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Webs for Flamingo Specht Modules

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ABSTRACT OF THE DISSERTATION

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A web basis of a representation of $\mathfrak{S}_n$ is a basis of the representation for which the action of $\mathfrak{S}_n$ can be understood through combinatorial rules called skein relations. In this thesis, we study web bases for two families of irreducible $\mathfrak{S}_n$ representations, indexed by the partitions $(d^2, 1^{n-2d})$ and $(d^3, 1^{n-3d})$. The first was introduced by Rhoades and is indexed by noncrossing set partitions of n. We use it to give a model for the top degree component of the fermionic diagonal coinvariant ring, and introduce another similar basis to model the entire fermionic diagonal coinvariant ring. We also give an embedding of the noncrossing set partition representation into an induction product of the Temperley-Lieb representation with a sign representation, thereby providing alternate proofs that the skein relations which define the

noncrossing set partition representation are in fact well defined. The second web basis is new, and simultaneously generalizes the SL_3 web basis of Kuperberg and the noncrossing set partition web basis. To define it and show it gives a basis, we draw on the combinatorics of Plabic graphs, jellyfish invariants, and weblike subgraphs.

Introduction

Representation theory of the symmetric group is one of the fields of mathematics which best showcases the power of *thinking combinatorially*. Rich algebraic structures are represented by simple diagrams, and algebraically defined operations on that structure are encoded via rules for manipulating the diagrams. These rules allow translation of algebraic reasoning into algorithmic reasoning about the manipulation of diagrams. The prototypical example of this paradigm is the classification of irreducible representations. Irreducible representations of $\mathfrak{S}_n$ are in bijection with integer partitions of n, which can be in turn viewed as *young diagrams*: left justified rows of boxes which decrease in length. Combinatorial rules on young diagrams express deeper algebraic structure. The Pieri rule, for example, states that the decomposition of a certain induction product into irreducible representations is given by all ways to add a number of boxes to a young diagram such that no column receives more than one new box. The structures of the irreducible representations themselves can be understood in a number of ways, each with their own benefits and insights. The standard construction is via *young tableaux*: fillings of the young diagrams with a natural number in each box. We will be interested in constructions which only exist for a select few irreducible representations called *web bases*.

The systematic study of web bases began with Kuperberg in 1996 [25], although some bases now considered web bases predate him. Kuperberg was interested in understanding the subspace of invariants of the action of a Lie algebra on an $n-fold$ tensor product of its defining representation and dual defining representation. When we take only the defining representation, there is a natural action of $\mathfrak{S}_n$ on this space, and Schur-Weyl duality guarantees it is irreducible. There are natural ways to move between these invariant subspaces for different n, namely tensor product and contraction. It can be shown that these two operations can be used to build all elements of these invariant subspaces from a finite list of starting invariants. Webs depict these two operations combinatorially. A web is a bipartite graph embedded in a disk with n boundary vertices. Interior vertices represent the starting invariants, edges between vertices represent tensor contraction, and concatenation of webs represents tensor product.

One benefit of the combinatorial interpretation webs provide is a simpler understanding of the relations between the operations of tensor product and contraction. These relations can be understood combinatorially through *skein relations*, which are ways of locally modifying webs to produce a linear combination of webs representing the same invariant. Perhaps the most important skein relation is the *uncrossing skein relation*, which gives a way to transform a web in which two edges cross each other into a sum of webs which no longer have this crossing. Thus, to understand the invariant subspace, one need only consider *planar webs*, webs in which no edges cross. An important problem in the study of webs is the question of how to prune down the set of webs further to a basis. Kuperberg addressed this for rank-two Lie algebras, including $\mathfrak{sl}_3$. For $\mathfrak{sl}_3$, planar webs in which no face has degree 4, called *nonelliptic webs* form a basis for the invariant space. A rotation-invariant answer to this question for $\mathfrak{sl}_4$ was recently announced in[13], and only non-canonical answers, in the sense they require choices to be made, are known for $\mathfrak{sl}_n$ [5].

Webs also serve to illuminate the structure of this invariant space considered as an $\mathfrak{S}_n$-module. The action of a permutation $\sigma \in \mathfrak{S}_n$ on a web is given by simply permuting the boundary vertices according to σ, then if a web basis is known, expanding into that basis using skein relations. It is for this reason that a rotation and reflection invariant basis of webs is desired. If a web basis is rotation and reflection invariant, then the action of any permutation in the dihedral group D_{2n} considered a subgroup of $\mathfrak{S}_n$ will again produce a basis web, and no expansion is necessary.

There are three web bases in particular we will focus on. The first, the Temperley-Lieb web basis, or $\mathfrak{sl}_2$ web basis, actually predates Kuperberg's development of the subject. Various aspects of it were studied by various authors, including the eponymous Temperley and Lieb [54]. The Temperley-Lieb basis is fairly simple combinatorially. It is indexed by noncrossing perfect matchings of $\{1,\ldots,n\}$, that is, ways to pair up n points around a circle with arcs, such that no two arcs cross. There is a single skein relation, given by replacing a crossing pair of arcs with both ways to uncross them. The Temperley-Lieb basis gives a basis for $(V^{\otimes n})^{SL_2}$, the space of SL_2-invariants of $V^{\otimes n}$, where V is the 2-dimensional defining representation of SL_2. Each arc represents a volume form on the two tensor factors indexed by those vertices which it connects. As an $\mathfrak{S}_n$ module, this invariant space is isomorphic to the Specht-module of shape (d,d) if $n = 2d$ and is 0 if n is odd.

The second web basis we will focus on is a generalization of the Temperley-Lieb web basis introduced by Rhoades [39]. Rhoades' web basis is indexed by singleton-free noncrossing set partitions rather than noncrossing matchings. This is a generalization of Temperley-Lieb since a noncrossing set partition in which every block is size 2 can be considered a noncrossing perfect matching. The single skein relation of the Temperley-Lieb action splits into three skein relations based on the sizes of the crossing set partitions. In Chapter 2, we connect Rhoades' basis to the theory of diagonal coinvariants by giving an explicit $\mathfrak{S}_n$-module isomorphism between the $\mathfrak{S}_n$ module spanned by noncrossing set partitions and the top degree piece of the fermionic diagonal coinvariant ring, FDR_n. To do so, we construct a set of operators on FDR_n which satisfy the skein relations of Rhoades' action, then show that these operators applied to a certain element create a basis of FDR_n. In doing so, we give better understanding of both the $\mathfrak{S}_n$ module structure of FDR_n and the resolution of crossings in Rhoades' set partition action. Chapter 3 gives a different but similar combinatorial basis for all of FDR_n rather than only the top graded piece, but at the cost of some of the nicer properties of the noncrossing set partition basis. In Chapter 4, we embed Rhoades' basis within the induction product of the Temperley-Lieb representation with a sign representation and use it to give alternate proofs of some of Rhoades' results.

The third web basis we will focus on is Kuperberg's $\mathfrak{sl}_3$ web basis, a modification of the Temperley-Lieb basis in a different direction. It is a basis for $(V^{\otimes n})^{SL_3}$, the SL_3 invariant subspace of $V^{\otimes n}$, where V is now the three-dimensional defining representation of SL_3. It is indexed by bipartite 3-regular planar graphs embedded in a disk with n boundary vertices and no interior faces of degree 4. There are now two main skein relations, an uncrossing skein relation similar to the one appearing in Temperley-Lieb, and a new skein relation which removes square faces.

The culmination of this thesis is the construction of a new web basis which combines the above two generalizations of Temperley-Lieb. Chapter 5 introduces *augmented webs*, which are similar to SL_3 webs, except one part of the bipartition of vertices may have degree three or more, rather than exactly three. Using them, we give a rotation invariant basis of the Specht module of shape $S^{(d^3,1^{n-3d})}$. Towards this end it is helpful to reinterpret these three web bases within a consistent framework using the theory of *plabic graphs* introduced by Postnikov [34] to study the totally nonnegative Grassmanian. Plabic graphs are planar graphs embedded in a disk, where vertices are bicolored, either black or white. We will be particularly interested in *normal plabic graphs*, plabic graphs in which the vertex coloring is proper, meaning adjacent vertices have different colors, and every white vertex has degree exactly three. Noncrossing matchings are naturally in bijection with normal plabic graphs where all vertices are degree exactly 2 (and thus there are no white vertices). Noncrossing set partitions are naturally in bijection with normal plabic graphs in which black vertices are degree at least two, and white vertices are degree exactly two (again, this implies there are no white vertices, we phrase it this way to match the description of SL_3 webs). Nonelliptic SL_3 webs are normal plabic graphs in which every vertex is degree exactly 3., and there exist no faces of degree 4. Somewhat surprisingly, the naive way to combine the objects indexing these two generalizations works out: *Augmented SL_3 webs* are normal plabic graphs with no faces of degree 4, all white vertices are degree exactly 3, and all black vertices are degree at least 3. Expanding on work of Fraser, Patrias, Pechenik, and Striker [11], we show how to associate SL_3 invariants to these objects to build our rotationally invariant basis. As an application, we use this basis to give a new cyclic sieving result for a q-analog of the hook-length formula for the partition $(d^3,1^{n-3d})$

One future goal of this project is in developing an alternative route towards solving the open problem of finding rotation invariant SL_r web bases for $r > 4$. Many of the ideas introduced to prove augmented webs form a basis generalize to $n > 4$, and finding a web basis for $S^{(d^r,1^{n-rd})}$ would include a web basis for $S^{(d^r)}$. Although this seems like a more complicated problem, our approach is not strictly harder, as augmented webs consist of linear combinations of SL_3 webs, and the search for a rotationally invariant SL_r web basis has been mostly restricted to looking for subsets of SL_r webs.

Chapter 1

Background

1.1 Representation theory of the symmetric group

Given a finite group G, a (complex) *representation* of G is a $\mathbb{C}$-vector space V and a group homomorphism $\rho : G \to \mathrm{End}(V)$. A *subrepresentation* of (V,ρ) is a subspace of V which is closed under $\rho(g)$ for all $g \in G$. A representation is *irreducible* if it contains no proper subrepresentation. Two representations (V_1,ρ_1) and (V_2,ρ_2) are *isomorphic* if there exists an invertible linear map $\phi : V_1 \to V_2$ such that $\phi^{-1} \circ \rho_2(g) \circ \phi = \rho_1(g)$ for all $g \in G$. Up to isomorphism, irreducible representations of G are in bijection with conjugacy classes of G. All representations are ismomorphic to a direct sum of irreducible representations.

When G is the symmetric group, conjugacy classes, and thus irreducible representations, are indexed by integer partitions of n. The irreducible indexed by a partition $\lambda \vdash n$ is denoted S^λ and is called the *Specht module* of shape λ. One way to construct the Specht modules is as follows. Let $\lambda \vdash n$, and let λ' be its transpose. Consider a matrix of $n\lambda'_1$ variables,

$$M = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & & \ddots & \vdots \\ x_{\lambda'_1,1} & x_{\lambda'_1,2} & \cdots & x_{\lambda'_1,n} \end{bmatrix}$$

The symmetric group $\mathfrak{S}_n$ acts on this matrix, and thus on $\mathbb{C}[M]$, by permuting columns of M.

Let $\pi = \{\pi_1, \pi_2, \ldots, \pi_{\lambda_1}\}$ denote a set partition of n with shape λ', i.e. the sizes of each part of the partition are given by the rows of λ'. For each such set partition π, let p_π be the polynomial

$$p_\pi = \prod_{i=1}^{\lambda_1} M^{\pi_i}_{[\lambda'_i]}$$

where $M^{\pi_i}_{[\lambda'_i]}$ denotes the matrix minor of M whose rows are indexed by $[\lambda'_i]$ and columns are indexed by π_i. Then the *Specht module* S^λ is the span of these polynomials as π ranges over all set partitions of shape λ.

Representations of the symmetric group are deeply connected with symmetric functions. The ring of symmetric functions consists of all formal power series in infinitely many variables $x_1, x_2, \ldots$ which are invariant under a permutation of the variables. For $i \in \mathbb{Z}^+$, the power sum symmetric function p_i $p_i = x_1^i + x_2^i + \cdots$. Given an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$. The Frobenius image of a representation, $\mathrm{Frob}(V)$, is the symmetric function given by

$$\sum_{\lambda \vdash n} z_\lambda^{-1} \mathrm{tr}(\rho(\sigma_\lambda)) p_\lambda$$

where σ_λ is any permutation whose cycle sizes match λ, and z_λ denotes the size of the centralizer of σ_λ. The resulting symmetric function does not depend on the choices of σ_λ.

The importance of the Frobenius image lies in the fact that the Frobenius images of the irreducible reprentations, called *Schur functions* $s_\lambda = \mathrm{Frob}(S^\lambda)$ are a basis for the ring of symmetric functions. Furthermore, if $V_1 \cong V_2 \oplus V_3$, then

$$\mathrm{Frob}(V_1) = \mathrm{Frob}(V_2) + \mathrm{Frob}(V_3)$$

Thus, determining the Frobenius image of a representation determines its decomposition into irreducible representations.

Multiplication in the ring of symmetric functions corresponds to induction product of representations. Given two representations V and W of $\mathfrak{S}_{m_1}$ and $\mathfrak{S}_{m_2}$ respectively, with

$m_1+m_2=n$, the induction product $V\circ W$ is given by

$$V\circ W=\mathrm{Ind}_{\mathfrak{S}_{m_1}\times\mathfrak{S}_{m_2}}^{\mathfrak{S}_n}V\otimes W$$

where $\mathfrak{S}_{m_1}\times\mathfrak{S}_{m_2}$ is identified with the parabolic subgroup of $\mathfrak{S}_n$ which permutes the first m_1 elements, $\{1,\ldots,m_1\}$, and last m_2 elements, $\{m_1+1,\ldots,n\}$, separately. Then

$$\mathrm{Frob}(V\circ W)=\mathrm{Frob}(V)\mathrm{Frob}(W)$$

When V is an irreducible representation S^μ for some partition μ of m_1 and W is a sign representation of $\mathfrak{S}_{m_2}$, the dual Pieri rule describes how to express $V\circ W$ in terms of irreducibles,

$$S^\mu\circ\mathrm{sign}_{\mathfrak{S}_{m_2}}\cong\sum_\lambda S^\lambda \tag{1.1.1}$$

where the sum is over all partitions λ whose young diagram can be obtained from that of μ by adding m_2 boxes, no two in the same row.

Another tool for determining the decomposition of a representation into irreducibles can be obtained by considering the action of the group algebra of the symmetric group. Given a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)\vdash n$, let $\mathfrak{S}_\lambda\subseteq\mathfrak{S}_n$ denote the Young subgroup $\mathfrak{S}_\lambda:=\mathfrak{S}_{\{1,\ldots,\lambda_1\}}\times\mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}}\times\cdots\times\mathfrak{S}_{\{n-\lambda_k,\ldots,n\}}$. To any subgroup $X\subseteq\mathfrak{S}_n$ we associate two group algebra elements $[X]_+$ and $[X]_-$ defined by $[X]_+=\sum_{w\in X}w$ and $[X]_+=\sum_{w\in X}\mathrm{sign}(w)w$. Then the following is true.

Lemma 1.1.1. *Let $\lambda,\mu\vdash n$. Then $[S_\lambda]_+$ kills S^μ unless $\lambda\preceq\mu$ and $[S_{\lambda'}]_-$ kills S^μ unless $\mu\preceq\lambda$.*

For an in depth discussion of this material, see [47].

1.1.1 Exterior algebras

We use $\wedge\{\Theta_n, \Xi_n\}$ to denote the exterior algebra over $\mathbb{C}$ generated by the $2n$ symbols $\theta_1, \dots, \theta_n, \xi_1, \dots, \xi_n$. Given subsets $S, T \subseteq [n]$ with $S = \{s_1 < \cdots < s_a\}, T = \{t_1 < \cdots < t_b\}$, we let $\theta_S \cdot \xi_T \in \wedge\{\Theta_n, \Xi_n\}$ denote the exterior monomial

$$\theta_S \cdot \xi_T := \theta_{s_1} \cdots \theta_{s_a} \cdot \xi_{t_1} \cdots \xi_{t_b}. \tag{1.1.2}$$

The set $\{\theta_S \cdot \xi_T \,:\, S, T \subseteq [n]\}$ is a basis of $\wedge\{\Theta_n, \Xi_n\}$. By declaring this basis to be orthogonal, we obtain an inner product $\langle -, - \rangle$ on the space $\wedge\{\Theta_n, \Xi_n\}$.

We will use a notion of *exterior differentiation* (or *contraction*). If $\Omega_m = (\omega_1, \omega_2, \dots, \omega_m)$ is an alphabet of fermionic variables, consider the rank m exterior algebra $\wedge\{\Omega_m\}$. We define a $\wedge\{\Omega_m\}$-module structure $\odot$ on $\wedge\{\Omega_m\}$ on by the rule

$$\omega_i \odot (\omega_{j_1} \cdots \omega_{j_r}) := \begin{cases} (-1)^{s-1} \omega_{j_1} \cdots \widehat{\omega_{j_s}} \cdots \omega_{j_r} & \text{if } j_s = i \\ 0 & \text{if } i \neq j_1, \dots, j_r \end{cases} \tag{1.1.3}$$

whenever $1 \leq j_1, \dots, j_r \leq m$ are distinct indices. The rule $(f_1 f_2) \odot g = f_1 \odot (f_2 \odot g)$ together with bilinearity yield $f \odot g \in \wedge\{\Omega_m\}$ for any $f, g \in \wedge\{\Omega_m\}$. We also define the *conjugate* $\bar{f} \in \wedge\{\Omega_m\}$ of an element $f \in \wedge\{\Omega_m\}$ by the rule

$$\overline{\sum_{1 \leq i_1 < \cdots < i_k \leq m} \alpha_{i_1, \dots, i_k} \cdot \omega_{i_1} \cdots \omega_{i_k}} = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \overline{\alpha_{i_1, \dots, i_k}} \cdot \omega_{i_k} \cdots \omega_{i_1} \tag{1.1.4}$$

where the α's are complex numbers and the bar on the right-hand side denotes complex conjugation.

We apply the $\odot$-action over the size $2n$ alphabet $(\theta_1, \dots, \theta_n, \xi_1, \dots, \xi_n)$ of variables in $\wedge\{\Theta_n, \Xi_n\}$. We leave the following simple proposition to the reader; its second part characterizes the $\odot$-action.

Proposition 1.1.2. *1. For any $w \in \mathfrak{S}_n$ and $f, g \in \wedge\{\Theta_n, \Xi_n\}$ we have*

$$\langle f, g \rangle = \langle w \cdot f, w \cdot g \rangle.$$

2. For any $f, g, h \in \wedge\{\Theta_n, \Xi_n\}$ we have

$$\langle f \cdot g, h \rangle = \langle g, \bar{f} \odot h \rangle.$$

3. Assume that $f, g \in \wedge\{\Theta_n, \Xi_n\}$ where f has homogeneous total degree d. For any $1 \leq i \leq n$ we have

$$\theta_i \odot (fg) = (\theta_i \odot f)g + (-1)^d f(\theta_i \odot g) \quad \textit{and} \quad \xi_i \odot (fg) = (\xi_i \odot f)g + (-1)^d f(\xi_i \odot g).$$

Proposition 1.1.2 (3) is a sign-twisted version of the Leibniz rule.

1.2 Noncrossing Matchings and Temperley-Lieb

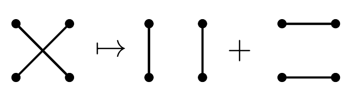
The special linear group SL_2 is the set of all 2×2 matrices with determinant 1. The special linear group SL_2 acts on a two-dimensional vector space V with basis $\{e_1, e_2\}$ by left multiplication, and acts on the n-fold tensor product $V^{\otimes n}$ diagonally. This section will be concerned with the SL_2 invariant subspace of $V^{\otimes n}$, $(V^{\otimes n})^{SL_2}$, which $\mathfrak{S}_n$ acts on via permuting tensor factors. When $n = 2$, the SL_2 invariant subspace of $V \otimes V$ is one-dimensional, generated by the element $e_1 \otimes e_2 - e_2 \otimes e_1$. Also, if $v_1 \in (V^{\otimes n_1})^{SL_2}$ and $v_2 \in (V^{\otimes n_2})^{SL_2}$, then the tensor product is also SL_2 invariant, $v_1 \otimes v_2 \in (V^{\otimes n_1 + n_2})^{SL_2}$. In fact, tensor product and linear combinations are enough to make all SL_2 invariants out of $e_1 \otimes e_2 - e_2 \otimes e_1$. The Temperley-Lieb web basis represents this combinatorially through *noncrossing perfect matchings*.

A *matching* of $[n]$ is a collection of disjoint size-two subsets of $[n]$. A matching is *perfect* if every element of $[n]$ appears in the matching, i.e. it is a set partition of $[n]$ into size two parts. We will often represent matchings via arc diagrams: drawings with the numbers 1 through n placed equally spaced around a circle, and arcs connecting matched elements. A matching is *noncrossing* if it does not contain two subsets $\{a,c\}$ and $\{b,d\}$ with $a < b < c < d$, or alternatively, if arcs do not cross in its arc diagram. Let $M(n)$ denote the set of all matchings of $[n]$, let $PM(n)$ denote the set of perfect matchings, and let $NCM(n)$ denote the set of all noncrossing matchings of $[n]$. To each perfect matching of $[n]$, we can associate an element of $(V^{\otimes n})^{SL_2}$ recursively: The unique matching when $n = 2$ corresponds to $e_1 \otimes e_2 - e_2 \otimes e_2$. If m_1 is a matching of n_1 corresponding to $v_1 \in (V^{\otimes n_1})^{SL_2}$ and m_2 is a matching of n_2 corresponding to $v_1 \in (V^{\otimes n_2})^{SL_2}$, then the matching obtained taking the union of m_1 with m_2 incremented by n_1 corresponds to $v_1 \otimes v_2$. If m is a matching corresponding to v and s_i is an adjacent transposition between two elements not matched to each other in m, then the matching obtained by swapping i and $i+1$ corrseponds to $-s_i \cdot v$.

We can then pullback the action of the symmetric group $\mathfrak{S}_n$ on $(V^{\otimes n})^{SL_2}$ to an action on $PM(n)$ as follows. If $\sigma \in \mathfrak{S}_n$ and $m = \{\{a_1, b_1\}, \ldots, \{a_n, b_n\}\}$ is a perfect matching, then

$$\sigma \circ m = \operatorname{sign}(\sigma)\{\{\sigma(a_1), \sigma(b_1)\}, \ldots, \{\sigma(a_n), \sigma(b_n)\}\}. \tag{1.2.1}$$

The invariants corresponding to perfect matchings do not form a basis for $(V^{\otimes n})^{SL_2}$. If m has a crossing, then its corrseponding invariant is equal to the sum of those corresponding to both ways to remove that crossing, shown below. This is called the *skein relation* or *Ptolemy relation*.



Let $\mathbb{C}[M(n)]$ denote the $\mathbb{C}$-vector space with basis given by perfect matchings. The quotient of this vector space by the skein relations is isomorphic to $(V^{\otimes n})^{SL_2}$, with basis given by perfect matchings. As an $\mathfrak{S}_n$ module, it is irreducible and isomorphic to $S^{(\frac{n}{2}, \frac{n}{2})}$.

We can extend this to all noncrossing matchings, rather than only perfect matchings. For any noncrossing matching m and adjacent transposition $s_i = (i, i+1)$, define

$$s_i \cdot m = \begin{cases} s_i \circ m & s_i \circ m \text{ is noncrossing} \\ m + m' & \text{otherwise.} \end{cases} \tag{1.2.2}$$

Here $\circ$ denotes the action on all matchings and m' is the matching where the subsets of m containing i and $i+1$, call them $\{i, a\}$ and $\{i+1, b\}$ have been replaced with $\{i, i+1\}$ and $\{a, b\}$ and all other subsets remain the same. In other words, $s_i \circ m$, m, and m' form a trio of matchings that differ only in a Ptolemy relation. It can be shown that this definition satisfies the braid relations and thus gives an action of the symmetric group on $\mathbb{C}[NCM(n)]$. There exists an $\mathfrak{S}_n$-equivariant linear projection $p_M : \mathbb{C}[M(n)] \to \mathbb{C}[NCM(n)]$ given for any matching m by

$$m \mapsto w^{-1} \cdot (w \circ m), \tag{1.2.3}$$

where w is any permutation for which $w \circ m$ is noncrossing. This projection can be thought of as a way to"resolve" crossings in a matching and obtain a sum of noncrossing matchings. The following proposition is not new, but we were unable to find a suitable reference and thus include a proof for completeness.

Proposition 1.2.1. *The kernel of the projection $p_M : \mathbb{C}[M(n)] \to \mathbb{C}[NCM(n)]$ is spanned by elements of the form*

$$\begin{aligned} &\{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}, \ldots, \{a_{2k-1}, a_{2k}\}\} \\ &\qquad + \{\{a_1, a_3\}, \{a_2, a_4\}, \{a_5, a_6\}, \ldots, \{a_{2k-1}, a_{2k}\}\} \\ &\qquad\qquad + \{\{a_1, a_4\}, \{a_2, a_3\}, \{a_5, a_6\}, \ldots, \{a_{2k-1}, a_{2k}\}\} \end{aligned} \tag{1.2.4}$$

for any $a_1, \ldots, a_{2k} \in [n]$, i.e. sums of three matchings which differ by a Ptolemy relation.

Proof. Let β denote the set of all elements of the form given in (1.2.4). To see that the span of β

is contained in the kernel of p_M, note that by the $\mathfrak{S}_n$-equivariance of p_M it suffices to check that applying p_M gives 0 in the case where $a_i = i$ for all i. In this case, we have

$$
\begin{aligned}
p_M(\{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\} + \{\{1,3\},\{2,4\}\ldots,\{2k-1,2k\}\} \\
+ \{\{1,4\},\{2,3\}\ldots,\{2k-1,2k\}\}) \\
= \{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\} + (2,3)\cdot(-\{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\}) \\
+ \{\{1,4\},\{2,3\},\ldots,\{2k-1,2k\}\} = 0 \quad (1.2.5)
\end{aligned}
$$

To see that the kernel is contained in the span, note that since p_M is a projection, the kernel is spanned by $m - p_M(m)$ for any matching m. Let t denote the minimum number of transpositions $s_{i_1},\ldots,s_{i_t}$ for which $(s_{i_1}\cdots s_{i_t})\circ m$ is noncrossing, and let $w = s_{i_1}\cdots s_{i_t}$. We will show by induction on t that $m - p_M(m) \in \text{span}(\beta)$. When $t = 0$, then $m - p_M(m) = 0$, so the claim is true. Otherwise, assume the claim holds for $t-1$. We have $m - p_M(m) = s_{i_1}\circ(s_{i_1}\circ m) - s_{i_1}\cdot p_M(s_{i_1}\circ m)$. By our inductive hypothesis, $s_{i_1}\circ m - p_M(s_{i_1}\circ m)$ lies in the span of β, so it suffices to verify for any $b\in\beta$, that if we apply $s_{i_1}\circ(-)$ to every crossing term of b and apply either $s_{i_1}\cdot(-)$ or $s_{i_1}\circ(-)$ to every noncrossing term of b, we remain in the span of β. This is true because β is closed under the $\circ$ action, and for every noncrossing matching m_1, either

$$s_{i_1}\circ m = s_{i_1}\cdot m$$

or

$$s_{i_1}\cdot m_1 = s_{i_1}\circ m_1 - (s_{i_1}\circ m_1 + m_1 + m_1')$$

where m_1' is obtained by replacing the sets $\{i,a\}$ and $\{i+1,b\}$ with the sets $\{i,i+1\}$ and $\{a,b\}$. In the second case, $s_{i_1}\circ m_1 + m_1 + m_1'$ is in β. □

1.3 Noncrossing set partitions and the skein action

A set partition of $[n]$ is a collection of disjoint subsets of $[n]$ whose union is $[n]$. A set partition is *noncrossing* if there do not exist distinct blocks A and B and elements $a,c \in A$, $b,d \in B$ with $a < b < c < d$. Let $\Pi(n)$ denote the set of all set partitions of n, and let $NCP(n)$ denote the set of all noncrossing set partitions of $[n]$. We can define an action of $\mathfrak{S}_n$ on $\mathbb{C}[\Pi(n)]$ analogous to the action on $\mathbb{C}[M(n)]$. Rhoades defined an action of $\mathfrak{S}_n$ on $\mathbb{C}[NCP(n)]$ as follows [39]. For any noncrossing set partition π and adjacent transposition s_i,

$$s_i \cdot \pi = \begin{cases} -\pi & i \text{ and } i+1 \text{ are in the same block of } \pi \\ -\pi' & \text{at least one of } i \text{ and } i+1 \text{ is in a singleton block of } \pi \\ \sigma(\pi') & i \text{ and } i+1 \text{ are in different size 2 or larger blocks of } \pi \end{cases}$$

where π' is the set partition obtained by swapping which blocks i and $i+1$ are in, and σ is defined for any almost-noncrossing (i.e. the crossing can be removed by a single adjacent transposition) partition π by $\sigma(\pi) = \pi + \pi_2 - \pi_3 - \pi_4$ where, if the crossing blocks in σ are $\{i, a_1, \ldots, a_k\}$ and $\{i+1, b_1, \ldots, b_l\}$, then π_2, π_3 and π_4 are obtained from π by replacing these blocks with

- $\{i, i+1\}$ and $\{a_1, \ldots, a_k, b_1, \ldots, b_l\}$ for π_2

- $\{i, i+1, a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_l\}$ for π_3

- $\{i, i+1, b_1, \ldots, b_l\}$ and $\{a_1, \ldots, b_k\}$ for π_3

when $k, l \geq 2$. If $k = 1$ then $\pi_4 = 0$ instead and if $l = 1$ then $\pi_3 = 0$ instead. The sum of partitions given by $\sigma(\pi)$ is best described with a picture, see Figure 4.1 in the introduction. The three possibilities (depending on whether $k, l \geq 2$) are the three skein relations mentioned in the introduction.

We again have an $\mathfrak{S}_n$-equivariant linear projection $p_\Pi : \mathbb{C}[\Pi(n)] \to \mathbb{C}[NCP(n)]$ given for any set partition π by

$$\pi \mapsto w^{-1} \cdot (w \circ \pi), \tag{1.3.1}$$

where w is any permutation for which $w \circ \pi$ (here $\circ$ denotes the action of $\mathfrak{S}_n$ on all set partitions) is noncrossing. We have the following proposition, analogous to Proposition 1.2.1, and with an analagous proof.

Proposition 1.3.1. *The kernel of the projection $p_\Pi : \mathbb{C}[\Pi(n)] \to \mathbb{C}[NCP(n)]$ is spanned by elements of the form*

$$w \circ (s_i \circ \pi + \sigma(\pi))$$

for any permutation w and singleton-free almost noncrossing set partition π, i.e. sums of set partitions which differ by a skein relation.

1.4 Jellyfish invariants

Jellyfish invariants were introduced in [29] and further developed in [11] in order to study the Specht module $S^{(d^r,1^{n-rd})}$. An (n,d,r)-jellyfish invariant is a certain element of $S^{(d^r,1^{n-rd})}$ attached to each ordered set partition of $[n]$ with d blocks and all blocks at least size r. We include the basic definitions from [11] below, see their chapter for examples and further exposition.

An *ordered set partition* of n is a set partition with a total order on its blocks. Two blocks A, B of an ordered set partition *cross* if there exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $a_1 < b_1 < a_2 < b_2$ or $b_1 < a_1 < b_2 < a_2$. An ordered set partition is *noncrossing* if no two of its blocks cross. Let $\mathscr{OP}(n,d,r)$ denote the set of all ordered set partitions with exactly d blocks and blocks of size at least r, and let $\mathscr{NCOP}(n,d,r)$ denote the set of all such partitions which are also noncrossing.

Definition 1.4.1. *Let $\pi = \{\pi_1, \dots, \pi_d\} \in \mathscr{OP}(n,d,r)$ be an ordered set partition. Define the set of r-*jellyfish tableaux*, $\mathscr{J}_r(\pi)$ to be the set of generalized tableau T with d columns and $n-(d-1)r$ rows with the following constriants:*

1. *$T_{ij} \in [n]$ or T_{ij} is nonempty*
2. *If $i \in [r], T_{ij}$ is nonempty.*
3. *If $i > r$, there exists exactly one j such that T_{ij} is nonempty*
4. *The nonempty entries in column j are exactly the elements of π_j in increasing order.*

For each $T \in \mathscr{J}_r(\pi)$, define a polynomial

$$J(T) = \prod_{i=1}^{d} M_{R_i(T)}^{\pi_i}$$

where $R_i(T)$ is the set of rows containing an entry in π_i.

*For each $\pi \in \mathscr{OP}(n,d,r)$, the r-*jellyfish invariant*, denoted $[\pi]_r$ is*

$$[\pi]_r = \sum_{T \in \mathscr{J}_r(\pi)} sign(T) J(T)$$

where $sign(T)$ denotes the sign of the reading word of T.

Fraser, Patrias, Pechenik, and Striker prove the following about r-jellyfish invariants:

Theorem 1.4.2 ([11, Theorem 4.24])**.** *For each ordered set partition $\pi \in \mathscr{OP}(n,d,r)$, the invariant $[\pi]_r$ lies in the flamingo Specht module $S^{(d^r,1^{n-rd})}$.*

Theorem 1.4.3 ([11, Proposition 5.11])**.** *For any ordered set partition $\pi \in \mathscr{OP}(n,d,r)$ and any permutation $w \in \mathfrak{S}_n$, we have*

$$w \cdot [\pi]_r = sign(w)[w \cdot \pi]_r$$

Note that this implies the span of jellyfish invariants is closed under the action of $\mathfrak{S}_n$, and must therefore be equal to $S^{(d^r,1^{n-rd})}$.

Theorem 1.4.4 ([11, Theorem 5.13]). *For each noncrossing set partition $\gamma \in \mathscr{NC}(n,d,r)$, order the blocks in any way to create a corresponding ordered set partition π_γ. Then the set $\{[\pi_\gamma]_r : \gamma \in \mathscr{NC}(n,d,r)\}$ is linearly independent.*

Fraser, Patrias, Pechenik, and Striker thus give a spanning set of $S^{(d^r,1^{n-rd})}$ indexed by all set partitions, and a linearly independent subset indexed by noncrossing set partitions. Thus, it is possible to choose a subset S of set partitions such that S indexes a basis and S contains all noncrossing set partitions. We will show how to do so in Section 3.

1.5 Noncrossing Tableaux

Noncrossing tableaux were introduced by P. Pylyavskyy in [35] to give a non-crossing counterpart to standard Young tableaux. Formally, noncrossing tableaux are set partitions; Pylyavskyy chose the name noncrossing tableaux to distinguish them from the more standard definition of noncrossing set partitions given in the previous subsection. As we will be using noncrossing tableaux in the context of set partitions, we will instead refer to these as *weakly* noncrossing set partitions. We will use a modification of this weaker condition to interpolate between strongly noncrossing set partitions and all set partitions.

Definition 1.5.1. *Let $A = \{a_1 < a_2 < \cdots < a_{|A|}\}$ and $B = \{b_1 < b_2 < \cdots < b_{|B|}\}$ be two disjoint subsets of $[n]$ with $|A| \leq |B|$. We say A and B are* weakly noncrossing *if for all $1 \leq i \leq |A| - 1$ we do not have*

$$a_i < b_i < a_{i+1} < b_{i+1}$$

or

$$b_i < a_i < b_{i+1} < a_{i+1}$$

and additionally, if $|A| < |B|$, *we do not have*

$$b_{|A|} < a_{|A|} < b_{|B|}$$

A set partition π *is* weakly noncrossing *if blocks of* π *are pairwise weakly noncrossing.*

Pylyavskyy showed that weakly noncrossing set partitions of shape $\lambda \vdash n$ are in bijection with standard Young tableaux of shape λ.

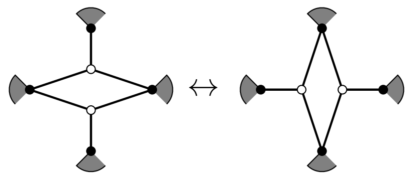
1.6 Plabic graphs

Plabic graphs were introduced by Postnikov in order to study the totally nonnegative Grassmanian. A textbook treatment can be found in [8]. We will only need combinatorial results about plabic graphs, which we list here.

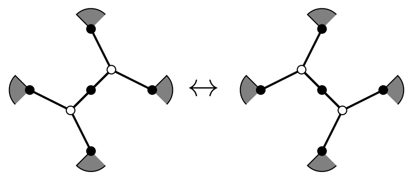
A *plabic graph* G is a planar graph embedded in a disk, possibly with loops and multiple edges between vertices, with interior vertices colored black and white and boundary vertices labelled clockwise 1 through n. A *normal* plabic graph is a plabic graph for which white vertices are degree three, boundary vertices only connect to black vertices, and same colored vertices do not share an edge. For this chapter, we consider only normal plabic graphs and state results only as they apply to normal plabic graphs, rather than including the full generality.

Given a normal plabic graph G , the *trip* at i is the walk in G starting at boundary vertex i which turns right at every black vertex and left at every white vertex until it reaches the boundary at a vertex we denote $\mathrm{trip}(i)$. The function defined by $i \mapsto \mathrm{trip}(i)$ is a permutation of $[n]$ and is called the *trip permutation* of G. The *exceedances* of G are the exceedances of this permutation, i.e. those trips for which $\mathrm{trip}(i) > i$.

Two normal plabic graphs are *normal move equivalent* if one can be obtained from the other via a seuquence of *normal urban renewal moves* and *normal flip moves*, which we now define. The normal urban renewal move is the move shown below, where filled in arcs represent any number of edges leading elsewhere in the graph



The normal flip move is



A normal plabic graph is *reduced* if it is not normal move equivalent to any plabic graph which contains a *forbidden configuration*, i.e. a face of degree two or a leaf vertex not adjacent to the boundary.

A *bad feature* of a normal plabic graph G is one of the following:

- A roundtrip: A cycle in G which turns left at every white vertex and right at every black vertex.
- An essential self-intersection: A trip in G which passes through the same edge twice.
- A bad double-crossing: Two trips in G which both pass through edge e_1 then edge e_2 in that order.

Theorem 1.6.1 ([8, Theorem 7.8.6])**.** *A normal plabic graph is reduced if and only if it does not contain any bad features.*

The more common use of this theorem is to test whether a plabic graph is reduced or not. The plabic graphs we are interested in, however, will be clearly reduced as their normal move equivalence class will have size 1. We will instead apply it to understand the structure of the trips of our plabic graphs.

1.7 SL_3 Webs

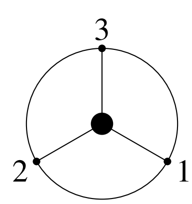
SL_3 webs, or A_2 webs, were introduced by Kuperberg to study SL_3 invariant tensors and the representation theory of the quantum group $U_q(\mathfrak{sl}_3)$ [25]. A sign string of length n is a string containing n letters, all each $+$ or $-$, e.g. $(++--++-)$. Given a sign string $s = s_1 s_2 \cdots s_n$, an SL_3 web of type s is a bipartite plabic graph with n boundary vertices in which every interior vertex has degree 3 and boundary vertex i is adjacent to a black vertex if $s_i = +$ and a white vertex if $s_i = -$. This is a slightly anachronistic version of the definition, as plabic graphs were defined after SL_3 webs, but the comparison will be useful for us later.

SL_3 webs have representation theoretic meaning. Let V be the three-dimensional defining representation of SL_3, with basis $\{e_1, e_2, e_3\}$, and let V^* denote its dual with dual basis $\{e_1^*, e_2^*, e_3^*\}$. An SL_3 web with sign string $(+++--++)$ e.g. represents an element of the space

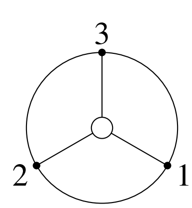
$$(V \otimes V \otimes V \otimes V^* \otimes V^* \otimes V \otimes V)^{SL_3}$$

of SL_3 invariant elements of $(V \otimes V \otimes V \otimes V^* \otimes V^* \otimes V \otimes V)$ where V is the three-dimensional defining representation of SL_3, $+$ correspond to copies of V and $-$ correspond to cpoies of V^*.

The unique SL_3 web of sign string $(+++)$



represents the tensor $\sum_{\sigma \in \mathfrak{S}_3} \text{sign}(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}$ and the unique SL_3 web of sign string $(---)$



represents the tensor $\sum_{\sigma\in\mathfrak{S}_3} \mathrm{sign}(\sigma) e^*_{\sigma(1)} \otimes e^*_{\sigma(2)} \otimes e^*_{\sigma(3)}$. Concatenation of webs represents tensor product, and an edge between vertices represents tensor contraction.

We can also give a purely combinatorial desciption of the invariant each web represents in terms of proper edge colorings. A *proper edge coloring* ℓ of an SL_3 web W is a labelling of the edges by the numbers $1,2,3$ such that no label appears more than once around each vertex. For each labelling, we get a simple basis tensor T_ℓ by taking the basis vector or dual basis vector e_j or e^*_j (depending on th sign string) at boundary vertex i whose incident edge has label j, and a sign $\mathrm{sign}(\ell)$ given by $(-1)^{cc(\ell)}$, where $cc(\ell)$ denotes the number of interior vertices for which $1,2,3$ appear in counterclockwise order in the labelling ℓ . The SL_3 invariant associated to W, which we denote $[W]_{SL_3}$ is

$$[W]_{SL_3} = \sum_{\text{proper labellings } \ell} \mathrm{sign}(\ell) T_\ell$$

A web is called *nonelliptic* if it contains no faces of degree 4 or less. The invariants for the set of all noneeliptic webs form a basis for the space of SL_3 invariant tensors.

1.8 Cyclic sieving

The cyclic sieving phenomenon was introduced by V. Reiner, D. Stanton, and D. White in order to unify a number of enumerative results in combinatorics [37].

Definition 1.8.1. *Let X be a finite set equipped with an action of the finite cyclic group $C \cong \mathbb{Z}/n\mathbb{Z}$ with generator c, let $X(q)$ be a polynomial, and let ζ be an n^{th} root of unity. The triple $(X, C, X(q)$ is said to exhibit the* cyclic sieving phenomenon *if $|X^{c^d}| = X(\zeta^d)$ for any integer $d > 0$, where X^{c^d} denotes the set of all elements of X fixed by c^d.*

One way of obtaining cyclic sieving results is via the following, which can be found in Sagan's survey [48] and follows from a result of Springer [49].

Theorem 1.8.2 ([48, Theorem 8.2], [49]). *Let W be a finite complex reflection group and let $C \leq W$ be cyclically generated by a regular element g. Let V be a W-module with a basis B such that $gB = B$. Then the triple*

$$(B, C, F^V(q))$$

exhibits the cyclic sieving phenomenon, where $F^V(q)$ denotes the fake degree polynomial for V.

See [48] for a complete definition of the fake degree polynomial, we will only need the following.

Proposition 1.8.3. *Let λ be a partition of n and let S^λ be the corresponding Specht module. The fake degree polynomial $F^{S^\lambda}(q)$ is given by*

$$F^{S^\lambda}(q) = q^{b(\lambda)} \frac{[n]!_q}{\prod_{(i,j)\in\lambda} [h_{ij}]_q}$$

where $b(\lambda) = 0\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots$ and h_{ij} denotes the hook length of box (i, j) in the Young diagram of λ.

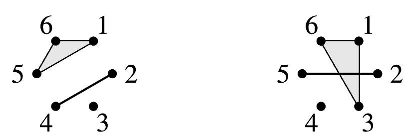
Chapter 2

Fermions, set partitions, skein relations

2.1 Introduction

This chapter concerns two modules over the symmetric group $\mathfrak{S}_n$. The first is combinatorial, involving skein relations which resolve crossings in set partitions of $[n] := \{1, \ldots, n\}$. The second is algebraic, arising from the ring of fermionic diagonal coinvariants. We describe the combinatorial module first.

A set partition π of $[n]$ is *noncrossing* if whenever $1 \leq a < b < c < d \leq n$ are four indices such that $a \sim c$ and $b \sim d$ in π, we have $a \sim b \sim c \sim d$ in π. Drawing the indices $1, 2, \ldots, n$ around a circle, this means that the convex hulls of the blocks of π do not intersect. A noncrossing and 'crossing' partition when $n = 6$ are shown below.

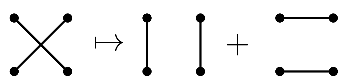


We write $\mathrm{NC}(n)$ for the family of noncrossing partitions of $[n]$ and $\mathrm{NC}(n,k)$ for the subfamily of noncrossing partitions of $[n]$ with k blocks. These sets are counted by the *Catalan* and *Narayana numbers*

$$|\mathrm{NC}(n)| = \mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n} \qquad |\mathrm{NC}(n,k)| = \mathrm{Nar}(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \tag{2.1.1}$$

The family $\Pi(n)$ of all set partitions of $[n]$ (noncrossing or otherwise) carries a natural action of the symmetric group $\mathfrak{S}_n$. Given $w \in \mathfrak{S}_n$ and $\pi \in \Pi(n)$, let $w(\pi) \in \Pi(n)$ be the set partition whose blocks are $w(B) = \{w(i) : i \in B\}$ where B is a block of π. Although the subset $\Pi(n,k) \subseteq \Pi(n)$ of k-block set partitions of $[n]$ is stable under this action of $\mathfrak{S}_n$ for any $k \leq n$, this action of $\mathfrak{S}_n$ **does not** preserve the noncrossing property: the sets $\mathrm{NC}(n)$ and $\mathrm{NC}(n,k)$ are not closed under this action. Despite this, Rhoades introduced [39] an action of $\mathfrak{S}_n$ on the linearized versions $\mathbb{C}[\mathrm{NC}(n)]$ and $\mathbb{C}[\mathrm{NC}(n,k)]$ of these sets[1]. We use a modified version of this action sketched as follows (for a precise formulation see Definition 2.3.2).

For $1 \leq i \leq n-1$, let $s_i := (i, i+1) \in \mathfrak{S}_n$ be the corresponding adjacent transposition. If $\pi \in \mathrm{NC}(n)$ is a noncrossing partition, the partition $s_i(\pi) \in \Pi(n)$ may or may not be noncrossing. If $s_i(\pi)$ is noncrossing, we set $s_i \cdot \pi := -s_i(\pi)$. If $s_i(\pi)$ is not noncrossing, we resolve the local crossing at $i, i+1$ using the skein relations shown in Figure 2.1. These relations come in three flavors, depending on whether the blocks of π being crossed at i and $i+1$ have exactly two or more than two elements. The top skein relation is the famous transformation



which appears in invariant theory, knot theory, and elsewhere. The lower two skein relations are less classical; to the knowledge of the authors they were not studied prior to [39]. The 2-term and 3-term skein relations are 'degenerations' of the 4-term skein relation in which one omits terms involving singleton blocks. We will make this more precise by means of certain 'block operators'; see the proof of Theorem 2.3.3.

The action $s_i \cdot \pi$ described above extends to an action of $\mathfrak{S}_n$ on the vector space $\mathbb{C}[\mathrm{NC}(n)]$. Since the skein relations in Figure 2.1 preserve the total number of blocks in a set partition, the subspace $\mathbb{C}[\mathrm{NC}(n,k)]$ is a submodule for this action. We have further submodules $\mathbb{C}[\mathrm{NC}(n,k,m)]$, where $\mathrm{NC}(n,k,m)$ is the family of k-block noncrossing set partitions of $[n]$ with m singletons. We refer to these modules collectively as *skein actions* of $\mathfrak{S}_n$, and their canonical bases $\mathrm{NC}(n), \mathrm{NC}(n,k)$, and $\mathrm{NC}(n,k,m)$ as *skein bases*.

[1]We work over $\mathbb{C}$ for convenience, but all of the results in this chapter hold over $\mathbb{Q}$.

