Barter Exchange: Modeling, Analysis, and Participant's Strategy

by

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Abstract

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In this study, we considered modern corporate barter platforms. In the first three chapters, we evaluated the probability of a participant on the platform being able to find an exchange partner(s) and his/her waiting time before being able to barter. Four stochastic models that characterize corporate barter platforms operating under different exchange mechanisms were analyzed. These models enable participants to concurrently barter with multiple partners in various exchange volumes. In addition, we prove that the platforms studied in the models are stable in terms of queueing theory, i.e., they do not expand such that there are infinite numbers of waiting participants. It was found that, under one of the exchange mechanisms (called *online with pairwise* exchange mechanism), the participants' waiting time is exponentially distributed. The expectation and variance (or their bounds) of waiting time were analyzed under other mechanisms also. Participants' preference was investigated based on the probability of being able to exchange and the corresponding waiting time under each exchange mechanism. Based on the participants' preference, suggestions for improving a platform's performance and profitability are provided accordingly.

Continued from the above results, in the next two chapters, participant's strategies under online with pairwise exchange mechanism are investigated. For a participant who implements EOQ model to purchase production materials, and who is willing to try barter to get rid of excessive inventories and barter-in production materials, a concise sufficient condition is developed to identify if it is beneficial to participate in a barter exchange. With the satisfaction of sufficient condition, if the participant cannot be matched with exchange partner(s) upon arrival, we suggest he always wait on the barter platform for future opportunities. While waiting, if his regular procurement and production plan does not permit back-order in production materials, a good strategy for the procurement of production materials during the waiting time is maintaining the usual practice with the regular supplier, and do not change the procurement activity until he is able to barter-in production materials. Additionally, if back-order in production materials is allowed during the waiting time on the barter platform, then a better temporary procurement plan with back-order may be available compared with no back-order allowed. We found a sufficient condition that justifies if back-order will be

beneficial, and the sufficient condition becomes necessary as well if the participant's cost function (including ordering cost, inventory holding cost, and back-order penalty cost) is convex. Last but not least, though simultaneous trade-in and trade-out is a common practice in barter, we recommend that participants separate trade-in and trade-out processes to reduce the total waiting time. The reduction in waiting time may outweigh negative effects from the separation of trade-in and trade-out activities.

To my family.

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Chapter 1

Introduction

1.1 Barter Exchange

Barter is a form of inventory exchange that does not involve money in the process. Common examples of corporate barter include exchanges between hotels and the media, where unoccupied hotel rooms are traded for free advertising space. In certain cases, employee discount programs result, in a broad sense, from corporate barter exchanges, where payments are partially disbursed by the service or goods provided. It is estimated that 65% of corporations listed on the NYSE use barter, and 20-25% of world trade comprises barter ([15]).

A "double coincidence of wants" is required to successfully conduct a barter exchange. Each participant has to possess inventories that exactly fulfill at least one other participant's demand, and all the participants' exchangeable demands have to be fulfilled simultaneously. The value of the inventories traded out should be of the same value as the commodities or services traded in, according to the participants' evaluation. In short, a barter exchange requires that the demand and supply of each participant be executed at the same time and be of the same value, as measured by the corresponding individuals. Traditionally, corporate barter is preferred during a monetary crisis or when foreign currency exchange problems persist. It was also applied when trust was absent between the trading parties. For example, when one person doubts the capital flow capability of the other party, he/she might require payment in goods, instead of cash. Because of its inefficiency and the lack of a common measure of value, barter used to be the least preferred option in the above undesirable situations. However, it has benefited from the development of the Internet and supply chains, and is now becoming an active option for many companies. Participants in modern corporate barter exchanges enjoy low inventory levels, increased sales, reduced pressure on cash flow, and minimized cost of acquiring debt ([22]). Among all these advantages, participants primarily benefit from moving surplus and obsolete inventories. Corporate barter participants, especially domestic participants, aim at reducing excess inventories by countertrade, rather than by using other alternatives such as selling the products at a low salvage value ([22]). It is claimed by IRTA (International Reciprocal Trade Association), one of the corporate barter platforms in North America, that its clients recovered $12 billion from previously lost and wasted revenues in 2008, and that a 15% annual increase was anticipated ([15] and [5]).

It is interesting that people are usually surprised on hearing for the first time that corporate barter is an extraordinarily large business involving tens of billions of dollars every year, and that the potential for growth is impressive as compared with other businesses at the same volume. Accompanied by continuing development of the Internet and global supply chains, as well as an increasing acknowledgment of the importance of corporate barter, it is anticipated that corporate barter will play an even more significant role in the future, and will constitute a noteworthy proportion of world trade. Aligned with this anticipation, studies focusing on corporate barter from the managerial perspective have started to emerge. Because choosing an appropriate exchange mechanism is one of the most important decisions that should be made when building a corporate barter platform, emerging managerial research of corporate barter platforms tends to concentrate on the choice of exchange mechanisms. A valid exchange mechanism defines when and how a barter exchange is executed. Widely used mechanisms embedded in modern corporate barter platforms include a combination of (a) an online or batch policy, which outlines when an exchange will be conducted, and (b) a pairwise or multi-way change type, which determines the manner in which the exchange is conducted. As a result, a total of four popular exchange mechanisms are available from which platform operators can choose, namely, the online with pairwise exchange, online with multi-way exchange, batch with pairwise exchange, and batch with multi-way exchange mechanisms. The definitions of the above policies (when to exchange) and exchange types (how to exchange) are as follows.

Online *vs* Batch The batch policy does not allow barter exchanges until a certain number of participants are waiting in the system, whereas the online policy allows every possible exchange to be executed whenever a new participant arrives.

Pairwise exchange *vs* Multi-way exchange Pairwise exchange allows only two participants in an exchange, i.e., a bilateral barter between A and B. Multi-way exchange permits a directed cycle of at least three participants in the barter.

As platform operators are free to choose any exchange mechanism, a natural question asked by not only barter platform practitioners but also participating companies and researchers concerns the efficiency of the current exchange mechanism adopted by the platform operator. Because participating companies are usually charged a commission for each transaction, an efficient exchange mechanism results in a win-win situation. The platforms enjoy increased revenue, and companies as potential customers feel more encouraged to participate. Key performance indicators of an efficient corporate barter exchange mechanism include the probability that a participant is able to exchange, the waiting time preceding a successful exchange, etc.

The extraordinary value and volume involved in modern corporate barter exchanges and the dynamics within a corporate barter platform motivated this study. Based on the fact that the participants' main aim is to reduce outdated and surplus products, in this study four stochastic models were developed to describe the countertrade taking place on barter platforms operating under different exchange mechanisms, i.e., one model for each of the popular exchange mechanisms described in the previous paragraph. The system and participants' behaviors were investigated using the models. We were particularly interested in the probability of a participant being able to trade and the time that he/she has to wait until the exchange is conducted. In this study, we investigated the above two performance measurements (waiting time and probability of exchange) in the case of multi-unit supplies and demands. To the best of our knowledge, this is the first study to report a multi-unit exchanges on a corporate barter platform. As the following section shows, a study based on single-unit exchanges exists ([2]). The transition from single-unit to multi-unit goods is not only a formal procedure; it is also a conceptual modification that relaxes several assumptions indirectly applied under single-unit exchange. An indivisible single-unit supply and demand exchange enforces the assumption that each participant exchanges with at most one other participant. However, it is not in general applied in corporate barter. For instance, a hotel can trade with several news media businesses simultaneously for free advertising spaces from different sources. The first contribution of our study is that a model for multi-unit goods is set, which allows exchanges with several participants simultaneously at various transaction amounts. The analytical results show that, if participants are not granted the option to withdraw, the online with pairwise exchange mechanism guarantees exchange opportunities for all participants, and the waiting time before they can conduct the exchange is exponentially distributed. Under other exchange mechanisms, because of the complexity, closed form distribution of waiting times is not available, but the expectation and variance (or the bounds of these statistics) of the waiting time are derived. Although with different model settings, our waiting time statistics conform with the bounds obtained in [2]. The results of a numerical study showed that under the online policy, a three-way exchange performs better than a pairwise exchange in terms of reduced waiting time and increased probability of all participants on the platform obtaining a barter opportunity. Under the batch policy, a three-way exchange is preferred when the probability of being able to barter is not too small; otherwise, a pairwise exchange surpasses a three-way exchange. Of the two policies, when the chance of being able to trade is not negligible, the online policy is always preferred by participants because of the short waiting time. In a case where the probability of obtaining a barter partner is extremely low, the online and batch policy are both reasonable options. The former guarantees that barter opportunities appear within a long but predictable time, whereas the latter leads to a result where the value recovered is not worse than the salvage value of the excessive inventory within a very short waiting time.

Based on the analytical result of online with pairwise exchange, we further investigated participant's strategies under this exchange mechanism. It is assumed that a participant implements EOQ model for the procurement of production materials, and that he would like to trade-out excessive inventories in exchange for production materials. Analytically, we developed sufficient condition that participating in barter is beneficial, and procurement strategies while waiting for an exchange opportunity are derived accordingly. Additionally, it is recommended that the participants separate trade-in and trade-out activities if possible.

	Corporate Barter	**Kidney Exchange**
Number of items in an exchange	Mostly multi-units	Single-unit
Number of exchange partner(s) in an exchange	Simultaneous exchange with multiple participants possible	Exchange with one other participant
Chain exchange	Chain exchange may not be welcomed or practical	Chain exchanges are welcomed and are beneficial
Motivation	To eliminate excessive inventories	To find a kidney
Participant unit	Single company	Patient-donor pair

Table 1.1: Differences between corporate barter and kidney exchange

The rest of the dissertation is organized as follows. In the remaining sections in this chapter, the related literature is reviewed; basic assumptions and model settings are stated as well. Chapter 2 and Chapter 3 present studies of online policy and batch policy, respectively. Chapter 4 delivers participant' strategies under online with pairwise exchange. Chapter 5 particularly investigates the cost function derived in Chapter 4.3. Finally, Chapter 6 concludes the study.

1.2 Literature review

Few research studies have focused on corporate barter from the managerial perspective, even though barter (not limited to corporate barter) is a very large topic in economics (see [8] for a comprehensive review not limited to economics). From the economics point of view, [33] investigated barter and monetary exchanges. The article demonstrates the efficiency of money by revealing that any barter exchange can be represented by a monetary exchange, but not vice versa, and thus, monetary exchange improves efficiency. Recent studies on barter covered exchanges involving houses/single-unit demand ([24]), timeshares ([35]), and tuition fees ([11]). It is noteworthy that the above papers all focus on the efficiency of resource allocation to a significant extent, a natural topic and approach considered by economics researchers. In contrast, the economics literature rarely mentions the efficiency of the barter platform operation, i.e., the profitability of its operation and the efficiency from the point of view of the participants (such as the resulting waiting times). In our study, alongside investigating the barter platform operation from the managerial perspective, we addressed its efficiency as well.

In the management science literature, the tendency was to consider that corporate barter was a subset of, or could be substituted by, kidney exchange. However, critical differences lie firmly between corporate barter and kidney exchange (Table 1.1 compares the two exchanges and is discussed in detail below). Regardless of all the fundamental differences, the simple fact that corporate barter contributes to a quarter of the world trade is more than sufficient to allow it to qualify as a serious and independent research topic. Rather than being a derivative of and eclipsed by kidney exchange, it deserves a well-defined model and an in-depth study. Despite the amount of studies devoted to kidney exchange, references to corporate barter are limited in management science. A corporate barter model was derived from kidney exchange models in the study of [2], where the dynamic behavior within a barter system was studied. In the model, it is assumed that the participant arrives at the beginning of each time unit in possession of an indivisible single item that he/she would like to trade out. The matching probability between any two participants is a constant. The model was built in a stochastic environment, because the participants arrive and depart over time. It is also characterized by its homogeneous property in that the matching probability is invariant over time and system status. The research focused on the participants' waiting time. The bound of the waiting time was derived under different exchange mechanisms and each mechanism was evaluated based on the resulting waiting time. It was concluded that, among all monotonic policies, the online policy is near optimal and that waiting time can be significantly reduced by allowing three cycles in exchanges. In addition, it was shown that waiting time could be further decreased by permitting chain exchanges.

In a comparison of corporate barter and kidney exchange, the similarities appear in the operations of the exchanges. Popular policies (online/batch) and the type of exchange (pairwise/multi-way) in kidney exchanges conform with those in a corporate barter. However, although similarities exist, it is important to recognize that corporate barter deviates significantly from kidney exchange. A donor can donate only one kidney, and thus, he/she can be matched to at most one compatible patient. However, corporate barter participants can and usually do possess different types of inventories in different volumes. This enables a participant to conduct exchanges with multiple participants simultaneously, involving various types of inventory and distinct amounts in the concurrent transactions. Therefore, the assumption of a single item in the study of [2] may not perfectly describe the exchange. Additionally, their study of chain exchange may not ideally characterize participants' behavior. Chain exchange in their study borrows the model settings of kidney chain exchanges initiated by altruistic donors (see [31], [10] and [23] for studies on chain exchanges). The introduction of an altruistic donor relaxes the constraints of the original donor-patient pairs, thus significantly easing the matching process, which is welcomed by both patients and doctors. In contrast, chain exchanges may not be welcomed on corporate barter platforms. Every chain exchange results in a participant who trades in from others but does not trade out (it is the final element of the chain). However, a one-way trade-in does not solve the problems associated with excessive inventories. As corporate barter participants aim primarily at moving surplus and obsolete inventories, there are not many incentives for them to trade without reducing inventory levels, not to mention the possibility of inventory levels being increased by trade-in activities. Although theoretically chain exchanges reduce the waiting time, managerially they may not be applicable, because no one would volunteer to be the final element of the chain.

The study of cycle exchanges in [2] was technically facilitated by applying continuous-time Markov chain and random graph theory. A random graph possesses properties that are similar to those of barter under batch policy. The graph randomly decides the connection between any two nodes, and thus, is able to simulate matching processes between participants in a barter system. The basic results from random graph theory of [7], [13], [16], and [21] imply that both the expected number of cycles in a (directed) random graph and the expected number of isolated vertices in the graph can be computed by identical indicators. Moreover, [12] showed a closed form distribution for the number of isolated vertices in an undirected random graph. Although many fascinating properties can be derived from random graph theory, our study is fundamentally based on the above identical indicator property and the closed form distribution derived by [12]. (See [14] for a brief but complete review; [17], [6], and [16] for cycle components in random graphs; and [20], [9], and [13] for the evolution of random (di)graphs.)

As there is a tendency to associate corporate barter with kidney exchange, we reviewed the literature related to kidney exchange. However, we reiterate that there are essential differences between the two exchanges. Studies of kidney exchanges focus on the number of matchings between incompatible patient-donor pairs. By swapping patient-donor pairs, a patient A with an incompatible donor can obtain a kidney from another donor B if patient A's donor is compatible with donor B's patient. Algorithms and policies for swapping patient-donor pairs have been studied and the efficiency of each policy has been discussed (see [27], [28], [30], [1], and [34] for studies on specific policies/algorithms). The frequently applied policies include the online policy and batch policy and the prevalent exchange types are pairwise and multi-way exchange. The policies and exchanges types are defined in the same manner as in corporate barter. A chain exchange is also possible when an altruistic donor presents in a kidney exchange system. Major studies on kidney exchange concluded that the number of successful matches can be increased by allowing chains and three-way cycles (for computational and simulation results, see [32]; for analytical results obtained in a static environment, see [29]; and for analytical results in a dynamic environment, see [3]). [29] also showed that matches with a size larger than four are not necessary in a static environment of kidney exchange, as they are not able to significantly improve the performance. [3] showed that, in a dynamic setting, as compared with the results from the online policy, the batch policy provides marginal benefits only. Studies focusing on topics other than policies and efficiency include that of [4], in which a mechanism was designed to motivate hospitals.

Our study was inspired by that of [2], and our model improves on their model for corporate barter systems. The original models of [2] are extended and further results are derived. Assumptions are relaxed in order to better characterize corporate barter systems. The participants' arrival process is now assumed to be random, and the number of exchanges in which a participant can simultaneously participate is also randomized. The expectation and variance of the waiting time, instead of the bound of waiting time, are derived in two out of the four models presented in this study. In particular, under the online policy with pairwise exchange, participants' waiting time is proven to be exponentially distributed. In addition to the differences in the assumptions and analytical results, our study differs from that of [2] in the approach taken to compare and evaluate exchange mechanisms. [2] invested significant effort in the derivation of bounds and ranked exchange mechanisms according to the bounds. In our study, on the contrary, we obtained the waiting time via concise approaches, and focused to a great extent on a numerical study to evaluate different exchange mechanisms comprehensively that involved performance measurements not limited to waiting time and the probability of being able to exchange.

1.3 Assumptions and model settings

In this study, we built stochastic models for corporate barter under different exchange mechanisms, and analyzed the system behavior in each model, then studied participant's strategies under one of the exchange mechanisms. In modeling and analysis of barter exchange, We were particularly interested in the probability that a participant can be matched and in his/her waiting time until obtaining a match under various policies and exchange types. The most frequently used exchange mechanisms include online with pairwise, online with multi-way, batch with pairwise, and batch with multi-way exchanges. Models for each of the above four exchange mechanisms are analyzed in the following two chapters. The study on multi-way exchange reported in Chapter 2.3 and Chapter 3.3 focused on three-way exchange. In all four models, we assume that participants bear heavy excess inventories. This assumption is justified by the fact that corporate barter participants aim primarily at reducing surplus and obsolete products, and that billions of previously wasted inventories are traded every year. We also assume that participants consider the elimination of inventories an urgent matter. Based on the above two fundamental assumptions regarding the participants' background, additional assumptions for modeling and analysis are made in four aspects in order to simplify the models, as follows.

Qualitative, not quantitative We focus on whether the trade can be conducted and ignore the actual transaction volume in each trade. Consequently, every barter exchange can be represented by a binary variable indicating whether or not trading opportunities between participants exist.

Trade We assume that exchanges are conducted whenever there are (i) bilateral matchings between a pair of participants, or (ii) directed matching cycles among three participants. This simplification enforces the assumption that a barter exchange is executed whenever there is an opportunity. As we assumed, participants feel it is urgent to move inventories, and hence, it is reasonable that a participant would choose to conduct an exchange whenever an opportunity arises.

Simultaneity We require that all participants execute a trade-in and out simultaneously. This assumption is reasonable in several aspects. As argued in the literature review, a one-way trade-in does not solve the problems generated by excessive inventories, and thus, provides limited motivation to participants. On the other hand, a one-way trade-out results in a (temporary) loss, as the participant does not recover any value from the excess inventory at the time that the transaction is completed. This outcome is in general worse than selling the product at salvage value. The fact that the participant chooses to barter rather than dispose of the inventories at salvage value implies that the participant expects a better return than the salvage value. Thus, the company has a strong incentive to avoid a one-way trade-out, as it results in a worse return. Collectively, we argue that a simultaneous trade-in and out is preferable to any one-way trade.

Departure from the system When a participant has found a barter opportunity or simultaneous multiple opportunities, he/she leaves the system. Additionally, under the batch policy, participants may be forced to leave the system without any exchange opportunities being found (see Chapter 3 for details). We do not assume that the participant has the option to leave/withdraw from the system when he/she does not find exchange opportunities and no longer wishes to participate. Therefore, participants experience a passive departure process.

In addition to the above four assumptions, we further define a *pool* as the sub-system in which existing participants are waiting for exchange opportunities.

Please note that the above assumptions are made in order to facilitate modeling and analysis of corporate barter exchange mechanisms. In Chapter 4 and Chapter 5 when participant's strategies are investigated, some of the assumptions may be relaxed or modified in order to achieve a better understanding of an optimal strategy.

As mentioned, in this study we attempted to create a model that allows multi-unit exchanges in the sense that each participant can potentially trade with multiple participants simultaneously. There are weaknesses in this qualitative model; however, we decided to retain the current model settings, as they are reasonably clear and convincing in terms of demonstrating managerial insights. Based on all the above assumptions, barter exchange opportunities can be represented by Bernoulli random variables. In the models, a homogeneous probability is defined to describe the chance that a participant can be matched with another participant in the system. The probability is defined as the chance of a one-way match, i.e., it describes that one participant's supply fulfills the demand of another. According to the simultaneity assumption, a participant trades only if both his/her supply and demand are matched by other participants. It should be noted that this results in the exclusion of a chain exchange. In fact, the combination of simultaneity and multi-unit exchange precludes chain exchanges on corporate barter platforms. For instance, when participants A and B have a pairwise exchange opportunity, and A's demand is compatible with altruistic donor C's supply, exchanges between A, B, and C can be conducted, provided that the demand of A is sufficiently large to allow him/her to take both B and C's supplies. In this case, an altruistic donor exists, but does not initiate a chain exchange. Indeed, an altruistic donor on the corporate barter platform performs as a secondary participant under simultaneity and multi-unit exchange assumptions. He/she will participate in an exchange if and only if a participant who has pairwise or multi-way exchange opportunities is willing to take the altruistic donor's supply. As one can see in this case, the altruistic donor does not fundamentally transform the exchange process, and thus, should not influence the participant's waiting time. In fact, the altruistic donor fulfills an important role in kidney exchange because of the single-unit restriction. A participant who takes an altruistic donor's kidney forfeits the right to participate in any bilateral exchanges with other participants. As all participants can trade with multiple participants simultaneously in our settings, an exchange with an altruistic donor does not preclude possibilities of trading with others. Therefore, the benefit gained from an altruistic donor or a chain exchange is marginal when a multi-unit exchange is permitted.

As mentioned, in modeling and analysis of exchange mechanisms, we focus on a participant's waiting time before obtaining a barter opportunity if there is any, and the probability that the participant can eventually find opportunities. These two measurements are able to define the operation efficiency of a corporate barter platform, and hence, determine the profitability of the barter group and the attractiveness of the platform to prospective customers. In Chapter 2 and Chapter 3, participants' waiting time and the probability of being able to exchange under different exchange mechanisms are explored. The specific assumptions applied to each mechanism are explained and justified in the corresponding chapters, where the underlying model is analyzed in detail.

Chapter 2

Online Exchange Policy

2.1 Online policy

As defined in Chapter 1.3, the online policy enforces the assumption that exchanges are triggered exclusively by individual arrivals. Assume that the participants arrive according to a Poisson process, with the corresponding inter-arrival time exponentially distributed with rate ν. We further assume that the probability of any one-way match is *iid* with probability p. This probability is invariant across system status and time. With the above two parameters, participant waiting time (or bound of waiting time) under different exchange types can be derived with a few more specific assumptions that apply to each exchange type. Meanwhile, the probability of being able to exchange under the online policy can be easily obtained based on the assumptions stated in Chapter 1.3. As participants are not granted the option of leaving the system before finding any opportunity, departures from an incapacitated system running under an online policy are associated with at least one exchange opportunity. In other words, all participants in the system eventually find a barter opportunity and leave the system, and thus, the probability of being able to exchange is always 1 under the online policy. However, here arises the issue of system stability if no capacity constraint is applied to the system. Clearly, an unstable system, i.e., one that almost surely expands to infinite customers waiting in the system, is not practically applicable in the real world, and therefore, it has to be guaranteed that the exchange mechanisms employed will result in a stable system. Therefore, in the study of the online policy, we primarily focus on the waiting time and system stability.

2.2 Pairwise exchange type

Consider an incapacitated corporate barter system operating under the online policy with pairwise exchange. By definition, the system considers any two participants with a bilateral matching opportunity can conduct a valid barter exchange. Consequently, a new arrival in the system can barter with a participant waiting in the pool with probability p^2. We do not assume any restriction on the number of exchanges in which a new arrival can potentially participate, and hence, the number of barter opportunities that he/she can find is a binomial random variable with parameters (i, p^2), where i is the number of participants waiting in the pool when the new arrival enters the barter platform. Multiple exchange opportunities are triggered when the new arrival is bilaterally matched by more than one in-pool participant. When this occurs, it does not necessarily imply that the supplies and/or demands of these participants are identical. Certainly, if the arrival has only one type of inventory and one type of demand, the matched participants should have an identical set of supply and demand. However, when the arriving participant bears several types of inventories and would like to trade in multiple types of commodities, then matched participants may possess various supplies and demands, provided that each of them matches a subset of the new arrival's supply and demand pairs. We do not (conceptually) combine participants with the same supply and demand types, because they arrive in the system at different times. Combining them would introduce bias into the participants' waiting time.

When no barter opportunity between the newly arriving participant and the existing participants exists, the new arrival enters the pool and waits for future opportunities. On the other hand, if this participant can exchange with at least one existing participant, he/she does not enter the pool, but directly leaves the system. Moreover, all current participants who have the ability to barter with the new arrival also leave the pool. This policy guarantees that there is no barter opportunity between any in-pool participants; otherwise; the exchange would have been executed when the corresponding participants arrived.

Under this policy and exchange type, the number of participants waiting in the pool, i.e., the state, is a continuous time Markov chain. State changes are exclusively triggered by every new arrival, and hence, the amount of time that the process spends in the same state before making a transition into a different state is exponentially distributed. When the process leaves state i, it next enters state $i+1$ with probability $(1-p^2)^i$, which is the probability of no barter opportunities. It enters state j, $0 \leq j \leq i-1$, with probability $B(i-j; i; p^2)$, which is the probability of $i-j$ exchange opportunities under binomial distribution with parameters (i, p^2).

Although the corporate barter system can be represented by a continuous time Markov chain, it is difficult to obtain a closed form of limiting probabilities by balancing equations. Nevertheless, the system possesses unique properties that allow the calculation of the waiting time via conditional probabilities. We found that the exact waiting time for those who enter the pool is exponentially distributed.

Theorem 1. *In-pool participants' waiting time is exponentially distributed with parameter* vp^2.

Proof. On the condition that new arrival m enters the pool, his waiting time is calculated by counting the number of new arrivals during his waiting time. Let T denote participant m's arrival time, and w denote his waiting time. Define Y_t as the number of arrivals during time $[T, T+t]$. Since the inter-arrival time is exponentially distributed, Y_t follows Poisson distribution.

$$\begin{aligned}
&P(w > t, Y_t = y | w > 0) \\
&= P(Y_t = y. \text{ No matches between participant } m \text{ and the } y \text{ new arrivals} \,| w > 0) \\
&= (1-p^2)^y \times e^{-\nu t} \frac{(\nu t)^y}{y!}.
\end{aligned} \tag{2.1}$$

Summing over all possible values of Y_t, we get

$$\begin{aligned}
&P(w > t | w > 0) \\
&= \sum_{Y_t=0}^{\infty} (1-p^2)^{Y_t} \times e^{-\nu t} \frac{(\nu t)^{Y_t}}{Y_t!} \\
&= \sum_{Y_t=0}^{\infty} (1-p^2)^{Y_t} e^{-\nu t(1-p^2)} e^{-\nu t p^2} \frac{(\nu t)^{Y_t}}{Y_t!} \\
&= e^{-\nu t p^2} \sum_{Y_t=0}^{\infty} e^{-\nu t(1-p^2)} \frac{[\nu t(1-p^2)]^{Y_t}}{Y_t!} \\
&= e^{-\nu t p^2}.
\end{aligned} \tag{2.2}$$

The above tail probability is identical to that of an exponential distribution with mean $1/\nu p^2$. □

The above theorem can be justified without calculation. The number of new arrivals for whom an in-pool participant has to wait until being able to exchange is geometrically distributed, and the inter-arrival time is of exponential distribution. Therefore, the total waiting time is a summation of exponential inter-arrival times up to a geometrically distributed random variable. As both exponential and geometric distributions exhibit the memoryless property, the summation should also be memoryless.

According to the participants' waiting time, we can further prove that the system does not expand to infinity, but remains stable.

Theorem 2. *An online with pairwise exchange system is always stable.*

Proof. The system can be characterized by participants' behavior as follows

- Participant arrives at rate ν, and has a *positive probability* to enter the pool;
- Every in-pool participant will stay an exponential time in the system with mean $1/\nu p^2$.

Now we create a new system which has the following properties:

- Participant arrives at rate ν, and will *absolutely* enter the pool;
- Every in-pool participant will stay an exponential time in the system with mean $1/\nu p^2$.

Obviously, the requirements for the latter system to be stable are sufficient to the online with pairwise system being stable. However, the latter system is indeed an $M/M/\infty$ queue, which is always stable as long as $p > 0$. Therefore, online with pairwise system is stable as well when the matching probability is nontrivial. $\square$

2.3 Three-way exchange type

Under the online policy with a three-way exchange system, barter opportunities are triggered if and only if the new arrival and in-pool participants can generate a directed exchange cycle of length three. The approach to exploring the online policy with a three-way exchange system comprises a combination of both analytics and a simulation. In this section, we describe a relaxed system that was built in order to obtain an upper bound of waiting time and to prove the stability of the original online policy with a three-way exchange system. Then, in Chapter 2.4, based on our numerical and analytical results, we claim that the expectation of participants' waiting time should not exceed $\frac{1}{\nu p^{1.5}}$, where p denotes the unilateral matching probability between any two participants and is assumed to be small. Collectively, we find that the online policy with a three-way exchange system is always stable; the expectation of participants' waiting time is well bounded by $\frac{1}{\nu p^{1.5}}$ when p is small and is bounded by the waiting time of the relaxed system when p is large. In the remaining content of this section, we present an analysis of the system and its relaxed model.

As in the model described in Chapter 2.2, the new arrival in an online policy with a three-way exchange system does not enter the pool if he/she is able to find at least one three-way exchange opportunity. Instead, this new participant immediately leaves the system with all the matched participants; otherwise, he/she enters the pool and waits for future opportunities. We assume that when a participant enters the pool, all the binary relationships, i.e., matched or not between this participant and existing participants, are preserved.

