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isomorphic to $Z \times Z_2$. By Theorem (9.5) of [2],

$$C \times D \approx \pi_1(M) \approx (Z \times Z_2) * G,$$

for some group G . By Folgerung 4 of [1], $G = 1$. Therefore $\pi_1(M) \approx Z \times Z_2$.

Lemmas (6a.2) and (6b.3) imply Theorem (6.1).

Theorems (4.1), (5.1) and (6.1) imply the theorem stated at the beginning of the paper.

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On Combinatorial Submanifolds of Differentiable Manifolds¹⁾

by MORRIS W. HIRSCH, California (USA)

§ 1. Introduction

The purpose of this work is to prove the following results relating combinatorial and differentiable manifolds.

(A) *A combinatorial submanifold V of a differentiable manifold M of the same dimension possesses a compatible differentiable structure.*²⁾

(B) *Every compact and contractible combinatorial manifold V possesses a compatible differentiable structure.*²⁾

(A differentiable structure on a combinatorial manifold M is called *compatible* if M has a rectilinear subdivision, each simplex of which is differentially imbedded.)

A. M. GLEASON has announced (unpublished) that a contractible *unbounded* combinatorial manifold has a compatible differentiable structure. Theorem (B) follows easily from this and Theorem (A). The proof of (B) given here is derived from JOHN STALLINGS' proof [11] of the generalized POINCARÉ conjecture.

(C) *The sequence*

$$\dots \xrightarrow{d} I^n \xrightarrow{j} \Theta^n \xrightarrow{k} A^n \xrightarrow{d} I^{n-1} \longrightarrow \dots \quad (1)$$

is well defined and exact.

This result was announced in [3]. Here I^n is the group of differentiable structures on S^n compatible with the usual combinatorial structure; Θ^n is the group of differentiable homotopy n -spheres modulo J -equivalence, and A^n the combinatorial analogue of Ω^n . Using powerful intrinsic methods, STEPHEN SMALE has shown [9, 10] that for $n \geq 8$ or $n = 5$, the map $j: I^n \rightarrow \Theta^n$ is an isomorphism, and using (B) (but not (C)) that $A^n = 0$, for $n \geq 8$ or $n = 6$.

In [8] SMALE proves that $I^3 = 0$, (this was proved independently by J. MUNKRES, and J. H. C. WHITEHEAD.) The fact that every combinatorial 3-manifold possess a unique (up to a diffeomorphism) compatible differentiable structure [6] implies that $k: \Theta^3 \rightarrow A^3$ is an isomorphism. Thus only the

¹⁾ Presented at the International Colloquium on Differential Geometry and Topology, Zürich, June 1960.

²⁾ Added in proof; The hypothesis of compactness is unnecessary.

subsequences $0 \rightarrow \Gamma^7 \rightarrow \Theta^7 \rightarrow \Lambda^7 \rightarrow \Gamma^6 \rightarrow 0$ and $0 \rightarrow \Lambda^5 \rightarrow \Gamma^4 \rightarrow \Theta^4 \rightarrow \Lambda^4 \rightarrow 0$ remain. In proving (C), SMALE's results are not used.

§ 2. Proof of (A)

In order to prove (A) it suffices to establish the stronger result 2.5 below.

If K is a subcomplex of complex N , the n th simplicial neighborhood of K is the union of the closed simplexes of the n 'th barycentric subdivision of N that meet K .

Lemma 2.1. *Let K be the boundary of a combinatorial manifold M . The second simplicial neighborhood A of K is combinatorially equivalent to $K \times I$, where I is the unit interval.*

Proof. This is a well known result. It follows e. g. from theorems 22 and 23 of [14], which state that any two "regular neighborhoods" of M (in the sense of [14]) in the same manifold are combinatorially equivalent and that A is a regular neighborhood of K in M . If K is identified with $K \times 0$, then $M' = M \cup K \times I$ is a manifold, and in M' both A and $K \times I$ are regular neighborhoods of K .

Now let V be a bounded combinatorial n -manifold imbedded as a subcomplex of an unbounded combinatorial n -manifold M . Assume M has a metric $d(x, y)$.

Lemma 2.2. *Let U be a neighborhood of V in M , and ϵ a positive continuous function on M . There is a semi-linear homeomorphism $h: M \rightarrow M$ with the following properties:*

- a) $h(V)$ is the second simplicial neighborhood of V in a subdivision of M ;
- b) $h(V) \subset U$;
- c) $h(x) = x$ if $x \in M - U$;
- d) $d(x, h(x)) < \epsilon(x)$ for all $x \in M$.