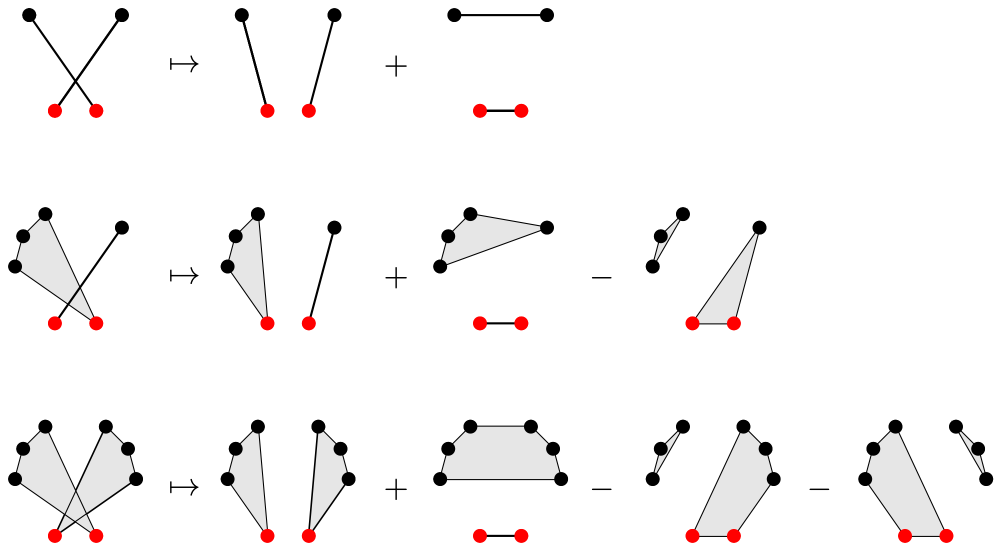


Figure 2.1. The three skein relations defining the action of $\mathfrak{S}_n$ on $\mathbb{C}[\mathrm{NC}(n)]$. The red vertices are adjacent indices $i, i+1$ and the shaded blocks have at least three elements. The symmetric 3-term relation obtained by reflecting the middle relation across the y-axis is not shown.

The skein action was introduced to give representation theoretic proofs of cyclic sieving results of Reiner-Stanton-White [37] and Pechenik [32] involving the rotational action of $\mathbb{Z}_n$ on various sets of noncrossing partitions of $[n]$. Skein bases generalize the Kazhdan-Lusztig cellular and $\mathfrak{sl}_2$-web bases (see [24, 33, 40, 45]) of symmetric group irreducibles labeled by 2-row rectangles.

The skein action has nice combinatorial properties. Permutations $w \in \mathfrak{S}_n$ have representing matrices in the skein basis with entries in $\mathbb{Z}$. 'Local symmetries' of noncrossing partitions are preserved: if $w \in \mathfrak{S}_n$ and $\pi \in \mathrm{NC}(n)$ are such that the set partition $w(\pi)$ is noncrossing, then $w \cdot \pi = \pm w(\pi)$ (Corollary 2.5.8). If we endow $\mathbb{C}[\Pi(n)]$ with a sign-twisted version of the permutation action of $\mathfrak{S}_n$, there is a $\mathfrak{S}_n$-equivariant projection

$$p : \mathbb{C}[\Pi(n)] \twoheadrightarrow \mathbb{C}[\mathrm{NC}(n)] \tag{2.1.2}$$

in which $p(\pi)$ is a $\mathbb{Z}$-linear combination of noncrossing partitions for any set partition π (Definition 2.5.1, Theorem 2.5.5). We regard $p(\pi)$ as a 'resolution of crossings' in the set partition π;

this generalizes the classical resolution of crossings in perfect matchings/chord diagrams. Before proceeding further, we issue a

> **Warning.** *The skein action used in this chapter differs from that in [39]. The fundamental relations in Figure 2.1 are unchanged, but the sign convention for applying s_i to a noncrossing set partition π for which $s_i(\pi)$ is also noncrossing differs. Our conventions yield sharper results, cleaner proofs, and give connections to the fermionic diagonal coinvariant ring described below.*

The skein action as presented in [39] had some drawbacks. The definition of this action was purely combinatorial and somewhat *ad hoc*; there was little algebraic reason 'why' these skein relations ought to hold. Checking that the action of the generators s_i extended to a well-defined action of $\mathfrak{S}_n$ involved extensive casework and a number of 'miraculous' 16-term identities[2]. The complicated nature of this action led to difficulty in computing the sign in the local symmetry formulas $w \cdot \pi = \pm w(\pi)$ described above. Finally, it was unclear how to extend the skein action from $\mathfrak{S}_n$ to a wider class of reflection groups W. In this chapter we address these issues by relating the skein action to fermionic diagonal coinvariants.

We turn to the algebraic module of study: the fermionic diagonal coinvariant ring. Let $\Theta_n = (\theta_1, \ldots, \theta_n)$ and $\Xi_n = (\xi_1, \ldots, \xi_n)$ be two lists of n anticommuting variables and let

$$\wedge\{\Theta_n, \Xi_n\} := \wedge\{\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n\} \tag{2.1.3}$$

be the exterior algebra generated by these symbols over $\mathbb{C}$. The ring $\wedge\{\Theta_n, \Xi_n\}$ has a bigrading

$$\wedge\{\Theta_n, \Xi_n\}_{i,j} := \wedge^i\{\theta_1, \ldots, \theta_n\} \otimes \wedge^j\{\xi_1, \ldots, \xi_n\}. \tag{2.1.4}$$

Adopting the language of physics, we refer to the variables θ_i, ξ_i as *fermionic* and general elements $f \in \wedge\{\Theta_n, \Xi_n\}$ as *fermions*[3].

[2] In fact, the intricacy of these identities led to a couple cases which were missed in [39]. A. Iraci [18] filled these gaps in his Master's Thesis at the University of Pisa.

[3] In physics, the equation $\theta_i^2 = 0$ is the *Pauli Exclusion Principle*: two identical fermions cannot occupy State i at the same time.

The ring $\wedge\{\Theta_n, \Xi_n\}$ carries a bigraded diagonal action of $\mathfrak{S}_n$ via

$$w \cdot \theta_i := \theta_{w(i)} \qquad w \cdot \xi_i := \xi_{w(i)} \qquad (w \in \mathfrak{S}_n,\ 1 \leq i \leq n). \tag{2.1.5}$$

If we let $\wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n}$ be the subsapce of $\mathfrak{S}_n$-invariants with vanishing constant term, the second author and Jongwon Kim introduced [23] the *fermionic diagonal coinvariant ring*

$$FDR_n := \wedge\{\Theta_n, \Xi_n\} / \langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n} \rangle. \tag{2.1.6}$$

The quotient FDR_n is a bigraded $\mathfrak{S}_n$-module.

The ring FDR_n is an anticommutative version of the Garsia-Haiman diagonal coinvariant ring DR_n which has an analogous definition [14] involving lists $X_n = (x_1, \ldots, x_n)$ and $Y_n = (y_1, \ldots, y_n)$ of commuting variables. Various authors [2, 4, 6, 23, 28, 30, 41, 42, 52, 53, 60, 61] have considered versions of DR_n involving mixtures of commuting and anticommuting variables.

Kim and Rhoades describe [23] the bigraded $\mathfrak{S}_n$-isomorphism type of FDR_n in terms of Kronecker products. In particular, the bigraded piece $(FDR_n)_{i,j}$ vanishes whenever $i + j \geq n$. If $i + j < n$ we have the Frobenius image

$$\mathrm{Frob}\,(FDR_n)_{i,j} = s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i-1,1^{i+1})} * s_{(n-j-1,1^{j+1})} \tag{2.1.7}$$

where $*$ denotes Kronecker product of Schur functions and we interpret $s_{(-1,1^n)} = 0$. Kim and Rhoades give a basis of FDR_n indexed by a certain collection of lattice paths, but the combinatorics of FDR_n was largely unexplored in [23].

Equation (2.1.7) implies that whenever $i + j < n$ we have

$$\dim\,(FDR_n)_{i,j} = \binom{n-1}{i}\binom{n-1}{j} - \binom{n-1}{i+1}\binom{n-1}{j+1} \tag{2.1.8}$$

so that for any $1 \leq k \leq n$

$$\dim (FDR_n)_{n-k,k-1} = \mathrm{Nar}(n,k) \quad \text{so that} \quad \sum_{k=1}^{n} \dim (FDR_n)_{n-k,k-1} = \mathrm{Cat}(n) \tag{2.1.9}$$

and FDR_n contains a natural Catalan-into-Narayana dimensional submodule by considering its extreme bidegrees. We isolate this submodule as follows.

Definition 2.1.1. *For $n \geq 0$, let $\overline{FDR_n}$ be the $\mathfrak{S}_n$-submodule of FDR_n given by*

$$\overline{FDR_n} := \bigoplus_{k=1}^{n} (FDR_n)_{n-k,k-1}. \tag{2.1.10}$$

The module FDR_n has dimension $\mathrm{Cat}(n)$ *and its constituent piece $(FDR_n)_{n-k,k-1}$ has dimension* $\mathrm{Nar}(n,k)$.

In this chapter we establish isomorphisms (Corollary 2.6.3) of $\mathfrak{S}_n$-modules

$$\mathbb{C}[\mathrm{NC}(n)] \cong \overline{FDR_n} \quad \text{and} \quad \mathbb{C}[\mathrm{NC}(n,k)] \cong (FDR_n)_{n-k,k-1} \tag{2.1.11}$$

thus giving an algebraic model for the skein action in terms of fermionic diagonal coinvariants. To do this, we attach (Definition 2.2.2) a fermion $f_\pi \in \wedge\{\Theta_n, \Xi_n\}$ to any set partition $\pi \in \Pi(n)$ and prove (Theorem 2.3.4) that the noncrossing fermions $\{f_\pi : \pi \in \mathrm{NC}(n,k)\}$ satisfy the skein relations and descend to a basis of $(FDR_n)_{n-k,k-1}$. This gives a basis (Theorem 2.6.2) of $\overline{FDR_n}$ tied to the combinatorics of set partitions. Furthermore, the algebraic model of fermions sharpens a number of results on the skein action in [39], as well as simplifying and clarifying their proofs. Finally, the methods in [23] extend naturally from $\mathfrak{S}_n$ to irreducible complex reflection groups W, thus giving an avenue for extending the skein action to other types.

The rest of the chapter is organized as follows. In **Section 2.2** we define two fermions F_π and f_π attached to any set partition π of $[n]$ (noncrossing or otherwise); the fermions F_π and f_π are related by a kind of differentiation. We also introduce the *block operators* ρ_B; these derivations of $\wedge\{\Theta_n, \Xi_n\}$ will be useful in our proofs. In **Section 2.3** we define the skein action and prove that the F_π and f_π satisfy the skein relations of Figure 2.1. **Section 2.4** studies submodules of the exterior algebra $\wedge\{\Theta_n, \Xi_n\}$. We prove that the combinatorial skein action is isomorphic to the space spanned by the F_π (as well as the space spanned by the f_π). **Section 2.5** applies the theory of fermions to resolve crossings in set partitions; this has a number of corollaries on the combinatorics of the skein action. **Section 2.6** studies submodules of the quotient space FDR_n and proves the isomorphisms (2.1.11). We close in **Section 2.7** with some open problems.

2.2 Fermions for set partitions

In this section we attach two fermions F_π and f_π to set partitions $\pi \in \Pi(n)$. These fermions are obtained by applying certain operators $\rho_{B_1}, \ldots, \rho_{B_k}$ indexed by the blocks $B_1, \ldots, B_k$ of π to the product $\theta_1 \cdots \theta_n$.

2.2.1 Block operators ρ and ψ, fermions F and f

Our key tool in defining F_π and f_π is a family of derivations of the ring $\wedge\{\Theta_n, \Xi_n\}$. For $B \subseteq [n]$ nonempty, define the *block operator* $\rho_B : \wedge\{\Theta_n, \Xi_n\} \to \wedge\{\Theta_n, \Xi_n\}$ by

$$\rho_B(f) := \sum_{\substack{i,j \in B \\ i \neq j}} \xi_i \cdot (\theta_j \odot f) \tag{2.2.1}$$

whenever $|B| > 1$ and

$$\rho_B(f) := \xi_i \cdot (\theta_i \odot f) \tag{2.2.2}$$

if $B = \{i\}$ is a singleton. For any permutation $w \in \mathfrak{S}_n$ we have

$$w \cdot (\rho_B(f)) = \rho_{w(B)}(w \cdot f) \tag{2.2.3}$$

which follows from the readily checked relation

$$w \cdot (g \odot f) = (w \cdot g) \odot (w \cdot f) \qquad (w \in \mathfrak{S}_n,\ g, f \in \wedge\{\Theta_n, \Xi_n\}). \tag{2.2.4}$$

A crucial property enjoyed by the block operators is as follows.

Lemma 2.2.1. *Let $A, B \subseteq [n]$ be two nonempty subsets. The operators ρ_A and ρ_B commute.*

Proof. The lemma reduces to the assertion that, for any fermion f, we have

$$\xi_a \cdot (\theta_{a'} \odot [\xi_b \cdot (\theta_{b'} \odot f)]) = \xi_b \cdot (\theta_{b'} \odot [\xi_a \cdot (\theta_{a'} \odot f)]). \tag{2.2.5}$$

Using sign-twisted Leibniz Rule of Proposition 1.1.2 (3) we compute

$$\xi_a \cdot (\theta_{a'} \odot [\xi_b \cdot (\theta_{b'} \odot f)]) = -(\theta_{a'}\theta_{b'}) \odot (\xi_a \xi_b f) \tag{2.2.6}$$

$$= -(\theta_{b'}\theta_{a'}) \odot (\xi_b \xi_a f) \tag{2.2.7}$$

$$= \xi_b \cdot (\theta_{b'} \odot [\xi_a \cdot (\theta_{a'} \odot f)]) \tag{2.2.8}$$

as required. □

By Lemma 2.2.1, for any set partition $\pi = \{B_1 / B_2 / \cdots / B_k\} \in \Pi(n)$, we have a well-defined linear operator $\rho_\pi : \wedge\{\Theta_n, \Xi_n\} \to \wedge\{\Theta_n, \Xi_n\}$ given by

$$\rho_\pi := \rho_{B_1} \circ \rho_{B_2} \circ \cdots \circ \rho_{B_k} \tag{2.2.9}$$

where the order of composition is immaterial. This facilitates the following definition.

Definition 2.2.2. *Let $\pi = \{B_1 / B_2 / \cdots / B_n\} \in \Pi(n)$ be a set partition. We define fermions $F_\pi, f_\pi \in \wedge\{\Theta_n, \Xi_n\}$ by*

$$F_\pi := \rho_\pi(\theta_1\theta_2\cdots\theta_n) = (\rho_{B_1} \circ \rho_{B_2} \circ \cdots \circ \rho_{B_k})(\theta_1\theta_2\cdots\theta_n)$$

and

$$f_\pi := (\xi_1 + \xi_2 + \cdots + \xi_n) \odot F_\pi.$$

As an example of these objects, for $\pi = \{1, 3 / 2\}$ we have

$$F_{\{1,3/2\}} = \rho_{\{1,3\}} \circ \rho_{\{2\}}(\theta_1\theta_2\theta_3) = \rho_{\{1,3\}}(-\xi_2 \cdot \theta_1\theta_3) = \xi_3\xi_2\theta_3 - \xi_1\xi_2\theta_1,$$
$$f_{\{1,3/2\}} = (\xi_1 + \xi_2 + \xi_3) \odot (\xi_3\xi_2\theta_3 - \xi_1\xi_2\theta_1) = \xi_2\theta_3 - \xi_3\theta_3 - \xi_2\theta_1 + \xi_1\theta_1.$$

The notation F_π and f_π is from calculus: the f's are the derivatives of the F's If $\pi \in \Pi(n,k)$ has k blocks, the fermion F_π has bidegree $(n-k,k)$ whereas f_π has bidegree $(n-k,k-1)$. These fermions have similar algebraic properties. We focus mainly on the cleaner F_π, but the f_π will be useful in the study of FDR_n.

Most of our results on these fermions will hold at the level of the block operators ρ_B. For example, the following result describes how $\mathfrak{S}_n$ acts on the F_π.

Proposition 2.2.3. *Let $\pi \in \Pi(n)$ and $w \in \mathfrak{S}_n$. We have*

$$w \cdot F_\pi = \mathrm{sign}(w) \cdot F_{w(\pi)} \qquad \textit{and} \qquad w \cdot f_\pi = \mathrm{sign}(w) \cdot f_{w(\pi)}.$$

Proof. Equation (2.2.3) gives the equality of operators

$$w \cdot (\rho_\pi(-)) = \rho_{w(\pi)}(w \cdot (-)). \tag{2.2.10}$$

Applying both sides of Equation (2.2.10) to $\theta_1\theta_2\cdots\theta_n$ yields the statement about the F's. A further application of $(\xi_1 + \xi_2 + \cdots + \xi_n) \odot (-)$ implies the statement about the f's. □

The skein action treats singleton blocks differently from larger blocks, and we will avoid casework with the following variant of the ρ-operators. Given $B \subseteq [n]$, define $\psi_B : \wedge\{\Theta_n, \Xi_n\} \to$

$\wedge\{\Theta_n, \Xi_n\}$ by

$$\psi_B = \begin{cases} \rho_B & |B| > 1, \\ 0 & |B| \leq 1. \end{cases} \tag{2.2.11}$$

Lemma 2.2.1 implies

$$\psi_A \circ \psi_B = \psi_B \circ \psi_A \text{ for all } A, B \subseteq [n] \tag{2.2.12}$$

so for any set partition $\pi = \{B_1 \,/\, B_2 \,/ \cdots /\, B_k\} \in \Pi(n)$ we have a well-defined linear operator $\psi_\pi : \wedge\{\Theta_n, \Xi_n\} \to \wedge\{\Theta_n, \Xi_n\}$ given by

$$\psi_\pi := \psi_{B_1} \circ \psi_{B_2} \circ \cdots \circ \psi_{B_k} \tag{2.2.13}$$

which does not depend on the order of composition factors. We have

$$\psi_\pi(\theta_1\theta_2\cdots\theta_n) = \begin{cases} F_\pi & \text{if } \pi \text{ has no singleton blocks,} \\ 0 & \text{if } \pi \text{ has at least one singleton block.} \end{cases} \tag{2.2.14}$$

It will be convenient to have a version $\psi_{A,B}$ of the ψ-operators which depend on two subsets $A, B \subseteq [n]$. These are defined by

$$\psi_{A,B}(f) := \sum_{\substack{a \in A \\ b \in B}} \xi_a \cdot (\theta_b \odot f) + \sum_{\substack{a \in A \\ b \in B}} \xi_b \cdot (\theta_a \odot f) \tag{2.2.15}$$

for any $f \in \wedge\{\Theta_n, \Xi_n\}$, so that $\psi_{A,B} = \psi_{B,A}$. When $A \cap B = \varnothing$, we have the useful identity

$$\psi_{A \sqcup B} = \psi_A + \psi_{A,B} + \psi_B. \tag{2.2.16}$$

Like the ρ-operators, the ψ-operators commute.

Lemma 2.2.4. *Let $A, B, C, D \subseteq [n]$ be four subsets. We have the following identities of linear* *operators on* $\wedge\{\Theta_n, \Xi_n\}$

$$\psi_A \circ \psi_B = \psi_B \circ \psi_A, \qquad \psi_{A,B} \circ \psi_C = \psi_C \circ \psi_{A,B}, \qquad \psi_{A,B} \circ \psi_{C,D} = \psi_{C,D} \circ \psi_{A,B}.$$

The proof of Lemma 2.2.4 is the same as that of Lemma 2.2.1 and is left to the reader.

Remark 2.2.5. *If* $X_n = (x_1, \ldots, x_n)$ *and* $Y_n = (y_1, \ldots, y_n)$ *are two lists of* commuting *variables, the* polarization operator *on the polynomial ring* $\mathbb{C}[X_n, Y_n]$ *acts by*

$$f \mapsto \sum_{i=1}^{n} y_i \cdot \frac{\partial f}{\partial x_i} \tag{2.2.17}$$

for any polynomial $f \in \mathbb{C}[X_n, Y_n]$*. This operator lowers x-degree by 1 while raising y-degree by 1. Similarly, the operators* $\rho_B, \psi_B, \psi_{A,B}$ *lower* θ*-degree by 1 while raising* ξ*-degree by 1. Polarization operators on commuting variables are* $\mathfrak{S}_n$*-equivariant; Equation* (2.2.3) *describes how the action of* $\mathfrak{S}_n$ *intertwines with block operators. Polarization on commuting variables has played a major role [1, 2, 14, 41] in the theory of diagonal symmetric group actions. Our work suggests that block operators might be useful objects when dealing with anticommuting variables.*

2.2.2 Antisymmetrization and the fermions F and f

Most of our results on the F_π and f_π will be provable at the level of the ρ and ψ operators. However, it will sometimes be useful to have a more explicit formula for these fermions. For a composition α, let $|\alpha|_{\text{odd}} := \alpha_1 + \alpha_3 + \cdots$ be the sum of the odd parts of α.

Definition 2.2.6. *Let* (w, α) *be a segmented permutation of size n where* $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ *has length k. We define* $G_{w,\alpha} \in \wedge\{\Theta_n, \Xi_n\}$ *by the formula*

$$G_{w,\alpha} := \text{sign}(w) \cdot (-1)^{|\alpha|_{\text{odd}}} \cdot (\theta_{w[1]_1} \theta_{w[1]_2} \cdots \theta_{w[1]_{\alpha_1 - 1}}) \cdots (\theta_{w[k]_1} \theta_{w[k]_2} \cdots \theta_{w[k]_{\alpha_k - 1}}) \times \xi_{w[1]} \cdots \xi_{w[k]}. \tag{2.2.18}$$

where

$$\xi_{w[i]} := \begin{cases} \xi_{w[i]_1} + \cdots + \xi_{w[i]_{\alpha_i - 1}} & \alpha_i > 1, \\ \xi_{w[i]_1} & \alpha_i = 1. \end{cases} \tag{2.2.19}$$

We define $g_{w,\alpha} \in \wedge\{\Theta_n, \Xi_n\}$ *by*

$$g_{w,\alpha} := (-1)^{n-k}(\xi_1 + \cdots + \xi_n) \odot G_{w,\alpha}. \tag{2.2.20}$$

The sign $\mathrm{sign}(w) \cdot (-1)^{|\alpha|_{\mathrm{odd}}}$ in Definition 2.2.6 are necessary to pass from segmented permutations to set partitions. The $(-1)^{n-k}$ in the definition of $g_{w,\alpha}$ occurs 'because' the derivative $(\xi_1 + \cdots + \xi_n) \odot (-)$ must commute past $n-k$ fermionic θ-variables; see Proposition 1.1.2 (3).

As an example of Definition 2.2.6, let $(w,\alpha) = 536 \cdot 7 \cdot 21 \cdot 84$. We have

$$\mathrm{sign}(w) = \mathrm{sign}(53672184) = +1 \quad \text{and} \quad |\alpha|_{\mathrm{odd}} = \alpha_1 + \alpha_3 = 3 + 2 = 5$$

so that

$$\begin{aligned} G_{w,\alpha} &= (+1) \cdot (-1)^5 \cdot (\theta_5\theta_3) \cdot 1 \cdot (\theta_2) \cdot (\theta_8) \cdot (\xi_5 + \xi_3) \cdot (\xi_7) \cdot (\xi_2) \cdot (\xi_8) \\ &= -(\theta_5\theta_3) \cdot 1 \cdot (\theta_2) \cdot (\theta_8) \cdot (\xi_5 + \xi_3) \cdot (\xi_7) \cdot (\xi_2) \cdot (\xi_8). \end{aligned}$$

Applying $(-1)^{8-4}(\xi_1 + \cdots + \xi_8) \odot (-)$ to both sides of this equation yields

$$g_{w,\alpha} = -(\theta_5\theta_3) \cdot 1 \cdot (\theta_2) \cdot (\theta_8) \times [2 \cdot \xi_7\xi_2\xi_8 - (\xi_5 + \xi_3)\xi_2\xi_8 + (\xi_5 + \xi_3)\xi_7\xi_8 - (\xi_5 + \xi_3)\xi_7\xi_2].$$

By antisymmetrizing the $g_{w,\alpha}$ and $G_{w,\alpha}$, we obtain our new formulation for the F and f fermions.

Definition 2.2.7. *Let* (w,α) *be a segmented permutation where* $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$. *We define* *elements* $\widetilde{F}_{w,\alpha}, \widetilde{f}_{w,\alpha} \in \wedge\{\Theta_n, \Xi_n\}$ *by*

$$\widetilde{F}_{w,\alpha} := \frac{[\mathfrak{S}_{w,\alpha}]^- \cdot G_{w,\alpha}}{(\alpha_1 - 1)! \cdots (\alpha_k - 1)!} \in \wedge\{\Theta_n, \Xi_n\} \tag{2.2.21}$$

and

$$\widetilde{f}_{w,\alpha} := \frac{[\mathfrak{S}_{w,\alpha}]^- \cdot g_{w,\alpha}}{(\alpha_1 - 1)! \cdots (\alpha_k - 1)!} \in \wedge\{\Theta_n, \Xi_n\}. \tag{2.2.22}$$

In our example $(w,\alpha) = 536 \cdot 7 \cdot 21 \cdot 84$ we have

$$\mathfrak{S}_{w,\alpha} = \mathfrak{S}_{\{5,3,6\}} \times \mathfrak{S}_{\{7\}} \times \mathfrak{S}_{\{2,1\}} \times \mathfrak{S}_{\{8,4\}}$$

so that

$$\widetilde{F}_{w,\alpha} = \frac{[\mathfrak{S}_{\{5,3,6\}}]^- \cdot [\mathfrak{S}_{\{7\}}]^- \cdot [\mathfrak{S}_{\{2,1\}}]^- \cdot [\mathfrak{S}_{\{8,4\}}]^- \cdot G_{w,\alpha}}{2! \cdot 0! \cdot 1! \cdot 1!}$$

and

$$\widetilde{f}_{w,\alpha} = \frac{[\mathfrak{S}_{\{5,3,6\}}]^- \cdot [\mathfrak{S}_{\{7\}}]^- \cdot [\mathfrak{S}_{\{2,1\}}]^- \cdot [\mathfrak{S}_{\{8,4\}}]^- \cdot g_{w,\alpha}}{2! \cdot 0! \cdot 1! \cdot 1!}.$$

Proposition 2.2.8. *Let* (w,α) *be a segmented permutation where* $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ *and let* $\pi = \Pi(w,\alpha) \in \Pi(n,k)$ *be the corresponding set partition. We have*

$$\widetilde{F}_{w,\alpha} = \begin{cases} F_\pi & k \equiv 0,3 \mod 4, \\ -F_\pi & k \equiv 1,2 \mod 4. \end{cases} \tag{2.2.23}$$

Proof. For $v \in \mathfrak{S}_n$, it follows from the definitions that $v \cdot G_{w,\alpha} = \mathrm{sign}(v) \cdot G_{vw,\alpha}$. Using this and the identity $[\mathfrak{S}_{vw,\alpha}]^- = v[\mathfrak{S}_{w,\alpha}]^- v^{-1}$, we calculate

$$v[\mathfrak{S}_{w,\alpha}]^- \cdot G_{w,\alpha} = [\mathfrak{S}_{vw,\alpha}]^- v \cdot G_{w,\alpha} = \mathrm{sign}(v)[\mathfrak{S}_{vw,\alpha}]^- G_{vw,\alpha}. \tag{2.2.24}$$

Dividing both sides by $(\alpha_1 - 1)!(\alpha_2 - 1)! \cdots (\alpha_k - 1)!$ implies

$$v \cdot \widetilde{F}_{w,\alpha} = \operatorname{sign}(v) \cdot \widetilde{F}_{vw,\alpha} \tag{2.2.25}$$

which agrees with the $\mathfrak{S}_n$ action on the F's in Proposition 2.2.3.

By Equation (2.2.25) and Proposition 2.2.3, it is enough to verify Equation (2.2.23) when the segmented permutation (w, α) has the form

$$(w,\alpha) = 1, 2, \ldots, \alpha_1 \cdot \alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_1 + \alpha_2 \cdot \cdots \cdot n - \alpha_k + 1, \ldots, n-1, n. \tag{2.2.26}$$

If we set

$$B_i := \{\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + 2, \ldots, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i\}, \tag{2.2.27}$$

then

$$F_\pi = (\rho_{B_k} \circ \cdots \circ \rho_{B_2} \circ \rho_{B_1})(\theta_1 \cdots \theta_n) \tag{2.2.28}$$

$$= \varepsilon \cdot [\mathfrak{S}_{w,\alpha}]^- \cdot (\xi_n \cdots \xi_{\alpha_1+\alpha_2} \xi_{\alpha_1} \cdot \theta_1 \theta_2 \cdots \theta_{\alpha_1 - 1} \theta_{\alpha_1+1} \theta_{\alpha_1+2} \cdots \theta_{\alpha_1+\alpha_2-1} \theta_{\alpha_1+\alpha_2+1} \cdots \theta_{n-1}) \tag{2.2.29}$$

where the sign ε is given by

$$\varepsilon = (-1)^{(k-1)\cdot\alpha_1 + (k-2)\cdot\alpha_2 + \cdots + 1\cdot\alpha_{k-1} + 0\cdot\alpha_k} = \begin{cases} (-1)^{|\alpha|_{\text{odd}}} & k \text{ even}, \\ (-1)^{|\alpha|_{\text{odd}} + n} & k \text{ odd}, \end{cases} \tag{2.2.30}$$

which simplifies to

$$\varepsilon = (-1)^{|\alpha|_{\text{odd}} + k\cdot n}. \tag{2.2.31}$$

Reversing the order of the k factors $\xi_n \cdots \xi_{\alpha_1+\alpha_2} \xi_{\alpha_1}$ introduces a sign change by $(-1)^{\binom{k}{2}}$ and moving these k factors past the $n-k$ factors $\theta_1\theta_2\cdots\theta_{\alpha_1-1}\theta_{\alpha_1+1}\theta_{\alpha_1+2}\cdots\theta_{\alpha_1+\alpha_2-1}\theta_{\alpha_1+\alpha_2+1}\cdots\theta_{n-1}$ changes sign by $(-1)^{k\cdot(n-k)}$. We conclude that

$$F_\pi = (-1)^{\binom{k}{2}+k\cdot(n-k)+k\cdot n} \cdot \widetilde{F}_{w,\alpha} = (-1)^{\binom{k}{2}-k^2} \cdot \widetilde{F}_{w,\alpha} \tag{2.2.32}$$

which is equivalent to the statement of the proposition. □

Proposition 2.2.8 implies that the $\widetilde{F}_{w,\alpha}$ and $\widetilde{f}_{w,\alpha}$ depend only on the set partition $\Pi(w,\alpha)$ rather than the segmented permutation (w,α) itself. This facilitates the definitions

$$\widetilde{F}_\pi := \widetilde{F}_{w,\alpha} \qquad \text{and} \qquad \widetilde{f}_\pi := \widetilde{f}_{w,\alpha} \tag{2.2.33}$$

where (w,α) is any segmented permutation such that $\Pi(w,\alpha) = \pi$. In this language, Proposition 2.2.8 informally reads

$$F_\pi = \pm\widetilde{F}_\pi \qquad f_\pi = \pm\widetilde{f}_\pi. \tag{2.2.34}$$

We will typically deal with set partitions having a fixed number of blocks, so the precise signs appearing in Equation (2.2.34) will usually not play a significant role in our work.

2.2.3 Restriction properties

Given any set partition π of $[n]$, we can form a set partition $\overline{\pi}$ of $[n-1]$ by removing n (and its block, if $\{n\}$ is a singleton). The effect of this operation on set partition fermions depends on the size of the block of π containing n. The answer is more attractive for the $\widetilde{F}$'s; applying $(\xi_1+\cdots+\xi_n)\odot(-)$ we can obtain a corresponding result for the $\widetilde{f}$'s.

Proposition 2.2.9. *Let $\pi \in \Pi(n,k)$ be a set partition of $[n]$ with k blocks. Let $\overline{\pi}$ be the set partition of $[n-1]$ obtained by removing n from π (and the block containing n, if $\{n\}$ is a* *singleton). Let B be the block of π containing n. The fermions F_π and $F_{\overline{\pi}}$ are related by*

$$\widetilde{F}_{\overline{\pi}} = \begin{cases} (-1)^n \theta_n \odot (\widetilde{F}_\pi \mid_{\xi_n=0}) & |B| \geq 3, \\ (-1)^{n-1} \theta_i \odot \widetilde{F}_\pi & |B| = 2 \text{ and } B = \{i,n\}, \\ (-1)^{k-1} \xi_n \odot \widetilde{F}_\pi & |B| = 1, \end{cases} \tag{2.2.35}$$

where $\widetilde{F}_\pi \mid_{\xi_n=0}$ evaluates $\widetilde{F}_\pi \in \wedge\{\Theta_n, \Xi_n\}$ at $\xi_n \to 0$.

Proof. Consider a segmented permutation (w,α) such that $\Pi(w,\alpha) = \pi$ which has the form

$$(na_1 \cdots a_j) \cdot (b_1 \cdots b_m) \cdot \ldots \cdot (c_1 \cdots c_p) \tag{2.2.36}$$

where the parentheses around segments are for readability. The fermion F_π is given by

$$\widetilde{F}_\pi = (-1)^{|\alpha|_{\text{odd}}} \text{sign}(na_1 \ldots a_j b_1 \ldots b_m \cdots c_1 \ldots c_p) \times \\ [\mathfrak{S}_{w,\alpha}]^- \cdot (\theta_n \theta_{a_1} \cdots \theta_{a_{j-1}}) \cdots (\theta_{c_1} \cdots \theta_{c_{p-1}})(\xi_n + \xi_{a_1} + \cdots + \xi_{a_{j-1}}) \cdots (\xi_{c_1} + \cdots + \xi_{c_{p-1}}). \tag{2.2.37}$$

If $|B| \geq 2$, then $j \geq 1$, a segmented permutation representing $\overline{\pi}$ is

$$(\overline{w}, \overline{\alpha}) := (a_1 \cdots a_j) \cdot (b_1 \cdots b_m) \cdot \cdots \cdot (c_1 \cdots c_p) \tag{2.2.38}$$

and since $|\alpha|_{\text{odd}} = |\overline{\alpha}|_{\text{odd}} - 1$ and $\text{sign}(\overline{w}) = (-1)^{n-1}\text{sign}(w)$ we have

$$\widetilde{F}_{\overline{\pi}} = (-1)^{|\alpha|_{\text{odd}}-1}(-1)^{n-1}\text{sign}(na_1 \ldots a_j b_1 \ldots b_m \cdots c_1 \ldots c_p) \times \\ [\mathfrak{S}_{\overline{w},\overline{\alpha}}]^- \cdot (\theta_{a_1} \cdots \theta_{a_{j-1}}) \cdots (\theta_{c_1} \cdots \theta_{c_{p-1}})(\xi_{a_1} + \cdots + \xi_{a_{j-1}}) \cdots (\xi_{c_1} + \cdots + \xi_{c_{p-1}}) \tag{2.2.39}$$

and comparing these formulas gives the result (In the case $|B| = 2$ and $j = 1$ the product $\theta_{a_1} \cdots \theta_{a_{j-1}}$ appearing in $F_{\overline{\pi}}$ is empty, while a permutation in $\mathfrak{S}_{w,\alpha}$ of negative sign is required to send $\xi_n + \xi_{a_1} + \cdots + \xi_{a_{j-1}} = \xi_n$ to ξ_i in F_π.)

If $|B| = 1$ and $j = 0$, a segmented permutation representing $\overline{\pi}$ is

$$(\overline{w}, \overline{\alpha}) := (b_1 \cdots b_m) \cdot \ldots \cdot (c_1 \cdots c_p) \tag{2.2.40}$$

and the relations $|\overline{\alpha}|_{\text{odd}} = n - |\alpha|_{\text{odd}}$ and $\text{sign}(\overline{w}) = (-1)^{n-1}\text{sign}(w)$ give

$$\begin{aligned} \widetilde{F}_{\overline{\pi}} = (-1)^{n-|\alpha|_{\text{odd}}}(-1)^{n-1}\text{sign}(nb_1 \ldots b_m \cdots c_1 \ldots c_p) \times \\ [\mathfrak{S}_{\overline{w},\overline{\alpha}}]^- \cdot (\theta_{b_1} \cdots \theta_{b_{m-1}}) \cdots (\theta_{c_1} \cdots \theta_{c_{p-1}})(\xi_{b_1} + \cdots + \xi_{b_{m-1}}) \cdots (\xi_{c_1} + \cdots + \xi_{c_{p-1}}) \end{aligned} \tag{2.2.41}$$

whereas

$$\begin{aligned} \widetilde{F}_{\pi} = (-1)^{|\alpha|_{\text{odd}}}\text{sign}(nb_1 \ldots b_m \cdots c_1 \ldots c_p) \times \\ [\mathfrak{S}_{\overline{w},\overline{\alpha}}]^- \cdot (\theta_{b_1} \cdots \theta_{b_{m-1}}) \cdots (\theta_{c_1} \cdots \theta_{c_{p-1}})(\xi_n)(\xi_{b_1} + \cdots \xi_{b_{m-1}}) \cdots (\xi_{c_1} + \cdots + \xi_{c_{p-1}}). \end{aligned} \tag{2.2.42}$$

Applying $\xi_n \odot (-)$ to remove the ξ_n from F_π involves k sign changes (the number of blocks of π, or the number of θ-variables). The top lines of Equations (2.2.41) and (2.2.42) differ in an additional factor of $(-1)^{2n-1} = -1$. □

2.3 Fermions and skein relations

2.3.1 Almost noncrossing partitions and the skein action

We present a modified version of the skein action of $\mathfrak{S}_n$ on $\mathbb{C}[\text{NC}(n)]$ defined in [39, Sec. 3]. The heart of this construction is a resolution of crossings in set partitions which are almost, but not quite, noncrossing.

A set partition $\pi \in \Pi(n)$ is *almost noncrossing* if π is not noncrossing but there exists an index $1 \leq i \leq n-1$ such that $s_i(\pi)$ is noncrossing. The index i is not always uniquely determined by π: if $\pi = \{1, 3, 5/2, 4\} \in \Pi(5)$ then both $s_1(\pi)$ and $s_3(\pi)$ are noncrossing, so that $\pi \in \mathrm{ANC}(5)$.

Let $\mathrm{ANC}(n)$ be the family of almost noncrossing set partitions of $[n]$. We define a set map

$$\sigma : \mathrm{ANC}(n) \longrightarrow \mathbb{C}[\mathrm{NC}(n)] \tag{2.3.1}$$

as follows; see Figure 2.1.

Definition 2.3.1. *Let $\pi \in \mathrm{ANC}(n)$ be such that $s_i(\pi) \in \mathrm{NC}(n)$. Then i and $i+1$ are in different blocks of π; let B_i be the block of π containing i and B_{i+1} be the block of π containing $i+1$. The blocks B_i and B_{i+1} both have size at least 2. We set*

$$\sigma(\pi) := \begin{cases} \pi_1 + \pi_2 & \text{if } |B_i| = |B_{i+1}| = 2, \\ \pi_1 + \pi_2 - \pi_3 & \text{if } |B_i| > 2 \text{ and } |B_{i+1}| = 2, \\ \pi_1 + \pi_2 - \pi_4 & \text{if } |B_i| = 2 \text{ and } |B_{i+1}| > 2, \\ \pi_1 + \pi_2 - \pi_3 - \pi_4 & \text{if } |B_i|, |B_{i+1}| > 2, \end{cases} \tag{2.3.2}$$

where the set partitions $\pi_1, \ldots, \pi_4 \in \mathrm{NC}(n)$ are obtained from π by replacing B_i and B_{i+1} with the new pair of blocks

- $(B_i - \{i\}) \cup \{i+1\}$ *and* $(B_{i+1} - \{i+1\}) \cup \{i\}$ *for* π_1,
- $(B_i \cup B_{i+1}) - \{i, i+1\}$ *and* $\{i, i+1\}$ *for* π_2,
- $B_i - \{i\}$ *and* $B_{i+1} \cup \{i\}$ *for* π_3, *and*
- $B_{i+1} - \{i+1\}$ *and* $B_i \cup \{i+1\}$ *for* π_4.

It is proven in [39, Lem. 3.3] that if $s_i(\pi)$ is noncrossing for more than one value of i, the above procedure yields the same element $\sigma(\pi) \in \mathbb{C}[\mathrm{NC}(n)]$. In other words, the set map σ is well-defined.

Definition 2.3.2. *For $1 \leq i \leq n-1$, the* skein action *of the adjacent transposition s_i on $\mathbb{C}[\mathrm{NC}(n)]$ is given by*

$$s_i \cdot \pi := \begin{cases} -s_i(\pi) & \textit{if } s_i(\pi) \textit{ is noncrossing,} \\ \sigma(s_i(\pi)) & \textit{otherwise.} \end{cases} \tag{2.3.3}$$

The sign conventions in Definition 2.3.2 are slightly different from those in [39, Eqn. (4.1)]. The action of s_i on $\pi \in \mathrm{NC}(n)$ in [39] did not introduce a sign when at least one of $i, i+1$ formed a singleton block of π. The calculation-intensive arguments of [39, Lem. 4.1, Lem. 4.2, Lem. 4.3] go through to show that the action of Definition 2.3.2 satisfies the Coxeter relations and we have an induced action of $\mathfrak{S}_n$ on $\mathbb{C}[\mathrm{NC}(n)]$. Fermions will give a more conceptual proof (Theorem 2.3.4, Theorem 2.4.6) that this action is well-defined.

2.3.2 Block operators and skein relations

In this subsection we prove our first major result: a link between fermions and skein relations. We first state our result at the level of the block operators ρ_π.

For notational convenience, if $\pi \in \mathrm{ANC}(n)$ is an almost noncrossing partition such that $s_i(\pi)$ is noncrossing, we define a linear operator

$$\rho_{\sigma(\pi)} : \wedge\{\Theta_n, \Xi_n\} \longrightarrow \wedge\{\Theta_n, \Xi_n\} \tag{2.3.4}$$

by the formula

$$\rho_{\sigma(\pi)} := \begin{cases} \rho_{\pi_1} + \rho_{\pi_2} & \text{if } |B_i| = |B_{i+1}| = 2, \\ \rho_{\pi_1} + \rho_{\pi_2} - \rho_{\pi_3} & \text{if } |B_i| > 2 \text{ and } |B_{i+1}| = 2, \\ \rho_{\pi_1} + \rho_{\pi_2} \qquad\quad - \rho_{\pi_4} & \text{if } |B_i| = 2 \text{ and } |B_{i+1}| > 2, \\ \rho_{\pi_1} + \rho_{\pi_2} - \rho_{\pi_3} - \rho_{\pi_4} & \text{if } |B_i|, |B_{i+1}| > 2, \end{cases} \tag{2.3.5}$$

where B_i is the block of π containing i, B_{i+1} is the block of π containing $i+1$, and $\pi_1, \ldots, \pi_4 \in$

NC(n) are as in Definition 2.3.1. The following result states that the block operators satisfy the skein relations.

Theorem 2.3.3. *Suppose $\pi \in \mathrm{ANC}(n)$ is an almost noncrossing partition such that $s_i(\pi)$ is noncrossing. We have*

$$\rho_\pi + \rho_{\sigma(\pi)} = 0 \tag{2.3.6}$$

as operators on $\wedge\{\Theta_n, \Xi_n\}$.

Proof. Suppose $A \sqcup \{i+1\}$ and $B \sqcup \{i\}$ are blocks of π. By the definition of ρ_π and $\rho_{\sigma(\pi)}$ and the commutativity statement in the last paragraph, it suffices to show the operator identity

$$\psi_{A\sqcup\{i+1\}} \circ \psi_{B\sqcup\{i\}} + \psi_{A\sqcup\{i\}} \circ \psi_{B\sqcup\{i+1\}} + \psi_{A\sqcup B} \circ \psi_{\{i,i+1\}} - \psi_A \circ \psi_{B\sqcup\{i,i+1\}} - \psi_{A\sqcup\{i,i+1\}} \circ \psi_B = 0 \tag{2.3.7}$$

where the ψ-operators avoid the branching in the definition of $\rho_{\sigma(\pi)}$. We prove Equation (2.3.7) by a sign-reversing involution.

In terms of the $\psi_{S,T}$-operators, the desired Equation (2.3.7) reads

$$\begin{aligned}(\psi_A + \psi_{A,\{i+1\}}) \circ (\psi_B + \psi_{B,\{i\}}) + (\psi_A + \psi_{A.\{i\}}) \circ (\psi_B + \psi_{B,\{i+1\}}) \\ + (\psi_A + \psi_{A,B} + \psi_B) \circ \psi_{\{i,i+1\}} - \psi_A \circ (\psi_B + \psi_{B,\{i\}s} + \psi_{B,\{i+1\}} + \psi_{\{i.i+1\}}) \\ - \psi_B \circ (\psi_A + \psi_{A,\{i\}} + \psi_{A,\{i+1\}} + \psi_{\{i.i+1\}}) = 0.\end{aligned} \tag{2.3.8}$$

Expanding the LHS of Equation (2.3.8), applying Lemma 2.2.4, and simplifying gives

$$\psi_{A,\{i+1\}} \circ \psi_{B,\{i\}} + \psi_{A,\{i\}} \circ \psi_{B,\{i+1\}} + \psi_{A,B} \circ \psi_{\{i,i+1\}} \tag{2.3.9}$$

so that given $f \in \wedge\{\Theta_n, \Xi_n\}$ the action of the LHS of Equation (2.3.8) on f is

$$\frac{1}{2} \sum_{(t_1,t_2,t_3,t_4)} \xi_{t_1} \cdot (\theta_{t_2} \odot (\xi_{t_3} \cdot (\theta_{t_4} \odot f))) \tag{2.3.10}$$

where the sum is over all quadruples $1 \le t_1, \ldots, t_4 \le n$ such that precisely one t_j lies in each of the four sets $A, B, \{i\}, \{i+1\}$. The factor of $\frac{1}{2}$ in the (2.3.10) arises from the double counting $(t_1, t_2) \leftrightarrow (t_3, t_4)$ involved in applying the ψ-expression (2.3.9) to f. Equation (2.3.8) and the theorem will be proved if we can show that the expression (2.3.10) vanishes. Anticommutativity yields

$$\xi_{t_1} \cdot (\theta_{t_2} \odot (\xi_{t_3} \cdot (\theta_{t_4} \odot f))) = -\xi_{t_3} \cdot (\theta_{t_2} \odot (\xi_{t_1} \cdot (\theta_{t_4} \odot f))) \tag{2.3.11}$$

which sets up a sign-reversing involution on the terms in (2.3.10). □

The sign-reversing involution in the proof Theorem 2.3.3 relied on anticommutativity in a crucial way. We regard this as evidence that fermions are a good setting for studying resolution of set partition crossings.