To better illustrate the last assumption, i.e., the preservation of relationships, we present a simple example. Consider a situation where two isolated in-pool participants, A and B, exist. When new participant C arrives, he/she finds one and only one unilateral exchange opportunity, i.e., A's supply matches his/her demand. As C cannot form a directed three-way exchange, he/she enters the pool. The binary relationships between C and the other participants remain the same afterward. Therefore, within the pool of three participants, a directed match from A to C can be observed and is the only match in the pool. The above policy and assumptions guarantee that no directed exchange cycles exist in the pool, but only segments of directed cycles.

The compatibility status of the pool (binary matching relationships between every pair of in-pool participants) can be represented by a directed graph. Vertices represent participants and directed edges correspond to directed matches. The evolution of the compatibility status (and associated directed graph) is illustrated in Figure 2.1.

Clearly, the evolution of a directed graph is a continuous time Markov chain. The state is represented by a digraph, and the transition between states is memoryless. However, to

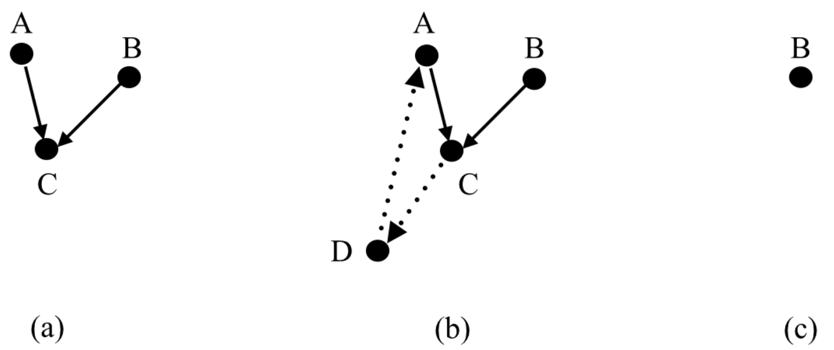

Figure 2.1: Evolution of compatibility status.

(a) Originally there are three in-pool participants. (b) New participant D arrives and forms a three-cycle with A and C; (c) A and C leave the system, while B remains. It should be noted that the edge between B and C disappears because C is no longer in the system.

derive the transition probability, it is necessary to know the exact shape of every possible digraph. This prerequisite for formulating the balance equation, in addition to the complex correlations between edges in the digraph, precludes every attempt to obtain the exact waiting time via an analytical approach. We present in Chapter 2.4 simulation results for customer waiting time based on the model settings of the original online policy with a three-way exchange system. An upper bound of waiting time is derived from a relaxed system in the remaining content of this section. Our numerical study showed that the upper bound is well performed when the matching probability is $p > 0.1$. Moreover, the relaxed system reveals an approach to proving the original system is stable.

In order to derive an upper bound, the preservation of relationships assumption is modified such that a newly entering participant has at most one edge. This edge, if it exists, is chosen (uniformly) randomly from directed edges between the new arrival and the isolated in-pool participants. To illustrate this relaxation, all the possible scenarios that a new arrival can encounter are enumerated as follows.

(a) Directed cycle(s) is(are) formed

(b) No directed cycles are formed, but at least one directed edge exists between the participant and isolated in-pool participants

(c) No directed cycles are formed, and no edges exist between the participant and isolated in-pool participants.

Under case (a), this newly arrived participant does not enter the pool. In the remaining two cases, this participant enters the pool with a different edge status. In the second case, this participant enters the pool with exactly one directed edge. This edge is uniformly

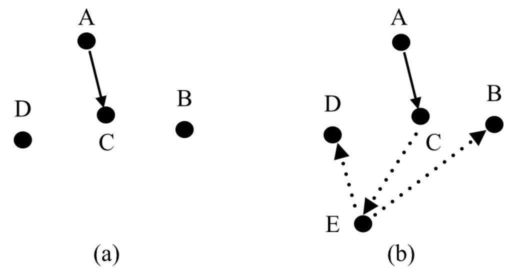

Figure 2.2: Edge status when entering the pool.

(a) There are four vertices originally; (b) New customer E arrives. As no directed cycle is formed, E enters the pool. Because E generates two edges with isolated vertices, namely, ED and EB, E enters the pool with exactly one of the edges, ED or EB, but not both. It should be noted that in this scenario, edge CE is ignored because C is not an isolated vertex.

chosen from the directed edges generated between this participant and isolated participants (see Figure 2.2 for details). In the third case, this participant enters the pool without any edges attached to him/her. Under these assumptions, no more than one edge is allowed among any three vertices in the digraph. This guarantees that each vertex is attached to at most one edge, and hence, the directed graph consists of entirely isolated vertices and pairs of vertices connected by a single edge. Consequently, no correlation between edges and vertices exists in the system, and every component in the digraph can be treated independently. This may be justified by an *exclusive partnership* formed between the new arrival and an isolated participant waiting in the pool. They leave the system together when a third party matches them both.

The simulation results show that the waiting time in the original system is well bounded by that in the relaxed system. It may be difficult to prove that the relaxed system indeed performs as a bound of the original system. Rather than the proof, we provide several arguments to justify the claim. First and most importantly, participants face competitions and restrictions when obtaining unilateral exchange opportunities in this relaxed system, while in the original system they are free to exploit any one-way exchange opportunities, and therefore, waiting time in the original system is reduced. It is possible that a connected vertex will become isolated in the original system and this potentially lengthens waiting time, but this situation occurs far less frequently than one would expect. A vertex m that has been connected becomes isolated if and only if it is the final one to leave the system as compared with all vertices that remain connected to m before they leave the system. For example, in Figure 2.1, C is the only vertex that connects to B before finding itself a three-way exchange opportunity, and B becomes isolated because C leaves earlier than B. Consequently, every disconnected subgraph that ever appeared in the compatibility status of the original system contributes at most one vertex that may become isolated. Clearly, the corresponding probability of a connected vertex returning to an isolated state is rather limited. Additionally, even if a vertex becomes isolated, it is far more efficient in obtaining an edge under the original online with three-way system than in the relaxed system. Collectively, we argue that the waiting time in the relaxed system provides an upper bound of that in the original system.

In the relaxed system, the new compatibility status is again a continuous time Markov chain. Define system state of the online with 3-way relaxed system by $(|E|, |IV|)$, where E is the set of edges, and IV is the set of isolated vertices in the digraph. An interior state $(|E|, |IV|)$ can be entered from $(|E|-1, |IV|+1)$, $(|E|, |IV|-1)$, and $(|E|+k, |IV|)$ where $k > 0$. Let $P_{(i,j)}$ denote the steady state distribution. Define $\mathbb{S}_1$ as the set of system states. $\mathbb{S}_1 = \{(i,j) \in \mathbb{I}^2 \,|\, i \geq 0, j \geq 0\}$. Assume that $P_{(i,j)} = 0$ whenever $(i,j) \notin \mathbb{S}_1$. The balance equation can be formulated as

$$
\begin{aligned}
P_{(i,j)} =& (1-p^2)^{i-1}[1-(1-p)^{2(j+1)}]P_{(i-1,j+1)} + (1-p^2)^i(1-p)^{2(j-1)}P_{(i,j-1)} \\
&+ \sum_{k\geq 1} \binom{i+k}{k} p^{2k}(1-p^2)^i P_{(i+k,j)},
\end{aligned} \tag{2.3}
$$

$$
\sum P_{(i,j)} = 1. \tag{2.4}
$$

Equation (2.3) can be easily justified. If the new arrival observes state $(i-1, j+1)$, with probability $(1-p^2)^{i-1}[1-(1-p)^{2(j+1)}]$ he cannot find 3-way exchange opportunities but can generate at least one directed edge with isolated vertices, therefore he forms exclusive partnership with one of these vertices; when the new arrival observes state $(i, j-1)$, with probability $(1-p^2)^i(1-p)^{2(j-1)}$ he will not form any 3-way exchange with paired participants nor generate any edge with isolated participants, hence he enters the system as an isolated vertex; in case a new arrival observes state $(i+k, j)$, there is a probability $\binom{i+k}{k} p^{2k}(1-p^2)^i$ such that the new arrival can find 3-way exchange opportunities with k pairs of in-pool participants, thus the state transits into (i, j).

As in Chapter 2.2, we first compute the expected waiting time before proceeding to prove system stability, because the waiting time again provides insights on the stability of the system. Though Little's Law can be used to compute the expected waiting time, it fails to provide a clear structure of the waiting process. We now present an alternative for waiting time calculation by a more structured approach. As will be shown at the end of this section, this approach is able to deliver a straightforward decomposition of the expected waiting time.

Before the computation, though, it is necessary to justify that arrivals of the system see time average. As is well known, PASTA applies to Poisson arrivals, but arrivals in this system significantly deviate from traditional continuous time Markov chain. Newly arrived participants in this system are event triggers that cause either birth or death process, while the classical Poisson arrival corresponds to birth process only. Nevertheless, following the identical procedure of the original proof of PASTA, it can be shown that arrivals to the barter platform indeed see time average.

Now consider an incapacitated pool. If a participant m enters the pool with an edge attached to him, or equivalently, he has chosen an available participant as an exclusive partner, then for each future arrival, there is a probability p^2 that the future arrival can form a 3-cycle with m and his partner. This is exactly the same as in Chapter 2.2, hence the distribution of participant m's waiting time is exponentially distributed with mean $1/\nu p^2$. Let w denote his waiting time, and I_p denote the event that he has a partner upon entering the pool. Define $P_{(i,j)}$ as the limiting probability that the relaxed system is in state (i, j). The conditional expected waiting time is

$$E[w|I_p = 1] = \frac{1}{\nu p^2} \tag{2.5}$$

with probability

$$P(I_p = 1) = \sum_{(i,j)\in\mathbb{S}_1} P_{(i,j)}(1-p^2)^i[1-(1-p)^{2j}], \tag{2.6}$$

which is the probability of a new arrival entering the pool with an edge.

On the other hand, if a new participant enters the pool as an isolated vertex, then he firstly waits for a new arrival to generate an edge (forming an exclusive partnership), then stays in the pool for another new arrival to match him and his partner. The latter step is the same as previously described, so we focus on his waiting time to generate an edge.

Assume that a new arrival m observes state (i_0, j_0) and enters the pool without an edge (and the state is now $(i_0, j_0 + 1)$). Whenever a new participant arrives, there is a probability p_s, which depends on the current state s, that participant m will get an edge, and probability $1 - p_s$ that he will keep isolated. Thus, the process of getting an edge follows a non-homogeneous geometric distribution associated with exponential inter-experiment time. The probability of success (getting an edge) at each experiment depends on the current state. We claim that the study of waiting time before obtaining a partner can be transformed into a study of semi-Markov chain with absorbing states. The latter system is able to characterize the non-homogeneous geometric distribution associated with exponential inter-experiment time. Specifically, the state of the semi-Markov chain changes after an exponentially distributed time (corresponding to a new arrival of the system). The embedded discrete time Markov chain consists of an absorbing state a, and transient states which are equivalent to the set $\mathbb{S}_1$. At each transient state (i, j), there is a probability $P_{(i,j),a}$ that next step will enter an absorbing state, and probability $1 - P_{(i,j),a}$ that the process continues. When the process enters an absorbing state, it indicates that customer m eventually generates an edge, otherwise it means that m is still isolated and waiting for a potential partner. Moreover, the number of transitions before entering absorbing state represents the number of new arrivals that customer m has to wait before achieving an exclusive partnership.

At transient state (i, j), the probability of transition into an absorbing state a is

$$P_{(i,j),a} = (1-p^2)^i[\sum_{k=0}^{2(j-1)} \binom{2(j-1)}{k} p^k(1-p)^{2(j-1)-k}(\frac{2p(1-p)}{k+1} + \frac{2p^2}{k+2})]. \tag{2.7}$$

The above equation can be justified as follows. When participant m is in state (i,j) as an isolated vertex, the next arrival will be able to form exclusive partnership with m on the condition that the new arrival cannot find 3-way exchange opportunities, and that the new arrival chooses the directed edge that he generates with m among all edges triggered between the new arrival and isolated vertices. The former condition occurs with probability $(1-p^2)^i$, and the latter condition further depends on the number of directed edges between the new arrival and m. When the new arrival triggers in total k directed edges with the other $j-1$ isolated vertices (which occurs with probability $\binom{2(j-1)}{k}p^k(1-p)^{2(j-1)-k}$), if the new arrival in addition generates one directed edge with m with probability $2p(1-p)$, then m will be selected as a partner with probability $1/(k+1)$. On the other hand, if there are two directed edges between the new arrival and m with probability p^2, then there is a chance of $2/(k+2)$ that the new arrival will choose one of the edges that he generated with m among all $k+2$ edges.

Transition into non-absorbing states from (i,j) occurs exclusively in the following three scenarios:

(a) transit into $(i-k,j)$ with probability

$$P_{(i,j),(i-k,j)} = \binom{i}{k} p^{2k}(1-p^2)^{i-k}. \tag{2.8}$$

(b) transit into $(i,j+1)$ with probability

$$P_{(i,j),(i,j+1)} = (1-p^2)^i(1-p)^{2j}. \tag{2.9}$$

(c) transit into $(i+1,j-1)$ with probability

$$\begin{aligned} P_{(i,j),(i+1,j-1)} = {} & (1-p^2)^i[\sum_{k=0}^{2(j-1)} \binom{2(j-1)}{k} p^k(1-p)^{2(j-1)-k}(\frac{k\cdot 2p(1-p)}{k+1}+\frac{k\cdot p^2}{k+2})] \\ & + (1-p^2)^i(1-p)^2[1-(1-p)^{2(j-1)}]. \end{aligned} \tag{2.10}$$

Define $P_{(i_1,j_1),(i_2,j_2)} = 0$ whenever $(i_k,j_k) \notin \mathbb{S}_1$ for $k=1,2$. Following classical reference, i.e., [18], [26] or [25], define $t_{(i,j)}$ as the expected number of time periods before the semi-Markov chain enters the absorbing state given that starts in (i,j). Then we have

$$t_a = 0, \tag{2.11}$$

$$t_{(i,j)} = 1 + \sum_{s\in\mathbb{S}_1\cup\{a\}} P_{(i,j),s}t_s, \quad \text{for any } (i,j)\in\mathbb{S}_1. \tag{2.12}$$

Solving the above linear equations yields $t_{(i,j)}$, and the conditional expected waiting time is

$$E[w|I_p=0] = \frac{t_{(i_0,j_0+1)}}{\nu} + \frac{1}{\nu p^2}. \tag{2.13}$$

Collectively, the expected total waiting time for a participant entering the system is

$$\begin{aligned}E[w] =& E[w|I_p = 0] \times P(I_p = 0) + E[w|I_p = 1] \times P(I_p = 1)\\ =& \sum_{(i,j)\in\mathbb{S}_1} (\frac{t_{(i,j+1)}}{\nu} + \frac{1}{\nu p^2})P_{(i,j)}(1-p^2)^i(1-p)^{2j} + \frac{1}{\nu p^2}\sum_{(i,j)\in\mathbb{S}_1} P_{(i,j)}(1-p^2)^i[1-(1-p)^{2j}]\\ =& \sum_{(i,j)\in\mathbb{S}_1} [\frac{t_{(i,j+1)}}{\nu}(1-p)^{2j} + \frac{1}{\nu p^2}]P_{(i,j)}(1-p^2)^i.\end{aligned} \tag{2.14}$$

Numerical result shows that Equation (2.14) provides an upper bound for participant's waiting time in the original unrelaxed system. This upper bound delivers a clear picture and structure. $P_{(i,j)}(1-p^2)^i$ when $(i,j)\in\mathbb{S}_1$ is the probability that the newly arrived participant will enter the pool due to no 3-way barter opportunities. Whoever enters the pool will have to wait at least a time duration with mean $\frac{1}{\nu p^2}$, which corresponds to the time spent in the pool after obtained a partner but before leaving the system. An additional waiting time of $\frac{t_{(i,j+1)}}{\nu}$ may apply if the participant enters the pool alone with conditional probability $(1-p)^{2j}$.

Another advantage of the bound derived in Equation (2.14) is that, both $\frac{t_{(i,j+1)}}{\nu}$ and $\frac{1}{\nu p^2}$ are independent of the limiting probabilities of the relaxed system. Consequently, if we are just interested in the conditional waiting time for those who enter the pool, it is not necessary to calculate the limiting probabilities of the relaxed system. It simply requires the stationary probabilities of a generic discrete time Markov Chain characterized by Equations (2.7) to (2.10). Since the corresponding Markov Chain is generic, it only has to be solved for once. Afterward, participants waiting time can be estimated in detail merely from their observations of the system before entering the pool.

Now it remains to prove the stability of the relaxed system, as well as the stability of the original online with three-way exchange system.

Theorem 3. *The online with three-way relaxed system is stable.*

Proof. When a new arrival of the system observes state (i,j), with probability $(1-p^2)^i$, the arrival will enter the pool (regardless of as an isolated vertex or as a partner); with probability $1-(1-p^2)^i$, the arrival will trigger at least two in-pool participants to leave the system. Thus, the birth rate of the system is $(1-p^2)^i\nu$ and the death rate is at least $2[1-(1-p^2)^i]\nu$ whenever the system is in state (i,j). Clearly, if a birth and death process with birth rate $\lambda_{(i,j)} = (1-p^2)^i\nu$ and death rate $\mu_{(i,j)} = 2[1-(1-p^2)^i]\nu$ is stable, then the relaxed system is stable as well.

Since

$$\sum_{n=1}^{\infty} \frac{\lambda_{(0,j_{(n,0)})}\lambda_{(1,j_{(n,1)})}\cdots\lambda_{(n-1,j_{(n,n-1)})}}{\mu_{(1,j_{(n,n)})}\mu_{(2,j_{(n,n+1)})}\cdots\mu_{(n,j_{(n,2n-1)})}} = \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n}(1-p^2)^{k-1}}{\prod_{k=1}^{n} 2[1-(1-p^2)^k]} = \sum_{n=1}^{\infty} a_n,$$
$$\forall j_{(b_1,b_2)} \geq 0, \text{and } b_1 > 0, 0 \leq b_2 < 2b_1, \tag{2.15}$$

and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(1-p^2)^n}{2[1-(1-p^2)^{n+1}]} = 0, \tag{2.16}$$

$\sum_{n=1}^{\infty} a_n$ converges. Additionally,

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_{(n,j)} a_n} > \frac{1}{\nu} \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty. \tag{2.17}$$

Therefore the birth and death process is stable, and so is the relaxed system. □

Based on Theorem 3, the stability of the original system can be proven.

Corollary 3.1. *The original online with three-way system is stable.*

The corollary can be easily justified. The two systems (not the pools) have the same arrival process, i.e., a Poisson process with rate ν. The original system has a larger death rate because of the freedom in obtaining resources (generating edges), and therefore, the recurrent time, i.e., the time required for returning to the same state, is shorter in the original system than in the relaxed system. As the relaxed system is stable, the original system is also stable.

2.4 Numerical study

Chapter 2.2 and Chapter 2.3 provide analytical results for participants' waiting time and stability of the systems, but do not provide a comparison of the exchange mechanisms. Analytical comparison and ranking reveal managerial insights less clearly. In this section, a numerical study is presented that was conducted in order to evaluate the models and obtain a general understanding of system behaviors. The results presented in this section, as well as in Chapter 3.4 and Chapter 3.5, provide corporate barter platform operators with guidelines for evaluating common exchange mechanisms and accordingly choosing the most efficient and profitable one in various situations. Because of the unavailability of real data from corporate barter platforms, we ran a number of generic simulations of the models covered in the last two sections. Without loss of generality, the inter-arrival rate in all simulations was set at $\nu = 1$. The simulations were conducted with 1,000,000 arrivals for extremely small matching probabilities, and with 30,000 arrivals for other matching probabilities. More arrivals are required in order to obtain accurate results when the matching probability p is extremely small. Under these circumstances, participants have to be (very) lucky to leave the system before 30,000 arrivals in all online models, and hence the average waiting time based on 30,000 arrivals would be biased and only count for short waiting times when p is extremely small. Our results showed that for $p \geq 0.02$, all online models become stable a considerable time before the 30,000th arrival. For the extremely small matching probabilities that we considered ($0.001 \leq p < 0.02$), 1,000,000 arrivals are sufficient to deliver an accurate

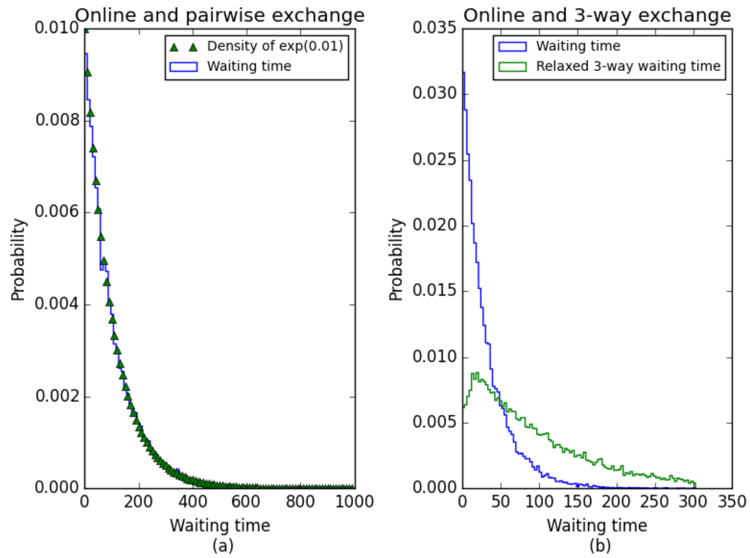

Figure 2.3: Histogram of waiting time under online policy.

comparative evaluation of the pairwise and three-way exchange types. Simulations intended to confirm the derivations and theorems presented in the last two sections are provided as well.

Numerical verification of theorems and derivations

In this subsection, we primarily investigate the customer waiting time and changes in the system state when $p = 0.1$. Since the essential behavior does not change with p, i.e., waiting time keeps to be exponentially distributed or bounded by a function of p and the system is always stable with any $p > 0$, the simulation result corresponding to a particular matching probability is a representative of the system performance under other matching probabilities.

Waiting time. Figure 2.3 (a) presents the histogram of waiting time for those who waited in the pool under online with pairwise exchange when $\nu = 1$ and $p = 0.1$. Note that it indeed follows an exponential distribution with rate 0.01.

Figure 2.3 (b) displays the histograms of waiting time for in-pool participants under the original online with 3-way model and its' corresponding relaxed model. It can be inferred that waiting time in the original system is upper bounded by the relaxed system. Later in this section, it shows that the relaxed system indeed provides an upper bound of waiting time for all matching probabilities.

State change. Figure 2.4 describes the number of participants waiting in the pool under the online policy with different exchange types. The data were collected by each arrival's observation. As justified earlier, PASTA keeps true in these systems, thus the observations are equivalent to steady state distribution. The result suggested that for all online models, state sizes are bounded in the simulations, and the systems are stable.

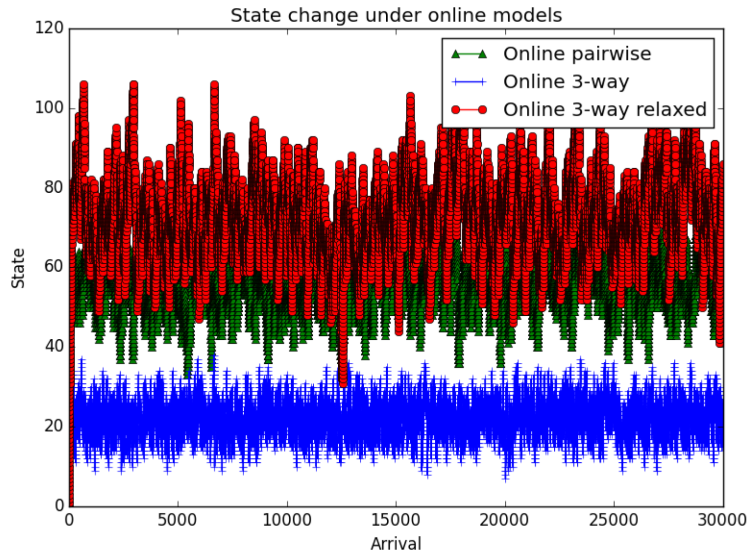

Figure 2.4: State changes under online models.

Evaluation of exchange mechanisms

The numerical results show that three-way exchange significantly surpasses pairwise exchange in most of the matching probabilities that we considered, and the results are very close to those of pairwise exchange when the latter exhibits advantages. Therefore, we conclude that three-way exchange is in practice better than pairwise exchange. We illustrate the results in three aspects, namely, pool size, waiting time, and number of successful matches.

Pool size. For pairwise exchange, it is observed that the state size is usually below $1/p^2$ (or the ceiling of $1/p^2$ for large p). It can be justified that, when there are more than $1/p^2$ in-pool participants, the expected number of successful matches for each new arrival is greater than or equal to one, representing a downward trend in the pool size.

In the original three-way exchange system, an estimation of the upper bound of the state size is also available. Clearly, we focus more on the state size when the matching probability is small, and are less concerned about large matching probabilities, as the former system is more prone to instability. We are able to derive a lower bound of the state size below which the system should remain in practice, namely, lower bound of the *upper bound of state size*,but do not prove yet that the lower bound indeed performs as the *upper bound of state size* when matching probability p is small.

Practical guideline of the maximum state size in a real system. In the online with three-way exchange system, there should be fewer than $1/p^{1.5}$ in-pool participants for most of the time when matching probability p is small.

To justify the bound, it can be argued that, firstly, there should be less than $1/p^2$ number of edges during most of the time, otherwise the new arrival on expectation can always trigger departure process; additionally, the average number of directed edges in $D(n,p)$ is $n(n-1)p$,

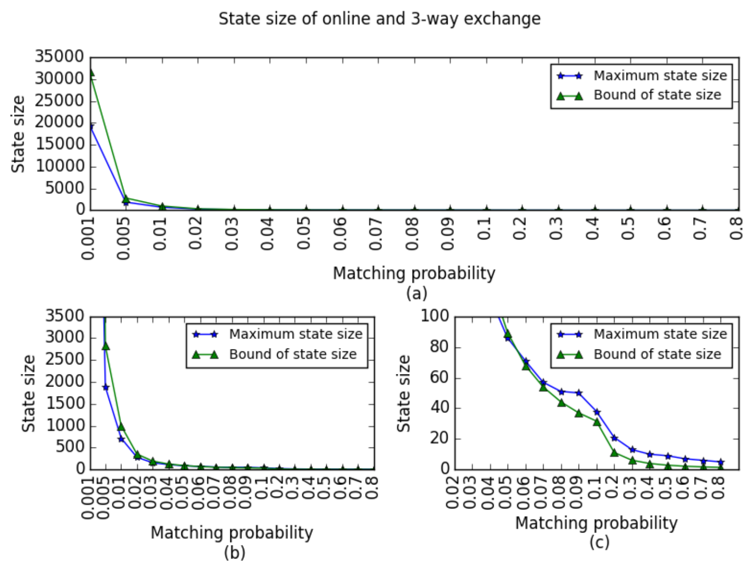

Figure 2.5: State changes under online with three-way models.

Comparison of the maximum state size in the simulations of the original online with three-way exchange system and the derived bound $1/p^{1.5}$. The axes are truncated in (b) and (c)

which is roughly n^2p for small p. Setting $1/p^2 = n^2p$ yields $n = 1/p^{1.5}$ as the result. However, the above process implies two relaxations. The first relaxation lies in the expectation $n(n-1)p$. The actual connections in the pool are looser than the expectation, because (a) an arrival without 3-cycle will enter the pool with less than (unconditionally) expected number of edges, as edges that can form 3-cycles are prohibited, and (b) each departure may delete a few connections between himself and still-in-pool participants. The second relaxation applies when n^2p is substituted for the expectation of connections, $n(n-1)p$. Therefore, $n = 1/p^{1.5}$ theoretically serves as a lower bound of the state size which the system should stay below for most of the time. The two relaxations above contribute to the error between the lower bound and the actual state size the system probabilistically keeping below of. Obviously, both errors due to the two relaxations become negligible as p decreases. The first error is upper bounded by a constant multiple of expected number of 3-cycles a new arrival can trigger, which eliminates when p is small; the second error equals np and clearly approaches to zero with p going to zero. Until now, the derivation process is rigorous, but we are not able to proceed forward to prove that $n = 1/p^{1.5}$ indeed performs as the upper bound of the state size for most of the time when the matching probability is small.

We observe the phenomenon that justifies our claim in Figure 2.5. The figure further shows that in the case where p is large, $n = 1/p^{1.5}$ still serves as a good estimation, because the difference between $n = 1/p^{1.5}$ and the actual maximum state size in the simulation is limited.

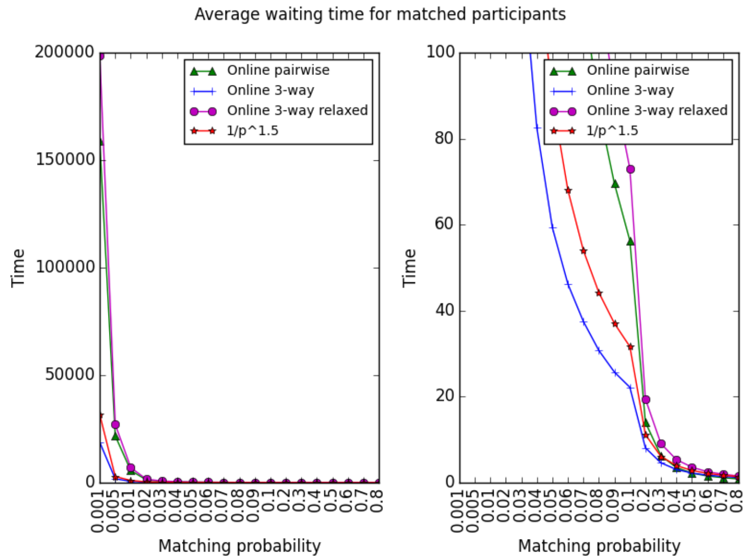

Figure 2.6: Waiting time under online models for matched participants.

