Submitted to Physical Review

UCRL-18813
Preprint

CRUDE DYNAMICAL ATTEMPTS IN SUGAWARA'S THEORY OF CURRENTS

Arnulf Rabl

March 18, 1969

AEC Contract No. W-7405-eng-48

LAWRENCE RADIATION LABORATORY
UNIVERSITY of CALIFORNIA BERKELEY

UCRL-18813

CRUDE DYNAMICAL ATTEMPTS IN SUGAWARA'S THEORY OF CURRENTS*

Arnulf Rabl

Lawrence Radiation Laboratory
University of California
Berkeley, California 94720

March 18, 1969

ABSTRACT

As a first step towards solving Sugawara's theory of currents we explore the possibility that certain spectral sums might be dominated by a single-particle state. We sandwich the equations of motion for the currents between single-particle states and evaluate the bilinear current terms by inserting a complete set of intermediate states. By approximating the intermediate states by a single particle we can derive nonlinear equations for the form factors. When we apply this method to the energy momentum tensor we find lower bounds for the particle masses in terms of their form factors.

I. INTRODUCTION

Dynamical current theories have recently attracted much attention. A particularly interesting and promising example has been proposed by H. Sugawara[1] in which the only fundamental dynamical variables are the vector and axial-vector current octets satisfying the SU(3) x SU(3) equal-time commutation relations

$$[V_0^{\;a}(\underset{\sim}{x}),\; V_\mu^{\;b}(\underset{\sim}{y})] = [A_0^{\;a}(\underset{\sim}{x}),\; A_\mu^{\;b}(\underset{\sim}{y})]$$

$$= if^{abc}\, V_\mu^{\;c}(\underset{\sim}{x})\, \delta(\underset{\sim}{x} - \underset{\sim}{y}) + i\, C\, \delta_{ab}\, \delta_{\mu k}\, \partial_k\, \delta(\underset{\sim}{x} - \underset{\sim}{y}) \quad , \tag{1.1a}$$

$$[V_0^{\;a}(\underset{\sim}{x}),\; A_\mu^{\;b}(\underset{\sim}{y})] = [A_0^{\;a}(\underset{\sim}{x}),\; V_\mu^{\;b}(\underset{\sim}{y})] = i\, f^{abc}\, A_\mu^{\;c}(\underset{\sim}{x})\, \delta(\underset{\sim}{x} - \underset{\sim}{y}) \quad , \tag{1.1b}$$

$$[V_i^{\;a}(\underset{\sim}{x}),\; V_k^{\;b}(\underset{\sim}{y})] = [A_i^{\;a}(\underset{\sim}{x}),\; A_k^{\;b}(\underset{\sim}{y})] = [V_i^{\;a}(\underset{\sim}{x}),\; A_k^{\;b}(\underset{\sim}{y})] = 0 \quad . \tag{1.1c}$$

Sugawara has shown that the energy momentum tensor $\Theta_{\mu\nu}$ is determined almost uniquely by Lorentz invariance to be

$$\Theta_{\mu\nu}(x) = \frac{1}{2C}\{[V^a_\mu(x),\; V^a_\nu(x)]_+ - g_{\mu\nu}\, V_\lambda^{\;a}(x)\, V_a^{\;\lambda}(x) + (V \longleftrightarrow A)\} \quad , \tag{1.2}$$

where C is the constant appearing in the Schwinger term of Eq. (1.1a). Repeated Lorentz or internal symmetry indices are to be summed over, throughout this paper. Using the explicit form of the Lorentz generators $P_\mu = \int d^3x\, \Theta_{0\mu}$ and the commutation relations (1.1), one can derive the equations of motion for the currents

$$\partial_\mu V_\mu^{\ a} = 0 = \partial^\mu A_\mu^{\ a} \quad , \tag{1.3a}$$

$$\partial_\mu V_\nu^{\ a}(x) - \partial_\nu V_\mu^{\ a}(x)$$

$$= \frac{1}{2C} f^{abc} \{[V_\mu^{\ b}(x), V_\nu^{\ c}(x)]_+ + [A_\mu^{\ b}(x), A_\nu^{\ c}(x)]_+\} \quad , \tag{1.3b}$$

$$\partial_\mu A_\nu^{\ a}(x) - \partial_\nu A_\mu^{\ a}(x)$$

$$= \frac{1}{2C} f^{abc} \{[V_\mu^{\ b}(x), A_\nu^{\ c}(x)]_+ + [A_\mu^{\ b}(x), V_\nu^{\ c}(x)]_+\} \quad . \tag{1.3c}$$

We do not concern ourselves very much with the symmetry breaking, for which several methods have been proposed.[2-5] Furthermore, in most of this paper we shall, for simplicity, restrict ourselves to the SU(2) version, i.e., we omit all axial currents and let the internal symmetry indices run from 1 to 3 only.

Whereas several studies of the formal properties[2-9] and of a few experimental consequences[10-12] of the theory have been conducted, no one has been able to solve it. Perturbation methods, for example, break down, as shown by Bardakci and Halpern.[3] In this paper we explore the simple (but probably unrealistic) possibility that certain spectral sums might be dominated by a single state, for instance one pion. This will alow us to derive approximate integral equations for the form factors.

