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Asymptotic behavior of a fluid model for bandwidth sharing with general file size distributions

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Asymptotic behavior of a fluid model for bandwidth sharing with general file size distributions

A dissertation submitted in partial satisfaction of the
requirements for the degree
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in

Mathematics

by

Yingjia Fu

Committee in charge:

Professor Ruth J. Williams, Chair
Professor Jelena Bradic
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Professor Tara Javidi
Professor Jason Schweinsberg

2021

The dissertation of Yingjia Fu is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2021

DEDICATION

The dissertation is dedicated to my parents, grandparents and my boyfriend.

EPIGRAPH

If I have seen further than others,

it is by standing upon the shoulders of giants.

—Isaac Newton

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ABSTRACT OF THE DISSERTATION

Asymptotic behavior of a fluid model for bandwidth sharing with general file size distributions

by

Yingjia Fu

Doctor of Philosophy in Mathematics

University of California San Diego, 2021

Professor Ruth J. Williams, Chair

We study the asymptotic behavior of solutions to a fluid model for a data communication network, where file sizes are *generally distributed* and the network operates under a *fair bandwidth sharing policy*, chosen from the family of (weighted) α-fair policies introduced by Mo and Walrand (2000). Solutions of the fluid model are measure-valued functions of time. Under law of large numbers scaling, Gromoll and Williams (2009) proved that these solutions approximate dynamic solutions of a flow level model for congestion control in data communication networks, introduced by Massoulié and Roberts (2000).

Our first result is the stability of the strictly *subcritical* version of this fluid model under

mild assumptions. For this, using a slight generalization of a Lyapunov function proposed by Paganini et al. (2012), and taking into account that some fluid model solution components may reach zero while others are positive, we prove that the Lyapunov function composed with a subcritical fluid model solution converges to zero as time goes to infinity. Our second result is on the asymptotic behavior (as time goes to infinity) of solutions of the *critical* fluid model, in which the nominal load on each network resource is less than or equal to its capacity and at least one resource is fully loaded. For this we introduce a new Lyapunov function, inspired by the work of Kelly and Williams (2004), Mulvany et al. (2019) and Paganini et al. (2012). Using this, under moderate conditions on the file size distributions, we prove that critical fluid model solutions converge uniformly to the set of invariant states as time goes to infinity, when started in suitable relatively compact sets. We expect that this result will play a key role in developing a diffusion approximation for the critically loaded flow level model of Massoulié and Roberts (2000). Furthermore, the techniques developed here may be useful for studying other stochastic network models with resource sharing.

Chapter 1

Introduction

1.1 Overview

The design and analysis of congestion control mechanisms for modern data networks such as the Internet is a challenging problem. Mathematical models at various levels have been introduced in an effort to provide insight into some aspects of this problem. In particular, Massoulié and Roberts [MR00] introduced a stochastic model called a flow level model that aimed to capture the connection level dynamics of file arrivals and departures in a network where bandwidth is dynamically shared amongst flows which correspond to continuous transfers of individual elastic files. A natural family of "fair" bandwidth sharing policies was introduced by Mo and Walrand [MW00] around the same time. These policies are often referred to as (weighted) α-fair policies, since a parameter $\alpha \in (0,\infty)$ (and optional weight parameters) is associated with the family. The cases $\alpha = 1$ (proportional fairness with equal weights) and $\alpha \to \infty$ (max-min fairness) have received particular attention.

1.1.1 Stability

One of the first natural questions to ask about the flow level model operating under an α-fair bandwidth sharing policy is "when is it stable?". Here we take stability to mean that a Markov process describing the model is positive Harris recurrent. Assuming Poisson arrivals and exponential file sizes, this is a solved problem. Indeed, under these assumptions, Lyapunov functions constructed by De Veciana et al. [DVLK01] for max-min fair and proportionally fair policies, and by Bonald and Massoulié [BM01] for α-fair policies ($\alpha \in (0,\infty)$), can be used to establish positive recurrence of the Markov chain that tracks the number of flows on each route, provided the network is subcritically loaded, i.e., the average load on each link is less than its capacity. Kelly and Williams [KW04] proved that subcriticality is necessary for positive recurrence of the Markov chain. Ye et al. [YOY05] generalized the stability result to where the arrival processes are stationary renewal processes, but the file sizes are still exponentially distributed, and the bandwidth sharing policies come from a class of utility based policies that include the α-fair policies.

When the interarrival time and file sizes are generally distributed, the process that records the number of flows on each route is usually not Markovian and a more complicated Markovian state descriptor is needed to track the dynamics of the model. Much less is known concerning stability in this general situation, although a few cases have been treated. Massoulié [Mas07] showed stability of subcritical networks under the proportionally fair policy with Poisson arrivals and phase-type distributions for file sizes. Bramson [Bra10] proved that subcritical networks operating under max-min fair policies and having general interarrival and file size distributions are stable, provided the file size distributions have finite pth moments for some $p > 2$.

One general approach to exploring stability of stochastic networks uses fluid models, solutions of which are obtained as functional law of large numbers limits from the original stochastic network. The idea of this approach is to first prove that the fluid model for a subcritical network is stable (i.e., all fluid model solutions converge towards the zero state) and then to use this to infer stability of the original stochastic model. This methodology has been successfully used to obtain sufficient conditions for stability of a variety of multiclass queueing networks (see [Bra08, Dai95] and the references therein) and was the approach used in the work by Massoulié [Mas07] mentioned above.

Gromoll and Williams [GW09] used a measure-valued process to track the dynamics of the flow level model with general interarrival and file size distributions when operating under a member of a family of fairly general bandwidth sharing policies that includes the α-fair policies of Mo and Walrand [MW00]. They showed that, under law of large numbers scaling, the measure-valued processes corresponding to a sequence of flow level models are tight and any weak limit point of the sequence is almost surely a continuous solution of a measure-valued fluid model. In [GW08], the same authors also established stability of the fluid model for α-fair bandwidth sharing policies ($\alpha \in (0,\infty)$), for linear networks and simple tree networks under subcritical loading. In this context, the zero state is the measure with each component equal to the zero measure on $[0,\infty)$.

Chiang et al. [CST06] obtained the same fluid model as [GW09] (but with a zero initial condition) from the flow level model via a different law of large numbers scaling limit in which the arrival rate and bandwidth capacity are allowed to grow to infinity proportionally, but the bandwidth per flow stays uniformly bounded. They used the fluid model to derive some conclusions concerning rate stability for the flow level model when file sizes have general distributions with compact support, and for bandwidth sharing policies that are a slight generalization of the α-fair policies of [MW00], in which the parameter $\alpha \in (0,\infty)$ is allowed to vary with the route. For their stability result, their α parameters need to be sufficiently small.

Paganini et al. [PTFA12] developed a Lyapunov function to study the stability of the fluid model introduced by Gromoll and Williams [GW09] for all α-fair policies ($\alpha \in (0,\infty)$). Using this function, under the assumptions that fluid model solutions are sufficiently smooth that they have densities that are strong solutions of a nonlinear parabolic partial differential equation, and that no fluid level on any route touches zero before all route levels reach zero, Paganini et al. [PTFA12] proved stability of the subcritical fluid model. The aim of Chapter 3 is to prove stability of the subcritical fluid model without the strong assumptions of Paganini et al. [PTFA12].

1.1.2 Critical Behavior

Beyond issues of stability, the performance of the flow level model when some resources are operating at or near capacity, is of particular interest. Indeed, as generally observed by Kelly and Laws [KL93], in the heavily loaded regime, important features of good control policies are often displayed in sharpest relief. Furthermore, system designers and managers often strive to position systems in this regime to achieve maximal utilization of resources. Diffusion approximations have provided useful and insightful measures of performance for various heavily loaded stochastic networks (see the survey article by Williams [Wil16] and references therein). For open multiclass queueing networks with head-of-the-line service, Bramson [Bra98] and Williams [Wil98] developed a modular approach to establishing diffusion approximations for these networks when heavily loaded. A key aspect of this approach was to analyze the asymptotic behavior of critical fluid model solutions and to use this analysis to establish a dimension reduction property called multiplicative state space collapse, which provided a crucial step in proving a diffusion approximation. (The fluid models associated with heavily loaded stochastic networks are called critically loaded, meaning that in the fluid model, the nominal load on each resource is less than or equal to its capacity and at least one resource is at capacity.) Various authors have expanded and adapted the approach of Bramson [Bra98] and Williams [Wil98], to establish diffusion approximations for a variety of other heavily loaded stochastic networks.

For the flow level model of Massoulié and Roberts [MR00], there are a few works establishing diffusion approximations under certain distributional, control or network assumptions. All of these use analysis of fluid models as a key ingredient. In general, it remains an open problem to establish a diffusion approximation for the flow level model with general interarrival time and file size distributions when operating under α-fair bandwidth sharing policies. We provide a brief summary of existing work in this area and then describe the main focus of Chapter 4.

With Poisson arrivals and exponentially distributed file sizes, Kelly and Williams [KW04] studied the asymptotic behavior of a critical fluid model for the flow level model operating under an α-fair bandwidth sharing policy, and proved uniform convergence of fluid model solutions to an invariant manifold when starting in a compact set. Subsequently, Kang et al. [KKLW09] used this analysis to prove multiplicative state space collapse, and, for $\alpha = 1$, combined the result of Kelly and Williams [KW04] with an invariance principle for reflected Brownian motion by Kang and Williams [KW07], to prove a diffusion approximation for the heavily loaded flow level model under a mild local traffic condition. The latter condition was subsequently weakened to a full rank condition on the network structure by Ye and Yao [YY12].

The fluid model considered by Kelly and Williams [KW04] focused on the fluid limit of the flow count process; the latter is a Markovian process when arrivals are Poisson and file sizes are exponentially distributed. As noted above, for more generally distributed arrivals and file sizes, a larger state descriptor is usually needed. A special case of the flow level model is when there is a single type of file and a single resource or communication link. In this case, bandwidth sharing is the same as processor sharing, and a natural state descriptor is a measure on the positive half line that keeps track of residual file sizes (plus a variable that tracks residual interarrival times). The modular approach of Bramson [Bra98] and Williams [Wil98] has been adapted to this case. Specifically, a fluid model for a GI/GI/1 processor sharing queue was developed by Gromoll et al. [GPW02], asymptotic analysis of the critical fluid model was carried out by Puha and Williams [PW04], and Gromoll [Gro04] subsequently used this to prove state space collapse and a heavy traffic diffusion approximation for the processor sharing queue. For the full flow level model of Massoulié and Roberts [MR00], operating under the proportional fair sharing discipline ($\alpha = 1$ and with equal weights), when arrivals are given by Poisson processes and file sizes have a phase-type distribution, Vlasiou et al. [VZZ14] used a critical fluid model analysis to study the steady-state distribution of the flow count process.

In Chapter 4, we analyze the asymptotic behavior (as time goes to infinity) of measure-valued solutions to the fluid model of Gromoll and Williams [GW09] for the α-fair bandwidth sharing policies of Mo and Walrand [MW00]. It is anticipated that this chapter will provide a crucial link in a modular approach to proving a diffusion approximation for the Massoulié-Roberts flow level model with general interarrival and file size distributions when operating under the aforementioned fair bandwidth sharing policies. The key to our analysis is a new Lyapunov function, the formulation of which was inspired by the work of Kelly and Williams [KW04], Mulvany et al. [MPW19] and Paganini et al. [PTFA12]. Using this, under moderate conditions on the file size distributions for the fluid model, we prove that critical fluid model solutions converge uniformly to the set of invariant states (called the invariant manifold) as time goes to infinity, when started in suitable relatively compact sets.

1.2 Notation and Terminology

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. For $x \in \mathbb{R}$, let $x^+ = \max(x, 0)$. Define $\mathbf{C}_b^1(\mathbb{R})$ (resp. $\mathbf{C}_b^1(\mathbb{R}_+)$) to be the set of once continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ (resp. $f : \mathbb{R}_+ \to \mathbb{R}$) that together with their first derivatives are continuous and bounded on $\mathbb{R}$ (resp. $\mathbb{R}_+$). Let $\mathbf{C}_c^\infty(\mathbb{R})$ be the set of infinitely differentiable functions defined on the real line that have compact support. Let $\mathbb{1}_A$ denote the indicator function of a set A and let $\mathbb{1} = \mathbb{1}_{\mathbb{R}_+}$.

Let $\mathbf{M}$ be the set of finite non-negative Borel measures on $\mathbb{R}_+$, endowed with the topology of weak convergence. If $\{\xi^n\}_{n=1}^\infty$ is a sequence in $\mathbf{M}$ converging (weakly) to $\xi \in \mathbf{M}$, we write $\xi^n \xrightarrow{w} \xi$ as $n \to \infty$. Given $\xi \in \mathbf{M}$, let $\mathbf{L}^1(\xi)$ denote the set of Borel measurable functions from $\mathbb{R}_+$ into $\mathbb{R}$ that are integrable with respect to ξ. For $f \in \mathbf{L}^1(\xi)$, let $\langle f, \xi \rangle = \int_{\mathbb{R}_+} f \, d\xi$. Also for any non-negative Borel measurable function $f \notin \mathbf{L}^1(\xi)$, let $\langle f, \xi \rangle = +\infty$. For $x \in \mathbb{R}_+$, let $\chi(x) = x$.

Define $\mathbf{M}_1 = \{\xi \in \mathbf{M} : \langle \chi, \xi \rangle < \infty\}$. Let $\mathbf{K} = \{\xi \in \mathbf{M} : \xi(\{x\}) = 0$ for all $x \in \mathbb{R}_+\}$, the set of continuous measures in $\mathbf{M}$, and let $\mathbf{K}_1 = \mathbf{M}_1 \cap \mathbf{K}$. Let $\mathbf{A}$ denote the elements of $\mathbf{M}$ that are absolutely continuous (with respect to Lebesgue measure).

Let $\mathbb{N}$ denote the set of positive integers. For $\mathbf{I} \in \mathbb{N}$, let $I = \{1, \ldots, \mathbf{I}\}$ and define

$$\mathbf{M}^{\mathbf{I}} = \{(\xi_1, \ldots, \xi_{\mathbf{I}}) : \xi_i \in \mathbf{M} \text{ for all } i \in I\},$$
$$\mathbf{M}_1^{\mathbf{I}} = \{(\xi_1, \ldots, \xi_{\mathbf{I}}) : \xi_i \in \mathbf{M}_1 \text{ for all } i \in I\},$$
$$\mathbf{K}^{\mathbf{I}} = \{(\xi_1, \ldots, \xi_{\mathbf{I}}) : \xi_i \in \mathbf{K} \text{ for all } i \in I\},$$
$$\mathbf{K}_1^{\mathbf{I}} = \{(\xi_1, \ldots, \xi_{\mathbf{I}}) : \xi_i \in \mathbf{K}_1 \text{ for all } i \in I\},$$
$$\mathbf{A}^{\mathbf{I}} = \{(\xi_1, \ldots, \xi_{\mathbf{I}}) : \xi_i \in \mathbf{A} \text{ for all } i \in I\}.$$

Here $\mathbf{M}^{\mathbf{I}}$ has its product topology and the other sets have the induced topologies as subsets of $\mathbf{M}^{\mathbf{I}}$. Fluid model solutions will take values in $\mathbf{M}^{\mathbf{I}}$ and we shall refer to the measure $\xi \in \mathbf{M}^{\mathbf{I}}$ that has ξ_i equal to the zero measure on $\mathbb{R}_+$ for all $i \in I$, as the zero measure (in $\mathbf{M}^{\mathbf{I}}$) or the zero state (for the fluid model). Given a real-valued Borel measurable function $f \geq 0$, for $\xi \in \mathbf{M}^{\mathbf{I}}$, define $\langle f, \xi \rangle = (\langle f, \xi_1 \rangle, \ldots, \langle f, \xi_{\mathbf{I}} \rangle)$.

With its topology of weak convergence, $\mathbf{M}$ is a Polish space (see [Pro56]), and a metric (called the Prokhorov metric) which induces this topology and under which $\mathbf{M}$ is complete and separable is defined as follows. For a Borel set $B \subset \mathbb{R}_+$ and $\varepsilon > 0$, define

$$B^{\varepsilon} = \{y \in \mathbb{R}_+ : \inf_{x \in B} |x - y| < \varepsilon\}.$$

For $\xi, \eta \in \mathbf{M}$, the Prokhorov distance between ξ and η is defined by

$$\mathbf{d}(\xi, \eta) = \inf\{\varepsilon > 0 : \xi(B) \leq \eta(B^{\varepsilon}) + \varepsilon \text{ and } \eta(B) \leq \xi(B^{\varepsilon}) + \varepsilon,$$
$$\text{for all closed sets } B \subset \mathbb{R}_+\}.$$

For $\xi, \eta \in \mathbf{M^I}$, define

$$\mathbf{d_I}(\xi, \eta) = \max_{i \in I} \mathbf{d}(\xi_i, \eta_i). \tag{1.1}$$

For any $\emptyset \neq \mathcal{B} \subset \mathbf{M^I}$ and $\xi \in \mathbf{M^I}$, define

$$\mathbf{d_I}(\xi, \mathcal{B}) = \inf_{\eta \in \mathcal{B}} \mathbf{d_I}(\xi, \eta).$$

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Chapter 1 is a combination of extracts from Section 1 of “Stability of a Subcritical Fluid Model for Fair Bandwidth Sharing with General File Size Distributions”, Stochastic Systems, Yingjia Fu and Ruth J. Williams, Volume 10, Number 3, 2020, and Section 1 of “Asymptotic Behavior of a Critical Fluid Model for Bandwidth Sharing with General File Size Distributions” by Yingjia Fu and Ruth J Williams, which has been submitted to a major journal in applied probability. The dissertation author was the co-author of these two papers.

Chapter 2

Fluid Model

Here we recall the fluid model developed by Gromoll and Williams [GW09] as a functional law of large numbers approximation to the flow level model of Massoulié and Roberts [MR00] operating under a bandwidth sharing policy such as one of the α-fair policies of [MW00]. This fluid model (with a zero initial condition) was also obtained by Chiang et al. [CST06] from the flow level model operating under a slight generalization of the α-fair policies of Mo and Walrand [MW00]. This used a different law of large numbers scaling limit from [GW09]; in particular, in the work of Chiang et al. [CST06], the arrival rate and bandwidth capacity were allowed to grow to infinity proportionally. We begin by introducing the fluid model parameters.

2.1 Parameters

Consider finitely many resources (e.g., links in a communication network) labelled by $j \in \mathcal{J} \equiv \{1,\dots,\mathbf{J}\}$, and a finite set of routes labeled by $i \in I \equiv \{1,\dots,\mathbf{I}\}$. A route $i \in I$ is simply a non-empty subset of $\mathcal{J}$ and is interpreted as the set of resources used by the route. Let R be the $\mathbf{J}\times\mathbf{I}$ incidence matrix satisfying $R_{ji} = 1$ if resource j is used by route i, and $R_{ji} = 0$ otherwise. Each resource $j \in \mathcal{J}$ has a fixed (bandwidth) capacity $C_j > 0$.