The fact that the F_π and f_π satisfy the skein relations is easily deduced from Theorem 2.3.3. In analogy with the case of block operators, if $\pi \in \mathrm{ANC}(n)$ is almost noncrossing and $s_i(\pi)$ is noncrossing, we define $F_{\sigma(\pi)} \in \wedge\{\Theta_n, \Xi_n\}$ by

$$F_{\sigma(\pi)} := \begin{cases} F_{\pi_1} + F_{\pi_2} & \text{if } |B_i| = |B_{i+1}| = 2, \\ F_{\pi_1} + F_{\pi_2} - F_{\pi_3} & \text{if } |B_i| > 2 \text{ and } |B_{i+1}| = 2, \\ F_{\pi_1} + F_{\pi_2} \qquad - F_{\pi_4} & \text{if } |B_i| = 2 \text{ and } |B_{i+1}| > 2, \\ F_{\pi_1} + F_{\pi_2} - F_{\pi_3} - F_{\pi_4} & \text{if } |B_i|, |B_{i+1}| > 2, \end{cases} \tag{2.3.12}$$

where B_i is the block of π containing i, B_{i+1} is the block of π containing $i+1$, and $\pi_1, \ldots, \pi_4 \in$

NC(n) are as in Definition 2.3.1. Similarly, we define $f_{\sigma(\pi)} \in \wedge\{\Theta_n, \Xi_n\}$ by

$$f_{\sigma(\pi)} := \begin{cases} f_{\pi_1} + f_{\pi_2} & \text{if } |B_i| = |B_{i+1}| = 2, \\ f_{\pi_1} + f_{\pi_2} - f_{\pi_3} & \text{if } |B_i| > 2 \text{ and } |B_{i+1}| = 2, \\ f_{\pi_1} + f_{\pi_2} \qquad - f_{\pi_4} & \text{if } |B_i| = 2 \text{ and } |B_{i+1}| > 2, \\ f_{\pi_1} + f_{\pi_2} - f_{\pi_3} - f_{\pi_4} & \text{if } |B_i|, |B_{i+1}| > 2. \end{cases} \tag{2.3.13}$$

Theorem 2.3.4. *Let $\pi \in \mathrm{NC}(n)$ and $1 \leq i \leq n-1$. Then*

$$s_i \cdot F_\pi := \begin{cases} -F_{s_i(\pi)} & \text{if } s_i(\pi) \text{ is noncrossing}, \\ F_{\sigma(s_i(\pi))} & \text{otherwise}. \end{cases} \tag{2.3.14}$$

and

$$s_i \cdot f_\pi := \begin{cases} -f_{s_i(\pi)} & \text{if } s_i(\pi) \text{ is noncrossing}, \\ f_{\sigma(s_i(\pi))} & \text{otherwise}. \end{cases} \tag{2.3.15}$$

Proof. Proposition 2.2.3 implies

$$s_i \cdot F_\pi = \mathrm{sign}(s_i) F_{s_i(\pi)} = -F_{s_i(\pi)} \tag{2.3.16}$$

so we are done if $s_i(\pi) \in \mathrm{NC}(n)$ is noncrossing. We therefore assume $s_i(\pi) \in \mathrm{ANC}(n)$ is almost noncrossing. The desired formula follows from applying both sides of the operator identity of Theorem 2.3.3 to $\theta_1 \cdots \theta_n$. □

Thanks to fermions and block operators, the proofs in this section were much faster and cleaner than the corresponding proofs in [39, Sec. 3]. The proofs in [39, Sec. 3] were brute force and involved extensive casework depending on block sizes; the ψ-operators in the proof of Theorem 2.3.3 unify this casework.

Theorem 2.3.3 yields other families of fermions labeled by set partitions which satisfy the skein relations. Suppose $h \in \wedge\{\Theta_n, \Xi_n\}$ is alternating, and let $\mathscr{T} := \{\rho_\pi(h) : \pi \in \Pi(n)\}$. Theorem 2.3.3 shows that span $\mathscr{T}$ is $\mathfrak{S}_n$-stable, and that the polynomials appearing in $\mathscr{T}$ satisfy the skein relations under the action of s_i. This construction also makes sense in the presence of more than two sets $\Theta_n, \Xi_n, \ldots, \Omega_n$ of fermionic variables; this might help in the multidiagonal context of Problem 2.7.3 below.

2.4 Noncrossing bases in $\wedge\{\Theta_n, \Xi_n\}$

2.4.1 The modules V and W

Given $n, k, m \geq 0$, we define six subspaces of $\wedge\{\Theta_n, \Xi_n\}$ as follows.

$$\begin{cases} W(n) := \mathrm{span}\{F_\pi : \pi \in \Pi(n)\}, \\ W(n,k) := \mathrm{span}\{F_\pi : \pi \in \Pi(n,k)\}, \\ W(n,k,m) := \mathrm{span}\{F_\pi : \pi \in \Pi(n,k,m),\} \end{cases} \qquad \begin{cases} V(n) := \mathrm{span}\{f_\pi : \pi \in \Pi(n)\}, \\ V(n,k) := \mathrm{span}\{f_\pi : \pi \in \Pi(n,k)\}, \\ V(n,k,m) := \mathrm{span}\{f_\pi : \pi \in \Pi(n,k,m)\}. \end{cases} \tag{2.4.1}$$

Degree considerations imply that the sums

$$W(n) = \bigoplus_{k=0}^{n} W(n,k) \qquad \text{and} \qquad V(n) = \bigoplus_{k=0}^{n} V(n,k) \tag{2.4.2}$$

of subspaces are direct. We shall see (Theorem 2.4.5) that the sums

$$W(n,k) = \sum_{m=0}^{k} W(n,k,m) \qquad \text{and} \qquad V(n,k) = \sum_{m=0}^{k} V(n,k,m) \tag{2.4.3}$$

are also direct. We record some additional structural properties of these spaces.

Proposition 2.4.1. *The six spaces $W(n), W(n,k), W(n,k,m), V(n), V(n,k)$, and $V(n,k,m)$ are closed under the action of $\mathfrak{S}_n$ on $\wedge\{\Theta_n, \Xi_n\}$. Furthermore, these spaces are spanned by* *noncrossing fermions. That is, we have*

$$\begin{cases} W(n) = \mathrm{span}\{F_\pi : \pi \in \mathrm{NC}(n)\}, \\ W(n,k) = \mathrm{span}\{F_\pi : \pi \in \mathrm{NC}(n,k)\}, \\ W(n,k,m) = \mathrm{span}\{F_\pi : \pi \in \mathrm{NC}(n,k,m)\}, \end{cases} \qquad \begin{cases} V(n) = \mathrm{span}\{f_\pi : \pi \in \mathrm{NC}(n)\}, \\ V(n,k) = \mathrm{span}\{f_\pi : \pi \in \mathrm{NC}(n,k)\}, \\ V(n,k,m) = \mathrm{span}\{f_\pi : \pi \in \mathrm{NC}(n,k,m)\}. \end{cases}$$

Proof. The $\mathfrak{S}_n$-closure follows from Proposition 2.2.3. To see that the noncrossing fermions span $W(n)$, we argue as follows. Let $\pi \in \Pi(n)$ be an arbitrary set partition. There exists $w \in \mathfrak{S}_n$ such that $w(\pi) \in \mathrm{NC}(n)$ is noncrossing. We have

$$F_\pi = \mathrm{sign}(w) \cdot w^{-1} \cdot F_{w(\pi)} \tag{2.4.4}$$

by Proposition 2.2.3. Writing w^{-1} as a product of adjacent transpositions s_i and applying them to $F_{w(\pi)}$ in succession, Theorem 2.3.4 guarantees the we obtain a $\mathbb{Z}$-linear combination of F_μ's for $\mu \in \mathrm{NC}(n)$ noncrossing. For the case of $W(n,k)$ and $W(n,k,m)$, observe that the skein relations in Figure 2.1 preserve the total number of blocks and the number of singleton blocks. The corresponding statements for the V-spaces follow from an application of $(\xi_1 + \cdots + \xi_n) \odot (-)$. □

We will see that the six spanning sets in Proposition 2.4.1 are in fact bases. The linear independence of these sets could in principle be established by examining expansions in the monomial basis $\theta_S \cdot \xi_T$ of $\wedge\{\Theta_n, \Xi_n\}$, but the coefficients involved obstruct this approach. We employ a more conceptual method hinging on a careful analysis of the singleton-free cases $W(n,k,0)$ and $V(n,k,0)$ of these $\mathfrak{S}_n$-modules.

2.4.2 Singleton-free partitions and flag-shaped irreducibles

A partition $\lambda \vdash n$ is *flag-shaped* if it is of the form $\lambda = (k,k,1^{n-2k})$ for some $1 \leq k \leq n/2$. The goal of this subsection is to show that the $\mathfrak{S}_n$-modules $W(n,k,0)$ and $V(n,k,0)$ spanned by singleton-free k-block fermions are flag-shaped irreducibles. We begin with a general criterion for when a Young symmetrizer or antisymmetrizer annihilates an $\mathfrak{S}_n$-irreducible. Here we compare partitions in dominance order.

Lemma 2.4.2. *Let $\lambda, \mu \vdash n$ be partitions. We have $[\mathfrak{S}_\mu]^+ \cdot S^\lambda \neq 0$ if and only if $\mu \leq \lambda$.*

Lemma 2.4.2 is equivalent to the fact that the *Kostka number* $K_{\lambda,\mu}$ counting semistandard tableaux of shape λ and content μ is nonzero if and only if $\mu \leq \lambda$ in dominance order.

Proposition 2.4.3. *For $k \leq n/2$, both of the $\mathfrak{S}_n$-modules $W(n,k,0)$ and $V(n,k,0)$ are isomorphic to the flag-shaped irreducible S^λ where $\lambda = (k,k,1^{n-2k})$.*

Proof. It follows from the hook-length formula and independent observations of O'Hara and Zeilberger that $\dim S^\lambda = |\mathrm{NC}(n,k,0)|$; see [32] for details. We verify that $[\mathfrak{S}_\lambda]^+$ **does not** annihilate $W(n,k,0)$, but that $[\mathfrak{S}_\mu]^+$ **does** annihilate $W(n,k,0)$ whenever $\lambda < \mu$.

To prove $[\mathfrak{S}_\lambda]^+ \cdot W(n,k,0) \neq 0$, by Proposition 2.2.8 it suffices to find a single $\pi_0 \in \Pi(n,k,0)$ with $[\mathfrak{S}_\lambda]^+ \cdot \widetilde{F}_{\pi_0} \neq 0$. We let

$$\pi_0 := \{1,\, 2k,\, 2k+1,\, \cdots,\, n-1,\, n\,/\,2,\, 2k-1\,/\,3,\, 2k-2\,/\,\cdots\,/\,k,\, k+1\} \tag{2.4.5}$$

so that

$$\begin{aligned}\widetilde{F}_{\pi_0} = C(\theta_2\xi_2 - \theta_{2k-1}\xi_{2k-1})(\theta_3\xi_3 - \theta_{2k-2}\xi_{2k-2})\cdots(\theta_k\xi_k - \theta_{k+1}\xi_{k+1})\times \\ [\mathfrak{S}_{\{1,2k,2k+1,\ldots,n-1,n\}}]^- \cdot (\theta_1\theta_{2k}\theta_{2k+1}\cdots\theta_{n-1})(\xi_1+\xi_{2k}+\xi_{2k+1}+\cdots+\xi_{n-1})\end{aligned} \tag{2.4.6}$$

where C is a nonzero constant. Equation (2.4.6) may be rewritten as

$$\begin{aligned}\widetilde{F}_{\pi_0} = C'(\theta_2\xi_2 - \theta_{2k-1}\xi_{2k-1})(\theta_3\xi_3 - \theta_{2k-2}\xi_{2k-2})\cdots(\theta_k\xi_k - \theta_{k+1}\xi_{k+1})\times \\ \left(e - \sum_{i\in\{1,2k,2k+1,\ldots,n-1\}}(i,n)\right)\cdot(\theta_1\theta_{2k}\theta_{2k+1}\cdots\theta_{n-1})(\xi_1+\xi_{2k}+\xi_{2k+1}+\cdots+\xi_{n-1})\end{aligned} \tag{2.4.7}$$

where $C' = (n-2k+1)! \cdot C$ is also nonzero. We claim that the coefficient of

$$\theta_1 \xi_1 \theta_2 \xi_2 \cdots \theta_k \xi_k \cdot \theta_{2k} \theta_{2k+1} \cdots \theta_{n-1} \tag{2.4.8}$$

in $[\mathfrak{S}_\lambda]^+ \cdot \widetilde{F}_{\pi_0}$ is nonzero. A permutation $w \in \mathfrak{S}_\lambda = \mathfrak{S}_k \times \mathfrak{S}_k$ can only contribute to the desired coefficient when $w(2k) = 2k$, and the permutation involved in the action on the second line of (2.4.7) is the identity e (rather than one of the transpositions (i,n)). We claim that all permutations w in this (nonempty) set contribute to the desired coefficient with the same sign. Indeed, for any such w, in order to contribute to the desired coefficient we must select the first term (with positive sign) in each of the $k-1$ factors on the top line of Equation (2.4.7), and then choose the first term in the ξ-sum on the bottom line of Equation (2.4.7). A uniform sign of $(-1)^{n-2k}$ is involved in moving the variables $\theta_{w(1)}$ and $\xi_{w(1)}$ next to each other in the second line. Once this is done, the factors $\theta_i \xi_i$ for $1 \leq i \leq k$ commute signlessly.

Now let $\mu \vdash n$ be a partition with $\lambda \leq \mu$ such that $[\mathfrak{S}_\mu]^+ \cdot W(n,k,0) \neq 0$. By Proposition 2.4.1 there exists $\pi \in \mathrm{NC}(n,k,0)$ such that $[\mathfrak{S}_\mu]^+ \cdot F_\pi \neq 0$. If $i \sim i+1$ in π, we have $(1+s_i) \cdot \pi = \pi - \pi = 0$ by Theorem 2.3.4. We argue that $\mu = \lambda$ as follows.

Since $\lambda \leq \mu$ we have $\mu_1 \geq k$. Because $[\mathfrak{S}_\mu]^+ \cdot F_\pi \neq 0$ and π has k blocks, this forces $\mu_1 = k$ and implies that $1, 2, \ldots, k$ are in distinct blocks of π. Since $\lambda \leq \mu$ we must also have $\mu_2 = k$ and furthermore $k+1, k+2, \ldots, 2k$ are in distinct blocks of π. Since π is noncrossing and has k blocks, this forces $\pi = \pi_0$. We have $\mu = (k, k, \mu_3, \ldots)$, and if $\mu_3 > 1$ then $s_{2k} \in \mathfrak{S}_\mu$ and so that $[\mathfrak{S}_\mu]^+ \cdot F_\pi = 0$, a contradiction. We conclude that $\lambda = \mu$. Lemma 2.4.2 applies to prove the $\mathfrak{S}_n$-isomorphism $W(n,k,0) \cong S^\lambda$. To obtain the corresponding statement about the $V(n,k,0)$, observe that the map $F \mapsto (\xi_1 + \cdots + \xi_n) \odot F$ is an $\mathfrak{S}_n$-equivariant surjection $W(n,k,0) \twoheadrightarrow V(n,k,0)$. By irreducibility, we have $V(n,k,0) \cong S^\lambda$, as well. □

Remark 2.4.4. *There is a faster proof of Proposition 2.4.3 relying on results in [39, Sec. 5]. Theorem 2.3.4 shows that $W(n,k,0)$ and $V(n,k,0)$ are nonzero quotients of $\mathbb{C}[\mathrm{NC}(n,k,0)]$. By [39, Prop. 5.2], we have $\mathbb{C}[\mathrm{NC}(n,k,0)] \cong S^\lambda$ where $\lambda = (k,k,1^{n-2k})$, and irreducibility forces*

$W(n,k,0) \cong V(n,k,0) \cong S^{\lambda}$. *We presented the argument above to illustrate how fermions give an easier proof than that of [39, Prop. 5.2].*

2.4.3 Linear independence

By Proposition 2.4.1, noncrossing fermions form a spanning set for the W- and V-modules. The next result states that they form a basis.

Theorem 2.4.5. *Given $n,k,m \geq 0$, the sets*

$$\{F_\pi : \pi \in \mathrm{NC}(n)\}, \quad \{F_\pi : \pi \in \mathrm{NC}(n,k)\}, \quad \textit{and} \quad \{F_\pi : \pi \in \mathrm{NC}(n,k,m)\} \tag{2.4.9}$$

are bases of the $\mathfrak{S}_n$-modules $W(n), W(n,k)$, and $W(n,k,m)$, respectively. Similarly, the sets

$$\{f_\pi : \pi \in \mathrm{NC}(n)\}, \quad \{f_\pi : \pi \in \mathrm{NC}(n,k)\}, \quad \textit{and} \quad \{f_\pi : \pi \in \mathrm{NC}(n,k,m)\} \tag{2.4.10}$$

are bases of $V(n), V(n,k)$, and $V(n,k,m)$, respectively.

Proof. By Proposition 2.4.1 it suffices to verify the linear independence of these six sets. We start with the case of $W(n,k)$.

Suppose that we have $c_\pi \in \mathbb{C}$ such that

$$\sum_{\pi \in \mathrm{NC}(n,k)} c_\pi \cdot F_\pi = 0. \tag{2.4.11}$$

For $1 \leq i \leq n$, let U_i be the subspace of $\wedge\{\Theta_n, \Xi_n\}$ spanned by monomials $\theta_S \cdot \xi_T$ for which $i \notin S$ but $i \in T$. There is a linear projection

$$\tau_i : \wedge\{\Theta_n, \Xi_n\} \twoheadrightarrow U_i \tag{2.4.12}$$

which fixes any monomial $\theta_S \cdot \xi_T \in U_i$ and sends any monomial $\theta_S \cdot \xi_T \notin U_i$ to zero.

For $\pi \in \mathrm{NC}(n,k)$, what does $\tau_i(F_\pi)$ look like? If $\{i\}$ is not a singleton in π, it follows from the definition of F_π that θ_i appears in any monomial of F_π whenever ξ_i does, so that $\tau_i(F_\pi) = 0$ in this case. If $\{i\}$ is a singleton in π, it is not hard to check that

$$\tau_i(F_\pi) = \pm \xi_i \cdot \overline{F}_{\pi^{(i)}} \tag{2.4.13}$$

where $\pi^{(i)}$ is the set partition of $\{1, \dots, \hat{i}, \dots, n\}$ obtained by removing the singleton $\{i\}$ from π and $\overline{F}$ is defined in the same way as F but over the variable set obtained by removing θ_i and ξ_i, $(\theta_1, \dots, \widehat{\theta}_i, \dots, \theta_n, \xi_1, \dots, \widehat{\xi}_i, \dots, \xi_n)$. Applying τ_i to both sides of Equation (2.4.11) gives

$$\sum_{\substack{\pi \in \mathrm{NC}(n,k) \\ \{i\} \text{ is a block of } \pi}} \pm c_\pi \cdot \xi_i \cdot \overline{F}_{\pi^{(i)}} = 0. \tag{2.4.14}$$

By induction on n and Equation (2.4.14), we have $c_\pi = 0$ whenever π has singleton blocks, so that Equation (2.4.11) reads

$$\sum_{\pi \in \mathrm{NC}(n,k,0)} c_\pi \cdot F_\pi = 0. \tag{2.4.15}$$

Proposition 2.4.3 implies $\dim W(n,k,0) = |\mathrm{NC}(n,k,0)|$ so that the spanning set $\{F_\pi : \pi \in \mathrm{NC}(n,k,0)\}$ of $W(n,k,0)$ must also be linearly independent. The coefficients c_π appearing in Equation (2.4.15) are therefore also zero and the set $\{F_\pi : \pi \in \mathrm{NC}(n,k)\}$ is linearly independent. Its subsets $\{F_\pi : \pi \in \mathrm{NC}(n,k,m)\}$ must also be linearly independent for any m, and the directness of the sums in (2.4.2) implies that $\{F_\pi : \pi \in \mathrm{NC}(n)\}$ is also linearly independent.

The proof for the f's and V's is almost identical to the proof for the F's and W's. One need only verify the identity

$$\tau_i(f_\pi) = \begin{cases} \pm \xi_i \cdot \overline{f}_{\pi^{(i)}} & \{i\} \text{ is a singleton block of } \pi, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.16}$$

where $\overline{f}_{\pi^{(i)}}$ is defined over the variable set $(\theta_1, \ldots, \widehat{\theta_i}, \ldots, \theta_n, \xi_1, \ldots, \widehat{\xi_i}, \ldots, \xi_n)$ and apply the same argument. □

2.4.4 Module structure

The fermion modules V and W coincide with the skein modules.

Theorem 2.4.6. *For any $n, k, m \geq 0$ we have isomorphisms of $\mathfrak{S}_n$ modules*

$$\begin{cases} \mathbb{C}[\mathrm{NC}(n)] \cong V(n) \cong W(n), \\ \mathbb{C}[\mathrm{NC}(n,k)] \cong V(n,k) \cong W(n,k), \\ \mathbb{C}[\mathrm{NC}(n,k,m)] \cong V(n,k,m) \cong W(n,k,m), \end{cases} \tag{2.4.17}$$

where $\mathbb{C}[\mathrm{NC}(n)], \mathbb{C}[\mathrm{NC}(n,k)]$, and $\mathbb{C}[\mathrm{NC}(n,k,m)]$ are endowed with the skein action. These isomorphisms are given by $\pi \leftrightarrow f_\pi \leftrightarrow F_\pi$ for π a noncrossing partition in each case.

Proof. Apply Theorem 2.3.4 and Theorem 2.4.5. □

We record the Frobenius images of the modules involved in Theorem 2.4.6.

Corollary 2.4.7. *For any $n, k, m \geq 0$ the Frobenius images of $\mathbb{C}[\mathrm{NC}(n,k,m)]$, $V(n,k,m)$, and $W(n,k,m)$ are given by the common symmetric function*

$$s_{(k-m,k-m,1^{n-2k+m})} \cdot s_{(1^m)} \tag{2.4.18}$$

These modules admit $\mathfrak{S}_n$-decompositions

$$\mathbb{C}[\mathrm{NC}(n)] = \bigoplus_{k=0}^{n} \mathbb{C}[\mathrm{NC}(n,k)] \qquad \text{and} \qquad \mathbb{C}[\mathrm{NC}(n,k)] = \bigoplus_{m=0}^{k} \mathbb{C}[\mathrm{NC}(n,k,m)], \tag{2.4.19}$$

$$V(n) = \bigoplus_{k=0}^{n} V(n,k) \qquad \text{and} \qquad V(n,k) = \bigoplus_{m=0}^{k} V(n,k,m), \tag{2.4.20}$$

$$W(n) = \bigoplus_{k=0}^{n} W(n,k) \qquad \textit{and} \qquad W(n,k) = \bigoplus_{m=0}^{k} W(n,k,m). \tag{2.4.21}$$

Proof. Proposition 2.4.3 shows $\mathbb{C}[\mathrm{NC}(n,k,0)] \cong S^{(k,k,1^{n-2k})}$. The definition of the skein action makes it clear that we have an induction product

$$\mathbb{C}[\mathrm{NC}(n,k,m)] \cong \mathbb{C}[\mathrm{NC}(n-m,k-m,0)] \circ \mathrm{sign}_{\mathfrak{S}_m} \tag{2.4.22}$$

so that

$$\mathrm{Frob}\,\mathbb{C}[\mathrm{NC}(n,k,m)] = s_{(k-m,k-m,1^{n-2k+m})} \cdot s_{(1^m)}. \tag{2.4.23}$$

Theorem 2.4.6 proves the first statement. The direct sum decompositions are clear for the skein modules; Theorem 2.4.6 implies their truth for the V-modules and W-modules as well. □

The Frobenius images in Corollary 2.4.7 may be easily calculated using the (dual) Pieri rule. As an example, we have

$$\mathrm{Frob}\,V(9,5,1) = s_{44} \cdot s_1 = s_{54} + s_{441},$$

$$\mathrm{Frob}\,V(9,5,2) = s_{331} \cdot s_{11} = s_{441} + s_{432} + s_{4311} + s_{3321} + s_{3313},$$

$$\mathrm{Frob}\,V(9,5,3) = s_{2211} \cdot s_{1^3} = s_{3321} + s_{331^3} + s_{32^3} + s_{32211} + s_{321^4} + s_{2^41} + s_{2^31^3} + s_{221^5},$$

$$\mathrm{Frob}\,V(9,5,4) = s_{1^5} \cdot s_{1^4} = s_{2^41} + s_{2^31^3} + s_{221^5} + s_{21^7} + s_{1^9}.$$

We have $\mathrm{Frob}\,V(9,5,0) = \mathrm{Frob}\,V(9,5,5) = 0$ since any 5-block set partition of $[9]$ has at least one and at most four singleton blocks. Summing these expressions gives the decomposition

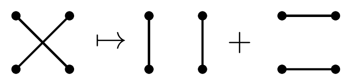
$$\mathrm{Frob}\,V(9,5) = s_{54} + 2s_{441} + s_{432} + s_{4311} + 2s_{3321} + 2s_{331^3} + s_{32^3} \\ + s_{32211} + s_{321^4} + 2s_{2^41} + 2s_{2^31^3} + 2s_{221^5} + s_{21^7} + s_{1^9}$$

of the Narayana-dimensional module $V(9,5)$. The reader may notice that the coefficients in

Frob $V(9,5)$ are all in the set $\{0,1,2\}$. This holds for any Frob $V(n,k)$, as may be verified with the dual Pieri rule. Similarly, if s_λ appears in Frob $V(n,k)$ we must have $\lambda_2 \geq \lambda_1 - 1$ and $\lambda_3 < 3$.

2.5 Resolution of crossings in set partitions

A *chord diagram* of size n (or a *perfect matching*) is a set partition of $[2n]$ consisting of n blocks, each of size two. In many mathematical contexts, one resolves crossings in chord diagrams by repeated applications of



Our skein action extends this technique to resolve crossings in arbitrary set partitions. Some of the definitions and results in this subsection are more precise and cleaner versions of the results in [39, Sec. 6, Sec. 7][4].

2.5.1 The crossing resolution p

Our resolution of a set partition as a linear combination of noncrossing set partitions is packaged as a projection map.

Definition 2.5.1. *We define a $\mathbb{C}$-linear map $p : \mathbb{C}[\Pi(n)] \to \mathbb{C}[\mathrm{NC}(n)]$ by setting*

$$p(\pi) := \sum_{\mu \in \mathrm{NC}(n)} c_{\pi,\mu} \cdot \mu$$

where $\pi \in \Pi(n)$ is a set partition and

$$F_\pi = \sum_{\pi \in \mathrm{NC}(n)} c_{\pi,\mu} \cdot F_\mu.$$

[4]The reader following along in [39] should note that the action of $\mathfrak{S}_n$ on $\mathbb{C}[\Pi(n)]$ denoted $\star$ there is different from – and more complicated than – our $\star$-action (which is simply the sign-twisted permutation action).

Given $\pi \in \Pi(n)$, we view $p(\pi) = \sum_{\mu \in \mathrm{NC}(n)} c_{\pi,\mu} \cdot \mu$, or its notationally abusive alias

$$\text{“}\, \pi = \sum_{\mu \in \mathrm{NC}(n)} c_{\pi,\mu} \cdot \mu \,\text{”} \tag{2.5.1}$$

as resolving the crossings in the set partition π. Theorem 2.4.5 guarantees that the linear map p is well-defined and that $p(\pi) = \pi$ whenever π is noncrossing.

Definition 2.5.1 can be cumbersome to apply since it involves basis expansions in $\wedge\{\Theta_n, \Xi_n\}$. However, there is a purely combinatorial algorithm for computing $p(\pi)$; this is presented after Corollary 2.5.10 and Observation 2.5.11 below. There is also an algebraic characterization of the linear map $p : \mathbb{C}[\Pi(n)] \to \mathbb{C}[\mathrm{NC}(n)]$. This is given after Theorem 2.5.5.

For arbitrary set partitions π, the coefficients $c_{\pi,\mu}$ appearing in Definition 2.5.1 are all integers. Indeed, if $w \in \mathfrak{S}_n$ is chosen such that $w(\pi) \in \Pi(n)$ is noncrossing, Proposition 2.2.3 yields

$$F_\pi = F_{w^{-1}(w(\pi))} = \mathrm{sign}(w) \cdot w^{-1} \cdot F_{w(\pi)}. \tag{2.5.2}$$

Since $w^{-1} \cdot w(\pi)$ is a $\mathbb{Z}$-linear combination of noncrossing partitions under the skein action and $\pi \mapsto F_\pi$ affords an isomorphism $\mathbb{C}[\mathrm{NC}(n)] \xrightarrow{\sim} W(n)$, we have $c_{\pi,\mu} \in \mathbb{Z}$ always.

By Theorem 2.4.6, for any $\pi \in \Pi(n)$ we have

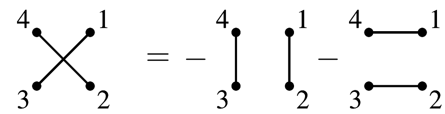
$$F_\pi = \sum_{\pi \in \mathrm{NC}(n)} c_{\pi,\mu} \cdot F_\mu \qquad \Leftrightarrow \qquad f_\pi = \sum_{\pi \in \mathrm{NC}(n)} c_{\pi,\mu} \cdot f_\mu \tag{2.5.3}$$

so that Definition 2.5.1 could have been stated using the f's rather than the F's. The F's are more convenient to use when computing $p(\pi)$. For our first example of crossing resolution, we revisit the classical setting of chord diagrams.

Example 2.5.2. (Chord diagrams) *Consider the crossing chord diagram $\pi = \{1,3/2,4\}$. The equation*

$$F_{\{1,3/2,4\}} = -F_{\{1,2/3,4\}} - F_{\{1,4/2,3\}}$$

gives rise to the crossing resolution

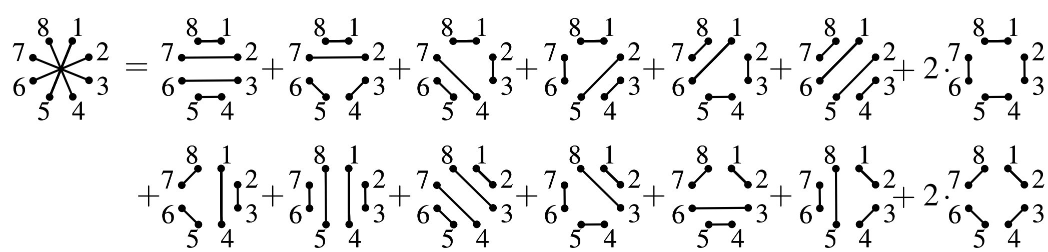


where the RHS is the image $p(\pi)$. The reader may be disturbed by the global minus sign appearing in $p(\pi)$. This is accounted for by Proposition 2.2.3 which yields

$$s_2 \cdot F_{\{1,2/3,4\}} = \mathrm{sign}(s_2) \cdot F_\pi = -F_\pi.$$

Since the expression $s_2 \cdot F_{\{1,2/3,4\}}$ is positive in the F-basis, the resolution $p(\pi)$ must be negative in the skein basis.

For a general chord diagram $\pi \in \Pi(2n)$, the resolution $p(\pi)$ agrees with the 'classical' chord diagram crossing resolution up to a global sign. To show that coefficients other than ± 1 can occur in this expansion, let $\pi = \{1,5/2,6/3,7/4,8\}$ be the 'asterisk of order 4'. Resolving the crossings in π results in



Our next example resolves crossings in set partitions π which are not chord diagrams. The relevant F-polynomials and basis expansion were calculated by computer.

Example 2.5.3. (Beyond Chord Diagrams) *If $\pi \in \Pi(n)$ is a set partition which is not a chord diagram, the resolution $p(\pi)$ can involve both positive and negative signs. For example, consider $\pi = \{1,2,6/3,4,8/5,7\} \in \Pi(8)$.*

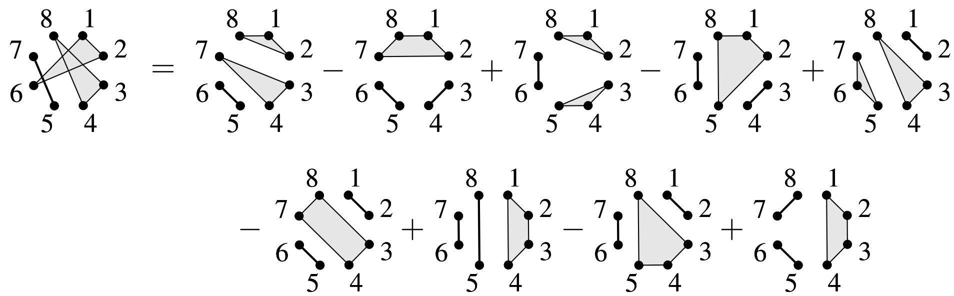


Figure 2.2. The crossing resolution for example 2.5.3

2.5.2 Two-block crossing resolution

Finding a combinatorial or algebraic interpretation of the coefficients c_μ in $p(\pi) = \sum_{\mu \in \mathrm{NC}(n)} c_\mu \cdot \mu$ for general set partitions $\pi \in \Pi(n)$ is an open problem. However, we can give such an interpretation when the set partition $\pi = \{A\,/\,B\}$ consists of just two blocks. This 'quadratic relation' may be useful in finding algebraic interpretations of the skein modules; see Problem 2.7.1 and the discussion thereafter.

To state our resolution for two-block set partitions, we need some notation. If $1 \leq i, j \leq r$, we let $[i, j]_n$ denote the (closed) cyclic interval from i to j in the cycle $(1, 2, \ldots, n)$. Explicitly, we have

$$[i, j]_n = \begin{cases} \{i, i+1, \ldots, j-1, j\} & i \leq j, \\ \{i, i+1, \ldots, n, 1, 2, \ldots, j-1, j\} & i > j. \end{cases} \tag{2.5.4}$$

If $\pi = \{A\,/\,B\}$ is a two-block set partition of $[n]$, there exist unique maximal nonempty cyclic intervals $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ in $(1, 2, \ldots, n)$ such that

$$A = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_m \qquad \text{and} \qquad B = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_m \tag{2.5.5}$$

and $(A_1, B_1, A_2, B_2, \ldots, A_m, B_m)$ is cyclically sequential. The set partition $\pi = \{A\,/\,B\}$ is noncrossing if and only if $m < 2$.

As an example of these concepts, let $\pi = \{A\,/\,B\} \in \Pi(16)$ where

$$A = \{1, 2, 4, 8, 9, 10, 12, 13, 14, 15, 16\} \quad \text{and} \quad B = \{3, 5, 6, 7, 11\}.$$

We have the disjoint union decompositions

$$A = A_1 \sqcup A_2 \sqcup A_3 \quad \text{and} \quad B = B_1 \sqcup B_2 \sqcup B_3$$

where the sets

$$A_1 = \{1, 2, 12, 13, 14, 15, 16\}, \quad A_2 = \{4\}, \quad A_3 = \{8, 9, 10\}$$

and

$$B_1 = \{3\}, \quad B_2 = \{5, 6, 7\}, \quad B_3 = \{11\}$$

are all cyclic intervals and the sets $(A_1, B_1, A_2, B_2, A_3, B_3)$ are cyclically sequential.

Proposition 2.5.4. *Let $\pi = \{A\,/\,B\} \in \Pi(n)$ and the decompositions $A = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_m$ and $B = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_m$ be as above. We have*

$$p(\pi) = \sum_{(S,T)} \varepsilon(S,T) \cdot \{S\,/\,T\} \tag{2.5.6}$$

where the sum is over all two-block noncrossing set partitions $\{S\,/\,T\} \in \mathrm{NC}(n)$ and the coefficient

$\varepsilon(S,T) \in \{+1,0,-1\}$ *is*

$$\varepsilon(S,T) = \begin{cases} 0 & \text{if } |S| < 2 \text{ or } |T| < 2, \\ +1 & \begin{array}{c}\text{if } |S|,|T| \geq 2 \text{ and } S,T \text{ are both unions of an } \textbf{odd} \text{ number} \\ \text{of sets in } (A_1,B_1,A_2,B_2,\ldots,A_m,B_m),\end{array} \\ -1 & \begin{array}{c}\text{if } |S|,|T| \geq 2 \text{ and } S,T \text{ are both unions of an } \textbf{even} \text{ number} \\ \text{of sets in } (A_1,B_1,A_2,B_2,\ldots,A_m,B_m).\end{array} \end{cases} \tag{2.5.7}$$

We give an example of Proposition 2.5.4 before proving it. Suppose $\pi = \{A\,/\,B\}$ is as before the statement of the proposition so that $m = 3$. The sets S,T involved in the expansion of $p(\pi)$ are complementary cyclic intervals in $(A_1,B_1,A_2,B_2,A_3,B_3)$. Collapsing the cyclic intervals A_i,B_i to points, this resolution $p(\pi)$ is shown in Figure 2.3. The partitions $\{B_1\,/\,[n]-B_1\},\{A_2\,/\,[n]-A_2\}$, and $\{B_3\,/\,[n]-B_3\}$ do not appear because the sets B_1,A_2, and B_3 are singletons so that $\varepsilon = 0$ for these terms. We now prove Proposition 2.5.4.

Proof. This is a more complicated version of the proof of Theorem 2.3.3. When $m < 2$, the set partition π is noncrossing, we have $p(\pi) = \pi$ and the proposition follows, so we assume $m \geq 2$. In particular, neither of the blocks of π are singletons.

Define four disjoint nonempty subsets $I,J,S,T \subseteq [n]$ by

$$I := A_1, \quad J := B_1, \quad S := A_2 \sqcup A_3 \sqcup \cdots \sqcup A_m, \quad T := B_2 \sqcup B_3 \sqcup \cdots \sqcup B_m. \tag{2.5.8}$$

We claim the following identity of linear endomorphisms of $\wedge\{\Theta_n,\Xi_n\}$:

$$\psi_{I\sqcup S} \circ \psi_{J\sqcup T} + \psi_{I\sqcup J} \circ \psi_{S\sqcup T} + \psi_{I\sqcup T} \circ \psi_{J\sqcup S} \\ - \psi_I \circ \psi_{J\sqcup S\sqcup T} - \psi_J \circ \psi_{I\sqcup S\sqcup T} - \psi_S \circ \psi_{I\sqcup J\sqcup T} - \psi_T \circ \psi_{I\sqcup J\sqcup S} = 0. \tag{2.5.9}$$

Expressed in terms of the $\psi_{S,T}$ operators, the LHS of Equation (2.5.9) is

$$\begin{aligned}(\psi_I+\psi_{I,S}+\psi_S)\circ(\psi_J+\psi_{J,T}+\psi_T)+(\psi_I+\psi_{I,J}+\psi_J)\circ(\psi_S+\psi_{S,T}+\psi_T)\\ +(\psi_I+\psi_{I,T}+\psi_T)\circ(\psi_J+\psi_{J,S}+\psi_S)\\ -\psi_I\circ(\psi_J+\psi_S+\psi_T+\psi_{J,S}+\psi_{J,T}+\psi_{S,T})-\psi_J\circ(\psi_I+\psi_S+\psi_T+\psi_{I,S}+\psi_{I,T}+\psi_{S,T})\\ -\psi_S\circ(\psi_I+\psi_J+\psi_T+\psi_{I,J}+\psi_{I,T}+\psi_{J,T})-\psi_T\circ(\psi_I+\psi_J+\psi_S+\psi_{I,J}+\psi_{I,S}+\psi_{J,S})\end{aligned} \tag{2.5.10}$$

which simplifies (by Lemma 2.2.4) to

$$\psi_{I,S}\circ\psi_{J,T}+\psi_{I,J}\circ\psi_{S,T}+\psi_{I,T}\circ\psi_{J,S} \tag{2.5.11}$$

and the claimed Equation (2.5.9) will be proved if we can show that (2.5.11) vanishes as an operator on $\wedge\{\Theta_n,\Xi_n\}$. To show this, let $f\in\wedge\{\Theta_n,\Xi_n\}$. The image of f under the operator (2.5.11) is

$$\frac{1}{2}\sum_{(a,b,c,d)}\xi_a\cdot(\theta_b\odot(\xi_c\cdot(\theta_d\odot f))) \tag{2.5.12}$$

where (a,b,c,d) range over all quadruples which have precisely one element in each of I,J,S,T. As in the proof of Proposition 2.2.8, the terms in (2.5.12) corresponding to (a,b,c,d) and (c,b,a,d) cancel so that (2.5.12) vanishes and Equation (2.5.9) is proven.

Equation (2.5.9) and Theorem 2.3.3 prove the proposition immediately when $m=2$; we may rearrange its terms as

$$\begin{aligned}\psi_{I\sqcup S}\circ\psi_{J\sqcup T}=-\psi_{I\sqcup J}\circ\psi_{S\sqcup T}-\psi_{I\sqcup T}\circ\psi_{J\sqcup S}\\ +\psi_I\circ\psi_{J\sqcup S\sqcup T}+\psi_J\circ\psi_{I\sqcup S\sqcup T}+\psi_S\circ\psi_{I\sqcup J\sqcup T}+\psi_T\circ\psi_{I\sqcup J\sqcup S}\end{aligned} \tag{2.5.13}$$

and apply both sides of Equation (2.5.13) to $\theta_1\theta_2\cdots\theta_n$. When $m>2$, we may still apply both sides of Equation (2.5.13) to $\theta_1\theta_2\cdots\theta_n$. The LHS evaluation is F_π while the RHS evaluation is a

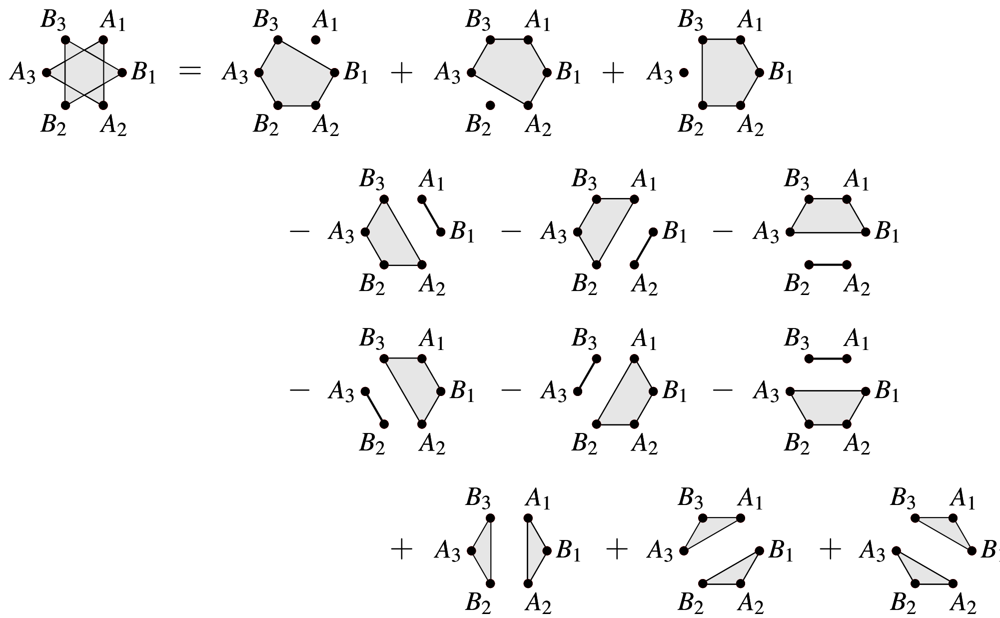


Figure 2.3. The noncrossing resolution of a two-block set partition $\pi = \{A\,/\,B\}$ where $A = A_1 \sqcup A_2 \sqcup A_3$, $B = B_1 \sqcup B_2 \sqcup B_3$, the sets B_1, A_2, B_3 are singletons, and the sets A_1, B_2, A_3 have more than one element.

linear combination of F_μ's for singleton-free (recall that $\psi_S = 0$ when S is a singleton) two-block set partitions $\mu \in \Pi(n)$ with strictly shorter sizes m of their cyclic interval decompositions. The proposition follows from induction on m. □

2.5.3 Equivariance and symmetries

The resolution projection p intertwines the skein action on $\mathbb{C}[\mathrm{NC}(n)]$ with the sign-twisted permutation action of $\mathfrak{S}_n$ on $\mathbb{C}[\Pi(n)]$.

Theorem 2.5.5. *Endow $\mathbb{C}[\Pi(n)]$ with the $\mathfrak{S}_n$-action*

$$w \star \pi := \mathrm{sign}(w) \cdot w(\pi) \tag{2.5.14}$$

and $\mathbb{C}[\mathrm{NC}(n)]$ with the skein action. The projection $p : \mathbb{C}[\Pi(n)] \twoheadrightarrow \mathbb{C}[\mathrm{NC}(n)]$ is $\mathfrak{S}_n$-equivariant.

Proof. By Proposition 2.2.3, the surjection $\mathbb{C}[\Pi(n)] \twoheadrightarrow W(n)$ given by $\pi \mapsto F_\pi$ is $\mathfrak{S}_n$-equivariant, so the theorem is a direct consequence of Definition 2.5.1. □

Theorem 2.5.5 characterizes p as follows. Suppose $f : \mathbb{C}[\Pi(n)] \to \mathbb{C}[\mathrm{NC}(n)]$ is a linear map such that

- we have $f(s_i \star \pi) = s_i \cdot f(\pi)$ for all $1 \leq i \leq n-1$ and $\pi \in \Pi(n)$ and
- for any partition $\lambda \vdash n$ we have $f(\pi_\lambda) = \pi_\lambda$ where

$$\pi_\lambda := \{1, 2, \cdots, \lambda_1 \,/\, \lambda_1 + 1, \lambda_1 + 2, \cdots, \lambda_1 + \lambda_2 \,/\, \cdots\}.$$

Since any $\pi \in \Pi(n)$ is $\mathfrak{S}_n$-conjugate to a set partition of the form π_λ, we have $f = p$. This characterization may be helpful in finding other applications of the map p; see Problem 2.7.1.

The set $\mathrm{NC}(n)$ of noncrossing partitions has a dihedral group of 'global' symmetries. That is, if $w_0 \in \mathfrak{S}_n$ is the long element satisfying $w_0(i) = n - i + 1$ and $c = (1, 2, \ldots, n) \in \mathfrak{S}_n$ is the long cycle, then $\mathrm{NC}(n)$ is closed under the action of w_0 and c. The following result is a generalization of [39, Prop. 6.1, Thm. 6.4]; thanks to fermions its proof is much shorter.

Corollary 2.5.6. *For any $\pi \in \mathrm{NC}(n)$, with respect to the skein action $w_0, c \in \mathfrak{S}_n$ act on π by*

$$w_0 \cdot \pi = (-1)^{\binom{n}{2}} \cdot w_0(\pi) \qquad \text{and} \qquad c \cdot \pi = (-1)^{n-1} \cdot c(\pi). \tag{2.5.15}$$

Proof. Apply Theorem 2.5.5 and the identities $\mathrm{sign}(w_0) = (-1)^{\binom{n}{2}}$ and $\mathrm{sign}(c) = (-1)^{n-1}$. □

We give an example to illustrate Corollary 2.5.6 and the power of the equivariance Theorem 2.5.5.