At the center of our investigation lie the equations of motion (1.3) which we sandwich between single-particle states, inserting a complete set of intermediate states in the bilinear current terms on the right-hand side. Since the matrix elements of currents can be expressed in terms of form factors, we obtain an infinite set of coupled nonlinear integral equations for the form factors. As a first approximation towards solving these equations we keep only one single-particle state on the right-hand side. We have, a priori, no idea about the validity of such an approximation, and can present only a weak plausibility argument why the single-particle term might dominate. For lack of any better technique we shall proceed to see how far this approximation will carry us.

In the pion case we obtain one nonlinear integral equation which can be solved for the spacelike pion form factor in terms of the pion mass and the Schwinger constant C. Alternatively, if the form factor is known, one can interpret this equation as a sum rule whose experimental success or failure tests the validity of our approximation. We have tried to solve the pion equation by iteration on a computer. It appears to be an eigenvalue problem, because the iteration algorithm converges to a not unreasonable solution if we choose C a factor 6 too small, but not for the physical value of $C \approx 0.02\ \mathrm{BeV}^2$ suggested by Nussinov.[12] The failure to converge for the physical C may be due to the neglect of higher states.

Although the situation for the pion does not look too bad, the corresponding approximation for the nucleon form factors (in Section III) produces what is probably nonsense by yielding four independent equations for two form factors. The hope that somehow the inconsistencies cancel for the actual values of the form factors is thoroughly defeated when we insert the experimental numbers. In some of the equations we find negative quantities on one side and positive ones on the other, and not even a different choice of C could save the situation.

In Section IV we derive sum rules from the energy momentum tensor $\Theta_{\mu\nu}$. We obtain a lower bound for the vacuum expectation value $\langle 0|\Theta_{\mu\nu}|0\rangle = \Lambda g_{\mu\nu}$ in terms of C and the pion form factor, and show that Λ would diverge if the ρ meson were stable. Finally we apply the one-particle approximation to $\langle \pi(p)|\Theta_{\mu\nu}|\pi(p)\rangle = 2p_\mu p_\nu(2\pi)^{-3}$ and find two equations; one is a rigorous inequality relating the form factor of the pion to its mass, whereas the other is an equation which cannot possibly be satisfied by the one-pion term alone. The corresponding sum rule for the nucleon mass and nucleon form factors can be compared with experiment and is badly violated. This as well as the experience with the nucleon form factors in Section III indicates that the one-particle approximation is unsuitable, even as a first ansatz.

II. THE PION FORM FACTOR

A. Derivation Of The Integral Equation

Let us sandwich the equation of motion (1.3b) between pion states normalized according to

$$\langle\pi(p', i')|\pi(p, i)\rangle = 2\, p_0\, \delta(\underset{\sim}{p}' - \underset{\sim}{p})\, \delta_{i'i} \tag{2.1}$$

and insert a complete set of intermediate states $1 = \sum_n |n\rangle\langle n|$ between the currents on the right-hand side. The relation

$$\langle p'|V(x)|p\rangle = e^{i(p'-p)\cdot x} \langle p'|V(0)|p\rangle$$

allows us to cancel the coordinate dependence on both sides of the equation, and we obtain[13]

$$i\, q_\mu \langle\pi(p', i')|V_\nu^{\;a}|\pi(p, i)\rangle - i\, q_\nu \langle\pi(p', i')|V_\mu^{\;a}|\pi(p, i)\rangle$$

$$= \frac{\epsilon^{abc}}{2C} \sum_n \langle\pi(p', i')|V_\mu^{\;b}|n\rangle\langle n|V_\nu^{\;c}|\pi(p, i)\rangle$$

$$+ \langle\pi(p', i')|V_\nu^{\;c}|n\rangle\langle n|V_\mu^{\;b}|\pi(p, i)\rangle \;, \tag{2.2}$$

where we have set $q = p' - p$ and $V = V(0)$. Graphically the situation can be represented by Fig. 1. The pion form factor $F(t)$ is defined by

$$\langle\pi(p', i')|V_\nu^{\,a}(0)|\pi(p, i)\rangle = \frac{i\epsilon^{i'ai}}{(2\pi)^3}(p' + p)_\nu F(t) ,$$

$$\text{with} \quad t = (p' - p)^2 . \qquad (2.3)$$

It is pure isovector, since the expectation value of the hypercharge part of the electromagnetic current between pion states vanishes. The boundary condition is, of course, $F(0) = 1$, representing a unit electric charge.

With the assumption that a single pion saturates the right-hand side of Eq. (2.2) we find

$$(p_\nu p'_\mu - p_\mu p'_\nu) F(t) = \frac{1}{4(2\pi)^3 C}\int \frac{d^3n}{2n_0}[(p' + n)_\mu (n + p)_\nu$$

$$- (p' + n)_\nu (n + p)_\mu]F(t_2) F(t_1) ,$$

$$\text{with} \quad t_2 = (p' - n)^2 \quad \text{and} \quad t_1 = (n - p)^2 . \qquad (2.4)$$

Let us note briefly that this approximation is gauge-invariant, because the contribution of each state on the right-hand side of Eq. (2.2) vanishes separately when multiplied by $q^\mu q^\nu$ (by antisymmetry).