Fix a vector $\nu = (\nu_1,\dots,\nu_{\mathbf{I}})$ where $\nu_i > 0$ for each $i \in I$, and a vector $\vartheta = (\vartheta_1,\dots,\vartheta_{\mathbf{I}})$

where for each $i \in I$, ϑ_i is a Borel probability measure on $\mathbb{R}_+$ that does not charge the origin and has finite mean, i.e., $\langle \chi, \vartheta_i \rangle < \infty$. For $i \in I$, the constant ν_i represents the mean arrival rate of files to route i and ϑ_i represents the distribution for the sizes of files arriving to route i.

For each $i \in I$, $\mu_i \equiv \frac{1}{\langle \chi, \vartheta_i \rangle}$ is the reciprocal of the mean of the distribution ϑ_i and $\rho_i \equiv \frac{\nu_i}{\mu_i}$ is interpreted as the *nominal load* (average bandwidth needed) on route i. For each $i \in I$, let ϑ_i^e be the *excess lifetime distribution* associated with ϑ_i. The probability measure ϑ_i^e is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+$ and has density

$$p_i^e(x) = \mu_i \langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle \quad \text{for all } x \in \mathbb{R}_+. \tag{2.1}$$

For each $i \in I$, we define $N_i(x) = \langle \mathbb{1}_{[0,x]}, \vartheta_i \rangle$, $\overline{N}_i(x) = 1 - N_i(x)$, $N_i^e(x) = \langle \mathbb{1}_{[0,x]}, \vartheta_i^e \rangle$, and $\overline{N}_i^e(x) = 1 - N_i^e(x)$ for each $x \in \mathbb{R}_+$. Note that $\mu_i^{-1} = \int_0^\infty \overline{N}_i(x)dx$ and $p_i^e(x) = \mu_i \overline{N}_i(x)$ for all $x \in \mathbb{R}_+$. For $\xi \in \mathbf{M}^{\mathbf{I}}$, for each $i \in I$, define $\overline{M}_\xi^i(x) = \langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle$ for each $x \in \mathbb{R}_+$.

2.2 Bandwidth Sharing Policy

We will consider a family of bandwidth sharing policies that were studied by Chiang et al. [CST06], and which are a slight generalization of the α-fair policies of Mo and Walrand [MW00].

The bandwidth allocations in the fluid model change dynamically as a function of the amount of fluid on each route. We will need the following notation to describe them. For each $z \in \mathbb{R}_+^{\mathbf{I}}$, let $I_+(z) = \{i \in I : z_i > 0\}$ and $O(z) = \{\psi \in \mathbb{R}_+^{\mathbf{I}} : \psi_i = 0 \text{ for all } i \notin I_+(z)\}$.

Fix parameters $\alpha_i > 0, \kappa_i > 0$, for each $i \in I$. Let $\alpha = (\alpha_1, \ldots, \alpha_{\mathbf{I}})$ and $\kappa = (\kappa_1, \ldots, \kappa_{\mathbf{I}})$. The following optimization problem will be used to define the bandwidth sharing policy associated with the pair of vector parameters (α, κ). Given $z \in \mathbb{R}_+^{\mathbf{I}}$, the vector of bandwidth allocations $\phi(z)$ associated with z is the unique value of $\psi \in O(z)$ that solves the following *utility maximization* *problem:*

$$\text{maximize} \sum_{i\in I_+(z)} \kappa_i z_i U_i\left(\frac{\psi_i}{z_i}\right) \text{ subject to} \sum_{i\in I} R_{ji}\psi_i \le C_j \text{ for all } j\in\mathcal{J},\ \psi\in O(z), \tag{2.2}$$

where for each i, $U_i : [0,\infty) \to [-\infty,\infty)$ is a utility function of the form

$$U_i(x_i) = \begin{cases} \frac{1}{1-\alpha_i} x_i^{1-\alpha_i} & \text{if } \alpha_i \neq 1, \\ \log(x_i) & \text{if } \alpha_i = 1. \end{cases}$$

Remark 2.2.1. *For $i \in I_+(z)$, we have $\phi_i(z) > 0$ because, either $U_i(0) = -\infty$ if $\alpha_i \ge 1$, or $U_i(0) = 0$ and $U_i'(x_i) \to +\infty$ as $x_i \to 0$ if $\alpha_i \in (0,1)$. Let $\mathcal{S}(z) = \{\psi \in \mathbb{R}_+^{\mathbf{I}} : \psi_i > 0$ for all $i \in I_+(z), \psi_i = 0$ for all $i \notin I_+(z)\}$. Then one can restrict the choice of ψ to the set $\mathcal{S}(z)$ for the utility maximization problem. The uniqueness of the maximizer follows from the strict concavity of the utility functions U_i, $i \in I_+(z)$. Furthermore, for $z \in \mathbb{R}_+^{\mathbf{I}}$, $\phi_i(\cdot)$ is continuous at z for each $i \in I_+(z)$. If $\alpha_i = \alpha \in (0,\infty)$ for all $i \in I$, this last statement was proved by Kelly and Williams [KW04]. When $\alpha_i \in (0,1)$ for all $i \in I$, it was noted by Chiang et al. [CST06] that a similar proof to that of [KW04] can be used to establish this result. Similar ideas can be used to give a proof for all $\alpha_i \in (0,\infty), i \in I$. For completeness, in Lemma A.1 in the Appendix, we give such a proof.*

2.3 Definition of Fluid Model Solutions

The fluid model of Gromoll and Williams [GW09], with the bandwidth sharing policy described in the previous section, is described below. For the remainder of this dissertation, the parameters $(R, C, \alpha, \kappa, \nu, \vartheta)$ are fixed and the bandwidth allocation function ϕ is as specified in the previous section.

Definition 2.3.1. *Given a continuous function $\zeta : [0,\infty) \to \mathbf{M}^{\mathbf{I}}$, define the auxiliary functions*

(z,Λ,τ,u,w) *by the following for all* $t \geq 0$*:*

$$\begin{aligned}
z(t) &= \langle \mathbb{1}, \zeta(t) \rangle, \\
\Lambda(t) &= \phi(z(t)), \\
\tau_i(t) &= \int_0^t \Big(\Lambda_i(s) \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) + \rho_i \mathbb{1}_{\{0\}}\big(z_i(s)\big) \Big) ds, \quad i \in I, \\
u(t) &= Ct - R\tau(t), \\
w(t) &= \langle \chi, \zeta(t) \rangle.
\end{aligned}$$

In Definition 2.3.1 the integrals defining $z(t)$ and $w(t)$ are to be interpreted componentwise. In particular, the i-th component of $w(\cdot)$ represents the fluid workload for route i, $w_i(t) = \langle \chi, \zeta_i(t) \rangle$, $t \geq 0$. The fluid workload per link is given by $\widetilde{w}_j(t) = \sum_{i \in I} R_{ji} w_i(t)$, $t \geq 0$.

A fluid model solution is defined through projections against test functions in the class

$$\mathcal{C} = \{ f \in \mathbf{C}_b^1(\mathbb{R}_+) : f(0) = f'(0) = 0 \}. \tag{2.3}$$

Definition 2.3.2. *A fluid model solution associated with the parameters* $(R, C, \alpha, \kappa, \nu, \vartheta)$ *is a continuous function* $\zeta : [0,\infty) \to \mathbf{M}^{\mathbf{I}}$ *that, together with its auxiliary functions* (z,Λ,τ,u)*, satisfies:*

(i) $\langle \mathbb{1}_{\{0\}}, \zeta(t) \rangle = 0$ *for all* $t \geq 0$*,*

(ii) the function u_j *is nondecreasing for all* $j \in \mathcal{J}$*,*

(iii) for each $f \in \mathcal{C}$*,* $i \in I$*, and* $t \geq 0$*,*

$$\langle f, \zeta_i(t) \rangle = \langle f, \zeta_i(0) \rangle - \int_0^t \langle f', \zeta_i(s) \rangle \frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}(z_i(s)) ds + \nu_i \langle f, \vartheta_i \rangle \int_0^t \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) ds. \tag{2.4}$$

Remark 2.3.1. *The auxiliary function* w *associated with* ζ *satisfies the following for all* $t \geq 0$ *for* *those i for which $w_i(0) < \infty$:*

$$w_i(t) = w_i(0) + \int_0^t \Big(\rho_i - \Lambda_i\big(z(s)\big)\Big) \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) ds; \tag{2.5}$$

see Lemma 3.3 of [GW09] and Lemma 4 of [CST06] for the method of proof.

Remark 2.3.2. *The third property in Definition 2.3.2 can be extended to hold for all functions $f \in \widetilde{\mathcal{C}} = \{f \in \mathbf{C}_b^1(\mathbb{R}_+) : f(0) = 0\}$. A proof of this is given in Lemma A.2 in Appendix A.*

Remark 2.3.3. *The fluid limit result proved by Gromoll and Williams [GW09] yields fluid model solutions which have initial states that are continuous measures and which have finite workload, i.e., for which $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$. Indeed, in order for fluid model solutions to be continuous functions of time, the initial condition cannot have any atoms. For our results in Chapter 3, we will be assuming that $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$, see Section 3.3.*

2.4 Additional Notation for Fluid Model Solutions

Suppose that $\zeta(\cdot)$ is a fluid model solution. We shall often use $\overline{M}_t^i(x)$ in place of $\overline{M}_{\zeta(t)}^i(x)$ to simplify notation. Let (z,Λ) be auxiliary functions associated with ζ, as in Definition 2.3.1. For each $i \in I$ and $0 \leq s < t < \infty$, let

$$S_{s,t}^i = \int_s^t \frac{\Lambda_i(r)}{z_i(r)} \mathbb{1}_{(0,\infty)}\big(z_i(r)\big) dr. \tag{2.6}$$

Note that this may take the value $+\infty$. However, if $z_i(r) > 0$ for all $r \in [s,t]$, then $S_{s,t}^i < \infty$, since $\Lambda_i(\cdot)$ is bounded and $z_i(\cdot)$ is continuous (hence it is bounded away from zero on the interval $[s,t]$). Indeed, $r \to S_{r,t}^i$ is continuously differentiable on $[s,t]$ because $\Lambda_i(\cdot) = \phi_i(z(\cdot))$ is continuous on $[s,t]$, since $z \to \phi_i(z)$ is continuous at points z where $z_i > 0$ (see Remark 2.2.1) and $r \to z_i(r)$ is continuous, and furthermore $r \to z_i(r)$ is continuous and bounded away from zero on $[s,t]$. We interpret $S^i_{s,t}$ as the cumulative amount of bandwidth per unit of fluid allocated to route i over the time interval $[s,t]$.

2.5 Acknowledgement

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Chapter 3

Stability of the Subcritical Fluid Model

In this chapter, we analyze the asymptotic behavior of solutions to the (strictly) *subcritical* fluid model for bandwidth sharing. In Section 3.1, we introduce assumptions on the parameters under which our results will be proved and in Section 3.2, we define the Lyapunov function H (as a function of measures); this is a slight variant of the function proposed by Paganini et al. [PTFA12]. We also introduce the composition $\mathcal{H}^{\zeta}$ of H with a fluid model solution ζ, and a function $\mathcal{K}^{\zeta}$ which is used to describe the density in time of $\mathcal{H}^{\zeta}$. Our main results for this chapter are stated in Section 3.3. The proofs of these main results are given in Sections 3.5 and 3.6. Some preliminary lemmas needed for our proofs are given in Section 3.4.

Our proofs have benefitted from prior works of others. In particular, our starting point is the clever Lyapunov function posited by Paganini et al. [PTFA12]. The preliminary results in Section 3.4 include three lemmas taken from [PTFA12] and four lemmas and a corollary giving some basic properties of fluid model solutions. The proofs of the fluid model solution results extend some techniques developed by Gromoll et al. [GPW02] for a critical fluid model of a single class processor sharing queue. The latter is a special case of a bandwidth sharing model with one route and one link. The final result in Section 3.4 is our proof of the continuity of the function $\mathcal{H}^{\zeta}(\cdot)$, which is a critical precursor to our proof of absolute continuity of this function.

The key new results, Theorem 3.3.1 and Corollary 3.3.1, are proved in Section 3.5. These rely on a result proved in Section 3.5.1, where we show that smoothed versions of each component of a fluid model solution satisfy certain parabolic partial differential equations on intervals of time where the fluid level for the component is not zero. This provides a rigorous formulation of a partial differential equation assumed to hold by Paganini et al. [PTFA12]. A similar smoothing technique was also used by Puha and Williams [PW16], in the study of the asymptotic behavior of critical fluid model solutions for a single class processor sharing queue. Our method is a little different from that of Puha and Williams [PW16] in that we smooth the entire fluid model solution, not just the initial condition. Theorems 3.3.2 and 3.3.3 are proved in Section 3.6. Having Theorem 3.3.1 and Corollary 3.3.1 in place, these proofs follow a similar line of argument to that of Paganini et al. [PTFA12]. However, we do generalize from having a common parameter α for all routes to the case where there is a separate α_i for each route $i \in I$, and we also establish uniformity of the convergence to the zero state under suitable conditions. Throughout, our proofs need to deal with the more complex bandwidth sharing model and especially to deal with the singular situation where the fluid level for some routes can reach zero while other route levels remain positive.

3.1 Assumptions

3.1.1 Subcritical Parameters

Henceforth in this chapter, we shall assume that the fluid model is subcritical, that is, the following assumption holds.

Assumption 3.1. *The parameters* (R, ρ, C) *satisfy*

$$\sum_{i \in I} R_{ji} \rho_i < C_j \quad \text{for all } j \in \mathcal{J}. \tag{3.1}$$

This condition means that the average load on each link is strictly less than its capacity. Under this condition, we can choose a sufficiently small $\delta > 0$ such that

$$\tilde{\rho}_i \equiv (1+\delta)\rho_i, \text{ for all } i \in I, \tag{3.2}$$

satisfies

$$\sum_{i \in I} R_{ji}\tilde{\rho}_i < C_j \text{ for all } j \in \mathcal{J} \quad \text{and} \quad (1-\delta)(1+\delta)^{\alpha_i+1} > 1 \text{ for all } i \in I. \tag{3.3}$$

We fix such a sufficiently small $\delta > 0$ henceforth and define

$$\theta_i(x) = \left(1 - \frac{m_i\mu_i}{\alpha_i}\int_0^x \overline{N}_i(u)du\right)^{-\alpha_i} \quad \text{for all } x \in [0,\infty),\ i \in I, \tag{3.4}$$

where $m_i \in (0,\alpha_i)$ is defined so that $\left(\frac{\alpha_i}{m_i}\right)^{\alpha_i} = (1-\delta)(1+\delta)^{\alpha_i+1}$ holds for all $i \in I$. Since $\mu_i \int_0^\infty \overline{N}_i(u)du = 1$, we have

$$\begin{aligned} 1 \geq 1 - \frac{m_i\mu_i}{\alpha_i}\int_0^x \overline{N}_i(u)du &= \frac{m_i}{\alpha_i}\left(1 - \mu_i\int_0^x \overline{N}_i(u)du\right) + 1 - \frac{m_i}{\alpha_i} \\ &= \frac{m_i\overline{N}_i^e(x)}{\alpha_i} + 1 - \frac{m_i}{\alpha_i} \\ &\geq 1 - \frac{m_i}{\alpha_i} > 0, \end{aligned}$$

and so $\theta_i(\cdot)$ is positive and bounded above and below on $[0,\infty)$ for all $i \in I$.

3.1.2 File Size Distributions

The following assumption will be used in Lemma 3.4.8 to prove continuity in time of $\mathcal{H}^\zeta$, the composition of the Lyapunov function H (defined below) with a suitable fluid model solution ζ. This continuity property ultimately features in our proof of the absolute continuity of $\mathcal{H}^\zeta$ as a function of time and the convergence of fluid model solutions to the zero state.

Assumption 3.2. *For each $i \in I$, the probability measure ϑ_i is in $\mathbf{K}_1$, that is, it has no atoms and has finite first moment.*

Remark 3.1.1. *We already assumed that ϑ_i has finite first moment, so the additional assumption here is that it has no atoms.*

The additional assumption below, will be used in showing that under suitable constraints on the initial conditions, fluid model solutions reach the zero state in finite time. Indeed, we will prove that the time can be chosen uniformly provided there is a uniform bound on the initial workload vector and on the p-th moments of the components of the initial state of the fluid model solutions.

Assumption 3.3. *There is $p \in (1,\infty)$ such that $B_{\vartheta,p} \equiv \max_{i \in I} \langle \chi^p, \vartheta_i \rangle < \infty$.*

3.2 Lyapunov Function

3.2.1 The Functions H and $\mathcal{H}^\zeta$

Definition 3.2.1. *Given $\xi \in \mathbf{M}^{\mathbf{I}}$, for each $i \in I$, define*

$$H_i(\xi) = \frac{\kappa_i}{\tilde{\rho}_i^{\alpha_i}} \int_0^\infty \left(\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle \right)^{\alpha_i + 1} \theta_i(x) dx, \tag{3.5}$$

and define

$$H(\xi) = \sum_{i \in I} \frac{H_i(\xi)}{\alpha_i + 1}. \tag{3.6}$$

The function H will be our Lyapunov function for studying stability in this chapter. It is a slight generalization of the one used by Paganini et al. [PTFA12], where we have made adjustments to allow for the fact that our α_i can depend on i. Note that for $\xi \in \mathbf{M}^{\mathbf{I}}$, $H(\xi) \in [0,\infty]$ and $H(\xi) = 0$ if and only if $\xi_i((0,\infty)) = 0$ for all $i \in I$. We shall ultimately be applying H to $\xi \in \mathbf{K}_1^{\mathbf{I}}$. For such

ξ, $H(\xi)$ is finite and such that $H(\xi)=0$ if and only if ξ_i is the zero measure on $\mathbb{R}_+=[0,\infty)$ for each $i\in I$.

Definition 3.2.2. *Given a fluid model solution $\zeta(\cdot)$, for each $t\geq 0$ and $i\in I$, define $\overline{M}_t^i(x)=\langle \mathbb{1}_{(x,\infty)},\zeta_i(t)\rangle$ for all $x\geq 0$ and*

$$\mathcal{H}_i^\zeta(t)=H_i\big(\zeta(t)\big)=\frac{\kappa_i}{\tilde{\rho}_i^{\alpha_i}}\int_0^\infty\big(\overline{M}_t^i(x)\big)^{\alpha_i+1}\theta_i(x)dx \text{ for all } i\in I, \tag{3.7}$$

and let

$$\mathcal{H}^\zeta(t)=H\big(\zeta(t)\big)=\sum_{i\in I}\frac{\mathcal{H}_i^\zeta(t)}{\alpha_i+1}. \tag{3.8}$$

The following provides a sufficient condition for $\mathcal{H}^\zeta(t)$ to be finite-valued for all $t\in[0,\infty)$.

