

# UC Riverside

## UC Riverside Previously Published Works

### Title

Killing fields generated by multiple solutions to the Fischer–Marsden equation II

### Permalink

<https://escholarship.org/uc/item/5q26h4th>

### Journal

International Journal of Mathematics, 27(10)

### ISSN

0129-167X

### Authors

Guan, Daniel  
Cernéa, Paul

### Publication Date

2016-09-01

### DOI

10.1142/s0129167x16500804

### Copyright Information

This work is made available under the terms of a Creative Commons Attribution-NonCommercial License, available at <https://creativecommons.org/licenses/by-nc/4.0/>

Peer reviewed

International Journal of Mathematics  
 Vol. 27, No. 9 (2016) 1650080 (6 pages)  
 © World Scientific Publishing Company  
 DOI: 10.1142/S0129167X16500804



## Killing fields generated by multiple solutions to the Fischer–Marsden equation II

Daniel Guan<sup>\*,†</sup> and Paul Cernéa<sup>†,§</sup>

<sup>\*</sup>*Department of Mathematics, University of California  
Riverside, CA 92521, USA*

<sup>†</sup>*Ooyala Video Technology, Santa Clara  
California, USA*

<sup>‡</sup>*zguan@math.ucr.edu*

<sup>§</sup>*pcernea@gmail.com*

Received 22 February 2016

Accepted 13 July 2016

Published

In the process of finding Einstein metrics in dimension  $n \geq 3$ , we can search metrics critical for the scalar curvature among fixed-volume metrics of constant scalar curvature on a closed oriented manifold. This leads to a system of PDEs (which we call the Fischer–Marsden Equation, after a conjecture concerning this system) for scalar functions, involving the linearization of the scalar curvature. The Fischer–Marsden conjecture said that, if the equation admits a solution, the underlying Riemannian manifold is Einstein. Counter-examples are known by Kobayashi and Lafontaine, and by our first paper. Multiple solutions to this system yield Killing vector fields. We showed in our first paper that the dimension of the solution space  $W$  can be at most  $n + 1$ , with equality implying that  $(M, g)$  is a sphere with constant sectional curvatures. Moreover, we also showed there that the identity component of the isometry group has a factor  $\text{SO}(\dim W)$ . In this second paper, we apply our results in the first paper to show that either  $(M, g)$  is a standard sphere or the dimension of the space of Fischer–Marsden solutions can be at most  $n - 1$ .

*Keywords:* Critical metrics; metrics with constant scalar curvature; sphere; elliptic PDE system; killing vector fields; isometry group; Fischer–Marsden solutions; cohomogeneity one metrics; scalar curvature functional; linearization of scalar curvature.

Mathematics Subject Classification 2010: 53B21, 53C10, 53C21, 53C24, 53C25

### 1. Introduction and Summary of Results

Let  $M$  be a closed, connected, orientable manifold of dimension  $n \geq 3$ . Consider the scalar curvature  $s$  as a function on the space  $\mathfrak{S}$  of Riemannian metrics of fixed (unit) volume and constant scalar curvature. Define the Laplacian as the trace of the Hessian  $\Delta = g^{ij} \nabla_i \nabla_j$ . Eigenvalues of the Laplacian are (necessarily non-negative) constants  $\lambda \geq 0$  for which there exist functions  $u \in C^\infty(M)$ , not identically zero,

*D. Guan & P. Cernéa*

such that

$$\Delta u + \lambda u = 0. \quad (1)$$

Beware that in Besse [1], for instance, the opposite sign convention is used for  $\Delta$ . From Koiso [8], we can conclude that, for any  $g \in \mathfrak{S}$ , if  $s/(n-1)$  is not a positive eigenvalue of the Laplacian, then, for any symmetric bilinear 2-tensor  $h$  such that

$$Lh := \nabla^i \nabla^j h_{ij} - \Delta(h_{ij} g^{ij}) - h_{ij} R^{ij} = 0 \quad \text{and} \quad \int_M h_{ij} g^{ij} d\mu = 0 \quad (2)$$

we can find a one-parameter family  $g(t)$  in  $\mathfrak{S}$  with  $g'(0) = h$ . Thus, for generic  $g \in \mathfrak{S}$ , the set of these  $h$  can be thought of as the tangent space of  $\mathfrak{S}$ .  $L$  is in fact the linearization of the scalar curvature, so that