Example 2.5.7. (Global symmetries) *Let $n = 6$ and $\pi = \{1, 5, 6 \,/\, 2, 4 \,/\, 3\}$. In order to compute $c \cdot \pi$ under the skein action, we write $c = s_1 s_2 s_3 s_4 s_5$ and apply each adjacent transposition successively using the skein relations.*

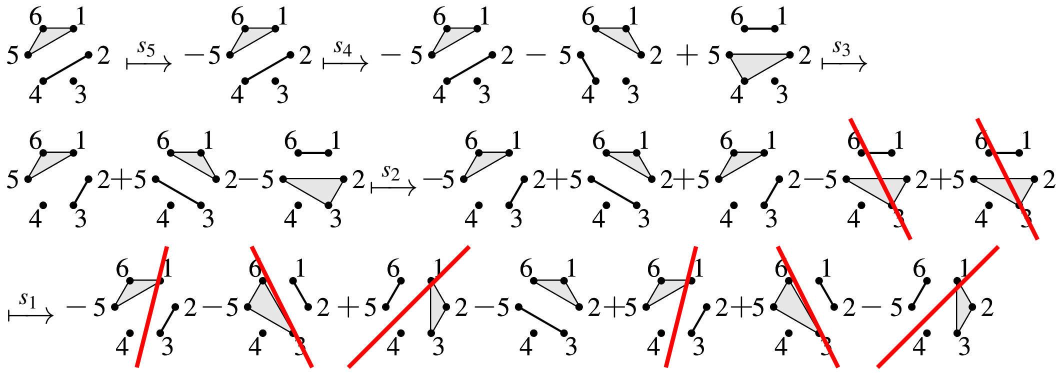


Figure 2.4. The computation of $c \cdot \pi$ in example 2.5.7

As predicted by Corollary 2.5.6, the set partition $c(\pi) = \{1, 2, 6 / 3, 5 / 4\} \in \mathrm{NC}(6)$ *is the only surviving term with sign* $(-1)^{6-1} = -1$. *We leave it for the reader to verify*

$$w_0 \cdot \pi = (-1)^{\binom{6}{2}} w_0(\pi) = -w_0(\pi) = -\{1, 2, 6 / 3, 5 / 4\}$$

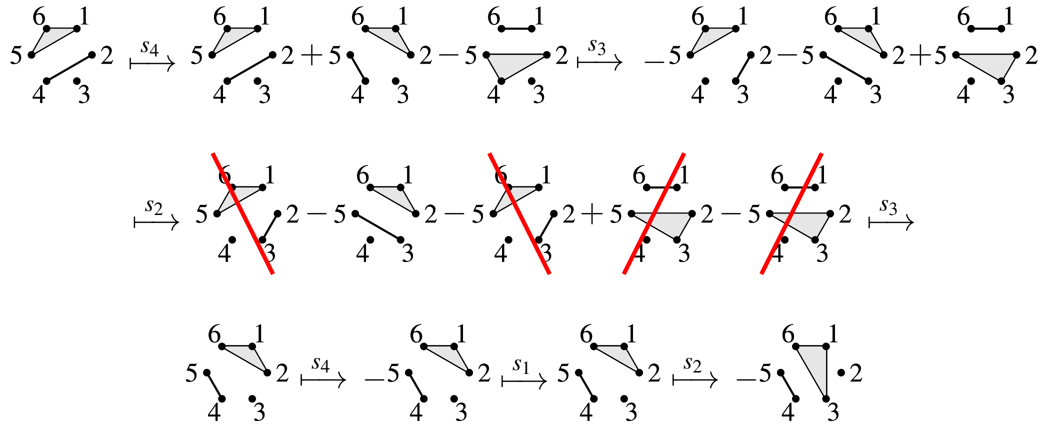
directly from the skein action.

Corollary 2.5.6 may be used in conjunction with Corollary 2.4.7 to obtain fixed-point counts for the action of the dihedral group $\langle w_0, c \rangle$ on the set $\mathrm{NC}(n)$. This gives rise to cyclic sieving phenomena for the rotational action of $\langle c \rangle$, as explained in [39]. Another immediate corollary of Theorem 2.5.5 is that the skein action respects 'local symmetries' of $\mathfrak{S}_n$ on $\mathrm{NC}(n)$. The following result is a sharpening of [39, Cor. 7.3].

Corollary 2.5.8. *Let* $w \in \mathfrak{S}_n$ *and* $\pi \in \mathrm{NC}(n)$ *be such that* $w(\pi) \in \mathrm{NC}(n)$. *We have the skein action*

$$w \cdot \pi = \mathrm{sign}(w) \cdot w(\pi). \tag{2.5.16}$$

Example 2.5.9. (Local symmetries) *Suppose* $\pi = \{1, 5, 6 / 2, 4 / 3\} \in \mathrm{NC}(6)$ *as above. Letting* $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 4 & 1 & 6 \end{pmatrix} \in \mathfrak{S}_6$, *the set partition* $w(\pi) = \{3, 1, 6 / 5, 4 / 2\} \in \Pi(6)$ *is noncrossing. Also we have* $\mathrm{sign}(w) = -1$. *Using the decomposition* $w = s_2 s_1 s_4 s_3 s_2 s_3 s_4$ *we calculate*



as predicted by Corollary 2.5.8.

2.5.4 Combinatorial crossing resolution

Proposition 2.5.4 gives a combinatorial way to calculate the resolution $p(\pi)$ for $\pi \in \Pi(n)$ which does not use F-fermions. The idea is to resolve a fixed pair of crossing blocks $A, B \in \pi$, resulting in a linear combination of partitions which are 'less crossing' than π. To describe this procedure, we need notation.

Let S be a finite set with a disjoint union decomposition $S = I \sqcup J$. If π_I is a set partition of I and π_J is a set partition of J, the (disjoint) union $\pi_I \sqcup \pi_J$ is a set partition of S. Conversely, if π is a set partition of S and I, J are unions of blocks of π we have the restrictions $\pi\mid_I$ and $\pi\mid_J$ of π to I and J.

If a finite set S has a total order *noncrossing partitions* and *cyclic intervals* of S are defined in the natural way. If π is a set partition of S, two blocks $A, B \in \pi$ are said to *cross* if $\pi\mid_{A \sqcup B}$ is not noncrossing. Roughly speaking, the next result states that we can locally resolve the crossing of A, B as in Proposition 2.5.4 while leaving the other blocks of π unchanged.

Corollary 2.5.10. *Let $\pi \in \Pi(n)$ and let A, B be blocks of π which cross. Write $A = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_m$ and $B = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_m$ where the A_i and B_i are maximal nonempty cyclic intervals in the ordered set $A \sqcup B$ such that the sequence $(A_1, B_1, A_2, B_2, \ldots, A_m, B_m)$ is cyclically sequential.*

Write $C := [n] - (A \sqcup B)$ for the union of the other blocks of π. We have

$$p(\pi) = \sum_{(S,T)} \varepsilon(S,T) \cdot p(\{S\,/\,T\} \sqcup \pi\,|_C) \tag{2.5.17}$$

where the sum is over all two-block noncrossing partitions $\{S\,/\,T\}$ of $A \sqcup B$ and the coefficient $\varepsilon(S,T)$ is defined as in Proposition 2.5.4.

Proof. When $[n] = A \sqcup B$ this is precisely Proposition 2.5.4. For $C = \{C_1\,/\,C_2\,/\,\cdots\,/\,C_r\}$, the desired equation will follow from the operator identity

$$\rho_A \circ \rho_B \circ \rho_{C_1} \circ \rho_{C_2} \circ \cdots \circ \rho_{C_r} = \sum_{(S,T)} \varepsilon(S,T) \cdot \rho_S \circ \rho_T \circ \rho_{C_1} \circ \rho_{C_2} \circ \cdots \circ \rho_{C_r} \tag{2.5.18}$$

where the conditions on (S,T) are the same as in the statement. As all sets A,B,S,T appearing in Equation (2.5.18) have size > 1, we have $\rho_A = \psi_A, \rho_B = \psi_B, \rho_S = \psi_S$, and $\rho_T = \psi_T$ so that Equation (2.5.18) is implied by

$$\psi_A \circ \psi_B = \sum_{(S,T)} \varepsilon(S,T) \cdot \psi_S \circ \psi_T. \tag{2.5.19}$$

Equation (2.5.19) is proven in the same way as Proposition 2.5.4. □

By Corollary 2.5.10, if $\pi \in \Pi(n)$ and A,B are blocks of π which cross, we may write $p(\pi)$ as a sum of elements of the form $\pm p(\pi')$ where π is obtained from π by replacing A,B with a new pair S,T of noncrossing, nonsingleton blocks. If all of the resulting partitions $\pi' \in \Pi(n)$ were noncrossing, this would give the resolution $p(\pi) \in \mathbb{C}[\mathrm{NC}(n)]$, but this need not be the case in general. However, the π' involved are 'less crossing' than π. We define the *tangle* $\tan(\pi)$ of a set partition $\pi = \{B_1\,/\,B_2\,/\,\cdots\,/\,B_k\}$ by

$$\tan(\pi) := |\{1 \leq i < j \leq k : \text{the blocks } B_i \text{ and } B_j \text{ cross}\}| \tag{2.5.20}$$

so that π is noncrossing if and only if $\tan(\pi) = 0$. Certainly any index pair of blocks S, T in the RHS of Equation (2.5.17) do not cross, whereas the blocks A, B on the LHS do. For the remaining blocks, we have the following

Observation 2.5.11. *Let $\pi \in \Pi(n)$ be a set partition, let $A, B \in \pi$ be blocks which cross, and let S, T index a term on the RHS of Equation* (2.5.17). *If D is a block of π other than A, B, then*

- *if D crosses just one of A, B, then D crosses at most one of S, T, and*
- *if D crosses neither A nor B, then D crosses neither S nor T.*

Given $\pi \in \Pi(n)$, Corollary 2.5.10 yields a 'greedy algorithm' to calculate $p(\pi) \in \mathbb{C}[\mathrm{NC}(n)]$.

1. If π is noncrossing, then $p(\pi) = \pi$. Otherwise, arbitrarily select two blocks A, B of π which cross.
2. Write $p(\pi) = \sum_{(S,T)} \varepsilon(S,T) \cdot p(\{S/T\} \sqcup \pi|_C)$ as in Equation (2.5.17). Go back to Step 1 for each $\{S/T\} \sqcup \pi|_C$ appearing on the RHS.

By Observation 2.5.11, any set partition $\pi' = \{S/T\} \sqcup \pi|_C$ involved in the RHS of Equation (2.5.17) satisfies $\tan(\pi') < \tan(\pi)$ so this algorithm terminates.

2.5.5 Quadratic ideals I and J

In this subsection we recast our work in the setting of *commutative* rings, ideals, and quotients. To any nonempty subset $B \subseteq [n]$ we associate a commuting variable y_B and let $R := \mathbb{C}[y_B : \varnothing \neq B \subseteq [n]]$ be the rank $2^n - 1$ polynomial ring in these variables.

We introduce two quadratic ideals $I, J \subset R$ as follows. For any pair (A, B) of nonempty disjoint subsets of $[n]$ which cross, the ideal I has a generator

$$y_A y_B - \sum_{(S,T)} \varepsilon(S,T) \cdot y_S y_T \tag{2.5.21}$$

where the pairs (S,T) and $\varepsilon(S,T)$ are as in Proposition 2.5.4.

The ideal I has formal similarities with the *Plücker ideals* of Schubert calculus. These are quadratic ideals in polynomial rings with variables Δ_B indexed by nonempty subsets $B \subseteq [n]$. The variable Δ_B corresponds to the top-justified minor in an $n \times n$ matrix $X = (x_{i,j})$ of variables with column set B. The analog of the generator (2.5.21) is another signed quadratic expression given by a determinantal identity due to Sylvester in 1851. See [12, Sec. 8.1] for a definition of the Plücker relations.

Although the relations (2.5.21) look somewhat like Plücker relations, there are two important differences. The sizes $\{|A|,|B|\}$ and $\{|S|,|T|\}$ of subsets involved in any Plücker relation are the same, but this homogeneity does not usually hold for the expressions (2.5.21). Furthermore, the sets pairs (A,B) of a given product $\Delta_A\Delta_B$ appearing in a Plücker relation can overlap, but all set pairs (A,B) and (S,T) appearing in (2.5.21) are disjoint.

The quotient R/I has infinite vector space dimension. To get Artinian quotients, we introduce the ideal $J \subset R$ given by

$$J = \langle y_A y_B \,:\, A \cap B \neq \varnothing \rangle. \tag{2.5.22}$$

Since the sets B indexing the variables y_B are nonempty, we have $y_B^2 \in J$ always. The quotient R/J has a basis consisting of monomials $y_{B_1} \cdots y_{B_k}$ for which the sets $B_1, \ldots, B_k$ are pairwise disjoint. Quotienting R by the larger ideal $I+J$ yields a more interesting basis of noncrossing disjoint subsets.

Proposition 2.5.12. *The quotient ring $R/(I+J)$ has a basis $\mathscr{B}$ consisting of monomials $y_{B_1} \cdots y_{B_k}$ for which the sets $B_1, \ldots, B_k \subseteq [n]$ are pairwise disjoint and noncrossing.*

Proof. Let $V \subset R$ be the vector subspace spanned by all monomials $y_{B_1} \cdots y_{B_k}$ for which $B_1, \ldots, B_k$ are pairwise disjoint. The monomials in $\mathscr{B}$ and the generators of I lie in V. Since $R = V \oplus J$ as vector spaces it suffices to show that $\mathscr{B}$ descends to a basis of $V/(I \cap V)$. To do this, we introduce direct sum decompositions as follows.

For any subset $U \subseteq [n]$, let $V_U \subseteq V$ be the subspace with basis given by monomials $y_{B_1} \cdots y_{B_k}$ with $B_1 \sqcup \cdots \sqcup B_k = U$. We have vector space direct sums

$$V = \bigoplus_{U \subseteq [n]} V_U \qquad \text{and} \qquad I \cap V = \bigoplus_{U \subseteq [n]} I \cap V_U \tag{2.5.23}$$

where the second direct sum is justified as I is spanned by elements of the form

$$\left(y_A y_B - \sum_{(S,T)} \varepsilon(S,T) \cdot y_S y_T \right) \cdot y_{C_1} \cdots y_{C_r} \tag{2.5.24}$$

where the union $A \cup B \cup C_1 \cup \cdots \cup C_r$ and the unions $S \cup T \cup C_1 \cup \cdots \cup C_r$ all equal the same set $U \subseteq [n]$. This gives rise to an identification

$$V/(I \cap V) = \bigoplus_{U \subseteq [n]} V_U/(I \cap V_U). \tag{2.5.25}$$

If we set $\mathscr{B}_U := \mathscr{B} \cap V_U$ we have the disjoint union

$$\mathscr{B} = \bigsqcup_{U \subseteq [n]} \mathscr{B}_U. \tag{2.5.26}$$

It suffices to prove the following

Claim: *$\mathscr{B}_U$ descends to a basis of $V_U/(I \cap V_U)$ for any $U \subseteq [n]$.*

Fix a subset $U \subseteq [n]$ and consider the exterior algebra $\wedge\{\Theta_U, \Xi_U\}$ over the set $\{\theta_u, \xi_u : u \in U\}$ of variables indexed by U. Lemma 2.2.1 states that the block operators satisfy $\rho_A \circ \rho_B = \rho_B \circ \rho_A$ for all subsets $A, B \subseteq [n]$. This endows $\wedge\{\Theta_U, \Xi_U\}$ with an R-module structure via

$$y_B \cdot f := \begin{cases} \rho_B(f) & B \subseteq U, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.27}$$

Restricting from R to V_U gives a bilinear map

$$V_U \times \wedge\{\Theta_U, \Xi_U\} \longrightarrow \wedge\{\Theta_U, \Xi_U\}. \tag{2.5.28}$$

Equation (2.5.18), applied over the variables indexed by U, implies that $I \cap V_U$ acts trivially on $\wedge\{\Theta_U, \Xi_U\}$, so we have an induced bilinear map

$$V_U/(V_U \cap I) \times \wedge\{\Theta_U, \Xi_U\} \longrightarrow \wedge\{\Theta_U, \Xi_U\}. \tag{2.5.29}$$

Theorem 2.4.5, again applied over $\wedge\{\Theta_U, \Xi_U\}$, implies that

$$\{(y_{B_1} \cdots y_{B_k}) \cdot (\theta_1 \cdots \theta_n) \,:\, y_{B_1} \cdots y_{B_k} \in \mathscr{B}_U\} \tag{2.5.30}$$

is linearly independent in $\wedge\{\Theta_U, \Xi_U\}$, so $\mathscr{B}_U$ is linearly independent in $V_U/(V_U \cap I)$. The fact that $\mathscr{B}_U$ spans $V_U/(V_U \cap I)$ follows from the relations (2.5.18) and the greedy algorithm following Observation 2.5.11. □

Placing the generator y_B corresponding to a subset $B \subseteq [n]$ in bidegree $(|B|, 1)$ gives R the structure of a bigraded ring $R = \bigoplus_{i,j \geq 0} R_{i,j}$. The ideals $I, J \subseteq R$ are both bihomogeneous, as are the quotients R/J and $R/(I+J)$. We close this section by recording their bigraded Hilbert series

$$\mathrm{Hilb}(R/J; q, t) = \sum_{m,k \geq 0} \binom{n}{m} \mathrm{Stir}(m,k) \cdot q^m t^k \tag{2.5.31}$$

$$\mathrm{Hilb}(R/(I+J); q, t) = \sum_{m,k \geq 0} \binom{n}{m} \mathrm{Nar}(m,k) \cdot q^m t^k \tag{2.5.32}$$

where $\mathrm{Stir}(m,k) := |\Pi(m,k)|$ is the *Stirling number of the second kind*.

2.6 Fermionic diagonal coinvariants

Up until this point, we have studied the fermions F_π and f_π as members of the exterior algebra $\wedge\{\Theta_n, \Xi_n\}$. In this section we study their images in $FDR_n = \wedge\{\Theta_n, \Xi_n\}/\langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n}\rangle$. Three members of the defining ideal of FDR_n are

$$\theta := \theta_1 + \cdots + \theta_n \qquad \xi := \xi_1 + \cdots + \xi_n \qquad \delta := \theta_1\xi_1 + \cdots + \theta_n\xi_n \tag{2.6.1}$$

where the dependence on n of $\theta, \xi, \delta \in \wedge\{\Theta_n, \Xi_n\}$ is suppressed and will be clear from context. In particular, we have

$$f_\pi = \pm \xi \odot F_\pi \tag{2.6.2}$$

for any set partition $\pi \in \Pi(n)$. Recall that $V(n,k) \subseteq \wedge\{\Theta_n, \Xi_n\}_{n-k,k-1}$ is the span of the set $\{f_\pi : \pi \in \mathrm{NC}(n,k)\}$. The following lemma states that multiplication by θ is an injective operation on $V(n,k)$.

Lemma 2.6.1. *For $1 \leq k \leq n$, the map $\theta \cdot (-) : V(n,k) \to \theta \cdot V(n,k)$ given by multiplication by θ is injective.*

Proof. Theorem 2.4.5 states that $\{f_\pi : \pi \in \mathrm{NC}(n,k)\}$ is a basis for $V(n,k)$ It suffices to show that $\{\theta \cdot f_\pi : \pi \in \mathrm{NC}(n,k)\}$ is a linearly independent subset of $\wedge\{\Theta_n, \Xi_n\}$. To this end, suppose

$$\sum_{\pi \in \mathrm{NC}(n,k)} c_\pi \cdot \theta \cdot f_\pi = 0 \tag{2.6.3}$$

for some coefficients $c_\pi \in \mathbb{C}$.

Let U_i and $\tau_i : \wedge\{\Theta_n, \Xi_n\} \twoheadrightarrow U_i$ be as in the proof of Theorem 2.4.5. If $\{i\}$ is a singleton of $\pi \in \mathrm{NC}(n,k)$, a direct computation shows

$$\tau_i(\theta \cdot f_\pi) = \pm \xi_i \cdot \theta^{(i)} \cdot \overline{f}_{\pi^{(i)}} \tag{2.6.4}$$

where $\overline{f}_{\pi^{(i)}}$ is as in the proof of Theorem 2.4.5 and $\theta^{(i)} := \theta_1 + \cdots + \widehat{\theta}_i + \cdots + \theta_n$. Furthermore, if $\{i\}$ is not a singleton of π, we compute $\tau_i(\theta \cdot f_\pi) = 0$. Applying τ_i to both sides of Equation (2.6.3) gives

$$\sum_{\substack{\pi \in \mathrm{NC}(n,k) \\ \{i\} \text{ a block of } \pi}} \pm c_\pi \cdot \xi_i \cdot \theta^{(i)} \cdot \overline{f}_{\pi^{(i)}} = 0 \tag{2.6.5}$$

which forces (since $\theta^{(i)} \cdot \overline{f}_{\pi^{(i)}}$ do not involve ξ_i) the relation

$$\sum_{\substack{\pi \in \mathrm{NC}(n,k) \\ \{i\} \text{ a block of } \pi}} \pm c_\pi \cdot \theta^{(i)} \cdot \overline{f}_{\pi^{(i)}} = 0. \tag{2.6.6}$$

By induction on n, we conclude that $c_\pi = 0$ whenever π has a singleton block, so that Equation (2.6.3) has the form

$$\sum_{\pi \in \mathrm{NC}(n,k,0)} c_\pi \cdot \theta \cdot f_\pi = 0. \tag{2.6.7}$$

Since the set $\{f_\pi : \pi \in \mathrm{NC}(n,k,0)\}$ is a basis for a flag-shaped $\mathfrak{S}_n$-irreducible and the map $\theta \cdot (-)$ is a nonzero $\mathfrak{S}_n$-homomorphism, the set $\{\theta \cdot f_\pi : \pi \in \mathrm{NC}(n,k,0)\}$ must be a basis for the same flag-shaped irreducible. Thus, the coefficients c_π appearing in Equation (2.6.7) all vanish and the lemma is proven. □

Orellana-Zabrocki [28] and Kim-Rhoades [23] independently proved

$$\langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n} \rangle = \langle \theta, \xi, \delta \rangle \tag{2.6.8}$$

as ideals in $\wedge\{\Theta_n, \Xi_n\}$, so the defining ideal of FDR_n is generated by three elements. We exploit this fact in the proof of the following theorem. Recall that $\overline{FDR_n} = \bigoplus_{k=1}^{n-1} (FDR_n)_{n-k,k-1}$ is the space of extreme bidegrees in FDR_n.

Theorem 2.6.2. *The set $\{f_\pi : \pi \in \mathrm{NC}(n,k)\}$ descends to a basis of $(FDR_n)_{n-k,k-1}$. Consequently, the set $\{f_\pi : \pi \in \mathrm{NC}(n)\}$ descends to a basis of $\overline{FDR_n}$.*

Proof. We define a subspace $U(n,k) \subseteq \wedge\{\Theta_n, \Xi_n\}$ by

$$U(n,k) := \langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n} \rangle \cap \wedge\{\Theta_n, \Xi_n\}_{n-k,k-1} \tag{2.6.9}$$

$$= \theta \cdot \wedge\{\Theta_n, \Xi_n\}_{n-k-1,k-1} + \xi \cdot \wedge\{\Theta_n, \Xi_n\}_{n-k,k-2} + \delta \cdot \wedge\{\Theta_n, \Xi_n\}_{n-k-1,k-2} \tag{2.6.10}$$

where the second line is justified by (2.6.8).

Claim: *We have $V(n,k) \cap U(n,k) = 0$ as subspaces of $\wedge\{\Theta_n, \Xi_n\}_{n-k,k-1}$.*

By Lemma 2.6.1, it suffices to show

$$(\theta \cdot V(n,k)) \cap (\theta \cdot U(n,k)) = 0. \tag{2.6.11}$$

Recall that $\langle -,- \rangle$ is the inner product on $\wedge\{\Theta_n, \Xi_n\}$ for which the monomial basis $\theta_S \cdot \xi_T$ is orthogonal. We show that $(\theta \cdot V(n,k)) \perp (\theta \cdot U(n,k))$ with respect to this inner product.

Fix a set partition $\pi \in \Pi(n,k)$. We need only show that

$$\langle \theta \cdot f_\pi, \theta \cdot \xi \cdot \theta_S \cdot \xi_T \rangle = 0 \text{ and } \langle \theta \cdot f_\pi, \theta \cdot \delta \cdot \theta_S \cdot \xi_T \rangle = 0$$

for any monomial $\theta_S \cdot \xi_T$. The inner product on the left is easier to verify: by the adjointness property in Proposition 1.1.2 (2) and the fact that $\bar{\theta} = \theta$ we have

$$\langle \theta \cdot f_\pi, \theta \cdot \xi \cdot \theta_S \cdot \xi_T \rangle = \pm \langle \theta \cdot (\xi \odot F_\pi), \theta \cdot \xi \cdot \theta_S \cdot \xi_T \rangle \tag{2.6.12}$$

$$= \mp \langle \xi \odot (\theta \cdot F_\pi), \theta \cdot \xi \cdot \theta_S \cdot \xi_T \rangle \tag{2.6.13}$$

$$= \mp \langle \theta \cdot F_\pi, \xi \cdot \theta \cdot \xi \cdot \theta_S \cdot \xi_T \rangle \tag{2.6.14}$$

$$= 0 \tag{2.6.15}$$

where the last line used $\xi^2 = 0$.

We turn to the argument that $\langle \theta \cdot f_\pi, \theta \cdot \delta \cdot \theta_S \cdot \xi_T \rangle = 0$ for any $S, T \subseteq [n]$. Using Proposition 1.1.2 (2) together with $\bar{\xi} = \xi$ and $\bar{\theta} = \theta$ we calculate

$$\langle \theta \cdot f_\pi, \theta \cdot \delta \cdot \theta_S \cdot \xi_T \rangle = \pm \langle \theta \cdot (\xi \odot F_\pi), \theta \cdot \delta \cdot \theta_S \cdot \xi_T \rangle \tag{2.6.16}$$

$$= \mp \langle \xi \odot (\theta \cdot F_\pi), \theta \cdot \delta \cdot \theta_S \cdot \xi_T \rangle \tag{2.6.17}$$

$$= \mp \langle \theta \cdot F_\pi, \xi \cdot \theta \cdot \delta \cdot \theta_S \cdot \xi_T \rangle \tag{2.6.18}$$

$$= \mp \langle F_\pi, \xi \cdot (\theta \odot \delta) \cdot \theta \cdot \theta_S \cdot \xi_T \rangle \pm \langle F_\pi, \xi \cdot \delta \cdot \theta \odot (\theta \cdot \theta_S \cdot \xi_T) \rangle \tag{2.6.19}$$

$$= \pm \langle F_\pi, \xi \cdot \delta \cdot \theta \odot (\theta \cdot \theta_S \cdot \xi_T) \rangle \tag{2.6.20}$$

where the last line used $\theta \odot \delta = \xi$ and $\xi^2 = 0$. The Claim is therefore reduced to showing

$$\langle F_\pi, \xi \cdot \delta \cdot \theta \odot (\theta \cdot \theta_S \cdot \xi_T) \rangle = 0 \tag{2.6.21}$$

for any $S, T \subseteq [n]$. We prove (2.6.21) by verifying the stronger claim that

$$\langle F_\pi, \xi \cdot \delta \cdot \theta_S \cdot \xi_T \rangle = 0 \tag{2.6.22}$$

for any $S, T \subseteq [n]$.

To see why Equation (2.6.22) holds, suppose that F_π and $\xi \cdot \delta \cdot \theta_S \cdot \xi_T$ have any monomials in common. From the definition of F_π, the sets S, T must satisfy the following three conditions.

1. For exactly one block B_1 of π, we have $B_1 \cap T = \varnothing$ and $|B_1 - S| = 2$.

2. For exactly one block B_2 of π, we have $B_2 \cap T = \varnothing$ and $|B_2 - S| = 1$.

3. For every other block B of π, we have $|B \cap T| = 1, |B - S| = 1$, and $B \cap T \subset B \cap S$ whenever B is not a singleton.

If S, T satisfy (1) - (3), the monomials shared between F_π and $\xi \cdot \delta \cdot \theta_S \cdot \xi_T$ have the form $\xi_i \theta_j \xi_j \cdot m$ where $i \in B_2 \cap S$ (or $i \in B_2$ if B_2 is a singleton) and $j \in B_1 - S$ and m is a monomial.

If $B_1 - S = \{j_1, j_2\}$, then $\xi_i \theta_{j_1} \xi_{j_1} m$ and $\xi_i \theta_{j_2} \xi_{j_2} m$ appear in F_π with opposite sign. We conclude that $\langle F_\pi, \xi \cdot \delta \cdot \theta_S \cdot \xi_T \rangle = 0$, completing the proof of the Claim.

We use the Claim to prove the result. By Equation (2.1.9) and Theorem 2.4.5, we have

$$\dim V(n,k) = \mathrm{Nar}(n,k) = \dim (FDR_n)_{n-k,k-1} \tag{2.6.23}$$

and by the definition of $U(n,k)$ we have

$$\dim \wedge\{\Theta_n, \Xi_n\}_{n-k,k-1} = \dim (FDR_n)_{n-k,k-1} + \dim U(n,k). \tag{2.6.24}$$

These dimension equalities combine with the Claim to give a direct sum decomposition

$$\wedge\{\Theta_n, \Xi_n\}_{n-k,k-1} = V(n,k) \oplus U(n,k) \tag{2.6.25}$$

so that the composite

$$V(n,k) \hookrightarrow \wedge\{\Theta_n, \Xi_n\}_{n-k,k-1} \twoheadrightarrow (FDR_n)_{n-k,k-1} \tag{2.6.26}$$

is a linear isomorphism. Theorem 2.4.5 finishes the proof. □

As a corollary, we obtain our promised identification between the skein action and the extreme bidegree components of the fermionic diagonal coinvariants.

Corollary 2.6.3. *We have an isomorphism of $\mathfrak{S}_n$-modules $\mathbb{C}[\mathrm{NC}(n,k)] \cong (FDR_n)_{n-k,k-1}$ for all $1 \leq k \leq n$.*

Proof. Apply the isomorphism (2.6.26) and Theorem 2.4.6. □

Equation (2.1.7), Theorem 2.4.6, Corollary 2.4.7, and Corollary 2.6.3 imply

$$\sum_{m=0}^{k} s_{(k-m,k-m,1^{n-2k+m})} \cdot s_{(1^m)} = \mathrm{Frob}\, \mathbb{C}[\mathrm{NC}(n,k)] \tag{2.6.27}$$

$$= \mathrm{Frob}\,(FDR_n)_{n-k,k-1} \tag{2.6.28}$$

$$= s_{(k,1^{n-k})} * s_{(n-k-1,1^{k-1})} - s_{(k-1,1^{n-k+1})} * s_{(n-k-2,1^{k+2})}. \tag{2.6.29}$$

The symmetric function identity

$$\sum_{m=0}^{k} s_{(k-m,k-m,1^{n-2k+m})} \cdot s_{(1^m)} = s_{(k,1^{n-k})} * s_{(n-k-1,1^{k-1})} - s_{(k-1,1^{n-k+1})} * s_{(n-k-2,1^{k+2})} \tag{2.6.30}$$

on the extreme ends of this chain of equalities relates an application of the dual Pieri rule to a difference of Kronecker products of hook Schur functions. It is possible (but tedious) to verify Equation (2.6.30) directly using rules for the Schur expansion of $s_\lambda * s_\mu$ where $\lambda, \mu \vdash n$ are hook shapes. One such rule, due to Rosas [43], implies that whenever a $\mathfrak{S}_n$-irreducible S^λ appears in $\mathbb{C}[\mathrm{NC}(n,k)]$, we must have $\lambda_3 < 3$. Rosas's rule also implies that the multiplicities of any irreducible S^λ in $\mathbb{C}[\mathrm{NC}(n,k)]$ lies in the set $\{0,1,2\}$.

2.7 Conclusion

In Section 2.5 we constructed a linear projection $p : \mathbb{C}[\Pi(n)] \twoheadrightarrow \mathbb{C}[\mathrm{NC}(n)]$ which resolves crossings in set partitions by means of the rank $2n$ exterior algebra $\wedge\{\Theta_n, \Xi_n\}$. It is natural to ask whether this crossing resolution is applicable more broadly. We will make the following vague problem more precise after its statement.

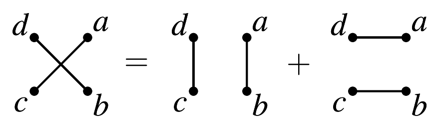
Problem 2.7.1. *Find instances of the crossing resolution p, or the skein relations in Figure 2.1, in other mathematical contexts.*

A classical application of the two-term skein relation is as follows. Let X be a $2 \times n$ matrix of variables and $\mathbb{C}[X]$ is the polynomial ring in these variables. The special linear group

SL_2 acts on the rows of X, and hence on the ring $\mathbb{C}[X]$, by linear substitutions. The invariant subring $\mathbb{C}[X]^{SL_2}$ is generated by the minors $\{\Delta_{ab} : 1 \leq a < b \leq n\}$ and the syzygy ideal of relations among these generators is generated by the Plücker relations

$$\Delta_{ac}\Delta_{bd} = \Delta_{ab}\Delta_{cd} + \Delta_{ad}\Delta_{bc} \qquad 1 \leq a < b < c < d \leq n. \tag{2.7.1}$$

The standard mnemonic for Equation (2.7.1) is the basic skein relation



Kung and Rota [24] gave a detailed combinatorial study of this invariant ring which has seen representation-theoretic application (e.g. [40, 45]).

Patrias, Pechenik, and Striker [29] gave an analogous invariant-theoretic interpretation of the three-term and four-term skein relations as follows. Let X be an $m \times n$ matrix of variables when $m = \ell + 2$ and let $\mathbb{C}[X]$ be the polynomial ring in these variables. Let $P \subseteq GL_m(\mathbb{C})$ be the parabolic subgroup of matrices of block form

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where A is 2×2 and C is $\ell \times \ell$ and consider the invariant subring $\mathbb{C}[X]^P$. This is the homogeneous coordinate ring of the two-step flag variety $\mathrm{Fl}(2, \ell)$ in $\mathbb{C}^n$. The authors of [33] use matrix minors to define elements of $\mathbb{C}[X]^P$ indexed by set partitions which satisfy the three-term and four-term skein relations. It may be useful to consider more general flag varieties in the context of skein theory.

In this chapter we gave a combinatorial interpretation of the bidegree components $(FDR_n)_{i,j}$ of FDR_n where $i + j = n - 1$ is maximal. It is natural to ask about bidegrees $(FDR_n)_{i,j}$. Future work of the first author will give such an interpretation related to *noncrossing* $(1,2)$-

configurations (see [55]). Another possible extension is as follows.

Problem 2.7.2. *Extend the skein action from type A to a wider class of reflection groups.*

As mentioned in the introduction, fermionic quotients suggest an avenue for Problem 2.7.2. More precisely, let W be an irreducible complex reflection group of rank n acting on its reflection representation $V \cong \mathbb{C}^n$. The action of W on V induces actions of W on

- the n-dimensional dual space V^*,
- the $2n$-dimensional direct sum $V \oplus V^*$, and
- the 2^{2n}-dimensional exterior algebra $\wedge(V \oplus V^*)$.

Kim and Rhoades defined [23] the *W-fermionic diagonal coinvariant ring* to be the quotient

$$FDR_W := \wedge(V \oplus V^*)/I \tag{2.7.2}$$

where I is the (two-sided) ideal in $\wedge(V \oplus V^*)$ generated by the W-invariants with vanishing constant term. By placing V in bidegree $(1,0)$ and V^* in bidegree $(0,1)$, the quotient FDR_W attains the structure of a bigraded W-module.

If $\theta_1, \ldots, \theta_n$ is a basis of V and $\xi_1, \ldots, \xi_n$ is the dual basis of V^*, Kim and Rhoades proved [23] that FDR_W may be modeled as

$$FDR_W \cong \wedge\{\Theta_n, \Xi_n\}/\langle \delta \rangle \tag{2.7.3}$$

where $\delta = \theta_1\xi_1 + \cdots + \theta_n\xi_n$. The bidegree component $(FDR_W)_{i,j}$ is nonzero if and only if $i + j \leq n$, and for $i + j \leq n$ we have $\dim(FDR_W)_{i,j} = \binom{n}{i}\binom{n}{j} - \binom{n}{i-1}\binom{n}{j-1}$ so that for any $0 \leq k \leq n$ we have

$$\dim(FDR_W)_{n-k,k} = \mathrm{Nar}(n+1, k+1) \tag{2.7.4}$$

and a plausible solution to Problem 2.7.2 could involve the quotient ring (2.7.3) and noncrossing partitions of $[n+1]$. One thing to note about such a solution is that it would involve an action of W on a space spanned by classical (type A) noncrossing partitions rather than the W-noncrossing partitions studied in Coxeter-Catalan theory, giving an action of type W on type A.

The presentation (2.7.3) holds somewhat beyond the realm of finite complex reflection groups. Indeed, the identification (2.7.3) holds whenever W is a subgroup of $\mathrm{GL}(V)$ such that the nonzero exterior powers $\wedge^0 V, \wedge^1 V, \ldots, \wedge^n V$ of V are pairwise nonisomorphic W-irreducibles [23]. One such subgroup W is $\mathrm{GL}(V)$ itself. It may be interesting to use (2.7.3) to analyze Problem 2.7.2 in this broader context.

Returning to the symmetric group, various authors [2, 6, 28, 30] have considered a 'multidiagonal' version of the fermionic coinvariants defined as follows. Consider an $k \times n$ matrix Θ of fermionic variables $\theta_{i,j}$ where $1 \leq i \leq k$ and $1 \leq j \leq n$. Let $\wedge\{\Theta\}$ be the exterior algebra over these variables, a $\mathbb{C}$-vector space of dimension 2^{nk}. The column permuting action of $\mathfrak{S}_n$ on the matrix Θ induces an action of $\mathfrak{S}_n$ on $\wedge\{\Theta\}$. The *fermionic multidiagonal coinvariant ring* is the quotient

$$FDR(n;k) := \wedge\{\Theta\}/I \tag{2.7.5}$$

where $I \subset \wedge\{\Theta\}$ is the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. We have $FDR(n;2) = FDR_n$. In general $FDR(n;k)$ is a k-fold graded $\mathfrak{S}_n$-module. Orellana and Zabrocki [28] found generators for the ideal I, as well as a combinatorial formula for the multigraded Frobenius image of the numerator $\wedge\{\Theta\}$.

Problem 2.7.3. *Give a combinatorial interpretation for the multigraded pieces of $FDR(n;k)$.*

This chapter (and future work of the first author) address Problem 2.7.3 when $k=2$. For $k>2$, one complicating feature in Problem 2.7.3 is that the supporting multidegrees of $FDR(n;k)$ are unknown. Specifically, if $\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{Z}_{\geq 0}^k$ it is conjectured [6] that $FDR(n,k)_{\mathbf{i}} \neq 0$ whenever $i_1 + \cdots + i_k < n$, but when $k>2$ there exist tuples $\mathbf{i}$ with $i_1 + \cdots + i_k \geq n$ and $FDR(n,k)_{\mathbf{i}} \neq 0$.

One clue that Problem 2.7.3 should have an interesting solution is a conjecture of F. Bergeron [2]. The column action of $\mathfrak{S}_n$ on Θ commutes with the row action of GL_k, so $\mathscr{G}(n;k) := \mathfrak{S}_n \times \mathrm{GL}_k$ acts on $\wedge\{\Theta\}$ and has an induced action on $FDR(n;k)$. The $\mathscr{G}(n;k)$-*character* of this module is

$$\mathrm{ch}_{\mathscr{G}(n;k)} FDR(n;k) := \sum_{\mathbf{i}} \mathrm{Frob}\, FDR(n;k)_{\mathbf{i}} \otimes (q_1^{i_1} \cdots q_k^{i_k}) \tag{2.7.6}$$

where the sum ranges over all $\mathbf{i} = (i_1, \dots, i_k) \in \mathbb{Z}_{\geq 0}^k$ and $q_1, \dots, q_k$ are variables. This character lies in $\Lambda(\mathbf{x}) \otimes \mathbb{C}[q_1, \dots, q_k]^{\mathfrak{S}_k}$.

Bergeron proved [2] that the limit as $k \to \infty$ of the character (2.7.6)

$$\mathscr{F}_n := \lim_{k \to \infty} \mathrm{ch}_{\mathscr{G}(n;k)} FDR(n;k) \tag{2.7.7}$$

is a well-defined element of $\Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{q})$ where $\mathbf{q} = (q_1, q_2, \dots)$. Bergeron defined [1] another element $\mathscr{E}_n \in \Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{q})$ in the same way as $\mathscr{F}_n$, but using matrices of *commuting* variables and conjectured [2] that

$$\mathscr{E}_n = \omega_{\mathbf{q}} \mathscr{F}_n \tag{2.7.8}$$

where the ω-involution acts on $\mathbf{q}$-variables alone. That is, fermionic multidiagonal coinvariants are expected to "contain all the information of" the commuting multidiagonal coinvariants.

Chapter 2 is a reprint of the material as it appears in International Math Research Notices, 2022, authored by the dissertation author and Brendon Rhoades.

Chapter 3

A combinatorial model for the fermionic diagonal coinvariant ring

3.1 Introduction

This chapter involves an algebraically defined $\mathfrak{S}_n$-module, and is concerned with modelling the $\mathfrak{S}_n$ action on this module via combinatorially defined objects. In particular, we will give a basis indexed by a certain type of noncrossing set partition for which the $\mathfrak{S}_n$ action has a nice combinatorial interpretation.

The module in question was introduced by Jongwon Kim and Rhoades [23], and is defined as follows. Let $\Theta_n = (\theta_1, \ldots, \theta_n)$ and $\Xi_n = (\xi_1, \ldots, \xi_n)$ be two sets of n anticommuting variables, and let

$$\wedge\{\Theta_n, \Xi_n\} := \wedge\{\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n\} \tag{3.1.1}$$

be the exterior algebra generated by these symbols over $\mathbb{C}$. The symmetric group $\mathfrak{S}_n$ acts on this exterior algebra via a diagonal action given by

$$w \cdot \theta_i := \theta_{w(i)} \qquad w \cdot \xi_i' := \xi_{w(i)}'. \tag{3.1.2}$$

for any permutation $w \in \mathfrak{S}_n$ and $1 \leq i \leq n$. Let $\wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n}$ denote the subspace of $\mathfrak{S}_n$-invariants with vanishing constant term. Then the fermionic diagonal coinvariant ring is defined as

$$FDR_n := \wedge\{\Theta_n, \Xi_n\}/\langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n}\rangle. \tag{3.1.3}$$

The ring FDR_n is a variant of the Garsia-Haiman diagonal coinvariant ring [14], which is defined analogously but with the anticommuting variables replaced with commuting ones. Several other variants involving more sets of variables or mixtures of anticommuting and commuting variables have been studied by other authors [2, 4, 6, 23, 28, 30, 41, 42, 52, 53, 60, 61].

The ring $\wedge\{\Theta_n, \Xi_n\}$ has a bigrading given by

$$(\wedge\{\Theta_n, \Xi_n\})_{i,j} := \wedge^i\{\theta_1, \dots, \theta_n\} \otimes \wedge^j\{\xi_1, \dots, \xi_n\}. \tag{3.1.4}$$

The invariant ideal $\langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n}\rangle$ is homogeneous, so FDR_n inherits the bigrading. In [23], Kim and Rhoades calculated the frobenius image of FDR_n to be given by

$$\mathrm{Frob}(FDR_n)_{i,j} = s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i-1,1^{i+1})} * s_{(n-j-1,1^{j+1})} \tag{3.1.5}$$

where $*$ denotes the Kronecker product of Schur functions. They remark that in the case when $i+j = n-1$, the above shows that the dimension of $(FDR_n)_{n-k,k-1}$ is given by the Narayana number $\mathrm{Nar}(n,k)$. Narayana numbers count noncrossing set partitions of $[n]$ into k blocks, and in joint work with Rhoades [21] we gave a combinatorial basis of $(FDR_n)_{n-k,k-1}$ indexed by set partitions for which the $\mathfrak{S}_n$-action was given by a skein action on noncrossing partitions first described by Rhoades in [39].

In this chapter we will give a similar result for all bidegrees, although our results will not give a combinatorial description for the full $\mathfrak{S}_n$-action. Instead, we will focus on the subgroup of $\mathfrak{S}_n$ consisting of permutations which leave n fixed (which we will abusively refer to as $\mathfrak{S}_{n-1}$). We will define a basis of $(FDR_n)_{i,j}$ indexed by a certain class of noncrossing set partitions defined in Section 3 for which the action of $\mathfrak{S}_{n-1}$ can be described via combinatorial manipulations of the indexing partitions and use this basis to give an expression for the Frobenius image

$$\mathrm{Frob}(\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(FDR_n)_{i,j}). \tag{3.1.6}$$

The rest of the chapter is organized as follows. Section 2 will describe an action of $\mathfrak{S}_{n-1}$ on certain set partitions and map this action into FDR_n. Section 3 will show that a restriction of this map is an isomorphism and use it to obtain a combinatorial basis of FDR_n. Section 4 will use the basis developed to calculate the bigraded $\mathfrak{S}_n$-structure of FDR_n. Section 5 will connect this basis to a cyclic sieving result of Thiel and address some avenues of further inquiry.

3.2 Set partitions and the action of $\mathfrak{S}_{n-1}$

The indexing set for our combinatorial basis will be a certain partially labelled subset $\Phi(n)$ of noncrossing set partitions of $[n]$.

Definition 3.2.1. *Let n,k,x,t be nonnegative integers. We define the following sets of set partitions:*

- *Let $\Psi(n)$ denote the set of all set partitions of n for which all blocks not containing n are size 1 or size 2, and blocks of size 1 not containing n are labelled with either a θ or a ξ'.*

- *Let $\Psi(n,k)$ be the set of partitions in $\Psi(n)$ in which the block containing n is size k.*

- *Let $\Psi(n,k,t,x)$ denote the set of partitions in $\Psi(n,k)$ which have exactly t singletons labelled θ and exactly x singletons labelled ξ'.*

- *Let $\Phi(n)$, $\Phi(n,k)$, and $\Phi(n,k,t,x)$ be the subsets of $\Psi(n)$, $\Psi(n,k)$, or $\Psi(n,k,t,x)$ respectively which consist only of the those set partitions which are noncrossing.*

For the rest of this chapter, when we refer to the singleton blocks of a partition $\pi \in \Psi(n)$, we only refer to those blocks of size 1 that do not contain n, even if the block containing n happens to be size 1. Similarly when we refer to the blocks of size two we refer to only the blocks of size two that do not contain n.

There is a natural action of $\mathfrak{S}_{n-1}$ on $\Psi(n)$, given by simply permuting elements between blocks and preserving labels of blocks. The sets $\Psi(n,k)$ and $\Psi(n,k,x,t)$ are closed under this action, but $\Phi(n)$ is not, as permuting the elements of a noncrossing permutation may introduce crossings. However, we can define an action of $\mathfrak{S}_{n-1}$ on the linearization $\mathbb{C}\Phi(n)$ by mapping $\mathbb{C}\Psi(n)$ into $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ in such a way that $\mathbb{C}\Phi(n)$ is $\mathfrak{S}_{n-1}$-invariant and pulling back the $\mathfrak{S}_{n-1}$-action.

Towards this goal, to each element $\pi \in \Psi(n)$ we will associate an element G_π of $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$. To define G_π we will make use of a tool we will call *block operators*. Let B be a block of a set partition $\pi \in \Psi(n)$, i.e. B is a nonempty subset of $[n]$ that either contains n or is size at most two. Define the *block operator* $\tau_B : \wedge\{\Theta_{n-1}, \Xi'_{n-1}\} \to \wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ by

$$\tau_B(f) = \begin{cases} (\prod_{i \in B \setminus \{n\}} \theta_i) \odot f & n \in B \\ \xi'_i \cdot (\theta_j \odot f) + \xi'_j \cdot (\theta_i \odot f) & n \notin B, B = \{i, j\} \\ f & B = \{i_\theta\} \\ \xi'_i \cdot (\theta_i \odot f) & B = \{i'_\xi\} \end{cases} \tag{3.2.1}$$

It will be important for what follows to note that block operators corresponding to blocks not containing n commute

Lemma 3.2.2. *Let A and B be two nonempty subsets of $[n-1]$ of size at most two. Then τ_A and τ_B commute.*

Proof. The lemma reduces to the fact that the operators $\{\xi'_1\cdot, \ldots, \xi'_{n-1}\cdot, \theta_1\odot, \ldots, \theta_{n-1}\odot\}$ all anticommute, and that each block operator is a degree two polynomial in these. □

Block operators also interact nicely with the action of $\mathfrak{S}_{n-1}$.

Lemma 3.2.3. *Let A be a subset of $[n-1]$ and let $\sigma \in \mathfrak{S}_{n-1}$. Then for any $f \in \wedge\{\Theta_{,n-1}, \Xi'_{n-1}\}$*

$$\sigma \cdot \tau_A(f) = \tau_{\sigma \cdot A}(\sigma \circ f)$$

where the action of $\mathfrak{S}_n$ on subsets is given by $\sigma \cdot \{a_1, \ldots, a_k\} = \{\sigma(a_1), \ldots, \sigma(a_k)\}$.

We can now define G_π.