To evaluate Eq. (2.4) we choose the special frame in which $p = (m, \underset{\sim}{0})$ and $q = (\nu, Q\hat{z})$; as a consequence

$$p' = (m + \nu, Q\hat{z}) , \qquad \nu = -t/(2m) , \qquad Q = \frac{1}{2m}[t(t - 4m^2)]^{\frac{1}{2}} ,$$

$$t_1 = (n - p)^2 = 2m(m - n_0) , \tag{2.5}$$

$$t_2 = (p' - n)^2 = t_1 - 2\nu n_0 + 2nQz ,$$

where n now stands for the length of the three-vector $|\underset{\sim}{n}| = \frac{1}{2m}[t_1(t_1 - 4m^2)]^{\frac{1}{2}}$, and $z = \cos(\hat{n}, \hat{q})$. We also see that $F(t_1)$ and $F(t_2)$ are azimuthally symmetric. In this frame the index pair $\mu = 0$, $\nu = 3$ gives us the only nontrivial equation

$$F(t) = \frac{1}{8(2\pi)^2 C m} \int_0^\infty \frac{n^2 dn}{n_0} \int_{-1}^{1} dz(m + n_0 - \frac{\nu}{Q} nz) F(t_1) F(t_2) , \tag{2.6}$$

while all other index combinations yield only $0 = 0$. The most convenient mass units are those for which $m = 0.5$ or $1 = 2m = 4m^2 = \cdots = 0.28$ BeV $= 0.0783$ BeV^2. Then the Schwinger constant is[12] $C = 0.02$ BeV$^2 = 0.25$, and the integral equation reads

$$F(t) = \frac{1}{4(2\pi)^2 C} \int_{-\infty}^{0} dt_1 \, n \, F(t_1) \int_{-1}^{1} dz \left(1 - t_1 - \left[\frac{t}{t - 1}\right]^{\frac{1}{2}} nz\right)$$

$$\times \; F(t_2) , \tag{2.7}$$

with $t_2 = t + t_1 - 2tt_1 + 2Qnz$, $Q = [t(t - 1)]^{\frac{1}{2}}$,

and $n = [t_1(t_1 - 1)]^{\frac{1}{2}}$.

A priori we have no idea about the validity of our single-particle approximation, and we can only give a weak plausibility argument why the off-diagonal terms might be less important. The charge $\int d^3x\, V_0{}^a(\underset{\sim}{x}, t)$ is diagonal and hence the terms $\langle n|V_0{}^a(0)|\pi(p)\rangle$, $\langle n| \neq$ pion, which occur in the off-diagonal contributions, vanish at $\underset{\sim}{n} = \underset{\sim}{p}$ (this is in the integration range), whereas $\langle\pi(n)|V_0{}^a(0)|\pi(p)\rangle = i\epsilon^{i'ai}(2\pi)^{-3}\,(n + p)_0\, F[(n - p)^2]$ does not. This might suppress the off-diagronal contributions in the range where they are presumably most important, i.e., at small momentum transfer. Also, we may note that the "kernel" of the diagonal term in Eq. (2.7), $\{1 - t_1 - [t/(t - 1)]^{\frac{1}{2}}\, nz\}$ is greater than 1 throughout the range of integration, whereas one can show that the corresponding off-diagonal terms will have kinematic zeroes and sign changes due to the difference in masses.

B. Solution Of The Integral Equation

In view of the uncertain basis of our approximation and for lack of sufficient understanding of nonlinear integral equations we do not delve into mathematical details or rigor. We restrict ourselves to a few simple observations and then try to obtain a numerical solution by iteration on a computer.

First we can show, under reasonable continuity and convergence assumptions to allow interchange of limit and integration, that

$$F(t) \to f\,(-t)^{-\gamma} \quad \text{as} \quad t \to -\infty\,, \tag{2.8}$$

with f and γ some constants, $\gamma > 3/2$ to assure convergence. At $t \to -\infty$ Eq. (2.7) becomes

$$1 = \lim_{t\to -\infty} \frac{1}{F(t)} \frac{1}{4(2\pi)^2 C} \int_{-\infty}^{0} dt_1 \; F(t_1) \; n \int_{-1}^{1} dz \; F(t_2)$$

$$\times \left[1 - t_1 - \left(\frac{t}{t-1}\right)^{\frac{1}{2}} nz\right] ,$$

where $t_2 \to t[1 - 2t_1 + 2z\left(t_1(t_1 - 1)\right)^{\frac{1}{2}}] \equiv t\, k(z, t_1)$,

and taking the limit inside the integral,

$$1 = \frac{1}{4(2\pi)^2 C} \int_{-\infty}^{0} dt_1 \; F(t_1) \; n \int_{-1}^{1} dz \; (1 - t_1 - nz) \lim_{t\to -\infty} \frac{F(tk)}{F(t)} .$$

In order for $\ell(k) \equiv \lim F(tk)/F(t)$ to exist, $\ell(k)$ must satisfy $\ell(k^x) = [\ell(k)]^x$, which has the solution $\ell(k) = k^{-\gamma}$, and this in turn implies $\lim F(t) = f\,(-t)^{-\gamma}$.

