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Journal

Physics Letters B, 96(3-4)

ISSN

0370-2693

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Publication Date

1980-11-03

DOI

10.1016/0370-2693(80)90776-5

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VORTICES IN TWO DIMENSIONAL CHIRAL SPIN SYSTEMS

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Received 29 July 1980

We show that vortex configurations generate a mass gap (finite correlation length) for $T > 0$ in non-abelian chiral $SU(N) \times SU(N)$ spin systems in two dimensions. These vortices and the fluctuations around them mutually stabilize each other in the infrared.

Non-abelian spin systems in two dimensions are believed to generate a mass gap for all nonzero couplings. In the language of statistical mechanics this implies a finite correlation length for all $T > 0$. Which spin configurations are responsible for generating this mass is still an open question for many systems. For the $O(3)$ nonlinear σ models it appears that a dense instanton configuration is responsible [1]. This is likely to be true for the CP^{N-1} models with the $O(3)$ σ model coinciding with CP^1 . This argument cannot be extended to higher $O(N)$ groups as these have no stable instanton solutions.

We shall study the chiral $SU(N) \times SU(N)$ [and $SU(N)/Z_N \times SU(N)/Z_N$] spin systems. Vortex configurations are responsible for generating the mass gap. For $N = 2$ this model is equivalent to the $O(4)$ nonlinear σ model and thus has no stable instanton solutions. Naively, these vortices have similar properties to those in the abelian $X-Y$ model and we might expect a mass gap to appear only at finite T [2]. However, unlike the abelian case, the spin wave fluctuations do have important interactions with the vortices and they mutually stabilize one another's infrared behavior. The action, partition function and correlation function for this model are

$$S = T^{-1} \int \text{Tr}(\partial_\mu g^\dagger \partial_\mu g) d^2x, \quad (1)$$

$$Z = \int [dg] \exp(-S), \quad (2)$$

$$\langle g^+(r)g(0) \rangle = Z^{-1} \int [dg] \exp(-S) g^+(r)g(0). \quad (3)$$

g is a unitary unimodular $N \times N$ matrix and $[dg]$ the invariant group integration over $SU(N)$. The action of (1) is invariant under $g \rightarrow U_L g U_R^\dagger$, with $U_{L,R}$ constant elements of $SU(N)$. This theory is asymptotically free in T [3]. We will provide a proof of this for a variant of the theory which can be transcribed to the present case. This model does have vortex solutions

$$g(r) = \exp(i\hat{r}\varphi). \quad (4)$$

$\hat{r}$ is an element of the algebra of $SU(N)$ such that $\exp(2\pi i n \hat{r})$ belongs to the center, Z_N , of the group; $n = 0, 1, \dots, N-1$. We have taken the vortex to be centered at the origin and φ is the azimuthal angle. The action for this configuration is logarithmically divergent both in the ultraviolet and the infrared. The first divergence is handled in the usual way by introducing a cut-off Λ reflecting an underlying discrete lattice structure of the theory. The infrared divergence is eliminated by considering configurations of many vortices with net vorticity zero. It is because of the latter point and difficulties in analyzing fluctuations around the solution of eq. (4) that we modify this model somewhat.

As g is an element of $SU(N)$,

$$A_\mu = i g^\dagger \partial_\mu g \quad (5)$$

is a pure gauge Yang-Mills potential. In terms of A_μ the action is

$$S = T^{-1} \int \text{Tr} A_\mu A_\mu d^2x. \quad (6)$$

To rewrite eqs. (2) and (3) in terms of A_μ would involve a complicated jacobian. Rather than this we will study a theory where the equations corresponding to eqs. (2) and (3) are

$$Z = \int [dA_\mu] \delta(F_{\mu\nu}) \exp(-S), \quad (7)$$

$$\begin{aligned} \langle g^+(r) g(0) \rangle &= Z^{-1} \int [dA_\mu] \delta(F_{\mu\nu}) \exp(-S) \\ &\times P \exp \left(i \int_0^r A \cdot dl \right). \end{aligned} \quad (8)$$

The δ -function guarantees that the integration is restricted to pure gauges as required by (5). Classically the theory described by (6) with the pure gauge constraint is equivalent to the previous one. The quantum mechanics (or statistical mechanics) is not the same. However, we feel that the essential features for small T are common to both. The symmetry is the same; the number of degrees of freedom is the same (only in two dimensions); both theories are asymptotically free with the same β function at small T . The last point is important as there exist other classically equivalent versions of the chiral models that are not asymptotically free [4].

To emphasize the above we shall now evaluate the one loop contribution to the renormalization group function. The technique we use is due to Polyakov [5]. We separate the potential A_μ into a slow part $\tilde{A}_\mu$ with Fourier components $|k| < \tilde{\Lambda}$, $\tilde{\Lambda} < \Lambda$, and a fast part $A^{(f)}$ with components $\tilde{\Lambda} < |k| < \Lambda$. We require both A and $\tilde{A}$ to be pure gauges. This forces

$$\begin{aligned} A_\mu^{(f)} &= D_\mu(\tilde{A}) \epsilon + \frac{1}{2} i [\epsilon, D_\mu(\tilde{A}) \epsilon] + O(\epsilon^3), \\ D_\mu(\tilde{A}) \epsilon &= \partial_\mu \epsilon - i [\tilde{A}, \epsilon]. \end{aligned} \quad (9)$$

For small T it is sufficient to keep only quadratic terms in ϵ . Integrating over ϵ we find that the theory with a cutoff Λ is the same as one with a cutoff $\tilde{\Lambda}$ and the temperatures related by

$$1/T(\tilde{\Lambda}) = 1/T(\Lambda) - (N/8\pi) \ln(\Lambda/\tilde{\Lambda}). \quad (10)$$

This is the same as the equivalent relation for a theory based on eqs. (1), (2) and (3).

