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IDEAL PROJECTIONS AND FORCING PROJECTIONS

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Abstract. It is well known that saturation of ideals is closely related to the "antichain-catching" phenomenon from Foreman–Magidor–Shelah [10]. We consider several antichain-catching properties that are weaker than saturation, and prove:

(1) If $\mathcal{I}$ is a normal ideal on ω_2 which satisfies *stationary antichain catching*, then there is an inner model with a Woodin cardinal;

(2) For any $n \in \omega$, it is consistent relative to large cardinals that there is a normal ideal $\mathcal{I}$ on ω_n which satisfies *projective antichain catching*, yet $\mathcal{I}$ is not saturated (or even strong). This provides a negative answer to Open Question number 13 from Foreman's chapter in the Handbook of Set Theory ([7]).

§1. Introduction. The notions of *antichain catching* and *self-genericity* first appeared in Foreman–Magidor–Shelah [10] and were used extensively by Woodin in his stationary tower arguments (see [18] or [7]); these topics are explored in detail in [7]. We consider several properties of ideals on uncountable cardinals related to antichain catching; these properties lie between saturation and precipitousness. For a normal ideal $\mathcal{I}$ on a regular uncountable κ, the main property of interest—which we call *ProjectiveCatch*$(\mathcal{I})$—is equivalent[1] to the statement that there is a normal ideal $\mathcal{J} \subset \wp(P_\kappa(H_\theta))$ (where θ is large relative to $\mathcal{I}$) such that:

> $\mathcal{J}$ projects canonically to $\mathcal{I}$ in the Rudin–Keisler sense, and the canonical Boolean homomorphism
> $$h_{\mathcal{I},\mathcal{J}} : \wp(\kappa)/\mathcal{I} \to \wp(P_\kappa(H_\theta))/\mathcal{J} \tag{1}$$
> is a *regular embedding*.

In the case where the completeness of $\mathcal{I}$ is at least ω_2, we also consider the "starred version" *ProjectiveCatch*$^*(\mathcal{I})$, which additionally requires that the dual of the ideal $\mathcal{J}$ from (1) concentrates on sets whose intersection with ORD is ω-closed.

In addition to *ProjectiveCatch*$(\mathcal{I})$, we also consider the stronger property *ClubCatch*$(\mathcal{I})$ and the weaker property *StatCatch*$(\mathcal{I})$. The property *ClubCatch*$(\mathcal{I})$ is equivalent to saturation of $\mathcal{I}$ (by Foreman [7]; see Theorem 3.2 below). The property *ProjectiveCatch*$(\mathcal{I})$ implies that $\mathcal{I}$ is precipitous;[2] if $\mathcal{I}$ is an ideal on ω_1,

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[1] By Lemmas 3.4 and 3.11.

[2] And *StatCatch*$(\mathcal{I})$ implies there exists some $T \in \mathcal{I}^+$ such that $\mathcal{I} \restriction T$ is precipitous.

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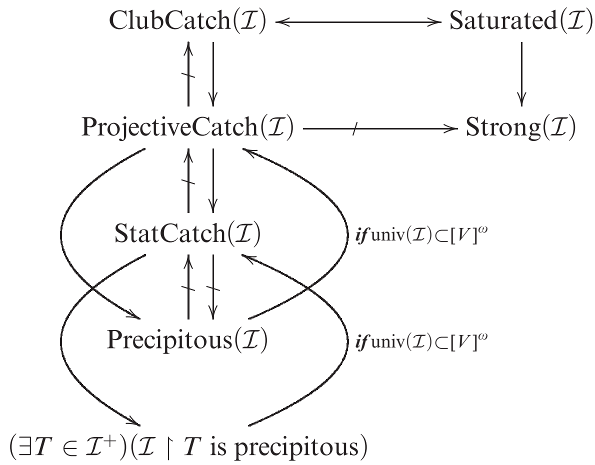


FIGURE 1. Implications and nonimplications.

then the converse also holds (see Theorem 3.8 below; we thank Ralf Schindler for pointing this out to us).

Figure 1 summarizes the implications and nonimplications among these concepts, which are proved in the present paper.

Theorems 1.1 and 1.2 below are the main results of the paper.

THEOREM 1.1. *If there is an $\mathcal{I}$ such that StatCatch$^*(\mathcal{I})$ holds, then there is an inner model with a Woodin cardinal.*

THEOREM 1.2. *Suppose κ is δ-supercompact for some inaccessible $\delta > \kappa$. Let $\mu < \kappa$ be regular. Then there is a forcing extension where $\kappa = \mu^+$, ProjectiveCatch$(\mathcal{I})$ holds for some ideal $\mathcal{I}$ on κ (and in fact the starred version ProjectiveCatch$^*(\mathcal{I})$ holds in the case where $\mu > \omega$), yet $\mathcal{I}$ is not a strong ideal;*[3] *in particular, $\mathcal{I}$ is not presaturated.*

One corollary of Theorem 1.2—see Section 5.5—is that for any regular uncountable κ, we have a negative solution to the $n = 0$ case of Open Question number 13 from Foreman [7], which asks:

QUESTION 1.3 (Foreman). *Suppose that $\mathcal{J}$ is an ideal on $Z \subseteq \wp(\kappa^{+(n+1)})$, and $\mathcal{I}$ is the projected ideal on the projection of Z to $Z' \subseteq \wp(\kappa^{+n})$. Suppose that the canonical homomorphism from $\wp(Z')/\mathcal{I}$ to $\wp(Z)/\mathcal{J}$ is a regular embedding. Is $\mathcal{I}$ $\kappa^{+(n+1)}$-saturated?*

Also, Theorem 1.1 and relative consistency results from [15] and [12][4] imply that, unlike the case for ideals on ω_1, precipitousness of an ideal $\mathcal{I}$ on ω_2 does *not* in general imply *ProjectiveCatch*$^*(\mathcal{I})$ (or even *StatCatch*$^*(\mathcal{I})$).

[3]An ideal $\mathcal{I}$ is *strong* iff it is precipitous and $\mathbb{B}_\mathcal{I}$ forces that the generic embedding sends μ to μ^{+V}, where μ is the completeness of $\mathcal{I}$. Every presaturated ideal on a successor cardinal μ is a strong ideal.

[4]Where it was shown, respectively, that precipitousness of $NS \restriction S^2_1$ can be forced from a model with a measurable cardinal and that precipitousness of $NS \restriction \omega_2$ can be forced from a model with a measurable cardinal of Mitchell order two.

Claverie–Schindler [21] proved that if there is a strong ideal then there is an inner model with a Woodin cardinal; this improved the earlier result by Steel [22], which reached essentially the same conclusion from a presaturated ideal. Theorem 1.2 shows that $StatCatch^*(\mathcal{I})$—the assumption used in our Theorem 1.1—does *not* imply that $\mathcal{I}$ is a strong ideal; so in particular, our Theorem 1.1 is not a special case of the result from [21].

The paper is organized as follows: Section 2 provides background and notation; Section 3 introduces *StatCatch* and *ClubCatch* and proves some basic facts about them; Section 4 proves Theorem 1.1; Section 5 proves Theorem 1.2 and the negative solution to Foreman's question; and Section 6 lists some open questions.

§2. Preliminaries.

Unless otherwise indicated, all notation agrees with Foreman [7]. If κ is regular and $\mu \subseteq H$, then $[H]^{<\mu}$ will denote $\{M \subseteq H \mid |M| < \mu\}$ and $\wp_\mu(H)$ will denote $\{M \in [H]^{<\mu} \mid M \cap \mu \in \mu\}$.

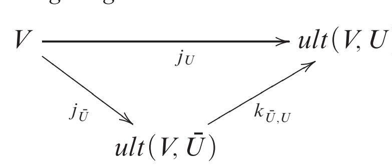
2.1. Ultrapowers. We will use some basic facts about ultrapowers:

FACT 2.1. *Suppose V is a model of set theory, $Z \in V$ is a set, and $U \subset \wp(Z) \cap V$ is an ultrafilter which is fine*[5] *and normal with respect to functions from V;*[6] *we do **not** require that $U \in V$. Let $H := \bigcup Z$ and suppose H is transitive. Let $j_U : V \to_U ult(V, U)$, and suppose the wellfounded part of $ult(V, U)$ has been transitivised. Also assume that each element of Z is extensional (so that it has a transitive collapse). Then:*

- *$j_U'' H \in ult(V, U)$ and is equal to $[id \restriction Z]_U$;*
- *$j_U \restriction H \in ult(V, U)$ and is equal to $[M \mapsto \sigma_M]_U$, where σ_M is the inverse of the transitive collapse map of M*

The following fact is about projections of ultrafilters and the resulting commutative diagram of ultrapowers; for more details (and much greater generality) see Section 4.4 of [7].

FACT 2.2. *Same assumptions as Fact 2.1. If $\bar{Z} \in V$ is another set such that $\bigcup \bar{Z} \subseteq \bigcup Z$ and the map $\pi : Z \to \bar{Z}$ is defined by $M \mapsto M \cap (\bigcup \bar{Z})$, then $\bar{U} := \{\bar{A} \in V \cap \wp(\bar{Z}) \mid \pi^{-1}{}'' \bar{A} \in U\}$ is an ultrafilter on $\wp(\bar{Z}) \cap V$ which is normal with respect to functions from V. Given any $f : \bar{Z} \to V$ (from V), let $F_f := f \circ \pi$. Then the map $k_{\bar{U},U} : ult(V, \bar{U}) \to ult(V, U)$ defined by $[f]_{\bar{U}} \mapsto [F_f]_U$ is well defined, elementary, and the following diagram commutes:*



We also remark:

FACT 2.3. *Same assumptions as Fact 2.2. Set $\bar{H} := \bigcup \bar{Z}$. Assume that $\wp(\bar{Z}) \in M$ for U-many M.*[7] *For each such M let H_M denote the transitive collapse of M*

[5] I.e., for every $a \in \bigcup Z$ the set $\{M \in Z \mid a \in M\}$ is an element of U.

[6] I.e., if $f : S \to V$ is a regressive function with $f \in V$ and $S \in U$, then f is constant on a set from U.

[7] For example, if U is fine and $\bar{Z} = \wp_\kappa(H_{\bar{\lambda}})$ and $Z = \wp_\kappa(H_\lambda)$ for some $\lambda >> \bar{\lambda}$.

and $\sigma_M : H_M \to M$ *denote the inverse of the collapsing map. Let* $\bar{Z}_M = \sigma_M^{-1}(\bar{Z})$ *and set*

$$\bar{\mathcal{U}}_M := \{\bar{a} \in H_M \cap \wp(\bar{Z}_M) \mid M \cap \bar{H} \in \sigma_M(\bar{a})\}$$

Then $\bar{U} \in ult(V, U)$ *and is equal to* $[M \mapsto \bar{\mathcal{U}}_M]_U$.

2.2. Ideals, ideal projections, and antichain catching. Suppose Z is a set and $F \subset \wp(Z)$ is a filter. The *universe of* F ($univ(\mathcal{F})$) is the set Z, and the *support of* F ($supp(F)$) is the set $\bigcup Z$. For example: suppose $\mu \leq \theta$ are regular cardinals, let $Z := \wp_\mu(H_\theta)$ (note $\bigcup Z = H_\theta$), and let F be the collection of $D \subseteq Z$ which contain a club; then F is a normal filter with support H_θ. *For the remainder of the paper, filter will always refer to a normal,*[8] *fine*[9] *filter*; similarly *ideal* will refer to a normal, fine ideal. Note that fineness of a filter implies that the support can be computed from the filter (i.e., if $\mathcal{F}$ is fine then $supp(\mathcal{F}) = \bigcup\bigcup\mathcal{F}$). If $\mathcal{F}$ is a filter then $\breve{\mathcal{F}}$ denotes its dual ideal; similarly if $\mathcal{I}$ is an ideal then $\breve{\mathcal{I}}$ denotes its dual filter. If Γ is a class, we say that a filter $\mathcal{F}$ *concentrates on* Γ iff there is an $A \in \mathcal{F}$ such that $A \subseteq \Gamma$; if $\mathcal{I}$ is an ideal we say that $\mathcal{I}$ *concentrates on* Γ iff its dual filter concentrates on Γ. A set $S \subseteq Z$ is $\mathcal{I}$-*positive* (written $S \in \mathcal{I}^+$) iff $S \notin \mathcal{I}$. If $S \in \mathcal{I}^+$ then $\mathcal{I} \restriction S$ denotes $\mathcal{I} \cap \wp(S)$. NS refers to the class of (weakly) nonstationary sets; that is, $A \in NS$ iff there exists an $F : [\bigcup A]^{<\omega} \to \bigcup A$ such that no element of A is closed under F; in many natural contexts this coincides with the notion of generalized (non)stationarity from Jech [14] (see [7] for more details on when these two notions coincide). Given a stationary set S, $NS \restriction S$ denotes $NS \cap \wp(S)$.

Definition 2.4. *Suppose* $\mathcal{I}'$ *is an ideal with support* Z', $\bigcup Z \subseteq \bigcup Z'$, *and the map* $\pi_{Z',Z} : Z' \to \wp(\bigcup Z)$ *is defined by* $M' \mapsto M' \cap (\bigcup Z)$. *The* ***canonical ideal projection of*** $\mathcal{I}'$ ***to*** Z *is*

$$\{A \subseteq Z \mid \pi_{Z',Z}^{-1}\text{"}A \in \mathcal{I}'\}$$

Example 2.5. *Let* $\lambda < \lambda'$ *be an uncountable cardinals,* $Z' := \wp_{\omega_1}(H_{\lambda'})$, $Z := \wp_{\omega_1}(H_\lambda)$, *and* $\mathcal{I}'$, $\mathcal{I}$ *be the collection of nonstationary subsets of* Z', Z *respectively. Note that* $H_{\lambda'} = supp(\mathcal{I}') = \bigcup Z'$ *and* $H_\lambda = supp(\mathcal{I}) = \bigcup Z$. *Then* $\mathcal{I}$ *is the canonical projection of* $\mathcal{I}'$ *to* $\wp_{\omega_1}(H_\lambda)$.

Example 2.6. *Let* $\mathcal{I}'$ *be as in Example* 2.5. *Let* $Z := \omega_1$ *and* $\mathcal{I}$ *be the nonstationary ideal on* ω_1. *Then* $\mathcal{I}$ *is the canonical ideal projection of* $\mathcal{I}'$ *to* ω_1. *Note here that* $univ(\mathcal{I}) = support(\mathcal{I}) = \omega_1$, *which was not the case in Example* 2.5).

We caution that if $\mu \leq \lambda < \lambda'$, $\pi : \wp_\mu(H_{\lambda'}) \to \wp_\mu(H_\lambda)$ is the map $M \mapsto M \cap H_\lambda$, and $S' \subset \wp_\mu(H_{\lambda'})$ is stationary, then it is **not** true in general that the canonical projection of $NS \restriction S'$ via π is equal to $NS \restriction \pi''S'$; in fact this canonical projection of $NS \restriction S'$ can even be the dual of an ultrafilter (see Fact 2.10 and Remark 2.11 below, and Section 4.4 of [7]).

If $\mathcal{I}$ is an ideal with universe Z, define an equivalence relation $\sim_\mathcal{I}$ on $\wp(Z)$ by $S \sim_\mathcal{I} T$ iff the symmetric difference of S with T is an element of $\mathcal{I}$. Define a relation $\leq_\mathcal{I}$ on $\wp(Z)$ by: $[S]_\mathcal{I} \leq_\mathcal{I} [T]_\mathcal{I}$ iff $S - T \in \mathcal{I}$; it is easy to check this is well-defined

[8] F is normal iff for every regressive $g : Z \to V$ there is an $S \in F^+$ such that $g \restriction S$ is constant.

[9] I.e., for every $b \in supp(F)$ there is an $A \in F$ such that $b \in M$ for all $M \in A$.

and that $\mathbb{B}_{\mathcal{I}} := (\wp(univ(\mathcal{I}))/\mathcal{I}, \leq_{\mathcal{I}})$ is a boolean algebra; $\mathbb{B}_{\mathcal{I}}$ is forcing equivalent to the nonseparative poset $(\mathcal{I}^+, \subset)$.[10]

FACT 2.7. *If $\mathcal{I}$ is a normal ideal on κ then $\mathbb{B}_{\mathcal{I}}$ is a κ^+-complete boolean algebra. Namely, if $Z \subset \mathbb{B}_{\mathcal{I}}$ is a set of size κ, then "the" diagonal union of Z does not depend (modulo $=_{\mathcal{I}}$) on the particular enumeration of Z used to form the diagonal union, and this diagonal union is the least upper bound of Z in $\mathbb{B}_{\mathcal{I}}$.*

If G is $(V, \mathbb{B}_{\mathcal{I}})$-generic then G is essentially an ultrafilter on $\wp(Z) \cap V$, which is normal with respect to functions from V (assuming $\mathcal{I}$ is normal, as we do throughout the paper).

FACT 2.8. *If $\mathcal{J}$ projects canonically to $\mathcal{I}$ then the map*

$$h_{\mathcal{I},\mathcal{J}} : \mathbb{B}_{\mathcal{I}} \to \mathbb{B}_{\mathcal{J}}$$

defined by

$$[S]_{\mathcal{I}} \mapsto [\{M \mid M \cap supp(\mathcal{I}) \in S\}]_{\mathcal{J}}$$

is a boolean homomorphism.

Suppose $\mathcal{J}$ projects canonically to $\mathcal{I}$ and that $G \subset \mathbb{B}_{\mathcal{J}}$ is generic; we will often identify G with $\{S \mid [S]_{\mathcal{J}} \in G\}$. Now G is a normal V-ultrafilter, and the upward closure of $h^{-1}_{\mathcal{I},\mathcal{J}}[G]$ is always a normal V-ultrafilter extending the dual of $\mathcal{I}$; let $\mathrm{proj}(G)$ denote this ultrafilter. However, $\mathrm{proj}(G)$ is **not** necessarily generic for $\mathbb{B}_{\mathcal{I}}$; in other words, the map $h_{\mathcal{I},\mathcal{J}}$ is not necessarily a regular embedding. The regularity of $h_{\mathcal{I},\mathcal{J}}$ is the central issue of this paper, which we will return to in Section 3.

Burke [3], building on work of Foreman (in the special case where $\mathcal{I}$ is maximal), shows that for *any* normal ideal $\mathcal{I}$ and any sufficiently large regular Ω, there is a smallest normal ideal $\mathcal{J}$ with support H_Ω such that $\mathcal{I}$ is the canonical ideal projection of $\mathcal{J}$ to $supp(\mathcal{I})$. Moreover, this $\mathcal{J}$ is easy to describe: for an $M \prec (H_\Omega, \in, \{\mathcal{I}\})$, say that M is $\mathcal{I}$-*good* iff $M \cap supp(\mathcal{I}) \in C$ for every $C \in M \cap \check{\mathcal{I}}$; then the $\mathcal{J}$ mentioned above is just the nonstationary ideal restricted to the collection of $\mathcal{I}$-good substructures of H_Ω (where Ω is sufficiently large relative to $\mathcal{I}$). We refer the reader to [7] for more information about the next few definitions and theorems.

DEFINITION 2.9. *For a regular Ω and an ideal $\mathcal{I}$ with transitive support, set:*

$$S^{Good}_{\mathcal{I},\Omega} := \{M \prec (H_\Omega, \in, \{\mathcal{I}\}) \mid M \text{ is } \mathcal{I}\text{-good}\}$$

Define

$$\Omega(\mathcal{I}) := (2^{univ(\mathcal{I})})^+ \tag{2}$$

$S^{Good}_{\mathcal{I}}$ denotes $S^{Good}_{\mathcal{I},\Omega(\mathcal{I})}$.

The following fact is proved in Proposition 4.20 of [7]:

FACT 2.10. *If $\mathcal{I}$ is an ideal then $S^{Good}_{\mathcal{I}}$ is stationary, and $NS \restriction S^{Good}_{\mathcal{I}}$ projects to $\mathcal{I}$ canonically and is the smallest such ideal (with universe $S^{Good}_{\mathcal{I},\Omega(\mathcal{I})}$) which has this property.*

---

[10] The latter is nonseparative because if $S \in \mathcal{I}^+$ and $T = S - \{x\}$ for some x, then typically $T \in \mathcal{I}^+$ yet every subset of T in $\mathcal{I}^+$ is still compatible with S in $(\mathcal{I}^+, \subset)$.

REMARK 2.11. *We caution that Fact* 2.10 *is quite special*; *it is* ***not*** *true in general that*: *if* $S \subset S^{Good}_{\mathcal{I}}$ *is stationary, then* $NS \restriction S$ *projects canonically to* $\mathcal{I} \restriction \{M \cap supp(\mathcal{I}) \mid M \in S\}$.[11]

DEFINITION 2.12. $NS \restriction S^{Good}_{\mathcal{I}}$ *is called the* conditional club filter relative to $\mathcal{I}$.

The following definitions go back to [10], and are explored in detail in [7].

DEFINITION 2.13. *Suppose* $\mathcal{I}$ *is an ideal with support* H *and* $M \prec (H_\Omega, \in, \{\mathcal{I}\})$ *for a regular* Ω.

- *If* $\mathcal{A}$ *is a maximal antichain in* $\mathcal{I}^+$, *we say* M catches $\mathcal{A}$ *iff there is an* $S \in \mathcal{A} \cap M$ *such that* $M \cap H \in S$.

Given a substructure $M \prec (H_\Omega, \in, \{\mathcal{I}\})$ such that $M \cap supp(\mathcal{I}) \in univ(\mathcal{I})$,[12] let $\sigma_M : H_M \to M \prec H_\Omega$ be the inverse of the transitive collapse of M, let $Z := univ(\mathcal{I})$, $Z_M := \sigma_M^{-1}(Z)$, $\mathcal{I}_M := \sigma_M^{-1}(\mathcal{I})$, and

$$\mathcal{U}_M := \{a \in H_M \cap \wp(Z_M) \mid M \cap supp(\mathcal{I}) \in \sigma_M(a)\}.$$

It is straightforward to check that $\mathcal{U}_M$ is an ultrafilter on $H_M \cap \wp(Z_M)$, and is normal with respect to functions from H_M. Let $j_{\mathcal{U}_M} : H_M \to_{\mathcal{U}_M} ult(H_M, \mathcal{U}_M)$ be the ultrapower embedding and define $k_M : ult(H_M, \mathcal{U}_M) \to H_\Omega$ by $[f]_{\mathcal{U}_M} \mapsto \sigma_M(f)(M \cap supp(\mathcal{I}))$. It is routine to show that k_M is well-defined, elementary, and $\sigma_M = k_M \circ j_{\mathcal{U}_M}$. M is called $\mathcal{I}$-*self-generic* iff $\mathcal{U}_M$ is generic over H_M for the poset $\sigma_M^{-1}(\mathbb{B}_{\mathcal{I}})$.