We might wonder about the uniqueness of the solution. Suppose F and G are two solutions of $F = \iint dt_1\, dz\, K\, F(t_1)\, F(t_2)$ with the same boundary condition $F(0) = 1 = G(0)$ and differing infinitesimally, $F(t) = G(t) + \epsilon(t)$; then one can easily show that ϵ has to obey the homogeneous linear integral equation $\epsilon = \int dt_1 \int dz\, K[\epsilon(t_1)\, G(t_2) + \epsilon(t_2)\, G(t_1)]$ subject to $\epsilon(0) = 0$. Setting $E = \max|\epsilon(t)|$, we find $|\epsilon| \leqslant E \iint dt_1\, dz |K| |G(t_1) + G(t_2)|$. If G is small enough to render $\iint dt_1\, dz|K| |G(t_1) + G(t_2)| < 1$, then we have $|\epsilon(t)| < E$, which is a contradiction. Hence $\epsilon \equiv 0$, and the solution is, if not unique, at least discrete.

Of physical interest is the analyticity of F. At first glance one might expect square root branch points at $t = 0$ and at $t = 1 = 4m^2$. The exact form factor should, of course, be analytic in the cut t plane, with a cut from $4m^2$ to $+\infty$. A careful evaluation of the limits $t \to 0$ and $t \to 1$ in Eq. (2.7) shows, however, that these square root singularities cancel and that a solution which is analytic at $t = 0$ and $t = 1$ is consistent. To see how this comes about, consider the point $t = 1$ and let $t = 1 + \eta + i\epsilon$, η and ϵ small and real and $\xi = \eta + i\epsilon$. Then

$$F(1 + \eta + i\epsilon) = \frac{1}{4(2\pi)^2 C} \int_{-\infty}^{0} dt_1 \, F(t_1) \, n \int_{-1}^{1} dz$$

$$\times F[1 - t_1 + 2nz(\xi)^{\frac{1}{2}}] \, [1 - t_1 - nz/(\xi)^{\frac{1}{2}}] \quad . \tag{2.9}$$

If we assume that $F(t_2)$ can be expanded about $t_2 = 1$,

$$F(t_2) = F[1 - t_1 + 2nz(\xi)^{\frac{1}{2}}] = F(1 - t_1) + 2nz(\xi)^{\frac{1}{2}} F'(1 - t_1) + \cdots ,$$

then all odd powers of $(\xi)^{\frac{1}{2}}$ cancel because $\int_{-1}^{1} dz \, z^n = 0$ for odd n, and we obtain

$$F(1 + \eta + i\epsilon) = \frac{1}{4(2\pi)^2 C} \int_{-\infty}^{0} dt_1 \, F(t_1) n \int_{-1}^{1} dz$$

$$\times [(1 - t_1) F(1 - t_1) - 2n^2 z^2 F'(1 - t_1) + \mathcal{O}(\eta + i\epsilon)] \quad .$$

On the other hand the assumption of a cut from $t = 4m^2$ to $+\infty$ also seems to be consistent, since Eq. (2.9) would be linear in the discontinuity across the cut. However, the equation by itself does not seem to possess a driving mechanism to produce such a cut.

We have written an iterative computer program which calculates $F^{(n+1)} = \iint F^{(n)} F^{(n)}$ and renormalizes to $F^{(n+1)}(0) = 1$ after each step by dividing $F^{(n)}$ by $[F^{(n+1)}(0)]^{\frac{1}{2}}$. We started from a dipole formula $F^{(0)}(t) = (t - 1)^{-2}$ and found the algorithm to converge to within 1% after five or six iterations. However, whether the function thus obtained is a solution turns out to depend on C. For the physical[12] value $C = 0.02$ BeV^2, $F^{(n+1)}(0)$ converges to 0.25 instead of 1. On the other hand, by choosing C by a factor 1/6 too small, we were able to obtain a solution (it is similar to the input $(t - 1)^{-2}$ but larger and decreasing more slowly at infinity). This experience suggests that Eq. (2.7) may present an eigenvalue problem with C as the eigenvalue. The failure to converge for the physical value of the Schwinger constant could of course be due to our neglect of the off-diagonal terms in Eq. (2.2).

The alternative interpretation of Eq. (2.7) would be to insert the experimental data for the pion form factor and see to what extent the one-pion term saturates it. Unfortunately the slow convergence of Eq. (2.7), coupled with the lack of experimental information at large $|t|$, precludes such an approach for the pion.[14]

III. THE NUCLEON FORM FACTORS

For the nucleon we take Eq. (2.2) with nucleons instead of pions, keeping only the one-nucleon state on the right-hand side. We introduce the form factor combinations which turn out to be most convenient here, $f = F_2^{I=1}$ and $g = F_1^{I=1} + 2MF_2^{I=1}$, via the standard definition

$$\langle p', s' | V_\mu^3(0) | p, s \rangle$$

$$= (2\pi)^{-3}\, \bar{u}(p', s')[\gamma_\mu\, g(t) - (p' + p)_\mu\, f(t)]\, u(p, s) \quad , \qquad (3.1)$$

with $t = (p' - p)^2$. We find

$$\bar{u}'[(q_\nu \gamma_\mu - q_\mu \gamma_\nu)g + 2(q_\mu p_\nu - q_\nu p_\mu)f]u$$