Proposition 3.2.1. *Let $\zeta(\cdot)$ be a fluid model solution. Suppose that $i\in I$ such that $w_i(0)=\langle\chi,\zeta_i(0)\rangle<\infty$. Then $\mathcal{H}_i^\zeta(t)$ is finite for all $t\geq 0$.*

Proof. Fix $t\geq 0$. By (2.5) we have that $w_i(t)$ is finite. Also, since $\overline{M}_t^i(x)\leq z_i(t)$ and $w_i(t)=\int_0^\infty\overline{M}_t^i(x)dx$, we have

$$\begin{aligned}\mathcal{H}_i^\zeta(t)&\leq\frac{\kappa_i\|\theta_i\|_\infty}{\tilde{\rho}_i^{\alpha_i}}(z_i(t))^{\alpha_i}\int_0^\infty\overline{M}_t^i(x)dx\\&=\frac{\kappa_i\|\theta_i\|_\infty}{\tilde{\rho}_i^{\alpha_i}}(z_i(t))^{\alpha_i}w_i(t)<\infty,\end{aligned}\tag{3.9}$$

where $\|\theta_i\|_\infty=\sup_{x\in[0,\infty)}|\theta_i(x)|$. $\square$

3.2.2 The Function $\mathcal{K}^\zeta$

In this section, we introduce the function $\mathcal{K}^\zeta$, which arises in taking the derivative of the function $t\to\mathcal{H}^\zeta(t)$.

Definition 3.2.3. *Suppose that $\zeta(\cdot)$ is a fluid model solution. Define for each $i \in I$ and $t \geq 0$,*

$$\begin{aligned}\mathcal{K}_i^\zeta(t) = \tilde{\rho}_i^{-\alpha_i}\Bigg(&-\kappa_i \Lambda_i(t)\big(z_i(t)\big)^{\alpha_i} \\ &- \kappa_i \int_0^\infty \big(\overline{M}_t^i(x)\big)^{\alpha_i} \left(\frac{\Lambda_i(t)}{z_i(t)} \mathbb{1}_{(0,\infty)}\big(z_i(t)\big)\right) \overline{M}_t^i(x)\theta_i'(x)dx \\ &+ \kappa_i(\alpha_i+1)\nu_i \int_0^\infty \big(\overline{M}_t^i(x)\big)^{\alpha_i} \overline{N}_i(x)\theta_i(x)dx\Bigg)\end{aligned} \tag{3.10}$$

and let

$$\mathcal{K}^\zeta(t) = \sum_{i \in I_+(z(t))} \frac{\mathcal{K}_i^\zeta(t)}{\alpha_i+1} \text{ for all } t \geq 0. \tag{3.11}$$

Remark 3.2.1. *In (3.10), if $z_i(t) = 0$, we interpret the right member of the equality to be zero and so $\mathcal{K}_i^\zeta(t) = 0$ in this case.*

Proposition 3.2.2. *Suppose that $\zeta(\cdot)$ is a fluid model solution. Then, for each $i \in I$ and $t \geq 0$,* $\sup_{s\in[0,t]} |\mathcal{K}_i^\zeta(s)| < \infty$.

Proof. Fix $i \in I$ and $t \geq 0$. For each $s \in [0,t]$, let

$$\begin{aligned} k_1(s) &= \tilde{\rho}_i^{-\alpha_i}\kappa_i\Lambda_i(s)\left(z_i(s)\right)^{\alpha_i}, \\ k_2(s) &= \tilde{\rho}_i^{-\alpha_i}\kappa_i \int_0^\infty \left(\overline{M}_s^i(x)\right)^{\alpha_i} \left(\frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}\big(z_i(s)\big)\right) \overline{M}_s^i(x)\theta_i'(x)dx, \\ k_3(s) &= \tilde{\rho}_i^{-\alpha_i}\kappa_i(\alpha_i+1)\nu_i \int_0^\infty \left(\overline{M}_s^i(x)\right)^{\alpha_i} \overline{N}_i(x)\theta_i(x)dx. \end{aligned}$$

Noting that $\theta_i'(x) = m_i\mu_i(\theta_i(x))^{\frac{\alpha_i+1}{\alpha_i}}\overline{N}_i(x)$ for all $x \in \mathbb{R}_+$, $\|\theta_i\|_\infty < \infty$, $\frac{\overline{M}_s^i(\cdot)}{z_i(s)}\mathbb{1}_{(0,\infty)}\big(z_i(s)\big) \leq 1$, $|\Lambda_i(\cdot)| \leq \max_j C_j$, $\overline{M}_s^i(\cdot) \leq z_i(s) < \infty$, and $\int_0^\infty \overline{N}_i(x)dx = \langle \chi, \vartheta_i\rangle = \mu_i^{-1} < \infty$, we have that $k_1(s)$,

$k_2(s)$, $k_3(s)$ are well defined, non-negative and finite for each $s \in [0,t]$. Indeed,

$$\sup_{s\in[0,t]} k_1(s) \leq \tilde{\rho}_i^{-\alpha_i} \kappa_i (\max_j C_j) \Big(\sup_{s\in[0,t]} z_i(s) \Big)^{\alpha_i} < \infty, \tag{3.12}$$

$$\sup_{s\in[0,t]} k_2(s) \leq \tilde{\rho}_i^{-\alpha_i} \kappa_i \Big(\sup_{s\in[0,t]} z_i(s) \Big)^{\alpha_i} (\max_j C_j)\, m_i \, \|\theta_i\|_\infty^{\frac{\alpha_i+1}{\alpha_i}} < \infty, \tag{3.13}$$

$$\sup_{s\in[0,t]} k_3(s) \leq \tilde{\rho}_i^{-\alpha_i} \kappa_i (\alpha_i+1) \nu_i \Big(\sup_{s\in[0,t]} z_i(s) \Big)^{\alpha_i} \langle \chi, \vartheta_i \rangle \|\theta_i\|_\infty < \infty. \tag{3.14}$$

Noting that

$$\mathcal{K}_i^\zeta(s) = -k_1(s) - k_2(s) + k_3(s), \quad \text{for all } s \in [0,t], \tag{3.15}$$

the result follows. $\square$

3.3 Main Results

Theorem 3.3.1. *Suppose that Assumptions 3.1 and 3.2 hold. Further suppose that $\zeta(\cdot)$ is a fluid model solution with $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$. For each $i \in I$, the function $\mathcal{H}_i^\zeta(\cdot)$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$, and $\mathcal{K}_i^\zeta(\cdot)$ is a density for $\mathcal{H}_i^\zeta(\cdot)$, that is, for each $t \geq 0$,*

$$\mathcal{H}_i^\zeta(t) - \mathcal{H}_i^\zeta(0) = \int_0^t \mathcal{K}_i^\zeta(s) ds. \tag{3.16}$$

Furthermore, for each $t \geq 0$,

$$\mathcal{K}_i^\zeta(t) \leq \kappa_i \big(z_i(t)\big)^{\alpha_i} \left(\frac{-\Lambda_i(t)}{\tilde{\rho}_i^{\alpha_i}} + \frac{\tilde{\rho}_i(1-\delta)}{\Lambda_i(t)^{\alpha_i}} \right) \mathbb{1}_{(0,\infty)}\big(z_i(t)\big), \tag{3.17}$$

where the right member is interpreted to be zero if $z_i(t) = 0$.

Corollary 3.3.1. *Under the assumptions of Theorem 3.3.1, $\mathcal{H}^\zeta(\cdot)$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$ and $\mathcal{K}^\zeta(\cdot)$ is a density for $\mathcal{H}^\zeta(\cdot)$. In addition, for all $t \geq 0$*

we have

$$\mathcal{K}^{\zeta}(t) \leq -\delta \sum_{i \in I_+(z(t))} \frac{\kappa_i \tilde{\rho}_i}{\alpha_i + 1} \left(\frac{z_i(t)}{\Lambda_i(t)} \right)^{\alpha_i}. \tag{3.18}$$

The proofs of Theorem 3.3.1 and Corollary 3.3.1 are presented in Section 3.5.

Theorem 3.3.2. *Suppose that Assumptions 3.1 and 3.2 hold. For any fluid model solution* $\zeta(\cdot)$ *with* $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$, $\mathcal{H}^{\zeta}(t)$ *decreases monotonically to zero as* $t \to \infty$. *Furthermore, for any* $W > 0$,

$$\lim_{t \to \infty} \sup\{\mathcal{H}^{\zeta}(t) : \zeta \text{ is a fluid model solution, } \zeta(0) \in \mathbf{K}_1^{\mathbf{I}},\ \max_{i \in I}(\langle \mathbb{1}, \zeta_i(0) \rangle, \langle \chi, \zeta_i(0) \rangle) \leq W\} = 0.$$

Consequently, $\zeta_i(t)$ *as a measure on* $(0,\infty)$ *converges vaguely*[1] *to the zero measure on* $(0,\infty)$ *as* $t \to \infty$, *for each* $i \in I$.

The following theorem shows that with the addition of Assumption 3.3 (with $p \in (1,\infty)$) to the assumptions of Corollary 3.3.1, and assuming the components of the initial fluid state have finite p-th moments, we have that the fluid model solution reaches the zero state in finite time, and the hitting time of the zero state is uniformly bounded for fluid model solutions starting in $\{\xi \in \mathbf{K}_1^{\mathbf{I}} : \max_{i \in I}(\langle \mathbb{1}, \xi_i \rangle,\ \langle \chi^p, \xi_i \rangle) \leq W\}$ for any fixed $W > 0$.

Theorem 3.3.3. *Suppose that Assumptions 3.1, 3.2 and 3.3 hold and let* $p \in (1,\infty)$ *be as in Assumption 3.3. For each* $W \geq 1$ *there exists* $T_W > 0$ *such that for all fluid model solutions* $\zeta(\cdot)$ *satisfying* $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$ *and* $\max_{i \in I}(\langle \mathbb{1}, \zeta_i(0) \rangle, \langle \chi^p, \zeta_i(0) \rangle) \leq W$, *we have* $\zeta(t) = \mathbf{0}$, *the zero measure in* $\mathbf{M}^{\mathbf{I}}$, *for all* $t \geq T_W$.

The proofs of Theorems 3.3.2 and 3.3.3 are given in Section 3.6.

[1] That is, $\langle f, \zeta_i(t) \rangle \to 0$ as $t \to \infty$ for each continuous function f with compact support in $(0,\infty)$.

3.4 Preliminary Lemmas

The following three lemmas are similar to Lemma 1, a result in Section III.C, and Lemma 5 in [PTFA12]. For the first lemma, Paganini et al. [PTFA12] indicated the idea for a proof. Here we provide more details, for completeness. For the other two lemmas, we provide the statements and the short proofs as a convenience to the reader.

Lemma 3.4.1. *Fix $z \in \mathbb{R}_+^{\mathbf{I}}$. Recall that $\phi(z)$ solves the maximization problem (2.2). Let ψ be a vector in $\mathbb{R}_+^{\mathbf{I}}$ such that $\psi_i > 0$ for all $i \in I_+(z)$ and $\sum_{i\in I} R_{ji}\psi_i \leq C_j$ for all $j \in \mathcal{J}$. Then*

$$\sum_{i\in I_+(z)} \kappa_i U_i'\left(\frac{\phi_i(z)}{z_i}\right)\big(\psi_i - \phi_i(z)\big) \leq 0, \tag{3.19}$$

where, for each $i \in I_+(z)$, $U_i'(x)$ is the derivative of $U_i(x)$ when $x > 0$.

Proof. Since (3.19) holds trivially for $z = 0$, we may assume that $z \neq 0$. Let $\tilde{\phi}(z) = \big(\phi_i(z) : i \in I_+(z)\big)$ and $\widetilde{\Psi} = \{\tilde{\psi} = (\tilde{\psi}_i : i \in I_+(z)), \tilde{\psi}_i > 0 \text{ for all } i \in I_+(z)\}$. For each $i \in I_+(z)$, U_i is a concave, continuously differentiable function on $(0,\infty)$. Then the following function is concave and continuously differentiable on $\widetilde{\Psi}$:

$$f(\tilde{\psi}) = \sum_{i\in I_+(z)} \kappa_i z_i U_i\left(\frac{\tilde{\psi}_i}{z_i}\right), \ \tilde{\psi} \in \widetilde{\Psi}.$$

Consider the set

$$\mathcal{F}(z) = \left\{\tilde{\psi} \in \widetilde{\Psi} : \sum_{i\in I_+(z)} R_{ji}\tilde{\psi}_i \leq C_j \text{ for all } j \in \mathcal{J}\right\}.$$

Then f achieves its maximum value on $\mathcal{F}(z)$ at $\tilde{\phi}(z)$. We claim that $\nabla f\big(\tilde{\phi}(z)\big) \cdot \big(\tilde{\psi} - \tilde{\phi}(z)\big) \leq 0$, for any $\tilde{\psi} \in \mathcal{F}(z)$. For a proof by contradiction, suppose there is $\tilde{\psi} \in \mathcal{F}(z)$ such that $\nabla f\big(\tilde{\phi}(z)\big) \cdot \big(\tilde{\psi} - \tilde{\phi}(z)\big) > 0$. Then for any $t \in [0,1]$, $\gamma(t) = t\tilde{\psi} + (1-t)\tilde{\phi}(z)$ is in $\mathcal{F}(z)$, since this set is convex, and $\frac{d}{dt} f\big(\gamma(t)\big)\big|_{t=0} = \nabla f\big(\tilde{\phi}(z)\big) \cdot \big(\tilde{\psi} - \tilde{\phi}(z)\big) > 0$, by our assumption. It follows that for all sufficiently small $t > 0$, we have $f\big(\gamma(t)\big) > f\big(\tilde{\phi}(z)\big)$, which contradicts the fact that $\tilde{\phi}(z)$ is the optimal solution of the maximization problem. Thus, $\nabla f\big(\tilde{\phi}(z)\big)\cdot\big(\tilde{\psi}-\tilde{\phi}(z)\big)\leq 0$. Computing the gradient of f, and using the fact that $\phi_i(z)=\tilde{\phi}_i(z)$ for all $i\in I_+(z)$, it follows that

$$\sum_{i\in I_+(z)}\kappa_i U_i'\left(\frac{\phi_i(z)}{z_i}\right)\big(\tilde{\psi}_i-\phi_i(z)\big)\leq 0 \text{ for all } \tilde{\psi}\in\mathcal{F}(z).$$

For a ψ satisfying the hypotheses of the lemma, $\tilde{\psi}=(\psi_i : i\in I_+(z))$ is in $\mathcal{F}(z)$ and so the inequality holds for it. Because the sum in this inequality does not involve $(\psi_i : i\notin I_+(z))$, it follows that (3.19) holds for ψ. □

Lemma 3.4.2. *Let $g(s)=s^a\big((a+1)q-bs\big)$ for $s\geq 0$ where a,b,q are fixed strictly positive real numbers. Then g has a maximum of $\left(\frac{aq}{b}\right)^a q$ at $s=\frac{aq}{b}$.*

Proof. Differentiating g with respect to $s>0$, we have:

$$g'(s)=(a+1)s^{a-1}\big(aq-bs\big),$$

which is zero on $(0,\infty)$ only when $s=\frac{aq}{b}$ and noting the sign of g' on either side of this value, we see that g has a local maximum at $s=\frac{aq}{b}$ with value $\left(\frac{aq}{b}\right)^a q$. Further noting that g is continuous on $[0,\infty)$ and is zero at $s=0$ and tends to $-\infty$ as $s\to\infty$, we see that the local maximum is the global maximum. □

Lemma 3.4.3. *For any strictly positive real numbers, a, b, q, we have*

$$-\frac{b}{q^a}+\frac{q}{b^a}\leq (a+1)\frac{q-b}{b^a}.$$

Proof. Let $f(x)=x^{a+1}$ for $x\geq 0$. Then f is a convex function. The tangent line at $x=q$ is a support line and so $b^{a+1}\geq q^{a+1}+(a+1)q^a(b-q)$. Dividing both sides by $q^a b^a$ yields the desired result. □

The remaining lemmas in this section contain various results for fluid model solutions that will be used in later sections in the chapter. The proof of Lemma 3.4.4 is the same as that of Proposition 4.2 in [GPW02], so we omit it. The proofs of Lemmas 3.4.5 and 3.4.6, are similar to those of Lemmas 4.1 and 4.3, respectively, from the work of Gromoll et al. [GPW02]. Since some details are a bit different, we provide the proofs for our context as a convenience to the reader.

Recall the third property in Definition 2.3.2. We now state a version of this property that holds for a class of functions of both time and space. Let $\mathbf{C}_b([0,\infty)\times\mathbb{R}_+)$ denote the set of continuous, bounded functions on $[0,\infty)\times\mathbb{R}_+$, and let $\mathbf{C}_b^1([0,\infty)\times\mathbb{R}_+)$ denote the set of once continuously differentiable functions defined on $[0,\infty)\times\mathbb{R}_+$ which, together with their first partial derivatives are bounded on $[0,\infty)\times\mathbb{R}_+$. That is, $f(s,x)$, $f_s(s,x)=\frac{\partial}{\partial s}f(s,x)$ and $f_x(s,x)=\frac{\partial}{\partial x}f(s,x)$ are continuous and bounded by a constant for all $(s,x)\in[0,\infty)\times\mathbb{R}_+$.

Lemma 3.4.4. *Let $\zeta:[0,\infty)\to\mathbf{M}^{\mathbf{I}}$ be continuous. Then for each $f\in\mathbf{C}_b([0,\infty)\times\mathbb{R}_+)$ and $i\in I$,*

$$t\to\langle f(t,\cdot),\zeta_i(t)\rangle$$

is a continuous function of $t\in[0,\infty)$.

Proof. The proof is the same as that of Proposition 4.2 in [GPW02]. □

The proofs of the next two lemmas are similar to those of Lemmas 4.1 and 4.3 in [GPW02]. However, because bandwidth sharing is more general than the processor sharing treated in [GPW02], and because special care is needed in our setting to treat the fact that $z_i(\cdot)$ can be zero at some times, we give the full proofs here. We note that the special case where $x=0$ in (3.24) follows from Appendix A in [BEZ14]. A dynamic equation for $z(t)$ for all $t\geq 0$ is also derived there.