DEFINITION 2.14. *For a regular* Ω *and an ideal* $\mathcal{I}$, *set*

$$S^{Self\,Gen}_{\mathcal{I},\Omega} := \{M \prec (H_\Omega, \in, \{\mathcal{I}\}) \mid M \text{ is } \mathcal{I}\text{-self generic}\},$$

$$S^{Self\,Gen,*}_{\mathcal{I},\Omega} := S^{Self\,Gen}_{\mathcal{I},\Omega} \cap \{M \mid M \cap ORD \text{ is } \omega\text{-closed}\}.$$

$S^{Self\,Gen}_{\mathcal{I}}$ *and* $S^{Self\,Gen,*}_{\mathcal{I}}$ *denote* $S^{Self\,Gen}_{\mathcal{I},\Omega(\mathcal{I})}$ *and* $S^{Self\,Gen,*}_{\mathcal{I},\Omega(\mathcal{I})}$, *respectively.*[13]

Finally, we recall the relationship between goodness, self-genericity, and antichain catching:

FACT 2.15. *Suppose* $\mathcal{I} \subset \wp(Z)$ *is an ideal. Fix any regular* $\theta >> |\wp(Z)|$ *and* $M \prec (H_\theta, \in, \{\mathcal{I}, Z\})$ *with* $M \cap supp(\mathcal{I}) \in Z$. *Then*:

- *If* M *is* $\mathcal{I}$-*self generic, then* M *is* $\mathcal{I}$-*good.*
- *The following are equivalent*:
  (1) M *is* $\mathcal{I}$-*self generic.*
  (2) M *catches every maximal* $\mathcal{I}$ *antichain which is an element of* M.

Note that if $\mathcal{I}$ *is an ideal on* ω_1, *then* $S^{SelfGen,*}_{\mathcal{I}} = \emptyset$ *because elements of* $S^{Good}_{\mathcal{I}}$ *cannot have* ω-*closed intersection with the ordinals.*[14]

[11] It might happen that there is a stationary $S \subset S^{Good}_{\mathcal{I}}$ and some $T \subset \{M \cap supp(\mathcal{I}) \mid M \in S\}$ such that $T \in \mathcal{I}^+$, yet $\{M \in S \mid M \cap supp(\mathcal{I}) \in T\}$ is nonstationary (though $\{M \in S^{Good}_{\mathcal{I}} \mid M \cap supp(\mathcal{I}) \in T\}$ is stationary, by Fact 2.10).

[12] For example, if $\mathcal{I}$ is an ideal on ω_1 this would just mean that $M \cap \omega_1 \in \omega_1$.

[13] Recall $\Omega(\mathcal{I})$ was defined in (2).

[14] Because if $M \in S^{Good}_{\mathcal{I}}$ then in particular $M \cap \omega_1 \in \omega_1$, so $M \cap ORD$ cannot be ω-closed.

We recall the following definitions:

DEFINITION 2.16. *Let $\mathcal{I}$ be a normal, fine ideal.*

- *$\mathcal{I}$ is* precipitous *iff* $\Vdash_{\mathbb{B}_{\mathcal{I}}}$ "*$ult(V, \dot{G})$ is wellfounded*".
- *$\mathcal{I}$ is* saturated *iff $\mathbb{B}_{\mathcal{I}}$ has the $|H|^+$-chain condition, where H is the support of $\mathcal{I}$ (so $\mathcal{I} \subset \wp(Z)$ where $H = \bigcup Z$).*
- *Suppose $\mathcal{I}$ is an ideal on κ. $\mathcal{I}$ is* strong *iff $\mathcal{I}$ is precipitous and* $\Vdash_{\mathbb{B}_{\mathcal{I}}}$ "$j_{\dot{G}}(\kappa) = \kappa^{+V}$".

Saturation and precipitousness are properties, which occur frequently in the set theory literature. Strongness (of an ideal) was introduced in Baumgartner–Taylor [2]; saturation (even presaturation) of $\mathcal{I}$ implies that $\mathcal{I}$ is a strong ideal. Baumgartner and Taylor conjectured that a strong ideal on ω_1 has the same consistency strength as a saturated ideal on ω_1 (namely, a Woodin cardinal). Their conjecture was recently confirmed in Claverie–Schindler [4], where it was shown that if there is a strong ideal on ω_1 then there is an inner model with a Woodin cardinal. Shelah (see [23]) had shown that one could force over a model with a Woodin cardinal to obtain a model where NS_{ω_1} is saturated (and thus strong). We caution that strongness in the sense of Baumgartner–Taylor [2] is not to be confused with the notion of κ being *ideally strong*, which was introduced in Claverie's PhD thesis and involves a sequence of ideals resembling an extender (the Claverie definition bears more resemblance to strong cardinals than does the Baumgartner–Taylor definition).

2.3. Duality Theorem. We will use a special case of Foreman's Duality Theorem ([7]). Suppose κ is regular and uncountable, $\mathbb{Q}$ is a partial order, and $\dot{U}$ is a $\mathbb{Q}$-name for a V-normal measure on κ. In V define $F(\dot{U})$ by:

$$S \in F(\dot{U}) \iff S \subseteq \kappa \text{ and } \Vdash_{\mathbb{Q}} \check{S} \in \dot{U}.$$

It is straightforward to check that $F(\dot{U})$ is a normal filter on κ. The following is Proposition 7.13 of Foreman [7]:

THEOREM 2.17 (Foreman). *Suppose κ is a regular uncountable cardinal, $\mathbb{Q}$ is a poset, and $\dot{U}$ is a $\mathbb{Q}$-name for a V-normal ultrafilter on κ such that*

$$\Vdash_{\mathbb{Q}} ult(V, \dot{U}) \text{ is wellfounded}.$$

Assume also that there are functions $f_{\mathbb{Q}}$, $(f_q)_{q\in\mathbb{Q}}$, and $f_{\dot{G}}$ with domain κ such that whenever G is $(V, \mathbb{Q})$-generic and $U := \dot{U}_G$ then:

- $j_U(f_{\mathbb{Q}})(\kappa) = \mathbb{Q}$.
- $j_U(f_{\dot{G}})(\kappa) = G$.
- *For each $q \in \mathbb{Q}$: $j_U(f_q)(\kappa) = q$.*

Then, the map

$$[S]_{F(\dot{U})} \mapsto [\![\check{S} \in \dot{U}]\!]_{RO(\mathbb{Q})}$$

is a dense embedding from $\mathbb{B}_{F(\dot{U})} \to RO(\mathbb{Q})$. Also the map

$$q \mapsto [S_q]_{\mathbb{B}_{F(\dot{U})}}$$

is a dense embedding from $\mathbb{Q} \to \mathbb{B}_{F(\dot{U})}$, where

$$S_q := \{\xi < \kappa \mid f_q(\xi) \in f_{\dot{G}}(\xi)\}.$$

§3. ***Catch*$(\mathcal{J},\mathcal{I})$, *StatCatch*$(\mathcal{I})$, and *ClubCatch*$(\mathcal{I})$.** The following definitions each say that, in some sense, the set $S_{\mathcal{I}}^{SelfGen}$ is large (recall $S_{\mathcal{I}}^{SelfGen}$ was defined in Definition 2.14):

Definition 3.1. *Let $\mathcal{I}$ be a normal fine ideal. We say*:

- *ClubCatch$(\mathcal{I})$ holds iff $S_{\mathcal{I}}^{SelfGen}$ is in the conditional club filter relative to $\mathcal{I}$.*[15]
- *ProjectiveCatch$(\mathcal{I})$ holds iff $S_{\mathcal{I}}^{SelfGen}$ "is positive over every $\mathcal{I}$-positive set"; that is, for every $\mathcal{I}$-positive set T, the set*

$$S_{\mathcal{I}}^{SelfGen} \searrow T := \{M \mid M \in S_{\mathcal{I}}^{SelfGen} \text{ and } M \cap supp(\mathcal{I}) \in T\}$$

  is stationary.
- *StatCatch$(\mathcal{I})$ holds iff $S_{\mathcal{I}}^{SelfGen}$ is (weakly) stationary.*[16]

If the completeness of $\mathcal{I}$ is at least ω_2, define ClubCatch$^*(\mathcal{I})$, StatCatch$^*(\mathcal{I})$, and ProjectiveCatch$^*(\mathcal{I})$ similarly, except using $S_{\mathcal{I}}^{SelfGen,*}$ instead of $S_{\mathcal{I}}^{SelfGen}$.

The following is just a reformulation of Lemma 3.46 of [7] to conform to the terminology of this paper:

Theorem 3.2. *$\mathcal{I}$ is saturated $\iff$ ClubCatch$(\mathcal{I})$ holds.*

There is an important difference between *ProjectiveCatch*$(\mathcal{I})$ and *StatCatch*$(\mathcal{I})$. *StatCatch*$(\mathcal{I})$ means that $S_{\mathcal{I}}^{SelfGen}$ is stationary; but by Remark 2.11, this does **not** imply that $NS \restriction S_{\mathcal{I}}^{SelfGen}$ projects canonically to $\mathcal{I}$. However, if the stronger *ProjectiveCatch*$(\mathcal{I})$ holds, then $NS \restriction S_{\mathcal{I}}^{SelfGen}$ **does** project canonically to $\mathcal{I}$. This is due to a more general fact: suppose $\mathcal{J}$ is an ideal which projects canonically to $\mathcal{I}$, and that S is a $\mathcal{J}$-positive set. If S is projective over $\mathcal{I}$—i.e. $S \searrow T$, is $\mathcal{J}$-positive for every $\mathcal{I}$-positive set T—then $\mathcal{J} \restriction S$ projects canonically to $\mathcal{I}$.

Let us define:

Definition 3.3. *Suppose $\mathcal{I}$ is a canonical ideal projection of some ideal $\mathcal{J}$ (in the sense of Definition 2.4). We say that $\mathcal{J}$* catches *$\mathcal{I}$ and write catch$(\mathcal{J},\mathcal{I})$ iff*:

- *the support of $\mathcal{J}$ contains $H_{\Omega(\mathcal{I})}$;*[17] *and*
- *$S_{\mathcal{I},supp(\mathcal{J})}^{SelfGen} \in \breve{\mathcal{J}}$; that is, there are $\mathcal{J}^+$-many $\mathcal{I}$-self-generic structures.*

Observe that the definition of *Catch*$(\mathcal{J},\mathcal{I})$ requires that the support of $\mathcal{J}$ be large relative to $\mathcal{I}$; in particular *catch*$(\mathcal{I},\mathcal{I})$ can never hold.

Lemma 3.4. *Let $\mathcal{I}$ be an ideal. The following are equivalent*:

(1) *ProjectiveCatch$(\mathcal{I})$.*
(2) *There exists an ideal $\mathcal{J}$ such that Catch$(\mathcal{J},\mathcal{I})$ holds.*

Proof. First assume *ProjectiveCatch*$(\mathcal{I})$ holds and set $\mathcal{J} := NS \restriction S_{\mathcal{I}}^{SelfGen}$. The definition of *ProjectiveCatch*$(\mathcal{I})$ easily implies that *Catch*$(\mathcal{J},\mathcal{I})$ holds.

Now assume there exists an ideal $\mathcal{J}$ such that *Catch*$(\mathcal{J},\mathcal{I})$ holds. Let $T \in \mathcal{I}^+$; by definition of *Catch*$(\mathcal{J},\mathcal{I})$:

$$S_{\mathcal{I},\mathrm{supp}(\mathcal{J})}^{SelfGen} \searrow T = \{M \in S_{\mathcal{I},\mathrm{supp}(\mathcal{J})}^{SelfGen} \mid M \cap supp(\mathcal{I}) \in T\} \in \mathcal{J}^+.$$

[15] See Definition 2.12 for the meaning of conditional club filter relative to $\mathcal{I}$.

[16] See the introduction to Section 2.2 for the definition of weakly stationary.

[17] The cardinal $\Omega(\mathcal{I})$ is defined in (2).

Recall that by "ideal" we always mean a normal, fine ideal; this implies that every set in $\mathcal{J}^+$ is stationary. So in particular, $S^{SelfGen}_{\mathcal{I}} \searrow T$ is stationary and the proof is finished. ⊣

There is a similar characterization of *ClubCatch*($\mathcal{I}$):

LEMMA 3.5. *Let $\mathcal{I}$ be an ideal. The following are equivalent*:

(1) *ClubCatch*($\mathcal{I}$) (*recall this is equivalent to saturation of $\mathcal{I}$ by Theorem* 3.2).
(2) *Catch*($\mathcal{J}, \mathcal{I}$) *holds, where $\mathcal{J}$ is the dual of the conditional club filter relative to $\mathcal{I}$.*

The following is a well-known argument:

LEMMA 3.6. *ProjectiveCatch*($\mathcal{I}$) *implies that $\mathcal{I}$ is precipitous. StatCatch*($\mathcal{I}$) *implies that there is some $T \in \mathcal{I}^+$ such that $\mathcal{I} \restriction T$ is precipitous.*

PROOF. First assume *ProjectiveCatch*($\mathcal{I}$). Suppose for a contradiction that $\mathcal{I}$ is not precipitous; then there is some $T \in \mathcal{I}^+$ which forces the $\mathcal{I}$-generic ultrapower to be ill- founded. By definition of *ProjectiveCatch*($\mathcal{I}$), $S^{SelfGen}_{\mathcal{I}} \searrow T$ is stationary. Now $H_{(2^{univ(\mathcal{I})})^+}$ is correct about the fact that T forces an illfounded generic ultrapower. Fix an $M \in S^{SelfGen}_{\mathcal{I}} \searrow T$ such that $M \prec (H_\theta, \in, \{\mathcal{I}, T\})$. As usual let $\sigma_M : H_M \to H_\theta$ be the inverse of the Mostowski collapse of M. Set $\bar{T} := \sigma_M^{-1}(T) = T \cap M$ and $\bar{\mathcal{I}} := \sigma_M^{-1}(\mathcal{I})$. By elementarity of σ_M, H_M believes that $\bar{T}$ forces the $\mathbb{P}_{\bar{\mathcal{I}}}$-generic ultrapower to be illfounded. But $M \in S^{SelfGen}_{\mathcal{I}}$, so the H_M-ultrafilter derived from σ_M is $(H_M, \mathbb{P}_{\bar{\mathcal{I}}})$-generic and $ult(H_M, U)$ is wellfounded. Note also that $\bar{T} \in U$ (since $M \cap supp(\mathcal{I}) \in T = \sigma_M(\bar{T})$). Contradiction.

Now assume only that *StatCatch*($\mathcal{I}$) holds; we want to show that there exists some $T \in \mathcal{I}^+$ such that $\mathcal{I} \restriction T$ is precipitous. Suppose this failed; then $1 \Vdash_{\mathbb{B}_{\mathcal{I}}}$ "the generic ultrapower is illfounded". Pick any $M \in S^{SelfGen}_{\mathcal{I}}$. Then, H_M believes *all* generic ultrapowers are illfounded, contradicting that $ult(H_M, \mathcal{U}_M)$ is wellfounded and $\mathcal{U}_M$ is generic over H_M. ⊣

The following lemma says that if *StatCatch* holds on some restriction of $\mathcal{I}$ then it holds on all of $\mathcal{I}$; in some sense this makes *StatCatch* much less interesting than *ProjectiveCatch*:

LEMMA 3.7. *StatCatch*($\mathcal{I}$) *holds* $\iff$ *StatCatch*($\mathcal{I} \restriction S$) *holds for some $\mathcal{I}$-positive S.*

PROOF. To see the nontrivial direction: suppose $S \in \mathcal{I}^+$ and *StatCatch*($\mathcal{I} \restriction S$) holds. We show:

$$S^{SelfGen}_{\mathcal{I} \restriction S} \cap \{M \mid M \prec (H_\theta, \in, \{\mathcal{I}, S\})\} \subseteq S^{SelfGen}_{\mathcal{I}}. \tag{3}$$

Suppose M is a model from the left side and $A \in M$ is a maximal antichain for $\mathcal{I}$. Then M sees that A can be refined to a maximal antichain of the form $A_S \cup A_{S^c}$, where A_S is a maximal antichain in $\mathcal{I} \restriction S$ and A_{S^c} is a maximal antichain in $\mathcal{I} \restriction S^c$.[18] Since $M \in S^{SelfGen}_{\mathcal{I} \restriction S}$ and $A_S \in M$ then there is some $T \in M \cap A_S$ such that $M \cap supp(\mathcal{I} \restriction S) = M \cap supp(\mathcal{I}) \in T$. But then $M \cap supp(\mathcal{I}) \in T'$, where T' is the unique element of A above T; note $T' \in M$. So we have shown that M catches all of its $\mathcal{I}$-maximal antichains. ⊣

[18]This is just a basic fact about boolean algebras: if A is a maximal antichain and b is an element of the boolean algebra, then $\{a \in A \mid a \leq b\} \cup \{a \in A \mid a \leq b^c\}$ is also a maximal antichain.

We thank Ralf Schindler for giving us permission to include the following theorem and proof, which in particular implies that the converse of Lemma 3.6 holds for ideals on ω_1. We discovered later that (unknown to Schindler) a special case of the theorem also essentially appeared in Ketchersid–Larson–Zapletal [17]:

THEOREM 3.8 (Schindler; Ketchersid–Larson–Zapletal [17]). *Let $\mathcal{I}$ be a normal ideal such that $univ(\mathcal{I})$ consists of countable sets.*[19] *Then, $\mathcal{I}$ is precipitous if and only if $ProjectiveCatch(\mathcal{I})$ holds.*

PROOF. Assume that $\mathcal{I}$ is precipitous; the other direction (that $ProjectiveCatch(\mathcal{I})$ implies precipitousness of $\mathcal{I}$) was already taken care of by Lemma 3.6. First we prove:

CLAIM 3.9. *Let $\mathcal{I}$ be an ideal such that $univ(\mathcal{I})$ consists of countable sets. Suppose H is a transitive set such that ${}^{<\omega}H \subset H$ (typically H will be a transitive ZF^- model), let $F : [H]^{<\omega} \to H$, and let ϕ be a function with domain ω such that $range(\phi) \in univ(\mathcal{I})$. Then there is a tree $T_{\phi,F,\mathcal{I}} \subseteq {}^{<\omega}H$ such that: $T_{\phi,F,\mathcal{I}}$ has an infinite branch iff there exists an $N \in S^{SelfGen}_{\mathcal{I}}$ such that $N \cap supp(\mathcal{I}) = range(\phi)$ and N is closed under F. Moreover, the construction of the tree $T_{\phi,F,\mathcal{I}}$ is absolute between any transitive ZF^- models which have ϕ, F, and $\mathcal{I}$ as elements.*

PROOF. (of Claim) Set $x := range(\phi)$. Let $T_{\phi,F,\mathcal{I}}$ be the set of all sequences $\langle a_0, a_1, \dots, a_n\rangle$ such that $n \in \omega$ and:

(1) $a_i \in H$ and a_i is finite, for each $i \leq n$
(2) $\phi(i) \in a_i$ for each $i \leq n$ (to ensure that a cofinal branch will contain x)
(3) $supp(\mathcal{I}) \cap (a_0 \cup a_1 \cup \dots \cup a_n) \subseteq x$ (to ensure that a branch will not contain any points in $supp(\mathcal{I}) - x$).
(4) For every $j < n$ and every $\vec{v} \in {}^{\leq j}(a_0 \cup a_1 \cup \dots \cup a_j)$: $F(\vec{v}) \in a_{j+1}$ (to ensure that the branch is closed under F)
(5) For each $i < n$: if a_i is a maximal $\mathcal{I}$-antichain then there exists a $S \in a_{i+1}$ such that $x \in S$ and $S \in a_i$ (to ensure that the branch is $\mathcal{I}$-self generic)
(6) For all $i < n$: $a_0 \cup a_1 \cup \dots \cup a_i \subseteq a_{i+1}$ (to ensure that the union of nodes in the branch will include the witnesses built in by the previous bullets).

Clearly $T_{\phi,F,\mathcal{I}}$ is a tree. It is straightforward to prove the claim now. $\dashv$

We now return to the proof of Theorem 3.8. Set $Z := univ(\mathcal{I})$. Let $\theta >> |Z|$, $F : [H_\theta]^{<\omega} \to H_\theta$, and $T \in \mathcal{I}^+$ be arbitrary. We need to find an $N \in [H_\theta]^\omega$ such that N is closed under F, N is $\mathcal{I}$-self generic, and $N \cap supp(\mathcal{I}) \in T$. Let $G \subset \mathbb{B}_\mathcal{I}$ be generic with $T \in G$, and $j : V \to_G ult(V, G)$ the well-founded generic ultrapower. Set $\mathcal{I}' := j(\mathcal{I})$, $H' := j(H_\theta)$, and $F' := j(F)$. By elementarity of $\mathcal{J}$, it suffices to show that $ult(V, G)$ believes there is an $\mathcal{I}'$-good, self-generic $N \in [H']^\omega$ which is closed under F' and such that $N \cap supp(\mathcal{I}') \in j_G(T)$. Now WLOG $supp(\mathcal{I})$ is transitive and so $x := j''_G supp(\mathcal{I}) = [id \restriction Z]_G$ is countable in $ult(V, G)$ (since we are assuming that Z consists only of countable sets); fix some $\phi \in ult(V, G)$ such that $\phi : \omega \to x$ is a bijection. Note also that since $T \in G$, that $x \in j_G(T)$. By Claim 3.9 it suffices to prove that the tree $T_{\phi,F',\mathcal{I}'}$ has an infinite branch in $ult(V, G)$; and since $ult(V, G)$ is wellfounded, it in turn suffices to prove that $T_{\phi,F',\mathcal{I}'}$ has an infinite branch in $V[G]$. Set $N := j''H^V_\theta \in V[G]$. It is easily checked, using Los

[19]For example, if $\mathcal{I}$ is a normal ideal on ω_1, or if $\mathcal{I}$ is a normal ideal on $[H_\theta]^\omega$.