The first three nodes of all online models are based on 1,000,000 arrivals. The remaining results are based on 30,000 arrivals.

Waiting time. Figure 2.6 shows the waiting time for all successfully matched participants (not only in-pool participants) under the online models. It can be observed that when the matching probability is small, a three-way exchange is significantly better in terms of shortened waiting time. The difference is eliminated when p increases, and pairwise exchange becomes slightly better when $p \geq 0.5$. When p is large, it is observed that the pool size is very small with no more than three participants for most of the time, and it is natural that pairwise exchange becomes slightly better, because it requires fewer in-pool participants to allow an exchange. We also find that the relaxed model provides good estimates when p is relatively large. We would comment that the relaxed model plays an important role in analytical derivation, as it proves the stability of the original three-way model, but it has limited power in estimating the waiting time under a three-way exchange when p is small. In fact, according to the practical upper bound of state size, using Little's law with arrival rate $\nu = 1$, it can be derived that the average waiting time of all participants should not exceed $1/p^{1.5}$ when p is small. Figure 2.6 indeed supports the hypothesis.

Number of matched participants. Figure 2.7 presents the number of matches among the first 30,000 arrivals in different models. It indicates that in a three-way exchange, even if the matching probability is extremely small, a significant proportion of participants is able to find exchange opportunities and leave the system before the 30,000th arrival. When $p \geq 0.005$, almost all participants in a three-way exchange are able to barter. However, in a pairwise exchange, almost all the participants have to wait for more than 30,000 arrivals when p is extremely small.

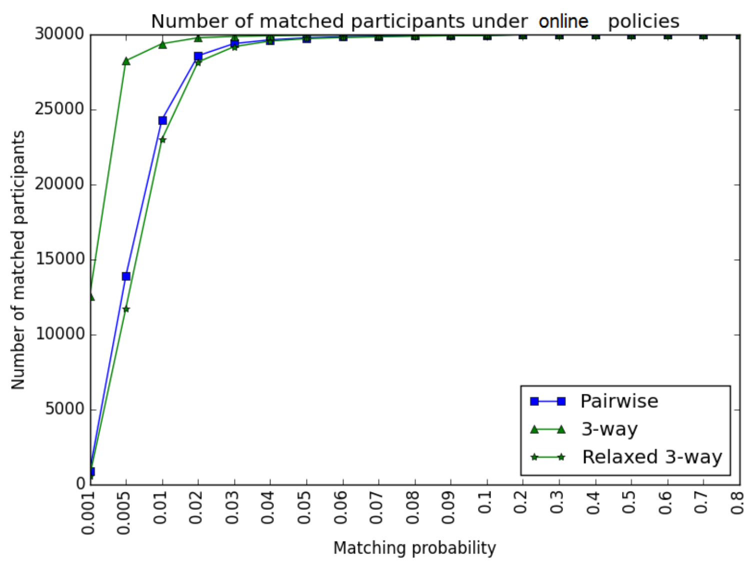

Figure 2.7: Number of matches in the first 30,000 arrivals.

Chapter 3

Batch Exchange Policy

3.1 Batch policy

Under the batch policy, new arrivals enter the pool directly and wait for other customers before exchanges can be performed. Assume that participants arrive according to a Poisson process with rate ν, and that the pool has a capacity C. Any one-way matching in the pool occurs with *iid* probability p. Whenever there are C customers in the pool, corporate barter exchanges are automatically conducted. Suppose that exchange opportunities can be found and issued instantaneously. Participants who are able to exchange leave the pool. The remaining participants remain and wait until there are sufficient customers to provide additional exchange opportunities. Apparently, there is an undesirable system state. When no exchange opportunity presents in the system, none of the participants leave the pool. Then, because the pool is always full, the system starts to execute an infinite number of exchange rounds. We assume that whenever this happens, the system discharges all existing participants immediately and restarts itself. By combining the above assumptions and those stated in Chapter 1.3, the expectation and variance of participants' waiting time can be obtained. In contrast to the online policy, the analysis of batch policy shows that system stability is guaranteed, but participants face the risk of being discharged before obtaining barter opportunities. Thus, in the analysis of batch policy, we focus on the waiting time and the probability of the participants being able to exchange.

3.2 Pairwise exchange type

By definition and assumptions, under the batch with pairwise exchange system, participants who can be bilaterally paired with another participant leave the pool. We assume that when there is at least one barter opportunity in the current round of exchange, any unilateral matching result between the unsuccessful participants is not carried into the next round. Therefore, the result of the next round is independent of that of the current round. The independence assumption turns out to be far less restrictive than expected and is justified

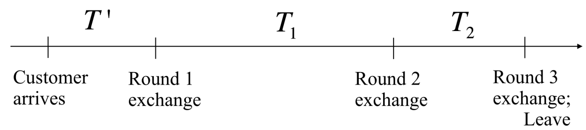

Figure 3.1: Illustration of exchange process.

when the random graph notation that models batch with pairwise exchanges is introduced. To facilitate a better understanding, the timeline of a customer who participated in three rounds of exchanges is described in Figure 3.1.

T' represents the time that the participant waited for the first round of exchange. T_i denotes the time between the $i-th$ and the $i+1st$ barter exchange. Because T' applies to every participant and acts as a buffer to increase the time spent in the system, it is ignored henceforth. We are interested in the total waiting time between the first and the last round of exchange that a participant is able to attend, or equivalently, the summation of T_i's that the participant experienced.

Similar to the assumptions in previous sections, the assumption here is that a participant can exchange with several participants simultaneously. Consequently, each round of exchange can be (independently and identically) modeled by an undirected random graph $G(C, p^2)$. It has a set of vertices $V = \{1, 2, 3, ..., C\}$ representing participants, and a set of undirected edges indicating bilateral matches between corresponding pairs of participants. Each edge is present in the graph independently and identically with probability p^2. The existence of an edge between two vertices indicates an exchange opportunity between the two participants. According to the policy, any vertices that are adjacent to edge(s) are deleted from the graph. Isolated vertices remain and wait for future participants in the next round of corporate barter exchange.

The well-known sharp threshold theory in random graph states that a random graph denoted by $G(n, p)$ almost surely has isolated vertices when $p < \frac{(1-\epsilon)\ln n}{n}$ and is almost surely connected when $p > \frac{(1+\epsilon)\ln n}{n}$. This theory facilitates our justification of the independence assumption. In the batch with pairwise exchange modeled by $G(C, p^2)$, whenever $p > \sqrt{\frac{\ln C}{C}}$, almost all participants in the pool can be connected and thus leave the system after the first round of exchange in which they participated. Consequently, the independence assumption does not affect the result when p is above the threshold. In Chapter 3.4 Numerical Study, a capacity of 100 is assumed in batch models. Figure 3.6 shows that when $p \geq 0.2$, almost all participants are able to depart from the system after the first exchange round in which they participated. Furthermore, it is observed that when $p \leq 0.001$, almost all participants are discharged from the system after the first round of exchange, because no opportunity exists in the pool. Therefore, in reality, with a pool size of 100, the independence assumption does not affect the result for most of the matching probabilities, i.e., when the matching probability is extremely small (≤ 0.001) or larger than relatively small (≥ 0.2). In the case where the probability is tricky and falls within (0.001,0.2), participants are on the border of paying the premium but being discharged. We may assume that under the advice of barter platform agents, participants actively update their profile, for example, type and volume of demand and volume and description of excessive inventories, in order to increase the probability of obtaining barter opportunities. As all participants update information after the current round, we can assume that future exchange opportunities are independent of the current situation. Collectively, the independence assumption does not affect the result in most cases, and can be justified when it does.

Recall that we are interested in the waiting time beyond the first exchange round in which one individual participated. Consider a participant m and his/her total waiting time between exchange rounds. All notations and their definitions are summarized as follows. Define

- X as the total number of rounds in which m is involved, which is geometrically distributed on $\{1, 2, 3, ...\}$.
- T_i, for $i \in \{1, 2, 3, ..., X-1\}$, as the time between the $i-th$ and the $i+1st$ round of exchange in which m participated. It should be noted that T_i's are *iid*.
- T as the total waiting time of m. $T = \sum_{i=1}^{X-1} T_i$ for $X > 1$, and $T = 0$ otherwise.
- N_i, for $i \in \{1, 2, 3, ..., X-1\}$, as the number of participants remaining after the $i-th$ round of exchange in which m participated. It should be noted that N_i 's are *iid*, and defined on $\{1, 2, 3, ..., C-2\}$
- $F(\cdot)$ as the *cdf* of T
- $H(\cdot)$ as the *cdf* of T_i.

We now present a few remarks about the above definitions. As any participant who enters the pool participates in at least one round of exchange, T actually describes the total waiting time beyond the first round of exchange. If m finds at least one exchange opportunity in the first round, i.e., $X = 1$, then $T = 0$ for m. In other words, T obtains a positive value only if $X > 1$. Random variable N_i denotes the number of isolated vertices after each round of exchange (except the last one) in which m is involved. Two conditions are applied. First, m him/herself is included in the N_i remainders for each $i < X$; otherwise, m would have already left the system before the $X-th$ round of exchange. Second, in each exchange round, except for the last one in which m is involved, there exists at least one exchange opportunity among the remaining $C-1$ participants; otherwise, again m would have left earlier because all participants would have been discharged from the system. Consequently, N_i's are defined on $\{1, 2, 3, ..., C-2\}$. $C-1$ is excluded from the domain, because under pairwise exchange it is impossible that a single individual departs from the system. Based on the above definitions, the distribution of T can be derived theoretically by conditional probability and convolution. The expectation and variance of T, on the other hand, can be obtained directly by conditioning without the knowledge of a closed form distribution of T. In the remaining content of this section, we focus on the derivation of $E[T]$ and $Var(T)$. Steps for obtaining $F(t)$ are presented at the end of this section.

Because $T \mid X > 1 = \sum_{i=1}^{X-1} T_i$, $T \mid (X = 1) = 0$, and T_i's are *iid*,

$$E[T|X] = (X-1)E[T_i], \tag{3.1}$$

$$Var(T|X) = (X-1)Var(T_i). \tag{3.2}$$

By conditional expectation, $E[T] = E[E[T \mid X]]$. Therefore we have

$$\begin{aligned} E[T] =& E[E[T \mid X]] = E[(X-1)E[T_i]] = E[T_i]E[X-1], \\ Var(T) =& E[Var(T \mid X)] + Var(E[T \mid X]) \\ =& E[(X-1)Var(T_i)] + Var((X-1)E[T_i]) \\ =& Var(T_i)E[X-1] + E[T_i]^2 Var(X). \end{aligned} \tag{3.3}$$

As discussed earlier, X is geometrically distributed. The terminating probability for each experiment is the summation of (a) $[1-(1-p^2)^{C-1}]$ representing the probability that m is able to exchange with another participant, and (b) $(1-p^2)^{C(C-1)/2}$ denoting that no exchange opportunities appear among all participants. Define $\tilde{p} = [1-(1-p^2)^{C-1}] + (1-p^2)^{C(C-1)/2}$; then,

$$E[X-1] = 1/\tilde{p} - 1, Var(X) = \frac{1-\tilde{p}}{\tilde{p}^2}. \tag{3.4}$$

Now, provided that $E[T_i]$ and $Var(T_i)$ are obtained, the expectation and variance of T are available. By the knowledge of $H(t \mid N_i = n) \sim \text{Gamma}(\nu, C-n)$,

$$\begin{aligned} E[T_i] =& E[E[T_i \mid N_i]] = E[\frac{C-N_i}{\nu}] = \frac{C-E[N_i]}{\nu}, \\ Var(T_i) =& E[Var(T_i \mid N_i)] + Var(E[T_i \mid N_i]) = E[\frac{C-N_i}{\nu^2}] + Var(\frac{C-N_i}{\nu}) \\ =& \frac{C-E[N_i]+Var(N_i)}{\nu^2}. \end{aligned} \tag{3.5}$$

The final piece of the puzzle is $E[N_i]$ and $Var(N_i)$. We state directly the results of these two statistics. Details are presented in the end of this section.

$$\begin{aligned} E[N_i] =& 1 + \frac{(C-1)(1-p^2)^{(C-2)}[1-(1-p^2)^{(C-1)/2}]}{1-(1-p^2)^{(C-1)(C-2)/2}}, \\ Var(N_i) =& \frac{1}{1-(1-p^2)^{(C-1)(C-2)/2}} \times \\ & \{(C-1)(1-p^2)^{(C-2)}[1+(C-2)(1-p^2)^{(C-3)} - (C-1)(1-p^2)^{(C-2)}] \\ & - (C-1)^2[1-(1-p^2)^{(C-2)}]^2[(1-p^2)^{(C-1)(C-2)/2} + \frac{(1-p^2)^{(3C-5)}}{1-(1-p^2)^{(C-1)(C-2)/2}}]\}. \end{aligned} \tag{3.6}$$

Substituting equations (3.4), (3.5), and (3.6) into (3.3), the expectation and variance of T can be easily obtained.

Distribution of waiting time under batch with pairwise exchange

Note that for any $t > 0$ (which implies that $X > 1$),

$$\begin{aligned} \bar{F}(t) &= \sum_{x=1}^{\infty} \Pr(T > t | X = x) P(X = x) \\ &= \sum_{x=2}^{\infty} \Pr(\sum_{i=1}^{x-1} T_i > t) P(X = x). \end{aligned} \tag{3.7}$$

In Equation (3.7), $\Pr(\sum_{i=1}^{x-1} T_i > t)$ can be obtained by convolution if $H(t)$, the distribution of T_i, is available; X is geometrically distributed with parameter $\tilde{p} = [1 - (1 - p^2)^{C-1}] + (1 - p^2)^{C(C-1)/2}$, which is the probability of participant m getting at least one exchange opportunity, or completely no opportunity in the entire system. Therefore, as long as $H(t)$ is available, the distribution of T will be accessible.

To derive $H(t)$, note that $T_i | N_i$ is a gamma distribution. Specifically,

$$H(t | N_i = n) \sim \text{Gamma}(\nu, C - n). \tag{3.8}$$

Hence by conditional probability,

$$H(t) = \Pr(T_i < t) = \sum_{n=1}^{C-2} \Pr(T_i < t | N_i = n) P(N_i = n). \tag{3.9}$$

Now the last missing piece is the distribution of N_i. We are able to derive the distribution of total waiting time T with the knowledge of N_i's distribution. As argued earlier in this section, N_i consists of participant m and the (conditionally) isolated participants among the remaining $C-1$ members. Define $\bar{N}_{C-1}$ as the number of isolated individuals among the $C-1$ participants (unconditionally). Let Bernoulli random variable E denote the opportunity to exchange among the $C - 1$ participants. $E = 0$ if there is no opportunity, and $E = 1$ otherwise. Obviously, $N_i = 1 + \bar{N}_{C-1} | (E = 1)$, and

$$P(N_i = n) = P(\bar{N}_{C-1} | (E = 1) = n - 1) = \frac{P(\bar{N}_{C-1} = n - 1)}{1 - P(\bar{N}_{C-1} = C - 1)}. \tag{3.10}$$

According to [12],

$$P(\bar{N}_{C-1} = n - 1) = \sum_{j=n-1}^{C-1} \binom{C-1}{j} \binom{j}{n-1} (-1)^{j-(n-1)} (1 - p^2)^{(C-1)j - j(j+1)/2}. \tag{3.11}$$

Now the distributions of N_i and T_i are available. Theoretically the distribution of T can be obtained by substituting Equation (3.8), (3.10) and (3.11) into Equation (3.9) and (3.7).

Derivation of $E[T]$ and $Var(T)$ under batch with pairwise exchange

Similar to earlier definitions, define $\bar{N}_{C-1}$ as the unconditional number of isolated individuals among the $C-1$ participants. Let E denote the opportunity to exchange within the smaller pool. $E = 0$ if there is no opportunity among the $C-1$ individuals, and $E = 1$ otherwise. Again, $N_i = 1 + \bar{N}_{C-1}|(E = 1)$, and

$$\begin{aligned} E[N_i] &= 1 + E[\bar{N}_{C-1}|E = 1], \\ Var(N_i) &= Var(\bar{N}_{C-1}|E = 1). \end{aligned} \tag{3.12}$$

By the formula of conditional expectation and variance,

$$\begin{aligned} E[\bar{N}_{C-1}] =& E[E[\bar{N}_{C-1}|E]], \\ Var(\bar{N}_{C-1}) =& Var(E[\bar{N}_{C-1}|E]) + E[Var(\bar{N}_{C-1}|E)]. \end{aligned} \tag{3.13}$$

Consequently,

$$\begin{aligned} E[\bar{N}_{C-1}|E = 1] =& \frac{E[\bar{N}_{C-1}] - E[\bar{N}_{C-1}|E = 0]P(E = 0)}{P(E = 1)}, \\ Var(\bar{N}_{C-1}|E = 1) =& \frac{Var(\bar{N}_{C-1}) - Var(E[\bar{N}_{C-1}|E]) - Var(\bar{N}_{C-1}|E = 0)P(E = 0)}{P(E = 1)}, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} Var(E[\bar{N}_{C-1}|E]) =& (E[\bar{N}_{C-1}|E = 0] - E[\bar{N}_{C-1}])^2 P(E = 0) \\ &+ (E[\bar{N}_{C-1}|E = 1] - E[\bar{N}_{C-1}])^2 P(E = 1). \end{aligned} \tag{3.15}$$

Now we are about to derive the intermediate elements required to obtain $E[\bar{N}_{C-1}|E = 1]$ and $Var(\bar{N}_{C-1}|E = 1)$. Clearly,

$$\begin{aligned} E[\bar{N}_{C-1}|E = 0] &= C - 1, \\ Var(\bar{N}_{C-1}|E = 0) &= 0, \end{aligned} \tag{3.16}$$

because all participants are isolated when $E = 0$. Moreover,

$$\begin{aligned} P(E = 0) &= (1 - p^2)^{(C-1)(C-2)/2}, \\ P(E = 1) &= 1 - (1 - p^2)^{(C-1)(C-2)/2}, \end{aligned} \tag{3.17}$$

since no edge is generated between any of the $\binom{C-1}{2}$ pairs of participants when $E = 0$.

To compute unconditional expectation $E[\bar{N}_{C-1}]$ and variance $Var(\bar{N}_{C-1})$, define $I_j = 1$ if participant j is isolated after a round of exchange within a pool of capacity $C-1$, and

$I_j = 0$ otherwise. Evidently, $P(I_j = 1) = (1-p^2)^{(C-2)}$. Furthermore,

$$\begin{aligned}
E[\bar{N}_{C-1}] =& E[\sum_{j=1}^{C-1} I_j] = (C-1)E[I_1] = (C-1)(1-p^2)^{C-2}, \\
Var(\bar{N}_{C-1}) =& Var(\sum_{j=1}^{C-1} I_j) = (C-1)Var(I_1) + (C-1)(C-2)Cov(I_1,\, I_2) \\
=& (C-1)(1-p^2)^{C-2}[1-(1-p^2)^{C-2}] \\
&+ (C-1)(C-2)[(1-p^2)^{(C-2)+(C-3)} - (1-p^2)^{2(C-2)}] \\
=& (C-1)(1-p^2)^{C-2}[1+(C-2)(1-p^2)^{C-3} - (C-1)(1-p^2)^{C-2}].
\end{aligned} \tag{3.18}$$

Substitute Equation (3.14) through (3.18) into (3.12) yields the final result.

3.3 Three-way exchange type

The three-way exchange type allows exactly three participants in an exchange. As in the batch with pairwise system, a future round of exchanges is independent of the current round if the current result contains at least one three-way exchange opportunity. Based on these assumptions, each round of three-way exchange can be independently and identically represented by a random digraph $D(C,\, p)$, where C is the number of vertices and p is the homogeneous probability that a directed edge presents in the digraph. More specifically, among all $C(C-1)$ potential directed edges formed by C vertices, each directed edge is present in the digraph with *iid* probability p. A one-way match between participant A's supply and B's demand is equivalent to a directed edge from A to B in $D(C,\, p)$, and a three-way exchange in the system is identical to a directed three-cycle formed by corresponding vertices in $D(C,\, p)$.

Under the batch policy with three-way exchange model, it is found that the procedures for obtaining the expectation and variance of the waiting time are the same as that in Chapter 3.2, but they are available only asymptotically. Because of the complexity in the correlations between nodes and edges in a digraph, an analysis can be performed only in the asymptotical sense. Before deriving the expectation and variance of the waiting time, we first state one lemma and two theorems regarding the distribution of the number of three-way exchanges in an exchange round, as well as the probability that a random participant can find exchange opportunities in the current round of exchange.

Define the $r-th$ factorial moment of a random variable X as $E_r[X] = E[X_r]$ where $X_r = X(X-1)...(X-r+1)$. We state the following lemma without a proof. The proof can be found in the literature, i.e, [7].

Lemma 4. *Let $\lambda = \lambda(n)$ be a non-negative bounded function on n. Suppose that the non-negative integer valued random variables $X_1, X_2, ...$ are such that $\lim_{n\to\infty}\{E_r(X_n) - \lambda^r\} = 0$, $r = 0, 1, 2, ...$ Then $X_n \xrightarrow{d} Poisson(\lambda)$*

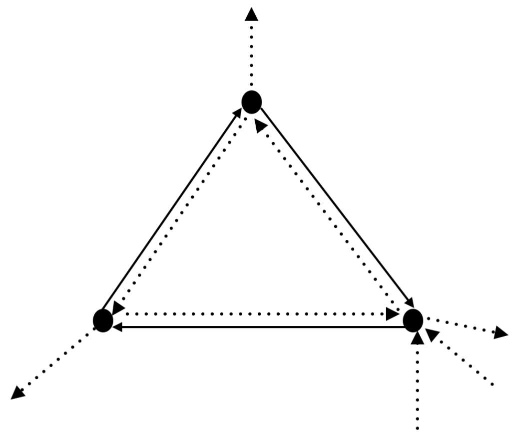

Figure 3.2: Illustration of 3-way exchange process.

Theorem 5. *Let $M(n)$ denote the total number of three-way exchange opportunities in the current round of the batch policy with three-way exchange system modeled by a random digraph $D(n, p)$. If $p = k/n$, where k is a positive constant, then $M(n) \xrightarrow{d} Poisson(\lambda)$, where $\lambda = k^3/3$.*

Proof. We will follow the standard approach in [7].

Evidently from Figure 3.2, the probability that a clockwise loop exists over a fixed set of three vertices is p^3, hence the probability of an exchange opportunity among any three participants is $2p^3 - p^6$. It is derived from the probability of a clockwise loop plus the probability of a counterclockwise loop minors the probability that both loops exist. Therefore,

$$E[M(n)] = \binom{n}{3}(2p^3 - p^6) \to \frac{n^3p^3}{3} = k^3/3 = \lambda. \tag{3.19}$$

By lamma 4, it is sufficient to show that for every non-negative r, $E_r[M(n)] \to \lambda^r$ as $n \to \infty$. Note that $E_r[M(n)]$ is the expected number of ordered r-tuples of 3-way exchanges in $D(n, p)$.

Let $E_r{}'[M(n)]$ denote the expected number of ordered r-tuples of 3-way exchanges with disjoint vertices. Then,

$$E_r{}'[M(n)] = \binom{n}{3}\binom{n-3}{3}...\binom{n-3(r-1)}{3}(2p^3 - p^6)^r \to \frac{n^{3r}p^{3r}}{3^r} = \lambda^r. \tag{3.20}$$

Therefore all we need is $E_r{}''[M(n)] = E_r[M(n)] - E_r{}'[M(n)] = o(1)$.

Let s denote the number of vertices that form ordered r-tuple of 3-cycles, and there is at least one overlap vertex among the r 3-cycles. Note that there must be more edges than nodes in the graph represented by the r 3-cycles, thus there are at least $s+1$ edges. For any 3 nodes that could form a 3-way exchange, there are potentially two distinct exchanges, i.e., clockwise exchange and counter clockwise exchange. Therefore

$$E_r''[M(n)] \leq \sum_{s=3}^{3r-1} \binom{n}{s} \{ \binom{s}{3} \times 2\}^r p^{s+1} = \sum_{s=3}^{3r-1} O(n^s p^{s+1}) = \sum_{s=3}^{3r-1} O(n^{-1}). \tag{3.21}$$

In particular, $E_r''[M(n)] = o(1)$, so the result follows. □

Theorem 6. *Let P_i denote the probability that an in-pool participant i is able to barter in the current round of exchange under the batch policy with three-way exchange system with system capacity n, homogeneous one-way matching probability $p = k/n$, and constant k. Then, $P_i = O(n^{-1})$.*

Proof. Let Y_n denote the number of participants that can find 3-way exchange opportunities in the current round.

$$E[Y_n] = \sum_{j=1}^{n} P_j = nP_i, \quad \text{for any } i. \tag{3.22}$$

Define set $\mathbb{S}_{Z(n)}$ as the collection of directed 3-cycles in $D(n,p)$ that share at least one vertex with other 3-cycles. Let $\mathbb{S}_{\bar{M}(n)}$ denote the set of ordered two tuples of 3-cycles such that the two elements in the same tuple share at least one vertex. $Z(n)$ and $\bar{M}(n)$ denote the cardinality of $\mathbb{S}_{Z(n)}$ and $\mathbb{S}_{\bar{M}(n)}$, respectively. Following the definition and notation in Theorem 5, we have $E[Z(n)] <= E[\bar{M}(n)] = E_2''[M(n)] = O(n^{-1})$. The first equation holds by definition, and the inequality can be justified as follows. Define set $\mathbb{S}_{M'(n)}$ that collects the unique first element in the ordered two tuples in $\mathbb{S}_{\bar{M}(n)}$, and let $M'(n)$ denote the corresponding cardinality. Clearly $\mathbb{S}_{M'(n)}$ is equivalent to $\mathbb{S}_{Z(n)}$, thus $Z(n) = M'(n)$. Considering the duplications in collecting the first elements that form $\mathbb{S}_{M'(n)}$, we have $\bar{M}(n) \geq M'(n)$, hence the relationship between $E[Z(n)]$ and $E_2''[M(n)]$ is justified. Moreover

$$E[Y_n] = 3E[M(n)] - aE[Z(n)] \to 3E[M(n)] = k^3, \tag{3.23}$$

where $0 \leq a \leq 3$, and $aE[Z(n)]$ performs as an adjustment of repeated counts. Note that each 3-cycle in $\mathbb{S}_{Z(n)}$ contributes at most three repeated counts, so Equation (3.23) holds. Then the result follows as $P_i = E[Y_n]/n = O(n^{-1})$ □

For waiting time analysis, we here state a few key intermediate elements for the final result, but ignore the details, as they are identical to those in Chapter 3.2. Following the same definition in Chapter 3.2, we assume $p = k/C$ for constant k. When the pool capacity C increases and p goes to zero, we have

$$\begin{aligned} &E[N_i] \to 1 + \frac{(C-1) - k^3 - (C-1)e^{-k^3/3}}{1 - e^{-k^3/3}}, \\ &Var(N_i) \to \frac{3k^3 - k^6 e^{-k^3/3}}{1 - e^{-k^3/3}} - (\frac{k^3 e^{-k^3/3}}{1 - e^{-k^3/3}})^2, \\ &\tilde{p} = k^3/C + e^{-k^3/3} \to e^{-k^3/3}. \end{aligned} \tag{3.24}$$

Please refer to the immediate following subsection for the derivation of these intermediate elements. Substituting Equation (3.4), (3.5), and (3.24) into (3.3) yields the asymptotical expectation and variance of participants' waiting time.

Derivation of asymptotic $E[N_i]$, $Var(N_i)$ and $\tilde{p}$ under batch with three-Way exchange

Notations and definitions remain the same as in the batch with pairwise exchange except for the following.

- N_i, for $i \in \{1, 2, 3, ..., X-1\}$, as the number of participants remained after the $i-th$ round of exchange that participant m participated in. Note that N_i 's are *iid* and are defined on $\{1, 2, 3, ..., C-3\}$;
- $\bar{N}_{C-1}$ as the unconditional number of participants remained among the $C-1$ participants. It is defined on $\{0, 1, 2, .., C-4, C-1\}$;
- E as the exchange opportunity among $C-1$ participants. $E = 1$ if there is at least one 3-way exchange, and $E = 0$ otherwise.