Lemma 3.4.5. *Suppose that $\zeta(\cdot)$ is a fluid model solution, $i \in I$, and $0 \le s < t < \infty$ are such that $\zeta_i(r) \neq 0$ for all $s < r < t$. Then for each $f \in \mathbf{C}_b^1([0,\infty) \times \mathbb{R}_+)$ such that $f(\cdot,0) \equiv 0$, we have that the following holds.*

$$\begin{aligned}\langle f(t,\cdot),\zeta_i(t)\rangle &= \langle f(s,\cdot),\zeta_i(s)\rangle + \int_s^t \langle f_r(r,\cdot),\zeta_i(r)\rangle dr \\ &\quad - \int_s^t \langle f_x(r,\cdot),\zeta_i(r)\rangle \frac{\Lambda_i(r)}{z_i(r)} \mathbb{1}_{(0,\infty)}\big(z_i(r)\big)dr + \nu_i \int_s^t \langle f(r,\cdot),\vartheta_i\rangle \mathbb{1}_{(0,\infty)}\big(z_i(r)\big)dr. \quad (3.20)\end{aligned}$$

Proof. Suppose that ζ, s,t, $i \in I$ and f are as in the statement of the lemma. Then for $r, r+h \in (s,t)$ we have

$$\begin{aligned}\langle f(r+h,\cdot),\zeta_i(r+h)\rangle - \langle f(r,\cdot),\zeta_i(r)\rangle &= \langle f(r+h,\cdot),\zeta_i(r+h)\rangle - \langle f(r,\cdot),\zeta_i(r+h)\rangle \\ &\quad + \langle f(r,\cdot),\zeta_i(r+h)\rangle - \langle f(r,\cdot),\zeta_i(r)\rangle. \qquad (3.21)\end{aligned}$$

In the following, for clarity, we write f_1 for the first partial derivative of f with respect to its first variable, and f_2 for its first partial derivative with respect to its second variable. The first difference on the right hand side of the equation (3.21) equals

$$\begin{aligned}\Big\langle \int_r^{r+h} f_1(u,\cdot)du, \zeta_i(r+h)\Big\rangle &= \Big\langle \int_0^1 f_1(r+hv,\cdot)hdv, \zeta_i(r+h)\Big\rangle \\ &= h\int_0^1 \langle f_1(r+hv,\cdot),\zeta_i(r+h)\rangle dv,\end{aligned}$$

where we have used Fubini's theorem to change the order of integration to obtain the last equality. For each $v \in [0,1]$, define a function $f^v : [0,\infty) \times \mathbb{R}_+ \to \mathbb{R}$ by $f^v(y,x) = f_1(r+(y-r)v,x)$ for $(y,x) \in [0,\infty) \times \mathbb{R}_+$. Then $f^v \in \mathbf{C}_b([0,\infty) \times \mathbb{R}_+)$, and so by Lemma 3.4.4, $y \to \langle f^v(y,\cdot),\zeta_i(y)\rangle$ is a continuous function of $y \in [0,\infty)$. Noting that $f^v(r+h,\cdot) = f_1(r+hv,\cdot)$, it follows that for each

$v \in [0,1]$,

$$\lim_{h\to 0}\langle f_1(r+hv,\cdot),\zeta_i(r+h)\rangle = \lim_{h\to 0}\langle f^v(r+h,\cdot),\zeta_i(r+h)\rangle = \langle f^v(r,\cdot),\zeta_i(r)\rangle = \langle f_1(r,\cdot),\zeta_i(r)\rangle.$$

Combining this and, because $f_1(\cdot,\cdot)$ is bounded by a constant and $\sup_{0\le u\le t} z_i(u)$ is finite by the continuity of $z_i(\cdot)$, using the bounded convergence theorem, we have

$$\lim_{h\to 0}\frac{\langle f(r+h,\cdot),\zeta_i(r+h)\rangle - \langle f(r,\cdot),\zeta_i(r+h)\rangle}{h} = \int_0^1 \langle f_1(r,\cdot),\zeta_i(r)\rangle dv = \langle f_1(r,\cdot),\zeta_i(r)\rangle.$$

Now consider the last difference on the right hand side of (3.21). For fixed $r \in (s,t)$, we can use Lemma A.2 with $f(r,\cdot)$ in place of $f(\cdot)$ there, to conclude that

$$\begin{aligned}\langle f(r,\cdot),\zeta_i(r+h)\rangle - \langle f(r,\cdot),\zeta_i(r)\rangle = &-\int_r^{r+h}\langle f_2(r,\cdot),\zeta_i(u)\rangle\frac{\Lambda_i(u)}{z_i(u)}\mathbb{1}_{(0,\infty)}\big(z_i(u)\big)du\\ &+\nu_i\langle f(r,\cdot),\vartheta_i\rangle\int_r^{r+h}\mathbb{1}_{(0,\infty)}\big(z_i(u)\big)du.\end{aligned}$$

Now, because $z_i(u) > 0$ for $u \in (s,t)$, we have that

$$u \to \langle f_2(r,\cdot),\zeta_i(u)\rangle\frac{\Lambda_i(u)}{z_i(u)}\mathbb{1}_{(0,\infty)}\big(z_i(u)\big),$$

is continuous on (s,t). Consequently, by the fundamental theorem of calculus,

$$\lim_{h\to 0}\frac{1}{h}\int_r^{r+h}\langle f_2(r,\cdot),\zeta_i(u)\rangle\frac{\Lambda_i(u)}{z_i(u)}\mathbb{1}_{(0,\infty)}\big(z_i(u)\big)du = \langle f_2(r,\cdot),\zeta_i(r)\rangle\frac{\Lambda_i(r)}{z_i(r)}\mathbb{1}_{(0,\infty)}\big(z_i(r)\big). \quad (3.22)$$

Finally, we note that $r \to \langle f(r,\cdot),\vartheta_i\rangle$ is continuous on $[0,\infty)$ by the bounded convergence theorem. Combining all of these and replacing $f_1(r,x), f_2(r,x)$ with $f_r(r,x), f_x(r,x)$, we can conclude that $r \to \langle f(r,\cdot),\zeta_i(r)\rangle$ is once continuously differentiable on (s,t), with continuous derivative given by

$$\langle f_r(r,\cdot),\zeta_i(r)\rangle - \langle f_x(r,\cdot),\zeta_i(r)\rangle \frac{\Lambda_i(r)}{z_i(r)} + \nu_i\langle f(r,\cdot),\vartheta_i\rangle, \quad r\in(s,t). \tag{3.23}$$

Integrating over any closed interval $[s_1,t_1]$ contained in (s,t), we obtain that (3.20) holds with s_1,t_1 in place of s,t, respectively. Then invoking Lemma 3.4.4 again for the continuity of $r\to\langle f(r,\cdot),\ \zeta_i(r)\rangle$ from the right at s and the left at t, and noting the boundedness of the integrands in the integrals in (3.20), we see that we can let $s_1\downarrow s$ and $t_1\uparrow t$ to obtain the desired result. $\square$

Lemma 3.4.6. *Suppose that $\zeta(\cdot)$ is a fluid model solution, $i\in I$ and $0\le s<t<\infty$ such that $\zeta_i(r)\neq 0$ for all $r\in[s,t]$. Then*

$$\overline{M}^i_t(x) = \overline{M}^i_s(x+S^i_{s,t}) + \nu_i\int_s^t \overline{N}_i(x+S^i_{u,t})du \quad \textit{for all } x\in\mathbb{R}_+. \tag{3.24}$$

Proof. Because $\zeta_i(\cdot)\neq 0$ on $[s,t]$, $z_i(\cdot)$ is strictly positive on $[s,t]$ and since it is continuous, there is $s_1\in[0,s]$, where $s_1<s$ if $s\neq 0$ and $s_1=0$ if $s=0$, such that $z_i(\cdot)$ is still strictly positive on $[s_1,t]$. Then $u\to S^i_{u,t}$ is continuously differentiable on $[s_1,t]$, with $\frac{dS^i_{u,t}}{du} = -\frac{\Lambda_i(u)}{z_i(u)}$ for $u\in[s_1,t]$. Consider $g\in\mathbf{C}^1_b(\mathbb{R})$ with $g(x)=0$ for all $x\le 0$. By the continuous differentiability of g, we must have $g'(x)=0$ for all $x\le 0$. Let

$$f(u,x) = g(x-S^i_{u,t}), \quad u\in[s_1,t],\ x\in\mathbb{R}_+.$$

Then, $f\in\mathbf{C}^1_b([s_1,t]\times\mathbb{R}_+)$ where for $u\in[s_1,t]$ and $x\in\mathbb{R}_+$,

$$f_u(u,x) = \frac{g'(x-S^i_{u,t})\Lambda_i(u)}{z_i(u)} \quad \text{and} \quad f_x(u,x) = g'(x-S^i_{u,t}).$$

Because $g(x)=0$ and $g'(x)=0$ for all $x\le 0$, we have for $u\in[s_1,t]$, $f(u,0)=0$ and $f_x(u,0)=0$. Let $\varepsilon\in(0,(t-s)/2)$. We wish to construct a function f^ε that satisfies the conditions in Lemma

3.4.5 and that equals f on $[s,t-\varepsilon]\times\mathbb{R}_+$. Let $h^\varepsilon\in\mathbf{C}^1_b([0,\infty))$ be such that

$$h^\varepsilon(u)=\begin{cases}1, & u\in[s,t-\varepsilon],\\ 0, & u\in[0,s_1)\cup[t,\infty).\end{cases}$$

If $s=0$, then $[0,s_1)=\emptyset$. When $[0,s_1)\neq\emptyset$, we have by continuity (from the left) of h_ε and h'_ε that $h_\varepsilon(s_1)=0$ and $h'_\varepsilon(s_1)=0$. Extend f to be identically equal to zero on $([0,s_1)\cup(t,\infty))\times\mathbb{R}_+$ and define

$$f^\varepsilon(u,x)=f(u,x)h^\varepsilon(u),\quad u\in[0,\infty),x\in\mathbb{R}_+.$$

Then, $f^\varepsilon\in\mathbf{C}^1_b([0,\infty)\times\mathbb{R}_+)$ with $f^\varepsilon(\cdot,0)\equiv 0$ and $f^\varepsilon=f$ on $[s,t-\varepsilon]\times\mathbb{R}_+\subset[s_1,t]\times\mathbb{R}_+$. On replacing f,t in (3.20) with $f^\varepsilon,t-\varepsilon$, respectively, we obtain

$$\begin{aligned}\langle f(t-\varepsilon,\cdot),\zeta_i(t-\varepsilon)\rangle =&\langle f(s,\cdot),\zeta_i(s)\rangle+\int_s^{t-\varepsilon}\langle g'(\cdot-S^i_{u,t}),\zeta_i(u)\rangle\frac{\Lambda_i(u)}{z_i(u)}du\\ &-\int_s^{t-\varepsilon}\langle g'(\cdot-S^i_{u,t}),\zeta_i(u)\rangle\frac{\Lambda_i(u)}{z_i(u)}du\\ &+\nu_i\int_s^{t-\varepsilon}\langle g(\cdot-S^i_{u,t}),\vartheta_i\rangle du\\ =&\langle g(\cdot-S^i_{s,t}),\zeta_i(s)\rangle+\nu_i\int_s^{t-\varepsilon}\langle g(\cdot-S^i_{u,t}),\vartheta_i\rangle du.\end{aligned}\tag{3.25}$$

Similar to Lemma 3.4.4, $u\to\langle f(u,\cdot),\zeta(u)\rangle$ is continuous on $[s,t]$ and so we can let $\varepsilon\to 0$ in (3.25) to obtain

$$\langle g(\cdot),\zeta_i(t)\rangle=\langle f(t,\cdot),\zeta_i(t)\rangle=\langle g(\cdot-S^i_{s,t}),\zeta_i(s)\rangle+\nu_i\int_s^t\langle g(\cdot-S^i_{u,t}),\vartheta_i\rangle du.\tag{3.26}$$

For $x\in\mathbb{R}_+$, to obtain (3.24) from (3.26), consider a sequence of non-negative functions $\{g_n\}_{n=0}^\infty\subset\mathbf{C}^1_b(\mathbb{R})$ satisfying $g_n(x)=0$ for all $x\le 0$ and all n, and such that g_n increases to $\mathbb{1}_{(x,\infty)}$ pointwise on $\mathbb{R}$ and apply the monotone convergence theorem. □

Lemma 3.4.7. *Suppose that Assumption 3.2 holds and let $\zeta(\cdot)$ be a fluid model solution. Suppose* *that* $0 \le s < t < \infty$ *and* $i \in I$ *such that* $\zeta_i(s) \in \mathbf{K}$ *and* $z_i(r) > 0$ *for all* $r \in (s,t)$. *Then* $\zeta_i(r) \in \mathbf{K}$ *for all* $r \in (s,t)$.

Proof. Fix $r \in (s,t)$. It suffices to show that $x \to \overline{M}^i_r(x)$ is continuous.

We first consider the case where $z_i(s) > 0$. From (3.24), we have for all $x \in \mathbb{R}_+$,

$$\overline{M}^i_r(x) = \overline{M}^i_s(x + S^i_{s,r}) + \nu_i \int_s^r \overline{N}_i(x + S^i_{u,r})du. \tag{3.27}$$

Because $\zeta_i(s) \in \mathbf{K}$, $y \to \overline{M}^i_s(y)$ is continuous and it follows that the first term on the right hand side of (3.27) is continuous as a function of x. From the assumption that $\vartheta_i \in \mathbf{K}$, we have that $y \to \overline{N}_i(y)$ is continuous (and bounded). It follows from the dominated convergence theorem that the second term on the right hand side of (3.27) is continuous as a function of x. This completes the proof when $z_i(s) > 0$.

Now suppose that $z_i(s) = 0$. Then for $s < s_0 < r < t_0 < t$, we have $z_i(\cdot) > 0$ on $[s_0, t_0]$ and so by (3.24) we have for all $x \in \mathbb{R}_+$,

$$\overline{M}^i_r(x) = \overline{M}^i_{s_0}(x + S^i_{s_0,r}) + \nu_i \int_{s_0}^r \overline{N}_i(x + S^i_{u,r})du. \tag{3.28}$$

Fix $x_0 \in \mathbb{R}_+$ and let $\varepsilon > 0$. Because $z_i(\cdot)$ is continuous and $z_i(s) = 0$, we can choose s_0 close enough to s so that $\overline{M}^i_{s_0}(\cdot) \le z_i(s_0) < \varepsilon/4$. It follows that the difference of two evaluations of the first term in the right hand side of (3.28), where the evaluations are at x_0 and $x \in \mathbb{R}_+$, has magnitude less than $\varepsilon/2$. For this fixed value of s_0, the last term in (3.28) is continuous as a function of x (because $\vartheta_i \in \mathbf{K}$). Combining the properties of the first and last terms in the right hand side of (3.28), it follows that there is $\delta > 0$ such that whenever $|x - x_0| < \delta$, we have

$$|\overline{M}^i_r(x) - \overline{M}^i_r(x_0)| \le \varepsilon.$$

Because $\varepsilon > 0$ and $x_0 \in \mathbb{R}_+$ were arbitrary, it follows that $x \to \overline{M}^i_r(x)$ is continuous when $z_i(s) =$

0. $\square$

Corollary 3.4.1. *Suppose that Assumption 3.2 holds and $\zeta(\cdot)$ is a fluid model solution with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}$. Then $\zeta(t) \in \mathbf{K}^{\mathbf{I}}$ for all $t > 0$.*

Proof. Fix $i \in I$ and $t > 0$. If $z_i(t) = 0$, then $\zeta_i(t) = 0$ is in $\mathbf{K}$. On the other hand, if $z_i(t) > 0$, then by the continuity of $z_i(\cdot)$, there is an open interval $V = (a,b)$ containing t such that $z_i(s) > 0$ on V and either $a = 0$ or $z_i(a) = 0$. In either case, $\zeta_i(a) \in \mathbf{K}$, and it follows from Lemma 3.4.7 that $\zeta_i(s) \in \mathbf{K}$ for all $s \in V$ and in particular, $\zeta_i(t) \in \mathbf{K}$. $\square$

Lemma 3.4.8. *Suppose that Assumption 3.2 holds. Let $\zeta(\cdot)$ be a fluid model solution with $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$. Then for each $i \in I$, $\mathcal{H}_i^{\zeta}(\cdot)$ as defined in (3.7) is continuous on $[0,\infty)$.*

Proof. Fix $i \in I$ and $t_0 \in [0,\infty)$.

We first consider the case where $z_i(t_0) = 0$. Then $\mathcal{H}_i^{\zeta}(t_0) = 0$. Note that $z_i(\cdot)$ is continuous. Also, because $w_i(0) = \langle \chi, \zeta_i(0) \rangle < \infty$ by assumption, it follows from (2.5), that $w_i(\cdot)$ is continuous. Then, because $z_i(t_0) = w_i(t_0) = 0$, it follows from the continuity of $z_i(\cdot)$, $w_i(\cdot)$ and (3.9), that $\mathcal{H}_i^{\zeta}(s)$ tends to zero as $s \to t_0$. So $\mathcal{H}_i^{\zeta}(\cdot)$ is continuous at t_0.

We now turn to the case where $z_i(t_0) > 0$. By the continuity of $z_i(\cdot)$, there is an interval $[s,t]$ containing t_0 on which $z_i(r) \neq 0$ for all $r \in [s,t]$, where we may choose $s < t_0 < t$ if $t_0 \neq 0$ and $s = t_0 < t$ if $t_0 = 0$. Because $\zeta(0) \in \mathbf{K}^{\mathbf{I}}$, it follows from Corollary 3.4.1 that for all $r \in [0,\infty)$, $x \to \overline{M}_r^i(x)$ is continuous. From Lemma 3.4.6, with r in place of t there, we have for each $r \in [s,t]$ that for each $x \in \mathbb{R}_+$,

$$\overline{M}_r^i(x) = \overline{M}_s^i(x + S_{s,r}^i) + \int_s^r \nu_i \overline{N}_i(x + S_{u,r}^i) du.$$

It then follows from the continuity of $\overline{M}_s^i(\cdot)$ and $\overline{N}_i(\cdot)$ (because ϑ_i is continuous) on $\mathbb{R}_+$, and the continuity of $r \to S_{u,r}^i$ for $r \in [s,t]$, for each fixed $u \in [s,t]$, that $r \to \overline{M}_r^i(x)$ is continuous for each

$x \in \mathbb{R}_+$. Now, for $r \in [s,t]$,

$$\mathcal{H}_i^{\zeta}(r) = \frac{\kappa_i}{\tilde{\rho}_i^{\alpha_i}} \int_0^\infty \left(\overline{M}_r^i(x)\right)^{\alpha_i} \theta_i(x)\overline{M}_r^i(x)dx,$$

where the integrand is dominated by $\|\theta_i\|_\infty(\sup_{u\in[s,t]} z_i(u))^{\alpha_i}\overline{M}_r^i(\cdot)$. By the generalized Lebesgue dominated convergence theorem and the fact that $w_i(r) = \int_0^\infty \overline{M}_r^i(x)dx$ is continuous as a function of r, we have that $\mathcal{H}_i^{\zeta}(r) \to \mathcal{H}_i^{\zeta}(t_0)$ as $r \to t_0$. □

3.5 Proofs of Theorem 3.3.1 and Corollary 3.3.1

3.5.1 Smooth Approximation of Measures

We use an approximation argument to prove Theorem 3.3.1. To prepare for this, for each positive integer n, let $\varphi_n \in \mathbf{C}_c^\infty(\mathbb{R})$ be such that $\varphi_n \geq 0, \varphi_n(x) = 0$ for all $x \in (-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, \infty)$, $\varphi_n(x) = \varphi_n(-x)$ for all $x > 0$, and $\int_{\mathbb{R}} \varphi_n(x)dx = 1$. Given $\xi \in \mathbf{M}$ and $n \in \mathbb{N}$, let ξ^n be the nonnegative, absolutely continuous Borel measure on $\mathbb{R}_+$ whose density is given by $d_n(x) = \int_{\mathbb{R}_+} \varphi_n(x-y)\xi(dy) = \int_{\mathbb{R}_+} \varphi_n(y-x)\xi(dy)$ for $x \in \mathbb{R}_+$, where we have used the symmetry of φ_n for the last equality. Note that $d_n(\cdot)$ is in $\mathbf{C}_b^\infty(\mathbb{R}_+)$, because φ_n is infinitely differentiable with compact support and ξ is a finite measure on $\mathbb{R}_+$. For any bounded, Borel measurable function f defined on $\mathbb{R}_+$, let $(f * \varphi_n)(y) = \int_{\mathbb{R}_+} \varphi_n(y-x)f(x)dx$ for $y \in \mathbb{R}_+$. Then, by Fubini's theorem,

$$\langle f, \xi^n \rangle = \int_{\mathbb{R}_+} f(x) \int_{\mathbb{R}_+} \varphi_n(y-x)\xi(dy)dx = \langle f * \varphi_n, \xi \rangle. \tag{3.29}$$

The following lemma can be proved in the same manner as Lemma 7.12 of Puha and Williams [PW16], so we omit the proof.