Definition 3.2.4. *Let $\pi \in \Psi(n)$ with blocks $B_1, \ldots, B_k$ and $n \in B_k$. Then*

$$G_\pi := \tau_{B_1} \cdots \tau_{B_k}(\theta_1\theta_2 \cdots \theta_{n-1}). \tag{3.2.2}$$

We can also give a description of the G_π not involving block operators as follows.

Proposition 3.2.5. *Let $\pi \in \Psi(n)$. Take the product of $\theta_i\xi'_i - \theta_j\xi'_j$ for every size two block $\{i, j\}$ of π with $i < j$. For each singleton block $\{i\}$ of π, multiply by θ_i or ξ'_i according to its label in increasing order. Then G_π is equal to the result multiplied by $(-1)^{inv}(\pi')$ where π' is the word formed by listing all size two blocks not containing n increasing within each block and by order of increasing minimal element, then listing all size one blocks not containing n in increasing order.*

For example, if $\pi = 1_\theta/2,5/3,4/5,6,8/7'_\xi$, then

$$G_\pi = (-1)^{\mathrm{inv}(253417)}(\theta_2\xi'_2 - \theta_5\xi'_5)(\theta_3\xi'_3 - \theta_4\xi'_4)\theta_1\xi'_7 \tag{3.2.3}$$

Proof. By Lemma 3.2.2 we can assume that all of the block operators corresponding to size two blocks appear before block operators according to singletons. Applying τ_{B_k} and any block operators corresponding to singletons to $(\theta_1\theta_2 \cdots \theta_{n-1})$ removes all θ_i indexed by elements of B_k and replaces θ_i indexed by ξ'-labelled singletons with ξ'_i. Note that $\tau_{\{i,j\}}\theta_i\theta_j = \theta_i\xi'_i - \theta_j\xi'_j$, and the proof follows. □

The $\mathfrak{S}_{n-1}$ action on these G_π matches the natural $\mathfrak{S}_{n-1}$ action on $\Psi(n)$, up to sign.

Proposition 3.2.6. *Let $\sigma \in \mathfrak{S}_{n-1}$ and $\pi \in \Psi(n)$. Then $\sigma \circ G_\pi = \operatorname{sign}(\sigma) G_{\sigma\circ\pi}$.*

Proof. Using the block operator definition of G_π and Lemma 3.2.3 we have,

$$\sigma \circ G_\pi = \sigma \circ (\tau_{B_1} \cdots \tau_{B_k}(\theta_1\theta_2\cdots\theta_{n-1})) \tag{3.2.4}$$

$$= \tau_{\sigma(B_1)} \cdots \tau_{\sigma(B_k)}(\sigma \circ (\theta_1\theta_2\cdots\theta_{n-1})) \tag{3.2.5}$$

$$= \tau_{\sigma(B_1)} \cdots \tau_{\sigma(B_k)}(\operatorname{sign}(\sigma)\theta_1\theta_2\cdots\theta_{n-1}) \tag{3.2.6}$$

$$= \operatorname{sign}(\sigma) G_{\sigma\circ\pi} \tag{3.2.7}$$

□

The goal of the remainder of this section is to show that $\operatorname{span}(\{G_\pi \mid \pi \in \Phi(n)\})$ is $\mathfrak{S}_{n-1}$ invariant. For this end we will need the following two relations of block operators.

Lemma 3.2.7. *Let $a,b,c,d \in [n-1]$. Then*

$$\tau_{\{a,b\}}\tau_{\{c,d\}} + \tau_{\{a,c\}}\tau_{\{b,d\}} + \tau_{\{a,d\}}\tau_{\{b,c\}} = 0 \tag{3.2.8}$$

as operators on the ring $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$

Proof. This is a straightforward calculation from the definition of τ. □

Lemma 3.2.8. *Let $A = \{a_1 < a_2\} \subset [n-1]$ and $B \subset [n]$ be two disjoint sets with $n \in B$. Let $b_1 < b_2 < \cdots < b_m$ be the elements of B that lie between a_1 and a_2, and suppose at least one such element exists. Then*

$$\tau_A\tau_B + \tau_{\{a_1,b_1\}}\tau_{B+a_2-b_1} + \sum_{i=1}^{m-1} \tau_{\{b_i,b_{i+1}\}}\tau_{B+a_1+a_2-b_i-b_{i+1}} + \tau_{\{b_m,a_2\}}\tau_{B+a_1-b_m} = 0 \tag{3.2.9}$$

as operators on the ring $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$

Proof. For any two cyclically consecutive elements c_1, c_2 of $a_1, b_2, \ldots, b_m, a_2$, and any third element $c_3 \in B \cup \{a_1, a_2\}$, the terms $\xi'_{c_1} \cdot \theta_{c_2} \odot \theta_{c_3} \odot$ and $\xi'_{c_1} \cdot \theta_{c_3} \odot \theta_{c_2} \odot$ will appear in the expansion of left hand side both exactly once or both exactly twice, depending on whether c_3 is also cyclically consecutive with c_1. In either case, anticommutativity results in the sum being 0. □

Together these lemmas allow us to demonstrate the $\mathfrak{S}_{n-1}$ invariance via a combinatorial algorithm.

Corollary 3.2.9. *Let $\sigma \in \mathfrak{S}_{n-1}$ and let $\pi \in \Phi(n)$. Then $\sigma \cdot G_\pi$ can be expressed as a linear combination of $\{G_\pi \mid \pi \in \Phi(n)\}$ via the following algorithm:*

1. *Apply σ to π, resulting in a set partition π' not necessarily in $\Phi(n)$.*

2. *If π' is contains any crossing two element blocks $\{a,c\}$, $\{b,d\}$, neither of which contain n, replace π' with minus the sum of the partitions obtained by replacing $\{a,c\},\{b,d\}$ with $\{a,b\},\{c,d\}$ and $\{a,d\},\{b,c\}$. Repeat on each new term of the sum until all terms of the sum do not contain crossing two element blocks.*

3. *For each term of the sum obtained in step 2, replace any two element set that crosses the block containing n as described by Lemma 3.2.8.*

4. *Replace each partition π'' in the sum obtained from step 3 with its corresponding $G_{\pi''}$ to express $\sigma \cdot G_\pi$ as a linear combination.*

Example 3.2.10. *Let $n = 8$ and let $\sigma \in \mathfrak{S}_{n-1}$ be the cycle (3576). Let $\pi \in \Phi(n)$ be the set partition $\{23/45/7_\theta/186\}$. An example of applying Corollary 3.2.9 to this situation is given in Figure 1.*

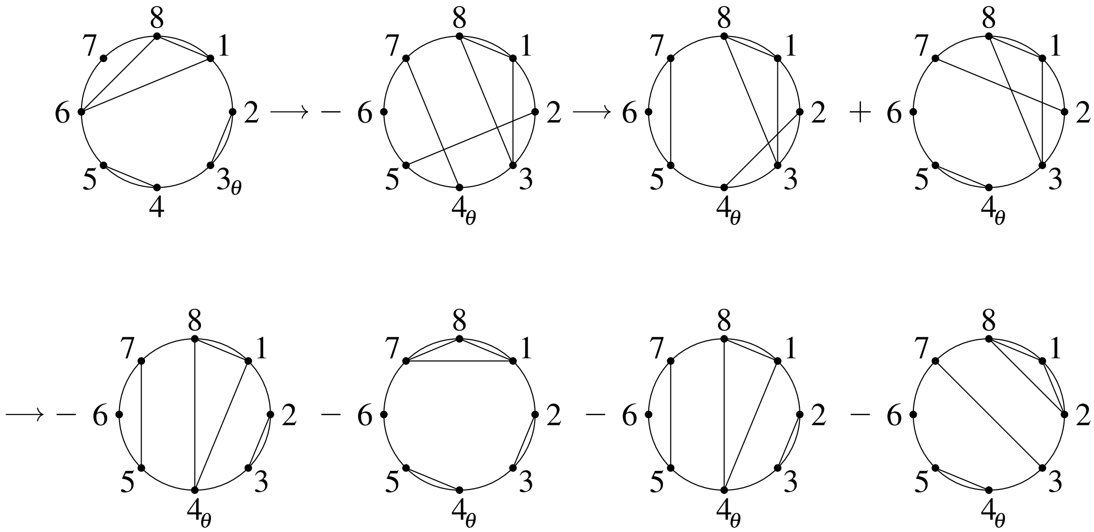


Figure 3.1. Applying Corollary 3.2.9

3.3 A combinatorial basis

We have shown that there is a mapping of $\mathfrak{S}_{n-1}$-modules $\mathbb{C}\Psi(n) \to \wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$. In this section we will show that the restriction of this mapping to $\mathbb{C}\Phi(n)$ is injective and becomes an isomorphism when composed with the quotient map $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\} \to FDR_n$, thereby proving the following.

Theorem 3.3.1. *The set $\{[G_\pi] \mid \pi \in \Psi(n)\}$ forms a basis for FDR_n, where $[f]$ denotes the equivalence class in FDR_n of $f \in \wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$.*

Proof. We begin with a dimension count; Kim and Rhoades [23] gave a basis of FDR_n indexed by a set $\Pi(n)_{>0}$ of Motzkin-like lattice paths defined as follows.

Definition 3.3.2. *Let $\Pi(n)_{>0}$ be the set of all lattice paths which start at $(0,0)$, take steps $(1,0),(1,1)$ or $(1,-1)$, only touch the x-axis at $(0,0)$ and have all $(1,0)$ steps labelled by θ or ξ'.*

The two indexing sets are in bijection.

Lemma 3.3.3. *There is a bijection between $\Pi(n)_{>0}$ and $\Phi(n)$.*

Proof. Given a Motzkin path in $\Pi(n)_{>0}$, draw a horizontal line extending to the right of each up step until it first intersects the path again. Label each step after the first 1 to $n-1$. Construct a set partition by placing every up step in a block with the down step it is connected to if such a down step exists, or in the block containing n otherwise. Place every horizontal step in a singleton block with the same label. The process can be reversed, and is therefore a bijection. □

The bijection is best described with a picture example as in Figure 2.

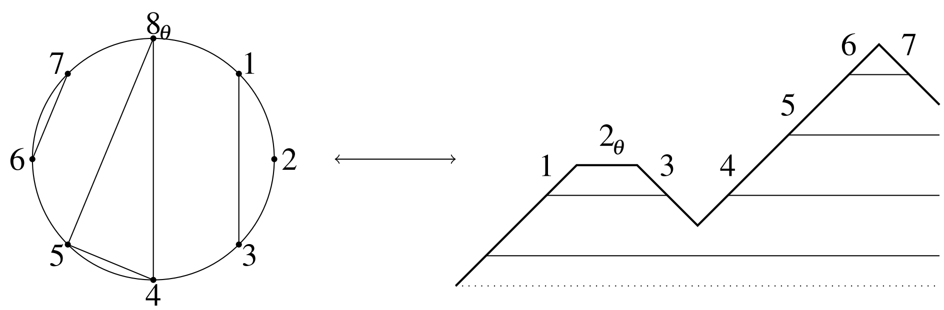


Figure 3.2. An example of Lemma 3.3.3

Therefore it suffices to show that $\{[G_\pi] \mid \pi \in \Psi(n)\}$ spans. By Corollary 3.2.9 and since FDR_n is defined as a quotient, it suffices to show that together, the sets

$$\beta := \{G_\pi \mid \pi \in \Psi(n)\}$$

and

$$\beta' := \{m(\theta_1\xi_1' + \cdots + \theta_{n-1}\xi_{n-1}') \mid m \text{ a monomial in } \wedge\{\Theta_{n-1}, \Xi_{n-1}'\}\}$$

span

$$\wedge\{\Theta_{n-1}, \Xi_{n-1}'\}.$$

To show that $\beta \cup \beta'$ spans, we will break $\wedge\{\Theta_{n-1}, \Xi_{n-1}'\}$ into many subspaces and show that each subspace is spanned.

Let S be a subset of $[n-1]$ of size $2k$ for some integer k. Let m denote a fixed monomial of $\wedge\{\Theta_{n-1},\Xi'_{n-1}\}$ such that the following conditions hold for all $i\in[n-1]$:

1. If $i\in S$, then neither ξ'_i nor θ_i appears in m.

2. If ξ'_i appears in m, then θ_i does not appear in m.

Let $V_{S,m}$ denote the subspace of $\wedge\{\Theta_{n-1},\Xi'_{n-1}\}$ which is spanned by monomials of the form $\theta_{s_1}\xi'_{s_1}\cdots\theta_{s_k}\xi'_{s_k}m$, where $s_1,\ldots,s_k\in S$. There are $\binom{2k}{k}$ such monomials, so

$$\dim(V_{S,m})=\binom{2k}{k}. \tag{3.3.1}$$

Consider the set of $\beta\cap V_{S,m}$. These will consist of all $\pi\in\Psi(n)'$ such that the size 1 parts of π and their labels correspond exactly with the monomial m, and the size two parts partition S. This set is therefore in bijection with noncrossing perfect matchings of S, so we have

$$|\beta\cap V_{S,m}|=\mathrm{Cat}(k) \tag{3.3.2}$$

where $\mathrm{Cat}(k)$ is the kth Catalan number. Consider as well the set $\beta'\cap V_{S,m}$. If m' is a degree $n-3$ monomial such that $(\theta_1\xi'_1+\cdots+\theta_{n-1}\xi'_{n-1})m'\in V_{S,m}$, then it must be the case that $m'=\theta_{s_1}\xi'_{s_1}\cdots\theta_{s_{k-1}}\xi'_{s_{k-1}}m$ for some choice of $s_1,\ldots,s_{k-1}\in S$. So we have

$$|\beta'\cap V_{S,m}|=\binom{2k}{k-1}. \tag{3.3.3}$$

Putting the above equations together we have

$$|(\beta\cup\beta')\cap V_{S,m}|=\mathrm{Cat}(k)+\binom{2k}{k-1}=\binom{2k}{k}=\dim(V_{S,m}) \tag{3.3.4}$$

and so it suffices to show that $(\beta\cup\beta')\cap V_{S,m}$ is a linearly independent set.

Let d be the degree of m, and let M be the set of monomials of degree $d+2k$ in

$\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ whose variables are in increasing numerical order with $\theta_1 < \xi_1 < \cdots < \theta_n < \xi_n$. Define an inner product $\langle -, - \rangle$ on the degree $d+2k$ part of $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ such that M is an orthonormal set. With respect to this inner product,

$$\beta \cap V_{S,m} \subseteq (\beta' \cap V_{S,m})^{\perp} \tag{3.3.5}$$

To see this, suppose that $f_\pi \in V_{S,m}$ and $(\theta_1\xi'_1 + \cdots + \theta_{n-1}\xi'_{n-1})m' \in V_{S,m}$ have monomials in common. Then m' must be equal to $\theta_{s_1}\xi'_{s_1}\cdots\theta_{s_{k-1}}\xi'_{s_{k-1}}m'$ where each s_i is in a distinct size 2 part of π. If this is the case, then $\theta_{s_1}\xi'_{s_1}\cdots\theta_{s_{k-1}}\xi'_{s_{k-1}}m'$ and f_π share exactly two monomials, corresponding to the two elements in the last size 2 part of π. These monomials will have coefficients of opposite sign in f_π and the same sign in $\theta_{s_1}\xi'_{s_1}\cdots\theta_{s_{k-1}}\xi'_{s_{k-1}}m'$, so the inner product will be 0. Therefore it suffices to show that $\beta \cap V_{S,m}$ and $\beta' \cap V_{S,m}$ are both individually linearly independent sets.

To see that $\beta \cap V_{S,m}$ is linearly independent, consider the lexicographic term order on monomials with respect to the variable order $\theta_1, \xi'_1, \theta_2, \xi'_2, \ldots$. With respect to this order, the leading term of f_π is $\theta_{s_1}\xi'_{s_1}\cdots\theta_{s_k}\xi'_{s_k}m$, where $s_1, \ldots, s_k$ are the numerically smaller elements of each size two block of π. Since π is noncrossing and m and S determine the singletons and block containing n, specifying the set of elements that are the smaller of their part uniquely determines π. Therefore the f_π contained in $V_{S,m}$ all have unique leading terms and are therefore linearly independent.

Kim and Rhoades proved [23] that in FDR_n, multiplication by $\theta_1\xi'_1 + \cdots + \theta_{n-1}\xi'_{n-1}$ is an injection, so $\beta' \cap V_{S,m}$ is also a linearly independent set and $V_{S,m}$ is spanned by $(\beta \cup \beta') \cap V_{S,m}$. Since every monomial is contained in some $V_{S,m}$, we therefore have that $\wedge\{\Theta_{n-1}, \Xi'_{n-1}\}$ is spanned by $\beta \cup \beta'$ and therefore $\{[G_\pi] \mid \pi \in \Psi(n)\}$ is a basis for FDR_n as desired. □

3.4 $\mathfrak{S}_{n-1}$ module structure

In this section we will describe the Frobenius image of each bigraded piece of FDR_n as an $\mathfrak{S}_{n-1}$ module. Consider the family of subspaces:

$$V(n,k,x,y) := \text{span}\{[G_\pi] \mid \pi \in \Phi(n,k,x,y)\} \subseteq FDR_n \tag{3.4.1}$$

These subspaces are in fact submodules of $\text{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(FDR_n)$, since they are closed under the action of $\mathfrak{S}_{n-1}$. To see this, note that no step of the algorithm described in Corollary 3.2.9 replaces a set partition with one with a different number of size two blocks, ξ'-labelled elements, or θ-labelled elements. Since $\Phi(n) = \oplus_{k,x,y}\Phi(n,k,x,y)$ the subspaces $V(n,k,x,y)$ make up all of FDR_n:

Proposition 3.4.1. *The i,j-graded piece of DR_n is a direct sum of $V(n,k,x,y)$:*

$$(FDR_n)_{i,j} = \bigoplus_{\substack{k,x,y \\ k+x=i \\ k+y=j}} V(n,k,x,y)$$

Proof. From the definition of G_π it is clear that if $\pi \in \Phi(n,k,x,y)$ then G_π has bidgree $(k+x,k+y)$. The result follows. □

To determine the structure of these modules we begin with $V(n,k,0,0)$. We first need a lemma

Lemma 3.4.2. *There exists a bijection from $\Phi(n,k,0,0)$ to $SYT(n-k-1,k)$, the set of standard Young tableau of shape $\lambda = (n-k-1,k)$.*

Proof. Define a function $g : \Phi(n,k,0,0) \to \binom{[n-1]}{[k]}$ by

$$g(\pi) = \{i \in [n-1] \mid i \text{ is in a block of size 2, and is the larger element in its block.}\} \tag{3.4.2}$$

For example, $g(14/23/78/569) = \{3,4,8\}$. Then g is injective, it is possible to recover the preimage of a set S under g by starting with the smallest i element of S, if $g(\pi) = S$, then for π to satisfy the noncrossing condition, $\{i-1,i\}$ must be a block of π. Then the next smallest element of S must be paired with the largest element smaller than it that is not already paired, and so on. This algorithm will produce a unique preimage iff S satisfies the condition that for any $k \in [n-1]$, $|S \cap [k]| \leq k/2$. Define another function $h : SYT(n-k-1,k) \to \binom{[n-1]}{k}$ by

$$h(T) = \{i \in [n-1] \mid i \text{ is in the second row of } T\} \tag{3.4.3}$$

Then h is also injective, and $S \in h(SYT(n-k-1,k))$ iff S satisfies the condition that for any $k \in [n-1]$, $|S \cap [k]| \leq k/2$. So the image of h and g are the same and the result follows. □

Proposition 3.4.3. *We have that* $V(n,k,0,0) \cong_{\mathfrak{S}_{n-1}} S^{(n-k-1,k)}$.

Proof. Let $\lambda = (n-k-1,k)$. By Theorem 5.2.9 and Lemma 3.4.2, the dimensions of the modules agree, so by Lemma 1.1.1 it suffices to show that $[\mathfrak{S}_\lambda]_+$ does not kill $V(k,0,0)$, but $[\mathfrak{S}_\mu]_+$ does kill $V(n,k,0,0)$ for all partitions $\mu \succ \lambda$.

We begin by showing that $[\mathfrak{S}_\lambda]_+$ does not kill $V(k,0,0)$. Let $\pi_0 \in \Phi(n,k,0,0)$ be the parition whose blocks are

$$\{n-1,n-2k\},\{n-2,n-2k+1\},\ldots,\{n-k,n-k-1\},\{1,2,3,\ldots,n-2k-1,n\}$$

Then using the block operator definition of F_{π_0} we have

$$[\mathfrak{S}_\lambda]_+ F_{\pi_0} = \sum_{\sigma \in \mathfrak{S}_\lambda} \sigma \cdot \tau_{\{n-1,n-2k\}} \cdots \tau_{\{n-k,n-k-1\}} \tau_{\{1,2,3,\ldots,n-2k-1,n\}} \theta_1 \cdots \theta_{n-1} \tag{3.4.4}$$

Consider the coefficient of $\theta_{n-1} \cdots \theta_{n-k} \xi'_{n-1} \cdots \xi'_{n-k}$ in the above expression. For a term to contribute to this coefficient, it must be the case that $\sigma \cdot \{1,2,3,\ldots,n-2k-1,n\} = \{1,2,3,\ldots,n-$

$2k-1,n\}$. If this is the case, then the summand corresponding to σ can be written as

$$\tau_{\{n-1,\sigma'(n-2k)\}} \cdots \tau_{\{n-k,\sigma'(n-k-1)\}} \theta_{n-2k} \cdots \theta_{n-1} \tag{3.4.5}$$

for some permutation σ' of $\{n-k-1, n-2k\}$. The coefficient of $\theta_{n-1} \cdots \theta_{n-k} \xi'_{n-1} \cdots \xi'_{n-k}$ in equation 3.4.5 above does not depend on σ', so all terms of the sum in equation 3.4.4 which contribute to the coefficient of $\theta_{n-1} \cdots \theta_{n-k} \xi'_{n-1} \cdots \xi'_{n-k}$ contribute the same sign, and thus the coefficient of $\theta_{n-1} \cdots \theta_{n-k} \xi'_{n-1} \cdots \xi'_{n-k}$ in $[\mathfrak{S}_\lambda]_+ F_{\pi_0}$ is nonzero. Thus $V(k,0,0)$ is not killed by $[\mathfrak{S}_\lambda]_+$.

Now let μ be any partition of $n-1$ such that $\lambda \succ \mu$, i.e. $\mu = (n-m, m-1)$ for any $m \leq k$. Let $\pi \in \Phi(n,k,0,0)$. Since $m-1 < k$, there must be at least two elements of i and j of $[n-m]$ in the same block in π. Then the transposition (i,j) acts on G_π via multiplication by -1, so $(1+(i,j))G_\pi = 0$. But $[\mathfrak{S}_\lambda]_+ = A(1+(i,j)$ for some symmteric group algebra element A, so indeed $[\mathfrak{S}_\lambda]_+ G_\pi = 0$, and the result follows. □

We can use $V(n,k,0,0)$ to determine the structure of $V(n,k,x,y)$ for any x,y.

Proposition 3.4.4. *We have that*

$$V(n,k,x,y) \cong_{\mathfrak{S}_{n-1}} \mathrm{Ind}_{\mathfrak{S}_{n-x-y-1} \otimes \mathfrak{S}_x \otimes \mathfrak{S}_y}^{\mathfrak{S}_{n-1}} S^{(n-x-y-k-1,k)} \otimes \mathrm{sign}_{\mathfrak{S}_x} \otimes \mathrm{sign}_{\mathfrak{S}_y}.$$

Proof. We can represent an element π of $\Phi(n,k,x,y)$ by the triple (X,Y,π'), where X is the set of singletons labelled by ξ', Y is the set of singletons labelled by θ, and π' is the set partition obtained by removing all singletons from π and decrementing indices. Let $F_{(X,Y,\pi')}$ denote G_π

for the corresponding π. The action of a transposition (i,j) on $F_{(X,Y,\pi')}$ is then given by

$$(i,j)\circ F_{(X,Y,\pi')} = \begin{cases} -F_{(X,Y,\pi')} & \{i,j\}\subset X \text{ or } \{i,j\}\subset Y \\ F_{(X,Y,(i,j)\circ\pi')} & \{i,j\}\subset (X\cup Y)^c \\ F_{(i,j)\circ X,(i,j)\circ Y,\pi'} \text{ otherwise} & \end{cases} \tag{3.4.6}$$

The proposition follows from the definition of induced representation. $\square$

Corollary 3.4.5. *The Frobenius image of $V(n,k,x,y)$ is given by $s_{(n-x-y-k-1,k)}s_{(1^x)}s_{(1^y)}$. The Frobenius image of $(FDR_n)_{i,j}$ is*

$$\sum_{\substack{k,x,y\\k+x=i\\k+y=j}} s_{(n-x-y-k-1,k)}s_{(1^x)}s_{(1^y)}$$

Proof. This follows directly from Proposition 3.4.4 and Proposition 3.4.1. $\square$

Corollary 3.4.6. *The bigraded Frobenius image of $Res^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(FDR_n)$ is given by*

$$\mathrm{grFrob}(\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(FDR_n);q,t) = (1-qt)\prod_{i=1}^{\infty}\frac{(1+x_iqz)(1+x_itz)}{(1-x_iz)(1-x_iqtz)}\bigg|_{z^{n-1}}$$

where the operator $(\cdots)\,|_{z^{n-1}}$ extracts the coefficient of z^{n-1}.

By Proposition 3.4.5 we have

$$\mathrm{grFrob}(\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(FDR_n);q,t) = \sum_i\sum_j\sum_{\substack{k,x,y\\k+x=i\\k+y=j}} s_{(n-x-y-k-1,k)}s_{(1^x)}s_{(1^y)}q^it^j. \tag{3.4.7}$$

Applying Jacobi-Trudi [47] to the $s_{(n-x-y-k-1,k)}$ terms on the right gives

$$\sum_i \sum_j \sum_{\substack{k,x,y \\ k+x=i \\ k+y=j}} s_{(n-x-y-k-1,k)} s_{(1^x)} s_{(1^y)} q^i t^j = \sum_i \sum_j \sum_{\substack{k,x,y \\ k+x=i \\ k+y=j}} (h_{n-x-y-k-1} h_k - h_{n-x-y-k} h_{k-1}) e_x e_y q^i t^j \tag{3.4.8}$$

and reindexing sums gives

$$\sum_i \sum_j \sum_{\substack{k,x,y \\ k+x=i \\ k+y=j}} h_{n-x-y-k-1} h_k e_x e_y q^i t^j = \sum_k h_k q^k t^k z^k \sum_x e_x q^x z^x \sum_y e_y q^y z^y \sum_m h_m z^m \bigg|_{z^{n-1}} \tag{3.4.9}$$

and

$$\sum_i \sum_j \sum_{\substack{k,x,y \\ k+x=i \\ k+y=j}} h_{n-x-y-k} h_{k-1} e_x e_y q^i t^j = \sum_k h_k q^{k+1} t^{k+1} z^k \sum_x e_x q^x z^x \sum_y e_y q^y z^y \sum_m h_m z^m \bigg|_{z^{n-1}} \tag{3.4.10}$$

from which the result follows.

3.5 Maximal bidegrees, cyclic sieving and further directions

Let X_n denote the subset of $\Phi(n)$ corresponding to bidegrees (i, j) where $i + j = n - 1$, in other words,

$$X_n = \bigcup_{2k+x+y=n-1} \Phi(n,k,x,y). \tag{3.5.1}$$

This set consists of noncrossing set partitions set partitions of $[n]$ in which n is in a block by itself, all other blocks are size 1 or 2, and singleton blocks other than n are labelled by θ or ξ'. The set $\{G_\pi \mid \pi \in X_n\}$ is invariant (up to sign changes) under the action of the cycle $(1, 2, \ldots, n-1)$, since n is in a block by itself and rotating all elements except n cannot introduce any new crossings. We therefore have the setup for a cyclic sieving result using Springer's theorem of regular elements (Theorem 3.5.1).

Theorem 3.5.1. *The triple* $(X_n, C_{n-1}, q^{\binom{n}{2}}\mathbf{fd}(FDR_n)_{i+j=n-1})$ *exhibits the cyclic sieving phenomenon where* C_{n-1} *is the cyclic group generated by* $(1,2,\ldots,n-1)$.

Proof. This follows directly from Theorem 3.5.1. □

Thiel [55] studied a version of this cyclic action in which rotation does not introduce a sign change, while in our setup it introduces a sign when n is odd. Thiel proved the following cyclic sieving.

Theorem 3.5.2 (Thiel, 2016)**.** *The triple* $(X_n, C_{n-1}, C_n(q))$ *exhibits the cyclic sieving phenomenon, where* C_{n-1} *is the cyclic group generated by* $(1,2,\ldots,n-1)$ *and* $C_n(q)$ *is the Mac-Mahon q-Catalan number, defined by*

$$C_n(q) := \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ q \end{bmatrix}_q .$$

Thiel proved his result via direct computation of $C_n(q)$ and enumeration of fixed points instead of using representation theory, so one might wonder if our basis could give an altenate algebraic proof of his result. The expression for Frobenius image given in Corollary 3.4.5 allows for the computation of the fake degree as

$$\mathbf{fd}((FDR_n)_{i+j=n-1}) = \sum_{\substack{k,x,y \\ 2k+x+y=n-1}} \begin{bmatrix} n-1 \\ 2k,x,y \end{bmatrix}_q C_k(q) q^{k+\binom{x}{2}+\binom{y}{2}} \tag{3.5.2}$$

Combining the two cyclic sieving results it must follow that $q^{\binom{n}{2}}\mathbf{fd}((FDR_n)_{i+j=n-1})$ is equivalent to $C_n(q)$ modulo $q^{n-1}-1$. We have had difficulty in determining this equivalence directly, however, so we propose the following problem:

Problem 3.5.3. *Is there a direct computational proof that* $q^{\binom{n}{2}}\boldsymbol{fd}((FDR_n)_{i+j=n-1})$ *and* $C_n(q)$ *are equivalent modulo* $q^n - 1$*?*

Such a proof would complete an alternative representation theoretic proof of Thiel's result.

In joint work with Rhoades [21] we developed a similar combinatorial model for the maximal bidegree components of FDR_n, with a basis indexed by all noncrossing set partitions. The action of S_n on that basis could be understood in terms of skein-like relations described by Rhoades [39]. Patrias, Pechenik, and Striker [29] independently discovered an alternate algebraic/geometric model for the irreducible submodule of this action generated by singleton-free noncrossing set partitions sitting in the coordinate ring of a certain algebraic variety. They associated to each partition a polynomial in this coordinate ring defined in terms of matrix minors, and showed that these polynomials satisfied the skein relations described in [39]. This suggests the following problem:

Problem 3.5.4. *Can our basis for $S^{(n-k-1,k)}$ be realized as a set of polynomials, similarly to the methods of Patrias, Pechenik, and Striker [29]?*

One reason for thinking an analogous model might exist is that the relation of block operators described in Lemma 3.2.7 also appears in the maximal bidegree model and corresponds to a certain identity of two-by-two matrix minors in the work of Patrias, Pechenik and Striker. Their construction therefore extends to give a model for the submodule generated by partitions in $\Phi(n)$ for which the block containing n is at most size two, but we have as yet been unable to discover a treatment of larger blocks satisfying our other relations.

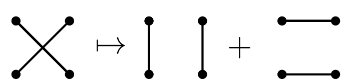
Chapter 3 is a reprint of the material as it appears in Combinatorial Theory, 2023. The dissertation author was the sole author.

Chapter 4

A skein action embedding

4.1 Introduction

This chapter concerns two actions of $\mathfrak{S}_n$. The first, due to Rhoades [39], is on the vector space with basis given by the set of noncrossing set partitions of $[n] := \{1, 2, \ldots, n\}$. We will refer to this action as the *skein action* on noncrossing set partitions as it is defined in terms of three *skein relations*, the simplest of which is the *Ptolemy relation* shown below.



The second is a well-known action on noncrossing matchings first studied by Rumer, Teller, and Weyl, then further developed by Temperley and Lieb, Kauffman, Kuperberg, and others [19, 25, 54, 58]. If V is the defining representation of SL_2, then the SL_2 invariants of $V^{\otimes n}$ have a basis, called the SL_2 *web basis* or *Temperley-Lieb basis*, indexed by noncrossing matchings. The $\mathfrak{S}_n$ action on $V^{\otimes n}$ which permutes tensor factors thus induces a $\mathfrak{S}_n$-action on the linear span of noncrossing matchings. Combinatorially, this action can be understood via the Ptolemy relation. A permutation in $\mathfrak{S}_n$ acts on a matching by swapping elements, then, if crossings were introduced, resolving those crossings via the Ptolemy relation.

The skein action on noncrossing set partitions was originally defined to provide a representation theoretic proof of a cyclic sieving result on noncrossing set partitions. Noncrossing set partitions of $[n]$ are counted by the Catalan numbers, and noncrossing set partitions of $[n]$ with exactly $n-k$ blocks are counted by the Narayana numbers:

$$N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

Reiner, Stanton and White [37] showed that a q-analogue of the Narayana numbers:

$$N(n,k,q) := \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q q^{k(k+1)}$$

exhibits the cyclic sieving phenomenon for the natural cyclic action on noncrossing set partitions with $n-k$ blocks. Their proof proceeded via direct calculation of $N(n,k,q)$ and sizes of fixed point sets; the skein action allowed for an alternate proof utilizing Springer's theorem on regular elements [39, 49]. The skein action has since been found within coinvariant rings and coordinate rings of certain partial flag varieties [21, 29], strengthening the claim that it is an action worth studying in its own right.

The skein action on noncrossing set partitions is defined combinatorially in an analogous way to the action on noncrossing perfect matchings. In fact, since noncrossing perfect matchings are a subset of noncrossing set partitions, it can be considered a generalization of the matching action to all noncrossing set partitions. To act by a transposition $(i,i+1)$ on a noncrossing matching, swap i and $i+1$, then if a crossing was introduced, use one of the following skein relations to resolve it, depending on the sizes of the blocks that cross:

Rhoades was able to determine the $\mathfrak{S}_n$-irreducible structure of the skein action on $mathbbC[NCP(n)]$, the span of noncrossing set partitions [39]. In particular, $\mathbb{C}[NCP(n)_0]$, the span of all singleton-free noncrossing set partitions with exactly k blocks is an $\mathfrak{S}_n$-irreducible of shape $(k,k,1^{n-2k})$, and the span of all noncrossing set partitions with exactly s singletons and exactly k non-singleton blocks is isomorphic to an induction product of $S^{(k,k,1^{n-2k-s})}$ with the sign representation of $\mathfrak{S}_s$. The structure of the noncrossing matching action is similar; the submodule

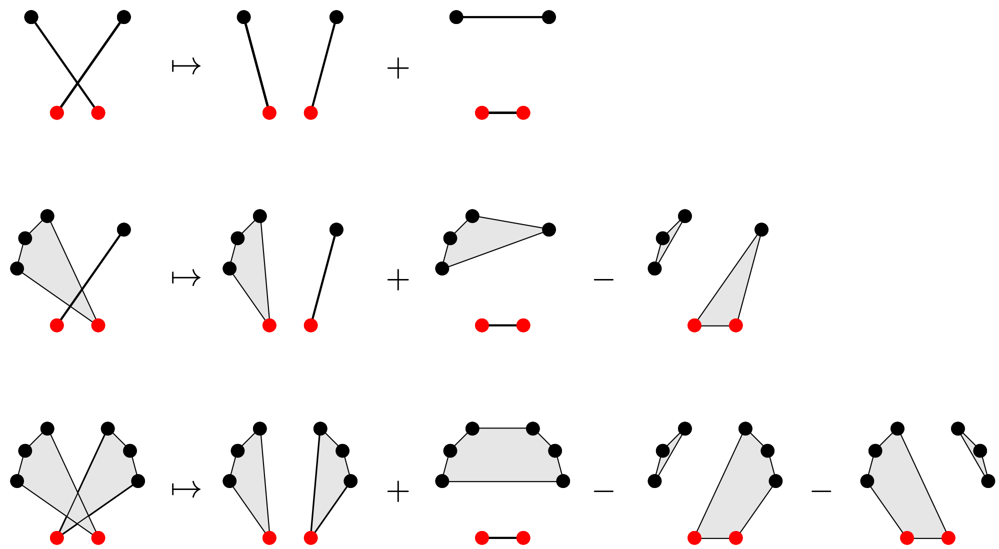


Figure 4.1. The three skein relations defining the action of $\mathfrak{S}_n$ on $\mathbb{C}[NCP(n)]$. The red vertices are adjacent indices $i, i+1$ and the shaded blocks have at least three elements. The symmetric 4-term relation obtained by reflecting the middle relation across the y-axis is not shown.

spanned by noncrossing matchings with exactly k pairs is isomorphic to the induction product of $S^{(k,k)}$ and a sign representation of $\mathfrak{S}_{n-2k}$. By the Pieri rule, this induction product is a direct sum of three irreducible submodules, one of which is isomorphic to $S^{(k,k,1^{n-2k})}$, so there exists a unique embedding of $\mathbb{C}[NCP(n)_0]$, the span of all singleton-free noncrossing set partitions in $\mathbb{C}[NCP(n)]$, into $\mathbb{C}[NCM(n)]$. The first main result of this chapter (appearing as Theorem 4.2.11 in section 3) explicitly describes the embedding as follows:

Theorem 4.1.1. *The linear map $f_n : \mathbb{C}[NCP(n)_0] \to \mathbb{C}[NCM(n)]$ defined by*

$$f_n(\pi) = \sum_{m \in M_\pi(n)} m$$

is an $\mathfrak{S}_n$-equivariant embedding of vector spaces. Here $M_\pi(n)$ is defined to be the set of all matchings m in $M(n)$ for which each block of π contains exactly one pair in m.

For an example of this map, let $\pi = \{\{1,2,3\},\{4,5\}\}$ then

$$f_n(\pi) = \{\{1,2\},\{4,5\}\} + \{\{1,3\},\{4,5\}\} + \{\{2,3\},\{4,5\}\}$$

is a sum of 3 matchings in $\mathbb{C}[NCM(n)]$. The proof of Theorem 4.1.1 also gives an alternate proof that the skein action on noncrossing set partitions is well-defined, see Remark4.2.12. The skein action being well-defined was originally shown through a laborious verification of the braid relations [39].

The second main result of this chapter (appearing as Theorem 4.3.4 in section 4) is to then describe the image of this map within $\mathbb{C}[NCM(n)]$. For this purpose, as well as the purpose of simplifying the proof of Theorem 4.1.1, it is helpful to introduce a multiplicative structure to $\mathbb{C}[NCM(n)]$, where multiplication corresponds to union when matchings are disjoint, and gives 0 otherwise. With this added structure, the image of f_n is a principal ideal:

Theorem 4.1.2. *Let H_n be the ideal of $\mathbb{C}[NCM(n)]$ generated by $f_n([n])$. Then*

$$\mathrm{im}(f_n) = H_n.$$

The SL_2 web basis has generalizations to other Lie groups. The first generalizations, to Lie groups of rank two and their quantum deformations was given by Kuperberg, with indexing sets given by certain planar diagrams embedded in a disk [25]. We propose a set of combinatorial objects which might serve as an analog of noncrossing set partitions for the SL_3 web basis, as their enumeration conjecturally matches the dimension of the Specht module $S^{(k^3,n-3k)}$.

The rest of the chapter is organized as follows. Section 2 will provide necessary background information. Section 3 will prove our first main result, the embedding from $\mathbb{C}[NCP(n)_0]$ to $\mathbb{C}[NCM(n)]$. Section 4 will determine the image of this embedding within $\mathbb{C}[NCM(n)]$. Section 5 will describe the conjectural analog for the SL_3 web basis.

4.2 The embedding

In order to prove that our map is an embedding, it will be helpful to introduce a multiplicative structure to work with. To do so we will introduce three commutative graded $\mathbb{C}$-algebras

R_n, A_n, and M_n, all with $\mathfrak{S}_n$-actions. If we forget the multiplicative structure, the underlying $\mathfrak{S}_n$-modules of R_n, A_n, and M_n will contain a copy $\mathbb{C}[\Pi(n)]$, $\mathbb{C}[M(n)]$, and $\mathbb{C}[NCM(n)]$ respectively. In the case of M_n, this copy will be all of M_n. The structure of this proof is best explained via a commutative diagram, see Figure 4.2. We will define a map $h_n \circ \iota_\Pi : \mathbb{C}[\Pi(n)_0] \to M_n$, and show that its kernel is equal to the kernel of p_Π. The desired embedding f_n then follows from the first isomorphism theorem.

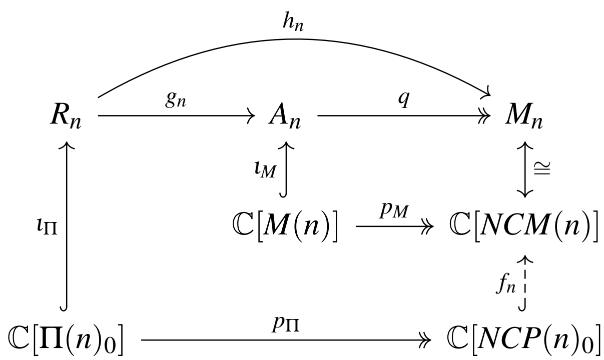


Figure 4.2. A commutative diagram of the maps used in the following proofs. All maps shown are $\mathfrak{S}_n$-equivariant linear maps. Maps between R_n, A_n, and M_n are also morphisms of $\mathbb{C}$-algebras. The desired embedding is shown as a dashed arrow.

We begin with the definition of R_n.

Definition 4.2.1. *Let $n \in \mathbb{N}$. Define R_n to be the unital commutative $\mathbb{C}$-algebra generated by nonempty subsets of $[n]$. Define a degree-preserving action of $\mathfrak{S}_n$ on R_n by*

$$\pi \cdot \{a_1, \ldots, a_k\} = \operatorname{sign}(\pi)\{\pi(a_1), \ldots, \pi(a_k)\}$$

for any permutation $\pi \in \mathfrak{S}_n$ and generator $\{a_1, \ldots, a_k\} \in R_n$.

The ring R_n can be thought of as the ring of multiset collections of subsets of $[n]$ with multiplication given by union of collections and addition purely formal. It is in this sense that it contains a copy of $\mathbb{C}[\Pi(n)]$, as set partitions of n are particular collections of subsets of $[n]$. To be precise, there exists an $\mathfrak{S}_n$-module embedding $\iota_\Pi : \mathbb{C}[\Pi(n)_0] \hookrightarrow R_n$, given by sending any singleton-free set partition π to the product of its blocks. For the proofs in this section, the main benefit of working with R_n instead of $\mathbb{C}[\Pi(n)]$ is that it allows us to work with only those two parts of a set partition which vary between terms in the skein relations, rather than carrying around excess notation for the unchanging parts.

The ring A_n is a subring of R_n designed to model matchings in much the same way which R_n models set partitions. It is defined as follows.

Definition 4.2.2. *Let $n \in \mathbb{N}$ and define A_n to be the $\mathfrak{S}_n$-invariant subalgebra of R_n generated by the size two subsets of $[n]$. The subring A_n is invariant under the $\mathfrak{S}_n$-action of R_n, and thus inherits a graded $\mathfrak{S}_n$-action from R_n.*

Like R_n, the ring A_n can be thought of as the ring of multiset collections of size-two subsets of $[n]$. As matchings are particular collections of size-two subsets of $[n]$, we again have an $\mathfrak{S}_n$-module embedding $\iota_M : \mathbb{C}[M(n)] \hookrightarrow A_n$, given by

$$\{\{a_1,b_1\},\ldots,\{a_k,b_k\}\} \longmapsto \{a_1,b_1\}\cdots\{a_k,b_k\}$$

for any matching $\{\{a_1,b_1\},\ldots,\{a_k,b_k\}\})$.

Our final ring, M_n, is defined as a quotient of A_n in the following way.

Definition 4.2.3. *Define I_n to be the ideal of A_n generated by elements of the following forms*

- $\{a,b\}\cdot\{a,b\}$

- $\{a,b\}\cdot\{a,c\}$

- $\{a,b\}\cdot\{c,d\}+\{a,c\}\cdot\{b,d\}+\{a,d\}\cdot\{b,c\}$

for any distinct $a,b,c,d \in [n]$. Then I_n is a $\mathfrak{S}_n$-invariant ideal of A_n, so define M_n to be the $\mathfrak{S}_n$-module $M_n := A_n/I_n$. Let $q : A_n \to M_n$ be the quotient map.

The first two types of elements listed in the definition of I_n serve the purpose of removing collections of size-two subsets which are not actually matchings. The third is the Ptolemy relation used to define the action of $\mathfrak{S}_n$ on $\mathbb{C}[NCM(n)]$, so quotienting by this ideal gives an $\mathfrak{S}_n$-module isomorphic to $\mathbb{C}[NCM(n)]$, as per the following argument.

Proposition 4.2.4. *There is an $\mathfrak{S}_n$-module isomorphism from $\mathbb{C}[NCM(n)]$ to M_n, given by*

$$\{\{a_1,b_1\},\ldots,\{a_k,b_k\}\} \longmapsto \{a_1,b_1\}\cdots\{a_k,b_k\}$$

for any noncrossing matching $\{\{a_1,b_1\},\ldots,\{a_k,b_k\}\}$.

Proof. Let $q : A_n \to M_n$ be the quotient map. Consider the map $q \circ \iota_M : \mathbb{C}[M(n)] \to M_n$. The kernel of $q \circ \iota_M$ is the preimage $\iota_M^{-1}(I_n)$. The image of ι_M is the span of all monomials consisting of nonintersecting generators, so $I_n \cap \iota_M$ is the linear span of elements of the form

$$(\{a,b\} \cdot \{c,d\} + \{a,c\} \cdot \{b,d\} + \{a,d\} \cdot \{b,c\})m$$

where $a,b,c,d \in [n]$ are distinct and m is a monomial not containing a,b,c,d. The kernel of $q \circ \iota_M$ is therefore spanned by the preimage of these elements. This is equivalent to the description of $\ker(p_M)$ given in Proposition 1.2.1, so the kernel of $q \circ \iota_M$ is equal to the kernel of p_M. The image of $q \circ \iota_M$ is all of M_n. To see this, note that products of generators of A_n form a vector space basis for A_n, and every such basis element is either in the image of ι_M or in I_n. We therefore have

$$\mathbb{C}[NCM(n)] \cong \mathbb{C}[M(n)]/\ker(p_M) = \mathbb{C}[M(n)]/\ker(q \circ \iota_M) \cong \operatorname{im}(q \circ \iota_M) = M_n \qquad (4.2.1)$$

where the isomorphism on the left is induced by the map p_M and the isomorphism on the right is induced by the map $q \circ \iota_M$. Composing these isomorphisms gives the stated map. $\square$

The following definition is the key idea behind our main theorem.

Definition 4.2.5. *Let $n \in \mathbb{N}$. Define the $\mathbb{C}$-algebra map $g_n : R_n \to A_n$ by*

$$g_n(A) = \sum_{\{a,b\} \subseteq A} \{a,b\}$$

for generators $A \in R_n$. Singleton sets are sent to 0 by g_n. Define $h_n := q \circ g_n$ where q is the quotient map $A_n \to M_n$.

We give the definition in terms of R_n, A_n, and M_n for simplicity and ease of proofs later, but the map we really care about is $h_n \circ \iota_\Pi : \mathbb{C}[\Pi(n)] \to M_n$. Under this map, a set partition π is sent to the product of its blocks, then each block is sent to the sum of all size-two subsets it contains. After distributing, we get a sum of all ways to pick a size two subset from each block. Composing with the isomorphism between M_n and $\mathbb{C}[NCM(n)]$ we get the sum of all matchings such that each block of π contains exactly one pair of the matching, as in Theorem 4.2.11.

We will now show that $h_n \circ \iota_\Pi$ factors through the projection map p_Π to produce an injective map. To do so, we will show that the kernels of these two maps agree. To show that the kernel of $h_n \circ \iota_\Pi$ contains the kernel of p_Π, we introduce an element of R_n modelling the five term skein relation depicted in Figure 4.1.