Theorem, that N is $\mathcal{I}'$-self-generic,[20] is closed under F', and $N \cap supp(\mathcal{I}') = x$. Then by Claim 3.9, $T_{\phi,F',\mathcal{I}'}$ has an infinite branch in $V[G]$. ⊣

Theorem 3.8 gives a nice characterization of precipitousness for NS_{ω_1}:[21]

COROLLARY 3.10. *Let* $\mathcal{I} := NS_{\omega_1}$. *Then*:

$$\begin{aligned} &\mathcal{I} \text{ is precipitous} && \iff S^{SelfGen}_{\mathcal{I}} \text{ is projective stationary} \\ &\mathcal{I} \text{ is somewhere precipitous} && \iff S^{SelfGen}_{\mathcal{I}} \text{ is stationary} \end{aligned}$$

The following (which essentially appears in [7]) is a standard application of Łoś Theorem; it says that if $catch(\mathcal{J}, \mathcal{I})$ holds then generics for $\mathbb{B}_{\mathcal{J}}$ project canonically to generics for $\mathbb{B}_{\mathcal{I}}$, and that this projection is an element of the generic ultrapower of V by $\mathcal{J}$.

LEMMA 3.11. *Suppose* $\mathcal{J}$ *projects canonically to* $\mathcal{I}$ *and that* $H_{\Omega(\mathcal{I})} \subseteq supp(\mathcal{J})$. *Let* $h_{\mathcal{I},\mathcal{J}} : \mathbb{B}_{\mathcal{I}} \to \mathbb{B}_{\mathcal{J}}$ *be the canonical boolean homomorphism from Fact* 2.8. *Then, the following are equivalent*:

(1) $catch(\mathcal{J}, \mathcal{I})$.
(2) *Whenever* G *is* $\mathbb{B}_{\mathcal{J}}$*-generic, then* $\bar{U} := h^{-1}_{\mathcal{I},\mathcal{J}}[G]$ *is* $(V, \mathbb{B}_{\mathcal{I}})$*-generic.*
(3) $h_{\mathcal{I},\mathcal{J}}$ *is a regular embedding.*

PROOF. The equivalence of item 1 with item 2 is a standard application of Los' Theorem, using Facts 2.1 and 2.3. The equivalence of item 2 with item 3 is a standard forcing fact. ⊣

COROLLARY 3.12. *Suppose* $\mathcal{J}_2$ *projects canonically to* $\mathcal{J}_1$, *and that* $\mathcal{J}_1$ *projects canonically to* $\mathcal{J}_0$. *Let* $h_{i,j} : \mathbb{B}_{\mathcal{J}_i} \to \mathbb{B}_{\mathcal{J}_j}$ *be the canonical boolean homomorphism* (*for* $i \le j$); *note these maps commute. If* $Catch(\mathcal{J}_2, \mathcal{J}_0)$ *holds then* $h_{0,2}$ *and* $h_{0,1}$ *are each regular embeddings.*

PROOF. That $h_{0,2}$ is a regular embedding follows from Lemma 3.11 (where $\mathcal{J}_2$ plays the role of $\mathcal{J}$ and $\mathcal{J}_0$ plays the role of $\mathcal{I}$). This, in turn, abstractly implies that $h_{0,1}$ is a regular embedding (if f and g are boolean homomorphisms and $f \circ g$ is a regular embedding, then g is also a regular embedding). ⊣

Finally a brief remark about the relationship between $StatCatch(\mathcal{I})$ and the Forcing Axiom for $\mathbb{B}_{\mathcal{I}}$; roughly, $StatCatch(\mathcal{I})$ is the requirement that the Forcing Axiom for $\mathbb{B}_{\mathcal{I}}$ holds in a very nice way. For a poset $\mathbb{P}$, $FA_\mu(\mathbb{P})$ means that for every μ-sized collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$, there is a filter on $\mathbb{P}$ which meets every element of $\mathcal{D}$. Note that $FA_\mu(\mathbb{P})$ is trivially true if $\mu = \omega$.

LEMMA 3.13. *Suppose* $\mathcal{I}$ *is an ideal on* μ^+ *where* μ *is regular. Then*:

$$StatCatch(\mathcal{I}) \implies FA_\mu(\mathbb{B}_{\mathcal{I}}). \tag{4}$$

PROOF. Suppose $StatCatch(\mathcal{I})$ holds, and let $\mathcal{D}$ be a μ-sized collection of dense subsets of $\mathbb{B}_{\mathcal{I}}$. Pick any $M \prec (H_\theta, \in, \{\mathcal{I}, \mathcal{D}\})$ such that $M \in S^{SelfGen}_{\mathcal{I}}$ and $\mu \subset M$. Since $M \in S^{SelfGen}_{\mathcal{I}}$ then the filter $g := \{T \in M \cap \wp(\mu^+) \mid M \cap \mu \in T\}$ is $(M, \mathbb{B}_{\mathcal{I}})$-generic (i.e., $g \cap D \cap M \neq \emptyset$ for each dense $D \in M$). Since $\mu \subset M$ and $\mathcal{D} \in M$, then $\mathcal{D} \subset M$ and so in particular $g \cap D \cap M \neq \emptyset$ for each $D \in \mathcal{D}$. ⊣

[20] Because G is the ultrafilter derived from the transitive collapse of N and is generic over H_θ for $\mathbb{B}_{\mathcal{I}}$.
[21] Note the $\Leftarrow$ directions of Corollary 3.10 are due to Lemma 3.6.

Remark 3.14. *Starting from just one measurable cardinal, Jech–Magidor–Mitchell–Prikry* [15] *proved that one can force $\mathbb{B}_{NS\restriction S^2_1}$ to have a σ-closed dense subset. Since $FA_{\omega_1}(\sigma$-closed$)$ is a theorem of ZFC, then $FA_{\omega_1}(\mathbb{B}_{NS\restriction S^2_1})$ holds in their model.*[22] *Combined with Theorem* 1.1 *of the current paper, it follows that the existence of an ideal $\mathcal{I}$ on ω_2 such that $StatCatch^*(\mathcal{I})$ holds is much stronger (in consistency strength) than the existence of an ideal $\mathcal{I}$ on ω_2 such that $FA_{\omega_1}(\mathbb{B}_{\mathcal{I}})$ holds.*

§4. Lower consistency bound of $StatCatch^*(\mathcal{I})$**.** In the following we focus on ideals on ω_2. Given a cardinal Ω and a structure $M \subseteq H_\Omega$, write

- $\alpha_M = M \cap \omega_2$, and
- $\tilde{\tau}_M = \sup(M \cap \omega_3)$.

We will focus on situations where $\alpha_M \in \omega_2$ and $\tilde{\tau}_M \in \omega_3$. The following theorem implies Theorem 1.1.

Theorem 4.1. *Let $\mathcal{I}$ be a normal fine ideal on ω_2 concentrating on $\omega_2 \cap cof(\omega_1)$ and for sufficiently large Ω let*

$S^*_{\mathcal{I}}$ = *the set of all $M \prec H_\Omega$ satisfying the following requirements*

(a) *M is self-generic with respect to $\mathcal{I}$.*
(b) *$\alpha_M \in \omega_2$ and $\tilde{\tau}_M \in \omega_3$.*
(c) $\mathrm{cf}(\alpha_M), \mathrm{cf}(\tilde{\tau}_M) > \omega$.

If $S^*_{\mathcal{I}}$ is stationary then there is a proper class inner model with a Woodin cardinal.

Proof. Assume there is no proper class inner model with a Woodin cardinal. We will use the core model theory as developed in [22]. In particular, we will assume that there is a measurable cardinal in **V** in order to simplify the situation.

As usual, instead of **K** we will work with a soundness witness W for $\mathbf{K}\,||\,\omega_3$. Thus, W is a thick proper class extender model, and $\mathbf{K}\,||\,\omega_3$ is contained in the Σ_1^W-hull of any thick class in W. We will make a substantial use of the following observation from [4].

$$\text{If } U \text{ is generic for } \mathbb{P}_{\mathcal{I}} \text{ over } \mathbf{V} \text{ and } M = \mathrm{Ult}(V, U) \text{ is well-founded then } W \text{ and } j(W) \text{ agree on the cardinal successor of } \omega_2. \tag{5}$$

We briefly sketch the proof of this fact. The point is that since $\mathbb{P}_{\mathcal{I}}$ is a small forcing, W is still thick in $\mathbf{V}[U]$ and witnesses the soundness of $(\mathbf{K}\,||\,\omega_3)^{\mathbf{V}}$. And since j is the ultrapower map associated with $\mathrm{Ult}(\mathbf{V}, U)$, also $j(W)$ is thick. Now W has the definability and hull property up to ω_2, so the same is true of $j(W)$ as the critical point of j is ω_2. All of the above implies that W and $j(W)$ coiterate to a common proper class extender model with no truncations on either side, and the critical point on the main branches of both sides of the coiteration are at least ω_2.

For each $M \in S^*_{\mathcal{I}}$ let H_M be the transitive collapse of M, $\sigma_M : H_M \to H_\Omega$ be the inverse to the Mostowski collapsing isomorphism, W_M be the collapse of $W\,||\,\Omega$, and $\tau_M = \alpha_M^{+W_M}$, where α_M was introduced above. We also write τ for ω_2^{+W}.

[22] Moreover the measurable cardinal is optimal; if $\mathcal{I}$ is an ideal such that $\mathbb{B}_{\mathcal{I}}$ has a σ-closed dense subset, then $\mathcal{I}$ is precipitous, which implies there is an inner model with a measurable cardinal. In fact Gitik–Shelah [13] showed that if $\mathbb{B}_{\mathcal{I}}$ is a proper poset then $\mathcal{I}$ is precipitous; and Balcar–Franek [1] showed that if $\mathbb{B}_{\mathcal{I}}$ is ω_1-preserving then $\mathcal{I}$ is somewhere precipitous.

We note that by Theorem 0.3 in [4], $\tau = \omega_3$. We will not need this fact, but we bring it to the attention as this fact is responsible for the need of our additional assumption that $\tilde{\tau}_M$ has uncountable cofinality.

Let U_M be the H_M-ultrafilter derived from the map $\sigma_M : H_M \to H_\Omega$. By our assumption on the self-genericity of M with respect to $\mathcal{I}$, the ultrafilter U_M is generic over H_M for the poset $\mathbb{P}^M_{\mathcal{I}} = \sigma_M^{-1}(\mathbb{P}_{\mathcal{I}})$. Let $\tilde{H}_M = \mathsf{Ult}(H_M, U_M)$ and $j_M : H_M \to \tilde{H}_M$ be the associated ultrapower map. We have $\mathsf{cr}(j_M) = \alpha_M$. Finally, let $k_M : \tilde{H}_M \to H_\Omega$ be the factor map between σ_M and j_M, that is, $k_M : [f]_{U_M} \mapsto \sigma_M(f)(\alpha_M)$. Since, $\alpha_M = (\omega_1^{\mathrm{V}})^{+H_M}$ we have $j_M(\alpha_M) = (\omega_1^{\mathrm{V}})^{+\tilde{H}_M}$, and since $k_M \restriction (\alpha_M + 1) = \mathsf{id} \restriction (\alpha_M + 1)$ the critical point of k_M is at least $j_M(\alpha_M)$. Write λ_M for $j_M(\alpha_M)$.

The statement in (5) can be expressed as a statement in the forcing language for $\mathbb{P}_{\mathcal{I}}$ in parameters $W, \mathbb{P}_{\mathcal{I}}$ and ω_2. (Here we actually replace W with its sufficiently long initial segment, in order that the parameter is an element of H_Ω.) By the elementarity of j_M, the same statement in the forcing language for $\mathbb{P}^M_{\mathcal{I}}$ holds in H_M at parameters $W_M, \mathbb{P}^M_{\mathcal{I}}$ and α_M. Since U_M is generic for $\mathbb{P}^M_{\mathcal{I}}$ over H_M, the models W_M and $\tilde{W}_M = j_M(W_M)$ agree on the cardinal successor of α_M, so $\alpha_M^{+\tilde{W}_M} = \tau_M$. By the condensation properties of extender models we have $W_M \,||\, \tau_M = \tilde{W}_M \,||\, \tau_M$, so in particular the models $W_M, \tilde{W}_M$ have the same subsets of α_M. This in turn implies that α_M is inaccessible in W_M and hence, λ_M is inaccessible in $\tilde{W}_M$. (More is true, see for instance [4], but we will not need more in our argument.) Now since k_M is the identity on λ_M the ordinal λ_M is a limit cardinal in W, $\alpha_M^{+W} = k_M(\tau_M) = \tau_M$, and $W \,||\, \tau_M = \tilde{W}_M \,||\, \tau_M = W_M \,||\, \tau_M$. Let F_M be the W_M-extender at (α_M, λ_M) derived from σ_M. Then, F_M is actually a W-extender, that is, it measures all sets in $\mathcal{P}(\alpha_M) \cap W$. We prove

$$F_M \in W. \tag{6}$$

This will yield a contradiction as follows. Since $k_M \restriction \lambda_M$ is the identity, F_M is also the extender at (α_M, λ_M) derived from j_M. The ultrapower map associated with $\mathsf{Ult}(W_M, F_M)$ agrees with j_M on $W_M \,||\, \tau_M = W \,||\, \tau_M$, so $H^W_{\lambda_M} = H^{\tilde{W}}_{\lambda_M} \subseteq \mathsf{Ult}(W_M \,||\, \tau_M, F_M) = \mathsf{Ult}(W \,||\, \tau_M, F_M)$. This says that F_M is a superstrong extender in W, which is impossible.

To see (6), we prove that for all but nonstationarily many structures $M \in S^*_{\mathcal{I}}$ the following holds.

$$\text{The phalanx } (W, \mathsf{Ult}(W, F_M), \lambda_M) \text{ is iterable.} \tag{7}$$

Here it is understood that wellfoundedness is part of the definition of iterability. The conclusion (6) then follows from the core model theory folklore that any extender that coheres to W and satisfies (7) is actually on the W-sequence. This is an instance of theorem 8.6 in [22]. That F_M coheres to W follows from the facts F_M coheres to $\tilde{W}_M$, $\mathsf{cr}(k) \geq \lambda_M$, and from the condensation properties of extender models which imply that the extender sequences of $\tilde{W}_M$ and W agree up to $\lambda_M^{+\tilde{W}_M} = j_M(\tau_M)$. The proof of (7) is a straigthforward adaptation of the frequent extension argument from [19] or its more specified instance in [20], and we will sketch the essentials of this adaptation below.

Let us recall the following terminology. Given two phalanxes (P, Q, λ) and (P', Q', λ') we say that a pair of maps (ρ, σ) is an embedding of (P, Q, λ) into

(P', Q', λ') if and only if $\rho : P \to P'$ and $\sigma : Q \to Q'$ are Σ_0-preserving and cardinal-preserving embeddings such that $\rho \restriction \lambda = \sigma \restriction \lambda$, $\sigma''\lambda \subseteq \lambda'$, and $\sigma(\lambda) \geq \lambda'$. In our argument below we will only make use of Σ_0-embeddings, as we will only be concerned with Σ_0-iterability. A straightforward copying construction yields the following: If P, Q are 1-small premice, (ρ, σ) is an embedding of the phalanx (P, Q, λ) into (P', Q', λ'), and $\mathcal{T}$ is an iteration tree on (P, Q, λ) then $\mathcal{T}$ can be copied onto an iteration tree $\mathcal{T}'$ on (P, Q, λ) via (ρ, σ) (of course, we only consider normal trees here). Thus, if (P', Q', λ') is iterable, then so is (P, Q, λ).

Instead of (7) we actually prove a stronger statement that for all but nonstationarily many $M \in S^*_{\mathcal{I}}$ the phalanx

$$(W, \mathsf{Ult}(W, G_M), \omega_2) \text{ is iterable} \tag{8}$$

where G_M is the W_M-extender at (α_M, ω_2) derived from σ_M. So assume for a contradiction that there is a stationary set $S \subseteq S^*_{\mathcal{I}}$ such that for all $M \in S$ the conclusion (8) fails, and let $\mathcal{T}_M$ be an iteration tree on $(W, \mathsf{Ult}(W, G_M), \omega_2)$ that witnesses the failure of iterability. Let ζ be large enough so that for each $M \in S$ the failure of iterability is already witnessed by $N = W \,||\, \zeta$, that is, when we view $\mathcal{T}_M$ as an iteration tree on $(N, \mathsf{Ult}(N_M, G_M), \omega_2)$ then either $\mathcal{T}_M$ has a last ill-founded model or $\mathcal{T}_M$ is of limit length and does not have a cofinal well-founded branch. Also, pick ζ to be a successor cardinal in W in order to simplify the calculations.

Let θ be a large regular cardinal such that the entire situation described above takes place in H_θ, and for each $M \in S$ let $Z_M \prec H_\theta$ be a countable elementary substructure such that $G_M, \mathcal{T}_M \in Z_M$. Fix the following notation.

- H^Z_M is the transitive collapse of Z_M and $\rho_M : H^Z_M \to H_\theta$ is the inverse to the Mostowski collapsing isomorphism.
- $\bar{N}_M, \bar{\mathcal{T}}_M, \bar{G}_M, \bar{\alpha}_M, \bar{\tau}_M, \bar{\delta}_M$ are the inverse images of $N_M, \mathcal{T}_M, G_M, \alpha_M, \tau_M, \omega_2$ under ρ_M.

Inside the structure H^Z_M the tree $\bar{\mathcal{T}}_M$ witnesses the noniterability of the phalanx $(\bar{N}_M, \mathsf{Ult}(\bar{N}_M, \bar{G}_M), \bar{\delta}_M)$. Since all premice we work with are 1-small, the argument from the proof of Lemma 2.4(b) in [22] shows that $\bar{\mathcal{T}}$ witnesses the noniterability of $(\bar{N}_M, \mathsf{Ult}(\bar{N}_M, \bar{G}_M), \bar{\delta}_M)$ in the sense of $\mathbf{V}$.

Recall that $\tau = \omega_2^{+W}$ and $\tilde{\tau}_M = \sup(\sigma''_M \tau_M)$. Let S' be the set of all $M' \prec H_\theta$ such that $M' \cap H_\Omega \in S$. Then, S' is a stationary set, and so is $S_1 = \{M' \cap H_\Omega \mid M' \in S'\}$. Given a model $M \in S_1$ we show that there is a set $a \in M$ such that $Y_M = \sigma''_M(Z \cap W \,||\, \tau_M) \subseteq a \subseteq M$. Obviously, Y_M is a countable subset of $W \,||\, \tilde{\tau}_M$ and $\tilde{\tau}_M \leq \tau$. If $\tau < \omega_3$ then there is a surjection $f : \omega_2 \to W \,||\, \tau$ such that $f \in M$. Otherwise, we use our assumption that $\tilde{\tau}_M$ has uncountable cofinality, so $\sup(Y_M) < \tilde{\tau}_M$. In this case pick any $\tau' \in M \cap \omega_3$ such that $\tau' > \sup(Y_M)$; then again there is some surjection $f : \omega_2 \to W \,||\, \tau'$ such that $f \in M$ (see our comments at the beginning of the proof. The case $\tau < \omega_3$ is actually vacuous, but we chose to include it here in order to demonstrate that the argument does not rely on the knowledge that $\omega_2^{+\mathbf{K}} = \omega_3$). Since $Y_M \subseteq M$ is countable and α_M has uncountable cofinality, there is some $\beta < \alpha_M$ such that $Y_M \subseteq f''\beta$. Letting $a = f''\beta$, it is clear that a satisfies the above requirements. Notice also that the conclusion $a \subseteq M$ follows immediately from the facts that $a \in M$, $\mathsf{card}(a) = \omega_1$, and $\omega_1 + 1 \subseteq M$.

Working in H_θ, assume $M \in S_1$ is of the form $M' \cap H_\Omega$ for some $M' \in S'$. Then, letting a be as in the previous paragraph, the set M witnesses the existential quantifier in the following statement.

$$H_\theta \models (\exists v \in S)(a \in v).$$

Since $M' \prec H_\theta$, there is some $\bar{M} \in S$ such that $a \in \bar{M}$. The last sentence in the previous paragraph applied to $\bar{M}$ in place of M yields $a \subseteq \bar{M}$. Thus, $Y_M \subseteq \bar{M}$. It follows that there is a regressive map $g : S_1 \to S$ such that $Y_M \subseteq g(M)$ for all $M \in S_1$. Press down and obtain a stationary $S^* \subseteq S_1$ and a structure $M^* \in S$ such that $g(M) = M^*$ for all $M \in S^*$. We thus have the following: the structure M^* is an element of S, the set $S^* \subseteq S$ is stationary, and $Y_M \subseteq M^* \subseteq M$ whenever $M \in S^*$. In the following we write α^* for α_{M^*}.

Given two structures $M, M' \in S$ such that $M \in M'$ there is a partial elementary map $\sigma_{M,M'} = \sigma_{M'}^{-1} \circ \sigma_M$ from M into M'. For $M \in S^*$ let

$$\tau_M^* = \sup((\sigma_{M^*,M}^{-1}) \circ \rho_M)''\bar{\tau}_M).$$

By the construction of M^* the map

$$\sigma_{M^*,M}^{-1} \circ \rho_M \restriction (\bar{N}_M \mid \bar{\tau}_M) : \bar{N}_M \mid \bar{\tau}_M \to W_{M^*} \mid \tau_M^*$$

is total. (Recall that $R \mid \beta$ denotes the initial segment of R of height β without the extender E_β^R as its top predicate, whereas $R \,||\, \beta$ denotes the corresponding initial segment with E_β^R as a top predicate.) Moreover, this map is Σ_0-preserving and cofinal. We can now apply the argument in the proof of the interpolation lemma (see [24], Lemma 3.6.10) to construct a premouse N_M^* such that $W_{M^*} \mid \tau_M^* \lhd N_M^*$ and $\tau_M^* = (\alpha^*)^{+N_M^*}$, along with Σ_0-preserving maps $\sigma_M^* : \bar{N}_M \to N_M^*$ and $\sigma_M' : N_M^* \to N$ such that σ_M^* extends $\sigma_{M^*,M}^{-1} \circ \rho_M \restriction (\bar{N}_M \mid \bar{\tau}_M)$, σ_M' extends $\sigma_{M^*,M} \restriction (W_{M^*} \mid \tau_M^*)$, and $\sigma_M' \circ \sigma_M^* = \rho_M$. Let us merely mention here that N_M^* is the ultrapower of $\bar{N}_M$ using the map $\sigma_{M^*,M}^{-1} \circ \rho_M \restriction (\bar{N}_M \,||\, \bar{\tau}_M)$, and σ_M' is the corresponding factor map. Here all premice are passive ZFC^--models, so N_M^* is a premouse, and both σ_M^* and σ_M' are actually fully elementary. Also, the map σ_M', when viewed as a map from N_M^* into W, is Σ_0-preserving.

