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Authors
Brenti, Francesco
Carnevale, Angela
Tenner, Bridget Eileen

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Odd diagrams, Bruhat order, and pattern avoidance

Francesco Brenti*†1, Angela Carnevale*2, and Bridget Eileen Tenner*‡3

[1] *Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, 00133 Roma, Italy*
brenti@mat.uniroma2.it
[2] *School of Mathematical and Statistical Sciences, National University of Ireland, Galway, Ireland*
angela.carnevale@nuigalway.ie
[3] *Department of Mathematical Sciences, DePaul University, Chicago, IL, U.S.A.*
bridget@math.depaul.edu

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Abstract. The odd diagram of a permutation is a subset of the classical diagram with additional parity conditions. In this paper, we study classes of permutations with the same odd diagram, which we call odd diagram classes. First, we prove a conjecture relating odd diagram classes and 213- and 312-avoiding permutations. Secondly, we show that each odd diagram class is a Bruhat interval. Instrumental to our proofs is an explicit description of the Bruhat edges that link permutations in a class.

Keywords. Odd diagram, Bruhat order, pattern avoidance, odd length

Mathematics Subject Classifications. 05A05, 05A15

1. Introduction

Odd analogues of well-known combinatorial objects and statistics associated with permutations (and, more generally, with Weyl and Coxeter group elements) have been recently considered and studied (see, for instance, [3, 4, 5, 6, 7, 10, 13, 14, 15]). In particular, odd analogues of permutation diagrams, called *odd diagrams*, were introduced and studied in [6]. It is well known that (classical) diagrams of permutations are in bijection with permutations themselves, and

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they capture interesting related features. For instance, the size of the diagram of a permutation equals its number of inversions (that is, its length), and the diagram constitutes a main tool in the definition of the Schubert variety associated with the permutation. The odd diagram of a permutation (see Definition 2.3) is a subset of the diagram, its size equals the number of *odd inversions*—its *odd length*, and can be used to define a corresponding *odd Schubert variety* [6].

It is easy to see that odd diagrams are not in bijection with permutations. As we show in this paper, however, when passing from diagrams to odd diagrams, we trade faithful encodings of permutations for a rich combinatorial structure within each *odd diagram class* (for a definition see Section 3). In this article, we carry out an in-depth analysis of these classes, relating them to well-studied notions such as pattern avoidance and Bruhat order in symmetric groups.

The following is our first main result, which implies in particular [6, Conjecture 6.1].

Theorem A. Every odd diagram class contains at most one permutation avoiding the pattern 213 and at most one avoiding 312. If these permutations exist, they are, respectively, the maximum and the minimum elements of the class with respect to the Bruhat order.

This is proved in Sections 3 and 4 (cf. Corollaries 3.4 and 4.13) as a consequence of a detailed analysis of *legal moves* carried out in Section 4, and a certain notion of *connectivity* within odd diagram classes.

Our second main result shows that odd diagrams partition each symmetric group in a particularly pleasant way (see also Theorem 6.1).

Theorem B. The subset of $\mathfrak{S}_n$ having a given odd diagram is a Bruhat interval.

The characterization of legal moves and Theorem B imply that, with respect to the right weak order, odd diagram classes exhibit the opposite behavior. Namely, in right weak order each odd diagram class is an antichain (cf. Corollary 6.11).

The paper is organized as follows. In Section 2, we collected some notation, definitions and preliminaries. The next sections focus on relationships between permutations having the same odd diagram. As we will see in Section 3, a key role in the study of this property is played by the permutation patterns 213 and 312. This sets the stage for Section 4, which develops a method for traversing the permutations in an odd diagram class. In this section we also complete the proof of Theorem A, resolving in particular the length conjecture [6, Conjecture 6.1]. In Section 5, we analyze the consequences of legality and study so-called *illegal patterns*. We prove Theorem B in the last section. We show that each odd diagram class has a unique Bruhat minimal and a unique Bruhat maximal element and we build on the theory of legal moves to show that in an odd diagram class there is always a maximal chain of elements within the class. We conclude the paper with some remarks and open questions regarding the intervals arising from odd diagram classes.

2. Notation and preliminaries

For $n \in \mathbb{N}$, we let $\mathfrak{S}_n$ denote the symmetric group of degree n. We regard $\mathfrak{S}_n$ as a Coxeter group generated by the simple transpositions $S = \{(i\, i+1) : i = 1, \dots, n-1\}$. The set of reflections of $\mathfrak{S}_n$ is $T = \{w^{-1}sw : w \in \mathfrak{S}_n,\ s \in S\} = \{(a\,b) : 1 \leqslant a < b \leqslant n\}$. The Coxeter length of a permutation $w \in \mathfrak{S}_n$ is denoted $\ell(w)$. It is well known (see, e.g., [2, Proposition 1.5.2]) that

$$\ell(w) = |\{(i, j) \in [n]^2 : i < j,\ w(i) > w(j)\}|.$$

The *Bruhat graph* of $\mathfrak{S}_n$ is the directed graph $B(\mathfrak{S}_n)$ having $\mathfrak{S}_n$ as its vertex set and where, for $u, v \in \mathfrak{S}_n$, $u \to v$ if and only if $v\,u^{-1} \in T$ and $\ell(u) < \ell(v)$. We say that $\{u, v\}$ is a *Bruhat edge* if either $u \to v$ or $v \to u$, and denote this by $u \leftrightarrow v$.

The *Bruhat order* on $\mathfrak{S}_n$ is the partial order, which we denote by $\leqslant$, which is the transitive closure of $B(\mathfrak{S}_n)$. Unless explicitly stated, we always regard $\mathfrak{S}_n$ as partially ordered by the Bruhat order. We follow [12, Chapter 3] for notation and terminology concerning posets. The following characterization of Bruhat order covering relations in the symmetric groups is well known (see, e.g., [2, Lemma 2.1.4]) and will be repeatedly used in the sequel.

Proposition 2.1. *Let $u, v \in \mathfrak{S}_n$. Then the following conditions are equivalent:*

- *u is covered by v in Bruhat order (written $u \lhd v$);*
- *there are $1 \leqslant i < j \leqslant n$ such that $v = u(i\,j)$, $u(i) < u(j)$, and there exists no k such that $i < k < j$ and $u(i) < u(k) < u(j)$.*

Recall that a permutation $u \in \mathfrak{S}_n$ is said to *contain the pattern* $\alpha = \alpha_1 \cdots \alpha_k$ if there exist $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that $u(i_1), \ldots, u(i_k)$ are in the same relative order as $\alpha_1, \ldots, \alpha_k$. A permutation $u \in \mathfrak{S}_n$ is said to *avoid* the pattern α if it does not contain the pattern α. We denote with $\mathrm{Av}_n(\alpha) = \{u \in \mathfrak{S}_n : u \text{ avoids } \alpha\}$ the set of permutations of degree n avoiding α.

We graph $w = w(1) \cdots w(n) \in \mathfrak{S}_n$ using matrix coordinates: the point $(i, w(i))$ appears in the ith row from the top of the grid and the $w(i)$th column from the left. We let $G(w)$ denote the graph of w.

Example 2.2. The permutation 41325 is graphed in Figure 2.1. Related objects appear in Figures 2.2 and 2.3.

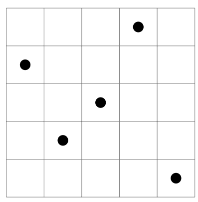


Figure 2.1: $G(41325)$.