Proof. By 2.1, the boundary of K of V has a neighborhood combinatorially equivalent to $K \times I$ in V , and another in $cl(M - V)$. The union B_0 of these two neighborhoods is again equivalent to $K \times I$. Moreover, we can take B_0 to be the second simplicial neighborhood of K in a subdivision of M ; if this subdivision is sufficiently fine, the second simplicial neighborhood B of B_0 will be in U . It will be clear that if the subdivision is sufficiently fine, d) will be satisfied. There is a combinatorial equivalence $u: B \rightarrow K \times I$ such that $u(B_0) = K \times [1/4, 3/4]$ and $u(K) = K \times 1/2$. We may assume that $h(B_0 \cap V) = K \times [0, 1/2]$. Let $f: I \rightarrow I$ be a semi-linear homeomorphism

such that $f(x) = x$ for x in a neighborhood of 0 and 1, and $f(1/2) = 3/4$. Define $g: K \times I \rightarrow K \times I$ by $g(x, t) = (x, f(t))$. Now define $h: M \rightarrow M$ by $h(x) = \begin{cases} x & \text{if } x \in M - B \\ u^{-1}gu(x) & \text{if } x \in B. \end{cases}$

Then h is the desired homeomorphism.

Now let M be a differentiable manifold. A combinatorial manifold A which is a subcomplex of a smooth triangulation of M is called a *combinatorial submanifold* of M . A vector field Φ on A in M is *transverse* if it is transverse to A in every coordinate system, in the sense of [13]. The following lemma is well known.

Lemma 2.3. *Let A be the boundary of the second simplicial neighborhood B of a subcomplex K of M . Then there is a transverse field on A .*

Proof. Each simplex ρ of the first simplicial neighborhood B' of K is the join $\sigma * \tau$ of unique simplices $\sigma \subset K$ and $\tau \subset M - K$. Each closed simplex α of A lies in such a join $\sigma * \tau$, disjoint from σ and τ , and each $x \in \alpha$ lies on a unique line $p * q$ with $p \in \sigma$, $q \in \tau$. It is easily seen that the unit tangent $\Phi(x)$ to $p * q$, directed from p to q , is transverse to α at x , and that Φ is continuous. Thus Φ is a transverse field on A .

Lemma 2.4. *Let M be an unbounded differentiable n -manifold, and $V \subset M$ a combinatorial submanifold, also of dimension n . Let U be a neighborhood of the boundary A of V , d a metric on M , and ϵ a positive continuous function on M . There is a homeomorphism $h: M \rightarrow M$ such that*

- a) h is a diffeomorphism on each closed simplex of a subdivision of M ;
- b) $h(A)$ has a transverse field;
- c) $h(x) = x$ if $x \in M - U$;
- d) $d(x, h(x)) < \epsilon(x)$ for all $x \in M$.

Proof. Apply 2.2 and 2.3.

Let V be a combinatorial submanifold of a differentiable unbounded n -manifold M . Assume either

- 1. V has dimension n ; or
- 2. V has dimension $n - 1$, is unbounded, and admits a transverse field.

Let U be a neighborhood of ∂V and ϵ a positive continuous function on M .

Theorem 2.5. *There is a homeomorphism $h: M \rightarrow M$ such that:*

- a) $h(V)$ is a differentiable submanifold of M , combinatorially equivalent to V ;

b) M has a smooth triangulation in which V is a subcomplex, every closed simplex of which is mapped diffeomorphically by h ;
 c) and d) as in 2.4.

Proof. Case 1) follows from 2.1 and case 2); Thus we assume 2).