Definition 4.2.6. *Let $i, j \geq 2$ and let $p_1, p_2, \ldots, p_i$ and $q_1, q_2, \ldots, q_j$ be distinct in $[n]$. Define $\kappa_n \in R_n$ by*

$$\begin{aligned} \kappa_n := \quad & \{p_1, \ldots, p_i\} \cdot \{q_1, \ldots, q_j\} - \{p_1, \ldots, p_{i-1}\} \cdot \{q_1, \ldots, q_j, p_i\} \\ & - \{p_1, \ldots, p_i, q_j\} \cdot \{q_1, \ldots, q_{j-1}\} + \{p_1, \ldots, p_{i-1}, q_j\} \cdot \{q_1, \ldots, q_{j-1}, p_i\} \\ & + \{p_1, \cdots, p_{i-1}, q_1, \cdots, q_{j-1}\} \cdot \{p_i, q_j\} \end{aligned} \tag{4.2.2}$$

Note that κ_n is implicitly depending on a choice of $p_1, \ldots, p_i$, and $q_1, \ldots, q_j$, we omit these from the notation to avoid clutter.

When $i, j > 2$, the element κ_n corresponds to the five-term skein relation depicted in

Figure 4.1. If i equals 2, then $\{p_1, \cdots p_{i-1}\} = \{p_1\}$ is a one element set and therefore sent to 0 by h_n, removing the term containing $\{p_1\}$ corresponds to the four-term skein relation depicted in Figure 4.1. Similarly, if j equals 2 or i and j both equal two, removing the terms in κ_n which are individually sent to 0 corresponds to the four or three-term skein relation depicted in Figure 4.1.

We have the following calculation.

Proposition 4.2.7. *The element $\kappa_n \in R_n$ lies in the kernel of h_n.*

Proof. Applying h_n to κ_n gives

$$\begin{aligned} h_n(\kappa_n) = & \sum_{\substack{\{a,b\}\subseteq\{p_1,\ldots,p_i\} \\ \{c,d\}\subseteq\{q_1,\ldots,q_j\}}} \{a,b\}\cdot\{c,d\} \\ & - \sum_{\substack{\{a,b\}\subseteq\{p_1,\ldots,p_{i-1}\} \\ \{c,d\}\subseteq\{q_1,\ldots,q_j,p_i\}}} \{a,b\}\cdot\{c,d\} \\ & - \sum_{\substack{\{a,b\}\subseteq\{p_1,\ldots,p_i,q_j\} \\ \{c,d\}\subseteq\{q_1,\ldots,q_{j-1}\}}} \{a,b\}\cdot\{c,d\} \\ & + \sum_{\substack{\{a,b\}\subseteq\{p_1,\ldots,p_{i-1},q_j\} \\ \{c,d\}\subseteq\{q_1,\ldots,q_{j-1},p_i\}}} \{a,b\}\cdot\{c,d\} \\ & + \sum_{\substack{\{a,b\}\subseteq\{p_1,\ldots,p_{i-1},q_1,\cdots,q_{j-1}\} \\ \{c,d\}\subseteq\{p_i,q_j\}}} \{a,b\}\cdot\{c,d\} \end{aligned} \tag{4.2.3}$$

Note that the pairs of sets defining the first and second summations in the above expression differ only in the location of p_i, and similarly for the third and fourth. Since these summations come with opposite signs, the $\{a,b\},\{c,d\}$ terms in the above expression will cancel unless one of a,b,c,d is equal to p_i. Similarly, comparing the first and third sums and the second and fourth sums we find cancellation unless at least one of a,b,c,d is equal to q_j. If the remaining two elements of a,b,c,d are both p's or both q's, then $\{a,b\}\cdot\{c,d\}$ also cancels. Therefore we have

$$h_n(\kappa_n) = \sum_{\substack{a\in\{p_1,\dots,p_{i-1}\}\\ b\in\{q_1,\dots,q_{j-1}\}}} \{a,p_i\}\cdot\{b,q_j\} + \{a,q_j\}\cdot\{b,p_i\} + \{a,b\}\cdot\{p_i,q_j\} \tag{4.2.4}$$

which is manifestly a sum of the definining relations of M_n. □

To show that the kernel of $h_n \circ \iota_\Pi$ is no larger than the kernel of p_π, we will show that the images of singleton-free noncrossing set partitions under $h_n \circ \iota_\Pi$ are linearly independent. To do so, we introduce a term order on M_n.

Definition 4.2.8. *Define a total order on the generators of M_n as follows by*

- *If $a < b$, $c < d$, and $a < c$, then $\{a,b\} < \{c,d\}$*
- *If $a < b$, $c < d$, $a = c$ and $b > d$, then $\{a,b\} < \{c,d\}$*

Let $\leq$ denote lexicographic order on monomials of M_n with respect to this order on the generators. Note that $b > d$ in the second condition is not a typo, earlier generators have small smallest element and large largest element, e.g. $\{1,n\}$ is the first in this total order.

With this monomial order we have the following.

Proposition 4.2.9. *The set $\{h_n \circ \iota_\Pi(\pi) \mid \pi \in NCP(n)_0\}$ is linearly independent.*

Proof. By Proposition 4.2.4, M_n has a basis consisting of monomials corresponding to noncrossing matchings. We claim that the leading term of $h_n \circ \iota_\Pi(\pi)$ when expanded in this basis is unique. By the definition of the term order, the leading term of $h_n \circ \iota_\Pi(\pi)$ is the noncrossing matching obtained by matching the smallest element of each block of π to the largest element of the same block. We can recover π be placing every unmatched element j in a block with the matched pair $\{i,k\}$ for which $i < j < k$ and $k - i$ is minimal, and the result follows. □

Corollary 4.2.10. *The kernel of $h_n \circ \iota_\Pi$ is spanned by the set of all elements of the form $w \circ (s_i \circ \pi + \sigma(\pi))$ (the skein relations) for any permutation w, adjacent transposition s_i, and singleton-free almost noncrossing set partition π.*

Proof. By Proposition 4.2.7, all such elements lie in the kernel. By Proposition 4.2.9 and a dimension count it is no larger. □

We can now prove our main result.

Theorem 4.2.11. *The linear map $f_n : \mathbb{C}[NCP(n)_0] \to \mathbb{C}[NCM(n)]$ defined by*

$$f_n(\pi) = \sum_{m \in M_\pi(n)} m$$

is a $\mathfrak{S}_n$-equivariant embedding of vector spaces. Here $M_\pi(n)$ is defined to be the set of all matchings m in $M(n)$ for which each block of π contains exactly one pair in m.

Proof. By Corollary 4.2.10 and Proposition 1.3.1, the kernel of $h \circ \iota_\Pi$ is equal to the kernel of p_Π. So we have

$$\mathbb{C}[NCP(n)_0] \cong \mathbb{C}[\Pi_0(n)]/\ker(p_\Pi) \cong \operatorname{im}(h \circ \iota_\Pi) \subset M_n \cong \mathbb{C}[NCM(n)] \tag{4.2.5}$$

where the isomorphism on the left is induced by p_Π and the isomorphism on the right is induced by $h \circ \iota_\Pi$. Chasing these isomorphisms and inclusions results in the map f_n. □

Remark 4.2.12. *Theorem 4.2.11 gives an alternate proof that the skein action is well defined. Instead of defining the skein action via the skein relations and checking that it satisfies the braid relations, we can instead define it as the pullback of the action on M_n through f_n. Corollary 4.2.10 shows that this pullback can then be interpreted via the skein relations.*

4.3 The image

We have an embedding $f_n : \mathbb{C}[NCP(n)_0] \hookrightarrow \mathbb{C}[NCM(n)]$, so it is a natural question to ask for a description of the image of f_n within $\mathbb{C}[NCM(n)]$. Via the commutative diagram in Figure 2,we have an isomorphism of images

$$\mathrm{im}(h_n) \cong \mathrm{im}(f_n). \tag{4.3.1}$$

So it is equivalent to describe the image of h_n, and the multiplicative structure of M_n will make describing the image of h_n easier. This section will show that the image of h_n has a simple description as a principal ideal, the proof of which will require the following lemmas.

Lemma 4.3.1. *Let $A \subseteq [n]$. Then $h_n(A)^2 = 0$.*

Proof. Applying the definition of h_n gives

$$h_n(A)^2 = \sum_{\substack{a,b\in[n]\\ a\neq b}} \sum_{\substack{c,d\in[n]\\ c\neq d}} \{a,b\} \cdot \{c,d\} \tag{4.3.2}$$

Using the defining relation of M_n that

$$\{a,b\} \cdot \{a,c\} = 0$$

we have

$$\sum_{\substack{a,b\in[n]\\ a\neq b}} \sum_{\substack{c,d\in[n]\\ c\neq d}} \{a,b\} \cdot \{c,d\} = \frac{1}{3} \sum_{\substack{a,b,c,d\in[n]\\ a,b,c,d \text{ distinct}}} \{a,b\} \cdot \{c,d\} + \{a,c\} \cdot \{b,d\} + \{a,d\} \cdot \{b,c\}. \tag{4.3.3}$$

The right hand side of the above equation equals 0 because

$$\{a,b\} \cdot \{c,d\} + \{a,c\} \cdot \{b,d\} + \{a,d\} \cdot \{b,c\} = 0$$

for any distinct $a, b, c, d \in [n]$. □

Lemma 4.3.2. *Let A, B be disjoint subsets of $[n]$. Then*

$$h_n(A) \cdot \left(\sum_{\substack{a \in A \\ b \in B}} \{a, b\} \right) = 0.$$

Proof. Applying h_n gives

$$h_n(A) \cdot \left(\sum_{\substack{a \in A \\ b \in B}} \{a, b\} \right) = \frac{1}{3} \sum_{\substack{a_1, a_2, a_3 \in A \\ b \in B}} \{a_1, a_2\} \cdot \{a_3, b\} + \{a_1, a_3\} \cdot \{a_2, b\} + \{a_2, a_3\} \cdot \{a_1, b\} = 0$$

□

Lemma 4.3.3. *Let $B_1, \ldots, B_k$ be the blocks of a singleton free set partition of $[n]$. Then*

$$h_n \left(\prod_{i=1}^{k} B_i \right) = h_n \left([n] \cdot \prod_{i=1}^{k-1} B_i \right)$$

Proof. We have the following calculation:

$$\begin{aligned} h_n \left([n] \cdot \prod_{i=1}^{k-1} B_i \right) &= \left(\sum_{\substack{a, b \in [n] \\ a < b}} \{a, b\} \right) \cdot h_n \left(\prod_{i=1}^{k-1} B_i \right) \\ &= \left(\left(\sum_{i=1}^{k} h_n(B_i) \right) + \left(\sum_{1 \leq i < j \leq k} \sum_{\substack{a \in B_i \\ b \in B_j}} \{a, b\} \right) \right) \cdot h_n \left(\cdot \prod_{i=1}^{k-1} B_i \right) \\ &= h_n(B_k) \cdot \left(\cdot \prod_{i=1}^{k-1} h_n(B_i) \right). \end{aligned}$$

The last line follows by the preceeding two lemmas. Lemma 4.3.2 shows that every term in the outer sum of

$$\sum_{1 \le i < j \le k} \sum_{\substack{a \in B_i \\ b \in B_j}} \{a, b\}$$

is annihilated by some term in the product

$$\prod_{i=1}^{k-1} h_n(B_k).$$

Similarly, Lemma 4.3.1 shows that every term except the $i = k$ term in the sum

$$\sum_{i=1}^{k} h_n(B_i)$$

is annihilated by some term in the product

$$\prod_{i=1}^{k-1} h_n(B_k).$$

□

We can now describe the image of h_n.

Theorem 4.3.4. *Let H_n be the ideal of M_n generated by $h_n([n])$. Then*

$$\mathrm{im}(h_n) = H_n.$$

Proof. It is immediate from Lemma 4.3.3 that the image of h_n is contained in H_n, so it suffices to show that H_n is no larger. We will do so by showing the dimension of H_n is no larger than the dimension of the image of h_n, i.e.

$$\dim(H_n) \le \dim(\mathrm{im}(h_n)) = \dim(\mathrm{im}(f_n)) = \#NCP(n)_0 \tag{4.3.4}$$

We begin by finding a spanning set for H_n: note that for any fixed $a \in [n]$,

$$h_n([n]) \cdot \left(\sum_{\substack{b \in [n] \\ b \neq a}} \{a,b\} \right) = \frac{1}{3} \sum_{\substack{b \in [n] \\ b \neq a}} \sum_{\substack{c \in [n] \\ c \neq a}} \sum_{\substack{d \in [n] \\ d \neq a}} (\{a,b\} \cdot \{c,d\} + \{a,c\} \cdot \{b,d\} + \{a,d\} \cdot \{b,c\}) = 0$$

so

$$h_n([n]) \cdot \{1,a\} = -h_n([n]) \cdot \left(\sum_{\substack{b \in [n] \\ b \neq a,1}} \{a,b\} \right).$$

Let $M_n^{(2)}$ denote the subspace of M_n spanned by noncrossing matchings of $\{2,\dots,n\}$. By the above computation, H_n is spanned by elements of the form

$$h_n([n]) \cdot m$$

for $m \in M_n^{(2)}$.

The dimension of H_n is thus the rank of the map $M_n^{(2)} \to M_n$ given by multiplication by $h_n([n])$. To give an upper bound for the rank, we give a lower bound on the nullity.

Let $\tilde{\pi}$ be a set partition of $\{2,\dots,n\}$. Consider the element $\tilde{f}_n(\tilde{\pi})$ of $M_n^{(2)}$ given by

$$\tilde{f}_n(\tilde{\pi}) := \prod_{B \in \tilde{\pi}} h_n(B)$$

for any singleton free noncrossing set partition $\tilde{\pi}$ of $\{2,\dots,n\}$. The notation is meant to highlight that this is an analogous definition to the definition of f. We will show that $\tilde{f}_n(\tilde{\pi})$ is in the kernel of the multiplication by $h_n([n])$ map. Indeed, let B_1 be the block of $\tilde{\pi}$ containing 2, and let π be the set partition of $[n]$ obtained by adding 1 to block B_1. We have

$$\begin{aligned}
h_n([n]) \cdot \tilde{f}_n(\tilde{\pi}) &= h_n([n]) \cdot \prod_{B \in \tilde{\pi}} h_n(B) \\
&= h_n(B_1) \cdot h_n \left([n] \cdot \prod_{\substack{B \in \pi \\ B \neq B_1 \cup \{1\}}} B \right) \\
&= h_n(B_1) \cdot h_n \left(\prod_{B \in \pi} B \right) \\
&= h_n(B_1) h_n(B_1 \cup \{1\}) h_n \left(\prod_{\substack{B \in \pi \\ B \neq B_1 \cup \{1\}}} B \right) \\
&= 0
\end{aligned}$$

The third equality follows from Lemma 4.3.3 and the final equality follows from the fact that

$$h_n(B_1) h_n(B_1 \cup \{1\}) = h_n(B_1)^2 + h_n(B_1) \left(\sum_{b \in B_1} \{1, b\} \right) = 0$$

which follows from Lemma 4.3.2 and Lemma 4.3.1. The collection of $\tilde{f}_n(\tilde{\pi})$ for singleton-free noncrossing set partitions π of $\{2, \ldots, n\}$ is linearly independent. To see this, note that any linear relation among the $\tilde{f}_n(\tilde{\pi})$ would also be a linear relation among $f_{n-1}(\pi)$ where π is the set partition of $[n-1]$ obtained by decrementing the indices in $\tilde{\pi}$. But f_{n-1} is an embedding and singleton-free noncrossing set partitions are linearly independent in $\mathbb{C}[NCP(n-1)_0]$. Thus, the dimension of the kernel of multiplication by $h_n([n])$ is at least the number of singleton-free noncrossing set partitions of $\{2, \ldots, n\}$.

The dimension of H_n is therefore bounded by

$$\begin{aligned}
\dim(H_n) \leq{} & \#\{\text{noncrossing matchings of } \{2, \ldots, n\}\} \\
& - \#\{\text{singleton-free noncrossing set partitions of } \{2, \ldots, n\}\}. \quad (4.3.5)
\end{aligned}$$

Noncrossing matchings of $\{2,\ldots,n\}$ are in bijection with noncrossing set partitions of $[n]$ in which only the block containing 1 may be a singleton (though it may be larger). Given a noncrossing set partition, take the matching that matches the largest and smallest element of each block not containing 1. Singleton-free noncrossing set partitions of $\{2,\ldots,n\}$ are in bijection with set partitions of $[n]$ in which $\{1\}$ is the unique singleton block. We therefore have

$$\begin{aligned}\#\{\text{singleton-free noncrossing set partitions of } [n]\} = &\\ &\#\{\text{noncrossing matchings of } \{2,\ldots,n\}\}\\ &-\#\{\text{singleton-free noncrossing set partitions of } \{2,\ldots,n\}\} \quad (4.3.6)\end{aligned}$$

and

$$\dim(H_n) \leq \#\{\text{singleton-free noncrossing set partitions of } [n]\}$$

as desired. □

4.4 Future directions

One of the goals motivating this chapter is to find new combinatorially nice bases for $\mathfrak{S}_n$-irreducibles which arise from existing bases in an analogous way to the skein action. More specifically, suppose we have a basis for S^λ which is indexed by certain structures on the set $[k]$, where $k = |\lambda|$ (e.g. noncrossing perfect matchings, in the case of this chapter). We can create a basis for the induction product of S^λ with a sign representation of $\mathfrak{S}_{n-k}$ indexed by all ways to put a certain structure on a k-element subset of $[n]$. The Pieri rule tells us which $\mathfrak{S}_n$ irreducibles this decomposes into. In particular, there will be one copy of $(\lambda, 1^{n-k})$. How do we isolate that irreducible?

It is perhaps optimistic to think that there will be a method that works in any sort of generality, but analogs may be found in some cases. For example, an analog might exist for the $SL(3)$-web basis for $S^{(k,k,k)}$ introduced by Kuperberg [25]. The web basis consists of planar bipartite graphs embedded in a disk with n boundary vertices all of degree 1, interior vertices are degree 3, all boundary vertices are in the same part of the bipartition, and no cycles of length less than 6 exist. One potential candidate for a basis for $S^{(k,k,k,1^{n-3k})}$ is as follows.

Conjecture 4.4.1. *Let A be the set of all planar bipartite graphs embedded in a disk for which the following conditions hold*

- *There are n vertices on the boundary of the disk, and there exists a bipartition in which all of these vertices are in the same part.*
- *Every interior vertex in the same part of the bipartition as the boundary vertices is degree 3. These are called negative interior vertices.*
- *Every interior vertex not in the same part of the bipartition as the boundary vertices is degree at least 3. These are called positive interior vertices.*
- *The number of positive interior vertices minus the number of negative interior vertices is exactly k.*
- *No cycles of length less than 6 exist.*

Then $|A|$ is equal to the dimension of $S^{(k,k,k,1^{n-3k})}$.

The set A can be thought of as consisting of webs for which the condition of interior vertices being degree 3 has been partially relaxed. The conjecture can be shown to hold for $k = 2$ and any n, as well as $n = 10, k = 3$ via direct enumeration. If the above conjecture is true, it suggests the following question.

Question 4.4.2. *Does there exist a combinatorially nice action of $\mathfrak{S}_n$ on $\mathbb{C}[A]$ which creates a $\mathfrak{S}_n$ module isomorphic to $S^{(k,k,k,1^{n-3k})}$? If so, what does the unique embedding into $S^{(k,k,k)}$ induced with a sign representation of $\mathfrak{S}_{n-3k}$ look like?*

A positive answer to this question might help elucidate how to apply similar methods more generally.

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Chapter 5

Augmented webs

5.1 Introduction

This chapter introduces a new *web basis* for a family of $\mathfrak{S}_n$ modules indexed by partitions $(d,d,d,1^{n-3d})$. The systematic study of web bases began with work of G. Kuperberg [25] in order to study the space of invariant tensors for simple Lie algebras and their quantum groups, though examples which are now considered web bases predate the term. What exactly constitutes a web basis differs somewhat between authors, we will use a list of properties laid out by C. Fraser, R. Patrias, O. Pechenik, and J. Striker in [11]. The properties they give are:

(1) Each basis element is indexed by a planar diagram with n boundary vertices, embedded in a disk.

(2) There is a topological criterion allowing identification of basis diagrams.

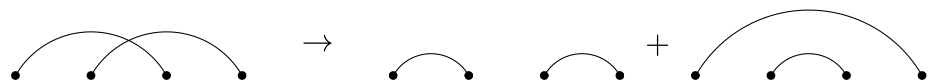
(3) The action of the long cycle $c = (12\ldots n)$ on the basis is by rotation of diagrams.

(4) The action of the long element $w_0 \in n(n-1)\ldots 1$ on the basis is by reflection of diagrams.

(5) There is a finite list of 'skein relations' describing the action of a simple transposition s_i on a basis diagram.

The simplest web basis is the Temperley-Lieb basis for two-row rectangle shapes (d,d), indexed by noncrossing perfect matchings of $2d$ vertices and studied by a variety of authors

[54, 58, 25, 38]. One useful property of the Temperley-Lieb basis is that it makes computation of the action of $\mathfrak{S}_n$ easy: to act by a permutation on a basis element, simply permute the matching, potentially introducing crossings, then resolve each crossing by replacing it with an uncrossing in both possible ways. This crossing resolution is called a skein relation, shown below.



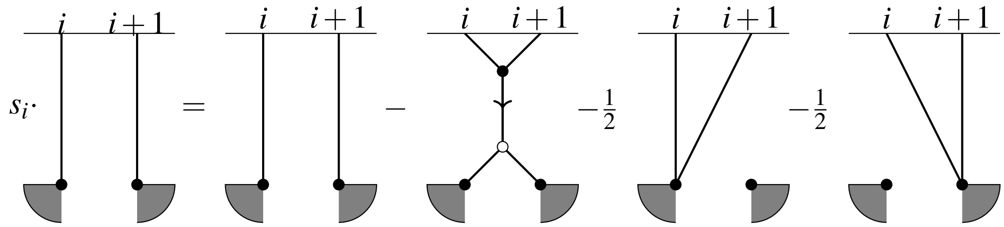
Kuperberg introduced similar bases for invariant spaces for rank-two Lie algebras. In this chapter we will primarily be interested in the type A_2; the Temperley-Lieb basis is type A_1. In type A_2, Kuperber's basis is indexed by bipartite trivalent planar graphs with no faces of degree less than 6, called nonelliptic SL_3 webs. For each nonelliptic SL_3 web, the corresponding element of $V^{\otimes n}$, where V is the three-dimensional defining representation of SL_3, is defined either recursively, in terms of tensor product and contraction, or combinatorially, in terms of a weighted sum over all proper edge coloring of the web with 3 colors. These form a basis for the space of SL_3 invariants of $V^{\otimes n}$, where SL_3 acts diagonally on $V^{\otimes n}$. The symmetric group $\mathfrak{S}_n$ acts on this invariant space by permuting tensor factors. As an $\mathfrak{S}_n$ module, the SL_3 invariant space of $V^{\otimes n}$ is irreducible and isomorphic to the Specht module of shape (d,d,d) where $n = 3d$, and is 0 if n is not a multiple of 3. Various ways to generalize this construction to SL_n have been studied [9, 5], though a rotation invariant version is known only for n up to 4 [13].

Rhoades generalized the Temperley-Lieb basis in a different direction, giving a web basis for shapes $(d,d,1^{n-2d})$ indexed by noncrossing set partitions with parts of size at least two [39]. In Rhoades' action, crossing resolution involved four skein relations based on the sizes of the crossing blocks. In previous work with Rhoades [22], we showed that this action of noncrossing set partitions could be found within the top degree component of the fermionic diagonal coinvariant ring. One main goal of this chapter is to generalize Kuperberg's SL_3 webs in an analogous way to Rhoades' generalization of the Temperley-Lieb basis.

To do so, we build upon work of R. Patrias, O. Pechenik, and J. Striker. In [29], they introduce *jellyfish invariants*, giving an alternate construction of Rhoades' basis within the homogeneous coordinate ring of a 2-step partial flag variety. Along with C. Fraser, they further develop these jellyfish invariants, reinterpreting them within the homogeneous coordinate ring of a Grassmanian [11]. They also extend their definitions to give jellyfish invariants living within a Specht modules indexed by a partition of shape $(d^r, 1^{n-rd})$. They dub these *flamingo Specht modules* as the partitions indexing them appear to stand on one leg. In the case $r > 2$, they do not give a basis for this module. Instead, they give a linearly independent set indexed by noncrossing set partitions and a spanning set indexed by all set partitions.

Our first result is to extend their linearly independent set to a basis of $S^{(d^r,1^{n-rd})}$ in the case $r > 2$. We do so by replacing the noncrossing condition with a weaker one based on ideas introduced by P. Pylyavskyy in [35], which we call *r-weakly noncrossing*. This basis fails to have the rotation and reflection invariance desired of a web basis, however. Our second result is to remedy the lack of rotation invariance in the case $r = 3$ by introducing a second basis indexed by a certain rotationally invariant set $AW(n,d)$ of normal plabic graphs we call *augmented SL_3 webs*, as they closely resemble Kuperberg's SL_3 webs with extra edges. This resolves a conjecture made in [21]. Plabic graphs were first introduced by A. Postnikov [34] in order to study the totally nonnegative Grassmanian; we use the combinatorial machinery developed for them to show that our indexing set has the correct enumeration. To define our basis, we use a modification of proper edge colorings for SL_3 webs which we call *consistent labellings*. Consistent labellings are closely related to the weblike subgraphs introduced by T. Lam in order to define SL_3 web immanants and later used by C. Fraser, T. Lam, and I. Le to introduce a higher rank version of Postnikov's boundary measurement map [26, 10], and we make this connection explicit.

Through consideration of the combinatorics of conistent labellings, we obtain skein relations for augmented SL_3 webs. These skein relations give a combinatorial description of the action of an adjacent transposition on an augmented web. The first skein relation, the crossing reduction rule shown below, shows how to expand the application of an adjacent transposition to an augmented web in the augmented web basis. The gray region represents an unknown number of edges connecting to other vertices of the web not depicted.



Note that not all terms on the right hand side of this relation are necessarily augmented webs, as black vertices of degree less than 3 or faces of degree 4 may be created. The remaining skein relations, which can be found in Section 6, show how to expand such terms when they arise.

One application of our rotationally invariant basis in the case $r = 3$ is that it gives us a cyclic sieving result on the indexing set. Let $X_{n,d}(q)$ denote the q-analog of the hook length formula for $\lambda = (d^3, 1^{n-3d})$,

$$X_{n,d}(q) := q^{3(d-1)+\binom{n-3(d-1)}{2}} \frac{[n]!_q}{\prod_{(i,j)\in\lambda}[h_{ij}]_q}$$

where h_{ij} denotes the hook length of a box (i, j) in the Young diagram for λ. We show that the triple $(AW(n,d), C, X_{n,d}(q))$ exhibits the cyclic sieving phenomenon when n is odd, and a signed version of cyclic sieving holds when n is even. Specializing to the case $n = 3d$ recovers a cyclic sieving result on SL_3 webs studied by T.K. Petersen, Pylyasky, and Rhoades in [33].

The rest of the chapter is organized as follows. In Section 2, we review necessary background information. In Section 3 we give a definition of r-weakly noncrossing set partitions and give a basis of $S^{(d^r,1^{n-rd})}$ which extends the jellyfish invariant basis. In Section 4, we define *augmented webs* as a certain subset of normal plabic graphs and give a combinatorial bijection between them and 3-weakly noncrossing set partitions. This bijection will draw on ideas developed by J. Tymoczko and H. Russell to give a bijection between SL_3 webs and objects called *m-diagrams*, a special case of our 3-weakly noncrossing set partitions [44, 56]. In Section 5, we define an SL_3-invariant polynomial attached to each normal plabic graph. We show that this definition extends jellyfish invariants and that the set of invariants attached to augmented webs satisfy properties (3) and (4) of a web basis. In Section 6, we show that skein relations hold for our plabic graph invariants. We use these skein relations to show that augmented web invariants are indeed a basis for $S^{(d^3,1^{n-3d})}$. In Section 7, we show that our augmented web invariants can be interpreted in terms of weblike subgraphs. In Section 8, we discuss the cyclic sieving result for augmented webs which arises from our rotationally invariant basis. In Section 9, we discuss some possible future directions for this work.

5.2 Weakly-noncrossing set partitions

In this section, we extend the linearly independent set given in [11] to a basis by introducing a weaker version of the noncrossing condition for ordered set partitions. The weaker version is similar to the noncrossing tableau defined by P. Pylyavskyy in [35]. Our version will differ in that it will depend on r.

Definition 5.2.1. *Let $A = \{a_1, a_2, \ldots, a_{|A|}\}$ and $B = \{b_1, b_2, \ldots, b_{|B|}\}$ be two subsets of $[n]$ each of size $\geq r$. We say that A and B are r-weakly noncrossing if the following holds:*

1. *For each $i = 1 \ldots, r-2$, The arc (a_i, a_{i+1}) does not cross the arc (b_i, b_{i+1}).*

2. *For any $j_1, j_2 \geq r$, the arc (a_{r-1}, a_{j_1}) does not cross the arc (b_{r-1}, b_{j_2}).*

An (ordered) set partition is r-weakly noncrossing if its blocks are pairwise r-weakly noncrossing.

One can think of this definition as being noncrossing in the sense of Pylyavskyy in the first $r-2$ entries, and noncrossing in the strong sense in the remaining entries.

Let $WNC(n,d,r)$ denote the set of all set partitions of $[n]$ into d blocks each of size at least r which are r-weakly noncrossing.

We first show that the set of r-weakly noncrossing set partitions is the correct size:

Proposition 5.2.2. *There is a bijection between standard Young tableaux of shape $(d^r, 1^{n-rd})$ and r-weakly noncrossing set partitions in $WNC(n,d,r)$.*

Proof. We show that both sets are in bijection with a set of rectangular tableaux filled with a subset of $[n]$:

Definition 5.2.3. *Let $T(n,d,r)$ denote the set of all tableaux of shape $\lambda = (d^r)$ filled with with integers in $[n]$ such that*

1. *Entries increase along rows and down columns.*

2. *No element of $[n]$ appears more than once.*

3. *For any i which does not appear in the tableaux, the number of entries $j < i$ appearing in row $r-1$ strictly exceeds the number of entries $j < i$ appearing in row r.*

Example 5.2.4. *Consider the two tableaux below.*

1	3	6	7
2	4	8	11
9	13	14	16

1	3	6	7
2	4	11	13
5	8	14	16

The tableau on the left is an element of $T(16,4,3)$. The tableau on the right is not in $T(16,4,3)$ because 9 does not appear as a filling and there are the same number of fillings less than 9 in the second and third rows, highlighted in gray.

The bijection between $SYT(d^r, 1^{n-rd})$ and $T(n,d,r)$ is as follows. Let t be a standard Young tableaux of shape $(d^r, 1^{n-rd})$.

1. If removing the blocks in rows larger than r (which we will refer to as the *tail*) produces a tableaux in $T(n,d,r)$, do so.

2. Otherwise, let i be the maximal element among the tail for which the number of elemenents $j < i$ in row $r-1$ equals the number of elements $j < i$ in row r. Remove the first block of row r and all blocks below it, shift all blocks in row r filled with $j < i$ one space to the left, and place a block filled with i in the newly formed opening.

The maximality of i will guarantee that the third property of $T(n,d,r)$ is satisfied for elements larger than i, and the shifting left will guarantee it is satisfied for elements smaller than i. Call the resulting tableau $f(t)$.

To reverse this process, let t' be a tableaux in $T(n,d,r)$. We obtain a standard Young tableaux of shape $(d^r, 1^{n-rd})$ as follows.

1. If the smallest element which does not appear in t' is larger than the entry in the first box of row r, simply append all integers not already appearing in the tableau in increasing order as the tail.

2. Otherwise, let i be the minimal filling in row r which is smaller than the filling of the box one space up and to the right, or the largest element of row r if no such element exists. Remove the box filled with i, shift all boxes to the left of it one space right, and insert the remaining entries in increasing order to form the tail.

The right shift will preserve the standard Young tableau property due to the minimality of i. Call the resulting tableau $g(t')$.

To verify that these two maps are indeed inverses, let $t \in SYT(d^r, 1^{n-rd})$. If removing the tail of t produces a tableau in $T(n,d,r)$, then it is clear that $g(f(t)) = t$. Otherwise, let i be the element inserted into row r to obtain $f(t)$. Before this insertion, there were the same number of elements less than i in row $r-1$ and row r of the tableau, so the filling one space up and to the right of i in $f(t)$ must be larger than i. Additionally, all boxes j to the left of i were shifted over, and since we started with a standard Young tableau, the filling one space up and to the right of them is smaller than j. Therefore, i is the filling removed by g, and $g(f(t)) = t$.

A similar argument shows that $f \circ g$ is also the identity. Indeed, if i was the element removed from row r by g, then there must be the same number of elements $j < i$ in row $r-1$ and r of $g(t')$, and no other element larger than i can have this propery as t' is in $T(n,d,r)$.

Example 5.2.5. *An example of this bijection is given below for $n = 16, d = 4, r = 3$. 12 is the maximal filling of the tail for which the second and third rows contain the same number of lesser fillings.*

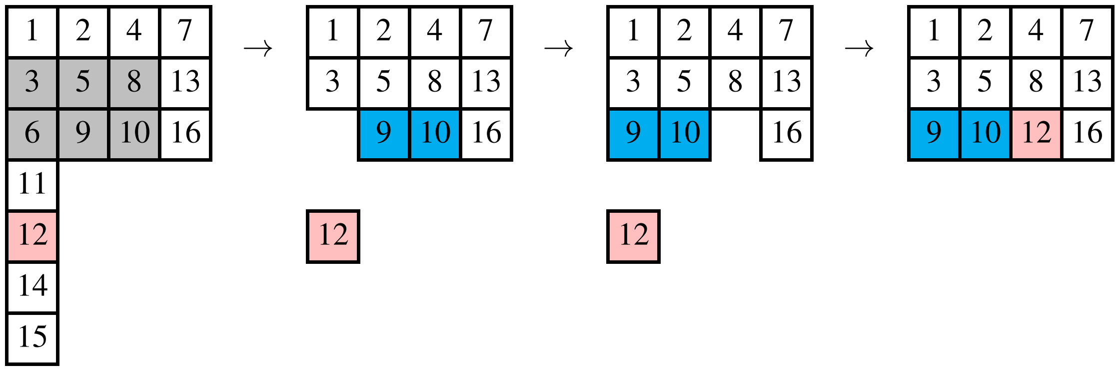


Figure 5.1. An example of the bijection between $SYT(d^r, 1^{n-rd})$ and $T(n,d,r)$.

The bijection between $T(n,d,r)$ and $WNC(n,d,r)$ is essentially repeated applications of the standard Catalan bijection between two row rectangular standard Young tableaux and noncrossing matchings. Given a tableau $t \in T(n,d,r)$, for $i = 1,\ldots,r$, let $R_i(t)$ denote the entries in row i of t. Place the numbers 1 through n in a line, and for each $i = 1,\ldots,r-1$, draw d arcs between elements of R_i and R_{i+1} such that

1. Elements of R_i are the left endpoints of arcs, and elements of R_{i+1} are the right endpoints of arcs.

2. There do not exist two arcs (a,b) and (c,d) such that $a < c < b < d$.

The standard Catalan bijection argument guarantees that this is uniquely possible. Then, for each positive integer m at most n not appearing in t, there is a unique shortest arc (a,b) created at step $r-1$ such that $a < m < b$. The third condition of $T(n,d,r)$ guarantees that such an arc exists, and the noncrossing condition above guarantees it is unique. Draw the arc (a,m). Finally, create a set partition π by placing all integers connected by arcs into the same block. Then $\pi \in WNC(n,d,r)$. To see that the noncrossing condition is satisfied, note that for $i = 1,\cdots,r-2$ if a_i and a_{i+1} are the i^{th} and $(i+1)^{th}$ smallest elements of a block of π, then they must necessarily be connected by an arc created at step i in the above process.

The inverse of this bijection is simple, given a set partition $\pi \in T(n,d,r)$ place the smallest element of each block in increasing order in row 1, the second smallest in row 2, and so on, up to row $r-1$. Finally, place the largest element of each block in row r.

Example 5.2.6. *Consider the tableau shown below.*

1	2	4	7
3	5	8	11
9	10	12	16

We get the following arc diagram. Arcs created by matching the first two rows are shown in red, arcs created by matching the second and third rows are shown in green, and arcs created by connecting missing entries are shown in blue.

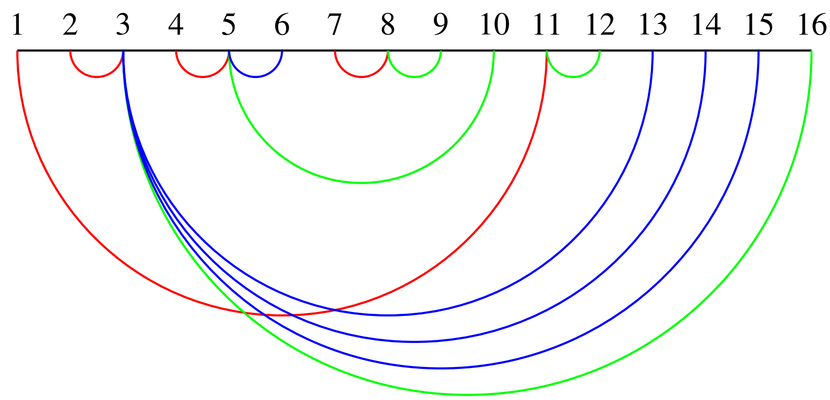


Figure 5.2. The arc diagram in the bijection from $T(n,d,r)$ to $WNC(n,d,r)$.

The resulting set partition is $\{\{1,11,12\},\{2,3,13,14,15,16\},\{4,5,6,10\},\{7,8,9\}\}$.

□

The second half of the proof of Proposition 5.2.2 also gives the following corollary, which we will need later:

Corollary 5.2.7. *A r-weakly noncrossing set partition γ is uniquely determined by the $r-1$ sets*

$$\{m \mid m \text{ is the } i^{th} \text{ smallest element of some block of } \gamma\}$$

for $i = 1, \dots, r-1$*, along with the set*

$$\{m \mid m \text{ is the largest element of some block of } \gamma\}$$

.

Proof. The information in these sets determines the elements of each row of the tableau in $T(n,d,r)$ as defined in the proof of Proposition 5.2.2. Placing elements in increasing order within each row recovers the tableau, and thus the set partition. □

Example 5.2.8. *Suppose* $n = 7$*,* $d = 3$*,* $r = 2$*, and thus* $\lambda = (2,2,2,1)$ *. There are fourteen standard Young tableaux of shape* λ*, and fourteen 2-weakly noncrossing set partitions. The bijection between them is shown in Figure 5.3, with the intermediary tableau in* $T(7,3,2)$ *and arc diagram shown as well.*

We can now define and prove our basis.

Theorem 5.2.9. *Let* $r \geq 2$*. Order each weakly noncrossing set partition* $\gamma \in WNC(n,d,r)$ *to create an ordered set partition* π_γ*. Then the set* $\{[\pi_\gamma]_r \mid \gamma \in WNC(n,d,r)\}$ *is a basis for the flamingo Specht module* $S^{(d^r,1^{n-rd})}$*.*

Proof. By Proposition 5.2.2 and Theorem 1.4.2, it suffices to show that $\{[\pi_\gamma]_r \mid \gamma \in WNC(n,d,r)\}$ is linearly independent. To do so, we introduce a monomial order and show that under this order, each $[\pi_\gamma]_r$ has a unique leading term.

Recall that the r-jellyfish invariant is a polynomial in the $\nu \times n$ variables:

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & & \ddots & \vdots \\ x_{\nu,1} & x_{\nu,2} & \cdots & x_{\nu,n} \end{bmatrix}$$

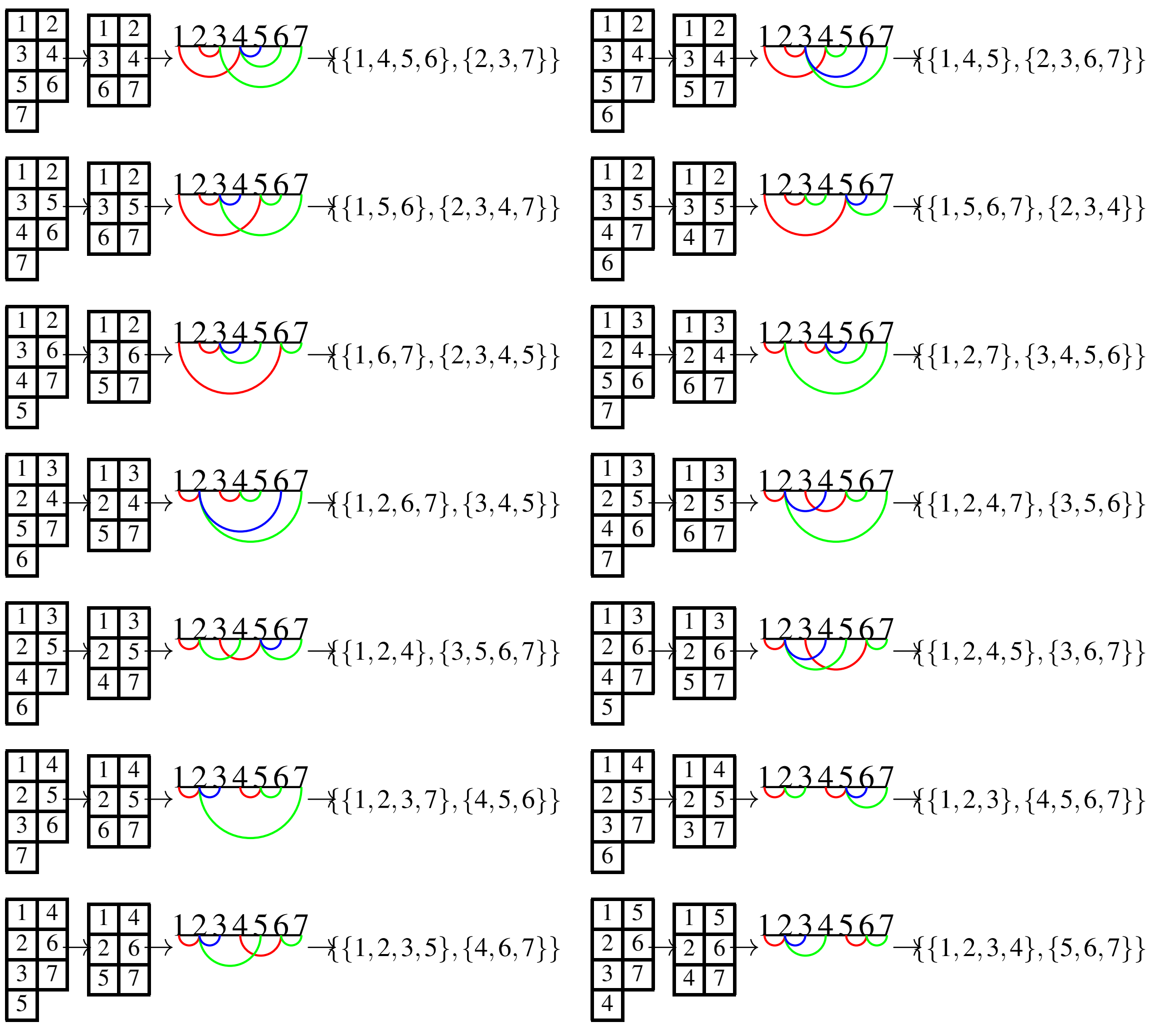


Figure 5.3. The bijection between $SYT(2,2,2,1)$ and $WNC(7,2,3)$.

We order these variables in a somewhat unusual way. Define an order on these variables by $x_{i_1,j_1} < x_{i_2,j_2}$ if and only if one of the following holds:

1. $i_1 < i_2$
2. $i_1 = i_2 \neq r$ and $j_1 < j_2$
3. $i_1 = i_2 = r$ and $j_1 > j_2$

In other words, we order them in reading order except we read the r^{th} row backwards. We then take the lexicogrpahic monomial order with respect to this ordering of variables. The unusual ordering is chosen to make use of Corollary 5.2.7. Without reversing the r^{th} row, lexicogrpahic leading terms are not unique.

Let the i^{th} block of γ be

$$\gamma_i := \{\gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,|\gamma_i|}\}$$

Let T be a jellyfish tableau associated to γ. Then the leading term of $J(T)$ is straightforward to compute from the definition, we have

$$\mathrm{lt}(J(t)) = \left(\prod_{i=1}^{d}\prod_{j=1}^{r-1} x_{j,\gamma_{i,j}}\right) \prod_{i=1}^{d} x_{r,\gamma_{i,|\gamma_i|}} \prod_{j=r+1}^{v} x_{j,u_j}$$

where u_j is the entry appearing in row j of $J(T)$. In words, for $i = 1, \ldots, r-1$, $x_{i,j}$ will appear if and only if j is the i^{th} smallest element of some block of γ, and $x_{r,j}$ will appear if and only if j is the largest element of some block of γ. The leading term of $[\pi_\gamma]_r$ will be the leading term of one of these $J(T)$, and we can see that the leading term contains the information of the sets described in Corllary 5.2.7. Thus, the leading term of $[\pi_\gamma]_r$ is unique and thus $\{[\pi_\gamma]_r \mid \gamma \in WNC(n,d,r)\}$ is linearly independent as desired. □

Remark 5.2.10. *The property of being r-weakly noncrossing is not preserved under rotation, for example* $\{1,4,5\},\{2,3,6\}$ *is weakly 3-noncrossing, but* $\{\{1,3,4\},\{2,5,6\}\}$ *is not. So the basis* *given in Theorem 5.2.9 is not rotation invariant as desired of a web basis. The next section will give a different basis which is rotation invariant in the case $r = 3$.*

5.3 Augmented Webs

For the rest of the chapter, we specialize to the case $r = 3$. We will introduce a new basis for this case which is rotation and reflection invariant. To index our basis, we introduce a subset normal plabic graphs which we call *augmented webs*. We call them augmentd webs to allude to the fact that they are similar to SL_3 webs, but potentially with vertices of higher degree. We will show that augmented webs are in bijection with 3-weakly noncrossing set partitions, and thus have the correct enumeration to index a basis of $S^{(d^3,1^{n-3d})}$. The benefit of working with augmented webs over 3-weakly noncrossing set paritions is that the set of augmented webs is rotation invariant.

Definition 5.3.1. *An* augmented web *is a normal plabic graph which contains no faces of degree less than 6 and no black vertices of degree less than 3. The* exceedance *of an augmented web is the number of black vertices minus the number of white vertices. Let $AW(n,d)$ denote the set of all augmented webs with n boundary vertices and exceedance d.*

Remark 5.3.2. *The term* exceedance *is chosen because the exceedance of an augmented web is also the number of exceedances in the trip permutation.*

Remark 5.3.3. *When an augmented web has no white vertices, it contains exactly the same information as a strongly noncrossing set partition, with the sets of all boundary vertices connected to a particular interior vertex forming the blocks.*

5.3.1 Combinatorial properties of augmented webs

In this subsection, we develop combinatorial results for augmented webs. Our first result is that all augmented webs are reduced plabic graphs.