The *diagram* $D(w)$ of a permutation w is

$$D(w) := \{(i, j) \in [n]^2 : j < w(i), w^{-1}(j) > i\}.$$

The diagram can be seen by drawing lines to the south (*legs*) and to the east (*arms*) of each point $(i, w(i)) \in G(w)$, and keeping the empty boxes that remain (see Figure 2.2).

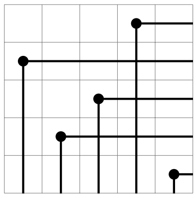


Figure 2.2: $D(41325)$ consists of the four empty boxes.

Definition 2.3. The *odd diagram* of a permutation w, as defined in [6], is the subset of $D(w)$ defined by
$$D_o(w) := \{(i,j) \in D(w) : i \not\equiv w^{-1}(j) \pmod 2\}.$$

We will often mark the elements of $D_o(w)$ by stars $*$, and refer to them as such. The odd diagram of $41325 \in \mathfrak{S}_5$ is depicted in Figure 2.3.

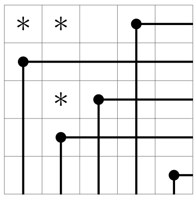


Figure 2.3: $D_o(41325)$ consists of the three boxes that are marked by $*$s.

3. Injectivity and permutation patterns

Permutations in $\mathfrak{S}_n$ can be partitioned by their odd diagrams, and we will indicate that two permutations are in the same odd diagram class by $\sim$; that is, we write $v \sim w$ if $D_o(v) = D_o(w)$. We begin by noting that if two permutations have the same odd diagrams, then the leftmost columns of their graphs must be the same.

Lemma 3.1. *If $v \sim w$ then $v^{-1}(1) = w^{-1}(1)$.*

Proof. If the leftmost column of $D_o(v)$ is empty, then clearly $v(1) = 1$ and the lemma follows. More generally, since we are looking at the leftmost column of the (odd) diagrams, no boxes are eliminated by arms. Therefore the lowest star in $D_o(v) = D_o(w)$ sits directly north of the leftmost point in the graphs of v and w. □

This enables us to prove something quite useful about permutations $v \sim w$.

Theorem 3.2. *If $v \sim w$ for $v \neq w$, then v has a 213-pattern and w has a 312-pattern, or conversely.*

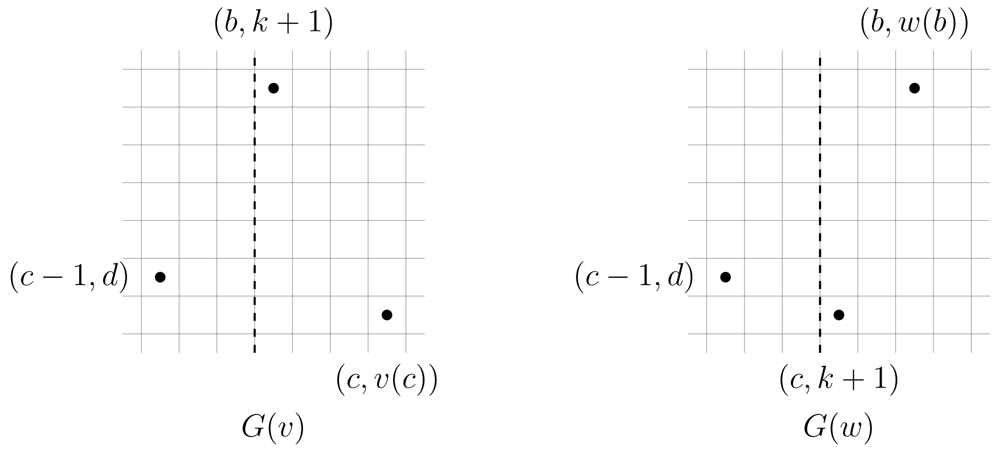


Figure 3.1: The graphs of the permutations v and w as described in the proof of Theorem 3.2, showing a 213-pattern in v and a 312-pattern in w. The graphs are identical to the left of the dashed line.

Proof. Let $k+1$ be minimal such that $v^{-1}(k+1) \neq w^{-1}(k+1)$. By Lemma 3.1, $k \geqslant 1$.

To ease notation, set

$$\begin{aligned} a &:= v^{-1}(k) = w^{-1}(k), \\ b &:= v^{-1}(k+1), \text{ and} \\ c &:= w^{-1}(k+1). \end{aligned}$$

So $(a, k) \in G(v) \cap G(w)$, while $(b, k+1) \in G(v)$ and $(c, k+1) \in G(w)$. Without loss of generality, assume that $b < c$.

Because $D_o(v) = D_o(w)$, the point $(c-1, k+1)$ must lie in the arm of $(c-1, d) \in G(w)$ for some $d \leqslant k$. By minimality of k, we have that $(c-1, d) \in G(v)$, as well. Again by minimality of k, it must also be the case that

$$v(c) > k+1 \text{ and } w(b) > k+1.$$

From this we find the desired patterns. $\square$

We illustrate the proof of Theorem 3.2 in Figure 3.1. Note that the point $(a, k) \in G(v) \cap G(w)$ is not needed for either of the patterns unless $d = k$.

A *vincular* permutation pattern is one in which pairs of letters may be required to appear consecutively. For example, the permutation 41325 contains the classical pattern 321 (demonstrated by the substring 432), but it does not contain the vincular pattern $\underline{32}1$ because there is no occurrence of 321 in which the first two letters are consecutive. In this language of vincular patterns, then, Theorem 3.2 actually shows a slightly stronger result.

Corollary 3.3. *Suppose that $v \sim w$ for $v \neq w$. Then, without loss of generality, v has a $2\underline{13}$-pattern and w has a $3\underline{12}$-pattern such that*

- *these two patterns occupy the same positions in v and w,*
- *the value of the "2" is the same in each pattern, and*
- *$G(v)$ and $G(w)$ coincide at all points (a, b), for b less than that shared value of "2."*

Theorem 3.2 also shows how the map $w \mapsto D_o(w)$ behaves on pattern classes, suggesting further significance to the patterns 213 and 312 that appear in the statement of the theorem, and verifying the first part of Conjecture 6.1 of [6].

Corollary 3.4. *(a) The map $w \mapsto D_o(w)$ is injective on $\mathrm{Av}_n(213)$. That is, if $v \neq w$ both avoid 213, then $D_o(v) \neq D_o(w)$. Moreover, we can replace 213 by $2\underline{13}$ in both statements.*

(b) The map $w \mapsto D_o(w)$ is injective on $\mathrm{Av}_n(312)$. That is, if $v \neq w$ both avoid 312, then $D_o(v) \neq D_o(w)$. Moreover, we can replace 312 by $3\underline{12}$ in both statements.

(c) For all other p with $|\mathrm{Av}_n(p)| > 1$, the map $w \mapsto D_o(w)$ is not injective on $\mathrm{Av}_n(p)$.

Proof. Parts (a) and (b) are clear. To prove (c), suppose first that p has one of the following three forms: $123 \cdots k$ (with $k > 2$), or $2134 \cdots k$ (with $k > 3$), or $3124 \cdots k$ (with $k > 3$). Then consider the permutations $v = n(n-1) \cdots 54213$ and $w = n(n-1) \cdots 54312$. These both avoid p, and they have the same odd diagrams.