$$= \frac{1}{2(2\pi)^3\, c} \int_0^\infty \frac{n^2 dn}{n_0} \int_{-1}^{1} dz \int_0^{2\pi} d\phi$$

$$\times\; \bar{u}'\{[\gamma_\nu g_2 - (p' + n)_\nu\, f_2]\, (\not{n} + M)[\gamma_\mu g_1 - (n + p)_\mu\, f_1]$$

$$-\; [\gamma_\mu g_2 - (p' + n)_\mu\, f_2](\not{n} + M)[\gamma_\nu g_1 - (n + p)_\nu\, f_1]\}u \;, \qquad (3.2)$$

where we have used the obvious abbreviations $u = u(p, s)$, $f = f(t)$, $f_1 = f(t_1)$, $f_2 = f(t_2)$, etc. To evaluate Eq. (3.2) we choose the same frame and the same notation as in the pion problem (see Eq. 2.5). After a great deal of tedious algebra we find four independent nontrivial equations (in contrast to the pion case, which gave us only one):

(a) for $\mu = 0$, $\nu = 3$, (3.3a)

$$g + (1 - t)f = \iint g_1 g_2 - \iint g_1 f_2[(1 - t_1)^2 - n^2 z^2 - 2tn_0 - 2Qnz]$$

$$- \iint f_1 g_2[(1 - t_1)^2 - n^2 z^2] + (1 - t)\iint f_1 f_2 (1 - t_1 + \frac{t}{Q} nz)^2;$$

(b) for $\mu = 0$, $\nu = 1,2$, (3.3b)

$$-tg = -\iint g_1 g_2 (1 - 2\frac{Q}{t} nz) + \iint g_1 f_2[t_1(t + t_1 - 1) - \frac{n^2}{2}(1 - z^2)$$

$$+ \frac{Q}{t}(t + t_1 - 1)nz] + \iint f_1 g_2[(1 - t_1)^2 - \frac{n^2}{2}(1 - z^2) + \frac{Q}{t}(t_1 - 1)nz]$$

$$+ t\iint f_1 f_2\left[\frac{n^2}{2}(1 - z^2)\right] \quad ;$$

(c) for $\mu = 3$, $\nu = 1,2$, (3.3c)

$$-tg = -2\iint g_1 g_2 t_1 + \iint g_1 f_2 \left[-\frac{n^2}{2}(1 + z^2) - Qnz + tt_1(1 + \frac{nz}{Q})\right]$$

$$+ \iint f_1 g_2 \left[-\frac{n^2}{2}(1 + z^2) + \frac{t}{Q}(t_1 - 1)nz\right] + t\iint f_1 f_2 \left[\frac{n^2}{2}(1 - z^2)\right] ;$$

(d) for $\mu = 1$, $\nu = 2$, (3.3d)

$$0 = \iint g_1 g_2 (t_1 - \frac{t}{Q} nz) + \iint g_1 f_2 \left[\frac{n^2}{2}(1 - z^2)\right]$$

$$+ \iint f_1 g_2 \left[\frac{n^2}{2}(1 - z^2)\right] .$$

We have employed the natural mass units $1 = 2M = 1.88$ BeV $= 3.53$ BeV^2, and $\iint$ stands for

$$\frac{1}{2(2\pi)^2\, C} \int_{-\infty}^{0} dt_1 \; n \int_{-1}^{1} dz \; .$$

We would have liked two equations for the two form factors, but found, alas, four inconsistent ones. To see to what extent the physical form factors satisfy these equations we have inserted the famous dipole fit[14]

$$G_E^{\,p}(t) = \frac{G_M^{\,p}(t)}{1 + \mu_p} = \frac{G_M^{\,n}(t)}{\mu_n} = \left[1 - \frac{1}{(0.71\ \mathrm{BeV})^2}\right]^{-2} \; ; \qquad (3.4)$$

The G's are related to the F's by $G_E = F_1 + \frac{t}{2M} F_2$ and $G_M = F_1 + 2M\, F_2$, and the I-spin decomposition is

$$F^p = F^{I=0} + F^{I=1}$$

$$F^n = F^{I=0} - F^{I=1} \; .$$

We have taken the liberty of neglecting $G_E^{\,n}$ altogether, which is justifiable in view of its smallness compared with the experimental uncertainties in $G_M^{\,n}$. We have evaluated Eqs. (3.3) for several values of t from 0 to -10 BeV^2 and found that the one-nucleon term, although generally of the right order of magnitude, fails to saturate the equations; in some cases even the sign comes out wrong.