Given a phalanx (W, Q, α^*) and a premouse (possibly a proper class one) Q', we write $Q' <_{\mathrm{S}} Q$ if and only if there is a normal iteration tree on (W, Q, α^*) such that Q' is an initial segment of the last model $M_\infty^{\mathcal{T}}$ of $\mathcal{T}$, and one of the following holds.

(a) W is on the main branch of $\mathcal{T}$.
(b) Q is on the main branch of $\mathcal{T}$ and there is a truncation on this main branch.
(c) Q is on the main branch of $\mathcal{T}$, there is no truncation on this main branch, and Q' is a proper initial segment of $M_\infty^{\mathcal{T}}$.

We will make heavy use of the following essential result; see [19], Lemma 3.2 or [20], proof of Theorem 3.4.

$$\text{The relation } <_{\mathrm{S}} \text{ is well-founded below } W. \tag{9}$$

That is, if we let $Q_0 = W$ then any sequence of models Q_n such that $Q_{n+1} <_{\mathrm{S}} Q_n$ is finite. Let us just stress that the conclusion in (9) may not be true for a general extender model W, but it is based, in a crucial way, on the fact that W is a soundness witness for an initial segment of $\mathbf{K}$, which is embeddable into $\mathbf{K}^c$.

Our initial assumption (precisely the fact that $M^* \in S$) guarantees that the phalanx $(W, \mathrm{Ult}(W, G_{M^*}), \omega_2)$ is not iterable. By (9) fix an $<_S$-minimal premouse Q below W with respect to $<_S$ witnessing the noniterability of $(W, \mathrm{Ult}(Q, G_{M^*}), \omega_2)$. That is, following hold.

(a) (W, Q, α^*) is iterable and $(W, \mathrm{Ult}(Q, G_{M^*}), \omega_2)$ is not iterable.
(b) If $Q' <_S Q$ then $(W, \mathrm{Ult}(Q', G_{M^*}), \omega_2)$ is iterable.

Notice that Q is a set size model, as the noniterability of a proper class model is witnessed by some if its proper initial segments.

By the construction of M^*, N^*_M and the maps σ^*_M, σ'_M, for every $a \in [\bar{\delta}_M]^{<\omega}$ and every $x \in [\bar{\alpha}_M]^{|a|}$ the following are equivalent for any $M \in S^*$.

- $x \in (\bar{G}_M)_a$.
- $\rho_M(x) \in (G_M)_{\rho_M(a)}$.
- $\rho_M(a) \in \sigma_M(\rho_M(x))$.
- $\rho_M(a) \in \sigma_{M^*}(\sigma^*_M(x))$.
- $\sigma^*_M(x) \in (G_{M^*})_{\rho_M(a)}$.

The usual copying argument then yields that $\rho'_M : [a, f]_{\bar{G}_M} \mapsto [\rho_M(a), \sigma^*_M(f)]_{G_{M^*}}$ is a Σ_0-preserving cardinal-preserving embedding from $\mathrm{Ult}(\bar{N}_M, \bar{G}_M)$ into $\mathrm{Ult}(N^*_M, G_{M^*})$; moreover $\rho'_M \restriction \bar{\delta}_M = \rho_M \restriction \bar{\delta}_M$ and $\rho'_M \circ \pi_{\bar{G}_M} = \pi_{G_{M^*}} \circ \sigma^*_M$ where $\pi_{\bar{G}_M}$ and $\pi_{G_{M^*}}$ are the corresponding ultrapower embeddings. Note also that $\rho'_M(\bar{\delta}_M) = \omega_2$. It follows that the pair (ρ_M, ρ'_M) is an embedding of the phalanx $(\bar{N}_M, \mathrm{Ult}(\bar{N}_M, \bar{G}_M), \bar{\delta}_M)$ into $(W, \mathrm{Ult}(N^*_M, G_{M^*}), \omega_2)$. This proves:

$$\text{The phalanx } (W, \mathrm{Ult}(N^*_M, G_{M^*}), \omega_2) \text{ is not iterable.} \tag{10}$$

Notice also that the phalanx (W, N^*_M, α^*) is iterable, because the pair (id, σ'_M) is an embedding of (W, N^*_M, α^*) into W.

The following reflection argument shows that the extender G_{M^*} can be replaced with an extender with shorter support; this will be needed below. Let θ' be large enough such that in $H_{\theta'}$ there is an iteration tree $\mathcal{R}$ witnessing the noniterability of the phalanx $(W \,||\, \tilde{\zeta}, \mathrm{Ult}(Q, G_{M^*}), \omega_2)$ for a suitable $\tilde{\zeta}$. Pick some countable elementary substructure X of $H_{\theta'}$ such that $\mathcal{R} \in X$; let H be the transitive collapse of X and $\sigma : H \to H_{\theta'}$ be the inverse to the Mostowski collapsing isomorphism. Then, $\mathcal{R}' = \sigma^{-1}(\mathcal{R})$ witnesses the noniterability of the phalanx $(W', \mathrm{Ult}(Q', G'), \beta')$ where, $\sigma(W', Q', \beta') = (W \,||\, \tilde{\zeta}, Q, \omega_2)$, again by the proof of Lemma 2.4(b) in [22]. Pick $M \in S^*$ such that $\alpha_M > \sup(X \cap \omega_2)$, and let $G = G_{M^*} \,|\, \alpha_M$. By the construction of the map σ'_M and by our choice of Q, the restriction of G to sets in Q agrees with the Q-extender derived from the map σ'_M. Since $x \in G'_a$ implies $\sigma(a) \in G_{\sigma(a)}$ for all $a \in [\beta']^{<\omega}$ and $x \in \mathcal{P}([\alpha']^{|a|}) \cap Q$, where $\alpha' = \sigma^{-1}(\alpha^*)$, the map $\sigma' : [a, f]_{G'} \mapsto [\sigma(a), \sigma(f)]_G$ maps $\mathrm{Ult}(Q', G')$ into $\mathrm{Ult}(Q, G)$ elementarily, $\sigma' \restriction \beta' = \sigma \restriction \beta' \subseteq \alpha_M$, and $\sigma'(\beta') = \pi_G(\alpha^*) \geq \alpha_M$; here of course π_G is the ultrapower embedding associated with $\mathrm{Ult}(Q, G)$. The pair (σ, σ') is thus an embedding of the phalanx $(W', \mathrm{Ult}(Q', G'), \beta')$ into $(W \,||\, \tilde{\zeta}, \mathrm{Ult}(Q, G), \alpha_M)$, witnessing that

$$\text{The phalanx } (W, \mathrm{Ult}(Q, G), \alpha_M) \text{ is not iterable.} \tag{11}$$

From now on the proof follows very closely the final argument in [19]. We work with M and Q picked above. Let $(\mathcal{U}, \mathcal{V})$ be the pair of iteration trees coming from the terminal coiteration of (W, Q, α^*) against (W, N_M^*, α^*), where $\mathcal{U}$ is on (W, Q, α^*) and $\mathcal{V}$ is on (W, N_M^*, α^*). The extender model W is thick as it is a soundness witness for an initial segment of $\mathbf{K}$, so W cannot be on the main branch on both sides of both trees.

We first argue that Q must be on the main branch $b^{\mathcal{U}}$ of $\mathcal{U}$. Otherwise, $M_\infty^{\mathcal{V}} <_{\mathsf{S}} Q$, and N_M^* is on the main branch $b^{\mathcal{V}}$ of $\mathcal{V}$. By the $<_{\mathsf{S}}$-minimality of Q the phalanx $(W, \mathsf{Ult}(M_\infty^{\mathcal{V}}, G_{M^*}), \omega_2)$ must be iterable. As W is thick there is no truncation on $b^{\mathcal{V}}$ and $M_\infty^{\mathcal{V}} \trianglelefteq M_\infty^{\mathcal{U}}$. The critical point of the iteration map $\pi_{b^{\mathcal{V}}}$ along the main branch of $\mathcal{V}$ is at least α^*, so the map $k : \mathsf{Ult}(N_M^*, G_{M^*}) \to \mathsf{Ult}(M_\infty^{\mathcal{V}}, G_{M^*})$ defined by $k : [a, f]_{G_{M^*}} \mapsto [a, \pi_{b^{\mathcal{V}}}(f) \restriction [\alpha^*]^{|a|}]_{G_{M^*}}$ is an elementary embedding with critical point strictly above ω_2, witnessing that the pair (id, k) is an embedding of the phalanx $(W, \mathsf{Ult}(N_M^*, G_{M^*}), \omega_2)$ into $(W, \mathsf{Ult}(M_\infty^{\mathcal{V}}, G_{M^*}), \omega_2)$. As we proved above that the former phalanx is not iterable, this shows that the latter phalanx cannot be iterable either, a contradiction.

Recall again that the pair (id, σ'_M) is an embedding of the phalanx (W, N_M^*, α^*) into W. Let $\mathcal{V}'$ be the iteration tree on W obtained by copying $\mathcal{V}$ via the pair (id, σ'_M), and let $\sigma_\infty : M_\infty^{\mathcal{V}} \to M_\infty^{\mathcal{V}'}$ be the map between the last models of $\mathcal{V}$ and $\mathcal{V}'$. Obviously, $\mathcal{V}'$ is a normal iteration tree on W with iteration indices strictly above α_M. By the agreement between the copy maps, $\sigma_\infty \restriction \nu = \sigma'_M \restriction \nu$, where ν is the first iteration index used in $\mathcal{V}$. In particular, σ_∞ agrees with σ'_M on all sets in $\mathcal{P}([\alpha^*]^{<\omega}) \cap N_M^* \,||\, \nu$.

We next show that either there is a truncation on $b^{\mathcal{U}}$ or $M_\infty^{\mathcal{V}}$ is a proper initial segment of $M_\infty^{\mathcal{U}}$. Otherwise, $M_\infty^{\mathcal{U}} \trianglelefteq M_\infty^{\mathcal{V}}$ and we have the iteration map $\pi_{b^{\mathcal{U}}} : Q \to M_\infty^{\mathcal{U}}$ along the main branch of $\mathcal{U}$. The critical point of $\pi_{b^{\mathcal{U}}}$ is at least α^*, so $\mathcal{P}([\alpha^*]^{<\omega}) \cap Q = \mathcal{P}([\alpha^*]^{<\omega}) \cap M_\infty^{\mathcal{U}}$. As pointed out above, the extender G restricted to the sets in Q agrees with the Q-exteder derived from σ'_M, so the same also holds when we replace Q with $M_\infty^{\mathcal{U}}$ and σ'_M with σ_∞. Let $W_\infty = \sigma_\infty(M_\infty^{\mathcal{U}})$. Standard arguments then show that the map $k : \mathsf{Ult}(M_\infty^{\mathcal{U}}, G) \to W_\infty$ defined by $k : [a, f]_G \mapsto \sigma_\infty(f)(a)$ is a Σ_0-preserving cardinal preserving embedding with critical point strictly above α_M. (We of course let $W_\infty = M_\infty^{\mathcal{V}'}$ if $M_\infty^{\mathcal{U}} = M_\infty^{\mathcal{V}}$.) It follows that the pair (id, k) is an embedding of the phalanx $(W, \mathsf{Ult}(M_\infty^{\mathcal{U}}, G), \alpha_M)$ into (W, W_∞, α_M). Now W_∞ is an initial segment of the last model on the normal iteration tree $\mathcal{V}'$ on W with indices strictly above α_M, and W, being a soundness witness for an initial segment of $\mathbf{K}$, is embeddable into $\mathbf{K}^c$. It follows that the phalanx (W, W_∞, α_M) can be embedded into a $\mathbf{K}^c$-generated phalanx, which is iterable by Theorem 6.9 in [22]. Hence, (W, W_∞, α_M) is also iterable, and so is $(W, \mathsf{Ult}(M_\infty^{\mathcal{U}}, G), \alpha_M)$. On the other hand, an argument similar to the one above in the proof that Q is on the main branch of $\mathcal{U}$ shows that, letting $k : \mathsf{Ult}(Q, G) \to \mathsf{Ult}(M_\infty^{\mathcal{U}}, G)$ be the map defined by $k : [a, f]_G \mapsto [a, \pi_{b^{\mathcal{U}}}(f) \restriction [\alpha^*]^{|a|}]_G$, the pair (id, k) is an embedding of $(W, \mathsf{Ult}(Q, G), \alpha_M)$ into $(W, \mathsf{Ult}(M_\infty^{\mathcal{U}}, G), \alpha_M)$. As we have seen that $(W, \mathsf{Ult}(Q, G), \alpha_M)$ is not iterable, neither is $(W, \mathsf{Ult}(M_\infty^{\mathcal{U}}, G), \alpha_M)$. This is a contradiction.

To summarize, we arrived at the conclusion that Q is on the main branch of $\mathcal{U}$, and either there is a truncation on the main branch $b^{\mathcal{U}}$ or $M_\infty^{\mathcal{V}}$ is a proper initial segment of $M_\infty^{\mathcal{U}}$. This means that $M_\infty^{\mathcal{V}} <_{\mathsf{S}} Q$, hence the phalanx $(W, \mathsf{Ult}(M_\infty^{\mathcal{V}}, G_{M^*}), \omega_2)$

must be iterable by the mininality of Q. On the other hand, we have seen in (10) that this phalanx is not iterable, which yields our final contradiction. ⊣

§5. Forcing models of *ProjectiveCatch*. In this section we investigate variations of the Kunen and Magidor constructions of saturated ideals from huge and almost-huge cardinals; in particular, what happens when their large cardinal assumptions are significantly weakened (roughly, weakened to slightly more than a supercompact cardinal). We ultimately prove that, starting from a κ which is δ-supercompact for some inaccessible $\delta > \kappa$, we can produce models of $ProjectiveCatch(\mathcal{I})$ (where $\mathcal{I}$ is nonstrong) on any successor of a regular cardinal (See Theorem 5.37).

5.1. Towers of supercompactness measures. First a few basic facts about towers of supercompactness measures (see e.g., Kanamori [16] for more details). Note that the definition of tower below allows for the possibility that the height of the tower is a successor ordinal; this is done in order to keep a uniform terminology for some of the later theorems.

Definition 5.1. *Let δ be an ordinal. A sequence $\vec{U} = \langle U_\gamma \mid \gamma < \delta \rangle$ is called a* $P_\kappa(-)$-tower of height δ *iff*:

(1) *For each $\gamma < \delta$: U_γ is a normal measure on $P_\kappa(\gamma)$.*
(2) *For each $\gamma < \gamma'$: U_γ is the projection of $U_{\gamma'}$ to γ.*

If $\vec{U}$ is a $P_\kappa(-)$-tower of height δ, there is a natural directed system and direct limit map $j_{\vec{U}} : V \to_{\vec{U}} ult(V, \vec{U})$.

Remark 5.2. *If the height of $\vec{U}$ is a successor ordinal $\beta + 1$, then the ultrapower by $\vec{U}$ is just the same as the ultrapower by the largest measure on the sequence; i.e., the ultrapower by U_β.*

Definition 5.3. *A $P_\kappa(-)$-tower $\vec{U}$ of height δ is called an* almost huge tower *iff δ is inaccessible and $j_{\vec{U}}(\kappa) = \delta$.*

We list some basic facts about towers; more details can be found in Kanamori [16].

Fact 5.4. *Suppose $\vec{U}$ is a $P_\kappa(-)$ tower of height δ. Then,*

(a) *$\kappa = crit(j_{\vec{U}})$, $j_{\vec{U}}(\kappa) \geq \delta$, and $ult(V, \vec{U})$ is closed under $< cf(\delta)$-sequences (so in particular is wellfounded if $cf(\delta) > \omega$).*
(b) *If $\delta = lh(\vec{U})$ is inaccessible, then the following are equivalent:*
   - *$j_{\vec{U}}$ is an almost huge embedding.*
   - *$j_{\vec{U}}(\kappa) = \delta$.*
(c) *If δ is inaccessible then $j_{\vec{U}}$"$H_\delta \in H_{\delta^+}$.*
(d) *If U is a normal measure on $P_\kappa(\delta)$ for some inaccessible $\delta > \kappa$, then the projections of U to $P_\kappa(\lambda)$ (for $\lambda < \delta$) form a tower of height δ. If δ is, for example, the least inaccessible or least weakly compact cardinal above κ, then this tower will **not** be an almost huge tower (i.e., $j_{\vec{U}}(\kappa) > \delta$).*
(e) *If $j : V \to N$ is some almost huge embedding with critical point κ such that $j(\kappa) = \delta$, then there is an almost huge tower $\vec{U}$ of height δ and a map $k : ult(V, \vec{U}) \to N$ such that $k \circ j_{\vec{U}} = j$.*
(f) *If δ is regular then $j_{\vec{U}}$ is continuous at δ.*
(g) *If $\vec{U}$ is almost huge and δ is Mahlo, then for almost every inaccessible $\gamma < \delta$, the system $\vec{U} \restriction \gamma$ is almost huge.*

(h) *If $\vec{U}'$ is a strict end-extension of $\vec{U}$ then there is a natural map $k := k_{\vec{U},\vec{U}'}$: $N_{\vec{U}} \to N_{\vec{U}'}$ such that $j_{\vec{U}'} = k \circ j_{\vec{U}}$. Let $\delta := ht(\vec{U})$; if δ is inaccessible then*:

$$crit(k) \in \{\delta, \delta^{+N_{\vec{U}}}\}. \tag{12}$$

Furthermore for any $\gamma < \delta$ and any $F : P_\kappa(\gamma) \to V$:

$$k(j_{\vec{U}}(F)(j_{\vec{U}}``\gamma)) = j_{\vec{U}'}(F)(j_{\vec{U}'}``\gamma). \tag{13}$$

PROOF. These facts are well-known, and we refer the reader to Kanamori [16]. Items (f) and (h)are very important for this paper, so we provide brief explanations. To see item (f): let $\eta < j_{\vec{U}}(\delta)$. Then, since ult$(V, \vec{U})$ is a direct limit, there is some $\lambda < \delta$ such that $\eta \in range(k_{U_\lambda,\vec{U}})$, where $k_{U_\lambda,\vec{U}}$ is the map from ult$(V, U_\lambda) \to$ ult$(V, \vec{U})$ in the direct limit diagram. Now δ is a fixed point of the map j_{U_λ}; so $k^{-1}_{U_\lambda,\vec{U}}(\eta) < \delta$. So pick any $\zeta \in (k^{-1}_{U_\lambda,\vec{U}}(\eta), \delta)$; then $j_{\vec{U}}(\zeta) \in (\eta, j_{\vec{U}}(\delta))$.

To see item (h): it is straightforward to see (by examining the directed systems for $\vec{U}$ and $\vec{U}'$) that $crit(k) \geq \delta$, where $k := k_{\vec{U},\vec{U}'}$ is the natural map from ult$(V, \vec{U}) \to$ ult$(V, \vec{U}')$; note that k is not to be confused with $k_{U'_\delta,\vec{U}'}$.[23] Moreover, since $\vec{U}'$ has height $> \delta$, then $N_{\vec{U}'}$ computes δ^+ correctly, whereas $N_{\vec{U}}$ does not (by item (c)). This implies that $crit(k) \leq \delta^{+N_{\vec{U}}}$. Since $crit(k)$ must be an $N_{\vec{U}}$-cardinal, this leaves δ and $\delta^{+N_{\vec{U}}}$ as the only possibilities for $crit(k)$. Each of these possibilities occur in nature.[24]

To see (13): fix some $\gamma < \delta$ and note that

$$|j_{\vec{U}}``\gamma|^{N_{\vec{U}}} = \gamma$$

which is $< crit(k)$ by (12). So $k(j_{\vec{U}}``\gamma) = k``(j_{\vec{U}}``\gamma)$. Then

$$k(j_{\vec{U}}(F)(j_{\vec{U}}``\gamma)) = k(j_{\vec{U}}(F))(k(j_{\vec{U}}``\gamma)) = j_{\vec{U}'}(F)(k``(j_{\vec{U}}``\gamma)) = j_{\vec{U}'}(F)(j_{\vec{U}'}``\gamma).$$

⊣

5.2. Review of regular embeddings. For a suborder $\mathbb{R}$ of a partial order $\mathbb{P}$, we say that $\mathbb{R}$ is a *regular suborder* of $\mathbb{P}$ iff $\leq_{\mathbb{R}}$ agrees with $\leq_{\mathbb{P}}$, $\perp_{\mathbb{R}}$ agrees with $\perp_{\mathbb{P}}$, and every maximal antichain in $\mathbb{R}$ is a maximal antichain in $\mathbb{P}$. It is well-known that this is equivalent to a Σ_0 statement about $\mathbb{R}$ and $\mathbb{P}$. Namely, given $p \in \mathbb{P}$ and $r \in \mathbb{R}$, we say that r is a pseudoprojection of p on $\mathbb{R}$ iff $r' ||_{\mathbb{P}} p$ for every $r' \leq_{\mathbb{R}} r$. Then:

FACT 5.5. *For a suborder $\mathbb{R}$ of $\mathbb{P}$, the following are equivalent*:

(1) *$\mathbb{R}$ is a regular suborder of $\mathbb{P}$.*
(2) *For every $p \in \mathbb{P}$ there exists an $r \in \mathbb{R}$ such that r is a pseudoprojection of p on $\mathbb{R}$.*

In particular, the statement "$\mathbb{R}$ is a regular suborder of $\mathbb{P}$" is Σ_0 and thus, absolute across transitive ZF^- models.

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[23]The domain of $k = k_{\vec{U},\vec{U}'}$ is the direct limit ult$(V, \vec{U})$, whereas the domain of $k_{U'_\delta,\vec{U}'}$ is the δ-supercompactness ultrapower ult(V, U'_δ).