Now, suppose that p does not have one of those three forms and consider the permutations $v = 213456 \cdots (n-1)n$ and $w = 312456 \cdots (n-1)n$. The only patterns these permutations contain are the three considered above. The two permutations v and w have the same odd diagram and, because p is not one of the three patterns listed above, they both avoid p. □

Let $o_n := |\{D_o(v) : v \in \mathfrak{S}_n\}|$ denote the number of distinct odd diagrams of permutations in $\mathfrak{S}_n$. It follows from [8, Propositions 1 and 3] that the number of permutations in $\mathrm{Av}_n(3\underline{12})$ is given by the nth Bell number B_n. Thus the previous result gives a lower bound for the number of odd diagrams in degree n.

Corollary 3.5. *Let $n \in \mathbb{N}$. Then $o_n \geqslant B_n$, where B_n is the nth Bell number.*

The first values of the sequence $\{o_n\}_{n \in \mathbb{N}}$ are: 1, 2, 5, 17, 70, 351, 2041, 13732, 103873, 882213 (cf. also [11, A335926]).

4. Legal moves

Definition 4.1. Recall from Section 2 that two permutations $v \neq \overline{v}$ are connected by a Bruhat edge if they agree on all but two values. A particular Bruhat edge $v \leftrightarrow \overline{v}$ is *legal* if $v \sim \overline{v}$. If $\overline{v} = vt \sim v$ we will also sometimes say that the transposition t is *legal for* v.

Because a transposition only changes the positions of two values in the permutation, legality depends only on the non-overlapping portions of the points, arms, and legs that were affected by that transposition.

Lemma 4.2. *Consider a Bruhat edge $v \leftrightarrow \overline{v}$ where the points that move are as indicated in red and blue in Figure 4.1. The Bruhat edge is legal if and only if none of the boxes that include only red or only blue in Figure 4.1 belong to $D_o(v)$ or to $D_o(\overline{v})$.*

Proof. Whether or not any of the other points is included in an odd diagram is not impacted by the points being swapped. □

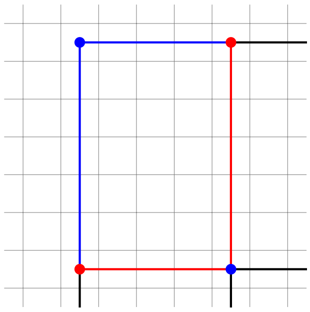


Figure 4.1: In a Bruhat edge $v \leftrightarrow \overline{v}$, the black arms and legs will arise for both permutations, whereas the red points and segments appear for only one permutation, and the blue points and segments only appear for the other.

The following result is fundamental in all that follows.

Theorem 4.3. *Let v be a permutation and $\overline{v} := v\,(ij)$ for $i < j$. Set $m := \min\{v(i), v(j)\}$ and $M := \max\{v(i), v(j)\}$. The Bruhat edge $v \leftrightarrow \overline{v}$ is legal if and only if the following requirements are met:*

(R1) i *and* j *have the same parity,*

(R2) $v(x) < m$ *for all* $x \in \{i+1, i+3, \ldots, j-1\}$*, and*

(R3) $v(y) \notin [m, M]$ *for all* $y \in \{j+1, j+3, \ldots\}$.

Proof. As suggested by Lemma 4.2, we prove the result by checking three things: the square marked by the blue point in the upper left of Figure 4.1, the squares marked by vertical colored segments in that figure, and the squares marked by horizontal colored segments in that figure.

- The upper-left square marked by the blue point will be in one (but not both) of the odd diagrams for v and $\overline{v}$ if and only if (R1) is not met.
- For the vertical colored segments to contribute no elements to either odd diagram, all boxes in those columns whose heights differ in parity to i (or j, by (R1)) must lie in the arms of points that appear to the left of m in the graph. This is equivalent to (R2).
- For the horizontal colored segments to contribute no elements to either odd diagram, there can be no boxes in those rows that are an odd distance above points in the graph. In light of (R2), this is equivalent to (R3).

Thus $v \sim \overline{v}$ if and only if (R1), (R2), and (R3) are met. $\square$

One impact of Theorem 4.3 is that if $v \sim \overline{v}$ is a legal Bruhat edge, then v and $\overline{v}$ differ by the change of a particular pattern.

Definition 4.4. Let $v \leftrightarrow \overline{v}$ be a (not necessarily legal) Bruhat edge. If the points in which v and $\overline{v}$ differ form the endpoints of a 213-pattern in one of the permutations and a 312-pattern in the other, then we call this a *pattern swap*, written $v \overset{\circ}{\leftrightarrow} \overline{v}$. The *type of the pattern swap for* v is the pattern (213 or 312) in v that gets changed to make $\overline{v}$.

So, for example, $5431627 \leftrightarrow 3451627$ is not a pattern swap, while $5431627 \leftrightarrow 5431726$ is a pattern swap of type 213 for 5431627 (and of type 312 for 5431726).

Corollary 4.5. *If a Bruhat edge is legal, then it must be a pattern swap.*

Proof. Suppose that $v \leftrightarrow \overline{v}$ is legal. By Theorem 4.3, (R1) and (R2) must be satisfied. Maintaining the notation from the proof of that theorem, (R1) says that $j \geqslant i+2$, and (R2) says that $v(j-1) = \overline{v}(j-1) < m$. Thus, the values in positions $\{i, j-1, j\}$ form a 213-pattern in one of the permutations, and a 312-pattern in the other, so $v \overset{\circ}{\leftrightarrow} \overline{v}$. $\square$

A pattern swap that results from a legal Bruhat edge will correspondingly be called a *legal* pattern swap. A pattern swap $v \overset{\circ}{\leftrightarrow} \overline{v}$ for which $D_o(v) \neq D_o(\overline{v})$ is an *illegal* pattern swap.

The properties discussed in Corollary 3.3 are highly localized, and the permutations might differ substantially otherwise. However, we can use these results to find much closer "neighbors" to a permutation, staying within the same class.

Definition 4.6. Consider $v \sim w$ with $v \neq w$. Maintaining notation from Theorem 3.2, let $\{(b, k+1), (c-1, d), (c, v(c))\}$ be the 213-pattern found in v, and let $\{(b, w(b)), (c-1, d), (c, k+1)\}$ be the corresponding 312-pattern found in w. Define permutations $\overline{v}_w := v\,(b\,c)$ and $\overline{w}_v := w\,(b\,c)$. In other words, $v \overset{\circ}{\leftrightarrow} \overline{v}_w$ via the 213-pattern found in v during Theorem 3.2, and $w \overset{\circ}{\leftrightarrow} \overline{w}_v$ via the 312-pattern found in w.

Example 4.7. Consider $v = 5431627$ and $w = 7461325$. Then $\overline{v}_w = 5461327$. Repeating the process with $u := \overline{v}_w$ and w produces $\overline{u}_w = 7461325 = w$.

Remark 4.8. Notice how the (classical) lengths of v and w compare to those of the permutations described in Definition 4.6:

$$\ell(v) < \ell(\overline{v}_w) \quad \text{and} \quad \ell(w) > \ell(\overline{w}_v). \tag{4.1}$$

These new permutations act as intermediaries, allowing us to travel between any two permutations in the same odd diagram class.

Theorem 4.9. *Suppose that* $v \sim w$ *with* $v \neq w$. *Then* $v \sim \overline{v}_w$ *and* $w \sim \overline{w}_v$.

