P = 3\%$, the corresponding uncertainty is 0.009. The former would lead to a bound

$$\left| -2\lambda \begin{pmatrix} 1.47 \\ -0.11 \\ 0.99 \end{pmatrix} + 16.2\delta x_W \right| < 0.053 \quad (3.38)$$

These regions are shown in Figures 3.6, 3.7, and 3.8.

3.6 Ratio of Leptonic to Hadronic Final States

The measurement of $R_\mu = \Gamma_\mu/\Gamma_{had} = B_\mu/B_{had}$ provides an independent test of the standard model. The deviation in R_μ is

$$\begin{aligned} \frac{\delta R_\mu}{R_\mu} &= \frac{\delta \Gamma_{\mu^-}}{\Gamma_{\mu^-}} \cdot \frac{\Gamma_{\mu^-}}{\Gamma_{\mu^-} + \Gamma_{\mu^+}} + \frac{\delta \Gamma_{\mu^+}}{\Gamma_{\mu^+}} \cdot \frac{\Gamma_{\mu^+}}{\Gamma_{\mu^-} + \Gamma_{\mu^+}} \\ &\quad - 2 \left(\frac{\delta \Gamma_u}{\Gamma_u} \frac{B^u}{B^{had}} + \frac{\delta \Gamma_{\bar{u}}}{\Gamma_{\bar{u}}} \frac{B^{\bar{u}}}{B^{had}} \right) \\ &\quad - 3 \left(\frac{\delta \Gamma_d}{\Gamma_d} \frac{B^d}{B^{had}} + \frac{\delta \Gamma_{\bar{d}}}{\Gamma_{\bar{d}}} \frac{B^{\bar{d}}}{B^{had}} \right) \end{aligned} \quad (3.39)$$

Here Γ_u represents the partial width for $Z \rightarrow u_L \bar{u}_r$. The coefficients 2 and 3 arise from the number of generations that contribute. Using the values in Tables 3.1 and 3.2,

$$\frac{\delta R_\mu}{R_\mu} = -2\lambda \begin{pmatrix} -0.42 \\ -0.42 \\ 0.07 \end{pmatrix} \delta x_W (-5.64) \quad (3.40)$$

The uncertainty in R_μ will be determined primarily by statistics. For $B_\mu = 3.3\%$, with $10^4, 10^5$, and 10^6 Zs, the fractional uncertainty in R_μ will be 6%, 1.7%, and 0.5%, respectively. The limits corresponding to 1.7% are shown in Figures 3.6, 3.7, and 3.8. Our ability to predict this ratio is of course limited by uncertainties in the QCD correction to the hadronic width. If $\alpha_s(m_Z^2)$ is uncertain by 0.03, the correction $\alpha_s(m_Z^2)/\pi$ introduces an uncertainty of about 1%.

3.7 W Mass

In Section 2.4 the W mass was determined as a function of α , G_F , m_Z , and Δr . The mass of the W has been determined at the CERN Sp $\bar{p}$ S Collider and at the Tevatron Collider, but only to about ± 2 GeV. The mass difference, $m_Z - m_W$ can be determined more precisely because some of the systemic uncertainties cancel. It seems likely that the difference will be measured soon to about ± 400 MeV. Together with a high precision measurement of m_Z , this will give the W mass to ± 400 MeV.

The measurement of the W mass from the observed spectrum of lepton transverse momenta involves the transverse momentum distribution of the produced W s. This is a subject that can be analyzed theoretically using QCD perturbation theory. Ultimately, however, the most reliable approach is to use the transverse momentum distribution measured in Z production as a guide to the distribution of the W s.

How good a test of the standard model will the mass of the W be? First let us interpret the W mass as a predictor of the Z mass. From the formula

$$m_W^2 = \frac{1}{2} \left(1 + \sqrt{1 - \frac{4\pi\alpha(1 + \Delta r)}{\sqrt{2}G_F m_Z^2}} \right) m_Z^2 \quad (3.41)$$

we find that an uncertainty δm_W produces an uncertainty in the predicted Z mass of

$$\delta m_Z = 0.81 \delta m_W \quad (3.42)$$

so that if $\delta m_W = 400$ MeV, $\delta m_Z = 324$ MeV. This compares favorably with the limits that could be set by asymmetry measurements.