[24]For example, if $\vec{U}'$ is almost huge of height δ', then $crit(k_{\vec{U}' \restriction \delta,\vec{U}'}) = \delta$ for almost every strong limit $\delta < \delta'$. On the other hand, if δ is the first inaccessible above κ and $\vec{U}'$ is a tower of height $\delta' > \delta$, then $k_{\vec{U}' \restriction \delta,\vec{U}'}$ fixes δ (because $N_{\vec{U}'}$ models "δ is the least inaccessible above κ") and so $crit(k_{\vec{U}' \restriction \delta,\vec{U}'})$ must be $\delta^{+N_{\vec{U}}}$.

The following convention will justify the notation in Theorem 5.12 and elsewhere.[25]

Fact 5.6. *Suppose* $\mathbb{R}, \mathbb{P}$ *are partial orders and* $\mathbb{R}$ *is a regular suborder of* $\mathbb{P}$. *Suppose* D *is a dense subset of* $\mathbb{P}$. *Let* $G \subset \mathbb{R}$ *be generic. In* $V[G]$ *define* $\frac{\mathbb{P}}{G} := \{p \in \mathbb{P} \mid p||_{\mathbb{P}} G\}$, *and* $\frac{D}{G} := \{p \in D \mid p||_{\mathbb{P}} G\}$ (*here* $p||_{\mathbb{P}} G$ *means that* p *is* $\mathbb{P}$-*compatible with each member of* G). *Then* $\frac{D}{G}$ *is a dense subset of* $\frac{\mathbb{P}}{G}$.

Proof. Let $p \in \frac{\mathbb{P}}{G}$. Let $\tilde{G}$ be a $(V[G], \frac{\mathbb{P}}{G})$-generic such that $p \in \tilde{G}$; it is standard that $G \subset \tilde{G}$ and that $\tilde{G}$ is $(V, \mathbb{P})$-generic. This implies that $\tilde{G}$ meets the set $D \cap p \downarrow_{\mathbb{P}}$ (because that set is dense below p and $p \in \tilde{G}$). Pick any $d \in \tilde{G} \cap D \cap p \downarrow_{\mathbb{P}}$. Then d, being in $\tilde{G} \supset G'$, is compatible with each member of G'. Thus d is an element of $\frac{D}{G}$ and $d \leq q$. ⊣

We also use:

Fact 5.7. *Suppose* $\mathbb{P}$ *is a poset*, $\dot{\mathbb{Q}}$ *and* $\dot{\mathbb{R}}$ *are* $\mathbb{P}$-*names for posets*, $\dot{e}$ *is a* $\mathbb{P}$-*name, and*

$$\Vdash_{\mathbb{P}} \dot{e} \text{ is a regular embedding from } \dot{\mathbb{Q}} \to \dot{\mathbb{R}}.$$

Define $\ell : \mathbb{P} * \dot{\mathbb{Q}} \to \mathbb{P} * \dot{\mathbb{R}}$ *by*

$$(p, \dot{q}) \mapsto (p, \dot{e}(\dot{q})).$$

Then ℓ *is a regular embedding.*

Proof. It is easy to see that ℓ is $\leq$ and $\perp$-preserving. To see regularity: let $(p, \dot{r})$ be an element of $\mathbb{P} * \dot{\mathbb{R}}$. Then p forces that $\dot{r}$ has a pseudoprojection via $\dot{e}$; so let $\dot{q}_{\dot{r}}$ be a name for this pseudoprojection. Now check that $(p, \dot{q}_{\dot{r}})$ is a pseudoprojection of $(p, \dot{r})$ via ℓ: let $(p', \dot{q}') \leq (p, \dot{q}_{\dot{r}})$. We need to show that $\ell(p', \dot{q}') = (p', \dot{e}(\dot{q}'))$ is compatible with $(p, \dot{r})$. Let g be generic for $\mathbb{P}$ with $p' \in g$, let $r := (\dot{r})_g$, $q_r := (\dot{q}_{\dot{r}})_g$, $q' := (\dot{q}')_g$, and $e := \dot{e}_g$. In $V[g]$, since $q' \leq q_r$ and q_r is a pseudoprojection of r via e, then $e(q')$ is compatible with r, as witnessed by some t. Then $(p', \dot{t})$ witnesses that $\ell(p', \dot{q}') = (p', \dot{e}(\dot{q}'))$ is compatible with $(p, \dot{r})$. ⊣

5.3. Generalization of Magidor's argument, and duality. Building on earlier work of Kunen and Laver (who used huge cardinals to produce saturated ideals on successor cardinals), Magidor proved that if $\mu < \kappa$ is a regular cardinal and $\vec{U}$ is an almost huge $P_\kappa(-)$-tower of height δ, then letting $\mathbb{P}$ be the appropriate $< \mu$-closed Kunen collapse which turns κ into μ^+, there is a saturated ideal on κ in the model $V^{\mathbb{P} * Col(\kappa, <\delta)}$. Recall that saturation of $\mathcal{I}$ is equivalent to $ClubCatch(\mathcal{I})$.

We aim to salvage much of the Magidor argument in the case where $\vec{U}$ is not necessarily almost huge. This serves several ends; it will enable us to:

(1) force instances of $ProjectiveCatch(\mathcal{I})$ for ideals on any successor cardinal from much weaker large cardinal assumptions than those used to force instances of $ClubCatch(\mathcal{I})$ (i.e., saturation of $\mathcal{I}$). Namely: whereas the only known models of saturated ideals on ω_2 start with almost huge embeddings, we will produce a model of $ProjectiveCatch(\mathcal{I})$ for an ideal $\mathcal{I}$ on ω_2, starting from only a κ which is supercompact up to (and including) an inaccessible.

[25] In Theorem 5.12 we have a regular embedding ι whose range is contained in $RO^N(j(\mathbb{P}))$ for some separative partial order $j(\mathbb{P})$. Fact 5.6 justifies dropping the RO^N part when forming quotients.

(2) Provide a general theory of ideals obtained from tower embeddings, where the height of the tower is turned into a successor cardinal.

The following assumptions are fixed for the remainder of the paper.

HYP 1. *$\vec{U}$ is a $P_\kappa(-)$-tower of inaccessible height δ, and $j : V \to_{\vec{U}} N$ is the ultrapower embedding.*

HYP 2. *$\mathbb{P} \subset V_\kappa$ is a κ-cc poset, μ is a regular cardinal below κ which remains a cardinal in $V^{\mathbb{P}}$, and $\Vdash_{\mathbb{P}} \kappa = \mu^+$. If $\vec{U}$ is **not** almost huge, we also require that $\mathbb{P}$ is $< \mu$-distributive.*

HYP 3. *In N there is a regular embedding $\iota : \mathbb{P} * Col(\kappa, < \delta) \to RO^N(j(\mathbb{P}))$ such that ι is the identity on $\mathbb{P}$.*[26]

HYP 4. *$G * H$ is a $(V, \mathbb{P} * Col(\kappa, < \delta))$-generic.*

If $\vec{U}$ is almost huge, then the standard example of such a $\mathbb{P}$ is the universal $< \mu$-closed Kunen collapse obtained via an amalgamated forcing; see Cummings [6] for details. If $\vec{U}$ is not almost huge—i.e., if $j(\kappa) > \delta$—then one could still use the $< \mu$-closed universal Kunen collapse; but in this case $\mathbb{P} := Col(\mu, < \kappa)$ would also work, since in that case $Col(\mu, < \kappa) * Col(\kappa, < \delta)$ is a $< \mu$-closed poset of size $< j(\kappa)$, and $j(\kappa)$ is inaccessible in N; so by standard absorption techniques of Levy collapses, N would have an ι as in HYP 3. For some of the later theorems dealing with $ProjectiveCatch$ we will place additional requirements on the poset $\mathbb{P}$ and the regular embedding ι.[27]

THEOREM 5.8. *Suppose $\hat{G}$ is $(V[G][H], j(\mathbb{P})/\iota``G * H))$-generic. Then, in $V[\hat{G}]$ there is an $\hat{H}$ which is $(N[\hat{G}], Col^{N[\hat{G}]}(j(\kappa), < j(\delta)))$-generic and an elementary embedding*

$$\tilde{j}_{\hat{G}} : V[G][H] \to N[\hat{G}][\hat{H}]$$

which extends j.

REMARK 5.9. *Theorem 5.8 is a slight improvement over the existing literature because:*

(1) *$\vec{U}$ is not required to be almost huge.*
(2) *The $\hat{H}$ constructed in $V[\hat{G}]$ is really an $(N[\hat{G}], Col^{N[\hat{G}]}(j(\kappa), < j(\delta)))$-generic object containing $\hat{j}``H$.*[28] *In the authors' view, this makes the subsequent "duality" computations conceptually simpler than the arguments in [11], [7], and [8]. In those papers, instead of finding an $\hat{H} \in V[\hat{G}]$ as in Theorem 5.8, a so-called "pseudo-generic tower" of conditions from $Col^{N[\hat{G}]}(j(\kappa), < j(\delta))$ is defined in $V[\hat{G}]$ in a way which decided enough of the generic embeddings—embeddings which they view as appearing in $V[\hat{G}]^{Col^{N[\hat{G}]}(j(\kappa), < j(\delta))}$ but not necessarily in $V[\hat{G}]$—in order to define a $V[G][H]$-normal ideal and compute its corresponding boolean algebra. However, both arguments ultimately provide liftings of embeddings in some small generic extension of $V[G][H]$.*

[26]More precisely: we require that $\iota(p, 1) = p$ for every $p \in \mathbb{P}$.

[27]Namely we will eventually add the following additional requirements (which are superfluous in the case where $\vec{U}$ is almost huge, i.e., when $j(\kappa) = \delta$). We will require that $range(\iota) \subset j(\mathbb{P}) \cap (H_{\delta^+})^N$, that $j(\mathbb{P}) \cap (H_{\delta^+})^N$ is regular in $j(\mathbb{P})$, and that V believes any generic for $j(\mathbb{P}) \cap (H_{\delta^+})^N$ will be extendable to an N-generic for $j(\mathbb{P})$. These additional requirements do hold for the examples of $\mathbb{P}$ given above.

[28]where $\hat{j} : V[G] \to N[\hat{G}]$ is the intermediate lifting which exists because $j``G \subset \hat{G}$.

Theorem 5.8 does not quite seem to suffice for our applications in Section 5.4, so we prove a more general version (Theorem 5.12) below. The generalized version uses the following technical definition:

Definition 5.10. *Given a transitive model W of ZFC, we will say that W resembles $V^{j(\mathbb{P})/\iota``G*H}$ iff:*

(1) *j is definable in W and there is some $\hat{g} \in W$ which is $(N[G][H], j(\mathbb{P})/\iota``G * H)$-generic (though $\hat{g}$ is not necessarily $(V[G][H], j(\mathbb{P})/\iota``G * H)$-generic).*
(2) *If $\vec{U}$ is almost huge then $N[\hat{g}]$ is $< \delta$-closed from the point of view of W.*
(3) *If $\vec{U}$ is not almost huge then $N[\hat{g}]$ is $< \mu$-closed from the point of view of W.*

We will say that such a $\hat{g}$ witnesses the resemblance *of W to $V^{j(\mathbb{P})/\iota``G*H}$.*

Remark 5.11. *If $\hat{G}$ is $(V[G][H], j(\mathbb{P})/\iota``G * H)$-generic,*[29] *then $\hat{G}$ witnesses that $W := V[\hat{G}]$ resembles $V^{j(\mathbb{P})/\iota``G*H}$ in the sense of Definition* 5.10. *Thus, Theorem* 5.8 *is a special case of Theorem* 5.12.

Proof. If $\vec{U}$ is almost huge then $j(\mathbb{P})$ is δ-cc in V, and standard arguments show that $N[\hat{G}]$ is $< \delta$-closed from the point of view of $V[\hat{G}]$.

If $\vec{U}$ is not almost huge then the $< \mu$-distributivity requirement in the Background Hypotheses from page 1267 implies that $N[\hat{G}]$ will be $< \mu$-closed from the point of view of $V[\hat{G}]$. ⊣

For expository purposes, *uppercase letters will be reserved for filters which are generic over $V[G][H]$, whereas lowercase letters are allowed to be merely generic over N or extensions of N. Also "hats" will typically indicate that the filter is on the j-image of posets*. In later sections we will be compelled to work with some $\hat{g} \in V[\hat{G}]$ which may not be generic over $V[G][H]$, so we state the following theorem in its full generality:

Theorem 5.12. *Suppose W resembles $V^{j(\mathbb{P})/\iota``G*H}$ (in the sense of Definition* 5.10) *and let $\hat{g} \in W$ witness this resemblance. Then in W there is an $\hat{h}$ which is $(N[\hat{g}], Col^{N[\hat{g}]}(j(\kappa), < j(\delta)))$-generic and an elementary embedding*

$$\tilde{j}_{\hat{g}} : V[G][H] \to N[\hat{g}][\hat{h}]$$

which extends j.

Proof. (of Theorem 5.12) We work inside W for the entire proof. Note that $G * H$ is the pointwise preimage of $\hat{g}$ via ι. Then, $G * H \in N[\hat{g}]$, since $\hat{g}$ and ι are elements of $N[\hat{g}]$. Also our assumptions on ι guarantee that

$$j``G \subset \hat{g}$$

and thus there is an elementary

$$\hat{j} : V[G] \to N[\hat{g}]$$

which extends j.

For each ordinal $\gamma < \delta$ let $H|\gamma$ denote $H \cap Col(\kappa, < \gamma)$ and set

$$m^H_\gamma := \bigcup(\hat{j}``H|\gamma).$$

[29] Recall that even though the range of ι may not be literally contained in $j(\mathbb{P})$, Fact 5.6 allows us to write $j(\mathbb{P})/\iota``G * H$ instead of the more cumbersome $RO^N(j(\mathbb{P}))/\iota``G * H$.

Since $G * H \in N[\hat{g}]$ and $j \restriction V_\gamma$ is an element of N for every $\gamma < \delta$, it follows that:

$$\forall \gamma < \delta \; \hat{j} \restriction V_\gamma[G] \in N[\hat{g}] \text{ and } m_\gamma^H \in N[\hat{g}]. \tag{14}$$

For any $p \in H|\gamma$, $|p|^{V[G]} < \kappa$ (by definition of the Levy collapse) and $\kappa = crit(\hat{j})$, so

$$(\forall \gamma < \delta)(\forall p \in H|\gamma)(\hat{j}(p) = \hat{j}\text{"}p \text{ and } |\hat{j}(p)|^{N[\hat{g}]} < \kappa). \tag{15}$$

It follows that $|m_\gamma^H|^{N[\hat{g}]} = |\bigcup(\hat{j}\text{"}H|\gamma)|^{N[\hat{g}]} \leq |\gamma|^{N[\hat{g}]}|\kappa|^{N[\hat{g}]} < \hat{j}(\kappa)$. So

$$(\forall \gamma < \delta)(\; m_\gamma^H \in Col^{N[\hat{g}]}(j(\kappa), < j(\gamma))). \tag{16}$$

Claim 5.13. *For each $\gamma < \delta$: $dom(m_\gamma^H) = \kappa \times j\text{“}\gamma$. Moreover, for any $\gamma < \gamma' < \delta$:*

$$m_{\gamma'}^H \restriction (j(\kappa) \times j(\gamma)) = m_{\gamma'}^H \restriction (\kappa \times j\text{"}\gamma) = m_\gamma^H. \tag{17}$$

Proof. These follow straightforwardly from (15). ⊣

Note that $\langle m_\gamma^H \mid \gamma < \delta \rangle$ is a descending sequence. It has the following important property:

Claim 5.14. *For any $\gamma < \delta$ and any $r \in Col^{N[\hat{g}]}(j(\kappa), < j(\gamma))$ such that $r \leq m_\gamma^H$: for every $\gamma' \in [\gamma, \delta)$: r is compatible with $m_{\gamma'}^H$ in $Col^{N[\hat{g}]}(j(\kappa), < j(\gamma'))$.*

Proof. This follows immediately from Claim 5.13. ⊣

Claim 5.15. *$N[\hat{g}]$ is closed under $< cf^W(\delta)$ sequences from W. Moreover:*

- *If $\vec{U}$ is not almost huge then $|\delta| = cf(\delta) = \mu$ from the point of view of both W and $N[\hat{g}]$.*
- *If $\vec{U}$ is almost huge then δ is regular from the point of view of both W and $N[\hat{g}]$.*

Proof. Suppose first that $\vec{U}$ is not almost huge; i.e., $j_{\vec{U}}(\kappa) > \delta$. Then, $|\delta|^{N[\hat{g}]} = cf^{N[\hat{g}]}(\delta) = \mu$. By Definition 5.10, $N[\hat{g}]$ and W have the same $< \mu$ sequences. So $cf^W(\delta) = cf^{N[\hat{g}]}(\delta)$.

If $\vec{U}$ is almost huge then $\delta = j_{\vec{U}}(\kappa)$ is regular in N and thus in $N[\hat{g}]$. By Definition 5.10, $N[\hat{g}]$ is closed under $< \delta$ sequences from W, so δ is regular in W as well. ⊣

For each $\eta \leq j(\delta)$ let $\mathbb{R}_{<\eta} := Col^{N[\hat{g}]}(j(\kappa), < \eta))$. In $N[\hat{g}]$ let

$$\mathcal{A} := \{A \subset \mathbb{R}_{<j(\delta)} \mid A \text{ is a maximal antichain}\}.$$

Since $N[\hat{g}]$ believes that $\mathbb{R}_{<j(\delta)}$ has the $j(\delta)$-cc and has cardinality $j(\delta)$, then $|\mathcal{A}|^{N[\hat{g}]} = j(\delta)$. For each $A \in \mathcal{A}$ let $D_A := \{r \in \mathbb{R}_{<j(\delta)} \mid \exists a \in A \; r \leq a\}$; now set $\mathcal{D} := \{D_A \mid A \in \mathcal{A}\}$. So $\mathcal{D} \in N[\hat{g}]$ is, in $N[\hat{g}]$, a $j(\delta)$-sized collection of all the relevant dense subsets of $\mathbb{R}_{<j(\delta)}$ (“relevant” in the sense that for a filter to be $(N[\hat{g}], \mathbb{R}_{<j(\delta)})$-generic, it suffices that the filter meets each element of $\mathcal{D}$).

Also, since $j(\delta)$ is inaccessible in $N[\hat{g}]$ then $N[\hat{g}]$ believes that $Col^{N[\hat{g}]}(j(\kappa), < j(\delta))$ has the $j(\delta)$-cc, so:

$$\forall D \in \mathcal{D} \; U_D := \{\eta < j(\delta) \mid D \cap \mathbb{R}_{<\eta} \text{ is dense in } \mathbb{R}_{<\eta}\} \text{ is unbounded (in fact club) in } j(\delta). \tag{18}$$

Using the following facts:

- $j(\delta) \in [\delta, \delta^{+V})$;[30]
- $\delta \leq j(\kappa)$;
- $j(\mathbb{P})$ adds a surjection from μ onto every ordinal $< j(\kappa)$;
- j is continuous at δ;[31] and
- j is definable in W (by definition of resemblance),

it follows that:

$$\lambda := |j(\delta)|^W = |\delta|^W = cf^W(\delta) = cf^W(j(\delta)). \tag{19}$$

Recall we are working in W. We now construct a descending sequence $\langle r_i \mid i < \lambda \rangle$ in $\mathbb{R}_{<j(\delta)}$ which will generate a $(N[\hat{g}], \mathbb{R}_{<j(\delta)})$-generic filter which contains $\hat{j}$"H; note that, in order for the filter generated by $\vec{r}$ to contain $\hat{j}$"H as a subset, it will suffice to arrange that m^H_γ is in the filter generated by $\vec{r}$ for cofinally many $\gamma < \delta$.

Let $\langle D_k \mid k < \lambda \rangle$ enumerate $\mathcal{D}$. Recursively construct a descending sequence $\langle r_k \mid k < \lambda \rangle$ in $\mathbb{R}_{<j(\delta)}$ and an increasing (not necessarily continuous) sequence $\langle \eta_k \mid k < \lambda \rangle$ of ordinals in $j(\delta)$ as follows. We maintain the following induction hypotheses:

$$r_k \in D_k \cap \mathbb{R}_{<j(j^{-1}\text{"}\eta_k)}, \tag{20}$$

$$r_k \leq m^H_{j^{-1}\text{"}\eta_k}. \tag{21}$$

Base step:

- Using (18), let η_0 be some ordinal $< j(\delta)$ such that $D_0 \cap \mathbb{R}_{<\eta_0}$ is dense in $\mathbb{R}_{<\eta_0}$.
- Observe that $m^H_{j^{-1}\text{"}\eta_0} \in \mathbb{R}_{<sup(j\text{"}(j^{-1}\text{"}\eta_0))} \subseteq \mathbb{R}_{<\eta_0}$. Let r_0 be some condition in $D_0 \cap \mathbb{R}_{<\eta_0}$ such that $r_0 \leq m_{j^{-1}\text{"}\eta_0}$.

Successor Step: Suppose $k < \lambda$ and $\langle r_i \mid i \leq k \rangle$ and $\langle \eta_i \mid i \leq k \rangle$ have been defined.

- Using (18), let η_{k+1} be some ordinal $< j(\delta)$ such that $D_{k+1} \cap \mathbb{R}_{<\eta_{k+1}}$ is dense in $\mathbb{R}_{<\eta_{k+1}}$ and such that $\eta_{k+1} > \sup(\{\eta_i \mid i \leq k\})$.[32]
- By (20), (21), and Claim 5.14, r_k and $m_{j^{-1}\text{"}\eta_{k+1}}$ are compatible in $\mathbb{R}_{<\eta_{k+1}}$; let r_{k+1} be a condition in $D_{k+1} \cap \mathbb{R}_{<\eta_{k+1}}$ below both of them. Clearly, the inductive hypothesis (21) is maintained. Also $j(j^{-1}\text{"}\eta_{k+1}) \geq \eta_{k+1}$ so the induction hypothesis (20) is also maintained.

Limit Case: Suppose k is a limit ordinal $< \lambda$ and that $\langle r_\ell \mid \ell < k \rangle$ and $\langle \eta_\ell \mid \ell < k \rangle$ have been constructed. Note that by Claim 5.15, these sequences are each elements of $N[\hat{g}]$. Set $r := \bigcup_{\ell<k} r_\ell$ and $\beta := \sup_{\ell<k} j(j^{-1}\text{"}\eta_\ell)$. Then, by the induction hypotheses (20) and (21):

$$r \in \mathbb{R}_{<\beta}, \text{ so } dom(r) \subset j(\kappa) \times \beta, \tag{22a}$$

$$r \supseteq \bigcup_{\ell<k} m^H_{j^{-1}\text{"}\eta_\ell}. \tag{22b}$$

[30] By item (c) of Fact 5.4.
[31] By item (f) of Fact 5.4.
[32] Note this supremum is $< j(\delta)$ because $k < \lambda$).