Proof. We show that $v \sim \overline{v}_w$. Maintain the notation of Theorem 3.2 and Definition 4.6. Since $v \sim w$, we have that $b \equiv c \pmod 2$ (else $(b, k+1) \in D_o(w) \setminus D_o(v)$). Also, $v(x) < k+1$ if $x \in \{b+1, b+3, \ldots, c-1\}$ (otherwise $w(x) > k+1$ by the minimality of k, so $(x, k+1) \in D_o(w) \setminus D_o(v)$). Finally, $v(x) \notin [k+2, v(c)-1]$ if $x \in \{c+1, c+3, \ldots\}$ (else $(c, v(x)) \in D_o(v) \setminus D_o(w)$). Hence, by Theorem 4.3, the Bruhat edge $v \leftrightarrow \overline{v}_w$ is legal and therefore $v \sim \overline{v}_w$. The proof that $w \sim \overline{w}_v$ is similar. □

Note an interesting consequence of Definition 4.6 and Theorem 4.9.

Corollary 4.10. *Consider permutations $v \sim w$ that agree for the first k, but not $k+1$, values (that is, $v^{-1}(i) = w^{-1}(i)$ for all $i \leqslant k$, but not for $i = k+1$). Then the permutations $\overline{v}_w$ and w agree for the first $k+1$ values, at least, as do $\overline{w}_v$ and v.*

From these results we see that the class of permutations with a given odd diagram is, in a sense, connected.

Definition 4.11. Let D be an odd diagram, and consider the set

$$\mathrm{Perm}_n(D) := \{w \in \mathfrak{S}_n : D_o(w) = D\}.$$

Define the *class graph* $G_{D,n}$ to have vertex set $\mathrm{Perm}_n(D)$, and an edge between vertices v and v' if $v' = \overline{v}_w$ for some permutation $w \in \mathrm{Perm}_n(D)$.

Corollary 4.12. *The class graph $G_{D,n}$ is connected.*

We conclude this section with the following result, which resolves the length conjecture [6, Conjecture 6.1] and concludes the proof of Theorem A.

Corollary 4.13.

(a) If $v \in \mathrm{Perm}_n(D)$ is 312-avoiding, then $v \leqslant u$ for all $u \in \mathrm{Perm}_n(D)$.

(b) If $w \in \mathrm{Perm}_n(D)$ is 213-avoiding, then $w \geqslant u$ for all $u \in \mathrm{Perm}_n(D)$.

Proof. Let u be a minimal (in Bruhat order) element of $\{z \in \mathrm{Perm}_n(D) : v \not\leqslant z\}$. Then, by Theorem 3.2 and our hypothesis, v has a 213-pattern and u has a 312-pattern. Thus $\overline{u}_v < u$ (notation as in Definition 4.6) which contradicts the minimality of u since, by Theorem 4.9, $u \sim \overline{u}_v$. The proof of (b) is analogous. □

5. Consequences of legality

While legality implies that a Bruhat edge is a pattern swap, not all pattern swaps are legal. In fact, as we show below, the potential for illegal pattern swaps is, in a sense, persistent in a class.

Definition 5.1. If v has an illegal pattern swap of type π, then we say that the pattern represented by π is an *illegal pattern* in v.

In other words, an illegal pattern is a 213- or 312-pattern in v for which the permutation v' obtained by swapping the left and right letters in the pattern, does not have the same odd diagram as v.

Theorem 5.2. *Suppose that $v \in \mathfrak{S}_n$ has an illegal pattern of type π. Then all $w \sim v$ also have illegal patterns of type π.*

Proof. Set $D := D_o(v)$. We prove this result recursively, showing that if v has this property, then every neighbor $\overline{v}_w$ of v in the graph $G_{D,n}$ defined above also has this property. By Corollary 4.12, this will prove the result.

Let $1 \leqslant i < h < j \leqslant n$ be the positions of the illegal pattern π in v. Set

$$m := \min\{v(i), v(j)\} \qquad \text{and} \qquad M := \max\{v(i), v(j)\}.$$

We first check whether there is a $\overline{v}_w$ with no illegal patterns of type π, in which $v(i) = \overline{v}_w(i)$ and $v(j) = \overline{v}_w(j)$. Suppose that there is such a $\overline{v}_w$. Note that if $\overline{v}_w \overset{\circ}{\leftrightarrow} v$ legally moved $(h, v(h))$, then for at least one of $h' \in \{h \pm 1\}$, we would have $v(h') < v(h)$ by (R2). Define the position

$$h^* := v^{-1}\Big(\min\big\{v(h-1), v(h), v(h+1)\big\}\Big),$$

and the pattern

$$\pi' := \Big\{\big(i, \overline{v}_w(i)\big), \big(h^*, \overline{v}_w(h^*)\big), \big(j, \overline{v}_w(j)\big)\Big\}$$

in $\overline{v}_w$. If π is illegal because $i \not\equiv j \pmod 2$, then π' is illegal in $\overline{v}_w$. We may therefore assume that $i \equiv j \pmod 2$. Now suppose that there exists $x \in \{i+1, i+3, \ldots, j-1\}$ such that $v(x) > m$. If $\overline{v}_w(x) = v(x)$, then π' is illegal in $\overline{v}_w$. Otherwise $\overline{v}_w = v(x\,z)$ for some $z \notin \{i, j, x\}$ with $v(z) < m$. But then either $(i, v(i))$ or $(j, v(j))$ would have violated (R2) for the supposedly legal $v \overset{\circ}{\leftrightarrow} v(x\,z)$. Therefore we can assume that there is no such x. Finally, suppose that there exists $y \in \{j+1, j+3, \ldots\}$ such that $v(y) \in [m, M]$. If $\overline{v}_w(y) = v(y)$, then π' is illegal in $\overline{v}_w$. Otherwise, $\overline{v}_w = v(y\,z)$ for some $z \notin \{i, j, y\}$ with $v(z) \notin [m, M]$. To be legal, we must have $z \equiv y \pmod 2$. If $z > j$, then π' will be illegal due to $(z, \overline{v}_w(z))$. On the other hand, if $z < j$, then multiplying by $(y\,z)$ would have failed (R2) because of the point $(v^{-1}(M), M)$, and hence would not have been a legal move. Therefore there is no such $\overline{v}_w$.

Now consider $\overline{v}_w$ in which, without loss of generality, $\overline{v}_w = v(i\,i')$ for some $i' \neq i$, and in particular $i \neq h' \neq j$ because $v(h') < m$. Note that $i' \neq j$ because the move $\overline{v}_w \leftrightarrow v$ is legal; on the other hand, the case of i' equalling h is not excluded.

Suppose that (R1) is not met in v. To fix this, the point at height i must move legally to form $\overline{v}_w$. By (R1), i and i' have the same parity. If $i = h'$, and t is the position of the middle value in the (necessarily) 312-pattern swapped to form $\overline{v}_w$, then

$$\Big\{\big(i, \overline{v}_w(i)\big), \big(t, \overline{v}_w(t)\big), \big(j, \overline{v}_w(j)\big)\Big\}$$

will be an illegal pattern in $\overline{v}_w$ failing (R2), so we can assume $i' \neq h$. Thus, in all but one case, we have that

$$\Big\{\big(I, \overline{v}_w(I)\big), \big(h, \overline{v}_w(h)\big), \big(j, \overline{v}_w(j)\big)\Big\}$$

is an illegal pattern of type π in $\overline{v}_w$, for at least one $I \in \{i, i'\}$, failing (R1). The only case where this might not hold is when $i < h < i'$ and $v(i') < v(h)$. For $v \overset{\circ}{\leftrightarrow} \overline{v}_w$ to have been legal, we must have had $v(i+1) < \min\{v(i), v(i')\}$, and thus

$$\Big\{\big(i, \overline{v}_w(i)\big), \big(i+1, \overline{v}_w(i+1)\big), \big(j, \overline{v}_w(j)\big)\Big\}$$

is an illegal pattern of type π in $\overline{v}_w$, again failing (R1). Thus we may assume that (R1) is satisfied, and hence $i' \equiv i \equiv j \pmod 2$.