IV. SUM RULES FROM $\Theta_{\mu\nu}$

A. The Vacuum Expectation Value

By Lorentz invariance the energy momentum tensor has the vacuum expectation value

$$\langle 0|\Theta_{\mu\nu}|0\rangle = \Lambda g_{\mu\nu} \quad . \tag{4.1}$$

For $\mu = \nu$ implies[13]

$$\Lambda = \frac{2}{C}\langle 0|V_0{}^a V_0{}^a|0\rangle = \frac{2}{C}\sum_n |\langle 0|V_0{}^a|n\rangle|^2 \quad . \tag{4.2}$$

Noting the positive definiteness of all contributions, we obtain a lower bound by keeping only the two-pion state, the lowest lying state allowed by the positive G parity of the vector current. (For simplicity we consider only the SU(2) part of the theory.) We can relate this expression to an integral over the timelike region of the pion form factor when we realize that

$$\langle 0|V_\mu{}^a|\pi^b(p)\,\pi^c(p')\rangle = \langle \pi^c(-p')|V_\mu{}^a|\pi^b(p)\rangle$$

$$= \frac{i\epsilon^{cab}}{(2\pi)^3}(-p' + p)_\mu F(t) \quad \text{with} \quad t = (p' + p)^2.$$

After some algebra and trivial angular integration we find

$$\Lambda \geqslant \Lambda_{2\pi} = \frac{6}{(2\pi)^4 C} \int_0^\infty \frac{p'^2 dp'}{p'_0} \int_0^\infty \frac{p^2\, dp}{p_0} \int_{-1}^{1} dz$$

$$\times (p_0 - p'_0)^2 |F(t)|^2 \quad , \qquad (4.3)$$

with $t = 2(m^2 + p_0 p'_0 - pp'z)$ and $z = \cos(\hat{p}, \hat{p}')$.

Power counting suggests that $F(t)$ has to decrease faster than t^{-a} with $a > 3/2$ as $t \to +\infty$ in order for this integral to converge. However, a finite lower bound for Λ may not be very meaningful, since Λ is likely to be infinite. For in the chiral version of the theory the single pion state contributes

$$\Lambda_\pi = \frac{2}{C} \sum_{a,b} \int \frac{d^3p}{2p_0} |\langle 0|A_0{}^a|\pi^b(p)\rangle|^2 \propto \frac{F_\pi{}^2}{C} \int d^3p\, p_0 \quad ,$$

which diverges if the pion decay constant, defined by $\langle 0|A_\mu{}^a|\pi^b(p)\rangle = F_\pi\, p_\mu\, \delta_{ab}$, does not vanish. A nonzero F_π requires, of course, some symmetry breaking to make $\partial^\mu A_\mu \neq 0$.[8,15] A similar argument shows that a stable rho-meson would give a divergent contribution to Λ.

B. Sum Rule For The Pion Mass

In this section we use the fact that

$$\langle \pi^a(p)|\Theta_{\mu\nu}|\pi^b(p)\rangle = 2(2\pi)^{-3}\, \delta_{ab}\, p_\mu p_\nu \quad , \qquad (4.4)$$

if we normalize according to Eq. (2.1). Strictly speaking this relation holds for the truncated tensor $\bar{\Theta}_{\mu\nu} = \Theta_{\mu\nu} - \Lambda g_{\mu\nu}$. The vacuum expectation contribution to the pion mass, $\Lambda g_{\mu\nu}\, 2p_0 \delta^3(\underset{\sim}{p} - \underset{\sim}{p})$, is singular but spurious; it is precisely the contribution of the disconnected diagrams in the sum over intermediate states $\sum \langle \pi(p)|V_\mu|n\rangle\langle n|V_\nu|\pi(p)\rangle$. Hence we hope that we may forget about the subtraction of the vacuum expectation value if we consistently neglect all disconnected diagrams.[13]

As before we work in the pion rest frame. For $\mu = \nu = 0$ we find

$$\frac{2m^2}{(2\pi)^3} = \frac{1}{2C} \sum_{\ell=0}^{3} \sum_{n} |\langle p|V_1{}^a|n\rangle|^2 , \tag{4.5}$$

the index 1 being summed with Euclidean metric. Since each state contributes a positive definitive amount we obtain a rigorous inequality for the pion mass,

$$m^2 \geqslant \frac{1}{(2\pi)^2\, C} \int_0^\infty dn\, n^2(m + n_0)|F(t_1)|^2 , \quad \text{with} \quad t_1 = 2m(m - n_0), \tag{4.6}$$

in terms of the spacelike pion form factor. Unfortunately the integral converges slowly, and for lack of experimental data at large $-t$ this relation is not very useful. It does, however, provide us with a new power bound, t^{-2}, for the form factor at infinity, if the integral is to converge. This bound is stronger than the exponent $-3/2$ found in Section II. The difference is due to the symmetry in the Lorentz indices of $\Theta_{\mu\nu}$ contrasted with the antisymmetry in the equation of motion.

Equation (4.4) implies some more relations. For $\mu = 0$, $\nu = n$, and $\mu = m \neq n = \nu$, we get only $0 = 0$, but for $\mu = \nu = m$ we find

$$0 = \int_0^\infty \frac{dn\, n^2}{n_0} \left[(m + n_0)^2 - \frac{n^2}{3}\right] |F(t_1)|^2 + \text{higher states} . \quad (4.7)$$

As the pion term is strictly greater than zero, this relation cannot possibly be satisfied by the pion alone. (The alternative $F(t) \equiv 0$ would contradict the boundary condition $F(0) = 1$ required by the fact that the pion has I-spin = 1.)