Lemma 3.5.1. *Let $\xi \in \mathbf{K}_1$. For each $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$, we have*

$$\left\langle \mathbb{1}_{(x+\frac{1}{n},\infty)}, \xi \right\rangle \leq \langle \mathbb{1}_{(x,\infty)}, \xi^n \rangle \leq \left\langle \mathbb{1}_{\left((x-\frac{1}{n})^+,\infty\right)}, \xi \right\rangle, \tag{3.30}$$

$$\langle \chi, \xi \rangle - \frac{\langle \mathbb{1}, \xi \rangle}{n} \leq \langle \chi, \xi^n \rangle \leq \langle \chi, \xi \rangle + \frac{\langle \mathbb{1}, \xi \rangle}{n}. \tag{3.31}$$

Furthermore, we have $\xi^n \in \mathbf{A}$ for each $n \in \mathbb{N}$ and as $n \to \infty$,

$$\xi^n \xrightarrow{w} \xi \quad \textit{and} \quad \langle \chi, \xi^n \rangle \to \langle \chi, \xi \rangle. \tag{3.32}$$

Given a fluid model solution $\zeta(\cdot)$, for each $t \geq 0$ and $i \in I$, let $\{\zeta_i^n(t)\}_{n=1}^{\infty}$ be the approximating sequence of measures for $\zeta_i(t)$, as defined above with $\zeta_i(t)$ in place of ξ. For any positive integer ℓ, let $\mathcal{C}_{0,\ell} = \{g \in \mathbf{C}_b^1(\mathbb{R}_+) : g = 0 \text{ on } [0, \frac{1}{\ell}]\}$. For $g \in \mathcal{C}_{0,\ell}$ and all $n > \ell$, we have $(g * \varphi_n)(0) = 0$ and $(g * \varphi_n)'(0) = 0$. It follows that $g * \varphi_n \in \mathcal{C}$. By (2.4), with $g * \varphi_n$ replacing f and noting that $(g * \varphi_n)'(\cdot) = (g' * \varphi_n)(\cdot)$, we have for any $t \geq 0$,

$$\begin{aligned}\langle g * \varphi_n, \zeta_i(t) \rangle = \langle g * \varphi_n, \zeta_i(0) \rangle &- \int_0^t \langle g' * \varphi_n, \zeta_i(s) \rangle \frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}(z_i(s)) ds \\ &+ \nu_i \langle g * \varphi_n, \vartheta_i \rangle \int_0^t \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) ds.\end{aligned} \tag{3.33}$$

Then, using (3.29), we can rewrite the above as

$$\langle g, \zeta_i^n(t) \rangle = \langle g, \zeta_i^n(0) \rangle - \int_0^t \langle g', \zeta_i^n(s) \rangle \frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}(z_i(s)) ds + \nu_i \langle g, \vartheta_i^n \rangle \int_0^t \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) ds. \tag{3.34}$$

For each positive integer n, $i \in I$, $t \geq 0$ and $x \in \mathbb{R}_+$, let

$$\overline{M}_t^{i,n}(x) = \langle \mathbb{1}_{(x,\infty)}, \zeta_i^n(t) \rangle, \qquad \overline{N}^{i,n}(x) = \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^n \rangle. \tag{3.35}$$

The following lemma is key to our proof of Theorem 3.3.1. It provides a rigorous formulation of the partial differential equation result assumed in Paganini et al. [PTFA12].

Lemma 3.5.2. *Assume that $\zeta(\cdot)$ is a fluid model solution. Suppose that $i \in \mathcal{I}$ and $0 \leq a < b < \infty$ are such that $z_i(t) \neq 0$ for all $t \in [a,b]$. Then, for each positive integer ℓ and all $n > \ell$, $t \to \overline{M}_t^{i,n}(x)$ is continuously differentiable on $[a,b]$ for each fixed $x \in \mathbb{R}_+$, and $x \to \overline{M}_t^{i,n}(x)$ is continuously differentiable on $[\frac{1}{\ell}, \infty)$ for each fixed $t \in [a,b]$, and furthermore,*

$$\frac{\partial \overline{M}_t^{i,n}(x)}{\partial t} = \frac{\Lambda_i(t)}{z_i(t)} \frac{\partial \overline{M}_t^{i,n}(x)}{\partial x} + \nu_i \overline{N}^{i,n}(x), \tag{3.36}$$

for $t \in [a,b]$, $x \geq \frac{1}{\ell}$, where the partial derivatives with respect to time at $t = a, b$ are from the right, left, respectively, and the partial derivative with respect to x at $x = 1/\ell$ is from the right.

Proof. For each $s \in [0,\infty)$, $i \in \mathcal{I}$ and fixed n, by the definition of $\zeta_i^n(s)$, $m_s^{i,n}(\cdot) = \int_{\mathbb{R}_+} \varphi_n(y - \cdot)\zeta_i(s)(dy)$ is the $\mathbf{C}_b^\infty$ density function for the measure $\zeta_i^n(s)$. Thus, $x \to \overline{M}_s^{i,n}(x)$ is continuously differentiable on $[0,\infty)$ with derivative function $-m_s^{i,n}(\cdot)$. By the finiteness of $\zeta_i(s)$, $\lim_{x\to\infty} m_s^{i,n}(x) = 0$. Using integration by parts, for any $g \in \mathcal{C}_{0,\ell}$ that has compact support in $\mathbb{R}_+$, we have for each $n > \ell$, using the facts that g is bounded, $g(\frac{1}{\ell}) = 0$, and g is zero outside some compact set, we have

$$\langle g', \zeta_i^n(s)\rangle = \int_{\frac{1}{\ell}}^\infty g'(x) m_s^{i,n}(x) dx = -\int_{\frac{1}{\ell}}^\infty g(x) \frac{dm_s^{i,n}(x)}{dx} dx. \tag{3.37}$$

Now suppose, as in the statement of the lemma, that $i \in \mathcal{I}$ and $0 \leq a < b < \infty$ such that $z_i(s) \neq 0$ for $s \in [a,b]$. Then we have from (3.34) that for any $t \in [a,b]$,

$$\langle g, \zeta_i^n(t)\rangle - \langle g, \zeta_i^n(a)\rangle = -\int_a^t \langle g', \zeta_i^n(s)\rangle \frac{\Lambda_i(s)}{z_i(s)} ds + \nu_i \langle g, \vartheta_i^n\rangle (t-a). \tag{3.38}$$

Fix $\ell, n > \ell$, $x_0 \geq \frac{1}{\ell}$ and $z > x_0$. Combining (3.37), (3.38), and considering a sequence $\{g_m\}_{m=1}^\infty$ of non-negative functions in $\mathcal{C}_{0,\ell}$ that have compact support and that converge monotonically upwards to $\mathbb{1}_{(x_0,z)}$, we obtain using monotone and dominated convergence (noting that $\frac{dm_s^{i,n}(x)}{dx} = -\langle \varphi_n'(\cdot - x), \zeta_i(s)\rangle$ is uniformly bounded for all $s \in [a,b]$ and $x \in \mathbb{R}_+$), that for all $t \in [a,b]$ and $z > x_0$,

$$\begin{aligned}
\langle \mathbb{1}_{(x_0,z)}, \zeta_i^n(t)\rangle - \langle \mathbb{1}_{(x_0,z)}, \zeta_i^n(a)\rangle &= \int_a^t \left\langle \mathbb{1}_{(x_0,z)}, \frac{dm_s^{i,n}}{dx} \right\rangle \frac{\Lambda_i(s)}{z_i(s)} ds + \nu_i \langle \mathbb{1}_{(x_0,z)}, \vartheta_i^n\rangle (t-a) \\
&= \int_a^t (m_s^{i,n}(z) - m_s^{i,n}(x_0)) \frac{\Lambda_i(s)}{z_i(s)} ds + \nu_i \langle \mathbb{1}_{(x_0,z)}, \vartheta_i^n\rangle (t-a).
\end{aligned}$$

We can let $z \to \infty$, using monotone and bounded convergence, plus the fact that $\lim_{z\to\infty} m_s^{i,n}(z) = 0$ for each $s \in [a,t]$, to conclude that for each $t \in [a,b]$ and $x_0 \geq \frac{1}{\ell}$,

$$\langle \mathbb{1}_{(x_0,\infty)}, \zeta_i^n(t)\rangle - \langle \mathbb{1}_{(x_0,\infty)}, \zeta_i^n(a)\rangle = -\int_a^t m_s^{i,n}(x_0) \frac{\Lambda_i(s)}{z_i(s)} ds + \nu_i \langle \mathbb{1}_{(x_0,\infty)}, \vartheta_i^n\rangle (t-a). \tag{3.39}$$

Rewriting, we have for all $t \in [a,b]$ and $x \geq \frac{1}{\ell}$,

$$\overline{M}_t^{i,n}(x) - \overline{M}_a^{i,n}(x) = \int_a^t \frac{\Lambda_i(s)}{z_i(s)} \frac{\partial \overline{M}_s^{i,n}(x)}{\partial x} ds + \nu_i \overline{N}^{i,n}(x)(t-a). \tag{3.40}$$

For fixed $x \geq \frac{1}{\ell}$, $s \to \frac{\partial \overline{M}_s^{i,n}(x)}{\partial x} = -m_s^{i,n}(x)$ is continuous, because the fluid model solution ζ_i is continuous as a function of time. Also, $s \to \frac{\Lambda_i(s)}{z_i(s)}$ is continuous on $[a,b]$, because $z_i(\cdot)$ is strictly positive there. It follows that $t \to \overline{M}_t^{i,n}(x)$ is continuously differentiable on $[a,b]$, and by differentiating (3.40), we obtain (3.36). Since all of the other properties have been verified, this completes the proof. □

3.5.2 Proof of Theorem 3.3.1

Proof. Assume that the hypotheses of Theorem 3.3.1 hold. Because $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$, we have by Corollary 3.4.1 and (2.5), that for each $t \geq 0$, $\zeta(t) \in \mathbf{K}_1^{\mathbf{I}}$. It follows that for each $i \in I$ and $t \geq 0$, $x \to \overline{M}_t^i(x)$ is continuous and integrable with respect to Lebesgue measure (with integral equal to

$\langle \chi, \zeta_i(t) \rangle < \infty$) on $[0,\infty)$. Also, $x \to \overline{N}_i(x)$ is continuous and integrable with respect to Lebesgue measure (with integral equal to $\langle \chi, \vartheta_i \rangle < \infty$) over $[0,\infty)$.

Fix $i \in I$. Because $\mathcal{K}_i^\zeta(\cdot)$ is bounded and measurable on $[0,t]$ for each $t \geq 0$, to prove the absolute continuity of $\mathcal{H}_i^\zeta(\cdot)$, it suffices to prove that (3.16) holds for each $t \geq 0$. We first prove that if $0 \leq a < b < \infty$ such that $z_i(s) \neq 0$ for all $s \in [a,b]$, then

$$\mathcal{H}_i^\zeta(b) - \mathcal{H}_i^\zeta(a) = \int_a^b \mathcal{K}_i^\zeta(s)ds. \tag{3.41}$$

Assume that $0 \leq a < b < \infty$ such that $z_i(s) \neq 0$ for all $s \in [a,b]$. For (3.42), we shall use the definition of $\mathcal{K}_i^\zeta(\cdot)$, the facts that $\Lambda_i(\cdot) \leq \max_j C_j$, $z_i(\cdot)$ is bounded on $[a,b]$, being continuous there, $\overline{M}_s^i(x) \leq z_i(s)$ for all $x \in \mathbb{R}_+$ and $s \in [a,b]$, $\left|\frac{\Lambda_i(\cdot)}{z_i(\cdot)}\right|$ is bounded on $[a,b]$ because $z_i(\cdot)$ is continuous and strictly positive there, $\theta_i'(x) = m_i\mu_i(\theta_i(x))^{\frac{\alpha_i+1}{\alpha_i}}\overline{N}_i(x)$ for all $x \in \mathbb{R}_+$, $\|\theta_i\|_\infty < \infty$, and $\int_0^\infty \overline{N}_i(x)dx = \langle \chi, \vartheta_i \rangle = \mu_i^{-1} < \infty$. With these we see that by dominated convergence,

$$\begin{aligned}\int_a^b \mathcal{K}_i^\zeta(s)ds = {}& -\tilde{\rho}_i^{-\alpha_i}\kappa_i \int_a^b \Lambda_i(s)(z_i(s))^{\alpha_i}ds \\ & + \lim_{\ell\to\infty} \tilde{\rho}_i^{-\alpha_i}\kappa_i \int_a^b \int_{\frac{1}{\ell}}^{\ell} (\overline{M}_s^i(x))^{\alpha_i} \left(\frac{-\Lambda_i(s)}{z_i(s)} \overline{M}_s^i(x)\theta_i'(x) + (\alpha_i+1)\nu_i \overline{N}_i(x)\theta_i(x) \right) dxds.\end{aligned} \tag{3.42}$$

Now, for positive integers ℓ and $n > \ell$, because $\vartheta_i \in \mathbf{K}_1$ and $\zeta_i(s) \in \mathbf{K}_1$ for all $s \in [a,b]$, by Lemma 3.5.1, we have that as $n \to \infty$, $\overline{N}_i^n(x) \to \overline{N}_i(x)$ for each $x \in (0,\infty)$ and $\overline{M}_s^{i,n}(x) \to \overline{M}_s^i(x)$ for each $x \in (0,\infty)$, $s \in [a,b]$. Moreover, $\overline{N}_i^n(x) \leq \overline{N}_i((x-1)^+)$ and $\overline{M}_s^{i,n}(x) \leq \overline{M}_s^i((x-1)^+) \leq z_i(s)$ for all $x \in \mathbb{R}_+$ and $s \in [a,b]$, where $x \to \overline{N}_i((x-1)^+)$ has integral on $(0,\infty)$ bounded by $\int_0^\infty \overline{N}_i(x)dx + 1 < \infty$ and there is a uniform bound on $z_i(\cdot)$ for all $s \in [a,b]$. It then follows by the dominated convergence theorem (using the boundedness of θ_i' and θ_i) that for each fixed positive integer ℓ,

$$\begin{aligned}&\int_a^b \int_{\frac{1}{\ell}}^{\ell} (\overline{M}_s^i(x))^{\alpha_i} \left(\frac{-\Lambda_i(s)}{z_i(s)} \overline{M}_s^i(x)\theta_i'(x) + (\alpha_i+1)\nu_i \overline{N}_i(x)\theta_i(x) \right) dxds \\ &= \lim_{n\to\infty} \int_a^b \int_{\frac{1}{\ell}}^{\ell} (\overline{M}_s^{i,n}(x))^{\alpha_i} \left(\frac{-\Lambda_i(s)}{z_i(s)} \overline{M}_s^{i,n}(x)\theta_i'(x) + (\alpha_i+1)\nu_i \overline{N}_i^n(x)\theta_i(x) \right) dxds. \quad (3.43)\end{aligned}$$

Using integration by parts on the first term, the expression is equal to

$$\begin{aligned}&\lim_{n\to\infty} \left(\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) [-(\overline{M}_s^{i,n}(\cdot))^{\alpha_i+1}\theta_i(\cdot)]_{\frac{1}{\ell}}^{\ell} ds \right. \\ &\quad \left. +(\alpha_i+1) \int_a^b \int_{\frac{1}{\ell}}^{\ell} (\overline{M}_s^{i,n}(x))^{\alpha_i} \left(\frac{\Lambda_i(s)}{z_i(s)} \frac{\partial \overline{M}_s^{i,n}(x)}{\partial x} + \nu_i \overline{N}_i^n(x) \right) \theta_i(x) dxds \right) \\ &= \lim_{n\to\infty} \left(\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) [-(\overline{M}_s^{i,n}(\cdot))^{\alpha_i+1}\theta_i(\cdot)]_{\frac{1}{\ell}}^{\ell} ds \right. \qquad (3.44) \\ &\quad \left. +(\alpha_i+1) \int_a^b \int_{\frac{1}{\ell}}^{\ell} (\overline{M}_s^{i,n}(x))^{\alpha_i} \left(\frac{\partial \overline{M}_s^{i,n}(x)}{\partial s} \right) \theta_i(x) dxds \right),\end{aligned}$$

where we have used Lemma 3.5.2 for the last equality. By Fubini's theorem (where the joint measurability of the integrand follows from (3.36) and the fact that the partial derivative with respect to x there is given by $-m_s^{i,n}(x)$), the quantity is equal to

$$\begin{aligned}&\lim_{n\to\infty} \left(\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) [-(\overline{M}_s^{i,n}(\cdot))^{\alpha_i+1}\theta_i(\cdot)]_{\frac{1}{\ell}}^{\ell} ds + (\alpha_i+1) \int_{\frac{1}{\ell}}^{\ell} \int_a^b (\overline{M}_s^{i,n}(x))^{\alpha_i} \left(\frac{\partial \overline{M}_s^{i,n}(x)}{\partial s} \right) \theta_i(x) dsdx \right) \\ &= \lim_{n\to\infty} \left(\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) \left[-(\overline{M}_s^{i,n}(\ell))^{\alpha_i+1}\theta_i(\ell) + \left(\overline{M}_s^{i,n}\left(\frac{1}{\ell}\right) \right)^{\alpha_i+1} \theta_i\left(\frac{1}{\ell}\right) \right] ds \right. \\ &\quad \left. + \int_{\frac{1}{\ell}}^{\ell} \left((\overline{M}_b^{i,n}(x))^{\alpha_i+1} - (\overline{M}_a^{i,n}(x))^{\alpha_i+1} \right) \theta_i(x) dx \right) \\ &= \int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) \left[-(\overline{M}_s^i(\ell))^{\alpha_i+1}\theta_i(\ell) + \left(\overline{M}_s^i\left(\frac{1}{\ell}\right) \right)^{\alpha_i+1} \theta_i\left(\frac{1}{\ell}\right) \right] ds \qquad (3.45) \\ &\quad + \int_{\frac{1}{\ell}}^{\ell} \left((\overline{M}_b^i(x))^{\alpha_i+1} - (\overline{M}_a^i(x))^{\alpha_i+1} \right) \theta_i(x) dx,\end{aligned}$$

where we have used bounded convergence to pass to the limit for the last equality. Observe that as $\ell \to \infty$, we have $\overline{M}_s^i(\ell) \to 0$, $\overline{M}_s^i\left(\frac{1}{\ell}\right) \to z_i(s)$, $\theta_i\left(\frac{1}{\ell}\right) \to 1$, and there is a uniform bound for $(s,x) \to \overline{M}_s^i(x)$ and $x \to \theta_i(x)$ for all $s \in [a,b], x \in \mathbb{R}_+$. Combining this with the fact that $(\overline{M}_s^i(x))^{\alpha_i+1} \le (z_i(s))^{\alpha_i}\overline{M}_s^i(x)$, which is integrable on $\mathbb{R}_+$ for $s = a,b$, we see that as $\ell \to \infty$, the expression after the last equals sign in (3.45) converges to

$$\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)}\right)(z_i(s))^{\alpha_i+1}ds + \int_0^\infty \left((\overline{M}_b^i(x))^{\alpha_i+1} - (\overline{M}_a^i(x))^{\alpha_i+1}\right)\theta_i(x)dx. \tag{3.46}$$

On substituting the above into (3.42), we obtain

$$\begin{aligned}\int_a^b \mathcal{K}_i^\zeta(s)ds &= -\tilde{\rho}_i^{-\alpha_i}\kappa_i \int_a^b \Lambda_i(s)(z_i(s))^{\alpha_i}ds \\ &\quad +\tilde{\rho}_i^{-\alpha_i}\kappa_i \int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)}\right)(z_i(s))^{\alpha_i+1}ds \\ &\quad +\tilde{\rho}_i^{-\alpha_i}\kappa_i \int_0^\infty \left((\overline{M}_b^i(x))^{\alpha_i+1} - (\overline{M}_a^i(x))^{\alpha_i+1}\right)\theta_i(x)dx \\ &= \mathcal{H}_i^\zeta(b) - \mathcal{H}_i^\zeta(a),\end{aligned} \tag{3.47}$$

as desired.