Based on eq. (10) the mass gap is expected to behave

as

$$m(T) = \Lambda \exp[-8\pi/NT(\Lambda)]. \quad (11)$$

The problem we address ourselves to is what configurations are responsible for this mass gap. As previously claimed, we will show that a gas of vortices, whose infrared behavior is stabilized by fluctuations, generate this behavior. Our treatment will be approximate and we will obtain the correct N and T dependence of $m(T)$; the constants [the 8π of eq. (11)] are beyond our powers.

We rewrite eqs. (7) and (8) using a gaussian representation for the δ -function

$$\begin{aligned} Z &= \lim_{g \rightarrow 0} \int [dA_\mu] \\ &\times \exp \left\{ -\text{Tr} \int d^2x [A^2/T + (2g^2)^{-1} F_{\mu\nu} F_{\mu\nu}] \right\}, \end{aligned} \quad (12)$$

with a similar expression in place of eq. (8). Rescaling the field A to A/g we obtain (limit $g \rightarrow 0$ implied)

$$\begin{aligned} Z &= \int [dA_\mu] \exp \left[-\int d^2x \text{Tr} \left(\frac{1}{2} F_{\mu\nu} F_{\mu\nu} + \mu^2 A_\mu A_\mu \right) \right], \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \quad \mu^2 = g^2/T. \end{aligned} \quad (13)$$

We recognize in (13) the action for a massive Yang–Mills theory with mass μ and coupling constant g . In the $g \rightarrow 0$ limit both vanish. This limit cannot be taken naively as it would result in a set of uncoupled abelian vector fields with no gauge fixing terms. The massive Yang–Mills theory is renormalizable and asymptotically free in T [6] with a renormalization group relation identical to (10).

Again the massive Yang–Mills theory in two dimensions has vortex solutions [7]

$$A_\mu^{(v)} = (i/g) \hat{\Phi}_\mu \mu K_1(\mu\rho). \quad (14)$$

$\hat{t}$ is as before; $\hat{\Phi}_\mu$ is the unit vector in the azimuthal direction and ρ is the radial coordinate. The normalization is such that the $1/\rho$ singularity is a pure gauge and does not contribute to the $F_{\mu\nu} F_{\mu\nu}$ term. The action of a vortex configuration is

$$S^{(v)} = (\pi/T) \text{Tr} \hat{t}^2 \ln(\Lambda^2/\mu^2). \quad (15)$$

As discussed earlier this action would have an infrared divergence in the $g \rightarrow 0$, implying $\mu \rightarrow 0$, limit. Likewise, the interaction among vortices would become long

ranged. If the a_μ denote spin fluctuations around the vortex, then the portion of the action quadratic in these is

$$S^{(fl)} = \int d^2x \operatorname{Tr} \left\{ \frac{1}{2} (D_\mu a_\nu - D_\nu a_\mu)^2 - ig a_\nu [F_{\mu\nu}^{(v)}, a_\mu] + \mu^2 a^2 \right\}, \quad (16)$$

$$D_\mu a_\nu = \partial_\mu a_\nu - ig [A_\mu^{(v)}, a_\nu].$$

This action is singular in the $g \rightarrow 0$ limit. Our argument is that the fluctuations and the vortices mutually stabilize one another. In addition to the explicit mass term μ in (16) the vortex acts as a damping term for the fluctuations. The important terms in (16) that act as a mass are

$$g^2 \operatorname{Tr} [a_\nu [A_\mu, [A_\mu, a_\nu]] - a_\nu [A_\mu, [A_\nu, a_\mu]]]. \quad (17)$$

The picture we have in mind is the plane filled with nonoverlapping vortices, namely their separation greater than $1/\mu$, and pointing in random directions in group space. The effective mass that such configurations contribute to the fluctuations is the average of (17) over the area a vortex acts. This area is proportional to $1/\mu^2$:

$$\mu^2 \delta_{\mu\nu} \delta^{\alpha\beta} \approx -\mu^2 \int d^2\rho dR R^{ll'} R^{jj'} t^l_t{}^{j'} f^{\alpha lm} f^{mj\beta} \times [\delta_{\mu\nu} - \hat{\Phi}_\mu \cdot \hat{\Phi}_\nu] \mu^2 K_1^2(\mu\rho), \quad (18)$$

where $t^l = \operatorname{Tr} \lambda^l \hat{t}$, $\lambda^l/2$ is a generator of $SU(N)$, $f^{\alpha\beta\gamma}$ are the structure constants of $SU(N)$ and $R^{ll'}$ a rotation in this group. dR denotes the invariant integration over the group. Evaluating (18) we obtain the consistency condition

$$1 \approx N \operatorname{tr} \hat{t}^2 \ln(\Lambda^2/\mu^2). \quad (19)$$

We see that the vortices become short ranged with a finite action [cf. eq. (15)]

$$S^{(v)} \sim (NT)^{-1}. \quad (20)$$

This result is true for both $SU(N) \times SU(N)$ and $SU(N)/Z_N \times SU(N)/Z_N$. Due to the generation of a finite size for the vortices their interactions do not play a significant role.

The contribution of the vortices to the correlation function, eq. (8), can be easily evaluated [8]:

$$\langle g^+(r) g(0) \rangle = \exp(-m(T)|r|), \quad (21)$$

with

$$m(T) = \Lambda \exp(-c/NT). \quad (22)$$

Our techniques are not powerful enough to evaluate c . Except for this point (22) agrees with (11) lending support to the claim that vortices stabilized by the fluctuations around them generate the mass gap.

This work was supported by a grant from the National Science Foundation.

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