Using (18), let η_k be some ordinal $< j(\delta)$ such that $D_k \cap \mathbb{R}_{<\eta_k}$ is dense in $\mathbb{R}_{<\eta_k}$ and such that $\eta_k > \sup\{\eta_\ell \mid \ell < k\}$. Note that $m^H_{j^{-1}\text{``}\eta_k} \upharpoonright j(\kappa) \times \beta = \bigcup_{\ell<k} m_{j^{-1}\text{``}\eta_\ell}$; this fact combined with (22a) and (22b) imply that r is compatible with $m^H_{j^{-1}\text{``}\eta_k}$. Let $r_k$ be some condition in $D_k \cap \mathbb{R}_{<\eta_k}$ which is below both $r$ and $m^H_{j^{-1}\text{``}\eta_k}$.

This completes the construction of the sequences $\vec{r}$ and $\vec{\eta}$. Note that $\langle \eta_k \mid k < \lambda \rangle$ will automatically be cofinal in $j(\delta)$, since for every $\zeta < j(\delta)$ there is some $D \in \mathcal{D}$ such that no $r \in D$ is an element of $\mathbb{R}_{<\zeta}$.[33] This, along with (21), guarantees that the upward closure of $\vec{r}$ contains every m^H_γ. Thus, the upward closure of $\vec{r}$ contains $\hat{j}\text{``}H$. ⊣

There is some freedom in Theorem 5.12 (depending on the enumeration of the dense sets in the proof), so for each $\hat{g}$ we just fix one lifting:

Definition 5.16. *Given a W and a $\hat{g} \in W$ as in the hypotheses of Theorem* 5.12, *we fix some $\hat{h}_{\hat{g}}$ and $\tilde{j}_{\hat{g}}$ as given by the conclusion of Theorem* 5.12. *We will often refer to $\tilde{j}_{\hat{g}}$ as "the" lifting given by Theorem* 5.12.

Definition 5.17. *Suppose $\gamma < \delta$ and $F \in V$ is some function with domain $P_\kappa(\gamma)$. In $V[G][H]$ pick any ϕ which is a surjection from $\kappa \to_{onto} \gamma$, and define $f_{F,\phi} : \kappa \to V[G][H]$ by:*

$$\xi \mapsto F(\phi\text{``}\xi)$$

for any ξ where this is defined.

Lemma 5.18. *Let $\gamma < \delta$ and $F \in V$ be any function with domain $P_\kappa(\gamma)$. Set $z := j(F)(j\text{``}\gamma)$. Let $\phi \in V[G][H]$ be any surjection from $\kappa \to_{onto} \gamma$ and let $f_{F,\phi}$ be as defined in Definition* 5.17.

Then for any model W which resembles $V^{j(\mathbb{P})/\iota\text{``}G*H}$ (in the sense of Definition 5.10) *and any $\hat{g} \in W$ which witnesses this resemblance, if $\tilde{j} = \tilde{j}_{\hat{g}}$ is the embedding given by Theorem* 5.12, *then*:

$$z = \tilde{j}(f_{F,\phi})(\kappa).$$

Proof. Fix such a model W and a $\hat{g} \in W$, and let $\tilde{j} := \tilde{j}_{\hat{g}}$ be the lifting of j. It is easy to see that $\tilde{j}(\phi)\text{``}\kappa = j\text{``}\gamma$. So:

$$\tilde{j}(f_{F,\phi})(\kappa) = f_{\tilde{j}(F),\tilde{j}(\phi)}(\kappa) = \tilde{j}(F)(\tilde{j}(\phi)\text{``}\kappa) = \tilde{j}(F)(j\text{``}\gamma) = j(F)(j\text{``}\gamma) = z \quad ⊣$$

Definition 5.19. *Let $z \in N$. Pick any representation $z = j(F)(j\text{``}\gamma)$ of z. In $V[G][H]$ pick any surjection $\phi : \kappa \to_{onto} \gamma$ and set $f_z := f_{F,\phi}$.*

Note that by Lemma 5.18, the choice of F and ϕ in the definition of f_z will not matter in terms of $\tilde{j}_{\hat{g}}(f_z)(\kappa)$ (where $\hat{g} \in W$ and W is any model resembling $V^{j(\mathbb{P})/\iota\text{``}G*H}$ in the sense of Definition 5.10). The following lemma is used in the next section:

Lemma 5.20. *Suppose $\vec{U}'$ is an end extension of $\vec{U}$ and $k : N_{\vec{U}} \to N_{\vec{U}'}$ is the function given by Fact* 5.4; *let $j' : V \to_{\vec{U}'} N_{\vec{U}'}$ be the ultrapower embedding. Suppose $\tilde{j}' : V[G][H] \to N_{\vec{U}'}[\hat{g}'][\hat{h}']$ is an elementary embedding which extends j'. Then for every $z \in N$:*

$$\tilde{j}'(f_z)(\kappa) = k(z), \tag{23}$$

[33] E.g., let E be the dense set $\{r \in \mathbb{R}_{<j(\delta)} \mid \zeta \in proj_1(dom(r))\}$, let A be a maximal antichain in E; then $A \in \mathcal{A}$ so D_A is the desired element of $\mathcal{D}$.

where f_z is the function in $V[G][H]$ as defined in Definition 5.19.

Proof. Say $z = j(F_z)(j"\gamma)$ and let $\phi_\gamma \in V[G][H]$ be a bijection from $\kappa \to \gamma$. Note that since the critical point of $\tilde{j}'$ is κ then $\tilde{j}'(\phi_\gamma)``\kappa = \tilde{j}'``\gamma$, and so:

$$\tilde{j}'(f_z)(\kappa) = \tilde{j}'(F_z)(\tilde{j}'(\phi_\gamma)``\kappa) = j'(F_z)(j'``\gamma) = k(j(F_z)(j``\gamma)) = k(z), \quad (24)$$

where the second equality uses the fact that $j' \subset \tilde{j}'$ and the next-to-last equation is by item (h) of Fact 5.4. ⊣

In particular, if $k(z) = z$ then the function f_z—although it is defined according to the map $j_{\vec{U}}$—will also represent z in ultrapowers derived from liftings of the map j'.

We also see that the tower embedding by $\vec{U}$ is turned into a simple ultrapower embedding by a measure on κ:

Corollary 5.21. *Let W resemble $V^{j(\mathbb{P})/\iota``G*H}$ as witnessed by $\hat{g} \in W$, and let $\tilde{j} := \tilde{j}_{\hat{g}} : V[G][H] \to N[\hat{g}][\hat{h}]$ be the embedding given by Theorem* 5.12. *Then $\tilde{j}$ is an ultrapower embedding by its derived measure on κ; i.e.*

$$N[\hat{g}][\hat{h}] = \{\tilde{j}(f)(\kappa) \mid f \in V[G][H] \cap {}^{\kappa}V[G][H]\}.$$

Moreover, for any $b \in N[G][H]$ there is a function $f_b \in V[G][H]$ that will always represent b in any such ultrapower; i.e., whenever W and $\hat{g} \in W$ are as above then it will always be the case that $b = \tilde{j}_{\hat{g}}(f_b)(\kappa)$.

Proof. Consider an arbitrary element $(j(F)(j``\gamma))_{\hat{g}*\hat{h}}$ of $N[\hat{g}][\hat{h}]$, where $F : P_\kappa(\gamma)$ maps into the $\mathbb{P} * Col(\kappa, < \delta)$ names. In $V[G][H]$ pick any surjection $\phi : \kappa \to_{onto} \gamma$ and define the function $h_F : \kappa \to V[G][H]$ by:

$$\xi \mapsto (F(\phi``\xi))_{G*H}.$$

Note that $\tilde{j}(G * H) = \hat{g} * \hat{h}$ by elementarity of $\tilde{j}$. Also $\tilde{j}(\phi)``\kappa = j``\gamma$ and so

$$\tilde{j}(h_F)(\kappa) = (h_{\tilde{j}(F)})^{N[\hat{g}][\hat{h}]}(\kappa) = (\tilde{j}(F)(\tilde{j}(\phi)``\kappa))_{\tilde{j}(G*H)} = (j(F)(j``\gamma))_{\hat{g}*\hat{h}}.$$

Thus, our arbitrary element of $N[\hat{g}][\hat{h}]$ has the correct form.

To see the "moreover" part of the corollary: let $b \in N[G][H]$, say $b = (j(F)(j"\gamma))_{G*H}$ and let $\phi \in V[G][H]$ be a bijection from $\kappa \to \gamma$. Recall the regular embedding $\iota : \mathbb{P} * Col(\kappa, < \delta) \to j(\mathbb{P})$ is assumed to be an element of N; let $f_\iota \in V[G][H]$ as defined in Definition 5.19. In $V[G][H]$ define a function $f_b : \kappa \to V[G][H]$ by

$$\xi \mapsto (F(\phi``\xi))_{f_\iota(\xi)^{-1}"G}. \quad (25)$$

Then if W resembles $V^{j(\mathbb{P})/\iota``G*H}$ as witnessed by some $\hat{g}$, then letting $\tilde{j} := \tilde{j}_{\hat{g}*\hat{h}}$:

$$\begin{aligned}\tilde{j}(f_b)(\kappa) &= (f_{\tilde{j}(b)})^{N[\hat{g}][\hat{h}]}(\kappa) = (\tilde{j}(F)(\tilde{j}(\phi)"\kappa))_{\tilde{j}(f_\iota)(\kappa)^{-1}"\tilde{j}(G)} \\ &= (j(F)(j"\gamma))_{\tilde{j}(f_\iota)(\kappa)^{-1}"\tilde{j}(G)} = (j(F)(j"\gamma))_{\iota^{-1}"\tilde{j}(G)} = (j(F)(j"\gamma))_{\iota^{-1}"\hat{g}} \\ &= (j(F)(j"\gamma))_{G*H} = b.\end{aligned}$$

⊣

The following definition is how we define an ideal in $V[G][H]$ using some poset whose forcing extension resembles $V^{j(\mathbb{P})/\iota``G*H}$. Of course, the most natural example of such a poset is $\frac{j(\mathbb{P})}{\iota"G*H}$, but we will need a more general definition for the following section.

Definition 5.22. *Suppose* $\mathbb{R} \in V[G][H]$ *is a poset such that* $V[G][H]^{\mathbb{R}}$ *resembles* $V^{j(\mathbb{P})/\iota``G*H}$ *in the sense of Definition* 5.10; *let* $\dot{\hat{g}}$ *be a* $\mathbb{R}$*-name witnessing this fact.*

In $V[G][H]$ *define* $F_{\dot{\hat{g}}} \subset P^{V[G][H]}(\kappa)$ *by:* $S \in F_{\dot{\hat{g}}}$ *iff* $\kappa \in \tilde{j}_{\dot{\hat{g}}_{G_{\mathbb{R}}}}(S)$ *for every* $G_{\mathbb{R}}$ *which is* $(V[G][H], \mathbb{R})$*-generic;*[34] *i.e.*

$$S \in F_{\dot{\hat{g}}} \iff [\![\kappa \in \tilde{j}_{\dot{\hat{g}}}(S)]\!]_{ro(\mathbb{R})} = 1_{\mathbb{R}}. \tag{26}$$

It is routine to see that $F_{\dot{\hat{g}}}$ is a normal filter on κ. We will use $\mathbb{B}_{F_{\dot{\hat{g}}}}$ to denote the boolean algebra $P^{V[G][H]}(\kappa)/F_{\dot{\hat{g}}}$.

We will need the following ad-hoc definition. Note the special case of the following definition, where $\mathbb{R} = \frac{j(\mathbb{P})}{\iota``G*H}$; unfortunately, this special case would not suffice for the arguments in the next section, so we must state the general version:

Definition 5.23. *Given a poset* $\mathbb{R} \in V[G][H]$, *we will say that* $\mathbb{R}$ *is* ***nice*** *iff* $\mathbb{R} \in N[G][H]$, $\mathbb{R}$ *is a regular suborder of* $\frac{j(\mathbb{P})}{\iota``G*H}$, *and there is some* $\mathbb{R}$*-name* $\dot{\hat{g}}$, *some* $b \in N[G][H]$, *and some formula* ϕ *such that* $1_{\mathbb{R}}$ *forces (over* $V[G][H]$*) that:*

(1) $\dot{\hat{g}}$ *witnesses the resemblance of* $V[G][H]^{\mathbb{R}}$ *to* $V^{j(\mathbb{P})/\iota``G*H}$.
(2) $\dot{G}_{\mathbb{R}}$ *is an element of* $N[\dot{\hat{g}}][\dot{\hat{h}}]$ *and is definable there via the formula* ϕ *and parameters* $\dot{\hat{g}}$, b *(i.e.,* $\dot{G}_{\mathbb{R}}$ *is the unique element* y *such that* $N[\dot{\hat{g}}][\dot{\hat{h}}] \models \phi(y, \dot{\hat{g}}, \check{b})$*).*

We will say that $\dot{\hat{g}}$, b, *and* ϕ *witness the niceness of* $\mathbb{R}$.

The following lemma gives a sufficient condition to apply Foreman's Duality Theorem.

Lemma 5.24. *Suppose* $\mathbb{R} \in V[G][H]$ *is nice, as witnessed by* $\dot{\hat{g}}$, b, *and* ϕ *(as in Definition* 5.23*). Then in* $V[G][H]$ *there are functions* $f_{\frac{j(\mathbb{P})}{\iota``G*H}}$, $f_{\hat{g}}$, $(f_p)_{p \in \frac{j(\mathbb{P})}{\iota``G*H}}$, f_{G*H}, $f_{\mathbb{R}}$, $(f_r)_{r \in \mathbb{R}}$, *and* $f_{G_{\mathbb{R}}}$, *each with domain* κ, *such that whenever* $G_{\mathbb{R}}$ *is* $(V[G][H], \mathbb{R})$*-generic then letting* $\hat{g} := \dot{\hat{g}}_{G_{\mathbb{R}}}$:

(1) $\tilde{j}(f_{\frac{j(\mathbb{P})}{\iota``G*H}})(\kappa) = \frac{j(\mathbb{P})}{\iota``G*H}$,
(2) $\tilde{j}(f_{\hat{g}})(\kappa) = \hat{g}$,
(3) $\tilde{j}(f_p)(\kappa) = p$ *for each* $p \in \frac{j(\mathbb{P})}{\iota``G*H}$,
(4) $\tilde{j}(f_{G*H})(\kappa) = G * H$,
(5) $\tilde{j}(f_{\mathbb{R}})(\kappa) = \mathbb{R}$,
(6) $\tilde{j}(f_r)(\kappa) = r$ *for each* $r \in \mathbb{R}$,
(7) $\tilde{j}(f_{G_{\mathbb{R}}})(\kappa) = G_{\mathbb{R}}$.

Proof. The existence of the functions $f_{\frac{j(\mathbb{P})}{\iota``G*H}}$, $(f_p)_{p \in \frac{j(\mathbb{P})}{\iota``G*H}}$, f_{G*H}, $f_{\mathbb{R}}$, and $(f_r)_{r \in \mathbb{R}}$ are guaranteed by the "moreover" part of Corollary 5.21, since the relevant objects are elements of $N[G][H]$ (recall part of the definition of niceness of $\mathbb{R}$ is that $\mathbb{R} \in N[G][H]$). The function $f_{\hat{g}}$ is defined to be the constant function with value G; then for any lifting $\tilde{j}$, the function $\tilde{j}(f_{\hat{g}})$ is the constant function with value $\tilde{j}(G) = \hat{g}$ (so in particular $\tilde{j}(f_{\hat{g}})(\kappa) = \hat{g}$).

To define the function $f_{G_{\mathbb{R}}}$. Let $f_b \in V[G][H]$ be the function given by the "moreover" part of Corollary 5.21, and let $f_{\hat{g}}$ be as defined in the previous paragraph. In $V[G][H]$ define $f_{G_{\mathbb{R}}} : \kappa \to V[G][H]$ by sending ξ to the unique y such

[34] Here we are implicitly fixing a $\mathbb{R}$-name for a particular lifting $\tilde{j}_{\dot{\hat{g}}}$ as in Definition 5.16.

that $\phi(y, f_b(\xi), f_{\hat{g}}(\xi))$. Then, for any $G_{\mathbb{R}}$ which is $(V[G][H], \mathbb{R})$-generic, letting $\hat{g} := \dot{\hat{g}}_{G_{\mathbb{R}}}$ and $\tilde{j} := \tilde{j}_{\hat{g}}$ be the lifting of j, then by elementarity, $\tilde{j}(f_{G_{\mathbb{R}}})(\kappa)$ is the unique element of $N[\hat{g}][\hat{h}]$ such that $N[\hat{g}][\hat{h}] \models \phi(y, \tilde{j}(f_b)(\kappa), \tilde{j}(f_{\hat{g}})(\kappa))$; i.e., the unique y such that $N[\hat{g}][\hat{h}] \models \phi(y, b, \hat{g})$. Of course, this unique element is, by assumption, $G_{\mathbb{R}}$. ⊣

Corollary 5.25. *Assume $\mathbb{R} \in V[G][H]$ is nice, as witnessed by $\dot{\hat{g}}$, b, and ϕ. Let $F_{\dot{\hat{g}}}$ be the filter from Definition* 5.22. *Let $\tilde{j}_{\dot{\hat{g}}}$ be the $\mathbb{R}$-name for the embedding from Definition* 5.16.

Then in $V[G][H]$ the map $\pi : \mathbb{B}_{F_{\dot{\hat{g}}}} \to RO(\mathbb{R})$ defined by

$$[S]_{F_{\dot{\hat{g}}}} \mapsto [\![\kappa \in \tilde{j}_{\dot{\hat{g}}}(S)]\!]_{RO(\mathbb{R})}$$

is a dense embedding.

There is also a natural dense embedding in the other direction: for each $r \in \mathbb{R}$ define

$$S_r := \{\xi < \kappa \mid f_r(\xi) \in f_{G_{\mathbb{R}}}(\xi)\}, \tag{27}$$

where f_r and $f_{G_{\mathbb{R}}}$ are the functions given by Lemma 5.24. *Then the map σ defined by $r \mapsto [S_r]_{F_{\dot{\hat{g}}}}$ is a dense embedding from $\mathbb{R} \to \mathbb{B}_{F_{\dot{\hat{g}}}}$.*

Proof. This follows directly from Foreman's Theorem 2.17 (viewing $V[G][H]$ as the ground model) and the existence of the functions $f_{\mathbb{R}}$, $(f_r)_{r\in\mathbb{R}}$, and $f_{G_{\mathbb{R}}}$ from Lemma 5.24. ⊣

Note that in the context of Corollary 5.25, the dense embedding $\sigma : \mathbb{R} \to \mathbb{B}_{F_{\dot{\hat{g}}}}$ can be used (inside $V[G][H]$) to characterize self-genericity as follows: for any $M \prec (H_\theta, \in, \{\sigma, F_{\dot{\hat{g}}}, \mathbb{R}\})$ with $\alpha_M := M \cap \kappa \in \kappa$:

$$\begin{aligned} M \in S^{SelfGen}_{F_{\dot{\hat{g}}}} &\iff \\ W := \{S_r \mid r \in M \cap \mathbb{R} \text{ and } \alpha_M \in S_r)\} &\text{ generates a } (M, \mathbb{B}_{F_{\dot{\hat{g}}}})\text{-generic} \iff \\ \sigma^{-1}\text{“}W &\text{ is } (M, \mathbb{R})\text{-generic} \iff \\ \{r \in M \cap \mathbb{R} \mid f_r(\alpha_M) \in f_{G_{\mathbb{R}}}(\alpha_M)\} &\text{ is } (M, \mathbb{R})\text{-generic} \end{aligned} \tag{28}$$

Corollary 5.26. *Assume $\mathbb{R} \in V[G][H]$ is nice, as witnessed by $\dot{\hat{g}}$ (and ϕ). Then, the following are equivalent:*

(1) *$F_{\dot{\hat{g}}}$ is saturated.*
(2) *$F_{\dot{\hat{g}}}$ is strong.*
(3) *$\mathbb{B}_{F_{\dot{\hat{g}}}}$ preserves κ^+.*
(4) *$\vec{U}$ is almost huge.*

(*In particular, this holds when $\mathbb{R} = \frac{j(\mathbb{P})}{\imath\text{“}G*H}$ and $\dot{\hat{g}}$ is the canonical name for the $\frac{j(\mathbb{P})}{\imath\text{“}G*H}$-generic object.*)

Proof. If $\vec{U}$ is almost huge, then $\frac{j(\mathbb{P})}{\imath\text{“}G*H}$ has the $\delta = \kappa^{+V[G][H]}$-cc (from the point of view of $V[G][H]$). By the assumed regularity of $e : \mathbb{R} \to \frac{j(\mathbb{P})}{\imath\text{“}G*H}$ (from Definition 5.23), then $\mathbb{R}$ also has the δ-cc. Then the dense embedding from $\mathbb{B}_{F_{\dot{\hat{g}}}} \to RO(\mathbb{R})$ given by Corollary 5.25 guarantees that $\mathbb{B}_{F_{\dot{\hat{g}}}}$ also has the δ-cc; so $F_{\dot{\hat{g}}}$ is saturated.

Now suppose $\vec{U}$ was **not** almost huge; then

$$j(\kappa) > \delta. \tag{29}$$

By Corollary 5.25, generic ultrapowers of $V[G][H]$ by $\mathbb{B}_{F_{\dot{g}}}$ are exactly those liftings of j of the form, $\tilde{j}_{\hat{g}}$ where $\hat{g} = (\dot{\hat{g}})_{G_{\mathbb{R}}}$ for some $(V[G][H], \mathbb{R})$-generic $G_{\mathbb{R}}$. In particular, by (29), such liftings always send κ strictly above $\delta = \kappa^{+V[G][H]}$. So $F_{\dot{g}}$ is not a strong filter in this case. ⊣

We will also use the following Lemma 5.27, which is simply a supercompact variation of Kunen's original construction of a saturated ideal from a huge cardinal. The proof of Lemma 5.27 is much simpler than the proof of Theorem 5.12 because of the presence of strong master conditions. Both Theorem 5.12 and Lemma 5.27 provide generic elementary embeddings with domain $V^{\mathbb{P}*Col(\kappa,<\delta)}$. The main difference is that in Theorem 5.12, δ was exactly the height of the tower whose embedding we were trying to lift; whereas in Lemma 5.27, δ is strictly smaller than the height of the tower whose embedding we are trying to lift.