We now turn to proving (3.16) for each $t \ge 0$. It clearly holds for $t = 0$, so we consider $t > 0$ fixed. If $z_i(s) \ne 0$ for all $s \in [0,t]$, then the result follows immediately from (3.41) with $a = 0$ and $b = t$. Therefore, we only need to treat the case where $z_i(s) = 0$ for some $s \in [0,t]$. Assuming this, let $s^* = \inf\{s \in [0,t] : z_i(s) = 0\}$ and $t^* = \sup\{s \in [0,t] : z_i(s) = 0\}$. Then, $0 \le s^* \le t^* \le t$, $z_i(s^*) = z_i(t^*) = 0$ and $z_i(s) > 0$ for $s \in (0,s^*) \cup (t^*,t)$. (The interval $(0,s^*)$ is empty if $z_i(0) = 0$ and (t^*,t) is empty if $z_i(t) = 0$.) In any event, we can write the open set $\mathcal{T}_t^i = \{s \in (0,t) : z_i(s) > 0\}$ as a (finite or countable) union of disjoint open intervals:

$$\mathcal{T}_t^i = (0,s^*) \cup \left(\bigcup_n (s_n,t_n)\right) \cup (t^*,t),$$

where $\bigcup_n (s_n,t_n) \subset (s^*,t^*)$ and $z_i(s_n) = z_i(t_n) = 0$ for each n.

For each fixed n, for $s_n < a < b < t_n$, we have that (3.41) holds. Then using the continuity of $\mathcal{H}_i^\zeta(\cdot)$ (see Lemma 3.4.8) and the boundedness of $\mathcal{K}_i^\zeta$ on $[s_n, t_n]$ that we can let $a \downarrow s_n$ and $b \uparrow t_n$ in the last equation, to obtain

$$\mathcal{H}_i^\zeta(t_n) - \mathcal{H}_i^\zeta(s_n) = \int_{(s_n,t_n)} \mathcal{K}_i^\zeta(s)ds.$$

Moreover, since $z_i(s_n) = z_i(t_n) = 0$, $\mathcal{H}_i^\zeta(s_n) = \mathcal{H}_i^\zeta(t_n) = 0$. Thus we have

$$\int_{(s_n,t_n)} \mathcal{K}_i^\zeta(s)ds = 0. \tag{3.48}$$

In a similar manner, we can obtain

$$\mathcal{H}_i^\zeta(s^*) - \mathcal{H}_i^\zeta(0) \;\; = \;\; \int_{(0,s^*)} \mathcal{K}_i^\zeta(s)ds, \tag{3.49}$$

where $\mathcal{H}_i^\zeta(s^*) = 0$, and

$$\mathcal{H}_i^\zeta(t) - \mathcal{H}_i^\zeta(t^*) = \int_{(t^*,t)} \mathcal{K}_i^\zeta(s)ds, \tag{3.50}$$

where $\mathcal{H}_i^\zeta(t^*) = 0$. Combining all of these and using the integrability of $\mathcal{K}_i^\zeta$ on $[0,t]$, the fact that $\mathcal{K}_i^\zeta(\cdot)$ is zero on $(0,t) \setminus \mathcal{T}_t^i$, and the disjointness of the intervals $\{(s_n, t_n)\}$, we have

$$\begin{aligned} \int_0^t \mathcal{K}_i^\zeta(s)ds &= \int_{(0,s^*)} \mathcal{K}_i^\zeta(s)ds + \sum_n \int_{(s_n,t_n)} \mathcal{K}_i^\zeta(s)ds + \int_{(t^*,t)} \mathcal{K}_i^\zeta(s)ds \\ &= -\mathcal{H}_i^\zeta(0) + 0 + \mathcal{H}_i^\zeta(t), \end{aligned}$$

which is the desired result (3.16).

We now prove (3.17). Since both sides are zero when $z_i(t) = 0$, it suffices to consider the case where $z_i(t) > 0$. In this case,

$$\begin{aligned}
\tilde{\rho}_i^{\alpha_i} \mathcal{K}_i^{\zeta}(t) &= -\kappa_i \Lambda_i(t)\big(z_i(t)\big)^{\alpha_i} \\
&\quad + \kappa_i \int_0^\infty \big(\overline{M}_t^i(x)\big)^{\alpha_i} \Big(-\frac{\Lambda_i(t)}{z_i(t)} \overline{M}_t^i(x) m_i \theta_i(x)^{\frac{\alpha_i+1}{\alpha_i}} \mu_i \overline{N}_i(x) + (\alpha_i+1)\nu_i \overline{N}_i(x)\theta_i(x)\Big) dx \\
&= -\kappa_i \Lambda_i(t)\big(z_i(t)\big)^{\alpha_i} \\
&\quad + \kappa_i \int_0^\infty \big(\overline{M}_t^i(x)\big)^{\alpha_i} \Big(-\frac{\Lambda_i(t)}{z_i(t)} \overline{M}_t^i(x) m_i \theta_i(x)^{\frac{1}{\alpha_i}} + (\alpha_i+1)\rho_i \Big) \mu_i \theta_i(x) \overline{N}_i(x) dx \\
&\leq -\kappa_i \Lambda_i(t)(z_i(t))^{\alpha_i} + \kappa_i \int_0^\infty \Big(\frac{\alpha_i}{m_i}\Big)^{\alpha_i} \rho_i^{\alpha_i+1} \Big(\frac{z_i(t)}{\Lambda_i(t)}\Big)^{\alpha_i} \mu_i \overline{N}_i(x) dx \\
&= -\kappa_i \Lambda_i(t)(z_i(t))^{\alpha_i} + \frac{\kappa_i \alpha_i^{\alpha_i} \rho_i^{\alpha_i+1} (z_i(t))^{\alpha_i}}{m_i^{\alpha_i} (\Lambda_i(t))^{\alpha_i}},
\end{aligned}$$

where we used Lemma 3.4.2, with $a = \alpha_i$, $q = \rho_i$, $b = \frac{\Lambda_i(t)}{z_i(t)} \theta_i(x)^{\frac{1}{\alpha_i}} m_i$, for the inequality, and the fact that $\int_0^\infty \mu_i \overline{N}_i(x) dx = 1$ for the last equality. Recall that $\delta > 0$ and $m_i \in (0, \alpha_i)$ were chosen so that $\big(\frac{\alpha_i}{m_i}\big)^{\alpha_i} = (1-\delta)(1+\delta)^{\alpha_i+1} > 1$ and $\tilde{\rho}_i = (1+\delta)\rho_i$ satisfies (3.3). Using that in the above expression, we obtain when $z_i(t) > 0$,

$$\begin{aligned}
\mathcal{K}_i^{\zeta}(t) &\leq \kappa_i (z_i(t))^{\alpha_i} \Big(-\frac{\Lambda_i(t)}{\tilde{\rho}_i^{\alpha_i}} + \frac{\tilde{\rho}_i(1-\delta)}{(\Lambda_i(t))^{\alpha_i}} \Big) \\
&= \kappa_i (z_i(t))^{\alpha_i} \Big(-\frac{\Lambda_i(t)}{\tilde{\rho}_i^{\alpha_i}} + \frac{\tilde{\rho}_i}{(\Lambda_i(t))^{\alpha_i}} - \delta \frac{\tilde{\rho}_i}{(\Lambda_i(t))^{\alpha_i}} \Big) \\
&\leq \kappa_i (z_i(t))^{\alpha_i} \Big((\alpha_i+1) \frac{\tilde{\rho}_i - \Lambda_i(t)}{(\Lambda_i(t))^{\alpha_i}} - \delta \frac{\tilde{\rho}_i}{(\Lambda_i(t))^{\alpha_i}} \Big),
\end{aligned} \tag{3.51}$$

where the last step follows by Lemma 3.4.3 with $a = \alpha_i$, $b = \Lambda_i(t)$ and $q = \tilde{\rho}_i$. The first inequality yields (3.17). We shall use the last inequality to prove Corollary 3.3.1. □

3.5.3 Proof of Corollary 3.3.1

Proof of Corollary 3.3.1. Given the results of Theorem 3.3.1, all that requires proof is the inequality. For fixed $t \geq 0$ and $i \in I_+(z(t))$, $U_i'\Big(\frac{\Lambda_i(t)}{z_i(t)}\Big) = \Big(\frac{z_i(t)}{\Lambda_i(t)}\Big)^{\alpha_i}$. Furthermore, $\tilde{\rho}$ has positive components and satisfies $\sum_{i\in I} R_{ji}\tilde{\rho}_i < C_j$ for all $j \in \mathcal{J}$. Then, by (3.51) and replacing $z, \psi, \phi(z)$ by $z(t), \tilde{\rho}, \Lambda(t)$, respectively, in Lemma 3.4.1, we obtain

$$\begin{aligned}
\mathcal{K}^{\zeta}(t) &= \sum_{i\in I_+(z(t))} \frac{\mathcal{K}_i^{\zeta}(t)}{\alpha_i + 1} \\
&\leq \sum_{i\in I_+(z(t))} \kappa_i \left(\frac{z_i(t)}{\Lambda_i(t)}\right)^{\alpha_i} \left(\tilde{\rho}_i - \Lambda_i(t)\right) - \delta \sum_{i\in I_+(z(t))} \frac{\kappa_i\tilde{\rho}_i}{\alpha_i+1}\left(\frac{z_i(t)}{\Lambda_i(t)}\right)^{\alpha_i} \\
&\leq -\delta \sum_{i\in I_+(z(t))} \frac{\kappa_i\tilde{\rho}_i}{\alpha_i+1}\left(\frac{z_i(t)}{\Lambda_i(t)}\right)^{\alpha_i}.
\end{aligned}$$

□

3.6 Proofs of Theorems 3.3.2 and 3.3.3

Theorem 3.3.1 and Corollary 3.3.1 are the main new results of this chapter. In particular, these results are given proofs that, in contrast to Paganini et al. [PTFA12], do not make strong smoothness assumptions on fluid model solutions and deal with the singular situation where some components of a fluid model solution may touch zero before all components reach zero. With these results in place, Theorems 3.3.2 and 3.3.3 follow in a similar manner to the arguments presented in [PTFA12]. However, we do generalize from having a common parameter α for all routes to the case where there is a separate α_i for each route $i \in I$. We also establish the uniformity of the convergence to the zero state under suitable conditions.

Proof of Theorem 3.3.2. Let $\zeta(\cdot)$ be a fluid model solution with $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$ and suppose that

$$\max_{i\in I}(\langle \mathbb{1}, \zeta_i(0)\rangle, \langle \chi, \zeta_i(0)\rangle) \leq W, \tag{3.52}$$

for some finite, positive constant W. By (3.9) and the fact that $w_i(t) \le w_i(0) + \rho_i t$, we have

$$\mathcal{H}_i^{\zeta}(t) \le \big(A_W + Bt\big)(z_i(t))^{\alpha_i} \quad \text{for all } t \ge 0,\ i \in I, \tag{3.53}$$

where

$$A_W = W \cdot \max_{i \in I} \left(\frac{\kappa_i \|\theta_i(\cdot)\|_\infty}{\tilde{\rho}_i^{\alpha_i}} \right) \quad \text{and} \quad B = \max_{i \in I} \left(\frac{\kappa_i \|\theta_i(\cdot)\|_\infty \rho_i}{\tilde{\rho}_i^{\alpha_i}} \right).$$

Let $\tilde{\rho}_\perp = \min_{i \in I} \tilde{\rho}_i$ and $C = \max_{j \in \mathcal{J}} C_j$. It follows from (3.53) that

$$\frac{\kappa_i \tilde{\rho}_i}{\alpha_i + 1} \left(\frac{z_i(t)}{\Lambda_i(t)} \right)^{\alpha_i} \ge \frac{\kappa_i \mathcal{H}_i^{\zeta}(t) \tilde{\rho}_\perp}{(\alpha_i + 1) C^{\alpha_i} (A_W + Bt)} \quad \text{for all } t \ge 0,\ i \in I. \tag{3.54}$$

Combining this with Corollary 3.3.1, we have for all $t \ge 0$,

$$\mathcal{K}^{\zeta}(t) \le -\delta \sum_{i \in I_+(z(t))} \frac{\kappa_i \tilde{\rho}_i}{\alpha_i + 1} \left(\frac{z_i(t)}{\Lambda_i(t)} \right)^{\alpha_i} \le -\frac{\delta D \tilde{\rho}_\perp}{A_W + Bt} \mathcal{H}^{\zeta}(t)$$

where $D = \min_{i \in I} \frac{\kappa_i}{C^{\alpha_i}}$, and we used the definition of $\mathcal{H}^{\zeta}(t)$ given in (3.8), as well as the fact that $\mathcal{H}_i^{\zeta}(t) = 0$ for $i \notin I_+(z(t))$.

Recall that $\mathcal{H}^{\zeta}(t) \ge 0$ for all $t \ge 0$. Since $\mathcal{K}^{\zeta}(\cdot)$ is the density (in time) for the absolutely continuous function $\mathcal{H}^{\zeta}(\cdot)$, we see from the above that $\mathcal{H}^{\zeta}(\cdot)$ is monotone decreasing with time and it is strictly decreasing on $\{s \ge 0 : \mathcal{H}^{\zeta}(s) > 0\}$. Let $\eta = \inf\{t \ge 0 : \mathcal{H}^{\zeta}(t) = 0\}$. Then for $0 \le t < \eta$, we have

$$\begin{aligned} \log \mathcal{H}^{\zeta}(t) &= \log \mathcal{H}^{\zeta}(0) + \int_0^t \frac{\mathcal{K}^{\zeta}(s)}{\mathcal{H}^{\zeta}(s)} ds \\ &\le \log \mathcal{H}^{\zeta}(0) - \int_0^t \frac{\delta D \tilde{\rho}_\perp}{A_W + Bs} ds. \end{aligned}$$

We observe that this holds for $t \ge \eta$ as well, because $\log \mathcal{H}^{\zeta}(t) = -\infty$ for such t. The last integral in the above expression diverges as $t \to \infty$. From this it follows that $\log \mathcal{H}^{\zeta}(t) \to -\infty$ as $t \to \infty$, and therefore, whether η is finite or infinite, we have that $\lim_{t\to\infty}\mathcal{H}^{\zeta}(t)=0$. Moreover, this convergence is uniform for all fluid model solutions satisfying $\zeta(0)\in\mathbf{K}_1^{\mathbf{I}}$ and (3.52). (Note that $\mathcal{H}_i^{\zeta}(0)$ is bounded by $A_W W^{\alpha_i}$ for this.)

In a similar manner to that in Remark 3 in [PTFA12], for each $i\in I$, because the weight function $\theta_i(\cdot)$ is bounded above and below on $[0,\infty)$, the convergence of $\mathcal{H}^{\zeta}(t)$ to zero as $t\to\infty$ implies that $\overline{M}_t^i(\cdot)$ converges to zero in $\mathbf{L}^{\alpha_i+1}$ (with Lebesgue measure) as $t\to\infty$, and because $\overline{M}_t^i(x)$ is monotone decreasing as a function of $x\in(0,\infty)$, it follows that $\overline{M}_t^i(x)$ converges to zero as $t\to\infty$ for each $x\in(0,\infty)$. Consequently, $\zeta_i(t)$ as a measure on $(0,\infty)$ converges vaguely to zero as $t\to\infty$ for each $i\in I$. □

We shall next prove Theorem 3.3.3. For the remainder of the section we shall assume that Assumptions 3.1, 3.2 and 3.3 hold and that $W\geq 1$ is fixed. Let $p\in(1,\infty)$ be such that $B_{\vartheta,p}<\infty$, as in Assumption 3.3. We shall need the following supporting propositions.

Proposition 3.6.1. *Suppose that $\zeta(\cdot)$ is a fluid model solution such that $\zeta(0)\in\mathbf{K}_1^{\mathbf{I}}$ and for each $i\in I$, $\langle\chi^p,\zeta_i(0)\rangle\leq W$. Then for each $i\in I$ and $t\geq 0$,*

$$\langle\chi^p,\zeta_i(t)\rangle\leq W+\nu_i t B_{\vartheta,p}. \tag{3.55}$$

Proof. By Remark 2.3.2, the fluid model equation (2.4) holds for ζ for all $f\in\widetilde{\mathcal{C}}=\{f\in\mathbf{C}_b^1(\mathbb{R}_+): f(0)=0\}$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $\widetilde{\mathcal{C}}$ such that $f_n(0)=0$, $f_n'\geq 0$ on $[0,\infty)$ for all n and $0\leq f_n\uparrow\chi^p$ on $[0,\infty)$ as $n\to\infty$. Equation (2.4) holds with f replaced by f_n and discarding the first integral term, which is a non-negative integral since $f_n'\geq 0$, we obtain for each $i\in I$ and $t\geq 0$,

$$\begin{aligned}
\langle f_n,\zeta_i(t)\rangle &\leq \langle f_n,\zeta_i(0)\rangle+\nu_i\langle f_n,\vartheta_i\rangle\int_0^t \mathbb{1}_{(0,\infty)}\big(z_i(s)\big)ds\\
&\leq \langle\chi^p,\zeta_i(0)\rangle+\nu_i\langle\chi^p,\vartheta_i\rangle t\\
&\leq W+\nu_i t B_{\vartheta,p}.
\end{aligned}$$

Letting $n \to \infty$ and using monotone convergence, we obtain

$$\langle \chi^p, \zeta_i(t) \rangle \quad \leq \quad W + \nu_i t B_{\vartheta,p},$$

as desired. $\square$

Proposition 3.6.2. *Under the conditions of Proposition 3.6.1, for each $i \in I$ and $t \geq 0$,*

$$w_i(t) \leq (W + \nu_i t B_{\vartheta,p})^{\frac{1}{p}} (z_i(t))^{\frac{1}{q}}, \tag{3.56}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Hölder's inequality and Proposition 3.6.1, we have

$$\begin{aligned} w_i(t) &= \langle \chi, \zeta_i(t) \rangle \\ &\leq \Big(\langle \chi^p, \zeta_i(t) \rangle \Big)^{\frac{1}{p}} \Big(\langle \mathbb{1}, \zeta_i(t) \rangle \Big)^{\frac{1}{q}} \\ &\leq (W + \nu_i t B_{\vartheta,p})^{\frac{1}{p}} (z_i(t))^{\frac{1}{q}}. \end{aligned}$$

$\square$

Proof of Theorem 3.3.3. Applying Proposition 3.6.2 to (3.9), we have for all fluid model solutions $\zeta(\cdot)$ satisfying $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$ and $\langle \chi^p, \zeta_i(0) \rangle \leq W$ for all $i \in I$,

$$\mathcal{H}_i^{\zeta}(t) \leq C^{\dagger} (A^{\dagger} + B^{\dagger} t)^{1-\beta} (z_i(t))^{\alpha_i + \beta}, \tag{3.57}$$

where $\beta = \frac{1}{q} = 1 - \frac{1}{p} \in (0,1)$, $A^{\dagger} = W \geq 1$, $B^{\dagger} = (\max_{i \in I} \nu_i) B_{\vartheta,p}$ and $C^{\dagger} = \max_{i \in I} \left(\frac{\kappa_i \|\theta_i(\cdot)\|_\infty}{\tilde{\rho}_i^{\alpha_i}} \right)$.