Following the relationship that $p = k/n$ in $D(n, p)$ for some constant k, and using the same notation and argument in the proof of Theorem 6, we have

$$E[\bar{N}_{C-1}] = (C-1) - E[Y_{C-1}] \to (C-1) - k^3. \tag{3.25}$$

To derive $Var(\bar{N}_{C-1})$, note that

$$E[Z(C-1)^2] = E_2[Z(C-1)] + E[Z(C-1)] \leq E_3''[M(C-1)] + E_2''[M(C-1)] = O(C^{-1}), \tag{3.26}$$

where $Z(C-1)$ and $M(C-1)$ follow the same definition of $Z(n)$ and $M(n)$ in Theorem 5 and Theorem 6 with $n = C-1$. $E_2[Z(C-1)], E_2''[M(C-1)]$ and $E_3''[M(C-1)]$ are identically defined as that in Theorem 5 and 6. The above inequality is true because

$$E_2[Z(C-1)] \leq E_3''[M(C-1)], \tag{3.27}$$

$$E[Z(C-1)] \leq E_2''[M(C-1)]. \tag{3.28}$$

Since inequality (3.28) is justified in Theorem 6, we will now focus on inequality (3.27). Clearly $E_2[Z(C-1)]$ represents the expected number of ordered 2-tuples of 3-cycles such that each element in the tuple has common vertex with other cycle(s) (not necessary with the cycle in the same tuple); while $E_3''[M(C-1)]$ denotes the expected number of 3-tuples of 3-cycles such that elements in the same tuple share vertex (or equivalently, the three 3-cycles in a tuple are formed by less than 9 distinct vertices). Define $\mathbb{S}_2$ as the set of ordered 2-tuples that contribute to $E_2[Z(C-1)]$, and $\mathbb{S}_3$ as set of ordered 3-tuples that contribute to $E_3''[M(C-1)]$. Let s denote an element in $\mathbb{S}_2$. If the two 3-cycles in s share vertex with each other, then adding any other 3-cycle into s will make $s \in \mathbb{S}_3$. On the other hand, if the two 3-cycles do not have common vertex, then there is at least one 3-cycle that is not in s but shares vertex with element in s. Adding this 3-cycle into s will again guarantee $s \in \mathbb{S}_3$. As a conclusion, $\mathbb{S}_2 \in \mathbb{S}_3$, thus inequality (3.27) holds. With (3.27) and (3.28), inequality (3.26) is justified. Therefore

$$Var(Z(C-1)) = E[Z(C-1)^2] - (E[Z(C-1)])^2 = O(C^{-1}), \tag{3.29}$$

which also implies that

$$Cov(M(C-1), Z(C-1)) = \rho\sqrt{Var(M(C-1))}\sqrt{Var(Z(C-1))} = O(C^{-1/2}), \tag{3.30}$$

where $M(C-1)$ converges to Poisson distribution with a constant parameter, and ρ represents the correlation between $M(C-1)$ and $Z(C-1)$. Then $Var(\bar{N}_{C-1})$ can be derived following the same argument of Equation (3.23)

$$\begin{aligned} Var(\bar{N}_{C-1}) &= Var((C-1) - Y_{C-1}) \\ &= Var((C-1) - (3M(C-1) - aZ(C-1))) \\ &= 9Var(M(C-1)) + a^2Var(Z(C-1)) - 6aCov(M(C-1), Z(C-1)) \\ &\to 3k^3. \end{aligned} \tag{3.31}$$

Moreover, from Theorem 5, we have

$$P(E=0) \to e^{-k^3/3}. \tag{3.32}$$

Finally, X, the number of rounds that participant m is involved in, is again a geometric distribution. The participant will leave the system if he obtains a 3-way exchange opportunity, or if there is no opportunity available in the entire system. Following Theorem 5 and Theorem 6, the corresponding probability is

$$\tilde{p} \to k^3/C + e^{-k^3/3} = O(C^{-1}) + e^{-k^3/3} \to e^{-k^3/3}. \tag{3.33}$$

With the knowledge of Equation (3.25), (3.31), (3.32) and (3.33), asymptotic $E[N_i]$ and $Var(N_i)$ can be derived following the same approach in Chapter 3.2.

3.4 Numerical study

The simulations were conducted with constant capacity $C = 100$. The selection of this capacity is due mainly to machine accuracy issues. We would have liked to start the simulation with $p < 1/C$, as $1/C$ is one of the performance thresholds for random graphs, and then gradually increase p. A large capacity results in a very small p value initially, but because of design issues, a computer is inherently incapable of handling very small decimal numbers

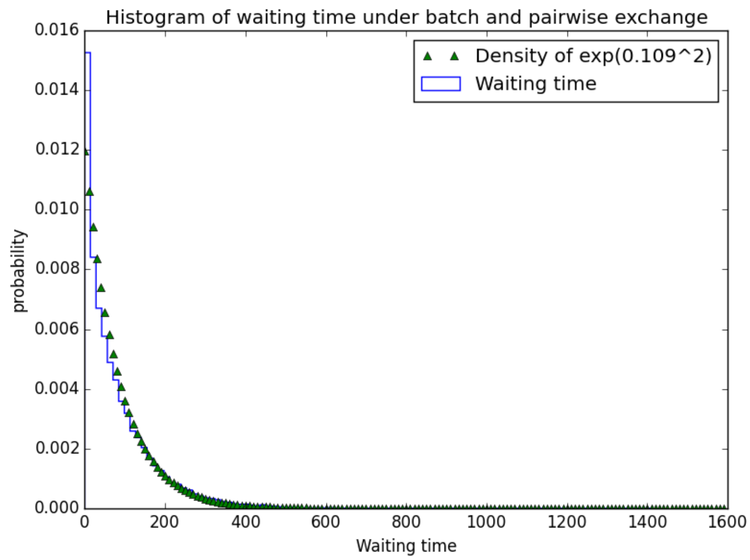

Figure 3.3: Histogram of waiting time under batch with pairwise exchange.

at a high accuracy level in general purpose computation. Thus, we decided to set the capacity at 100, and run the simulation with $p = 0.001$ up to $p = 0.8$. The arrival rate was again set at $\nu = 1$. Each simulation contained 3,000 rounds of batch exchanges. It should be noted that the waiting time to which we refer in this section is the time beyond the first round of exchange. We investigated the influence of the matching probability on the system performance, as well as the manner in which participants' preference would change under various matching probabilities. Simulations intended to confirm the derivations and theorems presented in the last two sections are provided as well.

Numerical verification of theorems and derivations

Distributions of participants' waiting time under both pairwise and 3-way exchange are studied when matching probability $p = 0.04$. This value of p is selected because the corresponding simulation provides the largest total waiting time among all participants.

Pairwise exchange. Figure 3.3 shows the histogram of waiting time under batch with pairwise exchange. The sample mean $\hat{\mu} = 83.58$ and sample variance $\hat{\sigma^2} = 8563.20$ are very close to theoretical mean $E[T] = 83.49$ and variance $Var(T) = 8392.33$. We also find that it is close to an exponential distribution with rate $\hat{\mu}$, which is the waiting time of online with pairwise exchange with matching probability $\hat{p} = \sqrt{1/\hat{\mu}} = 0.109$. Therefore, under batch with pairwise exchange, when $p = 0.04$, the waiting time beyond the first exchange is similar to the waiting time in online with pairwise exchange with matching probability $p = 0.109$. However, this kind of exponential approximation is not generally applied. Waiting time behavior under various values of p is discussed at the end of this subsection.

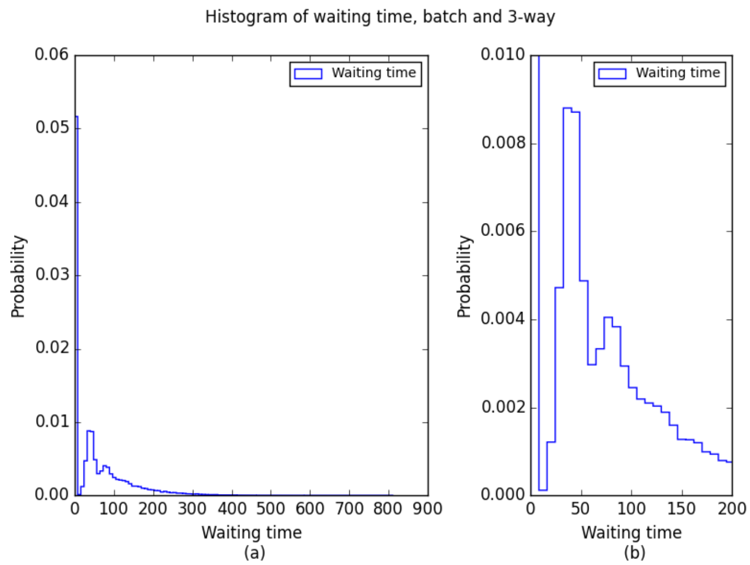

Figure 3.4: Waiting time under batch with 3-way exchange.

Axes truncated in (b)

3-way exchange. Figure 3.4 summarizes the histogram of waiting time under batch with 3-way exchange. It shows an empirical mean $\hat{\mu} = 56.94$ and variance $\hat{\sigma^2} = 5683.56$. The asymptotic theoretical mean and variance do not perform well in this case because the capacity is not large enough. However, Table 3.1 shows that sample mean and variance are very close to asymptotic statistics when p is smaller.

Waiting time distribution. It is observed from the numerical study that both pairwise and 3-way exchange exhibit similar behaviors in waiting time distribution.

When the matching probability p is extremely small, almost all participants are discharged. Histogram for the waiting time, in this case, behaves as if an impulse (Figure 3.5 (a)) at time zero. When p is relatively small, departures from the system are a good mixture of discharged and matched participants. During the transition from this phase to the next phase when no participants are discharged, the waiting time distribution can be approximated by an exponential distribution (Figure 3.5 (b)). While p becomes larger, all participants are able to find at least one opportunity, but a considerable proportion of participants has to wait for more than one exchange round. Figure 3.5 (c) displays waiting time histogram under this circumstance. It appears similar to a damped sinusoidal function. The $i - th$ peak represents the waiting time of participants who have to attend i rounds of exchange before matched. When p is large enough so that all participants will be able to exchange at the first round, waiting time distribution again looks like an impulse function (Figure 3.5 (d)).

Table 3.1: Comparison between empirical and asymptotically statistics

Matching probability	Sample Mean	Asymptotical mean
0.001	0.000890124894031	0.00100033338891
0.005	0.140708133014	0.130318363149
0.01	1.30579511782	1.39561242509

Matching probability	Sample variance	Asymptotical variance
0.001	0.00123483144785	0.00400333455587
0.005	0.622143479477	0.554546123762
0.01	8.39567653258	8.92579616649

Asymptotical result for $p > 0.01$ does not perform reasonably. In case when $0.01 < p < 0.04$, improvement is possible if the following terminating probability is used for the corresponding geometric distribution $\tilde{p} = k^3/C + e^{-k^3/3}$ where $k = p * C$ instead of the asymptotical probability $\tilde{p} \to e^{-k^3/3}$. However, it will not work if $p \geq 0.05$, because the corresponding k will lead to negative expectations. Nevertheless, it is unnecessary to calculate the statistics when $p \geq 0.08$, as the expected waiting time becomes negligible.

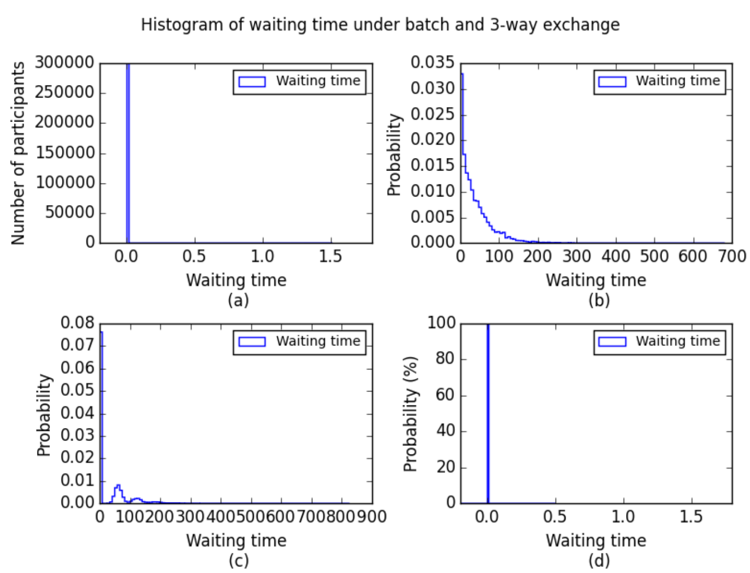

Figure 3.5: Matching result under batch with 3-way exchange.

(a)$p = 0.001$ (b)$p = 0.02$ (c)$p = 0.05$ (d)$p = 0.2$

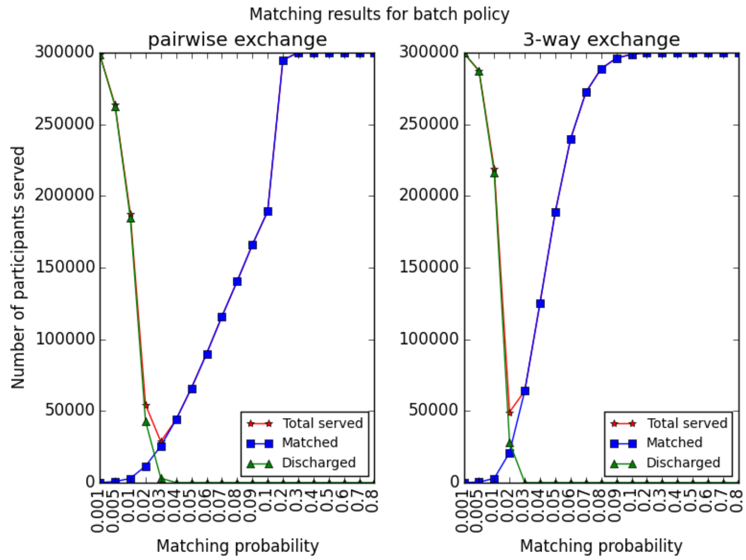

Figure 3.6: Matching results under batch policy.

Evaluation of exchange mechanisms

System performance. Figure 3.6 illustrates the matching results under the pairwise and three-way exchange systems. Similar behaviors are observed in both exchange types. When p is very small, almost all participants are discharged because no exchange opportunity exists in the entire system. When p is relatively small, only a limited number of participants can find an exchange opportunity because of the low matching probability. Meanwhile, the discharge probability is also considerably small, because the value of p is sufficient to guarantee at least one exchange in each round. As a result, the overall probability of participants leaving the system is limited, as also is the turnover rate of the system. The decreasing turnover rate explains the drop in the total number of customers served when p increases from extremely small to relatively small. While the matching probability continues to increase, suddenly almost all participants are able to exchange. Clearly, a system that performs beyond the sudden turning point is preferred, regardless of the exchange type.

The above observation conforms with the theoretical sharp threshold of a random graph. It has been stated (e.g., in [7]) that a random graph denoted by $G(n, p)$ almost surely (a) has no connected components of size larger than $O(log(n))$ when $p < 1/n$, (b) has a largest component that has a size of order $n^{2/3}$ when $p = 1/n$, (c) has a unique giant component containing a positive fraction of the vertices when $np \to C > 1$, (d) contains isolated vertices when $p < \frac{(1-\epsilon)\ln n}{n}$, and (e) is connected when $p > \frac{(1+\epsilon)\ln n}{n}$.

The above facts regarding a random graph support the conclusion that the system performance under both pairwise and three-way exchange can be improved by increasing the capacity or the matching probability. A higher matching probability not only increases the chance that a random participant can find exchange opportunities, but also reduces waiting

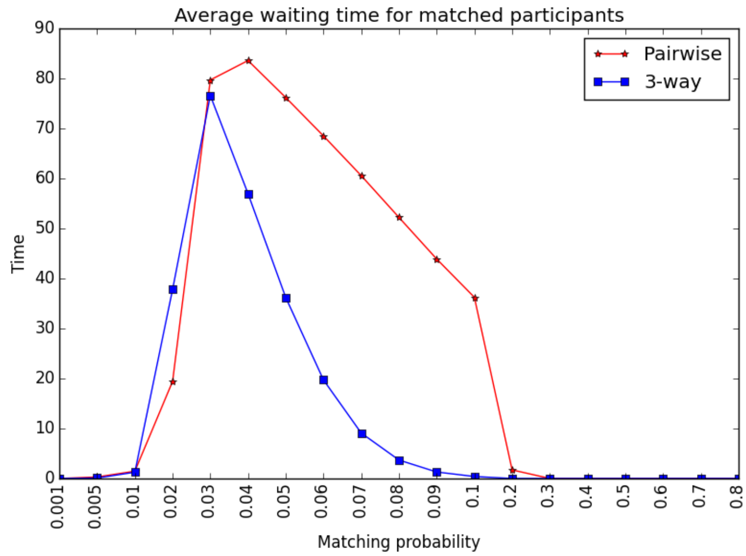

Figure 3.7: Average waiting time for matched participants.

time and increases the turnover rate so that more participants can be served by the corporate barter platform. In fact, because the thresholds are sharp, neither the capacity nor the matching probability has to be increased significantly in order to improve the performance. It is sufficient that $C * p$ be just above the threshold for a dramatically better performance to be observed.

Participants' preference. Figures 3.7 to 3.9 display the statistics for matched participants under both exchange types. It is observed that a three-way exchange results in a reduced waiting time and a higher chance of finding a match most of the time. However, when matching probability p is extremely small, pairwise exchange leads to a better result.

Intuitively, when p tends to zero, the corresponding random graph is extremely loosely connected. Pairwise exchange would then be preferred, because it requires two directed edges for a valid exchange, while a three-way exchange requires three edges. When p increases, the graph exhibits more diversity as well as more possibilities and then a three-way exchange surpasses a pairwise exchange, as the former is less restricted. Consider the case where participants A and B find a one-way exchange opportunity between themselves. Under a three-way exchange, A can essentially invite any one of the remaining $C-2$ participants to barter, as long as this participant is able to complete the directed exchange cycle. On the other hand, under pairwise exchange, A has to collaborate with B and there are no other backup options. Additionally, three-way exchange, denoted by $D(n,p)$, encounters the threshold earlier than pairwise exchange represented by $G(n,p^2)$, and therefore, in a certain range of p, $D(n,p)$ is almost surely connected with a considerable number of three-cycles, and meanwhile, $G(n,p^2)$ remains almost surely isolated vertices. Nevertheless, the difference in performance is eliminated when p continues to increase. When a participant can essentially

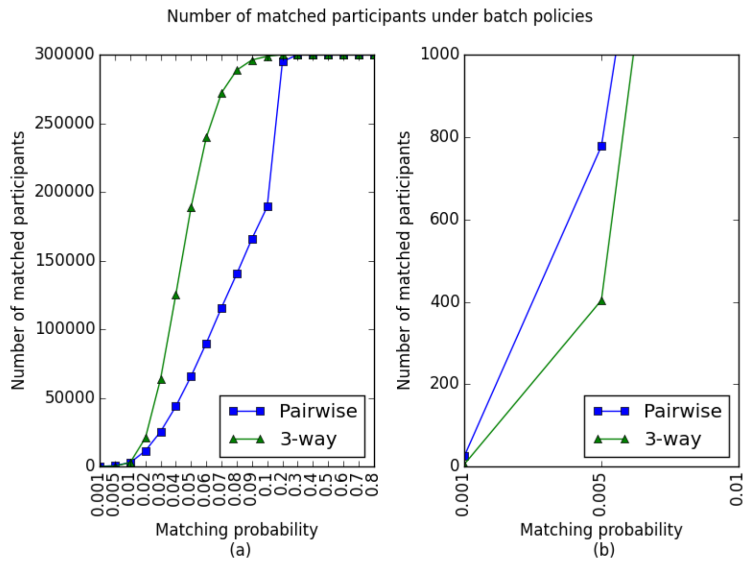

Figure 3.8: Number of matched participants.

The axes are truncated in (b)

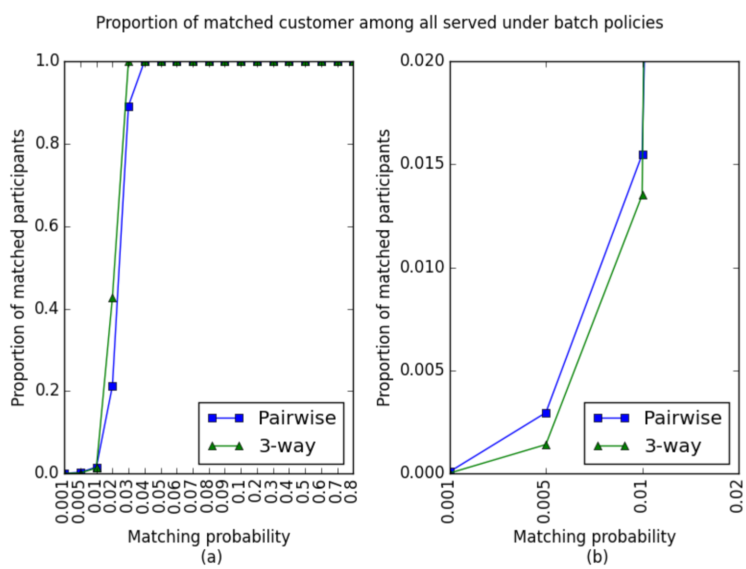

Figure 3.9: Proportion of matched participants.

The axes are truncated in (b)

exchange with any other participant, it makes no difference whether the exchange is pairwise or three-way. A mathematical analysis (not a rigorous proof) to justify the above intuition is provided below.

Let $D_i, i = 2, 3$, denote the event that participants are discharged from pairwise exchange when $i = 2$, and 3-way exchange when $i = 3$, and let $M_i, i = 2, 3$ denote the event that a random participant is able to find exchange opportunities in a round of pairwise exchange when $i = 2$, and 3-way exchange when $i = 3$. Again let p and C represent the homogeneous matching probability and system capacity, respectively. We maintain the relationship that $p = k/C$ with k constant. To justify the intuition, probabilities of leaving the system are compared. Firstly, the probability of being discharged from the system is

$$p(D_2) = (1 - p^2)^{C(C-1)/2} \to (1 - p^2)^{(k/p)^2/2} \to e^{(-k^2/2)}, \tag{3.34}$$

and

$$p(D_3) \to e^{(-k^3/3)} \tag{3.35}$$

by Theorem 5. It is clear that when $0 < k < 3/2$, $e^{(-k^2/2)} < e^{(-k^3/3)}$; when $k > 3/2$, $e^{(-k^3/3)} < e^{(-k^2/2)}$. Therefore potentially it is possible that for large enough C, $p(D_2) < p(D_3)$ for small k and $p(D_3) < p(D_2)$ for large k. On the other hand, for the chance of being involved in a valid exchange,

$$p(M_2) = 1 - (1 - p^2)^{C-1} \to Cp^2 = \frac{k^2}{C} \tag{3.36}$$

by Taylor Expansion, and

$$p(M_3) \to \frac{k^3}{C} \tag{3.37}$$

from Theorem 6. Again it could be the case that for a large enough C, $p(M_2) > p(M_3)$ when k is small and $p(M_3) > p(M_2)$ when k is large.

Collectively, when k is small, the pairwise exchange may result in a lower discharging probably and a higher chance of obtaining exchange opportunity, while when k becomes larger, the 3-way exchange may exhibit better performance.

3.5 Comparison of online and batch policies

We used waiting time as the criterion for comparing models. However, it should be noted that all participants in online models eventually find exchange opportunities, as there is no capacity limitation or discharge activities. For consistency reasons, only matched participants are considered in batch models, and the waiting time in this section is defined as the time between a participant's arrival and departure.

Figure 3.10 shows the average waiting time for matched participants under different models. It indicates that the average waiting time is very large for an extremely small matching probability under online models, although an exchange opportunity is guaranteed eventually. However, when p becomes relatively small, i.e., $p \geq 0.05$, the waiting time

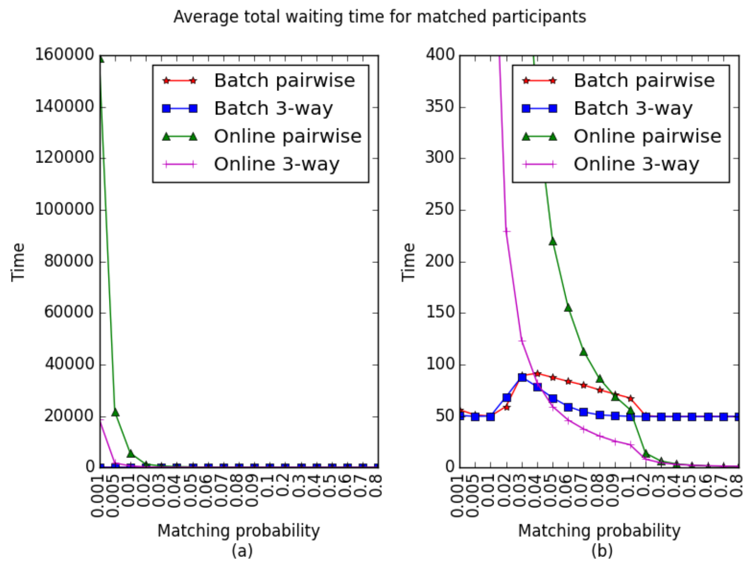

Figure 3.10: Waiting time under different models.

(a) The first three results of online models are based on 1,000,000 arrivals. (b) The axes are truncated. For extremely small matching probability p, the batch with three-way exchange system outperforms the batch with pairwise exchange system, which violates the conclusion in the previous subsection. It should be noted that only six participants were matched under the batch with three-way exchange system; the corresponding data point is highly likely to be inaccurate.

under the online with three-way exchange system surpasses that of batch models. This can be explained by the fact that when the value of p is reasonable, it is likely that most participants in the online with three-way exchange system no longer need to wait. A random arrival probably finds at least one exchange opportunity with a participant waiting in the pool, and thus, can leave the system immediately. In this case, the batch model provides no incentive to participants as they have to wait until there is a sufficiently large number of other participants in the pool. On the other hand, when the matching probability is extremely small, a batch exchange system does provide an incentive to participants, as the waiting time is bounded regardless of the matching result. Participants either find opportunities or are discharged within a short time. Therefore, when p is extremely small, i.e., $p \leq 0.001$, participants can try their luck in the batch with pairwise exchange option. The participant will not incur any disadvantage by attempting to use this option and it will not waste much of his/her time, as he/she will be informed of the result quickly. If the participant succeeds, he/she will obtain a better return than the salvage value, while if he/she fails, he/she can still revert to disposing of the outdated inventory at salvage value. When $0.005 \leq p \leq 0.05$, a time insensitive participant can choose to barter under the online with three-way exchange option, and it will be guaranteed that he/she will find an exchange opportunity within a reasonable time, while a time sensitive participant can alternatively barter under batch policy. When $p > 0.05$, as mentioned, the online policy is always recommended.

Chapter 4

Participant's Strategy under Online with Pairwise Exchange

4.1 Background and assumptions

The practice of trading out inventories in exchange for production materials has a long history. In 1973, when the Arab oil embargo cut off critical supplies of polystyrene, the building block of Styrofoam, Huntsman Chemical Corporation managed to barter in polystyrene for the production of Styrofoam by offering other chemicals that they had, and successfully avoided bankruptcy. A more recent and large-scale example results from the property bubble in China. Intended to have control over the real estate inventory and housing price, state-owned banks in China reduce or even stop providing loans to real estate companies. To survive, real estate company offers unsold houses and apartments in exchange for construction material. Barters related to crude oil is common as well. Besides the exchanges of crude oil and goods between countries, barter of refined oil and gas products for crude oil at the corporate level is the major type of transaction on oil barter exchange platform.

This study continues the first three chapters on barter exchange. In Chapter 2 and Chapter 3, we analyzed participants' waiting time under different exchange mechanisms. In this chapter, built on the above results, we try to study barter exchange from participant's perspective if the participant is provided the expected waiting time and corresponding variance of the waiting time. Particularly, assume that the barter exchange platform operates under online with pairwise exchange. As a recap, online exchange policy attempts to find all exchange opportunities whenever a participant arrives, and pairwise exchange type allows bilateral exchanges only. Major assumptions in this chapter remain the same as that in the previous ones, i.e., homogeneous one-way matching probability p_0, exponentially distributed inter-arrival time with rate ν, and no limits applied to the number of exchanges that one participant can simultaneously participate.

Under the above assumptions, it is revealed in the second chapter that the waiting time distribution of in-pool participant is exponentially distributed with rate $\mu = \nu p_0^2$. Now consider a new arrival to the barter platform. Under online with pairwise exchange, the platform operator can tell immediately if there are existing matching opportunities for the new arrival by a quick examination of the current in-pool participants. Thus we assume that the new arrival is informed instantaneously whether he can be matched upon arrival. If there are no current matching opportunities, the new arrival can either quit immediately or enter the pool for future opportunities. Otherwise, he leaves the system together with matched in-pool participants and negotiates off-line for the details of executing a barter exchange. The above process is slightly different from the online with pairwise exchange described in the second chapter. In Chapter 2, the participant has to enter the pool if there is no match upon arrival; while in this chapter he can choose to continue or quit. Please note that this modification does not affect the waiting time distribution for in-pool participants. Every in-pool participant still attempts to match with every new arrival independently, and the new arrival's decision regarding entering the pool or not does not affect the future matching activities of the current in-pool participants. The modification in the barter process only influences the probability that a new participant can be matched upon arrival. Since there are potentially fewer in-pool participants than the system described in the second chapter, every new arrival faces a higher probability of no matching when arrives the barter platform. However, once he decides to enter the pool for future opportunities, his waiting time remains to be exponentially distributed.