A more complete analysis can be done in the context of the models with an extra Z . The measurement of the W mass constrains only δx_W since it does not involve the neutral weak current, which is influence by λ . Since m_W^2 varies inversely with x_W ,

$$\frac{\delta x_W}{x_W} = -\frac{2\delta m_W}{m_W} \quad (3.43)$$

An uncertainty of 400 MeV in the W mass thus constrains δx_W to a region of width 0.00215. As seen in Figures 3.6, 3.7, and 3.8, this nicely complements the limits that can be obtained from measurements of the Z .

4 Summary

The standard model of electroweak interactions will soon be subjected to careful experimental scrutiny with the production and analysis of millions of Z s. These measurements will be sufficiently precise to require careful calculation of the radiative corrections, both those due to QED and those arising from the full electroweak model. It will be possible to set limits on possible extensions of the electroweak model, for example models with extra neutral gauge bosons. Indeed, only by proposing alternative models can we quantify the degree to which the standard model has been verified. In the instance of mixing with an extra Z it is essential to take into account the shift in the Z mass, which will cause an incorrect value of the weak mixing angle to be inferred.

Acknowledgments It is a pleasure to thank Prof. Chao Kuangta and his colleagues at Peking University for their hospitality and the opportunity to participate in the Heavy Flavor Physics Symposium in Beijing. The phenomenological analysis of Z mixing was begun with F. J. Gilman whom I thank for his contributions. I would also like acknowledge many useful suggestions offered by L. Randall and M. Suzuki.

References

- [1] For good coverage in a textbook, see C. Quigg, *Gauge Theories of Strong, Weak, and Electromagnetic Interactions*, Benjamin/Cummings, 1983 and T.-P. Cheng and L.-F. Li, *Gauge Theory of Elementary Particle Physics*, Oxford Univ. Press, 1984.
- [2] The presentation here follows R. N. Cahn, *Phys. Rev.* **36**, 2666 (1987). See also J. P. Alexander, *et al.*, *Phys. Rev.* **37**, 56 (1988) and references therein.
- [3] E. A. Kuraev and V. S. Fadin, *Yad. Fiz.* **41**, 4661 (1985).
- [4] D. Blockus *et al.*, *Proposal for Polarization at the SLC*, SLAC proposal, 1986.
- [5] C. Y. Prescott *et al.*, *Phys. Lett.* **B77**, 347 (1978).
- [6] B. L. Ioffe and V. Khoze, Leningrad Report 274 (1976) and *Sov. J. Nucl. Part.* **9**, 50 (1978).
- [7] J. D. Bjorken, *Proc. of Summer Institute on Particle Physics*, SLAC-198, p. 1, ed. M. Zipf.
- [8] R. N. Cahn, M. S. Chanowitz, and N. Fleishon, *Phys. Lett.* **82B**, 113 (1979).
- [9] H. Baer *et al.*, in *Physics at LEP*, CERN 86-02.
- [10] K. Fujikawa, B. W. Lee, and A. I. Sanda, *Phys. Rev.* **D6**, 2923 (1972).
- [11] A. Sirlin, *Phys. Rev.* **D22**, 971 (1980).
- [12] B. W. Lynn, M. E. Peskin, and R. G. Stuart, in *Physics at LEP*, CERN 86-02.
- [13] See, for example, R. N. Cahn *Lie Algebras and their Representations*, Benjamin/Cummings, 1984.
- [14] D. London and J. L. Rosner, *Phys. Rev.* **D34**, 1530 (1986).

- [15] See, for example, P. J. Franzini and F. J. Gilman, *Phys. Rev.* **D35**, 855 (1987) and references therein.
- [16] U. Amaldi et al., *Phys. Rev.* **D36**, 1385 (1987).
- [17] H. Abramowicz et al., *Phys. Rev. Lett.* **57**, 298 (1986).
- [18] J. V. Allaby et al., *Phys. Lett.* **177B**, 446 (1986).

LAWRENCE BERKELEY LABORATORY
TECHNICAL INFORMATION DEPARTMENT
1 CYCLOTRON ROAD
BERKELEY, CALIFORNIA 94720