Using this, for $i \in I_+(z(t))$, we have

$$\frac{\kappa_i \tilde{\rho}_i}{\alpha_i + 1} \left(\frac{z_i(t)}{\Lambda_i(t)} \right)^{\alpha_i} \geq \frac{\kappa_i \tilde{\rho}_i}{(\alpha_i + 1) C^{\alpha_i} (C^\dagger)^{\frac{\alpha_i}{\alpha_i + \beta}}} \left(\frac{\mathcal{H}_i^\zeta(t)}{(A^\dagger + B^\dagger t)^{1-\beta}} \right)^{\frac{\alpha_i}{\alpha_i + \beta}},$$

where $C = \max_j C_j$.

By Proposition 3.6.2, if $z_i(0) = \langle \mathbb{1}, \zeta_i(0) \rangle \leq W$ and $\langle \chi^p, \zeta_i(0) \rangle \leq W$ for all $i \in I$, then $w_i(0) = \langle \chi, \zeta_i(0) \rangle \leq W$ for all $i \in I$. Then, by Theorem 3.3.2, there is $T_1 < \infty$ such that for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$ and $\max_{i \in I}(\langle \mathbb{1}, \zeta_i(0) \rangle, \langle \chi^p, \zeta_i(0) \rangle) \leq W$, we have $\mathcal{H}^\zeta(t) \leq 1$ for all $t \geq T_1$. Then by the definition of $\mathcal{H}^\zeta(\cdot)$ and the fact that $A^\dagger \geq 1$, we have $\frac{\mathcal{H}_i^\zeta(t)}{(A^\dagger + B^\dagger t)^{1-\beta}} \leq 1$ for all $i \in I$ and $t \geq T_1$. On setting

$$\gamma = \min_{i \in I} \left\{ \frac{\kappa_i \tilde{\rho}_i}{(\alpha_i + 1) C^{\alpha_i} (C^\dagger)^{\frac{\alpha_i}{\alpha_i + \beta}}} \right\}, \qquad \alpha^\dagger = \max_{i \in I} \alpha_i \quad \text{and} \quad \gamma^\dagger = \frac{\gamma}{\mathbf{I}^{\frac{\alpha^\dagger}{\alpha^\dagger + \beta}}},$$

and noting that $x^{\frac{\alpha_i}{\alpha_i + \beta}} \geq x^{\frac{\alpha^\dagger}{\alpha^\dagger + \beta}}$ for all $i \in I$ when $0 < x \leq 1$, it follows from the above that for all

$t \geq T_1$,

$$\begin{aligned}
\sum_{i \in I_+(z(t))} \frac{\kappa_i \tilde{\rho}_i}{\alpha_i + 1} \left(\frac{z_i(t)}{\Lambda_i(t)} \right)^{\alpha_i} &\geq \gamma \sum_{i \in I_+(z(t))} \left(\frac{\mathcal{H}_i^{\zeta}(t)}{(A^{\dagger} + B^{\dagger} t)^{1-\beta}} \right)^{\frac{\alpha^{\dagger}}{\alpha^{\dagger}+\beta}} \\
&\geq \frac{\gamma}{(A^{\dagger} + B^{\dagger} t)^{\frac{\alpha^{\dagger}(1-\beta)}{\alpha^{\dagger}+\beta}}} \max_{i \in I} (\mathcal{H}_i^{\zeta}(t))^{\frac{\alpha^{\dagger}}{\alpha^{\dagger}+\beta}} \\
&\geq \frac{\gamma}{(A^{\dagger} + B^{\dagger} t)^{\frac{\alpha^{\dagger}(1-\beta)}{\alpha^{\dagger}+\beta}}} \max_{i \in I} \left\{ \left(\frac{\mathcal{H}_i^{\zeta}(t)}{\alpha_i + 1} \right)^{\frac{\alpha^{\dagger}}{\alpha^{\dagger}+\beta}} \right\} \\
&\geq \frac{\gamma}{(A^{\dagger} + B^{\dagger} t)^{\frac{\alpha^{\dagger}(1-\beta)}{\alpha^{\dagger}+\beta}}} \left(\frac{\sum_{i \in I} \frac{\mathcal{H}_i^{\zeta}(t)}{\alpha_i + 1}}{\mathbf{I}} \right)^{\frac{\alpha^{\dagger}}{\alpha^{\dagger}+\beta}} \\
&= \frac{\gamma^{\dagger} \left(\mathcal{H}^{\zeta}(t) \right)^{\frac{\alpha^{\dagger}}{\alpha^{\dagger}+\beta}}}{(A^{\dagger} + B^{\dagger} t)^{\frac{\alpha^{\dagger}(1-\beta)}{\alpha^{\dagger}+\beta}}}. \qquad (3.58)
\end{aligned}$$

Then, by Corollary 3.3.1, the density $\mathcal{K}^{\zeta}(\cdot)$ for $\mathcal{H}^{\zeta}(\cdot)$ satisfies for all $t \geq T_1$,

$$\mathcal{K}^{\zeta}(t) \leq -\delta \frac{\gamma^{\dagger} \left(\mathcal{H}^{\zeta}(t) \right)^{\frac{\alpha^{\dagger}}{\alpha^{\dagger}+\beta}}}{(A^{\dagger} + B^{\dagger} t)^{\frac{\alpha^{\dagger}(1-\beta)}{\alpha^{\dagger}+\beta}}}. \qquad (3.59)$$

Let $\eta = \inf\{t \geq 0 : \mathcal{H}^{\zeta}(t) = 0\}$. On $[0, \eta)$, $\left(\mathcal{H}^{\zeta}(\cdot) \right)^{\frac{\beta}{\alpha^{\dagger}+\beta}}$ is absolutely continuous with density given by the left-hand member of the following string of (in)equalities, which hold for all $T_1 \leq t < \eta$:

$$\frac{\beta}{\alpha^{\dagger} + \beta} \mathcal{H}^{\zeta}(t)^{\frac{-\alpha^{\dagger}}{\alpha^{\dagger}+\beta}} \mathcal{K}^{\zeta}(t) \leq \frac{\beta}{\alpha^{\dagger} + \beta} \left(\frac{-\delta \gamma^{\dagger}}{(A^{\dagger} + B^{\dagger} t)^{\frac{\alpha^{\dagger}(1-\beta)}{\alpha^{\dagger}+\beta}}} \right) = -\gamma^{\ddagger} \frac{d (A^{\dagger} + B^{\dagger} t)^{\frac{\beta(1+\alpha^{\dagger})}{\alpha^{\dagger}+\beta}}}{dt} \qquad (3.60)$$

where $\gamma^{\ddagger} = \frac{\delta\gamma^{\dagger}}{B^{\dagger}(1+\alpha^{\dagger})} > 0$. Integrating in time, we obtain for $T_1 \leq t < \eta$,

$$(\mathcal{H}^{\zeta}(t))^{\frac{\beta}{\alpha^{\dagger}+\beta}} \leq (\mathcal{H}^{\zeta}(T_1))^{\frac{\beta}{\alpha^{\dagger}+\beta}} - \gamma^{\ddagger}(A^{\dagger} + B^{\dagger}t)^{\frac{\beta(1+\alpha^{\dagger})}{\alpha^{\dagger}+\beta}} + \gamma^{\ddagger}(A^{\dagger} + B^{\dagger}T_1)^{\frac{\beta(1+\alpha^{\dagger})}{\alpha^{\dagger}+\beta}}. \tag{3.61}$$

The right-hand side of (3.61) goes to $-\infty$ as $t \to \infty$. Because $\mathcal{H}^{\zeta}(\cdot)$ is non-negative, it follows that $\mathcal{H}^{\zeta}(\cdot)$ reaches zero in finite time and stays there forever after. Assuming $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$, by Corollary 3.4.1 and (2.5), $\zeta(t) \in \mathbf{K}_1^{\mathbf{I}}$ for all $t \geq 0$, and it follows that $\zeta(t) = \mathbf{0}$ for all t such that $\mathcal{H}^{\zeta}(t) = 0$. Moreover, since $\mathcal{H}^{\zeta}(T_1)$ is bounded by one, and T_1 was chosen to be the same for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$ and $\max_{i\in I}(\langle \mathbb{1}, \zeta_i(0)\rangle, \langle \chi^p, \zeta_i(0)\rangle) \leq W$, it follows that there is a uniform bound $T_W < \infty$ for the time for these fluid model solutions to reach the zero state and stay there forever after. $\square$

3.7 Acknowledgement

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Chapter 4

Asymptotic Behavior of the Critical Fluid Model

In this chapter, we analyze the asymptotic behavior (as time goes to infinity) of measure-valued solutions to the *critical* fluid model for bandwidth sharing. In Section 4.1, we introduce the set-up and notation for the critical fluid model. In this chapter, we restrict to where α_i is a constant for all i, which we will denote by a scalar $\alpha \in (0,\infty)$. (Note that in this chapter, α will be a scalar rather than a vector in the previous chapter.) In Section 4.2, we introduce the characterization of its invariant states as developed by Gromoll and Williams [GW09]. We also recall some preliminary properties of fluid model solutions, taken from Chapter 3. In Section 4.3, we introduce key assumptions on fluid model parameters, under which our results will be proved. In Section 4.4, we reuse the symbol H from Chapter 3 to denote a new function for this chapter, and define functions K and $\underline{F}$, which are used in defining our Lyapunov function G in Section 4.7, and proving its properties. We reuse the symbol $\mathcal{H}^\zeta$ (respectively $\mathcal{K}^\zeta$) from Chapter 3 and define these new functions too, as the composition of H (respectively K) with a fluid model solution ζ.

Under our assumptions, the function $\mathcal{K}^\zeta$ will be shown to be the density in time of $\mathcal{H}^\zeta$. This relationship between $\mathcal{H}^\zeta$ and $\mathcal{K}^\zeta$, and a non-positive upper bound on $\mathcal{K}^\zeta$, is stated in the key result, Theorem 4.4.1, in Section 4.4.3. For the proof of this theorem, given in Section 4.9, we use a smooth approximation of fluid model solutions that was also used in Chapter 3, and which is similar to a smoothing used by Puha and Williams [PW16] and Mulvany et al. [MPW19]. For the proof of an associated lemma (Lemma 4.4.4), we also employ some inequalities (see Lemmas 3.4.1–3.4.3) from Chapter 3, which are similar to ones developed by Paganini et al. [PTFA12]. Conditions for sharpness of an inequality in Lemma 4.4.4 are new here and useful.

The function $\underline{F}$ is defined in Section 4.4.4 via an optimization problem, which is similar to one used by Kelly and Williams [KW04] for the case of Poisson arrivals and exponential file sizes. Section 4.5 presents a characterization of solutions of this optimization problem and of the optimization problem used to define the bandwidth sharing policy, and in Section 4.6, we give a further characterization of the invariant states for the fluid model. The proofs of these results are similar to those of results in [KW04]. Our Lyapunov function, G, and its composition, $\mathcal{G}^{\zeta}$, with a fluid model solution, ζ, is defined in Section 4.7. Key properties of G are stated there and proved in Section 4.10.3.

In Section 4.8, we state the main results of this chapter. These describe the asymptotic behavior of $\mathcal{G}^{\zeta}$ as time goes to infinity, i.e., that it decreases monotonically and converges uniformly to zero for all fluid model solutions starting in suitable relatively compact sets, and that fluid model solutions converge uniformly to the invariant manifold starting in such sets. The proofs of these main results are given in Section 4.11. These proofs draw on some arguments first introduced in [PW16], where the asymptotic behavior of a critical fluid model for a single class processor sharing queue was studied. These arguments were extended in [MPW19] to a critical fluid model of a multiclass processor sharing queue. However, for the bandwidth sharing (network) model considered here, key details for many parts of the arguments are more complicated than in either of these prior works. In particular, our Lyapunov function is different, we have a much more general bandwidth allocation policy, and we need to deal with the singular, but realistic, situation where the fluid level for some routes reaches zero.

In this chapter, in referencing arguments that we generalize from [PW16, MPW19], we shall generally refer to the first paper [PW16], from which the arguments were adapted for [MPW19]. In the course of proving the main results, along the way, in Lemma 4.11.1 we prove that when there is non-zero fluid flow on a route, the ratio of the total fluid mass on the route to the bandwidth allocated to that route is bounded for all time, and we use this to prove in Lemma 4.11.2 that any fluid model solution starting in one of our relatively compact sets stays within a (larger) relatively compact set from the same family for all time, where our relatively compact sets are more general than those in [PW16]. Besides the proof of properties of G, Section 4.10 develops some properties of resource level workload, the relationship between H and $\underline{F}$, and a bound on the total mass of fluid model solutions when started in suitable relatively compact sets, as preliminaries to the proofs of the main results. For reference, Appendix B gives some basic background on hazard rates.

4.1 Basic Assumptions for the Critical Fluid Model

In this chapter throughout, beyond the assumptions in Chapter 2, we assume here that the incidence matrix R has full row rank $\mathbf{J}$ and that the file size distributions $\vartheta_i, i \in \mathcal{I}$ have finite second as well as first moments. Furthermore, in this chapter, $\alpha \in (0,\infty)$ and we assume $\alpha_i = \alpha$ for all $i \in \mathcal{I}$. Note that α is now a scaler rather than the vector in Chapter 3.

Remark 4.1.1. *For each $i \in \mathcal{I}$, since ϑ_i has finite second moment, ϑ_i^e has finite mean given by*

$$\langle \chi, \vartheta_i^e \rangle = \frac{\mu_i}{2} \langle \chi^2, \vartheta_i \rangle. \tag{4.1}$$

Remark 4.1.2. *For the critical fluid model studied in this chapter, our proof of Theorem 4.8.1, which shows that the Lyapunov function constructed in this chapter decreases along fluid model solutions, extends to the situation where α_i depends on i. However, our proofs of Theorems*

4.8.2 and 4.8.3, which demonstrate that the Lyapunov function decreases to zero and fluid model solutions converge to the invariant manifold, depend on the scaling property that $\phi_i(rz) = \phi_i(z)$ for all $i \in I$, $z \in \mathbb{R}_+^{\mathbf{I}}$ and $r > 0$. However, this property does not hold when α_i depends on i.

Notice the fluid limit result proved by Gromoll and Williams [GW09] yields fluid model solutions which have initial states that are continuous measures and which have finite workload, i.e., for which $\zeta(0) \in \mathbf{K}^{\mathbf{I}} \cap \mathbf{M}_1^{\mathbf{I}}$. Indeed, in order for fluid model solutions to be continuous functions of time, the initial condition cannot have any atoms. For the analysis of Chapter 3, the initial conditions were required to satisfy $\zeta(0) \in \mathbf{K}^{\mathbf{I}} \cap \mathbf{M}_1^{\mathbf{I}}$. Here for our analysis of the critical case, we will ultimately assume that $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$ for some $\upsilon > 0$, where

$$\mathbf{K}_\upsilon^{\mathbf{I}} = \{\xi \in \mathbf{K}^{\mathbf{I}} : \langle \mathbb{1}_{[x,\infty)}, \xi_i \rangle \le \upsilon \langle \mathbb{1}_{[x,\infty)}, \vartheta_i^e \rangle \text{ for all } x \in \mathbb{R}_+, i \in I\}. \tag{4.2}$$

We note that since $\xi \in \mathbf{K}^{\mathbf{I}}$ and ϑ^e have no atoms, in (4.2), $\mathbb{1}_{[x,\infty)}$ can be replaced by $\mathbb{1}_{(x,\infty)}$ without changing the definition. So we can use the following alternative representation:

$$\mathbf{K}_\upsilon^{\mathbf{I}} = \{\xi \in \mathbf{K}^{\mathbf{I}} : \langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle \le \upsilon \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle \text{ for all } x \in \mathbb{R}_+, i \in I\}. \tag{4.3}$$

We shall define certain functions on

$$\mathbf{M}_\upsilon^{\mathbf{I}} = \{\xi \in \mathbf{M}^{\mathbf{I}} : \langle \mathbb{1}_{[x,\infty)}, \xi_i \rangle \le \upsilon \langle \mathbb{1}_{[x,\infty)}, \vartheta_i^e \rangle \text{ for all } x \in \mathbb{R}_+, i \in I\}, \tag{4.4}$$

which contains the closure of $\mathbf{K}_\upsilon^{\mathbf{I}}$. Note that in (4.4), we cannot replace $\mathbb{1}_{[x,\infty)}$ by $\mathbb{1}_{(x,\infty)}$, without changing the definition. Indeed, if $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$, then $\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle \le \upsilon \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle$ for all $x \in \mathbb{R}_+$, $i \in I$, but the converse is not true in general as ξ_i could have an atom at zero. Note that for any $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$, we have for each $i \in I$, $\langle \mathbb{1}, \xi_i \rangle \le \upsilon$ and

$$\langle \chi, \xi_i \rangle = \int_0^\infty \langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle dx \le \upsilon \int_0^\infty \overline{N}_i^e(x) dx = \upsilon \langle \chi, \vartheta_i^e \rangle = \frac{\upsilon \mu_i}{2} \langle \chi^2, \vartheta_i \rangle < \infty. \tag{4.5}$$

It follows that $\mathbf{M}_{\upsilon}^{\mathbf{I}}$ is compact as a subset of $\mathbf{M}^{\mathbf{I}}$ and so $\mathbf{K}_{\upsilon}^{\mathbf{I}}$, although not closed, is relatively compact as a subset of $\mathbf{M}^{\mathbf{I}}$; see Lemma 15.7.5 of [Kal83] for the method of proof.