For every barter participant, it is assumed that he would like to get rid of out-dated and slow-moving inventories in exchange for production materials. However, studies in the first three chapters analyze waiting time for in-pool participant but ignore actual exchange volume. Therefore, after an exponentially distributed time, an in-pool participant may be able to barter in sufficient production material and luckily trade out all obsolete inventories, or it is possible that he receives a limited amount of production material and results in recovering only a small proportion of the inventory value that was traded out. Obviously, the randomness in the exchange volume and waiting time contribute to the risk that an in-pool participant faces in a barter exchange. In this chapter, we would like to investigate how this risk could be reduced for an in-pool participant via two different methods, namely, by modifying the procurement process of the participant, and by conducting non-homogeneous barter exchange. The former method tries to hedge the randomness in exchange volume, while the latter one addresses the randomness in waiting time. In short, a careful temporary modification on the procurement process, i.e., the frequency of placing an order with the regular supplier and the amount of production material in each order, helps the participant reduce the chance of too much or too little production material on hand while waiting for a random one-time supply from barter exchange. On the other hand, a non-simultaneous barter exchange is proven (in Chapter 4.4) to be able to reduce total waiting time significantly. Naturally, a shortened waiting time brings down the possibility of running out of production materials while waiting on the platform and reduces the time for carrying out-dated inventories. Thus non-simultaneous exchange can be used as a tool to control the risk faced by barter participants. Please note that in this chapter we do not address the risk encountered by participants that can be matched upon arrival since they experience minimum risk if there is any.

Preliminary result shows that for a participant who implements the optimal EOQ model with no back-order for his procurement of production materials, upon satisfaction of certain conditions, a good strategy is maintaining the current EOQ procurement plan and always waiting for future exchange opportunities if he cannot be matched upon arrival. Since there is no change in the procurement plan (and no changes as well in other parts of the production), it won't hurt to barter, and potentially the participant can wait forever for a future exchange opportunity. It is also remarkable that, under some conditions, the above conclusion is independent of the exchange volume and the waiting time. More specifically, for any well-defined distributions of waiting time and exchange volume, if the saving in variable cost outweighs the increase in holding cost from trade-in activity, it is always true that participant should wait for future barter opportunity and maintain his regular procurement plan while staying on the platform. It implies as well that the above conclusion applies to any barter mechanism that the platform implements. In addition, if back-order is allowed while waiting on the barter platform, a better temporary procurement plan may be available and achieves a lower cost than that of the optimal procurement plan without back-order. We developed a sufficient condition that guarantees optimality of temporary procurement plan with back-order over no back-order, and proved that the sufficient condition is necessary as well if the participant's cost function for ordering cost, holding cost, and penalty cost is convex. We also find that non-simultaneous barter exchange can reduce waiting time. Since the waiting time does not fundamentally affect trade-in process (procurement) if the participant continues with the optimal EOQ, the significance of shortened waiting time lies in the trade-out process. Apparently, for a participant who feels urgent to get rid of obsolete inventories, a reduced waiting time for trade-out would be more than welcome. Hence we suggest non-simultaneous barter exchange for all participants if possible.

The next three sections dive into math analysis of each strategy mentioned above. Before we conclude this study, one additional chapter is provided to analyze the convexity of the cost function when back-order is allowed.

4.2 Temporary procurement plan with no back-order

This section presents analytical results of risk control from temporary procurement plan. We mainly focus on trade-in process, because it has a significant impact on production and can potentially lead to severe production disruption if the procurement plan is poorly coordinated with barter process. On the other hand, the trade-out process has a much less influence as these outdated inventories do not directly interact with current production plan; the obsolete inventories, in most cases, do not fit in the market and are waiting to be recycled. Also, trade-in process requires careful math analysis, while trade-out can be modeled easily. As shown in Chapter 4.4, the latter is mathematically equivalent to the difference between (a) (time discounted) expected trade-out volume/value and (b) (time discounted) linear holding cost with regard to time. Therefore in this section, we primarily focus on the coordination of procurement plan and trade-in process which work together to reduce the risk in a barter exchange. The term *Inventory* in this section refers to production material unless specified.

To simplify the model, assume that barter participant implements EOQ to procure production material from his regular supplier. Parameters and assumptions of EOQ model are listed as follows.

- C represents the cost per item.
- h represents constant holding cost per item per unit time
- D represents constant consumption rate of the production material.
- K represents ordering cost.
- Zero lead time assumed.
- No back order allowed.

Based on above parameters and EOQ settings, participant's minimum cost per unit time is $C^{(0)} = \sqrt{2KDh} + DC$. Clearly, $C^{(0)}$ performs as a baseline for the decision of entering the pool or not if the participant can not find matching opportunity upon arrival. If the expected cost per unit time during his stay on the barter platform is no more than $C^{(0)}$, the participant will benefit from barter exchanges; otherwise, it is better to quit once he is informed that there is no current opportunity when arrives.

To calculate the cost generated while waiting in the pool, assume that participant enters the pool and waits for future opportunities at time $T = 0$. For simplicity, we further assume that the participant has just used up production materials at $T = 0$ and is about to place a series of temporary orders to his regular supplier to hedge the risk while waiting for barter opportunity. These temporary orders follow the similar ordering pattern of EOQ but are not necessarily at the usual optimal order quantity. More specifically, each temporary order placed to the supplier has an order quantity Q, and order is placed every $t = Q/D$ units of time. The temporary orders are in effect until a trade-in happens. When the participant uses up inventories from both temporary orders and trade-in activity, he restores the regular procurement plan, namely, optimal EOQ with no back-order. Figure 4.1 shows the inventory changes when a trade-in opportunity is found at the third temporary order.

Benefited from the memoryless property of the waiting time, it is sufficient to model one temporary order, instead of a series of temporary orders. For instance, if Q denotes the order quantity of the temporary orders, then the participant places an order of Q units of production material at $t = 0$, the beginning of the waiting time for a barter exchange. If no exchange opportunity is found before $t = Q/D$, due to the memoryless property of the waiting time, the remaining waiting time is again exponentially distributed with rate $\mu = \nu p_0^2$. Therefore, the participant can place a second order with Q units of production material and wait for another $t = Q/D$ unit of time with the inventory from the second order. The temporary order denoted by Q can repeatedly be placed until an exchange opportunity

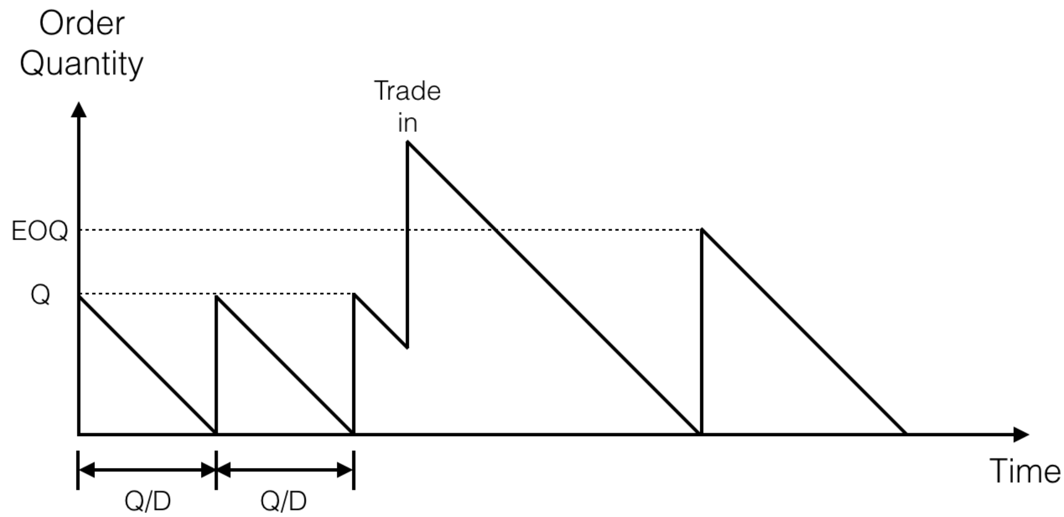

Figure 4.1: Changes in inventory position.

is found. We are intended to model the inventory changes and the corresponding cost in one temporary order. Nevertheless, the optimal strategy derived from one cycle remains optimal if it is repeated for the whole temporary period.

A remarkable fact of barter exchange is that, the participant does not pay for the items traded in. Therefore, it is assumed that the variable cost per item traded in is zero. Additionally, there is no fixed/setting-up cost for trade-in activities. Not all barter platforms ask for per transaction cost. Even if many platforms charge it, the cost can be re-structured so that this fixed cost applies to the trade-out activity. However, fixed cost and variable cost do apply to orders placed to the regular supplier. We assume that barter exchange can be conducted instantaneously when the opportunities are found. Let Y denote the actual trade-in volume of production materials and assume that it is bounded by constants. As a summary, the decision variables and additional parameters of the model are

- Decision variable $Q > 0$, denotes the constant order quantity in temporary orders.
- Parameter Y represents the volume of production material traded-in. Y is defined on $[Y_1, Y_2]$, where Y_1 and Y_2 are constants. Y is independent of Q.
- $g(\cdot)$ denotes pdf of Y.
- Parameter t_1 represents the time when barter opportunity is found. t_1 is independent of Q. From Chapter 2, t_1 follows exponential distribution with rate $\mu = \nu p_0^2$ if the exchange mechanism is online with pairwise exchange.
- $f_1(\cdot)$ and $\bar{F}_1(\cdot)$ represent pdf and tail distribution of t_1, respectively.

If the participant can find exchange opportunity before using up inventories from the current temporary order, the average cost per unit time within this inventory cycle is

$$C_1(Q, Y, t_1) = \frac{QCD + KD + \frac{1}{2}Q^2h + h(Q - t_1D)Y + \frac{1}{2}hY^2}{Q + Y}. \tag{4.1}$$

On the other hand, if no exchange opportunity can be found before the end of current inventory cycle, the average cost per unit time within the cycle is

$$C_2(Q) = \frac{KD}{Q} + \frac{1}{2}Qh + DC. \tag{4.2}$$

Particularly, define $Q^{EOQ} = \sqrt{\frac{2KD}{h}}$, the optimal EOQ quantity, then $C_2(Q^{EOQ}) = C^{(0)} = \sqrt{2KDh} + DC$.

The expected cost per unit time within a temporary order cycle is

$$E[C^{(1)}(Q)] = \int_{t_1=0}^{t_1=\frac{Q}{D}} \int_{Y=Y_1}^{Y=Y_2} C_1(Q, Y, t_1) g(Y) f_1(t_1) dY dt_1 + \int_{t_1=\frac{Q}{D}}^{t_1=\infty} C_2(Q) f_1(t_1) dt_1, \tag{4.3}$$

and the optimal strategy in this case can be modeled as

$$\begin{aligned} \min \quad & E[C^{(1)}(Q)] \\ s.t. \quad & Q > 0 \end{aligned} \tag{4.4}$$

Unfortunately, we were not able to derive a closed-form optimal solution for the above optimization problem. However, we do propose a simple but effective strategy for participants. That is, *no change if no barter opportunity*. We suggest that participant maintain his regular procurement activities, i.e., optimal EOQ, while waiting on the barter platform. A concise sufficient condition is developed to justify if the strategy is beneficial.

Theorem 7. *If*

$$\int_{t_1=0}^{t_1=\sqrt{\frac{2K}{hD}}} \int_{Y=Y_1}^{Y=Y_2} \frac{-(ht_1 + C)D + \frac{1}{2}hY}{\sqrt{\frac{2KD}{h}} + Y} Y g(Y) f_1(t_1) dY dt_1 < 0, \tag{4.5}$$

then $E[C^{(1)}(Q^{EOQ})] < C^{(0)}$.

Proof. Since

$$
\begin{aligned}
& E[C^{(1)}(Q^{EOQ})] - C^{(0)} \\
= & \int_{t_1=0}^{t_1=\frac{Q^{EOQ}}{D}} \int_{Y=Y_1}^{Y=Y_2} C_1(Q^{EOQ}, Y, t_1) g(Y) f_1(t_1) dY dt_1 \\
& + \int_{t_1=\frac{Q^{EOQ}}{D}}^{t_1=\infty} C_2(Q^{EOQ}) f_1(t_1) dt_1 - C^{(0)} \\
= & \int_{t_1=0}^{t_1=\frac{Q^{EOQ}}{D}} \int_{Y=Y_1}^{Y=Y_2} C_1(Q^{EOQ}, Y, t_1) g(Y) f_1(t_1) dY dt_1 \\
& + \int_{t_1=\frac{Q^{EOQ}}{D}}^{t_1=\infty} \int_{Y=Y_1}^{Y=Y_2} C_2(Q^{EOQ}) g(Y) f_1(t_1) dY dt_1 \\
& - \int_{t_1=0}^{t_1=\infty} \int_{Y=Y_1}^{Y=Y_2} C^{(0)} g(Y) f_1(t_1) dY dt_1 \\
= & \int_{t_1=0}^{t_1=\frac{Q^{EOQ}}{D}} \int_{Y=Y_1}^{Y=Y_2} [C_1(Q^{EOQ}, Y, t_1) - C^{(0)}] g(Y) f_1(t_1) dY dt_1 \\
= & \int_{t_1=0}^{t_1=\sqrt{\frac{2K}{hD}}} \int_{Y=Y_1}^{Y=Y_2} \frac{-(ht_1 + C)D + \frac{1}{2}hY}{\sqrt{\frac{2KD}{h}} + Y} Y g(Y) f_1(t_1) dY dt_1,
\end{aligned}
\tag{4.6}
$$

if the last line results in a negative value, then $E[C^{(1)}(Q^{EOQ})] < C^{(0)}$. □

It is remarkable to find that, when the sufficient condition is satisfied, maintaining the regular procurement activity is a good option, though not necessarily optimal. The result implies that it is always beneficial to barter upon satisfaction of the sufficient condition, and that the participant can simply continue with the optimal EOQ while waiting on the platform. Since there is no difference between the temporary and regular procurement processes, it indicates that, with the satisfaction of the condition, the participant should always wait for future opportunities if there is none upon arrival, and that the participant can potentially wait forever on the platform.

Additionally, the sufficient condition stated in Theorem 7 can be further simplified to a form that is independent of the waiting time and exchange volume.

Corollary 7.1. *If $\frac{1}{2}hY_2 < CD$, then $E[C^{(1)}(Q^{EOQ})] < C^{(0)}$ true for any probability distributions of t_1 and Y.*

Proof. If $\frac{1}{2}hY_2 < CD$, then $-(ht_1 + C)D + \frac{1}{2}hY < 0, \forall Y \in [Y_1, Y_2], t_1 > 0$. Hence,

$$
\int_{t_1=0}^{t_1=\sqrt{\frac{2K}{hD}}} \int_{Y=Y_1}^{Y=Y_2} \frac{-(ht_1 + C)D + \frac{1}{2}hY}{\sqrt{\frac{2KD}{h}} + Y} Y g(Y) f_1(t_1) dY dt_1 < 0 \tag{4.7}
$$

for any probability density functions $f_1(\cdot)$ and $g(\cdot)$. $\square$

The corollary delivers a clear insight. As C represents the cost per item purchased and D denotes the demand per unit time, CD stands for the total variable cost per unit time. On the other hand, $\frac{1}{2}hY_2$ presents an upper bound of additional holding cost incurred from trade-in activities. Hence, if the saving in variable cost outweighs the increase in holding cost, participants should always barter and maintain optimal EOQ procurement plan when waiting for an exchange opportunity, regardless of the waiting time distribution and exchange volume. Even through the waiting time distribution may not possess memoryless property anymore, the remaining waiting time is still a well-defined probability distribution. Therefore the cost described by (4.3) stays true for every single temporary order with the corresponding waiting time distribution, and Corollary 7.1 holds for the entire temporary period.

4.3 Temporary procurement plan with back-order

A natural extension from the last section is that whether back-order during the temporary period will be beneficial. If the answer to the question is positive, then we further would like to know if the optimal solution of EOQ with back-order remains a good option among the feasible temporary procurement plans. That is, can we expect the same strategy as in the last section, *no change if no barter opportunity*? Luckily, we are able to address the first question. In this section, we discuss, in general, if back-order during the temporary procurement plan would be better than no back-order allowed. However, we could not obtain a clear insight regarding the implementation of optimal EOQ with back-order as the temporary procurement plan, so the second question remains unanswered.

The model in this section is similar to that in Chapter 4.2 except for the possibility of back-order. Back-order is allowed with a penalty cost p per item per unit time. We model exactly one temporary order with order quantity Q. Assume that the length of a temporary order cycle is t units of time. t and Q are decision variables that we aim to optimize. For analytical simplicity, it is assumed that $Q/D \leq t \leq \alpha Q/D$, and $\underline{Q} \leq Q \leq \bar{Q}$, where $\alpha > 1$ is a constant, and $\underline{Q} \leq Q^{EOQ} \leq \bar{Q}$ are positive constants as well. For analytical reason, we further assume that Y_1, the lower bound of trade-in volume, is a constant that is larger than or equals to $\alpha\bar{Q}$. Finally, it is assumed that the trade in volume is uniformly distributed within $[Y_1, Y_2]$, i.e., $g(y) = 1/(Y_2 - Y_1), \forall y \in [Y_1, Y_2]$. Other parameters and variables remain the same as in Chapter 4.2.

Obviously, in this model, if the participant finds exchange opportunity before the end of the current cycle, it can happen before or after using up the inventories from the temporary order. The expected cost per unit time corresponding to a barter exchange conducted before inventory level reaches zero is

$$C_3(Q, Y, t_1) = \frac{QCD + KD + \frac{1}{2}Q^2h + \frac{1}{2}(Y + Q - t_1D)^2h - \frac{1}{2}(Q - t_1D)^2h}{Q + Y}, \tag{4.8}$$

and the expected cost per unit time when the exchange happens during back-order is

$$C_4(Q, Y, t_1) = \frac{QCD + KD + \frac{1}{2}Q^2 h + \frac{1}{2}(t_1 D - Q)^2 p + \frac{1}{2}(Y + Q - t_1 D)^2 h}{Q + Y}. \tag{4.9}$$

In case no opportunity can be found before time t, the corresponding cost is

$$C_5(Q, t) = \frac{tD^2 C + KD + \frac{1}{2}Q^2 h + \frac{1}{2}(tD - Q)^2 p}{tD}. \tag{4.10}$$

Hence the expected cost per unit time is

$$\begin{aligned} E[C^{(2)}(Q, t)] = & \int_{t_1=0}^{t_1=\frac{Q}{D}} \int_{Y=Y_1}^{Y=Y_2} C_3(Q, Y, t_1) g(Y) f_1(t_1) dY dt_1 \\ & + \int_{t_1=\frac{Q}{D}}^{t_1=t} \int_{Y=Y_1}^{Y=Y_2} C_4(Q, Y, t_1) g(Y) f_1(t_1) dY dt_1 \\ & + \int_{t_1=t}^{t_1=\infty} C_5(Q, t) f_1(t_1) dt_1. \end{aligned} \tag{4.11}$$

Now the model can be formulated as

$$\begin{aligned} \min \quad & E[C^{(2)}(Q, t)] \\ s.t. \quad & \frac{Q}{D} \le t \le \alpha \frac{Q}{D} \\ & \underline{Q} \le Q \le \bar{Q} \end{aligned} \tag{4.12}$$

A few remarks on (4.12) and (4.4). Similar to the model in Chapter 4.2, due to memoryless property of the waiting time, when an optimal solution is available, it leads to a temporary procurement plan that can be extended and repeatedly applied until a barter exchange opportunity appears. Additionally, $C_3(Q, Y, t_1) = C_1(Q, Y, t_1)$, and $C_5(Q, \frac{Q}{D}) = C_2(Q)$. Therefore, when (4.12) has an optimal solution on the boundary $t = \frac{Q}{D}$, the objective function $E[C^{(2)}(Q, \frac{Q}{D})]$ is equivalent to $E[C^{(1)}(Q)]$ in (4.4). It is clear that (4.12) and (4.4) implies the same optimal procurement plan if the optimal solution of (4.12) lies on the boundary $t = \frac{Q}{D}$. It further implies that if (4.12) has an optimal solution that does not lie on $t = \frac{Q}{D}$, it should have a better (smaller) objective function value compared with that in (4.4).

Due to the complexity of (4.12), it seems that there is no single statement that guarantees a strictly better result with back-order (defined by (4.12)) than no back-order (defined by (4.4)). Nevertheless, we are able to develop a concise sufficient condition such that (4.12) delivers better value than (4.4). Moreover, the sufficient condition becomes necessary if (4.12) is a convex optimization problem. Thus, it would be ideal if it could be proven that (4.12) is indeed a convex optimization problem. Unfortunately, it turns out that convexity is difficult to obtain. Therefore, in Chapter 5, we derived a set of sufficient (but not necessary) conditions such that the objective function of (4.12) will be convex within the feasible region if the conditions are satisfied.

Theorem 8. *$\forall (Q^c, t^c) \in \{(Q,t)|\underline{Q} \le Q \le \bar{Q}, t = \frac{Q}{D}\}$, if there exists a point $(Q^0, t^0) \in \{(Q,t)|\underline{Q} \le Q \le \bar{Q}, \frac{Q}{D} < t \le \alpha\frac{Q}{D}\}$, s.t.*

$$\lambda \underline{v} \cdot \nabla E[C^{(2)}(Q^c, t^c)] \le -a^0 < 0, \tag{4.13}$$

for an $a^0 > 0, 0 < \lambda \le 1$, and $\underline{v} \triangleq (Q^0, t^0) - (Q^c, t^c)$; then the optimal solution of (4.12) does not lie on $t = \frac{Q}{D}$.

Corollary 8.1. *If (4.12) is convex within the feasible region, and the optimal solution does not lie on $t = \frac{Q}{D}$, then $\forall (Q^c, t^c) \in \{(Q,t)|\underline{Q} \le Q \le \bar{Q}, t = \frac{Q}{D}\}$, there exists a point $(Q^0, t^0) \in \{(Q,t)|\underline{Q} \le Q \le \bar{Q}, \frac{Q}{D} < t \le \alpha\frac{Q}{D}\}$, s.t.*

$$\lambda \underline{v} \cdot \nabla E[C^{(2)}(Q^c, t^c)] \le -a^0 < 0, \tag{4.14}$$

for an $a^0 > 0, 0 < \lambda \le 1$, and $\underline{v} \triangleq (Q^0, t^0) - (Q^c, t^c)$

Before the rigorous proofs of the theorem and corollary, we would like to deliver a few remarks that facilitate the understanding of the sufficient condition stated in the theorem. The intuition for Theorem 8 and the corollary comes from Taylor expansion. Consider a point $(Q, t = \frac{Q}{D})$. If there exists a point (Q^0, t^0) in the feasible region that does not lie on $t = \frac{Q}{D}$, and that $[(Q^0, t^0) - (Q, t = \frac{Q}{D})] \cdot \nabla E[C^{(2)}(Q, t = \frac{Q}{D})] < 0$, then by Taylor expansion,

$$E[C^{(2)}(Q^0, t^0)] = E[C^{(2)}(Q, \frac{Q}{D})] + [(Q^0, t^0) - (Q, \frac{Q}{D})] \cdot \nabla E[C^{(2)}(Q, \frac{Q}{D})] + o(\cdot). \tag{4.15}$$

Since the last term $o(\cdot)$ is negligible, and the second term on the right hand side is negative by assumption, it can be concluded that $E[C^{(2)}(Q^0, t^0)] < E[C^{(2)}(Q, \frac{Q}{D})]$. Hence, for any point on $t = \frac{Q}{D}$, if the sufficient condition is satisfied, there must be a point not on the line $t = \frac{Q}{D}$ that provides strictly smaller (better) value than the point on the line. Thus any point on the line $t = \frac{Q}{D}$ can not be the optimal solution for (4.12), which implies that the optimal solution does not lie on $t = \frac{Q}{D}$.

Proof. Theorem 8

Since $E[C^{(2)}(Q,t)]$ is differentiable on $\{(Q,t)|Q > 0, t > 0\}$, by definition,

$$\lim_{||h|| \to 0} \frac{||E[C^{(2)}((Q,t) + h)] - E[C^{(2)}(Q,t)] - \nabla E[C^{(2)}(Q,t)] \cdot h||}{||h||} = 0. \tag{4.16}$$

Then by the definition of limit, $\forall \epsilon > 0, \exists \delta > 0, s.t. \forall ||h|| < \delta$, we have

$$||E[C^{(2)}((Q^c, t^c) + h)] - E[C^{(2)}(Q^c, t^c)] - \nabla E[C^{(2)}(Q^c, t^c)] \cdot h|| \le \epsilon \tag{4.17}$$

.

Define $h = \lambda \underline{v}$. Since

$$E[C^{(2)}((Q^c, t^c) + h)] - E[C^{(2)}(Q^c, t^c)] - \nabla E[C^{(2)}(Q^c, t^c)] \cdot h \tag{4.18}$$

is a scaler, we have

$$\begin{aligned}&E[C^{(2)}((Q^c,t^c)+\lambda\underline{v})]-E[C^{(2)}(Q^c,t^c)]-\nabla E[C^{(2)}(Q^c,t^c)]\cdot\lambda\underline{v}\leq\epsilon\\ \Rightarrow&E[C^{(2)}((Q^c,t^c)+\lambda\underline{v})]\leq E[C^{(2)}(Q^c,t^c)]+\nabla E[C^{(2)}(Q^c,t^c)]\cdot\lambda\underline{v}+\epsilon.\end{aligned}\tag{4.19}$$

Making $\epsilon < a^0$, then there is $E[C^{(2)}((Q^c,t^c)+\lambda\underline{v})] < E[C^{(2)}(Q^c,t^c)]$. Note that the feasible region is convex, therefore $(Q^c,t^c)+\lambda\underline{v}$ is feasible as well. □

Proof. Corollary 8.1

Let (Q^0,t^0) denote the optimal solution. Then $E[C^{(2)}(Q^0,t^0)] < E[C^{(2)}(Q^c,t^c)]$. By Taylor expansion,

$$E[C^{(2)}(Q^0,t^0)]=E[C^{(2)}(Q^c,t^c)]+\underline{v}\cdot\nabla E[C^{(2)}(Q^c,t^c)]+R(Q^1,t^1),\tag{4.20}$$

where $R(Q^1,t^1)$ denotes the remainder of Taylor expansion. Since the feasible region is convex, (Q^1,t^1) is feasible as well. Due to the convexity of the objective function, $R(Q^1,t^1)\geq 0$. Hence $\underline{v}\cdot\nabla E[C^{(2)}(Q^c,t^c)]<0$ □

Actually, Theorem 8 can be further simplified. $\forall(Q,t)\in\{(Q,t)|\underline{Q}<Q<\bar{Q},t=\frac{Q}{D}\}$, direction $[-1,D]^T$ is the only direction that cannot find a vector $\underline{v}$ pointing into the feasible region such that the dot product of the two vectors, $\underline{v}\cdot[-1,D]^T$, is negative (see Figure 4.2). Hence, for interior points on line $t=\frac{Q}{D}$, the sufficient condition in Theorem 8 is satisfied as long as $\nabla E[C^{(2)}(Q,t=\frac{Q}{D})]\neq[-1,D]^T$.

Corollary 8.2. *$\forall(Q^c,t^c)\in\{(Q,t)|\underline{Q}<Q<\bar{Q},t=\frac{Q}{D}\}$, if*

$$\nabla E[C^{(2)}(Q^c,t^c)]\neq\begin{bmatrix}-1\\D\end{bmatrix},\tag{4.21}$$

then (Q^c,t^c) is not optimal

Clearly, convexity of (4.12) can significantly simplify the problem. When the values of all parameters are available, convexity can be easily justified by plugging the values into (4.12). However, when the parameters are not all accessible, it may require considerable efforts to obtain convexity as shown in the Chapter 5.

4.4 Non-simultaneous barter exchange

This section studies the benefits from non-simultaneous barter exchange. As revealed in the first three chapters, a one-way exchange may worsen the inventory or financial situation one is facing, because a one-way trade-in does not solve problems associated with outdated inventory while a one-way trade-out does not recover value from the inventory traded out. Model in this section analyzes benefits from separate trade-in and trade-out processes for

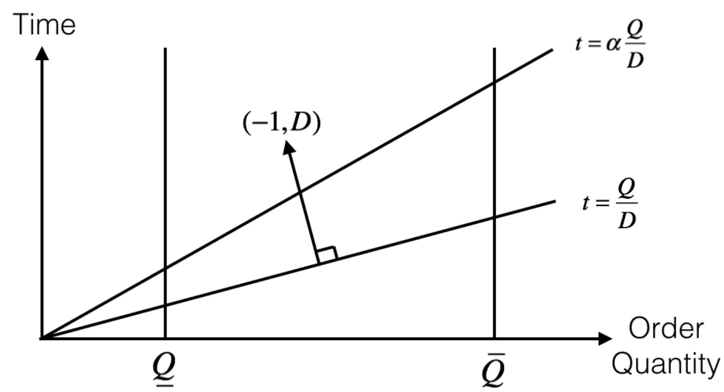

Figure 4.2: Simplification of Theorem 8.

in-pool participants and attempts to justify if the benefits outweigh side effects and reduce the risk faced by an in-pool participant.

Intuitively, a separate trade-in and trade-out should result in an overall reduced waiting time, because simultaneity restriction is relaxed. Analytically, in a pairwise exchange, if it is assumed that trade-in and trade-out are separate and independent processes, then by an almost identical proof of Theorem 1, an in-pool participant will wait for an exponentially distributed time with rate $\mu' = \nu p_0$ to complete his trade-in process; and at the same time, he has to wait an exponentially distributed time with again rate $\mu' = \nu p_0$ to finish the trade-out process. The total waiting time is the longer one between these two exponential waiting times. It should be much less than the waiting time in an exchange with simultaneous trade-in and trade-out, which is exponential with rate $\mu = \nu p_0^2$.

Since Chapter 4.2 shows that the procurement and production processes are not affected by waiting time if the participant continues with the optimal EOQ, in this section, we focus on trade-out process. The term *Inventory* in this section refers to the outdated inventory that the participant carries. Apparently, the model remains the same as in Chapter 4.2 except for the following parameters.