For uniformity we still keep the hypotheses in our Background Hypotheses from page 1267, though most of them are irrelevant to this lemma. Namely, we only consider the objects $\delta = lh(\vec{U})$, $\mathbb{P}$, and $G * H$ from those hypotheses.

Lemma 5.27. *Suppose $\vec{U}'$ is a $P_\kappa(-)$-tower of height strictly greater than δ.*[35] *Let $j' : V \to_{\vec{U}'} N'$ be the ultrapower.*

Assume there is some $r \in N'$ such that

$$r : \mathbb{P} * Col(\kappa, <\delta) \to RO^{N'}(j'(\mathbb{P}))$$

is a regular embedding and is the identity on $\mathbb{P}$.[36]

Let $\hat{G}'$ be $(V[G][H], \frac{j'(\mathbb{P})}{r\text{``}G*H})$-generic (recall $G * H$ was fixed in the Background Hypotheses on page 1267).

Let $\hat{j}' : V[G] \to N'[\hat{G}']$ be the lifting of j' which exists because $j'\text{``}G \subset \hat{G}'$. Then:

$$\hat{j}'\text{``}H \in N'[\hat{G}'] \tag{30}$$

and

$$m'_H := \bigcup \hat{j}'\text{``}H \in Col^{N'[\hat{G}']}(j'(\kappa), < j'(\delta)). \tag{31}$$

It follows that if $\hat{H}'$ is a $(V[\hat{G}'], Col^{N'[\hat{G}']}(j'(\kappa), < j'(\delta)))$-generic which has m'_H as an element, then in $V[\hat{G}'][\hat{H}']$ the map $\hat{j}'$ can be lifted to an elementary

$$\tilde{j}' : V[G][H] \to N'[\hat{G}'][\hat{H}'].$$

Finally:

$$\forall Z \in (H_{\delta^+})^{V[G][H]} : \tilde{j}'\text{``}Z \in N'[\hat{G}'][\hat{H}']. \tag{32}$$

Proof. First note that N' is closed under δ sequences, so $j' \restriction W \in N'$ for any $W \in H^V_{\delta^+}$. Second, $G * H$ is computed from $\hat{G}'$ via the map r and $r \in N'$, so $G * H \in N'[\hat{G}']$. From this it follows that, letting $\hat{j}'$ denote the intermediate lifting from $V[G] \to N'[\hat{G}']$:

$$\hat{j}' \restriction W[G] \in N'[\hat{G}'] \text{ for any } W \in H^V_{\delta^+}. \tag{33}$$

---

[35] Recall we allow the possibility that $height(\vec{U}') = \delta + 1$, so that $\vec{U}'$ is essentially a single normal measure on $P_\kappa(\delta)$.

[36] More precisely: we require that $r(p, 1) = p$ for every $p \in \mathbb{P}$.

Then (30) follows immediately. To see (31): each $s \in H$ has size $\leq \mu$, so $\hat{j}'(s) = \hat{j}'``s$. Thus $|\hat{j}(s)| < \kappa$ for each $s \in H$ and so in $N'[\hat{G}']$:

$$|m'_H| = |\bigcup \hat{j}'``H| = |\delta| \cdot |\kappa| = |\delta| < j'(\kappa)$$

(the last inequality is because $\delta < lh(\vec{U}')$). So m'_H has the right size in $N'[\hat{G}']$ to be a condition in the Levy collapse $Col(j'(\kappa), < j'(\delta))$. It is easily checked that m'_H is a function of the right form to be in this Levy Collapse.

Now let $\hat{H}'$ be $(V[\hat{G}'], Col^{N'[\hat{G}']}(j'(\kappa), < j'(\delta))$-generic with $m'_H \in \hat{H}'$. Then $\hat{j}'``H \subset \hat{H}'$ so $\hat{j}'$ can be extended to the map $\tilde{j}'$ as claimed. The map $\tilde{j}' \restriction W[G][H]$ will be an element of $N'[\hat{G}'][\hat{H}']$ for any $W \in H_{\delta^+}$. This completes the proof. ⊣

5.4. Interpolating posets and *ProjectiveCatch* **from supercompact towers.** Recall we are still assuming the Background Hypotheses from page 1267. Suppose $\mathbb{R} \in V[G][H]$ is any poset and $\dot{g}$ is a $\mathbb{R}$-name as in the assumptions of Lemma 5.24; for example, $\mathbb{R}$ could just be $\frac{j(\mathbb{P})}{\imath" G*H}$ and $\dot{g}$ could be the canonical name for the $\frac{j(\mathbb{P})}{\imath" G*H}$-generic object. Let $F := F_{\dot{g}}$ be the ideal on κ (in $V[G][H]$) defined in Definition 5.22. Recall from Corollary 5.26 that F is saturated $\iff$ F is strong $\iff$ $\vec{U}$ is almost huge. Therefore, if we want to obtain a situation where $V[G][H] \models$ "*ProjectiveCatch*(F) holds and F is not strong" then we must necessarily assume $\vec{U}$ is not almost huge. There is another reason for working with nonalmost huge $\vec{U}$: we would like to show that the large cardinal upper bound for *ProjectiveCatch* for ideals on ω_2 is significantly weaker than an almost huge cardinal (which is the best known upper bound for a saturated or even presaturated ideal on ω_2).

So assume $\vec{U}$ is not almost huge. In $V[G][H]$ consider some algebra $\mathcal{A} = (H_\theta[G][H], \dots)$. We would like to find, in $V[G][H]$, an F-self-generic substructure of $\mathcal{A}$. The idea is to take a generic ultrapower $\tilde{j} : V[G][H] \to N[\hat{g}][\hat{h}]$ (recall by Corollary 5.25 that all generic ultrapowers of $V[G][H]$ by F are of this form) and find a $\tilde{j}(F)$-self-generic structure in $N[\hat{g}][\hat{h}]$.

First we briefly describe the most natural attempt—namely, considering $Sk^{\tilde{j}(\mathcal{A})}(j"\gamma)$ for some $\gamma < \delta$—and show why such a structure *cannot* be $\tilde{j}(F)$-generic in the case where $\vec{U}$ is not almost huge. So assume $\vec{U}$ is not almost huge; this implies that, in $V[G][H]$, there is some $\mathbb{R}$-name $\dot{\psi}$ for a surjection from $\mu \to_{\text{onto}} \delta$. Fix a $\gamma < \delta$ and WLOG assume $\mathcal{A}$ extends $(H_\theta, \in, \{\dot{\psi}, \mathbb{R}\})$. Suppose toward a contradiction that $M' := Sk^{\tilde{j}(\mathcal{A})}(j"\gamma)$ were $\tilde{j}(F)$-self-generic in $N[\hat{g}][\hat{h}]$. Then $M' \cap j(\kappa) = \kappa$, and by (28) and elementarity of $\tilde{j}$, $N[\hat{g}][\hat{h}]$ believes that the following set is $(M', \tilde{j}(\mathbb{R}))$-generic:

$$K' := \{r' \in M' \cap \tilde{j}(\mathbb{R}) \mid f_{r'}^{N[\hat{g}][\hat{h}]}(\kappa) \in f_{\dot{G}_{\tilde{j}(\mathbb{R})}}^{N[\hat{g}][\hat{h}]}(\kappa)\}. \tag{34}$$

Note that $M' = \tilde{j}[Sk^{\mathcal{A}}(\gamma)]$; in particular $K' \subset \text{range}(\tilde{j})$ and so:

$$\begin{aligned} K' &= \{\tilde{j}(r) \mid r \in \mathbb{R} \text{ and } f_{\tilde{j}(r)}^{N[\hat{g}][\hat{h}]}(\kappa) \in f_{\tilde{j}(\dot{G}_{\mathbb{R}})}^{N[\hat{g}][\hat{h}]}(\kappa)\} \cap M' \\ &= \{\tilde{j}(r) \mid r \in \mathbb{R} \text{ and } \tilde{j}(f_r)(\kappa) \in \tilde{j}(f_{\dot{G}_{\mathbb{R}}})(\kappa)\} \cap M' \\ &= \{\tilde{j}(r) \mid r \in \mathbb{R} \text{ and } r \in G_{\mathbb{R}}\} \cap M' \\ &= \tilde{j}[G_{\mathbb{R}}] \cap \tilde{j}[Sk^{\mathcal{A}}(\gamma)]. \end{aligned}$$

Since K' is $(\tilde{j}[Sk^{\mathcal{A}}(\gamma)], \tilde{j}(\mathbb{R}))$-generic, then $G_{\mathbb{R}} \cap Sk^{\mathcal{A}}(\gamma)$ is $(Sk^{\mathcal{A}}(\gamma), \mathbb{R})$-generic. Since $\dot{\psi} \in Sk^{\mathcal{A}}(\gamma)$, $\mathrm{dom}(\psi) = \mu < \gamma \subset Sk^{\mathcal{A}}(\gamma)$, and $G_{\mathbb{R}}$ is $(Sk^{\mathcal{A}}(\gamma), \mathbb{R})$-generic, it follows that $\delta = \mathrm{range}(\psi) \subset Sk^{\mathcal{A}}(\gamma)$. But this is a contradiction, since

$$|Sk^{\mathcal{A}}(\gamma)|^{V[G][H]} = |\gamma|^{V[G][H]} < \delta.$$

We will instead find self-generic structures as follows. We know by Corollary 5.25 that if $\tilde{j} : V[G][H] \to N[\hat{g}][\hat{h}]$ is the lifting from Definition 5.16, then, the derived ultrafilter on κ is $(V[G][H], \mathbb{B}_F)$-generic. This implies that $\tilde{j}$"W is a $\tilde{j}(F)$-self-generic structure (from the point of view of $V[\hat{G}]$), where $W \in V[G][H]$ is any transitive ZF^- model with $F \in W$ and $P(\kappa) \subset W$. However, due to the limited closure of N, the object $\tilde{j}$"W is not an element of $N[\hat{g}][\hat{h}]$, so it is not clear if $N[\hat{g}][\hat{h}]$ has any $\tilde{j}(F)$-self-generic structures; thus it is not clear if $V[G][H]$ has any F-self-generic structures.

The idea for dealing with this issue is to assume there is a tower $\vec{U}'$ which properly end-extends $\vec{U}$, and somehow use the lifting $\tilde{j}'$ of the stronger embedding $j' : V \to_{\vec{U}'} N'$ given by Lemma 5.27 to obtain $\tilde{j}'(F)$-self-generic structures inside $N'[\hat{g}'][\hat{H}']$,[37] whose existence can then be pulled back to $V[G][H]$ via the elementarity of $\tilde{j}'$. More precisely, we would like to show that the ultrafilter on $P^{V[G][H]}(\kappa)$ derived from $\tilde{j}'$ is generic for $\mathbb{B}_F$, because this would guarantee that $\tilde{j}'$"W is $\tilde{j}'(F)$-self-generic (where W is as in the previous paragraph); and then, due to the high degree of closure of N', the object $\tilde{j}'$"W would be an element of $N'[\hat{g}'][\hat{H}']$ and thus we could pull back via $\tilde{j}'$ to get the existence of F-self-generic structures inside $V[G][H]$.

Showing that the ultrafilter derived from $\tilde{j}'$ is generic for $\mathbb{B}_F$ seems to require some sort of interpolation between the poset $j(\mathbb{P})$ and $j'(\mathbb{P})$. If $\vec{U}$ is almost huge, then $j(\mathbb{P})$ is an initial segment of $j'(\mathbb{P})$ and the interpolation is straightforward; namely, the map $k : N \to N'$ can be lifted to the relevant generic extensions; this was the key to the construction in [11] of layered ideals. However, in our situation where $\vec{U}$ is not almost huge, k **cannot** be lifted to have domain $N^{j(\mathbb{P})}$, because $crit(k) \in \{\delta, \delta^{+N}\}$ is not even a cardinal in $N^{j(\mathbb{P})}$.[38] The following definition provides a way around this issue.

Definition 5.28. *Working in V, suppose $\vec{U}'$ is a proper end-extension of $\vec{U}$. Let $j' : V \to_{\vec{U}'} N'$ and $k : N \to N'$ be the map from Fact* 5.4.

Let $\mathbb{Q}$ be a partial order. We will say that $\mathbb{Q}$ interpolates $j(\mathbb{P})$ and $j'(\mathbb{P})$ with respect to ι *iff*:

1. *$\mathbb{Q} \in N$ and is a subset of $(H_{\delta^+})^N$; in our application below it will actually be an element of $(H_{\delta^+})^N$.*
2. *$\mathbb{Q}$ is a regular suborder of $RO^N(j(\mathbb{P}))$.*
3. *The map ι from Hypothesis* 3 *on page* 1267 *maps regularly into $RO^N(\mathbb{Q})$.*
4. *Whenever $G * H$ is $\mathbb{P} * Col(\kappa, < \delta)$-generic, letting $\mathbb{R} := \frac{\mathbb{Q}}{\iota\text{"}G*H}$ (note this quotient makes sense by requirement* 3 *and Fact* 5.6*) then there is some $\mathbb{R}$-name $\dot{g}$ such that*:

[37] Where $\hat{H}'$ is generic for $\tilde{j}'(Col(\kappa, < \delta))$, as in Lemma 5.27.

[38] Because $j(\kappa)$ is the cardial successor of μ in $N^{j(\mathbb{P})}$.

(a) $\dot{\hat{g}}$ *witnesses that* $V[G][H]^{\mathbb{R}}$ *resembles* $V^{j(\mathbb{P})/\imath\text{"}G*H}$.
(b) $1_{\mathbb{R}}$ *forces that* $\dot{G}_{\mathbb{R}} = \dot{\hat{g}} \cap \mathbb{R}$.

(5) $k \restriction \mathbb{Q}$ *is an element of* N' *and maps* $\mathbb{Q}$ *regularly into* $RO^{N'}(j'(\mathbb{P}))$. *Note this is the only clause of the definition which mentions* j' *or* N'.

REMARK 5.29. *If* $\vec{U}$ *is almost huge and* $\mathbb{P} \subset V_\kappa$ *is* κ*-cc, then for any end-extension* $\vec{U}'$ *of* $\vec{U}$, *the poset* $j(\mathbb{P})$ *interpolates itself with* $j'(\mathbb{P})$ *with respect to the map* $\imath$. *The main interest in interpolating posets is when* $\vec{U}$ *is not almost huge.*

LEMMA 5.30. *Suppose* $\mathbb{Q}$ *interpolates* $j(\mathbb{P})$ *and* $j'(\mathbb{P})$ *with respect to* $\imath$. *Then*:

(1) $N \models$ "$\mathbb{Q}$ *has the* $crit(k)$*-cc*".
(2) *If* $crit(k) = \delta^{+N}$ *then* $k\text{"}\mathbb{Q} = \mathbb{Q}$.
(3) $k \circ \imath$ *maps* $\mathbb{P} * Col(\kappa, <\delta)$ *regularly into* $RO^{N'}(j'(\mathbb{P}))$ *and is the identity on* $\mathbb{P}$; *so the hypotheses of Lemma* 5.27 *are satisfied.*

PROOF. If $\mathbb{Q}$ did not have the $crit(k)$-cc in N, then there would be a maximal antichain $A \subset \mathbb{Q}$ in N of N-size $crit(k)$; thus $k(A) \supsetneq k\text{"}A$. Then $k(A)$ would be a maximal antichain in $j'(\mathbb{P})$ properly containing $k\text{"}A$, contradicting the assumption that k maps $\mathbb{Q}$ regularly into $j'(\mathbb{P})$.

If $crit(k) = \delta^{+N}$ then, since we assume $\mathbb{Q} \subset (H_{\delta^+})^N$, $k \restriction \mathbb{Q} = id$.

Item 3 just follows from the assumption that $\imath$ is the identity on $\mathbb{P}$, that $\mathbb{P} * Col(\kappa, <\delta) \subset V_\delta$, and that $crit(k) \geq \delta$ (by Fact 5.4). ⊣

The "starred" version of the function $f_{G_{\mathbb{R}}}$ and the set S_r appearing in the following lemma will turn out to be equivalent (modulo the relevant filter) to the unstarred versions from Lemma 5.24 and Corollary 5.25 (respectively). The purpose of introducing the starred versions is that they are more easily amenable to the elementarity arguments in Lemma 5.33 and Corollary 5.34 below.

LEMMA 5.31. *Suppose* $\mathbb{Q}$ *interpolates* $j(\mathbb{P})$ *and* $j'(\mathbb{P})$ *with respect to* $\imath$. *Let* $G * H$ *be* $(V, \mathbb{P} * Col(\kappa, <\delta))$*-generic and* $\mathbb{R} = \frac{\mathbb{Q}}{\imath\text{"}G*H}$. *Then* $\mathbb{R}$ *is nice* (*in the sense of Definition* 5.23).

Furthermore, the function $f^*_{G_{\mathbb{R}}}$ *defined by*:

$$\xi \mapsto G \cap f_{\mathbb{Q}}(\xi) \tag{35}$$

is $F_{\dot{\hat{g}}}$*-equivalent to the function* $f_{G_{\mathbb{R}}}$ *from Lemma* 5.24 (*they both always represent* $G_{\mathbb{R}}$ *in generic ultrapowers using* $F_{\dot{\hat{g}}}$).

Finally, for any $r \in \mathbb{R}$ *let*

$$S^*_r := \{\xi < \kappa \mid f_r(\xi) \in f^*_{G_{\mathbb{R}}}(\xi)\}. \tag{36}$$

Then $[S^*_r]_{F_{\dot{\hat{g}}}} = [S_r]_{F_{\dot{\hat{g}}}}$, *where* S_r *is the set defined in* (27).

PROOF. Since $\mathbb{Q}$ and $\imath$ are elements of N, then $\mathbb{R} \in N[G][H]$. Moreover, by requirement 4 in Definition 5.28, whenever $G_{\mathbb{R}}$ is $(V[G][H], \mathbb{R})$-generic then $G_{\mathbb{R}} = \hat{g} \cap \mathbb{R}$ (so in $V[G][H]$ the triple $\dot{\hat{g}}$, $\mathbb{R}$, and ϕ witness niceness of $\mathbb{R}$, where $\phi(y, u, v)$ is the formula $y = u \cap v$).

To see that $f^*_{G_{\mathbb{R}}}$ and $f_{G_{\mathbb{R}}}$ always represent the same object—namely $G_{\mathbb{R}}$—in generic ultrapowers by $F_{\dot{\hat{g}}}$—let $G_{\mathbb{R}}$ be an arbitrary $(V[G][H], \mathbb{R} = \frac{\mathbb{Q}}{\imath\text{"}G*H})$-generic, $\hat{g} := \dot{\hat{g}}_{G_{\mathbb{R}}}$, and $\tilde{j} := \tilde{j}_{\hat{g}}$. Then

$$\tilde{j}(f^*_{G_{\mathbb{R}}})(\kappa) = \hat{g} \cap \tilde{j}(f_{\mathbb{Q}})(\kappa) = \hat{g} \cap \mathbb{Q}. \tag{37}$$

Also, $\hat{g}$ is a filter for $\frac{j(\mathbb{P})}{\imath"G*H}$; this means that each element of $\hat{g}$ is $j(\mathbb{P})$-compatible with each element of $\imath"G*H$. Since $\perp_{\mathbb{Q}}$ and $\perp_{j(\mathbb{P})}$ agree and since $\imath$ maps into $\mathrm{RO}(\mathbb{Q})$ (by requirements 2 and 3 of Definition 5.28, respectively), then each element of $\hat{g}\cap\mathbb{Q}$ is $\mathbb{Q}$-compatible with each element of $\imath"G*H$. It follows that

$$\hat{g}\cap\mathbb{Q}=\hat{g}\cap\frac{\mathbb{Q}}{\imath"G*H}=\hat{g}\cap\mathbb{R}=G_{\mathbb{R}}. \tag{38}$$

Combining (38) with (37) yields

$$\tilde{j}(f^*_{G_{\mathbb{R}}})(\kappa)=G_{\mathbb{R}}. \tag{39}$$

Finally, $[S^*_r]_{F_{\dot{g}}}=[S_r]_{F_{\dot{g}}}$ follows from the definitions of S_r, S^*_r and the fact that $f_r=_{F_{\dot{g}}}f^*_r$. ⊣

Corollary 5.32. *If the hypotheses of Lemma* 5.31 *hold, then the map*

$$r\mapsto[S^*_r]_{F_{\dot{g}}} \tag{40}$$

is a dense embedding from $\mathbb{R}\to\mathbb{B}_{F_{\dot{g}}}$.

In other words, the statement of Corollary 5.25 *still holds when the set* S_r *from* (27) *is replaced by the set* S^*_r *from* (36).

Lemma 5.33. *Suppose* $\mathbb{Q}$ *interpolates* $j(\mathbb{P})$ *and* $j'(\mathbb{P})$ *with respect to* $\imath$. *Let* $G*H$ *be* $(V,\mathbb{P}*Col(\kappa,<\delta))$*-generic and set* $\mathbb{R}:=\frac{\mathbb{Q}}{\imath"G*H}$.