By standard approximation methods, it may be assumed that there is a differentiable non-zero vector field Φ on a neighborhood W of V contained in U , such that $\Phi|_V$ is transverse field. A generalization of the CAIRNS-WHITEHEAD theory of transverse fields [1, 13] shows that there is a submanifold C of dimension $n-1$ differentially imbedded in an arbitrary neighborhood of V , such that $\Phi|_C$ is transverse. (The CAIRNS-WHITEHEAD theory applies to a q -dimensional submanifold of EUCLIDEAN $(q+p)$ -space endowed with a transverse p -plane field. The present case follows, e.g., by imbedding M in R^{n+k} , and assigning to each point $x \in V$ the $k+1$ plane generated by $\Phi(x)$ and the k -plane normal to M at x . Alternatively, the methods of [13] simplify considerably in the special case where the submanifold has codimension 1, if transverse lines are replaced by the integral curves of a transverse vector field.³⁾ We can assume that each integral curve of Φ meets V in a unique point and C in a unique point. This establishes a map $f: V \rightarrow C$ which is a diffeomorphism on each closed simplex of V . Thus C is combinatorially equivalent to V . Let A be the region bounded by V and C which is fibered by the integral curves of Φ . Define $G: V \times I \rightarrow A$ by $G(x, 0) = x$, $G(x, 1) = f(x)$ and $G(x, t)$ is the point dividing the length of the integral curve joining x and $f(x)$ in the ratio $t|(1-t)$, for $0 < t < 1$. Then if Δ is a closed simplex of V , $G|_{\Delta \times I}$ is a diffeomorphism. Hence $G: K \times I \rightarrow M$ is a non-degenerate C^∞ subcomplex of M in the sense of [15]. By [15, p. 822, addendum] this triangulation of A can be extended to a smooth triangulation of M , after possible subdivision. An easy application of 2.1 (cf. proof of 2.2) establishes the desired extension $h: M \rightarrow M$ of $f: V \rightarrow C$. This completes the proof.

Remark. It can be shown that if a neighborhood in V of a closed subset $X \subset V$ is a differentiable submanifold of M , h can be chosen so that for some neighborhood Y of X in M , $h(x) = x$ if $x \in Y$.

The following theorem was announced by S. S. Cairns [16].

Theorem 2.6 (Cairns). *If M is a combinatorial manifold and if for some p , $M \times R^p$ has a compatible differentiable structure, then so has M .*

Proof. By induction on p . The case $p = 0$ is trivial. Let $F^p \subset R^p$ be a closed half-space. First assume M is unbounded. If $p > 0$ and if $M \times R^p$

³⁾ Cf. [18].

has a compatible differentiable structure, then so has $M \times F^p$, by (A). So therefore does its boundary $M \times R^{p-1}$, completing the induction. The case where M is bounded follows easily now from (A).

§ 3. Proof of (B)

Let V be an n -dimensional compact combinatorial manifold which is contractible. Let $\tilde{V}$ be the double of V , obtained by identifying two disjoint copies of V along their boundary. Then $\tilde{V}$ is a closed combinatorial n -manifold of the same homotopy type as S^n , and V is a submanifold. J. STALLINGS proves in [11] that if x is any point of $\tilde{V}$, then $\tilde{V} - x$ is combinatorially equivalent to EUCLIDEAN n -space, provided $n \geq 7$, and states that E. C. ZEEMAN has extended the result to the case $n \geq 5$. (If $n \leq 4$, theorem (B) is a consequence of well known results of CAIRNS [1, 2].) Thus we can assume that V is a submanifold of the differentiable manifold R^n . (Alternatively, $\tilde{V} - x$ is an unbounded contractible manifold, and one can apply GLEASON's theorem that $\tilde{V} - x$ has a compatible differentiable structure). Theorem (B) now follows from (A). Actually, this proves the following stronger result.

Theorem 3.1. *A compact, contractible, combinatorial n -manifold is combinatorially equivalent to a differentiable submanifold of R^n .*

GLEASON's theorem is proved by showing that an unbounded combinatorial contractible n -manifold can be immersed in R^n . This follows from the mere existence of a compatible differentiable structure by observing that such a structure is necessarily parallelizable, and applying a theorem of [4].⁴⁾

A plausible conjecture along these lines is that any combinatorial manifold all of whose cohomology groups vanish has a compatible differentiable structure. J. MUNKRES [7] has proved this except for compatibility.

§ 4. Proof of (C)

We must show that the sequence

$$\xrightarrow{d} \Gamma^n \xrightarrow{j} \Theta^n \xrightarrow{k} \Lambda^n \xrightarrow{d} \Gamma^{n-1} \longrightarrow \quad (1)$$

is well defined and exact.