- Parameter t_2 represents the time when trade-out opportunity is found. t_2 follows exponential distribution with rate μ' if the platform operates under online with pairwise exchange mechanism.
- $f_2(\cdot)$ and $\bar{F}_2(\cdot)$ represent pdf and tail distribution of t_2, respectively.
- Parameter Z represents the random trade-out volume.
- $l(\cdot)$ denotes pdf of Z.
- s represents the value recovered from each item traded out.

The potential gain from simultaneous barter exchange is

$$E[C_6] = s\int Zl(Z)dZ - h\int Zl(Z)dZ\int t_1 f_1(t_1)dt_1, \tag{4.22}$$

while for a non-simultaneous exchange, the expected gain is

$$E[C_7] = s\int Zl(Z)dZ - h\int Zl(Z)dZ\int t_2 f_2(t_2)dt_2. \tag{4.23}$$

Clearly the difference is denoted by

$$\begin{aligned} E[C_7] - E[C_6] &= h\int Zl(Z)dZ\int t_1 f_1(t_1)dt_1 - h\int Zl(Z)dZ\int t_2 f_2(t_2)dt_2 \\ &= (\frac{1}{\mu} - \frac{1}{\mu'})h\int Zl(Z)dZ \\ &= \frac{1}{\mu'}\frac{1-p_0}{p_0}h\int Zl(Z)dZ \\ &> 0. \end{aligned} \tag{4.24}$$

Therefore the non-simultaneous exchange is beneficial to all in-pool participants.

Chapter 5

Convexity of Participant's Cost Function with Back-Order

5.1 Sufficient condition for cost function being convex

In order to justify the convexity of model (4.12), we would like to firstly deliver a few calculation results. For the objective function in (4.12), by Leibniz rule, we have

$$\begin{aligned}
\frac{\partial E[C^{(2)}(Q,t)]}{\partial t} &= \int_{Y=Y_1}^{Y=Y_2} C_4(Q,Y,t)g(Y)f_1(t)dY + \frac{\partial C_5(Q,t)}{\partial t}\bar{F}_1(t) - C_5(Q,t)f_1(t), \\
&= \mu e^{-\mu t}\int_{Y_1}^{Y_2} \frac{QCD + KD + \frac{1}{2}(Q+Y)^2 h + \frac{1}{2}(p+h)(Q-tD)^2 - YtDh}{Q+Y} g(Y)dY \\
&+ \{-[K + \frac{Q^2(p+h)}{2D}]\frac{1}{t^2} + \frac{1}{2}DP\}e^{-\mu t} \\
&- \{(DC - QP) + [K + \frac{Q^2(p+h)}{2D}]\frac{1}{t} + \frac{1}{2}DPt\}\mu e^{-\mu t}
\end{aligned} \tag{5.1}$$

$$\begin{aligned}\frac{\partial E[C^{(2)}(Q,t)]}{\partial Q} &= \int_{t_1=0}^{t_1=\frac{Q}{D}}\int_{Y_1}^{Y_2}\frac{\partial C_3(Q,Y,t_1)}{\partial Q}g(Y)f_1(t_1)dYdt_1\\ &+\int_{t_1=\frac{Q}{D}}^{t_1=t}\int_{Y_1}^{Y_2}\frac{\partial C_4(Q,Y,t_1)}{\partial Q}g(Y)f_1(t_1)dYdt_1\\ &+\frac{\partial C_5(Q,t)}{\partial Q}\bar{F_1}(t)\\ &=\int_{t_1=0}^{t_1=t}\int_{Y=Y_1}^{Y=Y_2}[\frac{1}{2}h+\frac{YCD-KD+Yt_1Dh}{(Q+Y)^2}]g(Y)f_1(t_1)dYdt_1\\ &+\int_{t_1=\frac{Q}{D}}^{t_1=t}\int_{Y=Y_1}^{Y=Y_2}[\frac{1}{2}(p+h)-\frac{1}{2}\frac{(p+h)(Y+t_1D)^2}{(Q+Y)^2}]g(Y)f_1(t_1)dYdt_1\\ &+\frac{Q(p+h)-tDp}{tD}\bar{F_1}(t).\end{aligned} \tag{5.2}$$

After some calculation, we get

$$\begin{aligned}\frac{\partial E[C^{(2)}(Q,t)]}{\partial t}e^{\mu t} =&\mu\frac{QCD+KD+\frac{1}{2}(p+h)(Q-tD)^2+QtDh}{Y_2-Y_1}log\frac{Q+Y_2}{Q+Y_1}\\ &+\frac{1}{2}\mu h(Q-2tD)+\frac{1}{4}\mu h(Y_1+Y_2)\\ &-(\frac{2KD+(p+h)Q^2}{2t^2D}-\frac{Dp}{2})\\ &-\mu(DC-Qp+\frac{2KD+(p+h)Q^2}{2tD}+\frac{tDp}{2}),\end{aligned} \tag{5.3}$$

$$\begin{aligned}\frac{\partial E[C^{(2)}(Q,t)]}{\partial Q}e^{\mu t} =&\frac{h}{2}(e^{\mu t}-1)+(\frac{Q(p+h)}{tD}-p)\\ &-D[(QC+\frac{Qh}{\mu}+K)(e^{\mu t}-1)-Qht]\frac{1}{(Q+Y_1)(Q+Y_2)}\\ &+D[(C+\frac{h}{\mu})(e^{\mu t}-1)-ht]\frac{1}{Y_2-Y_1}log\frac{Q+Y_2}{Q+Y_1}\\ &-(p+h)\{\frac{D}{\mu}e^{\mu t-\mu\frac{Q}{D}}+[Q-D(\frac{1}{\mu}+t)]\}\frac{1}{Y_2-Y_1}log\frac{Q+Y_2}{Q+Y_1}\\ &+(p+h)\{-\frac{D^2}{\mu^2}e^{\mu t-\mu\frac{Q}{D}}+[\frac{Q^2}{2}+D(\frac{1}{\mu}+t)(\frac{D}{\mu}-Q)+\frac{t^2D^2}{2}]\}\frac{1}{(Q+Y_1)(Q+Y_2)}.\end{aligned} \tag{5.4}$$

Furthermore,

$$\begin{aligned}
&\frac{\partial}{\partial t}(\frac{\partial E[C^{(2)}(Q,t)]}{\partial t}e^{\mu t})\\
=&\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t^2}e^{\mu t}+\mu\frac{\partial E[C^{(2)}(Q,t)]}{\partial t}e^{\mu t}\\
=&\mu D\frac{(p+h)tD-pQ}{Y_2-Y_1}\log\frac{Q+Y_2}{Q+Y_1}-\mu Dh+(2K+\frac{(p+h)Q^2}{D})\frac{1}{t^3}+\mu[(k+\frac{(p+h)Q^2}{2D})\frac{1}{t^2}-\frac{Dp}{2}].
\end{aligned}\tag{5.5}$$

Therefore,

$$\begin{aligned}
&\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t^2}e^{\mu t}\\
=&\mu\{D[(p+h)tD-pQ]-\mu[QCD+KD+\frac{1}{2}(p+h)(Q-tD)^2+QtDh]\}\frac{1}{Y_2-Y_1}\log\frac{Q+Y_2}{Q+Y_1}\\
&+(2K+\frac{p+h}{D}Q^2)(\frac{1}{t^3}+\frac{\mu}{t^2}+\frac{\mu^2}{2t})+\mu^2 DC-\mu D(p+h)-\frac{1}{4}\mu^2 h(Y_1+Y_2)\\
&-\frac{1}{2}\mu^2 hQ-\mu^2 Qp+\mu^2 tDh+\frac{1}{2}\mu^2 tDp.
\end{aligned}\tag{5.6}$$

Additionally,

$$\begin{aligned}
&\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t\partial Q}e^{\mu t}=\frac{\partial}{\partial Q}(\frac{\partial E[C^{(2)}(Q,t)]}{\partial t}e^{\mu t})\\
=&\mu[CD+(p+h)(Q-tD)+tDh]\frac{1}{Y_2-Y_1}\log\frac{Q+Y_2}{Q+Y_1}\\
&-\mu[QCD+KD+\frac{1}{2}(p+h)(Q-tD)^2+QtDh]\frac{1}{(Q+Y_2)(Q+Y_1)}\\
&+\frac{1}{2}\mu(h+2p)-\frac{(p+h)Q}{tD}(\frac{1}{t}+\mu),
\end{aligned}\tag{5.7}$$

thus,

$$
\begin{aligned}
&(\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t \partial Q}e^{\mu t})^2 \\
=&\mu^2[CD+(p+h)Q-tDp]^2(\frac{1}{Y_2-Y_1}\log\frac{Q+Y_2}{Q+Y_1})^2 \\
&+\mu^2[QCD+KD+\frac{1}{2}(p+h)(Q^2+t^2D^2)-QtDp]^2\frac{1}{(Q+Y_2)^2(Q+Y_1)^2} \\
&+[\frac{1}{2}\mu(h+2p)-\frac{(p+h)Q}{tD}(\frac{1}{t}+\mu)]^2 \\
&-2\mu^2[CD+(p+h)Q-tDp][QCD+KD+\frac{1}{2}(p+h)(Q^2+t^2D^2)-QtDp] \\
&*\frac{1}{(Q+Y_2)(Q+Y_1)}\frac{1}{Y_2-Y_1}\log\frac{Q+Y_2}{Q+Y_1} \\
&+2\mu[CD+(p+h)Q-tDp][\frac{1}{2}\mu(h+2p)-\frac{(p+h)Q}{tD}(\frac{1}{t}+\mu)]\frac{1}{Y_2-Y_1}\log\frac{Q+Y_2}{Q+Y_1} \\
&-2\mu[QCD+KD+\frac{1}{2}(p+h)(Q^2+t^2D^2)-QtDp][\frac{1}{2}\mu(h+2p)-\frac{(p+h)Q}{tD}(\frac{1}{t}+\mu)] \\
&*\frac{1}{(Q+Y_2)(Q+Y_1)}
\end{aligned}
\tag{5.8}
$$

Finally,

$$
\begin{aligned}
&\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial Q^2}e^{\mu t}=\frac{\partial}{\partial Q}(\frac{\partial E[C^{(2)}(Q,t)]}{\partial Q}e^{\mu t}) \\
=&\frac{2}{(Q+Y_2)(Q+Y_1)}[-D(C+\frac{h}{\mu})(e^{\mu t}-1)-Dpt+(p+h)Q+(p+h)\frac{D}{\mu}(e^{\mu t-\mu\frac{Q}{D}}-1)] \\
&+\frac{2Q+Y_1+Y_2}{(Q+Y_2)^2(Q+Y_1)^2}[QD(C+\frac{h}{\mu})(e^{\mu t}-1)+KD(e^{\mu t}-1)+QDpt+(p+h)\frac{D^2}{\mu^2}(e^{\mu t-\mu\frac{Q}{D}}-1) \\
&-\frac{p+h}{2}(Q^2+t^2D^2+\frac{2tD^2}{\mu}-\frac{2QD}{\mu})] \\
&+(e^{\mu t-\mu\frac{Q}{D}}-1)\frac{p+h}{Y_2-Y_1}\log\frac{Q+Y_2}{Q+Y_1} \\
&+\frac{p+h}{tD}
\end{aligned}
\tag{5.9}
$$

In case we knew the exact values of the parameters, convexity could be verified by simply plugging-in the numbers into equation (5.6), (5.8), and (5.9), then justifying if the Hessian matrix is positive definite. However, when the values of the parameters are not fully available, it seems that there is no simple way of verifying convexity due to the complexity of the Hessian matrix derived from the objective function in (4.12). We are able to deliver a sufficient condition that guarantees convexity of (4.12). However, this sufficient condition could be very conservative. For notation simplification, define

$$\begin{aligned}
\mathbf{a} &= \mathbf{a}(Q,t) \\
&= (2K + \frac{p+h}{D}Q^2)(\frac{1}{t^3} + \frac{\mu}{t^2} + \frac{\mu^2}{2t}) + \mu^2 DC - \mu D(p+h) - \frac{1}{4}\mu^2 h(Y_1+Y_2) \\
&- \frac{1}{2}\mu^2 hQ - \mu^2 Qp + \mu^2 tDh + \frac{1}{2}\mu^2 tDp \\
&= (2K + \frac{p+h}{D}Q^2)(\frac{1}{t^3} + \frac{\mu}{t^2} + \frac{\mu^2}{2t}) \\
&+ \mu^2[DC - \frac{1}{4}h(Y_1+Y_2) + \frac{1}{2}(p+h)(tD-Q) + \frac{1}{2}tDh - \frac{1}{2}Qp] - \mu D(p+h) \\
\mathbf{b} &= \mathbf{b}(Q,t) = \mu\{D[(p+h)tD - pQ] - \mu[QCD + KD + \frac{1}{2}(p+h)(Q^2+t^2D^2) - QtDp]\} \\
\mathbf{c} &= \mathbf{c}(Q,t) = \frac{p+h}{tD} \\
\mathbf{d} &= \mathbf{d}(Q,t) \\
&= QD(C + \frac{h}{\mu})(e^{\mu t} - 1) + KD(e^{\mu t} - 1) + QDpt + (p+h)\frac{D^2}{\mu^2}(e^{\mu t - \mu\frac{Q}{D}} - 1) \\
&- \frac{p+h}{2}(Q^2 + t^2D^2 + \frac{2tD^2}{\mu} - \frac{2QD}{\mu}) \\
\mathbf{e} &= \mathbf{e}(Q,t) = -D(C + \frac{h}{\mu})(e^{\mu t} - 1) - Dpt + (p+h)Q + (p+h)\frac{D}{\mu}(e^{\mu t - \mu\frac{Q}{D}} - 1) \\
\mathbf{f} &= \mathbf{f}(Q,t) = (p+h)(e^{\mu t - \mu\frac{Q}{D}} - 1) \\
\mathbf{g} &= \mathbf{g}(Q,t) = \mu^2[CD + (p+h)Q - tDp]^2 \\
\mathbf{i} &= \mathbf{i}(Q,t) = \mu^2[QCD + KD + \frac{1}{2}(p+h)(Q^2+t^2D^2) - QtDp]^2 \\
\mathbf{j} &= \mathbf{j}(Q,t) = [\frac{1}{2}\mu(h+2p) - \frac{(p+h)Q}{tD}(\frac{1}{t} + \mu)]^2 \\
\mathbf{k} &= \mathbf{k}(Q,t) = \mu^2[CD + (p+h)Q - tDp][QCD + KD + \frac{1}{2}(p+h)(Q^2+t^2D^2) - QtDp] \\
\mathbf{l} &= \mathbf{l}(Q,t) = \mu[CD + (p+h)Q - tDp][\frac{1}{2}\mu(h+2p) - \frac{(p+h)Q}{tD}(\frac{1}{t} + \mu)] \\
\mathbf{r} &= \mathbf{r}(Q,t) = \mu[QCD + KD + \frac{1}{2}(p+h)(Q^2+t^2D^2) - QtDp][\frac{1}{2}\mu(h+2p) - \frac{(p+h)Q}{tD}(\frac{1}{t} + \mu)]
\end{aligned}$$

Please note that upper/lower case letters represent different components, so are bold/regular letters. For instance, K stands for the set-up cost in EOQ model, while $\mathbf{k}$ is a component in $(\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t \partial Q} e^{\mu t})^2$. Similarly, $g(y)$ is defined as the distribution of trade-in volume, while

$\mathbf{g}(Q,t)$ is again part of $(\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t \partial Q} e^{\mu t})^2$. With the above newly defined notations, we have

$$\begin{aligned}
\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t^2} e^{\mu t} =& \mathbf{a} + \mathbf{b} \frac{1}{Y_2 - Y_1} \log \frac{Q+Y_2}{Q+Y_1}, \\
\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial Q^2} e^{\mu t} =& \mathbf{c} + \mathbf{d} \frac{2Q+Y_1+Y_2}{(Q+Y_2)^2(Q+Y_1)^2} + \mathbf{e} \frac{2}{(Q+Y_2)(Q+Y_1)} + \mathbf{f} \frac{1}{Y_2-Y_1} \log \frac{Q+Y_2}{Q+Y_1}, \\
(\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t \partial Q} e^{\mu t})^2 =& \mathbf{g}(\frac{1}{Y_2-Y_1} \log \frac{Q+Y_2}{Q+Y_1})^2 + \mathbf{i} \frac{1}{(Q+Y_2)^2(Q+Y_1)^2} + \mathbf{j} \\
& - 2\mathbf{k} \frac{1}{(Q+Y_2)(Q+Y_1)} \frac{1}{Y_2-Y_1} \log \frac{Q+Y_2}{Q+Y_1} + 2\mathbf{l} \frac{1}{Y_2-Y_1} \log \frac{Q+Y_2}{Q+Y_1} \\
& - 2\mathbf{r} \frac{1}{(Q+Y_2)(Q+Y_1)}.
\end{aligned}$$

Eventually, we find that the analysis of the determinant of Hessian matrix with regard to the objective function in (4.12) is equivalent to the analysis of the following function

$$\begin{aligned}
& \frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t^2} e^{\mu t} * \frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial Q^2} e^{\mu t} - (\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t \partial Q} e^{\mu t})^2 \\
=& \mathbf{ac} - \mathbf{j} \\
& + (\mathbf{bf} - \mathbf{g})(\frac{1}{Y_2-Y_1} \log \frac{Q+Y_2}{Q+Y_1})^2 \\
& + [\mathbf{ad}(2Q+Y_1+Y_2) - \mathbf{i}] \frac{1}{(Q+Y_2)^2(Q+Y_1)^2} \\
& + [2\mathbf{be} + \mathbf{bd} \frac{2Q+Y_1+Y_2}{(Q+Y_2)(Q+Y_1)} + 2\mathbf{k}] \frac{1}{(Q+Y_2)(Q+Y_1)} \frac{1}{Y_2-Y_1} \log \frac{Q+Y_2}{Q+Y_1} \\
& + (\mathbf{af} + \mathbf{bc} - 2\mathbf{l}) \frac{1}{Y_2-Y_1} \log \frac{Q+Y_2}{Q+Y_1} \\
& + (2\mathbf{ae} + 2\mathbf{r}) \frac{1}{(Q+Y_2)(Q+Y_1)}.
\end{aligned} \tag{5.10}$$

Before the delivery of sufficient condition that guarantees convexity of (4.12), some lemmas will be presented first in order to facilitate the understanding of the sufficient condition. Please refer to Chapter 5.2 for the proofs of all the lemmas below.

Define $\mathbb{S}_4 = \{(Q,t) | \frac{Q}{D} \le t \le \alpha \frac{Q}{D}, \underline{Q} \le Q \le \bar{Q}\}$, and $\mathbb{S}_5 = \{(\underline{Q}, \frac{\underline{Q}}{D}), (\underline{Q}, \alpha \frac{\underline{Q}}{D}), (\bar{Q}, \frac{\bar{Q}}{D}), (\bar{Q}, \alpha \frac{\bar{Q}}{D})\}$. In short, $\mathbb{S}_4$ is the feasible region of (4.12), and $\mathbb{S}_5$ stands for the vertices of this feasible region.

Lemma 9. *If the following conditions are satisfied*

1. *Either* $2(p+h)(\frac{D^2}{\alpha^3 \bar{Q}^2} + \frac{\mu D}{\alpha^2 \bar{Q}} + \frac{\mu^2}{2\alpha}) - \mu^2 p - \frac{1}{2}\mu^2 h > 0$,
   or $2(p+h)(\frac{D^2}{\underline{Q}^2} + \frac{\mu D}{\underline{Q}} + \frac{\mu^2}{2}) - \mu^2 p - \frac{1}{2}\mu^2 h < 0$

2. Define $\tilde{\boldsymbol{a}}(x) = -\frac{3D^3}{x^4} - \frac{2\mu D^2}{x^3} - \frac{\mu^2 D}{2x^2}$.

   a) $\{[2K + \frac{(p+h)\underline{Q}^2}{D}]\tilde{\boldsymbol{a}}(\underline{Q}) * D + \frac{1}{2}\mu^2 Dp + \mu^2 Dh\} *$
   $\{[2K + \frac{(p+h)\underline{Q}^2}{D}]\tilde{\boldsymbol{a}}(\alpha\underline{Q}) * D + \frac{1}{2}\mu^2 Dp + \mu^2 Dh\} > 0.$

   b) $\{[2K + \frac{(p+h)\bar{Q}^2}{D}]\tilde{\boldsymbol{a}}(\bar{Q}) * D + \frac{1}{2}\mu^2 Dp + \mu^2 Dh\} *$
   $\{[2K + \frac{(p+h)\bar{Q}^2}{D}]\tilde{\boldsymbol{a}}(\alpha\bar{Q}) * D + \frac{1}{2}\mu^2 Dp + \mu^2 Dh\} > 0.$

   c) $[2K\tilde{\boldsymbol{a}}(\bar{Q}) - (p+h)\frac{D^2}{\bar{Q}^2} + \mu^2 h] * [2K\tilde{\boldsymbol{a}}(\underline{Q}) - (p+h)\frac{D^2}{\underline{Q}^2} + \mu^2 h] > 0.$

   d) $\{2K\tilde{\boldsymbol{a}}(\alpha\bar{Q}) * \alpha - (p+h)\frac{D^2}{\alpha^3\bar{Q}^2} + \frac{1}{2}\mu^2[\frac{p+h}{\alpha} + (p+h)(\alpha-1) + \alpha h - p]\} *$
   $\{2K\tilde{\boldsymbol{a}}(\alpha\underline{Q}) * \alpha - (p+h)\frac{D^2}{\alpha^3\underline{Q}^2} + \frac{1}{2}\mu^2[\frac{p+h}{\alpha} + (p+h)(\alpha-1) + \alpha h - p]\} > 0.$

Then $\bar{\boldsymbol{a}} = \max_{(Q,t)\in\mathbb{S}_4} \boldsymbol{a}(Q,t) = \max_{(Q,t)\in\mathbb{S}_5} \boldsymbol{a}(Q,t)$, *and* $\underline{\boldsymbol{a}} = \min_{(Q,t)\in\mathbb{S}_4} \boldsymbol{a}(Q,t) = \min_{(Q,t)\in\mathbb{S}_5} \boldsymbol{a}(Q,t)$.

Lemma 10. *If the following conditions are satisfied*

1. $\frac{p+h}{p} > \alpha$

2. $(D - \mu\underline{Q}\frac{h}{p+h})[D - \mu(\alpha - \frac{p}{p+h})\underline{Q}] > 0.$

3. $(D - \mu\bar{Q}\frac{h}{p+h})[D - \mu(\alpha - \frac{p}{p+h})\bar{Q}] > 0.$

4. $(Dh - \mu CD - 2\mu h\underline{Q})(Dh - \mu CD - 2\mu h\bar{Q}) > 0.$

5. $\{D[\alpha h + (\alpha-1)p] - \mu CD - \mu[(p+h)(1-\alpha)^2 + 2\alpha h]\underline{Q}\} *$
   $\{D[\alpha h + (\alpha-1)p] - \mu CD - \mu[(p+h)(1-\alpha)^2 + 2\alpha h]\bar{Q}\} > 0.$

Then $\frac{\partial \boldsymbol{b}}{\partial Q} < 0$, $\bar{\boldsymbol{b}} = \max_{(Q,t)\in\mathbb{S}_4} \boldsymbol{b}(Q,t) = \max_{(Q,t)\in\mathbb{S}_5} \boldsymbol{b}(Q,t)$, *and* $\underline{\boldsymbol{b}} = \min_{(Q,t)\in\mathbb{S}_4} \boldsymbol{b}(Q,t) = \min_{(Q,t)\in\mathbb{S}_5} \boldsymbol{b}(Q,t)$.

Lemma 11. *If the following conditions are satisfied*

1. $D(C + \frac{h}{\mu})(e^{\mu\frac{\underline{Q}}{D}} - 1) + \mu(\underline{Q}C + K)e^{\mu\frac{\underline{Q}}{D}} - \frac{2\mu\bar{Q}^2 h}{D + \mu\bar{Q}} > 0.$

2. $D(C + \frac{h}{\mu})(e^{\mu\frac{\alpha\underline{Q}}{D}} - 1) + \mu\alpha[\underline{Q}(C + \frac{h}{\mu}) + K]e^{\mu\frac{\alpha\underline{Q}}{D}} + 2\alpha\underline{Q}p - (p+h)\frac{D}{u}(\alpha - 1) + (p+h)(1 + \alpha^2)[\frac{D}{\mu(\alpha-1)} - \bar{Q}] > 0.$

Then $\frac{\partial \boldsymbol{d}}{\partial t} > 0$, $\bar{\boldsymbol{d}} = \max_{(Q,t)\in\mathbb{S}_4} \boldsymbol{d}(Q,t) = \boldsymbol{d}(\bar{Q}, \alpha\frac{\bar{Q}}{D})$, *and* $\underline{\boldsymbol{d}} = \min_{(Q,t)\in\mathbb{S}_4} \boldsymbol{d}(Q,t) = \boldsymbol{d}(\underline{Q}, \frac{\underline{Q}}{D})$.

Lemma 12. *If the following conditions are satisfied*

1. $-(C\mu + h) + (p+h)e^{-\mu\frac{\bar{Q}}{D}} > 0.$

2. $[-(C\mu+h)+(p+h)e^{-\mu\frac{\underline{Q}}{D}}]e^{\mu\alpha\frac{\bar{Q}}{D}} < p$.

Then $\boldsymbol{e} < 0$, $\frac{\partial \boldsymbol{e}}{\partial t} < 0, \frac{\partial \boldsymbol{e}}{\partial Q} < 0$, $\bar{\boldsymbol{e}} = \max_{(Q,t)\in\mathbb{S}_4} \boldsymbol{e}(Q,t) = \boldsymbol{e}(\underline{Q}, \frac{\underline{Q}}{D})$, *and* $\underline{\boldsymbol{e}} = \min_{(Q,t)\in\mathbb{S}_4} \boldsymbol{e}(Q,t) = \boldsymbol{e}(\bar{Q}, \alpha\frac{\bar{Q}}{D})$.

Lemma 13. *If* $\frac{p+h}{p} > \alpha$, *then* $\boldsymbol{g} > 0$, $\frac{\partial \boldsymbol{g}}{\partial t} < 0, \frac{\partial \boldsymbol{g}}{\partial Q} > 0$, $\bar{\boldsymbol{g}} = \max_{(Q,t)\in\mathbb{S}_4} \boldsymbol{g}(Q,t) = \boldsymbol{g}(\bar{Q}, \frac{\bar{Q}}{D})$, *and* $\underline{\boldsymbol{g}} = \min_{(Q,t)\in\mathbb{S}_4} \boldsymbol{g}(Q,t) = \boldsymbol{g}(\underline{Q}, \alpha\frac{\underline{Q}}{D})$.

Lemma 14. *If* $\frac{p+h}{p} > \alpha$, *then* $\boldsymbol{i} > 0$, $\frac{\partial \boldsymbol{i}}{\partial t} > 0, \frac{\partial \boldsymbol{i}}{\partial Q} > 0$, $\bar{\boldsymbol{i}} = \max_{(Q,t)\in\mathbb{S}_4} \boldsymbol{i}(Q,t) = \boldsymbol{i}(\bar{Q}, \alpha\frac{\bar{Q}}{D})$, *and* $\underline{\boldsymbol{i}} = \min_{(Q,t)\in\mathbb{S}_4} \boldsymbol{i}(Q,t) = \boldsymbol{i}(\underline{Q}, \frac{\underline{Q}}{D})$.

Lemma 15. *If* $\frac{\mu}{2}(h+2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha\bar{Q}}+\mu) < 0$, *then* $\boldsymbol{j} > 0$, $\frac{\partial \boldsymbol{j}}{\partial t} < 0, \frac{\partial \boldsymbol{j}}{\partial Q} > 0$, $\bar{\boldsymbol{j}} = \max_{(Q,t)\in\mathbb{S}_4} \boldsymbol{j}(Q,t) = \boldsymbol{j}(\underline{Q}, \frac{\underline{Q}}{D})$, *and* $\underline{\boldsymbol{j}} = \min_{(Q,t)\in\mathbb{S}_4} \boldsymbol{j}(Q,t) = \boldsymbol{j}(\bar{Q}, \alpha\frac{\bar{Q}}{D})$.

Lemma 16. *If the following conditions are satisfied,*

1. $\frac{p+h}{p} > \alpha$
2. $[-3p(p+h)\alpha + 2p^2 + (p+h)^2]D^2Q + (p+h)D^3C > 0$ *for* $\forall Q \in [\underline{Q}, \bar{Q}]$,
3. $[-\frac{3}{2}p(p+h)(1+\alpha^2) + 2p^2 + (p+h)^2]DQ^2 + (h-p)CD^2Q - KpD^2 > 0$ *for* $\forall Q \in [\underline{Q}, \bar{Q}]$,

then $\boldsymbol{k} > 0$, $\frac{\partial \boldsymbol{k}}{\partial t} > 0$, $\bar{\boldsymbol{k}} = \max_{(Q,t)\in\mathbb{S}_4} \boldsymbol{k}(Q,t) = \boldsymbol{k}(\bar{Q}, \alpha\frac{\bar{Q}}{D})$, *and* $\underline{\boldsymbol{k}} = \min_{(Q,t)\in\mathbb{S}_4} \boldsymbol{k}(Q,t) = \boldsymbol{k}(\underline{Q}, \frac{\underline{Q}}{D})$.