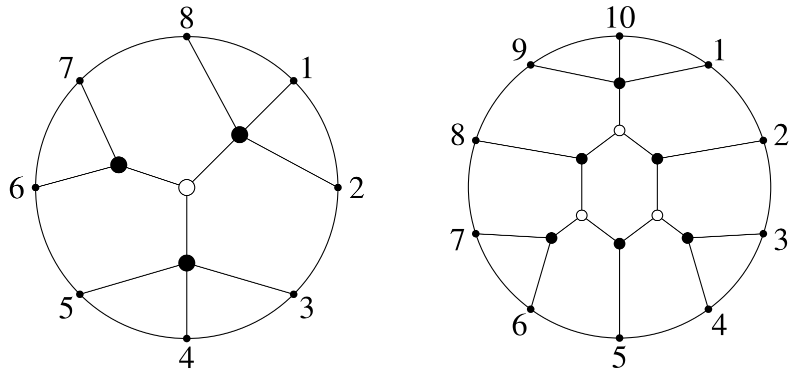


Figure 5.4. Examples of augmented in $A(8,2)$ and $A(10,3)$.

Proposition 5.3.4. *Let $W \in AW(n,d)$. Then W is reduced.*

Proof. As no square faces or vertices of degree two are present, normal plabic graph moves are not possible. Thus it suffices to check that W itself has no forbidden configurations and this is clear. □

Next, we show that augmented webs have an inductive structure we can exploit.

Lemma 5.3.5. *Let $W \in AW(n,d)$, and suppose W has at least one white vertex. Then for each connected component of W, there exists at least two black vertices each connected to exactly one white vertex.*

Proof. Let C be a maximal cycle in W, i.e. a cycle with no edge incident to an interior face of W. Since W is reduced by Proposition 5.3.4, it contains no round trips or essential self-intersections. Therefore there are at least two black vertices of W exterior to C which connect to a white vertex in C. Create a graph G with two types of vertices: a vertex for every vertex of W which is on the exterior of every cycle in W, and a vertex for every maximal cycle. Add an edge between a vertex v and a maximal cycle C whenever v is adjacent to a vertex in C. Then G is a forest, and since W had at least one white vertex, G has at least one edge. So G has two leaves, and these two leaves must be black vertices connected to exactly one white vertex. □

The use of Lemma 5.3.5 is that every augmented web with at least one white vertex can be built out of an augmented web with one fewer white vertices in the following way. Let u and v be the black vertex and its white neighbor identified by Lemma 5.3.5. If we remove vertices u and v, then connect the neighbors of v to the boundary by at least one edge each in a planar way, we get an augmented web W'.

We will need a notion of depth of a face, which we now define.

Definition 5.3.6. *Let $W \in AW(n,d)$. Let f be a face of W, and let f_0 be the face connected to the section of boundary between 1 and n. The* depth *of f is the number of exceedances which separate f from f_0.*

Let e be an edge of W. We say that e is a depth boundary edge *if the depth of the faces incident to e are not equal. Equivalently, e is a depth boundary edge if exactly one of the trips passing through e is an exceedance. We say e is a* left-to-right *depth boundary edge if, when oriented towads its black vertex endpoint, the depth of the face on the right is higher than the depth of the face on the left. Equivalently, e is a left-to-right depth boundary if only the trip passing through e from towards its black vertex is an exceedance. We define* right-to-left *deth boundary edges similarly.*

Lemma 5.3.7. *Let $W \in AW(n,d)$. Let v be an interior vertex of W. Then exactly two edges incident to v are depth boundary edges.*

Proof. Consider the set of all trips $t_1,\dots,t_k$ passing through v, ordered cyclically. Since W is reduced, the starts of all these trips must appear in the same cyclic order around the boundary of W, since otherwise we would introduce a bad double corssing. Similarly, the ends of these trips appear in the same cyclic order around W. Therefore, the set of exceedances passing through v is a proper nonempty subset of these trips which is cyclically consecutive around v. The first and last of these trips will contribute a depth boundary edge. □

Lemma 5.3.8. *Let $W \in AW(n,d)$. Let u and v be two adjacent vertices of W. Let t_u be any trip* *which passes through u but does not use edge (u,v), and let t_v be any trip which passes through v but does not use edge (u,v). Then trips t_u and t_v do not share any vertices.*

Proof. This follows from Euler's formula for planar graphs. Assume the contrary, that trips t_u and t_v meet at some vertex x. Let C be the cycle formed from t_u, t_v and edge (u,v), and suppose it is of length k. Consider the graph G containing all vertices and edges of W that are part of C or in its interior. Let V_{int} denote the number of vertices strictly in the interior of C, and let α be the average degree of these interior vertices. Then we have

$$|E(G)| \geq \frac{5}{4}k + \frac{\alpha}{2}V_{\text{int}} - 1$$

and thus by Euler's formula the number of faces of G not including the external face is at least $\frac{1}{4}k + \frac{\alpha-2}{2}V_{\text{int}}$. The total degree of these faces is

$$\frac{3}{2}k - 2 + \frac{\alpha}{V_{\text{int}}}$$

and thus their average degree is strictly less than 6, a contradiction. □

5.3.2 A bijection from tableaux to augmented webs

We can now show that augmented webs are in bijection with weakly 3-noncrossing set partitions, $WNC(n,d,3)$. To define our bijection, we first formally define the arc diagram used in the proof of Proposition 5.2.2. We call these m-diagrams, based on the objects of the same name developed by J. Tymoczko in [56].

Definition 5.3.9. *Let $\pi \in WNC(n,d,3)$. To form the m-*diagram *for π, place the vertices 1 through n equally spaced in a line, then for each block $B = \{b_1 < b_2 < \cdots < b_k\}$, draw a semicircular arc in the lower half plane from b_1 to b_2, and from b_2 to all other elements of B. We call the arc between b_1 and b_2 a first arc, and all other arcs second arcs. Note that the definition of weakly 3-noncrossing guarantees that first arcs do not cross first arcs, and second arcs do* *not cross second arcs. For visual clarity, we will often color first arcs in red and second arcs in black. The name m-diagram is due to the fact that in Tymoczko's definition, blocks were always size three and thus had a unique second arc, so the diagram appeared visually as a number of intersecting m shapes.*

The collection of first arcs and maximal second arcs of each block divide the lower half plane into a number of regions. We define the depth *of each region to be the number of first arcs and maximal second arcs which the region lies above.*

Example 5.3.10. *Let $\pi \in WNC(13,3,3)$ be the weakly 3-noncrossing set partition with three blocks, $\{\{1,4,6,7,8\},\{2,3,9,10\},\{5,11,12,13\}\}$. The m-diagram associated to π appears below.*

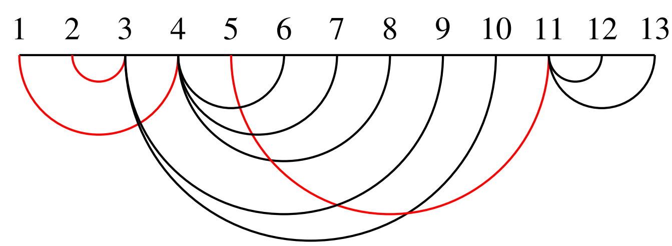


Figure 5.5. The m-diagram associated to $\{\{1,4,6,7,8\},\{2,3,9,10\},\{5,11,12,13\}\}$.

We can now define our bijection.

Definition 5.3.11. *The function $\varphi : WNC(n,d,3) \to AW(n,d)$ is defined as follows:*

Let $\pi \in WNC(n,d,3)$, and let M be its m diagram. For each block $B = \{b_1 < b_2 < \cdots < b_k\}$, introduce a black vertex v_B slightly above b_2, connected to b_2 by an edge. In a small region around b_2, modify the arcs connecting to b_2 to instead connect to v_B. Then, for every pair of blocks, if the first arc of one crosses some of the second arcs of the other, replace a small region containg all intersections as shown in figure 5.6.

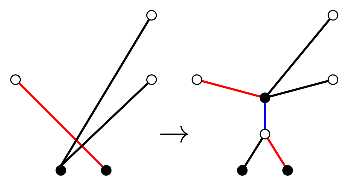


Figure 5.6. The replacement operation used in the definition of φ. The first arc is depicted in red, and the second arcs are depicted in black.

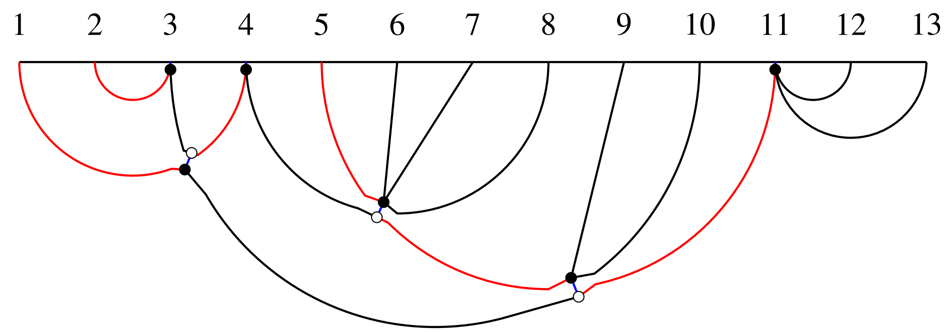


Figure 5.7. The web associated to $\{\{1,4,6,7,8\},\{2,3,9,10\},\{5,11,12,13\}\}$.

Example 5.3.12. *Continuing our prior example of* $\pi = \{1,4,6,7,8\},\{2,3,9,10\},\{5,11,12,13\}$*, the resulting web is depicted below.*

Proposition 5.3.13. *The function* φ *is well defined, i.e. if* $\pi \in WNC(n,d,3)$*, then* $\varphi(\pi)$ *is indeed in* $AW(n,d)$

Proof. We need to check that the resulting graph does not have a cycle of length 4, the other properties are clear. A 4 cycle would necessarily have two edges coming from first arcs and two edge coming from second arcs, such that no new edges are created. Orienting all edges in the m-diagram away from v_B for each block B, a 4 cycle would require that the arcs intersect with opposite orientations at each corner. But the the two first arcs would have to have different orientations, and this is not possible. $\square$

Lemma 5.3.14. *Let* $\pi \in WNC(n,d,3)$*. The first arcs and maximal second arcs of the m diagram for* π *divide the half plane into a number of regions. There is a depth preserving correspondence between faces of* $\varphi(\pi)$ *and these regions.*

Proof. Each replacement can be made so that edges coming from first arcs and maximal second arcs stay the same except in a ε radius region around each intersection. Each face of $\varphi(\pi)$ is thus contained (except for an ε-small portion) in a unique region. The trip starting at the first arc of each m will be an exceedance of $\varphi(\pi)$, which travels left to right along first arcs, maximal second arcs, and edges introduced by intersection replacement steps. These trips either cross at each intersection, using the new edge introduced at that intersection twice, or turn at from eachother at each intersection, using the new edge introduced at that intersection 0 times. Thus, the depth boundary paths consist exactly of those edges which come from first arcs and maximal second arcs, and thus the depth of each face matches the depth of the region it is contained in. □

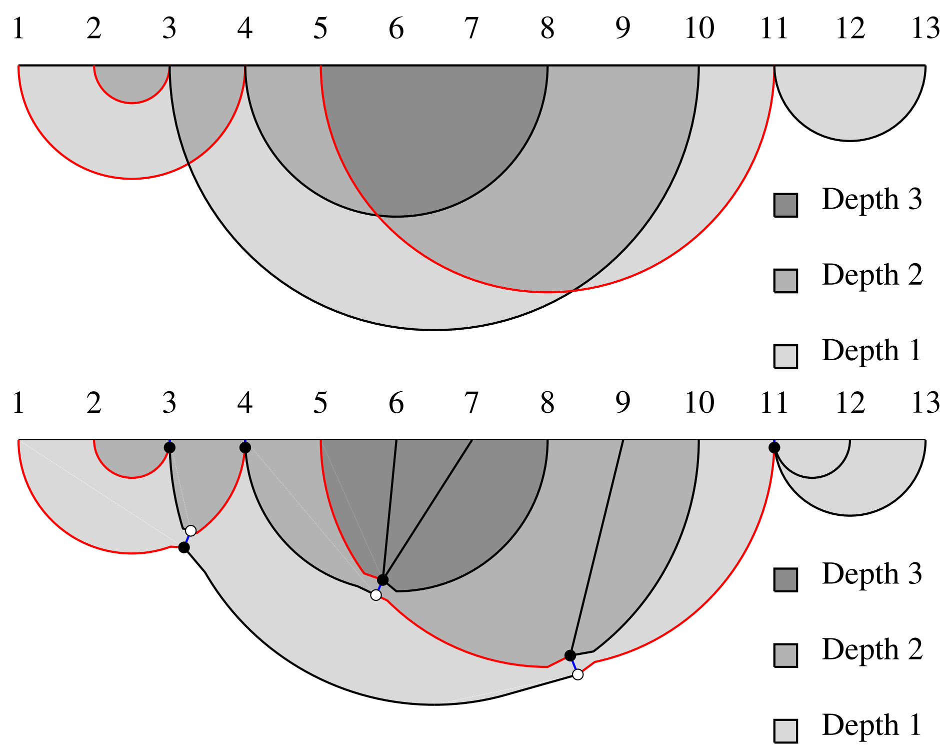


Figure 5.8. An example of the correspondence between m-diagram depth and augmented web depth. Above, an m-diagram with non-maximal second arcs removed and regions shaded by depth. Below, the corresponding augmented web with first and maximal second arcs mostly preserved, and faces shaded by depth.

Theorem 5.3.15. *The function $\varphi : WNC(n,d,3) \to AW(n,d)$ is a bijection.*

Proof. To show that φ is invertible, we introduce the following definition. The idea is that we will record extra information in the process of applying φ by way of coloring the edges. This extra information will allow us to invert the φ map. We will then show that the extra information was redundant, so φ is invertible.

Definition 5.3.16. *Given an augmented web $W \in AW(n,d)$, a* valid coloring *of W is a coloring of the edges of W with three colors, red, blue, and black such that the following conditions are satisfied:*

1. *Every interior vertex is incident to exactly one red edge, exactly one blue edge, and at least one black edge. Additionally, at each interior vertex, the incident red edge shares a face with the incident blue edge.*
2. *Right-to-left depth boundaries are colored red.*
3. *Left-to-right depth boundary edges incident to a boundary vertex are colored black.*
4. *No face has three consecutive edges colored red-blue-red.*

Note that we do not require this coloring to be proper, a vertex may have multiple black edges incident.

Given a weakly 3-noncrossing set partition $\pi \in WNC(n,d,3)$, we can create a valid coloring of $\varphi(\pi)$ by initially coloring first arcs red, second arcs black, and for each block b, the connection between v_b and b_2 blue. Then, at each replacement step, color the newly introduced edge blue and preserve all other colors. To see that the coloring obtained is a valid coloring, first note that initially and at each replacement step, the created vertices satisfy property 1 of a valid coloring. Property 2 of valid colorings holds by Lemma 5.3.14. To see that the third property of valid colorings holds, note that initially no such face exists and at no replacement step could such a face be created.

The interior blue edges contain the information of exactly which replacement steps have been performed, so given an augmented web $\varphi(\pi)$ and the valid coloring obtained through φ, we can recover π. Therefore, it suffices to show that every augmented web w admits exactly one valid coloring.

To do so, we will give an algorithm for finding a valid coloring and show that each step is forced. Let W be an augmented web.

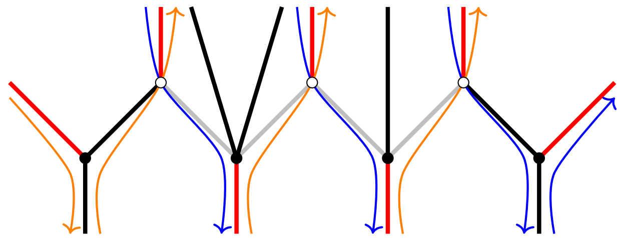
1. Consider the set of all right-to-left depth boundaries. In order to satisfy condition 2 of a valid coloring, we must color all such faces red. By Lemma 5.3.7, every vertex now has exactly one red edge. Similarly, color all left-to-right depth boundary edges which are incident to a boundary vertex black.

2. Every interior vertex is now incident to a red edge, so all remaining edges must be blue or black. Edges which do not share a face with the red edge at each of their vertices must be black by condition 1 of valid colorings, so color all such edges black.

3. Consider the set of yet uncolored edges adjacent to two red edges on the same face. By condition 3 of valid colorings, these edges must be black, so color all such edges black.

4. Consider the set of yet uncolored edges. Every interior vertex is incident to at most two of these edges, so their union consists of a set of disjoint paths and cycles. We claim that their union is in fact a disjoint union of paths with exactly one interior endpoint and odd length paths with two interior endpoints. Given this claim, there is exactly one way to satisfy condition 1 of valid colorings by coloring some of these edges blue and the rest black, that is, by coloring edges of a path alternating blue and black, starting at an interior endpoint of the path. So, if the claim holds, we are done.

To see that the claim holds, suppose towards a contradiction there is a cycle C among the uncolored edges after step 3. Let D_k be the maximal k depth boundary path sharing a vertex with C. Then D_k shares exactly one edge with C, as the uncolored edges incident to v cannot be on the same side of the depth boundary path passing through v. But this is a contradiction as that edge would have been colored black at step 3. Now suppose there is an even length path whose endpoints are both interior vertices. The path must be as shown below, though possibly longer or with the colors of vertices and trips reversed.



For the red edges to have been colored red in step 1, the trips drawn in orange must be exceedances, and the trip drawn in blue must not be. The two rightmost blue trips cannot cross more than once as W is reduced, so the face between n and 1 must lie to the right of the second rightmost blue trip. By Lemma 5.3.8, the leftmost orange trip and the three leftmost blue trips cannot cross eachother, and since W is reduced, the leftmost orange trip cannot cross itself. Thus, the leftmost orange trip cannot be an exceedance, but this is a contradiction. Lastly, suppose there is a boundary to boundary path among the uncolored edges. Consider the two black vertices incident to the endpoints of this path. The red edges adjacent to them must lie on the same side of the path, so one of the edges along this path incident to the boundary must be a left-to-right depth boundary edge, and this is a contradiction.

Thus, every augmented web has exactly one valid coloring, and φ is a bijection.

□

5.4 SL_3 invariants for augmented webs

In this section we introduce an invariant associated to each perfectly orientable normal plabic graph in such a way that the invariants associated to augmented webs form a basis of $S^{(d^3,1^{n-3d})}$. To do so, we first need to give an orientation to each plabic graph, which will determine the sign of our invariant.

5.4.1 Perfect orientations

A key idea in defining our augmented web invariants is that of a *perfect orientation*. Perfect orientations were introduced by A. Postnikov in [34]. Our definition will be slightly different in that our sink vertices will be interior vertices rather than boundary vertices, and we also include the information of a total order on the sinks.

Definition 5.4.1. *Let W be a normal plabic graph. A perfect orientation $\mathscr{O}$ of W is a choice of two things. First, an orientation of each edge of W such that each boundary edge is oriented away from the boundary, each interior white vertex has exactly one ingoing edge, each interior black vertex has* at most *one outgoing edge. There will then be a set of d black vertices with no outward edges, we refer to these as the sinks of $\mathscr{O}$ and denote them by $S_{\mathscr{O}}$. We also require that for every vertex v in W there is a directed path from v to a sink vertex. A perfect orientation also includes the information of a total order on the sinks, i.e. a bijection $f_{\mathscr{O}} : S_{\mathscr{O}} \to \{1, \dots, d\}$. For each perfect orientation, we call the set of edges which are oriented away from black vertices the independent set of $\mathscr{O}$ and denote it $I(\mathscr{O})$.*

If there exists at least one perfect orientation of W, we say that W is perfectly orientable.

Remark 5.4.2. *The set of perfectly orientable plabic under our definition differs slightly from the set of perfectly orientable plabic graphs under Postnikov's definition. For example, a white vertex connected by three edges to a single black vertex which is also connected to the boundary is perfectly orientable under Postnikov's definition but not ours. However, every plabic graph with an acyclic perfect orientation per Postnikov's definition will be perfectly orientable per our definition, and Postnikov, Speyer, and Williams show that all reduced plabic graphs have an acyclic perfect orientation [8, Lemma 3.2].*

We use this modified definition in order to allow for perfect orientations to be obtained from eachother via a sequence of small changes.

Definition 5.4.3. *Let W be an augmented web with perfect orientation $\mathscr{O}$. Let v be a white vertex* *of W. A* swivel move *is a change in orientation of exactly one ingoing edge and one outgoing edge at v which connect to distinct black vertices. We have necessarily removed one sink vertex and added one sink vertex, let the new sink be in the same position in the total order as the old sink. We call this a swivel move due to the fact that the set $I(\mathscr{O})$ is being rotated around this white vertex.*

Proposition 5.4.4. *Any two perfect orientations can be transformed into each other via a sequence of swivel moves and a reordering of the sink vertices.*

Proof. Let $\mathscr{O}_1$ and $\mathscr{O}_2$ be two perfect orientations. Consider the symmetric difference of independent sets of the two perfect orientations, $I(\mathscr{O}_1)\Delta I(\mathscr{O}_2)$ it is necessarily a union of disjoint cycles and paths between sinks of $\mathscr{O}_1$ and sinks of $\mathscr{O}_2$. We induct on the number of cycles present in $I(\mathscr{O}_1)\Delta I(\mathscr{O}_2)$. If there are no cycles, performing a swivel move at each white vertex along the paths from sinks of $\mathscr{O}_1$ and $\mathscr{O}_2$ using the edges of this path will transform $\mathscr{O}_1$ into a perfect orientation the same as $\mathscr{O}_2$ up to a reordering of its sinks. If there is a cycle in $I(\mathscr{O}_1)\Delta I(\mathscr{O}_2)$, find a walk in W which starts at a sink vertex of $\mathscr{O}_1$, travels to a white vertex of the cycle, travels around the cycle, then returns to the starting sink via the same path such that every other edge of this walk is in $I(\mathscr{O}_1)$. Performing a swivel move at each white vertex of this walk using the edges along this walk will result in a perfect orientation $\mathscr{O}_3$ for which $I(\mathscr{O}_3)\Delta I(\mathscr{O}_2)$ has one fewer cycles. □

We first check that this is a sensible definition of sign.

Proposition 5.4.5. *Let $\mathscr{O}_1, \mathscr{O}_2, \mathscr{O}_3$ be three perfect orientations of a web $W \in AW(n,d)$. Then we have*

$$sign(\mathscr{O}_1, \mathscr{O}_3) = sign(\mathscr{O}_1, \mathscr{O}_2)sign(\mathscr{O}_2, \mathscr{O}_3)$$

Proof. By Proposition 5.4.4, it suffices to check this holds when $\mathscr{O}_2$ and $\mathscr{O}_3$ differ by a single swivel move. Note that

$$I(\mathscr{O}_1)\Delta I(\mathscr{O}_3) = (I(\mathscr{O}_1)\Delta I(\mathscr{O}_2))\Delta(I(\mathscr{O}_2)\Delta I(\mathscr{O}_3)),$$

and $(I(\mathscr{O}_2)\Delta I(\mathscr{O}_3))$ is size two. We split into three cases.

- Case 1: $(I(\mathscr{O}_1)\Delta I(\mathscr{O}_2))$ and $(I(\mathscr{O}_2)\Delta I(\mathscr{O}_3))$ do not intersect. Then $(I(\mathscr{O}_1)\Delta I(\mathscr{O}_2))$ and $(I(\mathscr{O}_1)\Delta I(\mathscr{O}_2))$ are the same except one path is two edges longer, and $\sigma(\mathscr{O}_1,\mathscr{O}_2) = \sigma(\mathscr{O}_1,\mathscr{O}_3)$. Thus, $\text{sign}(\mathscr{O}_1,\mathscr{O}_3) = -\text{sign}(\mathscr{O}_1,\mathscr{O}_2)$

- Case 2: $(I(\mathscr{O}_1)\Delta I(\mathscr{O}_2))$ and $(I(\mathscr{O}_2)\Delta I(\mathscr{O}_3))$ intersect in a single edge. Then $(I(\mathscr{O}_1)\Delta I(\mathscr{O}_2))$ and $(I(\mathscr{O}_1)\Delta I(\mathscr{O}_2))$ are the same except for two paths. In this case, $\sigma(\mathscr{O}_1,\mathscr{O}_2)$ and $\sigma(\mathscr{O}_1,\mathscr{O}_3)$ differ by a single transposition and $\text{sign}(\mathscr{O}_1,\mathscr{O}_3) = -\text{sign}(\mathscr{O}_1,\mathscr{O}_2)$.

- Case 3: $(I(\mathscr{O}_1)\Delta I(\mathscr{O}_2))$ and $(I(\mathscr{O}_2)\Delta I(\mathscr{O}_3))$ interesect in two edges. Then $(I(\mathscr{O}_1)\Delta I(\mathscr{O}_2))$ and $(I(\mathscr{O}_1)\Delta I(\mathscr{O}_2))$ are the same except one path is two edges shorter, and $\sigma(\mathscr{O}_1,\mathscr{O}_2) = \sigma(\mathscr{O}_1,\mathscr{O}_3)$. Thus, $\text{sign}(\mathscr{O}_1,\mathscr{O}_3) = -\text{sign}(\mathscr{O}_1,\mathscr{O}_2)$

□

Proposition 5.4.5 allows for an alternative definition of relative sign, which we find more intuitive. The relative sign between two perfect orientations is given by the parity of the number of swivel moves and the sign of the permutation of the sinks in any sequence of swivel moves and a permutation of the sinks which transforms one orientation into the other. Proposition 5.4.5 guarantees this is well defined. In this sense, swivel moves can be thought of as playing a similar role to adjacent transpositions in determing the sign of a permutation.

5.4.2 Consistent Labellings

Our invariants will be defined in terms of consistent labellings, which we now define.

Definition 5.4.6. *Let W be a perfectly orientable normal plabic graph. A consistent labelling ℓ of W is a choice of a possibly empty subset $\ell(e)$ of $\{1,\dots,\nu\}$ for each edge e of W, such that the following hold:*

1. *At each interior white vertex, incident edge labels are disjoint and their union is $\{1,2,3\}$.*

2. *At each black vertex, incident edge labels are disjoint and their union contains $\{1,2,3\}$.*

3. *The label at each boundary edge has size 1, and for each $i \in \{4,\dots,\nu\}$, $\{i\}$ appears exactly once among boundary labels.*

The edges whose labels contain 1, 2, or 3 can be thought of as determining three dimer covers of W.

The boundary word of ℓ is the word given by reading off the labels at boundary edges in order, denoted

$$bd(\ell) = bd(\ell)_1 \cdots bd(\ell)_n$$

The boundary monomial of ℓ is the monomial

$$\mathbf{x}_{bd(\ell)} = x_{bd(\ell)_1,1} \cdots x_{bd(\ell)_n,n}$$

The weight of a consistent labelling is

$$wt(\ell) = \left(-\frac{1}{2}\right)^{\#\text{edge labels of size two}}$$

To each consistent labelling we also associate a sign, made up of a number of factors. Firstly, for each $1 \le i \le 3$, let E_i denote the set of edges of W whose label contains i. Consider the symmetric difference $E_i \Delta \mathscr{O}$. This will be a union of disjoint cycles and disjoint paths from sinks of $\mathscr{O}$ to boundary vertices whose incident edge is labelled i. For each boundary vertex b with label i, let the origin *of b be the sink vertex it connects to, denoted $origin(b)$. An origin* *inversion of w is a pair of boundary vertices $b_1 < b_2$ with origin$(b_1) >$ origin(b_2). For each i, we get a contribution to the sign of the labelling of*

$$(-1)^{\#\text{origin inversions between vertices labelled } i + \#\text{cycles of length 2 modulo 4 in } E_i \Delta \mathscr{O}}$$

We also have contributions to the sign of a consistent labelling coming from the number of edges of W with labels of size 2, the number of edges of $I(\mathscr{O})$ with an even size label, and inversions in the boundary word of ℓ. The sign of ℓ with respect to orientation $\mathscr{O}$ is given by

$$\left(\prod_{i=1}^{3}(-1)^{\#\text{cycles of length 2 modulo 4 in } E_i\Delta\mathscr{O}}\right)(-1)^{\#\text{origin inversions}+inv(bd(\ell))+\#\{e\in I(\mathscr{O})\,|\,|\ell(e)| \text{ is even}\}}.$$

We can also think of the sign contribution coming from origin inversions in a different way. Let the decorated boundary word *of ℓ, $\tilde{bd}(\ell)$, be the boundary word of ℓ with a subscript for the origin attached to each letter $1 \leq i \leq 3$. We can consider the decorated boundary word to be a permutation under the order*

$$1_1 < 1_2 < \cdots < 1_d < 2_1 < \cdots < 3_d < 4 < \cdots < v$$

Then the sign of ℓ is

$$\left(\prod_{i=1}^{3}(-1)^{\#\text{cycles of length 2 modulo 4 in } E_i\Delta\mathscr{O}}\right) sign(\tilde{bd}(\ell))(-1)^{\#\{e\in I(\mathscr{O})\,|\,|\ell(e)| \text{ is even}\}}.$$

times the sign of the decorated boundary word.

Remark 5.4.7. *The definition of weight of a labelling is a bit mysterious to us. It is chosen to make the Skein relations upcoming in Section 5.5 hold, and we lack any further explanation beyond that. The definition of sign of a labelling is chosen so that a change in orientation introduces a consistent change in sign among all possible consistent labellings.*

Example 5.4.8. *Consider the augmented web and consistent labelling shown below. Edges in*

$I(\mathscr{O})$ are highlighted in yellow, and the three sinks are labelled with their position in the total order on sinks.

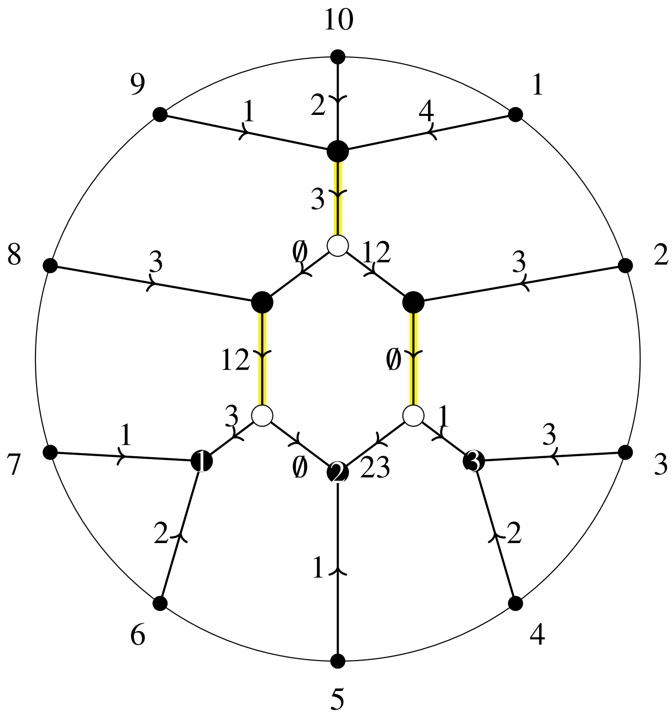


Figure 5.9. A web with a consistent labelling.

The boundary word of this labelling is 4332121312*, with 28 inversions. The origins of the boundary vertices for each label are:*

boundary vertices labelled 1:	*5*	*7*	*9*
origins:	*2*	*1*	*3*
boundary vertices labelled 2:	*4*	*6*	*10*
origins:	*3*	*1*	*2*
boundary vertices labelled 3:	*2*	*3*	*8*
origins:	*2*	*3*	*1*

The total number of origin inversions of ℓ is 5, one from label 1 and two each from labels 2 and 3. The decorated boundary word of ℓ is $43_23_32_31_22_11_13_11_32_2$, with 33 inversions. Our orientation is acyclic, so there are no cycles to consider, and two of the highlighted edges have an even size label. The sign of this labelling is therefore

$$(-1)^{33}(-1)^2 = -1$$

There are three edges with labels of size two, so the weight of this labelling is $(-\frac{1}{2})^3 = -\frac{1}{8}$.

Proposition 5.4.9. *Let W be a perfectly orientable normal plabic graph with consistent labelling ℓ. Let $\mathscr{O}_1$ and $\mathscr{O}_2$ be two distinct perfect orientations for ℓ. Then the sign of ℓ with respect to these orientations is related by*

$$sign(\ell, \mathscr{O}_1) = sign(\mathscr{O}_1, \mathscr{O}_2) sign(\ell, \mathscr{O}_2)$$

Proof. It suffices to show that this holds when $\mathscr{O}_1$ and $\mathscr{O}_2$ differ by a swivel move at a white vertex v. Let u_1 and u_2 be the sinks which vary between $\mathscr{O}_1$ and $\mathscr{O}_2$, and let u_3 denote the third neighbor of v. The origin of each boundary vertex is the same in $\mathscr{O}_1$ and $\mathscr{O}_2$ except for the the boundary vertices whose origin path travels along edge (v, u_3), which have swapped origins in $\mathscr{O}_1$ and $\mathscr{O}_2$. Either we have $|\ell(v, u_3)|$ is odd, in which case $|\ell(v, u_2)|$ and $|\ell(v, u_1)|$ are the same parity, or $|\ell(v, u_3)|$ is even, in which case $|\ell(v, u_2)|$ and $|\ell(v, u_1)|$ have opposite parity. In either case, we have

$$\text{sign}(\ell, \mathscr{O}_1) = -\text{sign}(\ell, \mathscr{O}_2)$$

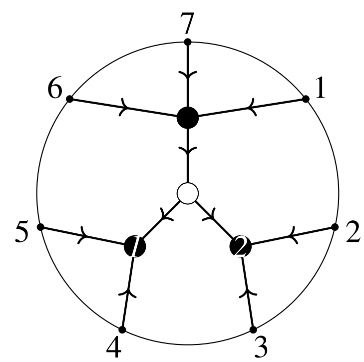
as desired. □

We can now define our invariants for normal plabic graphs.

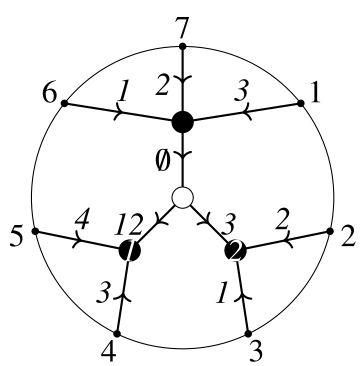
Definition 5.4.10. *Let W be a perfectly orientable normal plabic graph with perfect orientation $\mathscr{O}$. Let $CL(W)$ denote the set of all consistent labellings of W. Define an SL_3 invariant attached* *to W, denoted $[W, \mathscr{O}]$ by:*

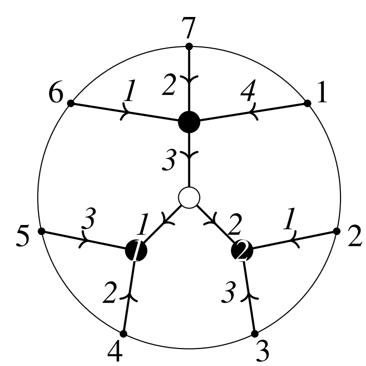
$$[W, \mathscr{O}] = \sum_{\ell \in CL(W)} sign(\ell, \mathscr{O}) wt(\ell) \mathbf{x}_{bd(\ell)}$$

Example 5.4.11. *Consider the augmented web $W \in AW(7,2)$ and perfect orientation $\mathscr{O}$ shown below.*



There are 288 consistent labellings in total, but only 2 up to graph automorphism (not necessarily boundary preserving) and permutation of $\{1,2,3\}$, shown below:





The left labelling has combined sign and weight of $-\frac{1}{2}$, and the right labelling has combined sign and weight 1. Let $Aut(W) \subset \mathfrak{S}_n$ denote the group of automorphisms of W identified with the corrseponding permutation of boundary vertices, which has size $6 \cdot 2 \cdot 2 \cdot 2 = 48$. Then we have

$$[W,\mathscr{O}] = \sum_{\sigma\in Aut(W)} \sum_{\omega\in A_3} sign(\sigma)\Big(-\frac{1}{2}x_{\omega(3),\sigma(1)}x_{\omega(2),\sigma(2)}x_{\omega(1),\sigma(3)}x_{\omega(3),\sigma(4)}x_{4,\sigma(5)}x_{\omega(1),\sigma(6)}x_{\omega(2),\sigma(7)}$$
$$+x_{4,\sigma(1)}x_{\omega(1),\sigma(2)}x_{\omega(3),\sigma(3)}x_{\omega(2),\sigma(4)}x_{\omega(3),\sigma(5)}x_{\omega(1),\sigma(6)}x_{\omega(2),\sigma(7)}\Big)$$

where A_3 is the alternating group on $\{1,2,3\}$.

To verify that this is a sensible definition, we first check that changing the orientation only introduces a change of sign.

Proposition 5.4.12. *Let W be a perfectly orientable normal plabic graph with perfect orientations $\mathscr{O}_1$ and $\mathscr{O}_2$. Then*

$$[W,\mathscr{O}_1] = sign(\mathscr{O}_1,\mathscr{O}_2)[W,\mathscr{O}_2]$$

Proof. By Proposition 5.4.5, we have

$$\begin{aligned}
[W,\mathscr{O}_1] &= \sum_{\ell\in CL(W)} \mathrm{sign}(\ell,\mathscr{O}_1)\mathrm{wt}(\ell)\mathbf{x}_{\mathrm{bd}(\ell)} \\
&= \mathrm{sign}(\mathscr{O}_1,\mathscr{O}_2) \sum_{\ell\in CL(W)} \mathrm{sign}(\ell,\mathscr{O}_2)\mathrm{wt}(\ell)\mathbf{x}_{\mathrm{bd}(\ell)} \\
&= \mathrm{sign}(\mathscr{O}_1,\mathscr{O}_2)[W,\mathscr{O}_2]
\end{aligned}$$

□

We call these invariants because the resulting polynomials will be invariant under a certain action of SL_3. Define an action of SL_3 which acts on the matrix of $3n$ variables

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ x_{3,1} & x_{3,2} & \cdots & x_{3,n} \end{bmatrix}$$

via left multiplication and leaves all other variables fixed. Then we have the following:

Proposition 5.4.13. *Let $X \in SL_3$ and $W \in AW(n,d)$ with perfect orientation $\mathscr{O}$. Then*

$$X \cdot [W, \mathscr{O}] = [W, \mathscr{O}]$$

This is clear for normal plabic graphs without interior white vertices and with only vertices of degree at least 3, i.e. jellyfish invariants. We will defer the proof for all augmented webs until Section 6, which will show that augmented web invariants live in $\mathfrak{S}_n$ closure of jellyfish invariants. The action of $\mathfrak{S}_n$ commutes with the action of SL_3, so the result will follow.

We next show that this definition generalizes jellyfish invariants.

Proposition 5.4.14. *Let $W \in AW(n,d)$ have no white vertices, let $\mathscr{O}$ be a perfect orientation of W, and let π be the corresponding ordered set partition. Then* $[W, \mathscr{O}] = [\pi]_3$.

Proof. Let $v_1, \ldots, v_d$ denote the interior vertices of W in order. We claim that a consistent labelling of W corresponds to a choice of jellyfish tableau for π as well as a choice of permutation for the elements of each block of π. Indeed, we can create a jellyfish tableau T_ℓ for π in the following manner. For each boundary vertex $1 \leq b \leq n$, if b has label i and is connected to interior vertex v_j, fill box i, j with the entry b. Then, for each interior vertex v_j, let $R_j(\ell)$ be the set of boundary vertices connected to v_j. Let $\sigma(v_j) \in \mathfrak{S}_n$ denote the permutation which reorders the elements of R_j to have increasing labels. We claim that

$$\operatorname{sign}(\ell)\mathbf{x}_{\operatorname{bd}(\ell)} = \operatorname{sign}(J(T_\ell)) \prod_{j=1}^{d} \operatorname{sign}(\sigma(v_j)) \left(\prod_{i \in R_j} x_{\operatorname{bd}(\ell)_i, i} \right) \tag{5.4.1}$$

The right side here represents one term in the monomial expansion of the product of determinants defining $J(T_\ell)$. From the definition of $\mathbf{X}_{\operatorname{bd}(\ell)}$ we see that the variables appearing on both sides of (5.4.1) agree, so the content of this claim is that the signs match. We show this in two parts. First, we claim that

$$\#\text{origin inversions of } \ell = \#\text{inversions within rows of } J(T_\ell) \tag{5.4.2}$$

Suppose (b_1, b_2) is an origin inversion of ℓ, with label i. Then b_1 appears in box $(i, \text{origin}(b_1))$ and b_2 appears in box $(i, \text{origin}(b_2))$. So (b_1, b_2) is also an origin inversion of T_ℓ. Next, we claim that

$$(-1)^{\text{inv}(\text{bd}(\ell))} = (-1)^{\text{\#inversions between rows of } J(T_\ell)} \prod_{j=1}^{d} \text{sign}(\sigma(v_j)) \tag{5.4.3}$$

or equivalently,

$$(-1)^{\text{inv}(\sigma(v_1)\sigma(v_2)\cdots\sigma(v_d)\cdot\text{bd}(\ell))} = (-1)^{\text{\#inversions between rows of } J(T_\ell)}. \tag{5.4.4}$$

If the k^{th} letter of $(\sigma(v_1)\sigma(v_2)\cdots\sigma(v_d)\cdot\text{bd}(\ell))$ is i, then a k appears in the i^{th} row of $J(T)$, so (5.4.4) holds. We therefore have

$$[W, \mathscr{O}] = [\pi]_3$$

as desired. □

Augmented web invariants satisfy the rotation and reflection invariance properties laid out in [11], as well as something slightly stronger.

Proposition 5.4.15. *Let $W \in AW(n,d)$ with perfect orientation $\mathscr{O}$ and let $\sigma \in \mathfrak{S}_n$. Let $\sigma \cdot W$ be the graph obtained by permuting the boundary vertices of W according to σ, i.e. if b is a boundary vertex and (v,b) is an edge of W, then $(v,\sigma(b))$ will be an edge of $\sigma \cdot W$. Suppose that $\sigma \cdot W$ is planar, so it is also in $AW(n,d)$. Then, abusively letting $\mathscr{O}$ also be a perfect orientation of $\sigma \cdot W$, we have*

$$\sigma \cdot [W, \mathscr{O}] = sign(\sigma)[\sigma \cdot W, \mathscr{O}]$$

Proof. For each consistent labelling ℓ of W, we get a corresponding consistent labelling $\sigma \circ \ell$ of $\sigma \circ W$. The decorated boundary word of $\sigma \cdot \ell$ is obtained by applying σ to the decorated boundary word of ℓ, so the result follows. □

Corollary 5.4.16. *Let c be the long cycle in $\mathfrak{S}_n$ and let w_0 be the long element. Given an augmented web $W \in AW(n,d)$ with orientation $\mathscr{O}_W$, let rot(W) and rot($\mathscr{O}_W$) be the web obtained* *by rotating W and $\mathscr{O}_W$ clockwise by $\frac{2\pi}{n}$. Let refl(W) and refl$(\mathscr{O}_W)$ be the web and orientation obtained by reflecting W and $\mathscr{O}_w$ across the perpendicular bisector between boundary vertices 1 and n. We have*

$$c \cdot [W, \mathscr{O}] = (-1)^{n-1} [rot(W), rot(\mathscr{O}_W)]$$

and

$$w_0 \cdot [W, \mathscr{O}] = (-1)^{n-1} [refl(W), refl(\mathscr{O}_W)]$$

Remark 5.4.17. *In the definition of $\sigma \cdot W$, we are implicitly using the fact that if a web is planar and the positions of its boundary vertices are fixed, it has a unique planar embedding in the disk up to boundary-preserving homeomorphism. This is due to a classical theorem of Whitney [59].*

Our main theorem regarding augmented web invariants is that they form a basis for the flamingo Specht module. We state it now but defer its proof until the end of the next section.

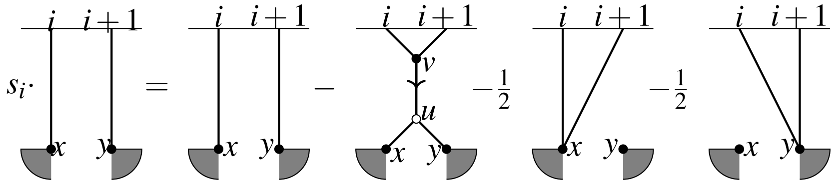
Theorem 5.4.18. *Choose a perfect orientation $\mathscr{O}_W$ for each augmented web $W \in AW(n,d)$. Then the set $\{[W, \mathscr{O}_W] \mid W \in AW(n,d)\}$ is a basis for the flamingo Specht module $S^{(d^3, 1^{n-3d})}$.*

5.5 Skein relations for augmented webs

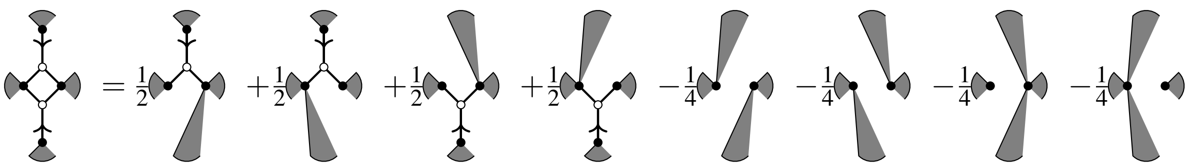
This section will introduce skein relations for normal plabic graph invariants, showing that they satsify property (5) of web bases. Furthermore, these skein relations will demonstrate that the span of augmented web invariants is an $\mathfrak{S}_n$ invariant module containing $S^{(d^3, n-3d)}$. Along with our combinatorial bijection from standard Young tableaux, Proposition 5.3.15, we will thus obtain a proof of Theorem 5.4.18.

We first give a diagrammatic representation of these relations. In each image below, shaded gray areas represent an unknown number of edges connecting to other vertices of the graph, and in the perfect orientation, edges are assumed to be oriented towards black vertices unless shown otherwise.

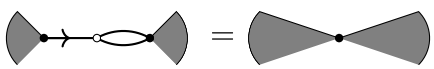
Crossing Reduction Rule



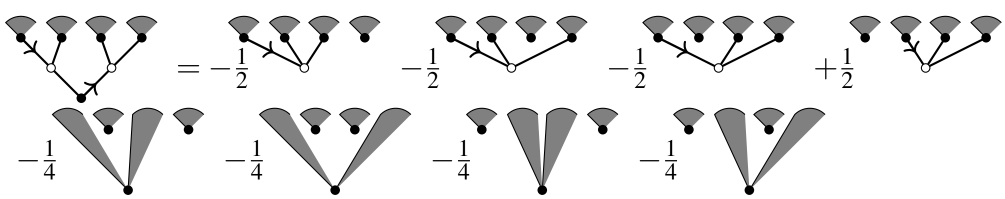
Square Reduction Rule



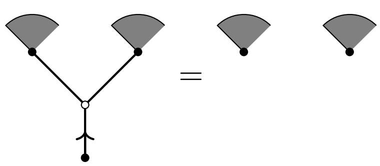
Double Edge Reduction Rule



Bivalent Vertex Reduction Rule



Leaf Reduction Rule



Boundary-Adjacent Bivalent and Leaf Reduction Rules

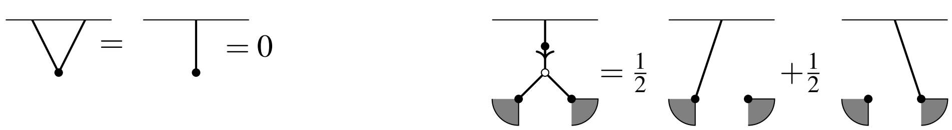


Figure 5.10. The skein relations for augmented webs.

Proposition 5.5.1 (Crossing reduction rule)**.** *Let W be a perfectly orientable normal plabic graph. Suppose W has two adjacent boundary vertices $i, i+1$ which connect to distinct interior vertices x and y respectively. Let W_I denote the web obtained from W by removing edges (x,i) and $(y,i+1)$, and attaching an "I" shape, i.e. adding an interior black vertex u and an interior white vertex v, then adding in edges $(x,v),(y,v)(v,u),(u,i)$ and $(u,i+1)$. We have the following relation:*

$$s_i \cdot [W] = [W] - [W_I] - -\frac{1}{2}[W_x] - \frac{1}{2}[W_y] \tag{5.5.1}$$

Proof. We divide consistent labellings for our webs into classes based on a fixed choice C of labels among edges in these webs other than those between x, y, u, v, i, and $i+1$, then show that within each class, equation 5.5.1 holds. Up to symmetry, there are five possible cases, we will explain the first in detail and give a table for the rest.