C. Sum Rule For The Nucleon Mass

Let us repeat the steps of Section IV B with the nucleon which satisfies[16]

$$\langle N(p,\ s)|\Theta_{\mu\nu}|N(p,\ s)\rangle = \frac{p_\mu p_\nu}{(2\pi)^3\ M} , \quad (4.8)$$

the difference from Eq. (4.4) being due to the different normalization for fermions $\langle N(p',\ s')|N(p,\ s)\rangle = M^{-1}\ p_0\ \delta(\underset{\sim}{p}' - \underset{\sim}{p})\ \delta_{s's}$. We find the lower bound for the nucleon mass[17]

$$M \geqslant \frac{3}{2(2\pi)^2\ C} \int_0^\infty \frac{dn\ n^2}{n_0} \{4(2n_0 - M)|g(t_1)|^2$$

$$- 2\left[(M + n_0)^2 + n^2\right] g(t_1)f(t_1) + (M + n_0)\left[(M + n_0)^2 + n^2\right] |f(t_1)|^2\} . \quad (4.9)$$

The convergence requirement that g be bounded by $t^{-3/2}$ and f by $t^{-5/2}$ is satisfied by the dipole form factors by an extra $t^{1/2}$, thus assuring convergence of Eq. (4.9) like $\int dt\, t^{-2}$. We have integrated Eq. (4.9) with the dipole fit,[14] and the result, which should be good to about 10%, disagrees strongly: the lower bound turns out to be 1.7 times as great as the nucleon mass itself.[17] Alternatively, to satisfy Eq. (4.9) C would have to be very much larger than $0.02\ \mathrm{BeV}^2$, which we have used.[12]

The symmetry breaking of Bardakci, Frishman, and Halpern (Ref. 2) might conceivably offer a way out of this dilemma. They add (in their Eq. 4.3) to $\Theta_{\mu\nu}$ a term $g_{\mu\nu}(\sigma^2 + \phi^a\phi^a - f_\pi m_\pi^{\ 2}\sigma + \frac{1}{4} f_\pi^{\ 2} m_\pi^{\ 2})$, where the $-f_\pi m_\pi^{\ 2}\sigma$ term produces PCAC as in the σ model. Since the quantity $\sigma^2 + \phi^a\phi^a + \frac{1}{4} f_\pi^{\ 2} m_\pi^{\ 2}$ is a c number it could be omitted. Now the $-f_\pi m_\pi^{\ 2}\sigma$ term, evaluated between nucleon states, might be negative enough to yield a reasonable lower bound for the nucleon mass when added to the right-hand side of Eq. (4.9).

V. CONCLUSIONS

In this paper we tested the one-particle approximation as a first step towards solving Sugawara's theory. The most straightforward result, a nonlinear integral equation for the pion form factor, does not appear to have a solution for the physical value of the Schwinger constant. The other equations, derived by keeping only single particle states, fare even worse: they are inconsistent with each other. All this indicates that the contributions of the higher-mass states are essential, even for the first ansatz. The algebraic complications due to the spin of these higher states would be prohibitive.

An interesting consequence of the sum rule (4.6) for the pion mass is the fact that m_π^2 is strictly greater than zero. Since the inequality (4.6) holds in the chiral symmetric theory as well, this shows that the pion cannot be a Goldstone boson.

That the pion cannot be a Goldstone boson in this theory has already been shown by Dashen and Frishman.[15] This of course leads to great difficulties in explaining PCAC and hadronic mass spectra, putting the entire burden for these phenomena on the symmetry-breaking dynamics. Such is not impossible,[18] of course, but would lose the elegance of the Goldstone approach.[19] On the other hand, it does have the ring of the "old fashioned" bootstrap idea, in which the pion mass sets the scale for the strong interactions and has no zero mass limit, PCAC arising dynamically in a way related to the smallness of the pion mass.

ACKNOWLEDGMENTS

I am indebted to Professor Martin B. Halpern for suggesting this problem, and for his advice and encouragement. I should also like to thank Ron Christensen for discussions and for help with the computer programming.

FOOTNOTES AND REFERENCES

* This work was supported in part by the U.S. Atomic Energy Commission.

1. H. Sugawara, Phys. Rev. 170, 1659 (1968).
2. K. Bardakci, Y. Frishman, and M. B. Halpern, Phys. Rev. 170, 1353 (1968).
3. K. Bardakci and M. B. Halpern, Phys. Rev. 172, 1542 (1968).
4. C. M. Sommerfield, Phys. Rev. 176, 2019 (1968).
5. H. Sugawara, Phys. Rev. Letters 21, 772 (1968).
6. D. J. Gross and M. B. Halpern, Berkeley preprint Sept. 1968.
7. H. Sugawara and M. Yoshimura, Phys. Rev. 173, 1419 (1968).
8. W. Bierter and K. Bitar, Nuovo Cimento Letters 1, 192 (1969).
9. B. Sakita, Wisconsin preprint, Sept. 1968.
10. D. J. Gross, Phys. Rev. Letters 21, 308 (1968).
11. C. G. Callan and D. J. Gross, Phys. Rev. Letters 21, 311 (1968).
12. S. N. Nussinov, Lawrence Radiation Laboratory Report UCRL-18396. Aug. 1968, finds a lower bound for the Schwinger constant $C \geqslant 0.0115\ \mathrm{BeV}^2$, and suggests the value $C = g_\rho{}^2/m_\rho{}^2 = 0.025$ or $0.018\ \mathrm{BeV}^2$ (depending on the uncertainty in the rho-width) using Weinberg's vector current sum rule [S. Weinberg, Phys. Rev. Letters 18, 507 (1967)].
13. Both in the equations of motion and in $\Theta_{\mu\nu}$ there are products of operators at the same point, a notoriously dangerous situation in field theory. S. Coleman, D. J. Gross, and R. Jackiw (Harvard University preprint, Jan. 1969) argue that one should separate the currents by a small distance ϵ, subtract the vacuum expectation