Particularly, the third condition is equivalent to

1. *If*
   - $-\frac{3}{2}p(p+h)(1+\alpha^2) + 2p^2 + (p+h)^2 \leq 0$, *or*
   - $-\frac{3}{2}p(p+h)(1+\alpha^2) + 2p^2 + (p+h)^2 > 0$ *and* $Q_{mid} = -\frac{(h-p)CD}{-3p(p+h)(1+\alpha^2)+4p^2+2(p+h)^2} \notin [\underline{Q}, \bar{Q}]$

   then $[-\frac{3}{2}p(p+h)(1+\alpha^2) + 2p^2 + (p+h)^2]D\bar{Q}^2 + (h-p)CD^2\bar{Q} - KpD^2 > 0$
   and $[-\frac{3}{2}p(p+h)(1+\alpha^2) + 2p^2 + (p+h)^2]D\underline{Q}^2 + (h-p)CD^2\underline{Q} - KpD^2 > 0$
2. *If* $-\frac{3}{2}p(p+h)(1+\alpha^2) + 2p^2 + (p+h)^2 > 0$ *and* $Q_{mid} \in [\underline{Q}, \bar{Q}]$,
   then $[-\frac{3}{2}p(p+h)(1+\alpha^2) + 2p^2 + (p+h)^2]DQ_{mid}^2 + (h-p)CD^2Q_{mid} - KpD^2 > 0$.

Lemma 17. *If* $\frac{p+h}{p} > \alpha$, *and* $\frac{\mu}{2}(h+2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha\bar{Q}}+\mu) < 0$, *then* $\boldsymbol{l} < 0$, $\frac{\partial \boldsymbol{l}}{\partial t} > 0$, $\max_{(Q,t)\in\mathbb{S}_4} \boldsymbol{l}(Q,t) \leq \bar{\boldsymbol{l}} = \mu\{CD + [h + (1-\alpha)p]\underline{Q}\}[0.5\mu(h+2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha\bar{Q}}+\mu)]$, *and* $\min_{(Q,t)\in\mathbb{S}_4} \boldsymbol{l}(Q,t) \geq \underline{\boldsymbol{l}} = \mu(CD + h\bar{Q})[0.5\mu(h+2p) - (p+h)(\frac{D}{\underline{Q}}+\mu)]$.

Lemma 18. *If the following conditions are satisfied*

1. $\frac{p+h}{p} > \alpha$, *and* $\frac{\mu}{2}(h+2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha\bar{Q}} + \mu) < 0$,

2. $\mu(\frac{1}{2}h + 2p - \frac{p+h}{\alpha}) + (4p - \frac{3(p+h)}{\alpha})\frac{1}{t} < 0$ *for* $\forall(Q,t) \in \mathbb{S}_4$,

3. *if* $\frac{h+2p}{2} < \frac{p+h}{\alpha}$, *then* $[(\alpha-1)p+\alpha h]D[\mu(\frac{h+2p}{2} - \frac{p+h}{\alpha})\bar{Q} - \frac{h+p}{\alpha^2}D] + \frac{h+p}{\alpha^2}D(\frac{2D}{\alpha} + \mu\underline{Q})[0.5h(1+\alpha^2) + 0.5p(1-\alpha)^2 + \frac{CD}{\bar{Q}} + \frac{KD}{\bar{Q}^2}] > 0$;
else if $\frac{h+2p}{2} \geq \frac{p+h}{\alpha}$, *then* $[(\alpha-1)p+\alpha h]D[\mu(\frac{h+2p}{2} - \frac{p+h}{\alpha})\underline{Q} - \frac{h+p}{\alpha^2}D] + \frac{h+p}{\alpha^2}D(\frac{2D}{\alpha} + \mu\underline{Q})[0.5h(1+\alpha^2) + 0.5p(1-\alpha)^2 + \frac{CD}{\bar{Q}} + \frac{KD}{\bar{Q}^2}] > 0$

then $\boldsymbol{r} < 0$, $\frac{\partial \boldsymbol{r}}{\partial t} > 0$, $\max_{(Q,t)\in\mathbb{S}_4} \boldsymbol{r}(Q,t) \leq \bar{\boldsymbol{r}} = \mu\{[0.5h(1+\alpha^2) + 0.5p(1-\alpha)^2]\underline{Q}^2 + CD\underline{Q} + KD\}[0.5\mu(h+2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha\bar{Q}} + \mu)]$, *and* $\min_{(Q,t)\in\mathbb{S}_4} \boldsymbol{r}(Q,t) \geq \underline{\boldsymbol{r}} = \mu[h\bar{Q}^2 + CD\bar{Q} + KD][0.5\mu(h+2p) - (p+h)(\frac{D}{\underline{Q}} + \mu)]$.

Above lemmas (9 to 18) primarily clarify and simplify the existence and location of global extreme points in each of the components defined by **a**, **b**, **d**, **e**, etc. These lemmas eliminate interior extreme points, thus facilitate us to locate the global extreme points on the boundary or on the four vertices in $\mathbb{S}_5$.

With the lemmas, we are ready to present the theorem that states the sufficient condition such that (4.12) is a convex optimization problem. Please refer to Chapter 5.2 for the proof.

Theorem 19. *Optimization problem (4.12) is strictly convex within the feasible region* $\mathbb{S}_4$ *if*

1. *Conditions in **Lemma** 9 to 18 hold, and*

2. $\boldsymbol{a} > 0$, $\boldsymbol{b} > 0$, $\boldsymbol{d} > 0$, *and*

3.

$$
\begin{aligned}
&-\bar{\boldsymbol{g}}(\frac{1}{Y_2-Y_1}\log\frac{\underline{Q}+Y_2}{\underline{Q}+Y_1})^2\\
&+[\underline{\boldsymbol{ad}}(2\underline{Q}+Y_1+Y_2)-\bar{\boldsymbol{i}}]\\
&*[\mathbb{1}_{\{\underline{\boldsymbol{ad}}(2\underline{Q}+Y_1+Y_2)-\bar{i}>0\}}\frac{1}{(\bar{Q}+Y_1)^2(\bar{Q}+Y_2)^2}+\mathbb{1}_{\{\underline{\boldsymbol{ad}}(2\underline{Q}+Y_1+Y_2)-\bar{i}<0\}}\frac{1}{(\underline{Q}+Y_1)^2(\underline{Q}+Y_2)^2}]\\
&+\underline{\boldsymbol{ac}}-\bar{\boldsymbol{j}}\\
&+[2\bar{\boldsymbol{b}}\underline{\boldsymbol{e}}+\underline{\boldsymbol{bd}}\frac{2\bar{Q}+Y_1+Y_2}{(\bar{Q}+Y_1)(\bar{Q}+Y_2)}+2\underline{\boldsymbol{k}}]\frac{1}{Y_2-Y_1}\\
&*[\mathbb{1}_{\{2\bar{b}\underline{\boldsymbol{e}}+\underline{\boldsymbol{bd}}\frac{2\bar{Q}+Y_1+Y_2}{(\bar{Q}+Y_1)(\bar{Q}+Y_2)}+2\underline{\boldsymbol{k}}>0\}}\log\frac{\bar{Q}+Y_2}{\bar{Q}+Y_1}*\frac{1}{(\bar{Q}+Y_1)(\bar{Q}+Y2)}\\
&+\mathbb{1}_{\{2\bar{b}\underline{\boldsymbol{e}}+\underline{\boldsymbol{bd}}\frac{2\bar{Q}+Y_1+Y_2}{(\underline{Q}+Y_1)(\underline{Q}+Y_2)}+2\underline{\boldsymbol{k}}<0\}}\log\frac{\underline{Q}+Y_2}{\underline{Q}+Y_1}*\frac{1}{(\underline{Q}+Y_1)(\underline{Q}+Y2)}]\\
&+(\underline{\boldsymbol{af}}+\underline{\boldsymbol{bc}}-2\bar{\boldsymbol{l}})\frac{1}{Y_2-Y_1}\log\frac{\bar{Q}+Y_2}{\bar{Q}+Y_1}\\
&+(2\bar{\boldsymbol{a}}\underline{\boldsymbol{e}}+2\underline{\boldsymbol{r}})[\mathbb{1}_{\{2\bar{a}\underline{\boldsymbol{e}}+2\underline{\boldsymbol{r}}>0\}}\frac{1}{(\bar{Q}+Y_1)(\bar{Q}+Y_2)}+\mathbb{1}_{\{2\bar{a}\underline{\boldsymbol{e}}+2\underline{\boldsymbol{r}}<0\}}\frac{1}{(\underline{Q}+Y_1)(\underline{Q}+Y_2)}]\\
&>0
\end{aligned}
\tag{5.11}
$$

5.2 Proofs

Proofs of lemmas

Proof. Lemma 9

Since

$$\frac{\partial \mathbf{a}}{\partial Q}=\frac{2Q(p+h)}{D}(\frac{1}{t^3}+\frac{\mu}{t^2}+\frac{\mu^2}{2t})-0.5\mu^2(2p+h), \tag{5.12}$$

we have

$$
\begin{aligned}
\frac{\partial \mathbf{a}}{\partial Q}&\geq\frac{2Q(p+h)}{D}(\frac{1}{(\alpha\frac{Q}{D})^3}+\frac{\mu}{(\alpha\frac{Q}{D})^2}+\frac{\mu^2}{2\alpha\frac{Q}{D}})-0.5\mu^2(2p+h)\\
&=2(p+h)(\frac{D^2}{\alpha^3Q^2}+\frac{\mu D}{\alpha^2 Q}+\frac{\mu^2}{2\alpha})-0.5\mu^2(2p+h)\\
&\geq 2(p+h)(\frac{D^2}{\alpha^3\bar{Q}^2}+\frac{\mu D}{\alpha^2\bar{Q}}+\frac{\mu^2}{2\alpha})-0.5\mu^2(2p+h),
\end{aligned}
\tag{5.13}
$$

and

$$
\begin{aligned}
\frac{\partial \mathbf{a}}{\partial Q} &\leq \frac{2Q(p+h)}{D}\left(\frac{1}{(\frac{Q}{D})^3}+\frac{\mu}{(\frac{Q}{D})^2}+\frac{\mu^2}{2\frac{Q}{D}}\right)-0.5\mu^2(2p+h) \\
&= 2(p+h)\left(\frac{D^2}{Q^2}+\frac{\mu D}{Q}+\frac{\mu^2}{2}\right)-0.5\mu^2(2p+h) \\
&\leq 2(p+h)\left(\frac{D^2}{\underline{Q}^2}+\frac{\mu D}{\underline{Q}}+\frac{\mu^2}{2}\right)-0.5\mu^2(2p+h).
\end{aligned} \tag{5.14}
$$

If

$$
2(p+h)\left(\frac{D^2}{\alpha^3\bar{Q}^2}+\frac{\mu D}{\alpha^2\bar{Q}}+\frac{\mu^2}{2\alpha}\right)-0.5\mu^2(2p+h)>0 \tag{5.15}
$$

or

$$
2(p+h)\left(\frac{D^2}{\underline{Q}^2}+\frac{\mu D}{\underline{Q}}+\frac{\mu^2}{2}\right)-0.5\mu^2(2p+h)<0, \tag{5.16}
$$

then there is no interior optimal point, and the optimal solution lies on the boundary of the feasible region.

For boundary $Q=\underline{Q}$,

$$
\frac{\partial \mathbf{a}}{\partial t}\Big|_{Q=\underline{Q}} = \left[2K+\frac{(p+h)\underline{Q}^2}{D}\right]\left(-\frac{3}{t^4}-\frac{2\mu}{t^3}-\frac{\mu^2}{2t^2}\right)+0.5\mu^2(2h+p)D, \tag{5.17}
$$

which is monotonic with regard to t. Therefore, if $\frac{\partial \mathbf{a}}{\partial t}|_{t=\frac{\underline{Q}}{D}} * \frac{\partial \mathbf{a}}{\partial t}|_{t=\alpha\frac{\underline{Q}}{D}} > 0$, it can be concluded that $\frac{\partial \mathbf{a}}{\partial t}$ does not change sign on the boundary $Q=\underline{Q}$. Hence the extreme point on this boundary lies on the endpoint(s), i.e., $(\underline{Q}, \frac{\underline{Q}}{D})$ or $(\underline{Q}, \alpha\frac{\underline{Q}}{D})$. Notice that

$$
\begin{aligned}
&\frac{\partial \mathbf{a}}{\partial t}\Big|_{t=\frac{\underline{Q}}{D}} * \frac{\partial \mathbf{a}}{\partial t}\Big|_{t=\alpha\frac{\underline{Q}}{D}} > 0 \\
\Rightarrow &\left\{\left[2K+\frac{(p+h)\underline{Q}^2}{D}\right]\tilde{\mathbf{a}}(\underline{Q}) * D+\frac{1}{2}\mu^2 Dp+\mu^2 Dh\right\} \\
&* \left\{\left[2K+\frac{(p+h)\underline{Q}^2}{D}\right]\tilde{\mathbf{a}}(\alpha\underline{Q}) * D+\frac{1}{2}\mu^2 Dp+\mu^2 Dh\right\} > 0
\end{aligned} \tag{5.18}
$$

which is indeed assumption $2(a)$ in Lemma 9

On the boundary $Q=\bar{Q}$,

$$
\frac{\partial \mathbf{a}}{\partial t}\Big|_{Q=\bar{Q}} = \left[2K+\frac{(p+h)\bar{Q}^2}{D}\right]\left(-\frac{3}{t^4}-\frac{2\mu}{t^3}-\frac{\mu^2}{2t^2}\right)+0.5\mu^2(2h+p)D. \tag{5.19}
$$

It is again monotonic with regard to t. If

$$\begin{aligned}
&\frac{\partial \mathbf{a}}{\partial t}|_{t=\frac{\bar{Q}}{D}} * \frac{\partial \mathbf{a}}{\partial t}|_{t=\alpha\frac{\bar{Q}}{D}} \\
=&\{[2K + \frac{(p+h)\bar{Q}^2}{D}]\tilde{\mathbf{a}}(\bar{Q}) * D + \frac{1}{2}\mu^2 Dp + \mu^2 Dh\} \\
&* \{[2K + \frac{(p+h)\bar{Q}^2}{D}]\tilde{\mathbf{a}}(\alpha\bar{Q}) * D + \frac{1}{2}\mu^2 Dp + \mu^2 Dh\} \\
>&0,
\end{aligned} \tag{5.20}$$

then again the extreme point of this boundary is located on the endpoint(s), namely, $(\bar{Q}, \frac{\bar{Q}}{D})$ or $(\bar{Q}, \alpha\frac{\bar{Q}}{D})$.

For the third boundary $t = \frac{Q}{D}$,

$$\frac{\partial \mathbf{a}}{\partial Q}|_{t=\frac{Q}{D}} = 2K(-\frac{3D^3}{Q^4} - \frac{2\mu D^2}{Q^3} - \frac{\mu^2 D}{2Q^2}) - (p+h)\frac{D^2}{Q^2} + \mu^2 h. \tag{5.21}$$

Due to monotonic property, if

$$\begin{aligned}
&\frac{\partial \mathbf{a}}{\partial Q}|_{t=\frac{\underline{Q}}{D}} * \frac{\partial \mathbf{a}}{\partial Q}|_{t=\frac{\bar{Q}}{D}} \\
=&[2K\tilde{\mathbf{a}}(\bar{Q}) - (p+h)\frac{D^2}{\bar{Q}^2} + \mu^2 h] * [2K\tilde{\mathbf{a}}(\underline{Q}) - (p+h)\frac{D^2}{\underline{Q}^2} + \mu^2 h] > 0,
\end{aligned} \tag{5.22}$$

then the extreme point of the boundary can be found at the endpoints.

Finally, on $t = \alpha\frac{Q}{D}$, we have

$$\frac{\partial \mathbf{a}}{\partial Q}|_{t=\alpha\frac{Q}{D}} = 2K(-\frac{3D^3}{\alpha^3 Q^4} - \frac{2\mu D^2}{\alpha^2 Q^3} - \frac{\mu^2 D}{2\alpha Q^2}) - (p+h)\frac{D^2}{\alpha^3 Q^2} + 0.5\mu^2[(2\alpha - 1 + \frac{1}{\alpha})h + (\alpha - 2 + \frac{1}{\alpha})p]. \tag{5.23}$$

Since it is monotonic with regard to Q, the extreme point of this boundary will be located on the endpoints if

$$\begin{aligned}
&\frac{\partial \mathbf{a}}{\partial Q}|_{t=\frac{\underline{Q}}{D}} * \frac{\partial \mathbf{a}}{\partial Q}|_{t=\frac{\bar{Q}}{D}} \\
=&\{2K\tilde{\mathbf{a}}(\alpha\bar{Q}) * \alpha - (p+h)\frac{D^2}{\alpha^3\bar{Q}^2} + \frac{1}{2}\mu^2[\frac{p+h}{\alpha} + (p+h)(\alpha - 1) + \alpha h - p]\} \\
&* \{2K\tilde{\mathbf{a}}(\alpha\underline{Q}) * \alpha - (p+h)\frac{D^2}{\alpha^3\underline{Q}^2} + \frac{1}{2}\mu^2[\frac{p+h}{\alpha} + (p+h)(\alpha - 1) + \alpha h - p]\} \\
>&0,
\end{aligned} \tag{5.24}$$

□

Proof. Lemma 10

Following the same concept in the proof of Lemma 9, we will firstly show that there is no interior optimal point for **b**, therefore the extreme point lies on the boundary. Since

$$\frac{\partial \mathbf{b}}{\partial t} = \mu^2 D^2 (p+h)(\frac{1}{\mu} + \frac{p}{p+h}\frac{Q}{D} - t) \tag{5.25}$$

and

$$\frac{\partial \mathbf{b}}{\partial Q} = -\mu^2 Dp(\frac{1}{\mu} + \frac{C}{p} + \frac{p+h}{p}\frac{Q}{D} - t), \tag{5.26}$$

it can be derived that

$$\begin{aligned} &\frac{\partial \mathbf{b}}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \mathbf{b}}{\partial Q} = 0 \\ \Rightarrow &(\frac{p}{p+h} - \frac{p+h}{p})\frac{Q}{D} = \frac{C}{p} \\ \Rightarrow &Q < 0. \end{aligned} \tag{5.27}$$

Hence there is no interior optimal points. Particularly, if $\frac{p+h}{p} > \alpha$, then for $\frac{\partial \mathbf{b}}{\partial Q}$, we have

$$\begin{aligned} &\frac{1}{\mu} + \frac{C}{p} + \frac{p+h}{p}\frac{Q}{D} - t \geq \frac{1}{\mu} + \frac{C}{p} + \frac{p+h}{p}\frac{Q}{D} - \alpha\frac{Q}{D} > 0 \\ \Rightarrow &\frac{\partial \mathbf{b}}{\partial Q} < 0. \end{aligned} \tag{5.28}$$

As shown, the extreme point is located on boundaries. We would like to further derive conditions such that the optimal point can be found at the endpoints of the boundaries.

On $Q = \underline{Q}$,

$$\frac{\partial \mathbf{b}}{\partial t}|_{Q=\underline{Q}} = \mu^2 D^2 (p+h)(\frac{1}{\mu} + \frac{p}{p+h}\frac{\underline{Q}}{D} - t). \tag{5.29}$$

Due to monotonic property, if

$$(\frac{1}{\mu} + \frac{p}{p+h}\frac{\underline{Q}}{D} - \frac{\underline{Q}}{D})(\frac{1}{\mu} + \frac{p}{p+h}\frac{\underline{Q}}{D} - \alpha\frac{\underline{Q}}{D}) > 0, \tag{5.30}$$

which is equivalent to

$$(D - \mu\underline{Q}\frac{h}{p+h})[D - \mu(\alpha - \frac{p}{p+h})\underline{Q}] > 0, \tag{5.31}$$

then the extreme point of this boundary is located at the endpoint(s).

For $Q = \bar{Q}$,

$$\frac{\partial \mathbf{b}}{\partial t}|_{Q=\bar{Q}} = \mu^2 D^2 (p+h)(\frac{1}{\mu} + \frac{p}{p+h}\frac{\bar{Q}}{D} - t). \tag{5.32}$$

It is monotonic over t. Thus the optimal point will be at the endpoint of this boundary if

$$(D-\mu\bar{Q}\frac{h}{p+h})[D-\mu(\alpha-\frac{p}{p+h})\bar{Q}]>0. \tag{5.33}$$

For $t=\frac{Q}{D}$,

$$\frac{\partial \mathbf{b}}{\partial Q}|_{t=\frac{Q}{D}}=\mu(Dh-\mu CD-2\mu hQ), \tag{5.34}$$

If

$$(Dh-\mu CD-2\mu h\underline{Q})(Dh-\mu CD-2\mu h\bar{Q})>0, \tag{5.35}$$

then the extreme point of the boundary is located at the endpoint.

Finally, on $t=\alpha\frac{Q}{D}$,

$$\frac{\partial \mathbf{b}}{\partial Q}|_{t=\alpha\frac{Q}{D}}=\mu\{D[(\alpha-1)p+\alpha h-\mu C]-\mu[(1-\alpha)^2(p+h)+2\alpha h]Q\}. \tag{5.36}$$

Based on the monotonic property, the condition for extreme point being on the endpoint is that

$$\begin{aligned}&\{D[\alpha h+(\alpha-1)p]-\mu CD-\mu[(p+h)(1-\alpha)^2+2\alpha h]\underline{Q}\}\\ *&\{D[\alpha h+(\alpha-1)p]-\mu CD-\mu[(p+h)(1-\alpha)^2+2\alpha h]\bar{Q}\}>0\end{aligned} \tag{5.37}$$

□

Proof. Lemma 11

Taking partial derivative with regard to t

$$\frac{\partial \mathbf{d}}{\partial t}=QD(C+\frac{h}{\mu})\mu e^{\mu t}+KD\mu e^{\mu t}+QDP+(p+h)\frac{D^2}{\mu}e^{\mu t}e^{-\mu\frac{Q}{D}}-(p+h)(tD^2+\frac{D^2}{\mu}). \tag{5.38}$$

Because

$$\frac{\partial^2 \mathbf{d}}{\partial t^2}=QD(C+\frac{h}{\mu})\mu^2e^{\mu t}+KD\mu^2e^{\mu t}+(p+h)D^2(e^{\mu t}e^{-\mu\frac{Q}{D}}-1)>0,\quad\forall(Q,t)\in\mathbb{S}_4 \tag{5.39}$$

it can be derived that

$$\begin{aligned}\frac{\partial \mathbf{d}}{\partial t}\geq&QD(C+\frac{h}{\mu})\mu e^{\mu\frac{Q}{D}}+KD\mu e^{\mu\frac{Q}{D}}+QDP+(p+h)\frac{D^2}{\mu}-(p+h)(QD+\frac{D^2}{\mu})\\ =&\mu D(QC+K)e^{\mu\frac{Q}{D}}+QDhe^{\mu\frac{Q}{D}}-QDh>0.\end{aligned} \tag{5.40}$$

Hence, the minimal point of $\mathbf{d}$ is located on $t=\frac{Q}{D}$, and the maximum point on $t=\alpha\frac{Q}{D}$.

On the boundary $t=\frac{Q}{D}$,

$$\frac{\partial \mathbf{d}}{\partial t}|_{t=\frac{Q}{D}}=D(C+\frac{h}{\mu})(e^{\mu\frac{Q}{D}}-1)+\mu(QC+K)e^{\mu\frac{Q}{D}}+Qh(e^{\mu\frac{Q}{D}}-2). \tag{5.41}$$

For $Qh(e^{\mu\frac{Q}{D}}-2)$, taking first order derivative then setting it to zero gives us

$$\begin{aligned}\frac{\partial[Qh(e^{\mu\frac{Q}{D}}-2)]}{\partial Q}&=h[(1+\frac{\mu Q}{D})e^{\mu\frac{Q}{D}}-2]=0\\ \Rightarrow e^{\mu\frac{Q}{D}}&=\frac{2D}{D+\mu Q}.\end{aligned} \tag{5.42}$$

Obviously, the second order derivative of $Qh(e^{\mu\frac{Q}{D}}-2)$ is positive. Therefore

$$Qh(e^{\mu\frac{Q}{D}}-2)\geq Qh(\frac{2D}{D+\mu Q}-2)=\frac{-2\mu Q^2h}{D+\mu Q}. \tag{5.43}$$

Next, for $\frac{-2\mu Q^2h}{D+\mu Q}$, its first order derivative can be calculated as

$$\frac{\partial[\frac{-2\mu Q^2h}{D+\mu Q}]}{\partial Q}=\frac{2\mu Qh(-2D-\mu Q)}{(D+\mu Q)^2}<0,\quad\forall(Q,t)\in\mathbb{S}_4. \tag{5.44}$$

Hence, it can be concluded that

$$Qh(e^{\mu\frac{Q}{D}}-2)\geq\frac{-2\mu Q^2h}{D+\mu Q}\geq\frac{-2\mu\bar{Q}^2h}{D+\mu\bar{Q}}. \tag{5.45}$$

Consequently, if

$$\begin{aligned}\frac{\partial\mathbf{d}}{\partial t}|_{t=\frac{Q}{D}}&=D(C+\frac{h}{\mu})(e^{\mu\frac{Q}{D}}-1)+\mu(QC+K)e^{\mu\frac{Q}{D}}+Qh(e^{\mu\frac{Q}{D}}-2)\\ &\geq D(C+\frac{h}{\mu})(e^{\mu\frac{\underline{Q}}{D}}-1)+\mu(\underline{Q}C+K)e^{\mu\frac{\underline{Q}}{D}}-\frac{2\mu\bar{Q}^2h}{D+\mu\bar{Q}}>0,\end{aligned} \tag{5.46}$$

then the minimum point of $\mathbf{d}$ is located at $(\underline{Q},\frac{\underline{Q}}{D})$

For boundary $t=\alpha\frac{Q}{D}$,

$$\begin{aligned}\frac{\partial\mathbf{d}}{\partial t}|_{t=\alpha\frac{Q}{D}}=&D(C+\frac{h}{\mu})(e^{\mu\alpha\frac{Q}{D}}-1)+\mu\alpha[Q(C+\frac{h}{\mu})+K]e^{\mu\alpha\frac{Q}{D}}+2\alpha Qp\\ &-(p+h)\frac{D}{\mu}(\alpha-1)+(p+h)\frac{D}{\mu}(\alpha-1)e^{\mu(\alpha-1)\frac{Q}{D}}-(p+h)(1+\alpha^2)Q.\end{aligned} \tag{5.47}$$

For $(p+h)\frac{D}{\mu}(\alpha-1)e^{\mu(\alpha-1)\frac{Q}{D}}-(p+h)(1+\alpha^2)Q$, taking first order derivative and setting it to zero,

$$\begin{aligned}&\frac{\partial[(p+h)\frac{D}{\mu}(\alpha-1)e^{\mu(\alpha-1)\frac{Q}{D}}-(p+h)(1+\alpha^2)Q]}{\partial Q}\\ =&(p+h)\frac{D}{\mu}(\alpha-1)\frac{\mu(\alpha-1)}{D}e^{\mu(\alpha-1)\frac{Q}{D}}-(p+h)(1+\alpha^2)=0\\ \Rightarrow& e^{\mu(\alpha-1)\frac{Q}{D}}=\frac{1+\alpha^2}{(\alpha-1)^2}.\end{aligned} \tag{5.48}$$

Since the second order derivative of $(p+h)\frac{D}{\mu}(\alpha-1)e^{\mu(\alpha-1)\frac{Q}{D}} - (p+h)(1+\alpha^2)Q$ is obviously positive, we have

$$\begin{aligned}
&(p+h)\frac{D}{\mu}(\alpha-1)e^{\mu(\alpha-1)\frac{Q}{D}} - (p+h)(1+\alpha^2)Q \\
\geq&(p+h)\frac{D}{\mu}(\alpha-1)\frac{1+\alpha^2}{(\alpha-1)^2} - (p+h)(1+\alpha^2)Q \\
=&(p+h)(1+\alpha^2)[\frac{D}{\mu(\alpha-1)} - \bar{Q}]
\end{aligned} \tag{5.49}$$

As a result, if

$$\begin{aligned}
\frac{\partial \mathbf{d}}{\partial t}|_{t=\alpha\frac{Q}{D}} =&D(C+\frac{h}{\mu})(e^{\mu\alpha\frac{Q}{D}}-1) + \mu\alpha[Q(C+\frac{h}{\mu})+K]e^{\mu\alpha\frac{Q}{D}} + 2\alpha Qp \\
&-(p+h)\frac{D}{\mu}(\alpha-1) + (p+h)\frac{D}{\mu}(\alpha-1)e^{\mu(\alpha-1)\frac{Q}{D}} - (p+h)(1+\alpha^2)Q \\
\geq&D(C+\frac{h}{\mu})(e^{\mu\alpha\frac{\underline{Q}}{D}}-1) + \mu\alpha[\underline{Q}(C+\frac{h}{\mu})+K]e^{\mu\alpha\frac{\underline{Q}}{D}} + 2\alpha\underline{Q}p \\
&-(p+h)\frac{D}{\mu}(\alpha-1) + (p+h)(1+\alpha^2)[\frac{D}{\mu(\alpha-1)} - \bar{Q}] \\
>&0,
\end{aligned} \tag{5.50}$$

then the maximum point of $\mathbf{d}$ is located at $(\bar{Q}, \alpha\frac{\bar{Q}}{D})$. □

Proof. Lemma 12

Taking partial derivative with regard to Q,

$$\frac{\partial \mathbf{e}}{\partial Q} = (p+h) - (p+h)e^{\mu(t-\frac{Q}{D})} < 0, \quad \forall (Q,t) \in \mathbb{S}_4. \tag{5.51}$$

Partial derivative with regard to t can be calculated as

$$\frac{\partial \mathbf{e}}{\partial t} = D[-(C\mu+h) + (p+h)e^{-\mu\frac{Q}{D}}]e^{\mu t} - Dp. \tag{5.52}$$

Assume $-(C\mu+h)+(p+h)e^{-\mu\frac{\bar{Q}}{D}} > 0$, then

$$\frac{\partial \mathbf{e}}{\partial t} \leq D[-(C\mu+h) + (p+h)e^{-\mu\frac{\underline{Q}}{D}}]e^{\mu\alpha\frac{\bar{Q}}{D}} - Dp. \tag{5.53}$$

We further assume that $[-(C\mu+h)+(p+h)e^{-\mu\frac{\underline{Q}}{D}}]e^{\mu\alpha\frac{\bar{Q}}{D}} < p$, then we have $\frac{\partial \mathbf{e}}{\partial t} < 0$. In this case, the maximum value of $\mathbf{e}$ is obtained on $t = \frac{Q}{D}$, and the minimum value on $t = \alpha\frac{Q}{D}$.