Now suppose that (R2) is not met in v. Let $x \in \{i+1, i+3, \ldots, j-1\}$ be maximal for which $v(x) \geqslant m$. To fix this in $\overline{v}_w$, the point at height i would have to move legally downward, using $i' \geqslant x$. In fact, because i and i' have the same parity, we must have $x \in \{i+1, i+3, \ldots, i'-1\}$, and $v(x) < \min\{v(i), v(i')\}$ to make this a legal move not violating (R2). Therefore, it must be that $m = v(j)$, and

$$\big\{(i, \overline{v}_w(i)), (h, v(h)), (j, v(j))\big\}$$

is an illegal pattern of type π in $\overline{v}_w$, again failing (R2) with this value of x.

Finally, suppose that (R3) is not met in v. In any legal Bruhat edge from v, we find that

$$\big\{(i, \overline{v}_w(i)), (j-1, v(j-1)), (j, v(j))\big\}$$

is an illegal pattern of type π in $\overline{v}_w$, again failing (R3).

The case $\overline{v}_w = v(j\, j')$ can be addressed by the same arguments. □

6. Odd diagram classes are Bruhat intervals

This section is devoted to proving the following result.

Theorem 6.1. *Let $D \subset [n]^2$ be an odd diagram. Then $\mathrm{Perm}_n(D) = [u, v]$ for some $u, v \in \mathfrak{S}_n$.*

We prove Theorem 6.1 in three main steps. The first is to show that given an odd diagram class, there exist one Bruhat-minimal and one Bruhat-maximal element in the class. Next, we will show that the class contains a maximal chain between those two extreme elements. Finally, we will use the "flip" operation of [1] to complete the argument.

For the rest of the paper, assume that $|\,\mathrm{Perm}_n(D)| \geqslant 2$. We start by showing that within a class, each value $k \in [n]$ can only appear in positions having the same parity.

Lemma 6.2. *Let $u, v \in \mathfrak{S}_n$ with $u \sim v$. Then $u^{-1}(k) \equiv v^{-1}(k) \pmod 2$ for all $k \in [1, n]$.*

Proof. Set $D := D_o(u) = D_o(v)$. Since $G_{D,n}$ is connected by Corollary 4.12, we may assume that u and v are connected by an edge. Hence there is $w \in \mathrm{Perm}_n(D)$, $w \neq v$ such that $u = \overline{v}_w$. This means that $u = v(b\, c)$ for some $b, c \in [n]$, and $u \leftrightarrow v$ is a legal Bruhat edge. Hence, by Theorem 4.3, $b \equiv c \pmod 2$.

Thus $u^{-1}(i) \equiv v^{-1}(i) \pmod 2$ for all $i \in [n]$, as desired. □

Note that the lemma implies, in particular, that we can talk about "admissible parity" of a column of the graph of a permutation in a class.

Definition 6.3. Fix an odd diagram D. Let the kth column be labeled $\varepsilon_k \in \{0,1\}$ according to the parity of $w^{-1}(k)$ for some (every, by Lemma 6.2) permutation $w \in \mathrm{Perm}_n(D)$. An arbitrary permutation v has *admissible parity* if $v^{-1}(k) \equiv \varepsilon_k \pmod 2$ for all k.

We exploit this notion in the next theorem to construct, starting from any permutation, the unique minimal element of its odd diagram class. The idea is to define a "smallest" permutation with the given odd diagram.

Theorem 6.4. *Fix an odd diagram D. There exists $u \in \mathrm{Perm}_n(D)$ such that, for all $w \in \mathrm{Perm}_n(D)$, we have $u \leqslant w$ in Bruhat order.*

Proof. Based on the odd diagram D, we will construct the permutation u by placing dots (points in the graph of u) in each column, from left to right. For each new column, we place a dot in the highest empty cell which does not have a star below it or to the right, and which has no dots already placed to its left, and which has admissible parity.

Consider an arbitrary $w \in \mathrm{Perm}_n(D)$. In order to show that the desired u exists, we will construct a sequence of permutations $w = w_1, w_2, \ldots, w_n$ such that

(i) $w_{k+1} \leqslant w_k$ in Bruhat order,

(ii) $w_{k+1}^{-1}(i) = w_k^{-1}(i)$ for all i, except for (at most) two values of $i \in [k+1, n]$, and

(iii) $w_k \sim w_{k+1}$.

Developing $w_2, w_3, \ldots, w_n$ will be based on the data of D, not of w. The only feature of w which could be considered relevant is $w^{-1}(1)$, but Lemma 3.1 means that this is forced by D itself.

Consider $k \geqslant 1$ and assume $w_1, \ldots, w_k$ have already been defined. Set i_{k+1} to be $\max\{i \in [n] : (i, k+1) \in D\}$ if the latter set is non-empty, and 0 otherwise. Let $\mathcal{A}_{k+1}$ be the set of $r \in [n]$ such that:

- $r > i_{k+1}$,
- $r \equiv \varepsilon_{k+1} \pmod 2$,
- $(r, s) \notin D$ if $k+1 < s \leqslant n$, and
- $r \notin \{w_k^{-1}(i) : i \in [k]\}$.

Informally, $\mathcal{A}_{k+1}$ is the set of admissible positions for an element in column $k+1$ of the graph of a permutation equivalent to w_k, and for which the values $1, \ldots, k$ have the same positions that they had in w_k. In particular, $w_k^{-1}(k+1) \in \mathcal{A}_{k+1}$, so this set is non-empty.

Let $b = \min \mathcal{A}_{k+1}$. If $b = w_k^{-1}(k+1)$ then $w_{k+1} := w_k$. Otherwise, set $c := w_k^{-1}(k+1)$. By definition, $b < c$ and $b \equiv c \pmod 2$. We set $w_{k+1} := w_k(b\,c)$. It is clear that w_{k+1} satisfies properties (i) and (ii). To show that $w_{k+1} \sim w_k$ it is enough to show that $(b\,c)$ is a legal transposition for w_k, using Theorem 4.3.

- (R1) follows from the parity condition in $\mathcal{A}_{k+1}$.

- (R2) translates to showing that $w_k(x) < w_k(c)$ for all $x \in \{b+1, b+3, \dots, c-1\}$. Indeed, if $w_k(x) > w_k(c) = k+1$ for such an x then $(x, k+1) \in D$, which would contradict the maximality of i_{k+1}.
- (R3) translates to showing that $w_k(y) \notin [w_k(c), w_k(b)]$ for all $y \in \{c+1, c+3 \dots\}$. If, instead, $w_k(y) \in [w_k(c), w_k(b)]$ then $(b, w_k(y)) \in D$, which would contradict $b \in \mathcal{A}_{k+1}$.

Therefore $w_{k+1} \sim w_k$.