$$\frac{\partial}{\partial t} (s_{g+th+O(t^2)})_{t=0} = Lh. \quad (3)$$

Following [1, p. 128], suppose  $g$  is a metric with  $s/(n-1)$  not a positive eigenvalue of the Laplacian (so  $s = 0$  is allowed). Define a metric  $g \in \mathfrak{S}$  to be *critical* for the Einstein–Hilbert action  $\mathcal{E}(g) = \int_M s_g d\mu$  if, given any one-parameter family  $g(t)$  in  $\mathfrak{S}$  with derivative  $g'(0) = h$  as above, we have  $\frac{d}{dt} \mathcal{E}(g)(0) = 0$ . Then (in [1, Remark 4.48], see also [1, 4.47])  $g$  is critical in this sense if and only if there exists some function  $f \in C^\infty(M)$ , such that

$$(L^* f)_{ij} := \nabla_i \nabla_j f - (\Delta f) g_{ij} - f R_{ij} = R_{ij} - \frac{s}{n} g_{ij}, \quad (4)$$

where  $L^*$  denotes the  $L^2$ -adjoint of  $L$ . Without assuming that  $s$  is constant, one could always prove that (e.g. see [2, Proposition 2.3])  $s$  is a non-negative constant, if there is nontrivial function in the kernel of  $L^*$ . Now, taking the trace of Eq. (4), we obtain

$$\Delta f + \frac{s}{n-1} f = 0 \quad (5)$$

so that, since  $s/(n-1)$  is not a positive eigenvalue, we must have  $f$  a constant (in fact, zero) and  $g$  must be an *Einstein manifold*  $R_{ij} = (s/n)g_{ij}$ . Besse [1, 4.48] goes further and asks, what if  $s/(n-1)$  is in the spectrum? if  $g$  obeys Eq. (4) (and so is formally critical), must  $g$  be Einstein? If  $g$  is not Einstein,  $f$  cannot be a constant.

In our work, we choose to focus on what happens if there are multiple solutions  $f_1$  and  $f_2$  to (4). Indeed, since  $f$  is an eigenfunction of the Laplacian, we can write  $u := 1 + f$  and rewrite (4) as the *critical metric equation*

$$\nabla_i \nabla_j u = u R_{ij} - \frac{s}{n-1} \left( u - \frac{1}{n} \right) g_{ij}. \quad (6)$$

If  $u = 1 + f_1$  and  $v = 1 + f_2$  are solutions to the above, then their difference  $x = f_1 - f_2$  solves the linear equation

$$\nabla_i \nabla_j x = x \left( R_{ij} - \frac{s}{n-1} g_{ij} \right) \quad (7)$$

and  $x$  is an eigenfunction of the Laplacian with eigenvalue  $s/(n-1)$ . We call  $x$  a *Fischer–Marsden solution*. The *Fischer–Marsden Conjecture* asked whether  $g$  that

*Killing fields generated by multiple solutions to the Fischer–Marsden equation II*

satisfy (7) are Einstein. Counterexamples to that have been found (see, for instance, Kobayashi [7] and Lafontaine [9] as well as our first paper [3]). We notice that, in almost all the known examples, the dimension of the solution space of (7) is at least 2.

If  $u$  and  $v$  are solutions to the critical metric equation (6), then  $(udv - vdu)^\sharp$  is a *conformal Killing field*. Even nicer, if  $x$  and  $y$  are Fischer–Marsden solutions, then

$$Y = x\nabla y - y\nabla x \quad (8)$$

is a *Killing field* (as observed in Lafontaine [10], where the situation in dimension  $n = 3$  is studied).

In our first paper [3], we proved

$$\text{Ric}(Y) = \rho Y \quad (9)$$

for some smooth function  $\rho$  defined, where  $Y \neq 0$  (depending on  $g$ , but not on choice of  $Y$ ).

One can prove that the space of Fischer–Marsden solutions has dimension less than or equal to  $(n + 1)$  with equality only if  $(M, g)$  has constant curvature (the dimension bound also appeared in [2, Corollary 2.4], but, it did not mention what happens when the dimension achieves the upper bound  $n + 1$  there). In fact, we proved in our first paper [3], the following stronger statement.