value, and then take the limit $\epsilon \to 0$. In a four-dimensional Sugawara model for fermions they have found that the Schwinger constant is singular like ϵ^{-2} in this limit. However, the explicit ϵ process is model-dependent, and we cannot find one without solving the theory anyway. In fact we have to search for an ϵ process during an approximate solution. In our approach we ignore these complications because everything remains regular in the limit when we relate the products of currents to products of form factors and neglect all disconnected terms. We can hope, with Nussinov (Ref. 12), that such singularities arise just from the disconnected terms and from the vacuum expectation values, and cancel each other.

14. For a good review and list of references on electromagnetic form factors see, e.g., R. Wilson, Phys. Today, Jan. 1969, p. 47-53.

15. R. F. Dashen and Y. Frishman, Institute of Advanced Studies preprint, Dec. 1968.

16. Our sum rule for the nucleon mass involves form factors and is quite different from the one derived by D. J. Gross (Ref. 10) in terms of scattering amplitudes.

17. The factor 3 in the numerator of the coefficient multiplying the integral of Eq. (4.9) comes from the SU(2) theory, where $\sum_{a=1}^{3} (\tau^a)^2 = 3$. In the SU(3) version it would be replaced by

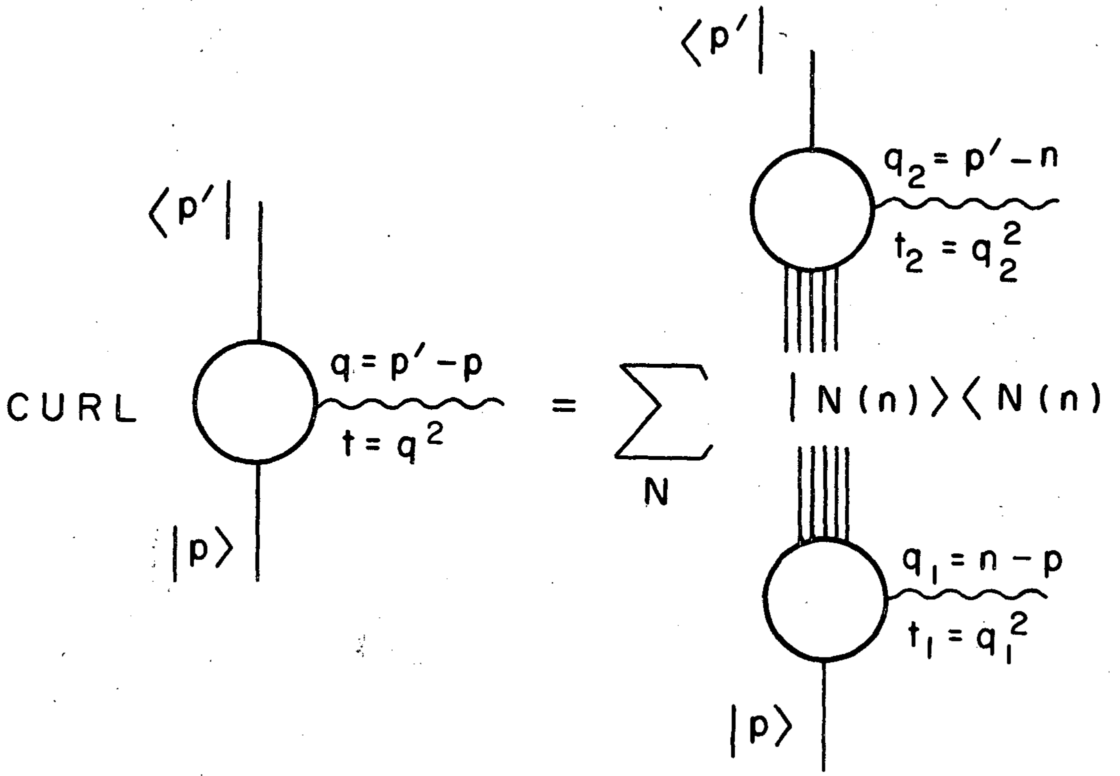
$$\sum_{a=1}^{8} (\lambda^a)^2 = 8,$$ and the discrepancy would be 8/3 as bad.

18. S. Fubini and G. Furlan, Massachusetts Institute of Technology preprint, Jan. 1969.

19. R. F. Dashen, Institute of Advanced Studies preprint, Jan. 1969, and R. F. Dashen and M. Weinstein, Institute of Advanced Studies preprint, Jan. 1969.

FIGURE CAPTIONS

Fig. 1. The equation of motion for the currents, when inserted between single-particle states, becomes a relation between form factors.



XBL693 - 2223

Fig. 1