The last permutation of the sequence $u := w_n$ is the minimal element of $\operatorname{Perm}_n(D)$. □

An analogue of the above result can be used to construct the maximal permutation in a class, as stated in the following.

Theorem 6.5. *Fix an odd diagram D. There exists $v \in \operatorname{Perm}_n(D)$ such that, for all $w \in \operatorname{Perm}_n(D)$, we have $v \geqslant w$ in Bruhat order.*

Proof. The proof follows similar lines to those of Theorem 6.4. Here the idea is to construct the maximal element by choosing, in every column, the largest admissible position. Keeping notation as in Theorem 6.4, we define the sequence of permutations $w_1, \dots, w_n$ such that

(i) $w_{k+1} \geqslant w_k$ in Bruhat order,

(ii) $w_{k+1}^{-1}(i) = w_k^{-1}(i)$ for all i, except for (at most) two values of $i \in [k+1, n]$, and

(iii) $w_k \sim w_{k+1}$.

At each step w_{k+1} is defined as in Theorem 6.4 except here we take $b = \max \mathcal{A}_{k+1}$. Arguing as in the previous theorem shows that $v := w_n$ is the maximal element of $\operatorname{Perm}_n(D)$. □

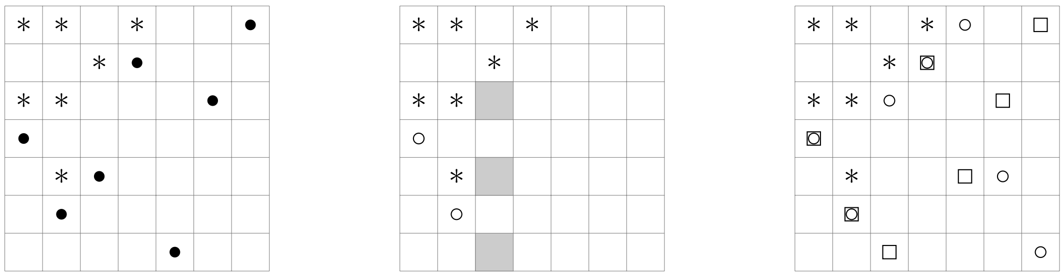
We demonstrate these results with an example.

Example 6.6. Let $w = 7461325 \in \mathfrak{S}_7$. The graph $G(w)$ and odd diagram $D = D_o(w)$ are depicted in Figure 6.1(A). Figure 6.1(B) illustrates a step in the construction of the minimal element of $\operatorname{Perm}_7(D)$. The shaded grey cells are the admissible positions in the third column ($\mathcal{A}_3$ in the proofs of Theorems 6.4 and 6.5). Figure 6.1(C) depicts the graphs of the minimal (5431627, whose graph is represented by $\circ$) and maximal (7461523, whose graph is represented by $\square$) elements in the class. $\operatorname{Perm}_7(D)$ is a Bruhat interval of size 18 and rank 5.

Theorems 6.4 and 6.5 imply that the minimal ("bottom") and maximal ("top") elements, u and v, of the class are unique and that $\operatorname{Perm}_n(D) \subseteq [u, v]$.

We now show that given an odd diagram class $\operatorname{Perm}_n(D)$ and its bottom and top elements u and v, there is a maximal chain (in Bruhat order) from u to v of elements within the class. We start by showing that given any permutation $w \neq v$, we can always find an element of $\operatorname{Perm}_n(D)$ covering w. We will sometimes call such a permutation a *legal cover* of w.

Proposition 6.7. *Let $w \in \mathfrak{S}_n$ and $D = D_o(w)$. Let the bottom and top elements of $\operatorname{Perm}_n(D)$ be u and v, respectively. Then there exists a maximal chain of elements in the class $\operatorname{Perm}_n(D)$ connecting u to v.*



(a) $G(w)$ and $D_o(w)$. (b) Admissible positions. (c) Min ($\circ$) and max ($\square$).

Figure 6.1: Building the minimal and maximal elements of $\mathrm{Perm}_7(D_o(7461325))$.

Our proof will rely on the following.

Lemma 6.8. *Let v denote the top element of $\mathrm{Perm}_n(D)$ and assume $w \neq v$. Then there exists a transposition t such that $w \leftrightarrow wt$ is a legal Bruhat edge and $w \lhd wt$.*

Proof. Since $w \sim v$ and $w \neq v$, Definition 4.6 and Theorem 4.9 ensure that there exists $\overline{w}_v \sim w$ which is obtained from w by applying a single (legal) transposition, say r. If k is the minimum column index in which w and v differ, then $r = (b\,c)$, where $b = w^{-1}(k)$ and $c = v^{-1}(k)$. Since $(b\,c)$ is legal for w, $b \equiv c \pmod 2$. Since v is the longest element of the class, $\ell(\overline{v}_w) < \ell(v)$ and it therefore follows from Remark 4.8 that $\ell(\overline{w}_v) > \ell(w)$ so $b < c$. If $\ell(\overline{w}_v) - \ell(w) = 1$ we are done.

If $\ell(\overline{w}_v) - \ell(w) > 1$, that is $\ell(\overline{w}_v) - \ell(w) \geqslant 3$, then $w(c) = k + i$ for some $i \geqslant 2$, $c - b \geqslant 2$ and, by Proposition 2.1, there exists $j \in [k+1, k+i-1]$ such that $d := w^{-1}(j) \in [b+1, c-1]$. At the level of the graph $G(w)$ this means that the points $(b, w(b))$ and $(c, w(c))$ are sufficiently far away from each other and that there is at least one point of the graph inside the rectangle they determine. Let $m := \min\{d \in [n] : (d, w(d)) \in [b+1, c-1] \times [k+1, k+i-1]\}$ so $(m, w(m))$ is the "highest" such dot. We claim that $t = (b\,m)$ is a legal transposition for w such that $w \lhd wt$. The transposition $(b\,c)$ is legal for w, so, by Theorem 4.3 $w(x) < w(b)$ for all $x \in \{b+1, b+3, \ldots, c-1\}$. Since $w(b) < w(m)$ this implies that $m \equiv b \pmod 2$, showing that both (R1) and (R2) from Theorem 4.3 hold. Finally, conditions (R2) and (R3) for $w \leftrightarrow w(b\,c)$ imply that $w(y) < w(b)$ if $y \in \{m+1, m+3, \ldots, c-1\}$ and that $w(y) \notin [w(b), w(m)]$ if $y \in \{c+1, c+3, \ldots\}$ so (R3) holds for $w \leftrightarrow w(b\,m)$.

Clearly, by minimality of m, the transposition t is also such that $w \lhd wt$. This proves the result. $\square$

Proof of Proposition 6.7. Our result follows by repeated application of Lemma 6.8 noting that, since $wt \in \mathrm{Perm}_n(D)$, $wt \leqslant v$ by Theorem 6.5. $\square$

We now show that any saturated 3-chain in an odd diagram class completes to a square in the same class. Let $x, y, z \in \mathfrak{S}_n$ be such that $x \lhd y \lhd z$. By [2, Lemma 2.7.3] the interval $[x, z]$ is isomorphic, as a poset, to a Boolean algebra of rank 2. Thus, there exists a unique permutation $w \in \mathfrak{S}_n$ such that $w \neq y$ and $x \lhd w \lhd z$. The proof of the following proposition relies on a case by case analysis that is based on an explicit description of w that we now provide.