**Proposition 1.** *Let  $W$  be the space of Fischer–Marsden solutions of (7), and  $\text{Iso}_0$  be the identity component of the isometry group of  $(M, g)$ . Then  $\text{Iso}_0$  is locally  $\text{SO}(\dim W) \times G_1$  with a compact Lie group  $G_1$ , which is the kernel of the action of  $\text{Iso}_0$  on  $W$ . Moreover, all the  $\text{SO}(\dim W)$  orbits are either  $S^{\dim W - 1}$  or its fixed points.*

Therefore, if  $\dim W = n$ , then the  $\text{SO}(\dim W)$  orbits are either  $S^{n-1}$  or fixed points. In particular,  $M$  is cohomogeneity one with  $S^{n-1}$  as the generic orbits. There is an  $\text{SO}(\dim W)$  equivariant dense open set  $N$  of  $M$  such that  $N = I \times S^{n-1}$  and  $g = dt^2 + (f(t))^2 g_0$  with  $g_0$  the standard metric on  $S^{n-1}$ . We now state our main result.

**Theorem.** *Let  $(M, g)$  be a compact Riemannian manifold with a dimension  $n \geq 3$ , and let  $W$  be the space of the nonconstant solutions of the Fischer–Marsden equation. If  $\dim W \geq n$ , then  $(M, g)$  is a standard sphere.*

In Sec. 2, all Riemann metrics are assumed to have positive constant scalar curvatures. We notice that, if (7) holds,  $s$  is a constant. Since  $s$  is an eigenvalue, it is non-negative. If  $s = 0$ , our solutions are constants. Therefore  $xR_{ij} = 0$  and  $g$  is Ricci flat. In particular, if  $s = 0$ ,  $g$  is Einstein.

We notice that our situation here is a special case of the cohomogeneity one case. We should deal with the cohomogeneity one case in the near future. Our method of warp products could be regarded as a reformulation of the reduction method in [10]. We shall apply our new method to the general case.

*D. Guan & P. Cernéa*

## 2. The Proof

In this section, we state and prove several lemmas which will establish the main Theorem, so we assume that we are working under the conditions in the Theorem.

By choosing the right orthonormal solutions  $x$  and  $y$  in Proposition 4 in our first paper [3], we might assume that  $Yx = \beta y = -y$  and  $Yy = -\beta x = x$ , i.e.  $\beta = -1$ .

**Lemma 1.** *Let  $Y = x \nabla y - y \nabla x$  be a Killing vector field generated by orthonormal Fischer–Marsden solutions  $x$  and  $y$ . Then  $|Y|^2 = x^2 + y^2$ .*

**Proof.** By the proof of Proposition 1 as well as [3, Proposition 4], we have

$$|Y|^2 = xY(y) - yY(x) = x^2 + y^2. \quad (10)$$

This proves the lemma.  $\square$

Recall on  $N \subset M$   $g$  has the form  $dt^2 + (f(t))^2 g_0$  with  $g_0$  the standard metric on  $S^{n-1}$ .

We also notice that  $Y$  acts on each of the  $\text{SO}(\dim W)$  orbits and does not depend on  $t$ .  $W$  gives us an  $\text{SO}(\dim W)$  equivariant conformal map from each orbit to its image with a metric  $(h(t))^2 g_0$ . Every function in  $W$  is just a linear function in the image. Therefore,  $x = h(t)x_0$  for some function  $h(t) > 0$  and some linear function  $x_0$  on  $S^{n-1}$ . Moreover, we have.

**Lemma 2.** *Let  $x_0$  be a Fischer–Marsden solution on the standard  $S^{n-1}$ . We have that  $x = f(t)x_0$  is a solution on  $(M, g)$ .*

**Proof.** We have

$$|Y|^2 = (f(t))^2 |Y|_0^2 = (f(t))^2 (x_0^2 + y_0^2) = x^2 + y^2.$$

Since  $(x, y)$  is proportional to  $(x_0, y_0)$ , we have  $x = f(t)x_0$ .  $\square$

**Lemma 3.**  *$ff'' - (f')^2 + 1 = 0$  and  $1 - (f')^2 = \frac{s}{n(n-1)} f^2$ .*

**Proof.** We have  $\Delta x + \frac{s}{n-1}x = 0$  and  $\Delta_0 x_0 + (n-1)x_0 = 0$ . We choose an orthonormal basis

$$\partial_t, e_1/f, \dots, e_{n-1}/f$$

near our considered point on  $M$  with  $e_j, j = 1, \dots, n-1$ , an orthonormal basis on the standard  $S^{n-1}$ .