Let $r:=k\circ\imath$. *Then*

$$k \text{ maps } \mathbb{R}=\frac{\mathbb{Q}}{\imath"G*H} \text{ regularly into } \frac{j'(\mathbb{P})}{(k\circ\imath)"G*H}. \tag{41}$$

Let $f^*_{G_{\mathbb{R}}}$ *be the function defined in the statement of Lemma* 5.31. *Suppose* $\tilde{j}':V[G][H]\to N'[\hat{G}'][\hat{H}']$ *is some elementary embedding which extends* j' *and such that*:

$$\tilde{j}'(G)=\hat{G}' \tag{42}$$

For each $b\in N$ *let* f_b *be the function in* $V[G][H]$ *given by Definition* 5.19.[39] *Define* $G_{\mathbb{R}}:=\mathbb{Q}\cap k^{-1}``\hat{G}'$. *Then*:

$$\text{If } \hat{G}' \text{ is } (V,j'(\mathbb{P}))\text{-generic then } G_{\mathbb{R}} \text{ is } (V[G][H],\mathbb{R})\text{-generic}, \tag{43}$$

$$\tilde{j}'(f_b)(\kappa)=k(b) \text{ for all } b\in N, \tag{44}$$

$$\tilde{j}'(f^*_{G_{\mathbb{R}}})(\kappa)=\hat{G}'\cap k(\mathbb{Q}). \tag{45}$$

Moreover, if we also assume $\mathbb{Q}\in(H_{\delta^+})^N$ *and* $crit(k)=\delta^{+N}$ *then* $k(\mathbb{Q})=k``\mathbb{Q}=\mathbb{Q}$ *and*

$$G_{\mathbb{R}}=\mathbb{Q}\cap\hat{G}', \tag{46}$$

$$\tilde{j}'(f_r)(\kappa)=r \text{ for all } r\in\mathbb{R} \text{ (Note } \mathbb{R}\subset\mathbb{Q}\subset N), \tag{47}$$

$$\tilde{j}'(f^*_{G_{\mathbb{R}}})(\kappa)=G_{\mathbb{R}}. \tag{48}$$

Proof. The statement (43) follows from (41), which in turn follows from requirements 3 and 5 of Definition 5.28. Equation (44) follows from Lemma 5.20.

[39]Note that even though f_b is defined even for $b\in N[G][H]$ by Corollary 5.21, the expression $k(b)$ will only make sense for $b\in N$ because, as remarked above, k cannot be extended to have domain $N[G][H]$ in the case that $\vec{U}$ is not almost huge.

Since the function $f^*_{G_{\mathbb{R}}}$ is defined (in $V[G][H]$) by

$$\xi \mapsto f_{\hat{g}}(\xi) \cap f_{\mathbb{Q}}(\xi) = G \cap f_{\mathbb{Q}}(\xi) \tag{49}$$

then by (42) and elementarity of $\tilde{j}'$:

$$\tilde{j}'(f^*_{G_{\mathbb{R}}})(\kappa) = \hat{G}' \cap \tilde{j}'(f_{\mathbb{Q}})(\kappa) = \hat{G}' \cap k(\mathbb{Q}) \tag{50}$$

where the last equation is by Lemma 5.20 (note $\mathbb{Q}$ is an element of N). This proves (45).

Finally, suppose we also assume that $k(\mathbb{Q}) = \mathbb{Q}$ and $k \restriction \mathbb{Q} = id$. Then clearly (44) implies (47), and moreover

$$\hat{G}' \cap k(\mathbb{Q}) = \hat{G}' \cap \mathbb{Q} = \mathbb{Q} \cap k^{-1}``\hat{G}'. \tag{51}$$

This, combined with (43), implies (46). Also (50) and (51) imply (48). $\dashv$

The following corollary is the key point of interpolating posets; it essentially says that liftings by j and liftings by j' yield the same ultrafilters on $\wp^{V[G][H]}(\kappa)$:

Corollary 5.34. *Suppose $\mathbb{Q}$ interpolates $j(\mathbb{P})$ and $j'(\mathbb{P})$ with respect to $\imath$ and that*

$$\mathbb{Q} \in (H_{\delta^+})^N \text{ and } crit(k) = \delta^{+N}. \tag{52}$$

Let $G * H$ be $(V, \mathbb{P} * Col(\kappa, <\delta))$-generic and $\mathbb{R} := \frac{\mathbb{Q}}{\imath\text{''}G*H}$. For each $r \in \mathbb{R}$ let S^*_r be the subset of κ defined in (36).

Let:

- *$G_{\mathbb{R}}$ be $(V[G][H], \mathbb{R})$-generic.*
- *$\hat{g} := \dot{\hat{g}}_{G_{\mathbb{R}}}$ (where $\dot{\hat{g}}$ is the $\mathbb{R}$-name witnessing resemblance of $V[G][H]^{\mathbb{R}}$ to $V^{j(\mathbb{P})/\imath\text{''}G*H}$).*
- *$\tilde{j} := \tilde{j}_{\hat{g}} : V[G][H] \to N[\hat{g}][\hat{h}]$ be the lifting as in Definition* 5.16.
- *$\hat{G}'$ be $(V[G][H][G_{\mathbb{R}}], \frac{j'(\mathbb{P})/\imath\text{''}G*H}{G_{\mathbb{R}}})$-generic (note $\mathbb{R}$ is a regular subalgebra of $j'(\mathbb{P})/\imath\text{''}G * H$ by assumption* (52) *and Lemma* 5.30).
- *$\hat{H}'$ be $(V[\hat{G}'], Col^{N'[\hat{G}']}(j'(\kappa), < j'(\delta)))$-generic with $\bigcup \hat{j}'``H \in \hat{H}'$, and in $V[\hat{G}'][\hat{H}']$ let $\tilde{j}' : V[G][H] \to N'[\hat{G}'][\hat{H}']$ be the lifting of j' given by Lemma* 5.27.

Then for any $r \in \mathbb{R}$:

$$\kappa \in \tilde{j}(S^*_r) \iff r \in G_{\mathbb{R}} \iff \kappa \in \tilde{j}'(S^*_r). \tag{53}$$

It follows that the ultrafilter on $P^{V[G][H]}(\kappa)$ derived from $\tilde{j}$ is the same as the ultrafilter derived from $\tilde{j}'$ and, furthermore, this ultrafilter is $(V[G][H], \mathbb{B}_{F_{\dot{\hat{g}}}})$-generic.

Proof. Corollary 5.32 implies that $r \in G_{\mathbb{R}} \iff \kappa \in \tilde{j}(S^*_r)$. Items (47) and (48) of Lemma 5.33 imply that $r \in G_{\mathbb{R}} \iff \kappa \in \tilde{j}'(S^*_r)$. $\dashv$

Finally we give examples of interpolating posets.

Lemma 5.35. *Suppose $\mathbb{P} = Col(\mu, <\kappa)$. Let $\mathbb{Q} := Col(\mu, <\delta + 1)$.*[40] *Then*:

1. *We can WLOG assume that the $\imath \in N$ from Hypothesis* 3 *on page* 1267 *maps regularly into $\mathbb{Q}$.*
2. *$\mathbb{Q}$ satisfies item* 4 *from Definition* 5.28.

[40]This poset is forcing equivalent to $Col(\mu, \delta)$.

Proof. If $\vec{U}$ is almost huge then the lemma is trivial (since $\mathbb{Q}$ is a regular end-extension of $j(\mathbb{P})$ in that case). So assume that $\vec{U}$ is not almost huge. First we show the "WLOG" part; i.e., it can be arranged that ι maps into $RO^N(\mathbb{Q})$ and be the identity on $\mathbb{P}$. Note that

$$\mathbb{Q} \simeq \mathbb{P} \times Col(\mu, [\kappa, \delta + 1)) \tag{54}$$

and that each factor is computed the same in V and $V^{\mathbb{P}}$. Also, by standard absorption theory for Levy collapses:

$$\Vdash_{\mathbb{P}} Col^{V^{\mathbb{P}}}(\kappa, < \delta) \text{ regularly embeds into } RO^{V^{\mathbb{P}}}(Col(\mu, [\kappa, \delta + 1))). \tag{55}$$

Let $\dot{r}$ be a $\mathbb{P}$-name for a regular embedding witnessing (55). Then, by Fact 5.7, the map

$$\ell : \mathbb{P} * Col^{V^{\mathbb{P}}}(\kappa, < \delta) \to \mathbb{P} * RO^{V^{\mathbb{P}}}(Col(\mu, [\kappa, \delta + 1)))$$

defined by

$$(p, \dot{q}) \mapsto (p, \dot{r}(\dot{q}))$$

is a regular embedding.

Let $D := \{(p, \check{q}) \mid q \in Col(\mu, [\kappa, \delta + 1))\}$. D is dense in the target poset of ℓ, i.e., D is dense in $\mathbb{P} * RO(Col(\mu, [\kappa, \delta + 1)))$. Define $\ell_D : \mathbb{P} * Col^{V^{\mathbb{P}}}(\kappa, < \delta) \to D$ by

$$(p, \dot{q}) \mapsto sup(\{d \in D \mid \ell(p, \dot{q}) \geq d\}).$$

Note that D is closed under arbitrary suprema in the poset $\mathbb{P} * RO(Col(\mu, [\kappa, \delta+1)))$; this is just due to the fact that the underlying set of $\mathbb{Q}$ is closed under arbitrary intersections.[41] So ℓ_D is well-defined, maps into D, and is a regular embedding. Moreover, it is easy to see that ℓ_D acts as the identity on $\mathbb{P}$; i.e., $\ell_D(p, 1) = (p, 1)$ for all $p \in \mathbb{P}$. Let $\phi : D \to \mathbb{Q}$ be the isomorphism defined by $(p, \check{q}) \mapsto p \cup q$. Then, $\phi \circ \ell_D$ is a regular embedding from $\mathbb{P} * Col^{V^{\mathbb{P}}}(\kappa, < \delta) \to \mathbb{Q}$ such that $\phi(p, 1) = p$ for all $p \in \mathbb{P}$.

To see that $\mathbb{Q}$ satisfies item 4 from Definition 5.28: Let $G_{\mathbb{Q}}$ be $(V[G][H], \frac{\mathbb{Q}}{\iota``G*H})$-generic. Since N is closed under $< \delta$ sequences and $\mathbb{Q}$ is $< \mu$-distributive, then:

$$V[G_{\mathbb{Q}}] \models N[G_{\mathbb{Q}}] \text{ is closed under } < \mu \text{ sequences.} \tag{56}$$

Consider the poset $\mathbb{Q}' := Col(\mu, [\delta + 1, j(\kappa))$; this is computed the same in all models and

$$\mathcal{A} := \{A \in N[G_{\mathbb{Q}}] \mid A \text{ is maximal antichain in } \mathbb{Q}'\}$$

has size $j(\kappa)$ in $N[G_{\mathbb{Q}}]$ and thus size μ in $V[G_{\mathbb{Q}}]$ (since $j(\kappa) > \delta$). Then $V[G_{\mathbb{Q}}]$ can pick a μ-enumeration of $\mathcal{A}$ and use (56) to construct a $g_{\mathbb{Q}'}$ which is $(N[G_{\mathbb{Q}}], \mathbb{Q}')$-generic. Thus, by the Product Lemma, $G_{\mathbb{Q}} \times g_{\mathbb{Q}'}$ is $(N, \mathbb{Q} \times \mathbb{Q}')$-generic. Let $\phi : \mathbb{Q} \times \mathbb{Q}' \leftrightarrow Col(\mu, < j(\kappa))$ be the standard isomorphism given by $(q, q') \mapsto q \cup q'$. Then $\hat{g} := \phi``(G_{\mathbb{Q}} \times g_{\mathbb{Q}'})$ is $(N, Col(\mu, < j(\kappa))$-generic and $\hat{g} \cap \mathbb{Q} = G_{\mathbb{Q}}$. ⊣

Lemma 5.36. *Suppose $\vec{U}'$ is a proper end-extension of $\vec{U}$. Let $j' : V \to_{\vec{U}'} N'$ and $k : N \to N'$ be the map from Fact* 5.4. *Let $\mathbb{P} = Col(\mu, < \kappa)$ and $\iota \in N$ be as in*

[41] I.e., if $Z \subset D$, then the supremum of Z in $\mathbb{P} * RO(Col(\mu, [\kappa, \delta + 1)))$ is exactly $(p^*, \check{q}^*)$ where p^* is the intersection of all the first coordiantes of elements of Z and q^* is the intersection of all the second coordinates of elements of Z.

Lemma 5.35. *Let* $\mathbb{Q} := Col(\mu, < \delta + 1)$. *Suppose* $\vec{U}$ *is* ***not*** *almost huge, and that* $crit(k) = \delta^{+N}$. *Then* $\mathbb{Q}$ *interpolates* $j(\mathbb{P})$ *and* $j'(\mathbb{P})$ *w.r.t.* ι.

Proof. $\mathbb{Q} \in (H_{\delta^+})^N$ and is a regular suborder of $Col(\mu, < \eta)$ for any $\eta \geq \delta + 1$. Since $crit(k) = \delta^{+N}$ then $k(\mathbb{Q}) = \mathbb{Q}$ is a regular suborder of $Col(\mu, < j'(\kappa)) = j'(\mathbb{P})$. That $\mathbb{Q}$ satisfies the other requirements of interpolation was proved in Lemma 5.35. ⊣

Finally we use these to prove the main theorem of this section:

Theorem 5.37. *Suppose* $\kappa < \delta$ *are inaccessible,* κ *is* δ*-supercompact, and* δ *is the least inaccessible cardinal above* κ. *Let* $\mu < \kappa$ *be a regular cardinal. Then the model* $V^{Col(\mu,<\kappa)*Col(\kappa,<\delta)}$ *believes there is a normal ideal* $\mathcal{F}$ *on* $\kappa = \mu^+$ *such that* $ProjectiveCatch(\mathcal{F})$ *holds and* $\mathcal{F}$ *is not a strong ideal.*

If $\mu > \omega$ *then the starred version* $ProjectiveCatch^*(\mathcal{F})$ *holds.*

Proof. Let U be a normal measure on $P_\kappa(\delta)$. Let $\vec{U}$ be the projection of U to a tower of height δ. To conform to the terminology above, let $\vec{U}' := \vec{U} \cup \{(\delta, U)\}$ (so ultrapowers by U are the same as ultrapowers by $\vec{U}'$). Let $j : V \to_{\vec{U}} N$, $j' : V \to_{\vec{U}'} N'$, and $k : N \to N'$ as usual. Since N and N' are both correct about δ being the least inaccessible cardinal above κ, then $k(\delta) = \delta$, $\vec{U}$ is not almost huge, and:

$$crit(k) = \delta^{+N}. \tag{57}$$

Let μ be any regular cardinal below κ, and let $\mathbb{P} := Col(\mu, < \kappa)$. Let $\iota \in N$ be a regular embedding from $\mathbb{P} * Col(\kappa, < \delta) \to RO^N(Col(\mu, < \delta + 1))$ given by Lemma 5.35. Let $\mathbb{Q} := Col(\mu, < \delta + 1)$. By Lemma 5.36, $\mathbb{Q}$ interpolates $j(\mathbb{P})$ and $j'(\mathbb{P})$ w.r.t. the map ι.

Let $G * H$ be $(V, \mathbb{P} * Col(\kappa, < \delta))$-generic and $\mathbb{R} := \frac{\mathbb{Q}}{\iota``G*H}$. Let $\mathcal{F} := \mathcal{F}_{\dot{g}}$, where $\dot{g}$ is from Definition 5.28. Let $S \in V[G][H]$ be $\mathcal{F}$-positive. By Corollary 5.32 there is an $r \in \mathbb{R}$ such that $0 < [S^*_r]_\mathcal{F} \leq [S]_\mathcal{F}$.

In $V[G][H]$ consider an arbitrary algebra $\mathcal{A} = (H_{\delta^+}[G][H], \in, \{\mathbb{B}_\mathcal{F}\} \dots)$. We need to show that, in $V[G][H]$, there is some $M \prec \mathcal{A}$ such that $M \cap \kappa \in S^*_r$ and M is $\mathcal{F}$-self-generic.

Let $G_\mathbb{R}$ be $(V[G][H], \mathbb{R})$-generic with $r \in G_\mathbb{R}$. Now pick any $\hat{G}'$ which is $(V[G][H][G_\mathbb{R}], \frac{j'(\mathbb{P})}{G_\mathbb{R}})$-generic and let $\hat{H}'$ be $(V[\hat{G}'], Col^{N'[\hat{G}']}(j'(\kappa), < j'(\delta)))$-generic with $\bigcup \hat{j}'``H \in \hat{H}'$, and in $V[\hat{G}'][\hat{H}']$ let $\tilde{j}' : V[G][H] \to N'[\hat{G}'][\hat{H}']$ be the lifting of j' given by Lemma 5.27. Then $\kappa \in \tilde{j}'(S^*_r)$, and by (57) and Corollary 5.34:

$$\text{The ultrafilter on } P^{V[G][H]}(\kappa) \text{ derived from } \tilde{j}' \text{ is } (V[G][H], \mathbb{B}_\mathcal{F})\text{-generic.} \tag{58}$$

In $V[G][H]$ fix some transitive W such that $\delta \subset W \prec \mathcal{A}$, $|W| = \delta$, and ${}^\omega W \subset W$.[42] Set $M' := \tilde{j}'[W]$. Then $M' \prec \tilde{j}'(\mathcal{A})$, and $M' \cap \tilde{j}'(\kappa) = \kappa$. Also, by (58) the ultrafilter derived from $\tilde{j}'$ is $(W, \mathbb{B}_\mathcal{F})$-generic; this is equivalent to saying

[42]This is possible because $\delta^\omega = \delta$ in $V[G][H]$.

that M' is $\tilde{j}'(\mathcal{F})$-self-generic. Thus $V[\hat{G}'][\hat{H}']$ models:

$$\begin{aligned} &M' \prec \tilde{j}'(\mathcal{A}),\\ &M' \text{ is } \tilde{j}'(\mathcal{F})\text{-self-generic},\\ &M' \cap j'(\kappa) \in \tilde{j}'(S_r^*). \end{aligned} \tag{59}$$

Since $|W| = \delta$ then $M' = \tilde{j}'[W]$ is an element of $N'[\hat{G}'][\hat{H}']$; furthermore the statements appearing in (59) are just Σ_0 statements, so they are also true in $N'[\hat{G}'][\hat{H}']$. So by elementarity of $\tilde{j}'$:

$$V[G][H] \models (\exists M)(M \prec \mathcal{A} \ \& \ M \text{ is } \mathcal{F}\text{-self-generic} \ \& \ M \cap \kappa \in S_r^*).$$

Finally, note that in the case where $\mu > \omega$, then $\text{crit}(\tilde{j}') > 2^\omega$. In this case the ω-closure of W transfers over to ω-closure of M' from the view of $N'[\hat{G}'][\hat{H}']$. It follows that in $V[G][H]$ we would obtain ProjectiveCatch$^*(\mathcal{F})$, not merely ProjectiveCatch$(\mathcal{F})$. ⊣

5.5. Negative solution to Open Question 13 from [7]. Theorem 5.37 of the previous section implies that the hypothesis of the following lemma is consistent (relative to large cardinals), for any regular uncountable κ:

Lemma 5.38. *Suppose $\mathcal{J}_0$ is a normal ideal on a regular uncountable κ such that:*

- *ProjectiveCatch$(\mathcal{J}_0)$ holds; yet*
- *$\mathcal{J}_0$ is not a strong ideal.*

Then, there is a normal ideal $\mathcal{J}_1$ projecting to $\mathcal{J}_0$ such that the pair $\mathcal{J}_1, \mathcal{J}_0$ witnesses a "no" answer to Open Question number 13 from Foreman [7]. More precisely, $\mathcal{J}_1 \subset \wp\wp(\kappa^+)$, $\mathcal{J}_1$ projects canonically to $\mathcal{J}_0$, the canonical homomorphism $h_{\mathcal{J}_0,\mathcal{J}_1} : \mathbb{B}_{\mathcal{J}_0} \to \mathbb{B}_{\mathcal{J}_1}$ is a regular embedding, yet $\mathcal{J}_0$ is not saturated.

Proof. By Lemma 3.4, there is a $\mathcal{J}_2$ (with a large support relative to $\mathcal{J}_0$) such that *Catch*$(\mathcal{J}_2, \mathcal{J}_0)$ holds. Let $\mathcal{J}_1$ be the canonical projection of $\mathcal{J}_2$ to κ^+. Then, $\mathcal{J}_2$ projects canonically to $\mathcal{J}_1$, and $\mathcal{J}_1$ projects canonically to $\mathcal{J}_0$. By Corollary 3.12, the canonical homomorphism from $\mathbb{B}_{\mathcal{J}_0} \to \mathbb{B}_{\mathcal{J}_1}$ is a regular embedding. Since $\mathcal{J}_0$ is not strong, then it is not saturated. ⊣

Remark 5.39. *For the special case where $\kappa = \omega_1$, the negative answer to Foreman's question also follows from Theorem 3.8 and the fact that precipitousness does not imply strongness. More precisely: if $\mathcal{J}_0$ is a precipitous ideal on ω_1, then ProjectiveCatch$(\mathcal{J}_0)$ holds by Theorem 3.8; so if $\mathcal{J}_0$ is not strong*[43] *then $\mathcal{J}_0$ satisfies the hypotheses of Lemma* 5.38.

§6. Concluding remarks and questions.

Question 6.1. *The Proper Forcing Axiom (PFA) implies there is no presaturated ideal on ω_2 (Foreman-Magidor [9]). Is PFA consistent with an ideal $\mathcal{I}$ on ω_2 such that StatCatch$(\mathcal{I})$ or ProjectiveCatch$(\mathcal{I})$ holds? It is known (see Cox [5]) that, relative to a huge supercompact cardinal, PFA is consistent with an ideal $\mathcal{I}$ on $[\lambda]^{\omega_1}$ (with completeness ω_2) such that ProjectiveCatch$^*(\mathcal{I})$ holds.*

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[43]The Jech–Magidor–Mitchell–Prikry example of a precipitous ideal in $V^{Col(\omega,<\kappa)}$ where κ is measurable is not a strong ideal.

QUESTION 6.2. *Set $S^2_1 := \omega_2 \cap cof(\omega_1)$. Building on work of Kunen and Magidor, Woodin proved that it is consistent relative to an almost-huge cardinal that $NS \restriction S$ is saturated for* some *stationary $S \subset S^2_1$. It is a well-known open problem whether $NS \restriction S^2_1$ can be saturated. Since ProjectiveCatch is a weakening of saturation, it also makes sense to ask: Can ProjectiveCatch$(NS \restriction S^2_1)$ hold? What about ProjectiveCatch$^*(NS \restriction S^2_1)$?*

QUESTION 6.3. *By a well known theorem of Shelah, if $\mathcal{I}$ is an ideal whose dual concentrates on $\omega_2 \cap cof(\omega)$, then $\mathcal{I}$ is not presaturated. Can ProjectiveCatch$(\mathcal{I})$ hold for such an $\mathcal{I}$? What about when $\mathcal{I}$ is the nonstationary ideal restricted to $\omega_2 \cap cof(\omega)$?*

Note that the answer to Questions 6.2 and 6.3 is "yes" if we replace *ProjectiveCatch* with *StatCatch*; this is because of Lemma 3.7 and the fact that it is consistent (by Woodin; see [7]) for some *restriction* of NS_{ω_2} to be saturated.

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