Let $y = x(a\,b)$ for some $a < b$ and $z = y(c\,d)$ for some $c < d$. We have $x(a) < x(b)$ and $y(c) < y(d)$ and there are five cases:

- $|\{a, b, c, d\}| = 4$: In this case $w = x(c\,d) = z(a\,b)$.
- $a < b = c < d$: In this case $x(a) < x(d)$, $w = x(c\,d)$ if $x(b) < x(d)$ and $w = z(a\,b)$ if $x(b) > x(d)$.
- $a < b = d > c$: In this case $x(a) > x(c)$, $w = z(a\,b)$ if $c < a$ and $w = x(c\,d)$ if $a < c$.
- $b > a = c < d$: In this case $x(b) < x(d)$, $w = z(a\,b)$ if $d > b$ and $w = x(c\,d)$ if $d < b$.
- $b > a = d > c$: In this case $x(c) < x(b)$, $w = z(a\,b)$ if $x(c) > x(a)$ and $w = x(c\,d)$ if $x(c) < x(a)$.

Proposition 6.9. *Let D be an odd diagram, with $x \lhd y \lhd z$ all in $\mathrm{Perm}_n(D)$. Then we have $[x, z] \subseteq \mathrm{Perm}_n(D)$.*

Proof. We keep the notation introduced before the statement of Proposition 6.9. We will show that either $x \leftrightarrow w$ or $w \leftrightarrow z$ is a legal Bruhat edge. We need to consider several cases, depending on $|\{a, b, c, d\}|$, parities and the relative order of positions and values involved in the swaps.

Case 1. Suppose $|\{a, b, c, d\}| = 4$, that is $(a\,b)$ and $(c\,d)$ commute. We claim that in this case $(c\,d)$ is a legal move for x such that $w = x(c\,d)$ covers x. Clearly, the parity condition (R1) from Theorem 4.3 holds. By our assumption, $x(a) < x(b)$ and $x(c) < x(d)$. To show that $x \leftrightarrow w$ is a legal move, we need to show that (R2) and (R3) hold.

- If $a \equiv c \pmod 2$ then x, y, w and z coincide in all positions $i \in [n]$ with $i \not\equiv a \pmod 2$. But these are the only values involved in the requirements for the legality of the relevant moves. So (R2) and (R3) for $y \leftrightarrow z$ imply the analogous conditions for $x \leftrightarrow w$ independently of the relative order of a, b, c and d.
- For $a \not\equiv c \pmod 2$, we will consider $a < c$ (all other cases work similarly).

  If $a < b < c < d$ then clearly (R2) and (R3) hold for $y \leftrightarrow z$ if and only if they hold for $x \leftrightarrow w$, since positions and values involved in the swaps are the same.

  The case in which the two transpositions interlace, that is $a < c < b < d$, cannot occur. Indeed, for $(a\,b)$ to be legal for x, $x(c) < x(a)$ should hold, and for $(c\,d)$ to be legal for y, $x(a) = y(b) < y(c) = x(c)$ should hold.

  Finally, suppose $a < c < d < b$. Our assumptions and condition (R2) for $x \leftrightarrow y$ imply $x(c) < x(d) < x(a) < x(b)$. This in turn shows that both (R2) and (R3) hold for $x \leftrightarrow w$ if and only if they hold for $y \leftrightarrow z$.

Case 2. Suppose $|\{a, b, c, d\}| = 3$. The parity condition is clearly always satisfied in this case.

Suppose $a < b = c < d$. By assumption, we have $x(a) < x(b)$ and $x(a) < x(d)$ (since $x \leqslant y \leqslant z$).

- If $x(a) < x(b) < x(d)$, then $w = x(c\,d)$ and (R2) and (R3) for $y \leftrightarrow z$ imply (R2) and (R3) for $x \leftrightarrow w$. This is because $\min\{x(a), x(d)\} = x(a) < x(b) = \min\{x(b), x(d)\}$ and $\max\{x(b), x(d)\} = \max\{x(a), x(d)\}$.
- If $x(a) < x(d) < x(b)$ then $w = z(a\,b)$ and we claim that z is a legal cover of w, or, equivalently, that $x \leftrightarrow x(a\,d)$ is legal. We show the latter. Condition (R2) for $x \leftrightarrow y$ and for $y \leftrightarrow z$ implies $x(i) < x(a)$ for all $i \in \{a+1, a+3, \dots, d-1\}$; that is, (R2) holds for $x \leftrightarrow x(a\,d)$. Similarly, condition (R3) for $y \leftrightarrow z$ implies (R3) for $x \leftrightarrow x(a\,d)$.

Suppose now that $a = c < b < d$. Then, since $x \leqslant y \leqslant z$, $x(a) < x(b) < x(d)$, and $w = z(a\,b)$. We claim that then $w \leftrightarrow z$ is a legal move. Indeed, condition (R2) for $z \leftrightarrow z(a\,b)$ holds if and only if it holds for $x \leftrightarrow y$. Moreover, if $i \in \{b+1, b+3, \dots\}$ then $z(i) \notin [x(a), x(b)]$ because condition (R3) holds for $x \leftrightarrow y$ and because $i \neq d$. While $z(i) \notin [x(b)+1, x(d)]$ if $i < d$ because $y \lhd z$ and if $i > d$ because condition (R3) holds for $y \leftrightarrow z$. Hence condition (R3) holds for $z \leftrightarrow z(a\,b)$.

Suppose now that $a = c < d < b$. Then $x(a) < x(b) < x(d)$ and $w = x(c\,d)$. We claim that in this case $x \leftrightarrow w$ is a legal move. Indeed, condition (R2) for $x \leftrightarrow x(c\,d)$ follows from condition (R2) for $x \leftrightarrow y$. Let $i \in \{d+1, d+3, \dots\}$, so $i \neq b$. Then $x(i) \notin [x(a), x(b)]$ if $i < b$ because $x \lhd y$ and if $i > b$ because condition (R3) holds for $x \leftrightarrow y$. While $x(i) \notin [x(b)+1, x(d)]$ because condition (R3) holds for $y \leftrightarrow z$. Hence condition (R3) holds for $x \leftrightarrow w$.

The proofs of the cases $a < c < b = d$, $c < a < b = d$ and $c < a = d < b$ are all analogous, and are therefore omitted. $\square$

Let $u, v \in \mathfrak{S}_n$ be such that $u \to v$ in the Bruhat graph, and set $\lambda(u, v) := v\,u^{-1} \in T$. If $(x_0, \dots, x_d) \in \mathfrak{S}_n^d$ is a saturated chain, then define

$$\lambda(x_0, \dots, x_d) := (\lambda(x_0, x_1), \dots, \lambda(x_{d-1}, x_d)) \in T^d.$$

Let $\preceq$ be the lexicographic order on T, so $(1\,2) \preceq (1\,3) \preceq \cdots \preceq (1\,n) \preceq (2\,3) \preceq \cdots \preceq (n{-}1\,n)$. We use the same notation for the lexicographic order on T^d for any $d \in \mathbb{N}$. Given two saturated chains of the same length $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{S}_n^d$, we write $\mathcal{C}_1 \preceq \mathcal{C}_2$ and say that $\mathcal{C}_1$ is *lexicographically smaller* than $\mathcal{C}_2$ if $\lambda(\mathcal{C}_1) \preceq \lambda(\mathcal{C}_2)$. It is well known, and easy to see (see, e.g., [2, Chapter 5, Exercise 20]), that $\preceq$ is a reflection order. (We refer the reader to, e.g., [2, §5.2], for the definition of and further information about reflection orderings.) A saturated chain $(x_0, \dots, x_d) \in \mathfrak{S}_n^d$ is *increasing* if $\lambda(x_0, x_1) \preceq \cdots \preceq \lambda(x_{d-1}, x_d)$.