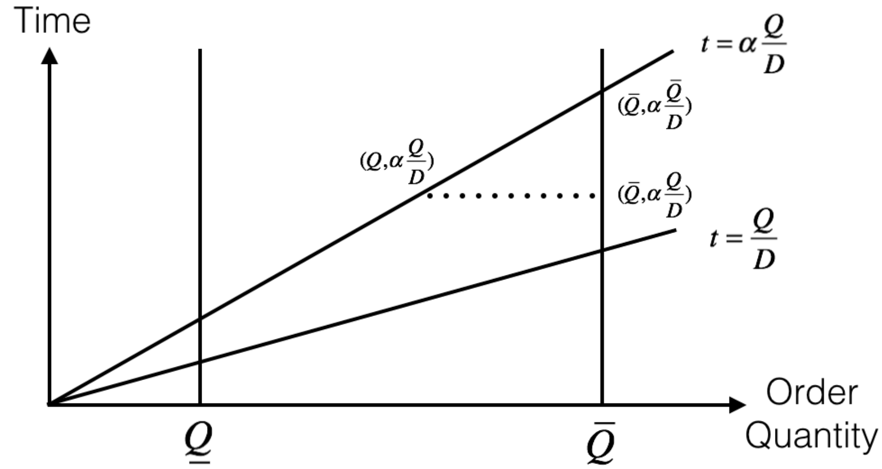

Figure 5.1: For the proof of Lemma 12,*class 1*.

On $t = \frac{Q}{D}$,

$$\frac{\partial \mathbf{e}}{\partial Q}|_{t=\frac{Q}{D}} = -D(C + \frac{h}{\mu})\frac{\mu}{D}e^{\mu\frac{Q}{D}} + h = -C\mu e^{\mu\frac{Q}{D}} - he^{\mu\frac{Q}{D}} + h < 0,, \quad \forall (Q,t) \in \mathbb{S}_4. \tag{5.54}$$

Hence the maximum point can be found at $(\underline{Q}, \frac{\underline{Q}}{D})$. Particularly,

$$\mathbf{e}(\underline{Q}, \frac{\underline{Q}}{D}) = -D(C + \frac{h}{\mu})(e^{\mu\frac{\underline{Q}}{D}} - 1) + h\underline{Q}. \tag{5.55}$$

Since $e^{\mu\frac{\underline{Q}}{D}} - 1 \geq \mu\frac{\underline{Q}}{D}$, it can be shown that

$$\mathbf{e}(\underline{Q}, \frac{\underline{Q}}{D}) \leq -D(C + \frac{h}{\mu})\mu\frac{\underline{Q}}{D} + h\underline{Q} = -\mu C\underline{Q} < 0. \tag{5.56}$$

The minimum point of $\mathbf{e}$, which locates on $t = \alpha\frac{Q}{D}$, can be identified by an easier approach. Points on boundary $t = \alpha\frac{Q}{D}$ can be classified into two groups. The first group contains points that horizontally correspond to points on $Q = \bar{Q}$ (Figure 5.1). This group can be defined as $\mathbb{S}_6 = \{(Q,t)|t = \alpha\frac{Q}{D}, \alpha Q \geq \bar{Q}, \underline{Q} \leq Q \leq \bar{Q}\}$. The second group holds points that are horizontally associated with points on $t = \frac{Q}{D}$ (Figure 5.2). A precise definition of the second group is $\mathbb{S}_7 = \{(Q,t)|t = \alpha\frac{Q}{D}, \alpha Q < \bar{Q}, \underline{Q} \leq Q \leq \bar{Q}\}$.

For $\forall (Q,t) \in \mathbb{S}_6$, we know that $\mathbf{e}(Q,t) > \mathbf{e}(\bar{Q},t)$ as $\frac{\partial \mathbf{e}}{\partial Q} < 0$. Furthermore, $\mathbf{e}(\bar{Q},t) > \mathbf{e}(\bar{Q}, \alpha\frac{\bar{Q}}{D})$, since by assumption $\frac{\partial \mathbf{e}}{\partial t} < 0$ (details shown in Figure 5.1). Additionally, for

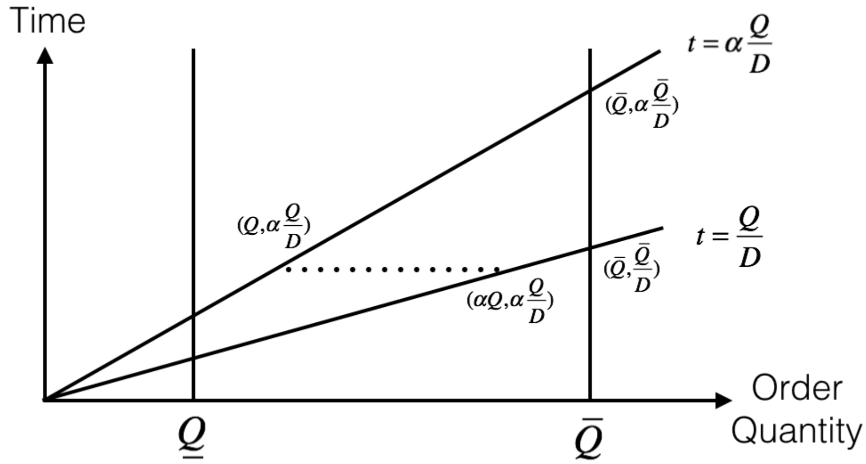

Figure 5.2: For the proof of Lemma 12,*class 2*.

$\forall(Q,t) \in \mathbb{S}_7$, since it was proven that $\frac{\partial \mathbf{e}}{\partial Q}|_{t=\frac{Q}{D}} < 0$, we have $\mathbf{e}(Q,t) > \mathbf{e}(tD,t) > \mathbf{e}(\bar{Q},\frac{\bar{Q}}{D}) > \mathbf{e}(\bar{Q},\alpha\frac{\bar{Q}}{D})$ (refer to Figure 5.2 for details). As a conclusion, the minimum point of $\mathbf{e}$ is obtained at $(\bar{Q},\alpha\frac{\bar{Q}}{D})$. $\square$

For the remaining lemmas, define

$$\begin{aligned}
\tilde{\mathbf{g}} &= \tilde{\mathbf{g}}(Q,t) = CD + (p+h)Q - tDp \\
\tilde{\mathbf{i}} &= \tilde{\mathbf{i}}(Q,t) = QCD + KD + \frac{1}{2}(p+h)(Q^2+t^2D^2) - QtDp \\
\tilde{\mathbf{j}} &= \tilde{\mathbf{j}}(Q,t) = \frac{1}{2}\mu(h+2p) - \frac{(p+h)Q}{tD}(\frac{1}{t}+\mu)
\end{aligned}$$

Proof. Lemma 13

For $\tilde{\mathbf{g}}$, taking partial derivatives with regard to t and Q, respectively,

$$\frac{\partial \tilde{\mathbf{g}}}{\partial t} = -pD < 0, \tag{5.57}$$

and

$$\frac{\partial \tilde{\mathbf{g}}}{\partial Q} = p+h > 0. \tag{5.58}$$

Furthermore, if $\frac{h+p}{p} > \alpha$, then we will always have $\tilde{\mathbf{g}} > 0$ within the feasible region.

Since

$$\frac{\partial \mathbf{g}}{\partial t} = 2\mu^2\tilde{\mathbf{g}}\frac{\partial \tilde{\mathbf{g}}}{\partial t} < 0, \quad \text{and} \quad \frac{\partial \mathbf{g}}{\partial Q} = 2\mu^2\tilde{\mathbf{g}}\frac{\partial \tilde{\mathbf{g}}}{\partial Q} > 0, \tag{5.59}$$

the maximum point of $\mathbf{g}$ is located on $t = \frac{Q}{D}$, and the minimum on $t = \alpha\frac{Q}{D}$. Clearly, for the boundary $t = \frac{Q}{D}$,

$$\begin{aligned} \mathbf{g}(Q, t = \frac{Q}{D}) &= \mu^2[CD + (p+h)Q - pQ]^2 = \mu^2(CD + hQ)^2 \\ &\leq \mu^2(CD + h\bar{Q})^2 = \mathbf{g}(\bar{Q}, t = \frac{\bar{Q}}{D}); \end{aligned} \tag{5.60}$$

on boundary $t = \alpha\frac{Q}{D}$

$$\begin{aligned} \mathbf{g}(Q, t = \alpha\frac{Q}{D}) &= \mu^2[CD + (p+h)Q - \alpha pQ]^2 = \mu^2\{CD + [h + (1-\alpha)p]Q\}^2 \\ &\geq \mu^2\{CD + [h + (1-\alpha)p]\underline{Q}\}^2 = \mathbf{g}(\underline{Q}, t = \alpha\frac{\underline{Q}}{D}). \end{aligned} \tag{5.61}$$

Note that the last inequality holds when $\frac{h+p}{p} > \alpha$. □

Proof. Lemma 14

For $\tilde{\mathbf{i}}$, taking partial derivative with regard to t,

$$\frac{\partial \tilde{\mathbf{i}}}{\partial t} = (p+h)tD^2 - pDQ. \tag{5.62}$$

Since $tD \geq Q$, we have $\frac{\partial \tilde{\mathbf{i}}}{\partial t} > 0$ within the feasible region. Therefore

$$\tilde{\mathbf{i}}(Q, t) \geq \tilde{\mathbf{i}}(Q, \frac{Q}{D}) = QCD + KD + hQ^2 > 0. \tag{5.63}$$

Hence, $\frac{\partial \mathbf{i}}{\partial t} = 2\mu^2 \tilde{\mathbf{i}} \frac{\partial \tilde{\mathbf{i}}}{\partial t} > 0$.

Additionally,

$$\frac{\partial \tilde{\mathbf{i}}}{\partial Q} = CD + (p+h)Q - pDt \geq CD + (p+h)Q - \alpha pQ, \quad \forall (Q, t) \in \mathbb{S}_4. \tag{5.64}$$

If $\frac{h+p}{p} > \alpha$, then $\frac{\partial \tilde{\mathbf{i}}}{\partial Q} > 0$, and $\frac{\partial \mathbf{i}}{\partial Q} = 2\mu^2 \tilde{\mathbf{i}} \frac{\partial \tilde{\mathbf{i}}}{\partial Q} > 0$

Obviously, the maximum point of $\mathbf{i}$ can be found on $t = \alpha\frac{Q}{D}$, and the minimum on $t = \frac{Q}{D}$. Therefore for the maximum value,

$$\begin{aligned} \mathbf{i}(Q, t = \alpha\frac{Q}{D}) &= \mu^2[QCD + KD + \frac{1}{2}p(1-\alpha)^2Q^2 + \frac{1}{2}h(1+\alpha^2)Q^2]^2 \\ &\leq \mu^2[\bar{Q}CD + KD + \frac{1}{2}p(1-\alpha)^2\bar{Q}^2 + \frac{1}{2}h(1+\alpha^2)\bar{Q}^2]^2 \\ &= \mathbf{i}(\bar{Q}, t = \alpha\frac{\bar{Q}}{D}); \end{aligned} \tag{5.65}$$

for the minimum value,

$$\begin{aligned}\mathbf{i}(Q, t = \frac{Q}{D}) &= \mu^2[QCD + KD + hQ^2]^2 \geq \mu^2[\underline{Q}CD + KD + h\underline{Q}^2]^2 \\ &= \mathbf{i}(\underline{Q}, t = \frac{\underline{Q}}{D}).\end{aligned} \tag{5.66}$$

$\square$

Proof. Lemma 15

Taking partial derivatives with regards to t and Q, respectively,

$$\begin{aligned}\frac{\partial \tilde{\mathbf{j}}}{\partial t} &= \frac{(p+h)Q}{t^2 D}(\frac{1}{t} + \mu) + \frac{(p+h)Q}{t^3 D} > 0, \\ \frac{\partial \tilde{\mathbf{j}}}{\partial Q} &= -\frac{(p+h)}{tD}(\frac{1}{t} + \mu) < 0.\end{aligned} \tag{5.67}$$

Hence the maximum value of $\tilde{\mathbf{j}}$ is located on $t = \alpha\frac{\bar{Q}}{D}$, and

$$\tilde{\mathbf{j}} \leq 0.5\mu(h + 2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha Q} + \mu) \leq 0.5\mu(h + 2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha\bar{Q}} + \mu) \tag{5.68}$$

If it is assumed that $0.5\mu(h + 2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha\bar{Q}} + \mu) < 0$, then we have $\tilde{\mathbf{j}} < 0$. Moreover, $\frac{\partial \mathbf{j}}{\partial t} = 2\tilde{\mathbf{j}}\frac{\partial\tilde{\mathbf{j}}}{\partial t} < 0$, and $\frac{\partial \mathbf{j}}{\partial Q} = 2\tilde{\mathbf{j}}\frac{\partial\tilde{\mathbf{j}}}{\partial Q} > 0$. Hence the maximum value of $\mathbf{j}$ can be found on $t = \frac{\bar{Q}}{D}$, and the minimum on $t = \alpha\frac{\bar{Q}}{D}$. Particularly,

$$\begin{aligned}\mathbf{j}(Q, t = \frac{Q}{D}) &= [\frac{1}{2}\mu(h + 2p) - (p+h)(\frac{D}{Q} + \mu)]^2 \\ &\leq [\frac{1}{2}\mu(h + 2p) - (p+h)(\frac{D}{\underline{Q}} + \mu)]^2 = \mathbf{j}(\underline{Q}, t = \frac{\underline{Q}}{D}) \\ \mathbf{j}(Q, t = \alpha\frac{Q}{D}) &= [\frac{1}{2}\mu(h + 2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha Q} + \mu)]^2 \\ &\geq [\frac{1}{2}\mu(h + 2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha\bar{Q}} + \mu)]^2 = \mathbf{j}(\bar{Q}, t = \alpha\frac{\bar{Q}}{D})\end{aligned} \tag{5.69}$$

$\square$

Proof. lemma 16

Taking second order partial derivative with regard to t,

$$\begin{aligned}\frac{\partial^2 \mathbf{k}}{\partial t^2}/\mu^2 &= \frac{\partial}{\partial t}(\frac{\partial \tilde{\mathbf{g}}}{\partial t}\tilde{\mathbf{i}} + \tilde{\mathbf{g}}\frac{\partial \tilde{\mathbf{i}}}{\partial t}) = -3pD^3(p+h)t + 2p^2D^2Q + (p+h)^2D^2Q + (p+h)D^3C \\ &\geq -3p(p+h)D^2\alpha Q + 2p^2D^2Q + (p+h)^2D^2Q + (p+h)D^3C \\ &= [-3p(p+h)\alpha + 2p^2 + (p+h)^2]D^2Q + (p+h)D^3C, \quad \forall (Q,t) \in \mathbb{S}_4\end{aligned} \tag{5.70}$$

Assume $[-3p(p+h)\alpha+2p^2+(p+h)^2]D^2Q+(p+h)D^3C > 0, \forall Q \in [\underline{Q}, \bar{Q}]$, then the minimum value for $\frac{\partial \mathbf{k}}{\partial t}$ is located on $t = \frac{Q}{D}$, and

$$\begin{aligned}&\frac{\partial \mathbf{k}}{\partial t}/\mu^2 \geq \frac{\partial \mathbf{k}}{\partial t}|_{t=\frac{Q}{D}}/\mu^2 \\ &= [-1.5p(p+h)(1+\alpha^2)+2p^2+(p+h)^2]DQ^2+(h-p)CD^2Q-KpD^2\end{aligned} \tag{5.71}$$

If it is further assumed that $[-1.5p(p+h)(1+\alpha^2)+2p^2+(p+h)^2]DQ^2+(h-p)CD^2Q-KpD^2 > 0, , \forall Q \in [\underline{Q}, \bar{Q}]$, then we know that the maximum value of $\mathbf{k}$ is located on $t = \alpha\frac{Q}{D}$, and the minimum on $t = \frac{Q}{D}$. More specifically,

$$\begin{aligned}\mathbf{k}(Q,t) \geq \mathbf{k}(Q, \frac{Q}{D}) &= \mu^2[CD+hQ][QCD+KD+hQ^2] \\ &\geq \mu^2[CD+h\underline{Q}][\underline{Q}CD+KD+h\underline{Q}^2] = \mathbf{k}(\underline{Q}, \frac{\underline{Q}}{D})\end{aligned} \tag{5.72}$$

For the maximum value, assume that $\frac{h+p}{p} > \alpha$.

$$\begin{aligned}&\mathbf{k}(Q,t) \leq \mathbf{k}(Q, \alpha\frac{Q}{D}) \\ &= \mu^2\{CD+[h+(1-\alpha)p]Q\}\{QCD+KD+[0.5(p+h)(1+\alpha^2)-p\alpha]Q^2\} \\ &\leq \mu^2\{CD+[h+(1-\alpha)p]\bar{Q}\}\{\bar{Q}CD+KD+[0.5(p+h)(1+\alpha^2)-p\alpha]\bar{Q}^2\} \\ &= \mathbf{k}(\bar{Q}, \alpha\frac{\bar{Q}}{D})\end{aligned} \tag{5.73}$$

Note that the inequality holds because $0.5(p+h)(1+\alpha^2)-p\alpha \geq 0.5h(1+\alpha^2)+0.5p(1+\alpha^2-2\alpha) > 0$ □

Proof. Lemma 17

If $\frac{p+h}{p} > \alpha$, and $\frac{\mu}{2}(h+2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha Q}+\mu) < 0$, then

$$\frac{\partial \mathbf{l}}{\partial t} = \mu\frac{\partial \tilde{\mathbf{g}}}{\partial t}\tilde{\mathbf{j}} + \mu\tilde{\mathbf{g}}\frac{\partial \tilde{\mathbf{j}}}{\partial t} > 0. \tag{5.74}$$

Therefore,

$$\begin{aligned}\mathbf{l}(Q,t) \geq \mathbf{l}(Q, \frac{Q}{D}) &= \mu(CD+hQ)[0.5\mu(h+2p)-(p+h)(\frac{D}{Q}+\mu)] \\ &\geq \mu(CD+h\bar{Q})[0.5\mu(h+2p)-(p+h)(\frac{D}{\underline{Q}}+\mu)].\end{aligned} \tag{5.75}$$

The last step holds since $\tilde{\mathbf{j}} < 0$. Similarly, when $\frac{p+h}{p} > \alpha$,

$$\begin{aligned}\mathbf{l}(Q,t) \leq \mathbf{l}(Q, \alpha\frac{Q}{D}) &= \mu\{CD+[h+(1-\alpha)p]Q\}[0.5\mu(h+2p)-\frac{p+h}{\alpha}(\frac{D}{\alpha Q}+\mu)] \\ &\leq \mu\{CD+[h+(1-\alpha)p]\underline{Q}\}[0.5\mu(h+2p)-\frac{p+h}{\alpha}(\frac{D}{\alpha\bar{Q}}+\mu)].\end{aligned} \tag{5.76}$$

□

Proof. Lemma 18

Taking second order partial derivative with regard to t,

$$\begin{aligned}\frac{\partial^2 \mathbf{r}}{\partial t^2}/\mu =&(p+h)D^2[0.5\mu(h+2p) - \frac{(p+h)Q}{tD}(\frac{1}{t}+\mu)] \\ &+ 2[(p+h)tD^2 - pDQ][\frac{(p+h)Q}{D}(\frac{2}{t^3}+\frac{\mu}{t^2})] \\ &- [QCD + KD + 0.5(p+h)(Q^2+t^2D^2) - pDtQ][\frac{(p+h)Q}{D}(\frac{6}{t^4}+\frac{2\mu}{t^3})] \\ =&0.5\mu(p+h)(2p+h)D^2 \\ &+ \frac{(p+h)Q^2}{t^2}[p(\frac{4}{t}+\mu) - 0.5(p+h)\frac{Q}{D}(\frac{6}{t^2}+\frac{2\mu}{t}) - (C+\frac{K}{Q})(\frac{6}{t^2}+\frac{2\mu}{t})].\end{aligned} \tag{5.77}$$

Since $\frac{Q}{D} \leq t \leq \frac{Q}{D}$,

$$\begin{aligned}\frac{\partial^2 \mathbf{r}}{\partial t^2}/\mu \leq& 0.5\mu(p+h)(2p+h)D^2 \\ &+ (p+h)D^2[p(\frac{4}{t}+\mu) - 0.5(p+h)\frac{1}{\alpha}(\frac{6}{t}+2\mu) - (C+\frac{K}{Q})(\frac{6}{t^2}+\frac{2\mu}{t})] \\ \leq&(p+h)D^2[\mu(0.5h+p) + p(\frac{4}{t}+\mu) - 0.5(p+h)\frac{1}{\alpha}(\frac{6}{t}+2\mu)] \\ =&(p+h)D^2[\mu(0.5h+2p-\frac{p+h}{\alpha}) + (4p - \frac{3(p+h)}{\alpha})\frac{1}{t}]\end{aligned} \tag{5.78}$$

Assume that $\mu(0.5h+2p-\frac{p+h}{\alpha}) + (4p-\frac{3(p+h)}{\alpha})\frac{1}{t} < 0$ for $\forall (Q,t) \in \mathbb{S}_4$, then

$$\begin{aligned}&(\frac{\partial \mathbf{r}}{\partial t}|_{(Q,t)})/\mu \geq (\frac{\partial \mathbf{r}}{\partial t}|_{(Q,\alpha\frac{Q}{D})})/\mu \\ =&[(\alpha-1)p+\alpha h]D[\mu(\frac{h+2p}{2} - \frac{p+h}{\alpha})Q - \frac{p+h}{\alpha^2}D] \\ &+ \frac{p+h}{\alpha^2}D(\frac{2D}{\alpha}+\mu Q)[0.5h(1+\alpha^2) + 0.5p(1-\alpha)^2 + \frac{CD}{Q} + \frac{KD}{Q^2}]\end{aligned} \tag{5.79}$$

When the assumptions stated in Lemma 18 hold, we have $\frac{\partial \mathbf{r}}{\partial t} > 0$. Therefore,

$$\begin{aligned}
&\mathbf{r}(Q,t) \geq \mathbf{r}(Q, \frac{Q}{D})\\
=&\mu[hQ^2 + CDQ + KD][0.5\mu(h+2p) - (p+h)(\frac{D}{Q} + \mu)]\\
\geq&\mu[h\bar{Q}^2 + CD\bar{Q} + KD][0.5\mu(h+2p) - (p+h)(\frac{D}{\underline{Q}} + \mu)]\\
&\mathbf{r}(Q,t) \leq \mathbf{r}(Q, \alpha\frac{Q}{D})\\
=&\mu\{[0.5h(1+\alpha^2) + 0.5p(1-\alpha)^2]Q^2 + CDQ + KD\}[0.5\mu(h+2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha Q} + \mu)]\\
\leq&\mu\{[0.5h(1+\alpha^2) + 0.5p(1-\alpha)^2]\underline{Q}^2 + CD\underline{Q} + KD\}[0.5\mu(h+2p) - \frac{p+h}{\alpha}(\frac{D}{\alpha\bar{Q}} + \mu)]
\end{aligned} \tag{5.80}$$

□

Proof of Theorem 19

Proof. Theorem 19

When Lemma 9 to Lemma 18 hold, $\boldsymbol{a} > 0$, and $\boldsymbol{b} > 0$, we have $\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t^2} > 0$. Additionally, (5.11) is the lower bound of (5.10). Convexity follows since both $\frac{\partial^2 E[C^{(2)}(Q,t)]}{\partial t^2}$ and (5.11) are positive. □

Chapter 6

Conclusion

A huge volume of corporate barter exchanges is executed every year, and billions of dollars are recovered. Thus, the issue deserves a well-defined model and an in-depth study. This study investigates the dynamics of a corporate barter platform, and explores the participants' waiting time and the system performance under different policies and exchange types. The behaviors of the systems are investigated. We focus in particular on the probability that a participant will find an exchange opportunity and in his/her corresponding waiting time before being able to barter. Extended from the modeling and analysis of online with pairwise exchange, participant's strategies are developed.

In this study, models are built to describe the ability of participants to conduct an exchange under an online/batch policy and pairwise/three-way exchange types. Under the online policy, customers are guaranteed to be able to (eventually) find exchange partners, and the exchange system is always stable to provide service. The waiting time was proven to be exponentially distributed in a pairwise exchange, and the study showed that the waiting time in a three-way exchange is well upper bounded by that in a pairwise exchange in the majority of the matching probabilities, and is very close to that in pairwise exchange in the remaining cases. In addition to waiting time, a three-way exchange also surpasses a pairwise exchange in that a significantly larger proportion of participants are able to conduct an exchange when the matching probability is small. Under the batch policy, customers are not guaranteed an exchange opportunity. The overall chance of being able to conduct an exchange varies according to the system parameters. We show that the probability of finding exchange partner(s), as well as the system performance and profitability, can be dramatically improved if the corresponding parameters exceed the sharp thresholds, and this can be achieved by increasing the system capacity and/or enlarging the homogeneous matching probability. Unlike under the online policy, the exchange system is always stable under the batch policy because of the capacity constraint (pool size). The expectation and variance of customer waiting time are derived for a pairwise exchange, and asymptotical behaviors are analyzed for a three-way exchange. The results show that pairwise exchange performs better when the matching probability is extremely small, but three-way exchange is preferable in all other cases, although the difference is eliminated as the matching probability increases.

We recommend that in cases of extremely small matching probability, participants attempt to use the batch policy with pairwise exchange option, because the waiting time is well bounded, and the resulting return is always no worse than the salvage value. For time insensitive participants, the online policy with three-way exchange is an option, as it guarantees exchange opportunities within a predictable time. When the matching probability is greater than 0.05, the online is always superior to the batch policy.

Extended from modeling and analysis of corporate barter exchange mechanisms, we explored participants' strategies on a barter platform. Assume that a participant implements EOQ model for the procurement of production materials, and that he is intended to trade-out excessive inventories in exchange for production materials. If the barter platform is operated under online with pairwise exchange mechanism, it is suggested that, upon satisfaction of certain conditions, the participant should try barter exchange and wait for future opportunities if he cannot be matched to exchange partner(s) upon arrival to the platform. We found that a good procurement strategy of production materials for the participant while waiting on the barter platform, in case back-order in production materials is not permitted, is to maintain the regular procurement plan with the usual supplier. In other words, there should be no changes if no barter opportunity is found. Particularly, this strategy is the reason we recommend online with pairwise barter exchange for participants implementing EOQ model for procurement activities and wishing to trade-in production materials. Since there is no change in the procurement plan, it barely incurs any cost to barter, and participants can potentially wait forever for a future opportunity. As summarized in the second chapter, under online with pairwise exchange mechanism, barter opportunity is guaranteed if the participant is willing to wait. Therefore, under the circumstance described above, it is always beneficial to barter and wait for future opportunities if there is none upon arrival.

On the other hand, if back-order is allowed in production materials during the waiting time on the platform, there may be a better strategy available than the optimal one when no back-order is allowed. The strategy, in this case, is better in terms of reduced total cost including ordering cost, holding cost, and penalty cost for back-order. We are able to find a concise sufficient condition that justifies if the optimal strategy with back-order is more appealing than that with no back-order. Moreover, it is proven that the sufficient condition is necessary as well if the cost function is convex.

Finally, we suggest that participants separate trade-in and trade-out activities. Though simultaneous trade-in and trade-out is an important property of barter exchange, and breaking the simultaneity may introduce negative effects, we found that separation of trade-in and trade-out processes significantly reduces the total waiting time, and the reduction in waiting time may outweigh the negative effects resulted from non-simultaneous activities.

To the best of our knowledge, this study is the first attempt to investigate qualitatively corporate barter platforms with a stochastic arrival process and random simultaneous exchange opportunities, and provide strategies to participants accordingly. While presenting practical strategies for participants, the study does leave some holes unsolved. We did not find closed form solutions for the optimal procurement strategy, though it is fairly easy to find a good strategy when back-order is not permitted, and to judge if back-order would be beneficial than no back-order. Convexity of the cost function with back-order is difficult to justify, and the current approach is not fully developed. These two aspects of the study deserve a careful investigation and a deeper dive in order to facilitate a better understanding of the participant's strategy. Another potential research direction is to involve virtual currencies in corporate barter exchanges. On modern corporate barter platforms, such as the National Association of Trade Exchanges and IRTA, virtual currencies are created. Participants can obtain virtual currency by trading out excessive inventory, and then purchase the desired commodities from other companies using virtual currency. Clearly, a common measure is created in the exchanges, and the inevitable requirement of simultaneity in a traditional barter is weakened in this case. With these advantages, it can be anticipated that virtual currency will be the trend in corporate barters, and thus, an investigation of the system is worthwhile.

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