The group Θ^n is defined as follows. An element of Θ^n is an equivalence class $[M]$ of oriented, closed differentiable manifolds M which have the homo-

⁴⁾ GLEASON's theorem follows from 2.6 and [17], in which it is proved that if M^n is a contractible combinatorial unbounded manifold, then $M^n \times R^p$ is combinatorially equivalent to R^{n+p} for some p .

topology type of the n -sphere S^n , under the relation of J -equivalence. Two oriented differentiable manifolds M_0, M_1 are J -equivalent if there is an oriented differentiable manifold N whose boundary is (diffeomorphic to) the disjoint union of M_1 and $-M_0$ (where $-M_0$ means M_0 with the opposite orientation), and such that both M_0 and M_1 are deformation retracts of N . Addition in Θ^n is defined by $[A] + [B] = [A \# B]$ where $A \# B$ is the connected sum of A and B . This is defined by removing the interior of an n -ball from each of A and B and joining the two boundary $(n-1)$ -spheres by an orientation reversing diffeomorphism which is extendable to the whole n -ball, and then smoothing the resulting corner. It can be shown that the diffeomorphism class of $A \# B$ is independent of the choices made, and the J -equivalence class $[A \# B]$ is independent of the representatives of $[A]$ and $[B]$ that are chosen. See [5] for details. Define $-[A] = [-A]$, and Θ^n becomes an abelian group, with $[S^n]$ as identity element.

Using combinatorial instead of differentiable manifolds, J -equivalence and connected sum are analogously defined, and A^n is the group of J -equivalence classes $\langle M \rangle$ of oriented combinatorial closed n -manifolds M which are homotopy spheres.

The elements of Γ^n are (diffeomorphism classes of) oriented differentiable manifolds which are combinatorially equivalent to the boundary of an $(n+1)$ -simplex. Addition is defined using $\#$, and the inverse of $M \in \Gamma^n$ is $-M$. Using these definitions, Γ^n is an abelian group, with S^n for 0, although this is not obvious. It is a consequence of the following result.

Theorem 4.1. (MUNKRES-THOM). *There is at most one compatible differentiable structure on a contractible combinatorial n -manifold, up to a diffeomorphism.*

Proof. See [6, 12]

The only difficulty about proving that Γ^n is a group is showing

$$M \# (-M) = S^n.$$

We can obtain $M \# (-M)$ by removing the interior E of an n -ball from M , and taking the boundary V of $(M - E) \times I$ and smoothing the corner. But $(M - E) \times I$ is then a differentiable manifold combinatorially equivalent to $A^n \times I$, where A^n is an n -simplex, and by 4.1 its boundary $V = M \# (-M)$ is diffeomorphic to S^n , which is the zero element of Γ^n .

The map $k: \Theta^n \rightarrow A^n$ is defined as follows. If M is a differentiable manifold, let kM be the corresponding combinatorial manifold, i. e., kM is a simplicial complex L such that there is a homeomorphism $t: L \rightarrow M$ which is a smooth triangulation of M . Such an L exists and is unique up to combinatorial equivalence [15].

Now define $k: \Theta^n \rightarrow A^n$ by $k[M] = \langle kM \rangle$. It is obvious that k preserves sums and J -equivalence, so k is a well defined homomorphism.

The map $j: \Gamma^n \rightarrow \Theta^n$ is defined by $jM = [M]$.

To define $d: A^n \rightarrow \Gamma^{n-1}$, let M represent an element of A^n , and let E be the interior of an n -simplex of M . Then $M - E$ is contractible, and by (B) possesses a compatible differentiable structure, which is unique by 4.1, up to diffeomorphism. The combinatorial structure of $M - E$ is independent of E , and dM is defined to be the boundary of $M - E$. We shall see shortly that if A and B are J -equivalent, then $dA = dB$.

Lemma 4.2. a) $d(M \# N) = dM + dN$

b) $d(-M) = -dM$.

Proof. Let C and D be closed n -simplices in M and N respectively. In forming $M \# N$, remove the interiors of n -simplices disjoint from C and D . Now M -int C and N -int D have unique compatible differentiable structures by (B) and 4.1, and then $(M \# N) - (\text{int } C \cup \text{int } D) = (M\text{-int } C) \# (N\text{-int } D)$ has a compatible differentiable structure. Now join C to D in $M \# N$ by a simple differentiable arc, meeting C and D only at its end points. A tubular neighborhood Q of this arc can be chosen so that $C \cup D \cup Q$ is a combinatorial n -cell in $M \# N$. Then $M\text{-int } (C \cup D \cup Q)$ has $d(M \# N)$ for its boundary (after smoothing). On the other hand, this boundary is diffeomorphic to $\partial(M\text{-int } C) \# \partial(N\text{-int } D) = d(M) \# d(N)$, which proves a). The proof of b) is obvious.