Let $\mathcal{C} := (x_0, \dots, x_d)$ be a saturated chain. Let $i \in [d-1]$, and let $y_i \in \mathfrak{S}_n$ be the unique element such that $x_{i-1} \lhd y_i \lhd x_{i+1}$ and $y_i \neq x_i$. Following [1, §6] we define the *flip* of $\mathcal{C}$ at i to be

$$\mathrm{flip}_i(\mathcal{C}) := (x_0, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_d).$$

Note that $\mathrm{flip}_i(\mathrm{flip}_i(\mathcal{C})) = \mathcal{C}$. The following result is essentially known. However, for lack of an adequate reference, and for completeness, we include its proof here.

Proposition 6.10. *Let $u, v \in \mathfrak{S}_n$, $u \leqslant v$. Then any two maximal chains in $[u, v]$ are related by a sequence of flips.*

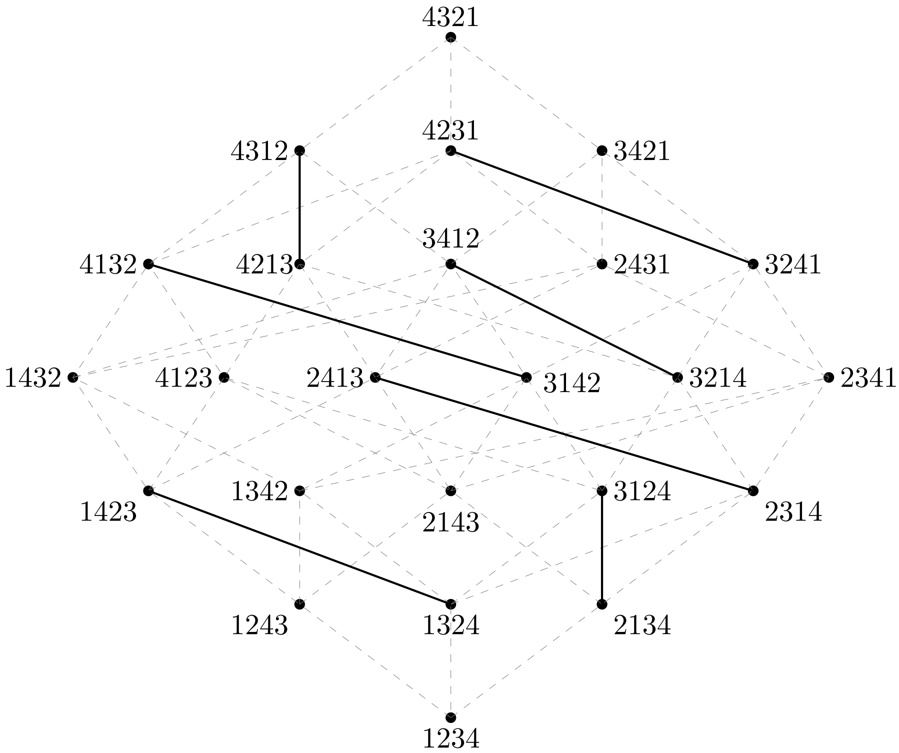


Figure 6.2: The partition of the symmetric group $\mathfrak{S}_4$ into odd diagram classes. Solid edges connect permutations within an odd diagram class. Each class in $\mathfrak{S}_4$ is either a singleton or a rank 1 Bruhat interval.

Proof. It is well known (see, e.g., [9, Proposition 4.3]) that there is a unique increasing maximal chain $\mathcal{Z}$ in $[u, v]$, and that it is lexicographically first among all maximal chains in $[u, v]$. Let $\mathcal{C} = (x_0, \dots, x_d)$ be a maximal chain in $[u, v]$. It is enough to show that $\mathcal{C}$ and $\mathcal{Z}$ are connected by a sequence of flips. We prove this by induction on the number of maximal chains that are lexicographically smaller than $\mathcal{C}$. If $\mathcal{C} \neq \mathcal{Z}$ then there is $i \in [d-1]$ such that $\lambda(x_{i-1}, x_i) \succ \lambda(x_i, x_{i+1})$. Let $(x_0, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_d) := \mathrm{flip}_i(\mathcal{C})$. Then, since in $[x_{i-1}, x_{i+1}]$ there is a unique increasing maximal chain, $\lambda(x_{i-1}, y_i) \prec \lambda(y_i, x_{i+1})$, and since this increasing maximal chain is lexicographically first among all maximal chains in $[x_{i-1}, x_{i+1}]$, $\lambda(x_{i-1}, y_i) \prec \lambda(x_{i-1}, x_i)$. Hence $\mathrm{flip}_i(\mathcal{C})$ is lexicographically smaller than $\mathcal{C}$, and this concludes the proof. □

Proof of Theorem 6.1. By Theorems 6.4 and 6.5, there exist $u, v \in \mathrm{Perm}_n(D)$ such that $\mathrm{Perm}_n(D) \subseteq [u, v]$. By Proposition 6.7 there is a maximal chain $\mathcal{C}$ in $[u, v]$ such that $\mathcal{C} \subseteq \mathrm{Perm}_n(D)$. By Proposition 6.9 the flip of any maximal chain in $[u, v]$ that is contained in $\mathrm{Perm}_n(D)$ is still contained in $\mathrm{Perm}_n(D)$. Hence, by Proposition 6.10, all maximal chains in $[u, v]$ are contained in $\mathrm{Perm}_n(D)$, so $[u, v] \subseteq \mathrm{Perm}_n(D)$. □

Figure 6.2 shows the partition of $\mathfrak{S}_4$ into Bruhat intervals arising as odd diagram classes.

We conclude with a curious consequence of Theorem 6.1 that is, in a sense, dual to it, and with a conjecture. It is clear from the parity condition in Theorem 4.3 that no legal cover in

Bruhat order is a covering relation in right weak order, as these are given by adjacent transpositions. Theorem 6.1 has the following stronger consequence.

Corollary 6.11. *Every odd diagram class* $\mathrm{Perm}_n(D)$ *is an antichain in right weak order.*

Proof. We denote by $\leqslant_R$ the right weak order and by $\lhd_R$ the corresponding covering relation. Suppose $u \sim v$ and $u \leqslant_R v$. Then u cannot be covered by v, so there exists w such that $u \lhd_R w \leqslant_R v$. Since the same chain of relations holds in Bruhat order as well, by Theorem 6.1 this implies $w \sim u$, which is impossible. □

Computations with SageMath [16] suggest that few isomorphism types of Bruhat intervals arise as odd diagram classes. Moreover, based on evidence for $n \leqslant 10$, we formulate the following.

Conjecture 6.12. $\mathrm{Perm}_n(D)$ is rank-symmetric for any odd diagram D.

Bruhat intervals arising as odd diagram classes are not, however, self-dual in general. For example, if $D = \{(1,1),(1,2),(1,3),(1,5),(2,4),(3,1),(3,2),(3,3),(5,2),(5,3),(7,3)\}$ and $n = 9$ then $\mathrm{Perm}_9(D) = [654172839, 958172634]$ and one can check that this interval is not self-dual.

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