Case 1: Among the fixed labels of edges incident to x, 2 and 3 are present. Among the fixed labels of edges incident to y, $1,2$, and 3 are present. The missing labels around the boundary are 1 and 4. Then there is exactly one way to label the remaining edges of W, edge (x,i) must have label 1 and edge $(y,i+1)$ must have label 4. Call this labelling ℓ. There are two ways to label W_I, both with weight $\frac{1}{2}$: edge (x,u) must have label 1, edge (y,u) must be unlabelled, edge (u,v) must have label $\{2,3\}$, and edges (v,i) and $(v,i+1)$ must have labels 1 and 4 in either order. Call these labellings $\ell_{I,1}$ and $\ell_{I,2}$. There are two ways to label W_x. There are no ways to label W_y. Note that origin inversions do not change between these labellings, and the chosen orientation of each web is compatible with the specified labelling, so the relative sign is given only by the relative change in the boundary word. Thus, there exist a fixed monomial m such that

$$s_i \cdot \sum_{\substack{\ell \in CL(W) \\ \ell \text{ extends } C}} \operatorname{sign}(\mathscr{O}, \mathscr{O}_\ell) \operatorname{sign}(\ell, \mathscr{O}_\ell) \operatorname{wt}(\ell) \mathbf{x}_{\operatorname{bd}(\ell)} = (x_{4,i} x_{1,i+1}) m$$

$$\sum_{\substack{\ell \in CL(W) \\ \ell \text{ extends } C}} \operatorname{sign}(\mathscr{O}, \mathscr{O}_\ell)\operatorname{sign}(\ell, \mathscr{O}_\ell)\operatorname{wt}(\ell)\mathbf{x}_{\operatorname{bd}(\ell)} = (x_{1,i}x_{4,i+1})m$$

$$\sum_{\substack{\ell \in CL(W_I) \\ \ell \text{ extends } C}} \operatorname{sign}(\mathscr{O}, \mathscr{O}_\ell)\operatorname{sign}(\ell, \mathscr{O}_\ell)\operatorname{wt}(\ell)\mathbf{x}_{\operatorname{bd}(\ell)} = \frac{1}{2}(x_{1,i}x_{4,i+1} - x_{4,i}x_{1,i+1})m$$

$$\sum_{\substack{\ell \in CL(W_x) \\ \ell \text{ extends } C}} \operatorname{sign}(\mathscr{O}, \mathscr{O}_\ell)\operatorname{sign}(\ell, \mathscr{O}_\ell)\operatorname{wt}(\ell)\mathbf{x}_{\operatorname{bd}(\ell)} = (x_{1,i}x_{4,i+1} - x_{4,i}x_{1,i+1})m$$

$$\sum_{\substack{\ell \in CL(W_y) \\ \ell \text{ extends } C}} \operatorname{sign}(\mathscr{O}, \mathscr{O}_\ell)\operatorname{sign}(\ell, \mathscr{O}_\ell)\operatorname{wt}(\ell)\mathbf{x}_{\operatorname{bd}(\ell)} = 0$$

Therefore, since

$$(x_{4,i}x_{1,i+1}) = (x_{1,i}x_{4,i+1}) - \frac{1}{2}(x_{1,i}x_{4,i+1} - x_{4,i}x_{1,i+1}) - \frac{1}{2}(x_{1,i}x_{4,i+1} - x_{4,i}x_{1,i+1}) - 0,$$

among classes of labellings which fit into this case, equation 5.5.1 holds.

The remaining cases are as follows. To read the following table, first note that to condense information, we have replaced the monomial $x_{a,i}x_{b,i+1}$ with the word ab. Then let C denote a fixed way to label the edges of W, W_I, W_x and W_y other than those between x, y, u, v, i, and $i+1$. The fixed labels of C at x, y, and the missing labels of C around the boundary fit into one of the cases listed in the rows of this table, up to a permutation of $\{1,2,3\}$ and $\{x,y\}$. Then there is a fixed monomial m such that for each column headed by a web, if the entry in that row and column headed by $s_i \cdot W$ is a, then there exists a monomial m such that

$$\sum_{\substack{\ell \in CL(W) \\ \ell \text{ extends } C}} \operatorname{sign}(\mathscr{O}, \mathscr{O}_\ell)\operatorname{sign}(\ell, \mathscr{O}_\ell)\operatorname{wt}(\ell)\mathbf{x}_{\operatorname{bd}(\ell)} = am$$

The first row is Case 1.

□

Table 5.1. The table for the proof of the crossing reduction rule.

Labels at x	Labels at y	Boundary	$s_i \cdot W$	W	W_I	W_x	W_y
$\{2,3\}$	$\{1,2,3\}$	$\{1,4\}$	41	14	$\frac{1}{2}(14-41)$	$14-41$	0
$\{2,3\}$	$\{2,3\}$	$\{1,1\}$	11	11	0	0	0
$\{2,3\}$	$\{1,3\}$	$\{1,2\}$	21	12	$12-21$	0	0
$\{1,2,3\}$	$\{1,2,3\}$	$\{4,5\}$	$54-45$	$45-54$	$45-54$	$45-54$	$45-54$
$\{1\}$	$\{1,2,3\}$	$\{2,3\}$	0	0	$\frac{1}{2}(21-12)$	$12-21$	0

Proposition 5.5.2 (Square reduction rule)**.** *Let W be a perfectly orientable normal plabic graph. Suppose W has a face of degree 4. Let v_1, v_2 denote the white vertices of this face, and let u_1, u_2, u_3, u_4 denote the neighbors of v_1, v_2, connected by edges $(v_1,u_1),(v_1,u_2),(v_1,u_4)$ and $(v_2,u_2),(v_2,u_3),(v_3,u_4)$. Let π be a noncrossing set partition of $\{1,2,3,4\}$ in which no block contains both u_2 and u_4. Let W_π be the web obtained from W by first deleting u_1 or u_2 if they connect to two vertices v_i, v_j whose indices lie in the same block of π, then identifying all vertices among $v_1,\dots,v_4$ whose indices are in the same block in π. We will write these set partitions without brackets and with vertical bars between blocks, e.g. $W_{1|23|4}$. Then we have the following relation:*

$$\sum_{\pi} \left(-\frac{1}{2}\right)^{4-\#\,\textit{blocks of}\,\pi} [W_\pi] = 0 \tag{5.5.2}$$

Proof. The proof is similar to that of the crossing rule, but there are many more cases. Again, we explain one case in detail and give a table for the rest.

Let C be a fixed way to label edges of the W_π other than those incident to v_1 and v_2. Then the boundary monomial of any consistent labelling extending C is fixed, so we need to check that the coefficients satisfy equation 5.5.2. We proceed casewise, based on the fixed labels of C present at each of the four black vertices u_1, u_2, u_3, and u_4.

Case 1: The fixed labels at v_1 contain $1,2$, and 3. The fixed labels at v_2 contain 1 and 2 but not 3. The fixed labels at v_3 contain 3 but not 1 or 2. The fixed labels at v_4 do not contain 1, 2, or 3.

There are two consistent labellings of $W_{1|2|3|4}$ which extend C, as shown below.

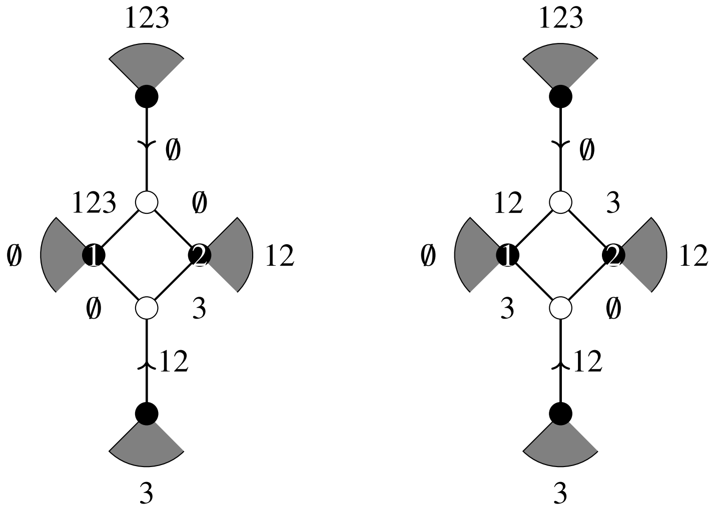


Figure 5.11. The two consistent labellings of $W_{1|2|3|4}$.

Relative to eachother, the left labelling has weight and sign $-\frac{1}{2}$, and the right labelling has weight and sign $-\frac{1}{4}$.

There is one consistent labelling of each of $W_{1|23|4}$, $W_{1|2|34}$, $W_{14|2|3}$, and $W_{14|23}$ which extends C, and none for the remaining W_{π}.

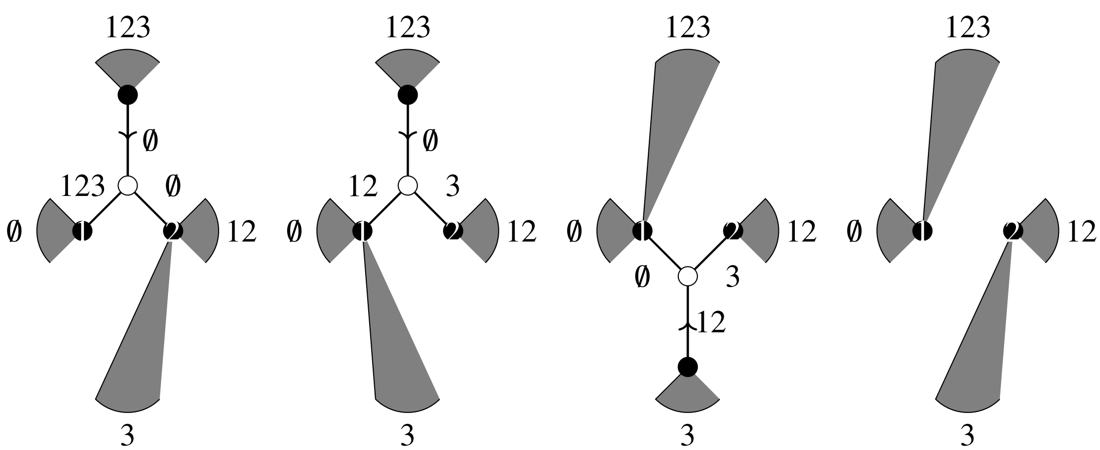


Figure 5.12. The consistent labellings of $W_{1|23|4}, W_{1|2|34}, W_{14|2|3}$, and $W_{14|23}$.

Relative to our earlier labellings, these carry weights and sign -1, $-\frac{1}{2}$, $\frac{1}{2}$ and 1, respectively. We check that these satsify equation 5.5.2:

$$0 = (-\frac{1}{2} - \frac{1}{4}) + (-\frac{1}{2})(-1 + \frac{1}{2} - \frac{1}{2}) + (\frac{1}{4})(1)$$

The remaining cases our included in the following table, case 1 appears in row 4. Cases which can be obtained by a permutation of $\{1,2,3\}$ or a symmetry of the square are not included.

Table 5.2. The table for the proof of the square reduction rule.

Labels at u_1	u_2	u_3	u_4	$W_{1\|2\|3\|4}$	$W_{1\|23\|4}$	$W_{1\|2\|34}$	$W_{12\|3\|4}$	$W_{14\|2\|3}$	$W_{14\|23}$	$W_{12\|34}$	$W_{123\|4}$	$W_{134\|2}$
$\{1,2,3\}$	$\{1,2,3\}$	$\emptyset$	$\emptyset$	-1	-1	-1	0	1	1	0	0	1
$\{1,2,3\}$	$\{1,2\}$	$\{3\}$	$\emptyset$	$-\frac{3}{4}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	0
$\{1,2,3\}$	$\{1,2\}$	$\emptyset$	$\{3\}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0	0
$\{1,2,3\}$	$\{1\}$	$\{2,3\}$	$\emptyset$	$-\frac{3}{4}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	0
$\{1,2,3\}$	$\{1\}$	$\{2\}$	$\{3\}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0
$\{1,2\}$	$\{1,2,3\}$	$\{3\}$	$\emptyset$	$-\frac{1}{4}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	1
$\{1,2\}$	$\{1,2,3\}$	$\emptyset$	$\{3\}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	1	0	0	1
$\{1,2\}$	$\{1,2\}$	$\{3\}$	$\{3\}$	$-\frac{1}{4}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1	0	0	0
$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\emptyset$	0	0	-1	0	1	0	0	0	0
$\{1,2\}$	$\{1,3\}$	$\{2\}$	$\{3\}$	$-\frac{3}{4}$	$-\frac{1}{2}$	-1	0	$\frac{1}{2}$	1	0	0	0
$\{1,2\}$	$\{1,3\}$	$\{3\}$	$\{2\}$	$\frac{1}{2}$	0	1	0	0	0	0	0	0
$\{1,2\}$	$\{1\}$	$\{2,3\}$	$\{3\}$	$-\frac{3}{4}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	0	0	0
$\{1,2\}$	$\{3\}$	$\{1,2\}$	$\{3\}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	0	0
$\{1,2\}$	$\{3\}$	$\{1,2\}$	$\{3\}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	0	0
$\{1\}$	$\{1,2,3\}$	$\{1\}$	$\{2,3\}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	0	0
$\{1\}$	$\{1,2,3\}$	$\{2\}$	$\{3\}$	$\frac{1}{4}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	1
$\{1\}$	$\{1,2,3\}$	$\emptyset$	$\{2,3\}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	0	0	1
$\{1\}$	$\{1,2\}$	$\{3\}$	$\{2,3\}$	$\frac{1}{4}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	0	0	0
$\{1\}$	$\{2,3\}$	$\{1\}$	$\{2,3\}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	0	0
$\emptyset$	$\{1,2,3\}$	$\emptyset$	$\{1,2,3\}$	1	1	1	1	1	1	1	1	1

□

Proposition 5.5.3 (Double edge reduction rule)**.** *Let W be a perfectly orientable normal plabic graph with a white vertex v with exactly two neighbors u_1, connected by 1 edge, and u_2, connected by 2 edges. Let $\mathscr{O}$ be a perfect orientation such that $I(\mathscr{O})$ contains (v,u_1). Let W' and $\mathscr{O}'$ be the plabic graph and orientation obtained by contracting v, u_1, and u_2. Then*

$$[W,\mathscr{O}] = [W',\mathscr{O}']$$

Proof. Fix a labelling of edges other than those incident to v. We have 4 cases up to a permutation of $\{1,2,3\}$

- Case 1: Among the fixed labels at u_1, $\{1,2,3\}$ appear. Then the union of the labels of the two edges between v and u_1 is $\{1,2,3\}$. There are two ways to label these edges with one of size 3, and these come with each with relative sign and weight -1. There are 6 ways to label the edges with one label of size 2 and the other of size 1, each with relative sign and weight $frac12$, for a total weight and sign of 1.

- Case 2: Among the fixed labels at u_2, $\{1,2\}$ appear. Then (v,u_1) has label 3 and the edges between v and u_2 have labels 1 and 2 split between them. There are two ways to have a label of size 2, these appear with relative sign and weight $-\frac{1}{2}$. There are two ways to have two labels of size 1, these appear with relative sign and weight 1, for a total weight and sign of 1.

- Case 3: Among the fixed labels at u_1, only $\{1\}$ appears. Then (v,u_1) has label $\{2,3\}$, and one of the edges to u_2 has label $\{1\}$. There are two ways to do this, each with relative sign and weight 1.

- Case 4: The fixed labels at u_1 are all empty. Then there is only one way to label the edges incident to v, with weight one.

In all cases W' has no choices to be made, and thus has relative sign and weight 1, so the result holds. □

Proposition 5.5.4 (Leaf vertex removal)**.** *Let W be a perfectly orientable plabic graph with a black vertex u of degree 1 connected to white vertex v. Let $\mathscr{O}$ be a perfect orientation wth the edge (u,v) oriented towards u. Let W' be the plabic graph obtained by removing vertices u and v and $\mathscr{O}'$ be the resulting perfect orientation. Then $[W,\mathscr{O}] = [W',\mathscr{O}']$.*

Proof. If we take a consistent labelling of W, and remove vertices u and v, we get a consistent labelling of W' with the same sign and weight, so the result follows. □

Proposition 5.5.5 (Boundary adjacent leaf removal)**.** *Let W be a perfectly orientable normal plabic graph with a black vertex of degree one or two only connected to the boundary. Then* $[W] = 0$.

Proof. There are no consistent labellings of W, so the result follows. □

Proposition 5.5.6 (Boundary adjacent bivalent vertex removal)**.** *Let W be a perfectly orientable normal plabic graph with a degree 2 black vertex u connected to one boundary vertex and one white vertex v Let the other neighbors of v be x and y. Let $\mathscr{O}$ be a perfect orientation of W in which (u,v) is oriented towards v. Let W_x be the graph obtained by removing u and v and connecting x to the boundary, and let $\mathscr{O}_x$ be the orientation obtained from $\mathscr{O}_x$ in the same fashion. Let W_y and $\mathscr{O}_y$ be analogous. Then we have*

$$[W,\mathscr{O}] = \frac{1}{2}[W_x,\mathscr{O}_x] + \frac{1}{2}[W_y,\mathscr{O}_y]$$

Proof. A consistent labelling of W must have an edge label of size one on the boundary edge incident to u, and edge label of size 2 on the edge (u,v), and an edge label of size one on exactly one of the edges (u,x) and (u,y). Thus, consistent labellings of W are in bijection with the disjoint union of consistent labellings of W_x and W_y, and each carries relative sign and weight $\frac{1}{2}$. Thus

$$[W,\mathscr{O}] = \frac{1}{2}[W_x,\mathscr{O}_x] + \frac{1}{2}[W_y,\mathscr{O}_y]$$

as desired. □

Proposition 5.5.7 (Bivalent vertex reduction rule)**.** *Let W be a normal plabic graph with a black vertex u of degree 2 connected to two white vertices v_1 and v_2. Let u_1,u_2,u_3,u_4 be the other neighbors of v_1 and v_2. Let $\mathscr{O}$ be a perfect orientation such that (u_1,v_1) and (u,v_2) are in $I(\mathscr{O})$. For* $1 \leq i \leq 8$*, let W_i denote the plabic graphs with orientation $\mathscr{O}_i$ as shown in the bivalent vertex* *reduction rule above. We have*

$$[W,\mathscr{O}] = -\frac{1}{2}[W_1,\mathscr{O}_1] - \frac{1}{2}[W_2,\mathscr{O}_2] - \frac{1}{2}[W_3,\mathscr{O}_3] + \frac{1}{2}[W_4,\mathscr{O}_4] \\ -\frac{1}{4}[W_5,\mathscr{O}_5] - \frac{1}{4}[W_6,\mathscr{O}_6] - \frac{1}{4}[W_7,\mathscr{O}_7] - \frac{1}{4}[W_8,\mathscr{O}_8]$$

Proof. As per the proof of the square rule, we have four cases to consider as shown in the following table

Table 5.3. The table for the proof of the bivalent vertex reduction rule.

Labels at u_1	u_2	u_3	u_4	W	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\emptyset$	1	0	-1	-1	1	0	1	1	0
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2\}$	$\{3\}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
$\{1,2,3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$-\frac{1}{2}$	0	0	0	-1	0	0	0	0
$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$	$\{3\}$	$-\frac{1}{4}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	-1	0

□

We can now give a proof of Theorem 5.4.18, which states the set $\{[W,\mathscr{O}_W] \mid W \in AW(n,d)\}$ is a basis of $S^{(d^3 1^{n-3d})}$.

Proof. Let $W \in AW(n,d)$. Apply the crossing reduction rule to rewrite $s_i \cdot [W,\mathscr{O}_w]$ as a sum of invariants for normal plabic graphs G_i with coefficients c_i, i.e.

$$s_i \cdot [W,\mathscr{O}_w] = \sum_i c_i [G_i,\mathscr{O}_{G_i}]$$

If any of the G_i are not augmented webs, i.e. they have a face of degree four or a black vertex of degree less than 3, we can apply one of the other skein relations to rewrite $[G_i,\mathscr{O}_{G_i}]$. Each time we do so, we replace a plabic graph with k white vertices by a linear combination of plabic graphs with strictly fewer than k white vertices, so by iterating this process we eventually rewrite

$s_i \cdot [W, \mathscr{O}_w]$ as a linear combination of augmented web invariants. Consequently,

$$\operatorname{span}(\{[W, \mathscr{O}_W] \mid W \in AW(n,d)\})$$

is closed under the action of $\mathfrak{S}_n$. By Theorem 1.4.2 and Proposition 5.4.14, $\operatorname{span}(\{[W, \mathscr{O}_W] \mid W \in AW(n,d)\})$ has a nonzero intersection with $S^{(d^3, 1^{n-3d})}$. By Theorem 5.3.15, the dimension of $\operatorname{span}(\{[W, \mathscr{O}_W] \mid W \in AW(n,d)\})$ is at most the dimension of $S^{(d^3, 1^{n-3d})}$. Since $S^{(d^3, 1^{n-3d})}$ is irreducible, we have

$$\operatorname{span}(\{[W, \mathscr{O}_W] \mid W \in AW(n,d)\}) = S^{(d^3, 1^{n-3d})}$$

and Theorem 5.3.15 shows that $\{[W, \mathscr{O}_W] \mid W \in AW(n,d)\}$ is indeed a basis. □

5.6 Augmented web invariants via weblike subgraphs

In this section we explain how to interpret our augmented web invariants in terms of the weblike subgraphs introduced by T. Lam in [26]. To do so, we first need to reinterpet out augmented web invariants as tensors rather than polynomials in $n \times \nu$ variables.

Let $V_0 \cong \mathbb{C}^\nu$ be a ν dimensional vector space with basis $\{e_1, \ldots, e_\nu\}$, let V be the span of the first three basis vectors, $V = \operatorname{span}(\{e_1, e_2, e_3\})$. Consider the space W consisting of the direct sum of all tensor products of $3d$ copies of V and $n - 3d$ copies of $\mathbb{C}$, e.g. when $n = 4$ and $d = 1$, W is

$$(V \otimes V \otimes V \otimes \mathbb{C}) \oplus (V \otimes V \otimes \mathbb{C} \otimes V) \oplus (V \otimes \mathbb{C} \otimes V \otimes V) \oplus (\mathbb{C} \otimes V \otimes V \otimes V)$$

W injects into $V_0^{\otimes n}$ by replacing the $n - 3d$ copies of $\mathbb{C}$ with $\bigwedge^{n-3d} \operatorname{span}(\{e_4, \ldots, e_\nu\})$, e.g. if $v_1, v_2, v_3 \in V$,

$$1 \otimes v_1 \otimes v_2 \otimes 1 \otimes v_3 \mapsto e_4 \otimes v_1 \otimes v_2 \otimes e_5 \otimes v_3 - e_5 \otimes v_1 \otimes v_2 \otimes e_4 \otimes v_3$$

There is also a natural injection of $V_0^{\otimes n}$ into the polynomial ring generated by the $n \times \nu$ variables

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & & \ddots & \vdots \\ x_{\nu,1} & x_{\nu,2} & \cdots & x_{\nu,n} \end{bmatrix}$$

given by

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \mapsto x_{i_1,1} x_{i_2,2} \cdots x_{i_n,n}$$

Recall that our augmented web invariants live in this polynomials ring, and furthermore, they live in the span of monomials which contain exactly one variable from each column, each with degree one. Additionally, in each augmented web invariant, the tensor factors corresponding to basis vectors $e_4, \ldots, e_\nu$ are alternating. Thus, augmented web invariants live in the image of the injection $\iota : W \hookrightarrow \mathbb{C}[x_{1,1}, \ldots, x_{\nu,n}]$. Denote the preimage under this injection of $[W, \mathscr{O}]$ by $\widetilde{[W, \mathscr{O}]}$.

We can make ι into an $\mathfrak{S}_n$ homomorphism by pulling back the action of $\mathfrak{S}_n$ on the vector space $\mathbb{C}[x_{1,1}, \ldots, x_{\nu,n}]$ to W. Note that this pullback is *not* just simply permuting tensor factors. An adjacent transposition s_i acts on simple basis tensors by

$$s_i \cdot (v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) = \begin{cases} -s_i \cdot (v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) & v_1, v_{i+1} \in \mathbb{C} \\ s_i \cdot (v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) & \text{otherwise} \end{cases},$$

i.e. it picks up a sign if both tensor factors come from $\mathbb{C}$.

The benefit of this viewpoint is that W is more well-studied in terms of webs. Kuperberg's work [25] gives a basis for W in terms of SL_3 webs with 0 clasps, i.e. SL_3 webs with $n - 3d$ boundary vertices without edges. We will explain how to expand $\widetilde{[W, \mathscr{O}]}$ into this clasped web basis.

As introduced by Lam, given a normal plabic graph W a 3-weblike subgraph is an assignment of a nonnegative integer to each edge such that the sum of edges around each interior vertex is 3. A weblike subgraph can be turned into an SL_3 web with 0-clasps (i.e. unused boundary vertices) by deleting each edge assigned 0 or 3, and contracting each path of edges alternately assigned 1's and 2's to a single edge. A consistent labelling ℓ gives rise to a weblike subgraph $W'(\ell)$ via the sizes of its edge labels. We have the following

Proposition 5.6.1. *Let W be a normal plabic graph with perfect orientation $\mathscr{O}$. Let W' be a 3-weblike subgraph of W with $d(W')$ edges of multiplicity 2. Then*

$$\iota^{-1}\Big(\sum_{\substack{\ell\in CL(G)\\ W'(\ell)=W'}} sign(\ell,\mathscr{O})wt(\ell)\mathbf{x}_{bd(\ell)}\Big) = \pm\left(-\frac{1}{2}\right)^{d(W')}[W']_{SL_3}$$

where $[W']_{SL_3}$ denotes the usual SL_3 web invariant.

Proof. Up to a reordering of labels larger than 3, consistent labellings ℓ with $W'(\ell) = W'$ are in bijection with proper edge labellings of W'. So it suffices to check that the difference in the definition of sign for consistent labellings and proper edge labellings is the same among all such labellings. Any two proper edge labellings of W' can be transformed into eachother via swapping the labels of any path alternately labelled $i, j, \ldots, i, j$ for $1 \leq i \leq j$. Swapping such a path will introduce a sign change both in the definition of sign for proper edge labellings and for consistent labellings. □

Corollary 5.6.2. *We thus have*

$$\widetilde{[W,\mathscr{O}]} = \sum_{\text{weblike subgraphs } W' \text{ of } W} \pm\left(\frac{1}{2}\right)^{d(W')}[W']_{SL_3}$$

Remark 5.6.3. *Note that when a normal plabic graph is an SL_3 web, i.e. all vertices are degree 3, our invariants do not match the usual SL_3 invariants. Instead, we get a sum over all SL_3 webs which appear as a weblike subgraph in W.*

5.7 Cyclic sieving for augmented webs

The basis $\{[W,\mathscr{O}_W] \mid W \in AW(n,d)\}$ given in Theorem 5.2.9 has all the necessary properties to obtain a cyclic sieving result via Springer's theorem of regular elements. The only detail left is that we need to be careful about the orientations. To address orientation, we need the following Lemma:

Lemma 5.7.1. *Let $W \in AW(n,d)$ with perfect orientation $\mathscr{O}$. Suppose W is fixed by rotation by $i \geq 2$, i.e. $rot^i(W) = W$. Then i divides n and exactly one of the following holds:*

- $\frac{n}{i} \mid d$
- $\frac{n}{i} \mid d-1$
- $\frac{n}{i} = 3$ *and* $\frac{n}{i} \mid d+1$

Let $k = \frac{di}{n}, \frac{(d-1)i}{n}, \frac{(d+1)i}{n}$ depending on which of the cases above holds. Then we have

$$sign(\mathscr{O}, rot^i(\mathscr{O})) = (-1)^{(\frac{n}{i}-1)k}$$

The relevance here is that this sign depends only on n,d, and i, not on W itself.

Proof. Note that by Lemma 5.4.5 it suffices to prove this for some orientation $\mathscr{O}$. We proceed by induction on the number of interior white vertices. If there are no interior white vertices, then a perfect orientation is simply a total order on the black vertices. Rotation by i induces a permutation on the black vertices with at most one cycle of size 1 and cycles of size $\frac{n}{i}$, and thus $\frac{n}{i} \mid d$ or $\frac{n}{i} \mid d-1$. Thus,

$$\text{sign}(\mathscr{O}, \text{rot}^i(\mathscr{O})) = (-1)^{(\frac{n}{i}-1)k}$$

If there is a single white vertex v, it is necessarily fixed by rotation by i, and thus $\frac{n}{i} = 3$. Rotation by i induces a permutation of the $d+1$ black vertices into cycles of size 3, and thus

$3 \mid d+1$. The orientation $\mathscr{O}_W$ differs from $\mathrm{rot}^i(W)$ via a swivel move at v, a transposition of the two sink vertices adjacent to v, and a 3-cycle applied to each other orbit of sinks. Thus,

$$\mathrm{sign}(\mathscr{O}, \mathrm{rot}^i(\mathscr{O})) = 1 = (-1)^{(\frac{n}{i}-1)k}$$

If there is more than one white vertex, then by Lemma 5.3.5 and our rotation invariance assumption, we can find three black vertices each connected to exactly one interior white vertex such that these three white vertices are distinct. Remove these 6 vertices and connect their neighbors to the boundary in a planar and rotationally invariant way to get a web W'. From any perfect orientation $\mathscr{O}'$ of W' we can build a perfect orientation $\mathscr{O}$ of W by orienting the removed edges from white vertex to black vertex. We then have by inductive hypothesis

$$\mathrm{sign}(\mathscr{O}, \mathrm{rot}^i(\mathscr{O})) = \mathrm{sign}(\mathscr{O}', \mathrm{rot}^i(\mathscr{O}')) = (-1)^{(\frac{n}{i}-1)k}.$$

So the result follows by induction. □

We can now state our cyclic sieving result.

Theorem 5.7.2. *Let $C = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group with generator c acting on $AW(n,d)$ by rotation. Let $X_{n,d}(q)$ be the fake degree polynomial for $S^(d^3, 1^{n-3d})$, i.e.*

$$X_{n,d}(q) = q^{3(d-1)+\binom{n-3(d-1)}{2}} \frac{[n]!_q}{\prod_{(i,j)\in\lambda}[h_{ij}]_q}$$

If n is odd, then the triple

$$(AW(n,d), C, X_{n,d}(q))$$

exhibits the cyclic sieving phenomenon.

If n is even, then we have cyclic sieving up to sign, i.e.

$$|AW(n,d)^{c^i}| = |X_{n,d}(\zeta^i)|$$

where $AW(n,d)^{c^i}$ dentoes the fixed point set of $AW(n,d)$ under the action of c^i, and ζ is a primitive n^{th} root of unity.

Proof. If n is odd, then we can choose orientations $\mathscr{O}_W$ for each web $W \in AW(n,d)$ such that

$$c \cdot [W, \mathscr{O}_W] = [\mathrm{rot}(W), \mathscr{O}_{\mathrm{rot}(W)}]$$

To do so, select a web W from each C-orbit and pick any orientation $\mathscr{O}_W$ for it. For $1 \leq i \leq n$, let $\mathscr{O}_{\mathrm{rot}^i(W)} = \mathrm{rot}^i(\mathscr{O}_W)$. Lemma 5.7.1 guarantees that this is possible even if W has rotational symmetry, as $(-1)^{(\frac{n}{i}-1)k} = 1$.

Thus, $S^{(d^3,1^{n-3d})}$, $\{[W,\mathscr{O}_W] \mid W \in AW(n,d)$ and the rotation action of C satisfy the hypotheses of Theorem 3.5.1 and the result follows.

If n is even, choose any orientation for each web $W \in AW(n,d)$. Then $\{[W,\mathscr{O}_W] \mid W \in AW(n,d)$ is not necessarily fixed by the action of c, but c will act via a *signed* permutation matrix. Lemma 5.7.1 shows that the diagonal of c^i will either contain only 0's and 1's or only 0's and -1's. In either case, $|tr(c^i)| = |AW(n,d)^{c^i}|$, and the proof of Theorem 3.5.1 [48] shows that

$$|AW(n,d)^{c^i}| = |X_{n,d}(\zeta^i)|$$

holds as desired. □

Example 5.7.3. *When $n = 10$, $d = 3$, then $X_{10,3}(q)$ has a rather nice form. The hook lengths are*

6	4	3
5	3	2
4	2	1
1		

and thus $X_{10,3}(q) = q^{12} \begin{bmatrix} 10 \\ 4 \end{bmatrix}_q$. *Since* $X_{10,3}(q)$ *is a single q-binomial (*$n = 10, d = 3$ *is the only case for which this is true), we can verify the cyclic sieving result in this case by checking that the orbits of* $AW(10,3)$ *under rotation are in size-preserving bijection with the orbits of size 4 subsets of* $\{1,\ldots,10\}$ *under cyclic permutation. There are two orbits of size 5 for each set, the orbits containing sequences* $\{1,2,6,7\}$ *and* $\{1,3,6,8\}$ *and the orbits containing webs shown below*

The remaining 20 orbits are all size 10, one web from each is shown below

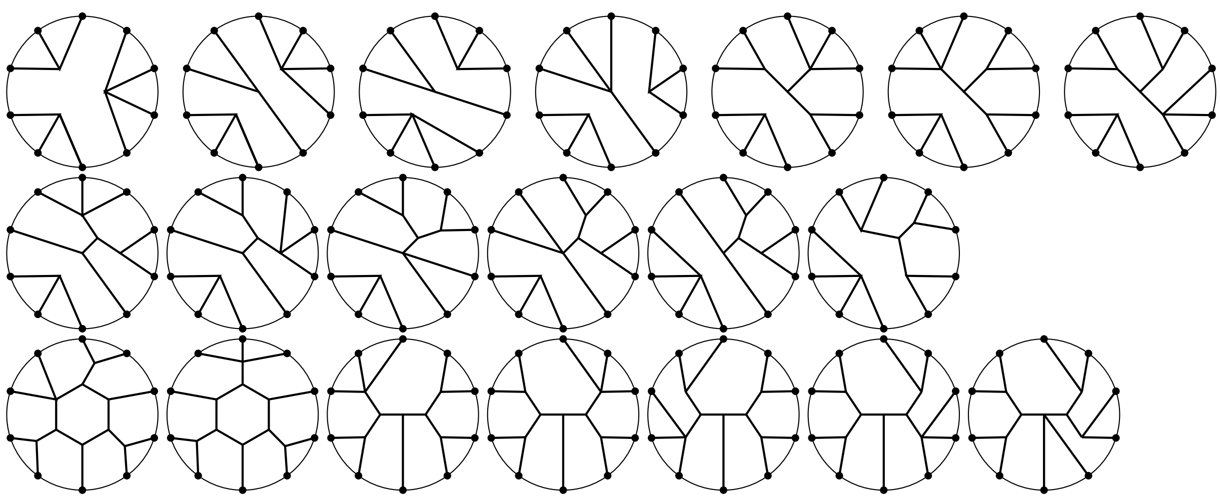


Figure 5.13. The rotation orbits of $AW(10,3)$.

5.8 Future Directions

We have given a rotationally invariant basis for $S^{(d^3,1^{n-3d})}$, and a natural question to ask is whether there is a way to generalize these results to $r > 3$. Many of the results in Section 5 as well as the crossing reduction rule in Section 6 readily generalize when r is odd (when r is even, a different treatment of signs is needed, as a change in orientation will not give a global change in sign). We can thus obtain spanning set for an $\mathfrak{S}_n$ invariant submodule containing $S^{d^r,1^{n-rd}}$ and the question remains as to how to prune it down to a basis, as the results of Section 4 do not seem to readily generalize. There are two directions in which one might approach this problem. The first is to begin by looking for a rotationally invariant set of the right size via $WNC(n,d,r)$:

Problem 5.8.1. *Is there a combinatorially nice injection of $WNC(n,d,r)$ for $r > 3$ into the set of normal plabic graphs, such that the image is closed under rotation?*

The second approach is to determine skein relations first, and use those to reduce the set of normal plabic graphs to a basis.

Problem 5.8.2. *Extend the definitions from Section 5 to $r > 3$ for r odd. What are the corresponding skein relations?*

We expect these questions to likely be quite difficult, as answering both would encompass constructing a rotationally invariant basis of S^{d^r}, a question which was only recently answered in the case $r = 4$ by C. Gaetz, O Pechenik, S. Pfannerer, J. Striker, and J. Swanson [13] and remains open for $r > 4$. However, most investigation into this question has been concerned with finding a *subset* of SL_n webs which forms a basis. Towards this end, it is perhaps a feature, rather than a bug of our construction that it does not consist of genuine SL_r webs, but rather linear combinations of ones with the same underlying simple graph and its minors, as it gives a new place to search.

If we do consider the difference between our augmented web invariants and SL_3 web invariants to be something to be fixed, we can do so by constructing a poset on classical SL_3 webs with $W \leq V$ whenever W is a weblike subgraph of V, which is equivalent to the graph minor poset restricted to SL_3 webs. By Corollary 5.6.2 can thus write

$$[W] = \sum_{V \leq W} h(V,W)[V]_{SL_3}$$

where h is an element of the incidence algebra for this poset which is defined up to sign in Corollary 5.6.2, and the sign is defined implicitly in the preceeding exposition. Inverting h would then recover the classical SL_3 web invariants. We thus propose the following:

Problem 5.8.3. *Extend the definition of h to the poset of perfectly orientable normal plabic graphs with order given by graph minors. Is there a simple combinatorial description of the inverse of h in the incidence algebra?*

One important property of the m-diagram construction of SL_3 webs is that, as shown by Petersen, Pylyavskyy, and Rhoades, it intertwines promotion on rectangular tableaux and rotation of webs [33], thereby giving an algebraic proof of the cyclic sieving phenomenon for promotion on three-row rectangles. This is not the case for $n > 3d$, however, as promotion for tableaux of shape $(d^3, 1^{n-3d})$ for $n > 3d$ is not so well-behaved and the order of promotion does not divide n in general. It may be interesting to investigate if there is a variant of promotion which our bijections do intertwine.

Problem 5.8.4. *Give a combinatorial description similar to promotion of the cyclic action on standard Young tableaux of shape $(d^3, 1^{n-3d})$ given by the pullback of rotation on webs. Does the combinatorial description have a natural extension to other shapes? If so, which shapes have order dividing n?*

A combinatorially defined cyclic action with order dividing n for another family of partition shapes would be good evidence for the existence of a web basis for those shapes. It is not clear that the bijection we give is necessarily the most natural, so in answering this question one may want to consider other possible bijections between standard Young tableaux and augmented webs.

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Bibliography

[1] F. Bergeron. Multivariate Diagonal Coinvariant Spaces for Complex Reflection Groups. *Adv. Math.*, **239** (2013), 97–108.

[2] F. Bergeron. The bosonic-fermionic diagonal coinvariant modules conjecture. Preprint, 2020. `arXiv:2005.00924`.

[3] M.J. Bergvelt and J.M. Rabin. Supercurves, their Jacobians, and super KP equations. *Duke Math. J.*, **98(1)**, (1999), 1–57.

[4] S. Billey, B. Rhoades, and V. Tewari. Boolean product polynomials, Schur positivity, and Chern plethysm. *Int. Math. Res. Not. IMRN*, Volume 2021, Issue 21, 16634–16668, rnz261.

[5] Sabin Cautis, Joel Kamnitzer, and Scott Morrison, *Webs and quantum skew howe duality*, Mathematische Annalen 360 (2012).

[6] M. D'Adderio, A. Iraci, and A. Wyngaerd. Theta operators, refined Delta conjectures, and coinvariants. *Adv. Math.*, **376**, (2021), 107477.

[7] Sergey Fomin and Pavlo Pylyavskyy, *Tensor diagrams and cluster algebras*, Advances in Mathematics **300** (2016), 717–787, Special volume honoring Andrei Zelevinsky.

[8] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky, *Intro to cluster algebras. chapter 7*, (2021).

[9] Bruce Fontaine, *Generating basis webs for* sl_n , Advances in Mathematics **229** (2012), no. 5, 2792–2817.

[10] Chris Fraser, Thomas Lam, and Ian Le, *From dimers to webs*, Transactions of the AMS (2017).

[11] Chris Fraser, Rebecca Patrias, Oliver Pechenik, and Jessica Striker, *Web invariants for flamino specht modules*, preprint (2023).

[12] W. Fulton. *Young Tableaux*. London Mathematical Society Student Texts, no. 35. Cambridge University Press, 1997.

[13] Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Joshua Swanson, *Rotation invariant web bases from hourglass plabic graphs*, preprint (2023).

[14] M. Haiman. Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.* **3**, (1994), 17–76.

[15] M. Hara and J. Watanabe. The determinants of certain matrices arising from the Boolean lattice. *Discrete Math.*, **308 (23)**, (2008), 5815–5822.

[16] C. Haske and R. O. Wells, Jr. Serre duality on complex supermanfolds. *Duke Math. J.*, **54(2)**, (1987), 493–500.

[17] L. Hodgkin. Problems of fields on super Riemann surfaces. *J. Geom. and Phys.*, **6**, (1989), 333–348.

[18] A. Iraci. *Cyclic Sieving for Noncrossing Partitions.* Master's Thesis, Università di Pisa, 2016.

[19] L. Kauffman. State models and the Jones polynomial. *Topology*, **26** (1987), no. 3, 395–407.

[20] J. Kim. A combinatorial model for the fermionic diagonal coinvariant ring. To appear, *Combin. Theory*, 2022. `arXiv:2204.06059`.

[21] Jesse Kim, *An embedding of the skein action on set partitions into the skein action on matchings*, Electronic journal of Combinatorics (2024).

[22] Jesse Kim and Brendon Rhoades, *Set partitions, fermions, and skein relations*, International Mathematics Research Notices **2023** (2022), no. 11, 9427–9480.

[23] J. Kim and B. Rhoades. Lefschetz theory for exterior algebras and fermionic diagonal coinvariants. *Int. Math. Res. Not. IMRN*, (2020).

[24] J. P. S. Kung and G.-C. Rota. The invariant theory of binary forms. *Bull. Amer. Math. Soc.*, **19 (1)** (1984), 27–85.

[25] Greg Kuperberg, *Spiders for rank 2 lie algebras*, Communications in Mathematical Physics **180** (1996), 109–151.

[26] Thomas Lam, *Dimers, webs, and positroids*, Journal of the London Mathematical Society **92** (2014).

[27] D. Mumford. *Abelian Varieties*, pp. 22-24. Oxford University Press, London 1970.

[28] R. Orellana and M. Zabrocki. A combinatorial model for the decomposition of multivariate polynomial rings as an S_n-module. *Electron. J. Combin.*, **27 (3)**, 2020, P3.24.

[29] Rebecca Patrias, Oliver Pechenik, and Jessica Striker, *A web basis of invariant polynomials from noncrossing partitions*, Advances in Mathematics **408** (2022), 108603.

[30] B. Pawlowski, E. Ramos, and B. Rhoades. Spanning line configurations and representation stability. Preprint, 2019. `arXiv:1907.07268`.

[31] Kyle Petersen, Pavlo Pylyavskyy, and Brendon Rhoades, *Promotion and cyclic sieving via webs*, Journal of Algebraic Combinatorics **30** (2008).

[32] O. Pechenik. Cyclic sieving of increasing tableaux and small Schröder paths. *J. Combin. Theory Ser. A*, **125**, 2014, 357–378.

[33] K. Petersen, P. Pylyavskyy, and B. Rhoades. Promotion and cyclic sieving via webs. *J. Algebraic Combin.*, **30**, 2009, 19–41.

[34] Postnikov, *Total positivity, grassmanians, and networks*, (2006).

[35] Pavlo Pylyavskyy, *Non-crossing tableaux*, Annals of Combinatorics **13** (2009), 323–339.

[36] J.M. Rabin. Super elliptic curves. *J. Geom. Phys.*, **15** (1995), 252–280.

[37] V. Reiner, D. Stanton, and D. White. The cyclic sieving phenomenon. *J. Combin. Theory Ser. A*, **108**, 2004, 17–50.

[38] Brendon Rhoades, *Cyclic sieving, promotion, and representation theory*, Journal of Combinatorial Theory, Series A **117** (2010), no. 1, 38–76.

[39] B. Rhoades. A skein action of the symmetric group on noncrossing partitions. *J. Algebraic Combin.*, **45 (1)**, 2017, 81–127.

[40] B. Rhoades. The polytabloid basis expands positively into the web basis. *Forum Math. Sigma*, **7**, 2019, e26.

[41] B. Rhoades and A. T. Wilson. Vandermondes in superspace. *Trans. Amer. Math. Soc.*, **373 (6)**, (2020), 4483–4516.

[42] B. Rhoades and A. T. Wilson. Set superpartitions and superspace duality modules. Preprint, 2021. `arXiv:2104.05630`.

[43] M. Rosas. The Kronecker product of Schur functions indexed by two-row shapes or hook shapes. *J. Algebraic Combin.*, **14 (2)**, (2001), 153–173.

[44] Heather Russell, *An explicit bijection between semistandard tableaux and non-elliptic* sl_3 *webs*, Journal of Algebraic Combinatorics **38** (2013), 851–862.

[45] H. Russell and J. Tymoczko. The transition matrix between the Specht and web bases is unipotent with additional vanishing entries. *Int. Math. Res. Not. IMRN*, 2019 (5), 1479–1502.

[46] C. Sabbah and C. Schnell. The MHM Project (Version 2). 2023.

[47] B. Sagan. *The Symmetric Group,* Springer, New York, 2001

[48] Bruce Sagan, *The cyclic sieving phenomenon: A survey*, (2010).

[49] T. Springer. Regular elements of finite reflection groups. *Inventiones math. 25, 159-198*, 1974.

[50] J. H. Silverman. *The Arithmetic of Elliptic Curves,* 2nd Edition, Appendix B. Springer, New York 2009.

[51] R. Stanley. Variations on differential posets. In *Invariant Theory and Tableaux* (D. Stanton, ed.), The IMA Volumes in Mathematics and its Applications, vol. 19, Springer-Verlag, New York, 1990, 145–165.

[52] J. Swanson. Tanisaki witness relations for harmonic differential forms. Preprint, 2021. `arXiv:2109.05080`.

[53] J. Swanson and N. Wallach. Harmonic differential forms for pseudo-reflection groups I. Semi-invariants. To appear, *J. Combin. Theory Ser. A*, 2021. `arXiv:2001.06076`.

[54] H. N. V. Temperley and E. H. Lieb, *Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the 'percolation' problem*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences **322** (1971), no. 1549, 251–280.

[55] M. Thiel. A new cyclic sieving phenomenon for Catalan objects. *Discrete Math.*, **340 (3)**, (2017), 426–429.

[56] Julianna Tymoczko, *A simple bijection between standard* $3 \times n$ *tableaux and irreducible webs for* sl_3, Journal of Algebraic Combinatorics **35** (2012), 611–632.

[57] B. Westbury. Invariant tensors and the cyclic sieving phenomenon *The Electronic Journal of Combinatorics* **22**, (2015). `arXiv:0912.1512`

[58] H. Weyl, G. Rumer, and E. Teller, *Eine für die valenztheorie geeignete basis der binären vektorinvarianten*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse **1932** (1932), 499–504.

[59] Hassler Whitney, *2-isomorphic graphs*, American Journal of Mathematics **55** (1933), no. 1, 245–254.

[60] M. Zabrocki. A module for the Delta conjecture. Preprint, 2019. `arXiv:1902.08966`.

[61] M. Zabrocki. Coinvariants and harmonics. Blog for Open Problems in Algebraic Combinatorics 2022.