We notice that

$$\text{grad}(x) = f'x_0\partial_t + f^{-1}\text{grad}_0(x_0).$$

By Proposition 35 in [6, p. 206], we have

$$\begin{aligned} \nabla_{\partial_t}(f'x_0\partial_t + f^{-1}\text{grad}_0(x_0)) &= f''x_0\partial_t + (f'/f)f^{-1}\text{grad}_0(x_0) - f'/f^2\text{grad}_0(x_0) \\ &= x_0f''\partial_t, \end{aligned} \quad (11)$$

*Killing fields generated by multiple solutions to the Fischer–Marsden equation II*

where we applied the Leibniz rule to the second term. We also have the component of the second covariant derivatives tangent to the orbits:

$$\tan(\nabla_{e_j/f}(f'x_0\partial_t + f^{-1}\text{grad}_0x_0)) = x_0(f')^2/f^2e_j + f^{-2}\nabla_{e_j}^0\text{grad}_0x_0.$$

The Laplace of  $x$  is:

$$\begin{aligned}\nabla_i\nabla_ix &= x_0f'' + (n-1)((f')^2/f)x_0 + (\Delta_0x_0)/f \\ &= x_0[f'' + (n-1)(f')^2/f - (n-1)/f],\end{aligned}\tag{12}$$

where  $i$  sums from 0 to  $n-1$ .

Therefore, since  $\Delta x = -sx/(n-1) = -sfx_0/(n-1)$ , we have:

$$ff'' + (n-1)((f')^2 - 1) = -sf^2/(n-1).\tag{13}$$

On the other hand, by Corollary 43 in [6, p. 211], we have  $\text{Ric}(\partial_t, \partial_t) = -(n-1)f''/f$ ,

$$\text{Ric}(e_j/f, e_j/f) = (n-2)/f^2 - f''/f - (n-2)(f')^2/f^2.$$

That is,

$$\begin{aligned}s &= (n-1)[-f''/f + ((n-2)/f^2 - f''f - (n-2)(f')^2/f^2)] \\ &= -(n-1)[2f''/f - (n-2)(1 - (f')^2)/f^2].\end{aligned}\tag{14}$$

Compare these two equations, we get our Lemma.  $\square$

**Lemma 4.**  $f = A^{-1}\sin(At + B)$  with  $A^2 = \frac{s}{n(n-1)}$  and  $A > 0$ .

**Proof.** We have  $f' = \pm\sqrt{1 - (Af)^2}$ . But  $f$  is not constant, so we have

$$Af = \sin(\pm At + B).\tag{15}$$

But

$$\sin(-At + B) = -\sin(At - B) = \sin(At - B + \pi)\tag{16}$$

which proves the lemma.  $\square$

With all these in place, we can establish our main Theorem.

### Acknowledgments

Thanks to Professor Michael T. Anderson for making the second author aware of problems involving the linearization of the scalar curvature. We are also very grateful for the kind comments from the referees, which made this paper much better. After we submitted this paper, the first author gave a talk in the differential geometry seminar in the University of California at Riverside. Dr. He, Chenxu kindly told us their related recent work [4]. We thank him here for sharing their very interesting works with us.

*D. Guan & P. Cernéa*

### References

- [1] A. L. Besse, *Einstein Manifolds* (Springer-Verlag, 1987).
- [2] J. Corvino, Scalar curvature deformation and a gluing construction for the Einstein constraint equations, *Comm. Math. Phys.* **214** (2000) 137–189.
- [3] P. Cernéa and D. Guan, Killing fields generated by multiple solutions to Fischer–Marsden equation, *Internat. J. Math.* **26**(4) (2015), doi: 10.1142/S0129167X15400066.
- [4] C. He, P. Petersen and W. Wylie, Warped product rigidity, *Asian J. Math.* **19** (2015) 135–170.
- [5] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I, II* (John Wiley & Sons Inc, 1996).
- [6] B. O’Neill, *Semi-Riemannian Geometry* (Academic Press Inc, 1983).
- [7] O. Kobayashi, A differential equation arising from scalar curvature function, *J. Math. Soc. Japan* **34** (1982) 665–675.
- [8] N. Koiso, A decomposition of the space  $\mathcal{M}$  of Riemannian metrics on a manifold, *Osaka J. Math.* **16** (1979) 423–429.
- [9] J. Lafontaine, Sur la geometrie d’une generalisation de l’equation differentielle d’Obata, *J. Math. Pures Appl.* **62** (1983) 63–72.
- [10] J. Lafontaine, A remark about static space times, *J. Geom. Phys.* **59**(1) (2009) 50–53.