Lemma 4.3. *Let M be a closed oriented combinatorial homotopy n -sphere. If M has a compatible differentiable structure, $dM = 0$.*

Proof. If E is the interior of an n -simplex A of M , and B the interior of an n -ball differentially imbedded in E , then $M - E$ and $M - B$ are combinatorially equivalent. Assuming M has a compatible differentiable structure, take A to be a simplex of a smooth triangulation of M . Then $M - B$ is a differentiable submanifold of M and hence $\partial(M - B) = \partial B = S^{n-1} = 0 \in \Gamma^{n-1}$. By 4.1, compatible differentiable structures on $M - E$ and $M - B$ are diffeomorphic; hence $\partial(M - E) = d(M) = 0$.

Corollary 4.4. *If M bounds a contractible manifold, $d(M) = 0$.*

Proof. By (A), M has a compatible differentiable structure and 4.3 applies.

Theorem 4.5. $d: A^n \rightarrow \Gamma^{n-1}$ defined by $d\langle M \rangle = dM$ is a well defined homomorphism.

Proof. We must show first that if $\langle M \rangle = \langle N \rangle$, then $dM = dN$. If M is J -equivalent to N , then $M \# (-N)$ is J -equivalent to $\partial \Delta^{n+1}$. Since

$\partial\Delta^{n+1}$ bounds Δ^{n+1} , $M \# (-N)$ bounds a contractible manifold. By 4.4 $d(M \# (-N)) = O$, and by 4.2, $d(M \# (-N)) = d(M) - d(N)$. Thus $d(M) - d(N) = O$, so $d: I^n \rightarrow I^{n-1}$ is well defined, and 4.4 proves d to be homomorphism.

Now we prove that the sequence (1) is exact. We leave the proof that $jd = kj = dk = O$ to the reader as an exercise; the last equality, for example, follows from 4.3.

Let M be an element of I^n such that $j(M) = O$. This means M bounds a contractible differentiable manifold V . Since $\partial V = M$ is combinatorially equivalent to $\partial\Delta^{n+1}$, $V \cup \Delta^{n+1}$ is a combinatorial homotopy sphere and hence represents an element λ of I^{n+1} . It is obvious that $d(\lambda) = \partial(V\text{-int } \Delta^{n+1}) = M$. This establishes the exactness of the sequence jd .

Let $\langle M \rangle \in I^n$ be such that $d\langle M \rangle = O$. This means for some n -simplex Δ in M , $M\text{-int } \Delta$ has a compatible differentiable structure making $\partial(M\text{-int } \Delta)$ diffeomorphic to S^{n-1} . Choosing such a diffeomorphism, attach the n -ball D^n to $M\text{-int } \Delta$ to obtain a differentiable manifold N which is combinatorially equivalent to M . Thus $k[M] = \langle N \rangle$ and dk is exact.

Finally let M represent an element of Θ^n annihilated by k . This means M is J -equivalent to $\partial\Delta^{n+1}$ in the combinatorial sense. Let V be a combinatorial $(n+1)$ -manifold realizing this J -equivalence. Let T be a «tube» joining M to $\partial\Delta^{n+1}$ in V , i.e., T is a equivalent to $I \times \Delta^n$ with $O \times \Delta^n \subset M$ and $1 \times \Delta^n \subset \partial\Delta^{n+1}$, and no other points of T in ∂V . (Such a T can easily be constructed by first putting a compatible differentiable structure on the contractible manifold $V \cup \Delta^{n+1}$.) Then $V\text{-int } T$ is contractible if T is «unknotted», which is always true if $n+1 > 3$ and which is the case for $n+1 = 3$ provided T is chosen properly. (In fact, $\Theta^2 = O$, so this case is unnecessary.) Thus $V\text{-int } T$ has a compatible differentiable structure, by (B). By 2.5, we can assume that $M\text{-int } (O \times \Delta^n)$ is a differentiable submanifold A of the boundary of $V\text{-int } T$. The closure of the complement of A is combinatorially equivalent to $\partial\Delta^{n+1}$, and hence is diffeomorphic to D^n , while A is combinatorially equivalent, and hence diffeomorphic to $M - E$, where E is the interior of an n -ball, $\bar{E}$ differentially imbedded in M . Thus there is a diffeomorphism $f: \partial(M - E) \rightarrow S^{n-1}$ such that $\partial(V\text{-int } T)$ is diffeomorphic to $(M - E) \cup_j D^n$. Let P be the manifold $\bar{E} \cup_j D^n$. Then P is an element of I^n , and $\partial(V\text{-int } T)$ is the same as $M \# (-P)$. Since $V\text{-int } T$ is contractible, $[M - P] = O$, and so $[M] = [P] = jP$. This establishes the exactness of the sequence (1).

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