A small comment on notation is in order here. In this chapter, we only refer to $\mathbf{M}_{\upsilon}^{\mathbf{I}}$ with general $\upsilon > 0$. Consequently, when we refer to $\mathbf{M}_{1}^{\mathbf{I}}$, we do not mean $\mathbf{M}_{\upsilon}^{\mathbf{I}}$ with $\upsilon = 1$.

4.2 Invariant States

Under a natural condition on the parameters R, C, ν, ϑ, there exist fluid model solutions that are time invariant. Following Section 6 of [GW09], we call these invariant states for the fluid model.

Definition 4.2.1. *A vector of measures $\xi \in \mathbf{M}^{\mathbf{I}}$ is an invariant state for the fluid model if there is a fluid model solution ζ satisfying $\zeta(t) = \xi$ for all $t \geq 0$.*

To help describe invariant states, let

$$\mathcal{P} = \{z \in \mathbb{R}_{+}^{\mathbf{I}} : \phi_i(z) = \rho_i \text{ for all } i \in I_{+}(z)\}. \tag{4.6}$$

Theorem 6.3 of [GW09] gives necessary and sufficient conditions for the existence of invariant states for the fluid model and a representation for the invariant states. For convenience, we formulate these results as a proposition here and refer readers to [GW09] for the proof.

Proposition 4.2.1. *There exist invariant states for the fluid model if and only if*

$$R\rho \leq C. \tag{4.7}$$

When (4.7) holds, the set of invariant states is given by

$$\mathcal{M} = \{\xi \in \mathbf{M}^{\mathbf{I}} : \xi_i = z_i \vartheta_i^e, \textit{ for all } i \in I \textit{ and some } z \in \mathcal{P}\}. \tag{4.8}$$

Remark 4.2.1. *We call $\mathcal{M}$ the invariant manifold for the fluid model.*

4.2.1 Some Properties of Fluid Model Solutions

The first two propositions in this subsection are the same as Corollary 3.4.1 and Lemma 3.4.6 in Section 3.4 of Chapter 3, respectively. For later use, we state the results here without proof.

Proposition 4.2.2. *Suppose that $\vartheta \in \mathbf{K}^{\mathbf{I}}$ and that ζ is a fluid model solution with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}$. Then $\zeta(t) \in \mathbf{K}^{\mathbf{I}}$ for all $t \geq 0$.*

Remark 4.2.2. *The assumption on ϑ is in addition to the basic requirements that its components do not charge the origin and have finite first and second moments. The assumption on ϑ in Proposition 4.2.2 is automatically satisfied if our Assumption 4.2 (stated in Section 4.3) holds.*

Proposition 4.2.3. *Suppose that ζ is a fluid model solution, $i \in I$ and $0 \leq s < t < \infty$ such that $\zeta_i(r) \neq 0$ for all $r \in [s,t]$. Then*

$$\overline{M}_t^i(x) = \overline{M}_s^i(x + S_{s,t}^i) + \nu_i \int_s^t \overline{N}_i(x + S_{u,t}^i)du \quad \text{for all } x \in \mathbb{R}_+. \tag{4.9}$$

Remark 4.2.3. *If ζ is a fluid model solution, $i \in I$ and $0 \leq s_0 < t < \infty$ such that $\zeta_i(r) \neq 0$ for all $r \in (s_0,t]$ and $\zeta_i(s_0) = 0$, then (4.9) holds for $s \in (s_0,t]$ and letting $s \downarrow s_0$, since $\overline{M}_s^i(x + S_{s,t}^i) \leq \overline{M}_s^i(0) = z_i(s) \to z_i(s_0) = 0$ as $s \to s_0$, by taking the limit as $s \to s_0$ in (4.9), we obtain*

$$\overline{M}_t^i(x) = \nu_i \int_{s_0}^t \overline{N}_i(x + S_{u,t}^i)du \quad \text{for all } x \in \mathbb{R}_+. \tag{4.10}$$

4.3 Key Assumptions

In this section, we first state additional assumptions on fluid model parameters for the critical case and on file size distributions needed for our analysis.

4.3.1 Critical Parameters

For our main results, we shall assume that the fluid model is critical, that is, the parameters (R, ρ, C) satisfy the following assumption.

Assumption 4.1. *We assume that*

$$\sum_{i \in I} R_{ji} \rho_i \leq C_j \quad \text{for all } j \in \mathcal{J}, \tag{4.11}$$

and that $\mathcal{J}_* = \{j \in \mathcal{J} : \sum_{i \in I} R_{ji} \rho_i = C_j\}$ is non-empty. Furthermore, without loss of generality, we assume that the first $\mathbf{J}_* = |\mathcal{J}_*|$ elements of $\mathcal{J}$ correspond to the set $\mathcal{J}_*$.

Assumption 4.1 requires that the average load on each resource is less than or equal to its capacity and that there exists at least one resource that is fully loaded.

Remark 4.3.1. *The Lyapunov function defined later in this chapter could also be applied when $\mathcal{J}_*$ is empty. Since the stability result for that strictly subcritical case has already been shown in Chapter 3 with weaker assumptions, we focus only on the critical case here, where at least one resource is fully loaded.*

4.3.2 File Size Distributions

The following assumption will be used in the proofs of Lemmas 4.3.1 and 4.4.1, which are used to prove Lemma 4.4.2. The latter gives the continuity in time of $\mathcal{H}^{\zeta}$, the composition of the function H (defined below) with a suitable fluid model solution ζ. This continuity property ultimately features in our proof of the absolute continuity of $\mathcal{H}^{\zeta}$ as a function of time and the convergence of fluid model solutions to the invariant manifold.

Assumption 4.2. *For each $i \in I$, assume the file size distribution ϑ_i is continuous and there is a* *finite constant C_ϑ such that*

$$\overline{N}_i(x) \le C_\vartheta \overline{N}_i^e(x) \text{ for all } x \in [0,\infty), i \in I. \tag{4.12}$$

Remark 4.3.2. *We already assumed in Section 4.1 that ϑ_i has finite first and second moments and Assumption 4.2 is in addition to this. Condition (4.12) is equivalent to ϑ_i^e having bounded hazard rate, which implies the support of ϑ_i^e (and hence of ϑ_i) is unbounded. A sufficient condition for ϑ_i^e to have bounded hazard rate is that ϑ_i is absolutely continuous with bounded hazard rate. For the definition and some examples related to hazard rate, see Appendix B.*

Assumption 4.2 is used to prove the following lemma, which will help us to analyze the asymptotic behavior of fluid model solutions.

Lemma 4.3.1. *Suppose that Assumption 4.2 holds. Fix $T > 0$ and $\upsilon > 0$. For any fluid model solution ζ with $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, we have $\zeta(t) \in \mathbf{K}_{\upsilon_T^*}^{\mathbf{I}}$ for all $t \in [0,T]$, where $\upsilon_T^* = \upsilon + C_\vartheta T \max_{i\in I} \nu_i$.*

Proof. Let ζ be a fluid model solution with $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$. For $t \in (0,T]$ and $i \in I$, either $z_i(t) = 0$ or $z_i(t) \neq 0$. If $z_i(t) = 0$, then $\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} = 0$ for all $x \in [0,\infty)$. If $z_i(t) \neq 0$, let $t_0^i = \sup\{s \in [0,t) : z_i(s) = 0\}$ where $\sup\emptyset = 0$. We consider the case where $t_0^i > 0$ first. Then $\zeta_i(\cdot)$ is nonzero on $(t_0^i, t]$, $\zeta_i(t_0^i) = 0$ and by Remark 4.2.3 and Assumption 4.2, for all $x \in [0,\infty)$,

$$\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \le \nu_i \frac{\overline{N}_i(x)}{\overline{N}_i^e(x)}(t - t_0^i) \le \nu_i C_\vartheta (t - t_0^i). \tag{4.13}$$

If $t_0^i = 0$, then $\zeta(\cdot)$ is nonzero on $(0,t]$. In this case, by (3.24), for all $s \in (0,t)$, we have for all $x \in [0,\infty)$,

$$\overline{M}_t^i(x) \le \overline{M}_s^i(x) + \nu_i \overline{N}_i(x)(t-s).$$

On letting $s \downarrow 0$ and using the facts that $s \to \zeta(s)$ is continuous and $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, together with

Assumption 4.2, we obtain

$$\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \leq \upsilon + \nu_i C_\vartheta t, \text{ for all } x \in [0,\infty).$$

Combining the above with the fact that $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, we obtain for any $t \in [0,T]$, $\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \leq \upsilon_{i,T}^*$ for all $x \in [0,\infty)$, where $\upsilon_{i,T}^* = \upsilon + \nu_i C_\vartheta T$. The desired result follows from this, Proposition 4.2.2 and the alternative representation of $\mathbf{K}_{\upsilon_T^*}^{\mathbf{I}}$ (see (4.3)). $\square$

Remark 4.3.3. *In Lemma 4.3.1, υ_T^* depends on T. Later, after more results have been developed, we shall prove in Lemma 4.11.2, with the addition of Assumption 4.1, that υ_T^* can be chosen not to depend on T.*

4.4 Functions for Fluid Model Analysis

In this section, we reuse the following symbols: H, $\mathcal{H}^\zeta$, $\mathcal{K}^\zeta$ and $H_i, \mathcal{H}_i^\zeta, \mathcal{K}_i^\zeta$ for each $i \in I$, to define functions in preparation for the Lyapunov function to be defined in Section 4.7. For critical fluid model analysis, we also define function K and $\underline{F}$, which together with H, are used in defining our Lyapunov function and establishing its properties. We describe some properties of $\mathcal{H}^\zeta$ and $\mathcal{K}^\zeta$, the compositions of H and K, respectively, with a fluid model solution, ζ. In particular, we give the relationship between $\mathcal{H}^\zeta$ and $\mathcal{K}^\zeta$, and some properties of $\underline{F}$.

We shall define functions H and K on $\bigcup_{\upsilon>0} \mathbf{M}_\upsilon^{\mathbf{I}}$ and then apply them to fluid model solutions ζ with initial conditions in $\bigcup_{\upsilon>0} \mathbf{K}_\upsilon^{\mathbf{I}}$ to obtain functions $\mathcal{H}^\zeta$ and $\mathcal{K}^\zeta$ of time. The larger domain for H and K is needed for the proof of Theorem 4.8.2.

4.4.1 The Functions H and $\mathcal{H}^\zeta$

Definition 4.4.1. *Given $\xi \in \cup_{\upsilon>0}\mathbf{M}^{\mathbf{I}}_{\upsilon}$, for each $i \in I$, define*

$$H_i(\xi) = \frac{\kappa_i}{\rho_i^\alpha}\int_0^\infty \left(\frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e\rangle}\right)^{\alpha+1} \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e\rangle dx, \tag{4.14}$$

and define

$$H(\xi) = \frac{1}{\alpha+1}\sum_{i\in I} H_i(\xi). \tag{4.15}$$

Remark 4.4.1. *For $\xi \in \bigcup_{\upsilon>0} \mathbf{M}^{\mathbf{I}}_{\upsilon}$, if $\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e\rangle = 0$, then $\langle \mathbb{1}_{(x,\infty)}, \xi_i\rangle = 0$ and we interpret the integrand in (4.14) at x as being zero. Note that when Assumption 4.2 holds, $\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e\rangle > 0$ for all $x \in [0,\infty)$, since the support of ϑ_i is unbounded in this case.*

The function H will be used in defining our new Lyapunov function. For $\xi \in \cup_{\upsilon>0}\mathbf{M}^{\mathbf{I}}_{\upsilon}$, there is $\upsilon > 0$ such that $\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon}$ and then $H_i(\xi) \le \frac{\kappa_i \upsilon^{\alpha+1}}{\rho_i^\alpha}\langle \chi, \vartheta_i^e\rangle < \infty$ for all $i \in I$. It follows that $H_i(\xi)$, $i \in I$, and $H(\xi)$ are finite. Furthermore, we have the following lemma.

Lemma 4.4.1. *The functions $H_i, i \in I$, and H are continuous, non-negative, real-valued functions on $\mathbf{M}^{\mathbf{I}}_{\upsilon}$ for each $\upsilon > 0$.*

Proof. The non-negative, real-valued property follows from observation and the last paragraph before this lemma. For the continuity, fix $\upsilon > 0$. Suppose that $\{\xi_n\}_{n=1}^\infty$ is a sequence in $\mathbf{M}^{\mathbf{I}}_{\upsilon}$ converging (weakly) to $\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon}$. Then as $n \to \infty$, $\langle \mathbb{1}_{(x,\infty)}, \xi_n\rangle \to \langle \mathbb{1}_{(x,\infty)}, \xi\rangle$ for almost every $x \in [0,\infty)$. Since $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathbf{M}^{\mathbf{I}}_{\upsilon}$, the sequence of integrands in the definition of $H_i(\xi_n)$ is dominated by $\upsilon^{\alpha+1}\overline{N}_i^e(\cdot)$, which is integrable because $\langle \chi, \vartheta_i^e\rangle < \infty$. Thus, by the dominated convergence theorem, $H_i(\xi_n) \to H_i(\xi)$ as $n \to \infty$ for each $i \in I$. It follows that $H_i, i \in I$ and H are continuous on $\mathbf{M}^{\mathbf{I}}_{\upsilon}$. $\square$

Remark 4.4.2. *The form of H is largely inspired by two prior works: Mulvany et al. [MPW19] and Paganini et al. [PTFA12]. In [MPW19], building on work of Puha and Williams [PW16],* *Mulvany et al. considered a relative entropy functional for comparing the probability measure on $\mathbb{R}_+$ with density proportional to $p_i(x) = \langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle$ to the probability measure on $\mathbb{R}_+$ with density proportional to $q_i(x) = \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle$. When normalized to be probability densities, p_i and q_i are the densities of excess lifetime distributions associated with ξ_i and ϑ_i^e, respectively. The relative entropy employed by Mulvany et al. [MPW19] uses $u \to u\ln(u)$ in place of the function $f(u) = u^{\alpha+1}$ that we have used in the integral in (4.14). The form of $H_i(\xi)$ used here is proportional to the so-called f-divergence [Csi67] for the two finite measures on $\mathbb{R}_+$ that have densities p_i and q_i. Further inspiration for our use of f in place of $u \to u\ln(u)$ comes from Paganini et al. [PTFA12]; see also Chapter 3, as in [FW20], for the inclusion of the weights κ_i. In those works, for the strictly subcritical case, f was applied directly to the function p_i (no quotient) and integrated with a reference density θ_i that involved ϑ_i^e, to give the i-th Lyapunov function component. In fact, if one formally takes the limit in the Lyapunov function in [PTFA12] and Chapter 3 as critical loading is approached on all resources, one obtains the H_i and H in (4.14) and (4.15) for the case where equality holds in (4.7) (all resources are fully loaded).*

Definition 4.4.2. *Suppose that Assumption 4.2 holds. Given a fluid model solution ζ with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}^{\mathbf{I}}_{\upsilon}$, for each $t \geq 0$ and $i \in I$, define*

$$\mathcal{H}_i^{\zeta}(t) = H_i\big(\zeta(t)\big) = \frac{\kappa_i}{\rho_i^{\alpha}} \int_0^{\infty} \left(\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \right)^{\alpha+1} \overline{N}_i^e(x)dx \quad \text{for all } i \in I, \tag{4.16}$$

and let

$$\mathcal{H}^{\zeta}(t) = H\big(\zeta(t)\big) = \frac{1}{\alpha+1} \sum_{i \in I} \mathcal{H}_i^{\zeta}(t). \tag{4.17}$$

Lemma 4.4.2. *Suppose that Assumption 4.2 holds. Let ζ be a fluid model solution with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}^{\mathbf{I}}_{\upsilon}$. Then for each $i \in I$, $\mathcal{H}_i^{\zeta} : [0,\infty) \to [0,\infty)$ is well defined and continuous on $[0,\infty)$.*

Proof. This follows immediately on combining Lemmas 4.3.1, 4.4.1 and the fact that $t \to \zeta(t)$ is continuous. □

4.4.2 The Functions K and $\mathcal{K}^\zeta$

In this section, we introduce the functions K and $\mathcal{K}^\zeta$. The latter arises in taking the derivative of the function $\mathcal{H}^\zeta(\cdot)$.

Definition 4.4.3. *Given $\xi \in \cup_{\upsilon>0}\mathbf{M}_\upsilon^{\mathbf{I}}$, for each $i \in I$, define*

$$\begin{aligned}K_i(\xi) = \kappa_i\rho_i^{-\alpha}\Bigg(&-\phi_i(\langle 1\!\!1,\xi\rangle)(\langle 1\!\!1,\xi_i\rangle)^\alpha \\ &+\int_0^\infty \left(\frac{\langle 1\!\!1_{(x,\infty)},\xi_i\rangle}{\langle 1\!\!1_{(x,\infty)},\vartheta_i^e\rangle}\right)^\alpha \langle 1\!\!1_{(x,\infty)},\vartheta_i\rangle\Big(-\frac{\alpha\phi_i(\langle 1\!\!1,\xi\rangle)}{\langle 1\!\!1,\xi_i\rangle}\frac{\langle 1\!\!1_{(x,\infty)},\xi_i\rangle}{\langle 1\!\!1_{(x,\infty)},\vartheta_i^e\rangle\langle\chi,\vartheta_i\rangle}+\nu_i(\alpha+1)\Big)1\!\!1_{(0,\infty)}(\langle 1\!\!1,\xi_i\rangle)dx\Bigg).\end{aligned} \tag{4.18}$$

Then with $z = \langle 1\!\!1,\xi\rangle$, define

$$K(\xi) = \frac{1}{\alpha+1}\sum_{i\in I_+(z)} K_i(\xi). \tag{4.19}$$

Remark 4.4.3. *For $\xi \in \bigcup_{\upsilon>0}\mathbf{M}_\upsilon^{\mathbf{I}}$, if $x \in \mathbb{R}_+$ such that $\langle 1\!\!1_{(x,\infty)},\vartheta_i^e\rangle = 0$, then $\langle 1\!\!1_{(x,\infty)},\xi_i\rangle = 0$ and we interpret the integrand in the integral in (4.18) as being zero at x. In (4.18), if $\xi_i = 0$, we interpret the right member of the equality to be zero and so $K_i(\xi) = 0$ in this case. If $\xi_i \neq 0$, there is $\upsilon > 0$ such that $\langle 1\!\!1_{(x,\infty)},\xi_i\rangle \leq \upsilon\langle 1\!\!1_{(x,\infty)},\vartheta_i^e\rangle$ for all $x \in [0,\infty)$. Then noticing $\int_0^\infty\langle 1\!\!1_{(x,\infty)},\vartheta_i\rangle dx = \langle\chi,\vartheta_i\rangle < \infty$, we have $|K_i(\xi)| < \infty$. Note that (4.19) can also be written as $K(\xi) = \sum_{i\in I} K_i(\xi)/(\alpha+1)$.*

The following property of the K_i and K will be used in proving our main results.

Lemma 4.4.3. *Fix $\upsilon > 0$. The functions K_i, $i \in I$, and K are real-valued, upper semicontinuous functions on $\mathbf{M}_\upsilon^{\mathbf{I}}$. Furthermore, if $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$ and $i \in I$ such that $z_i = \langle 1\!\!1,\xi_i\rangle \neq 0$, then K_i is continuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$ at ξ.*

Proof. The real-valuedness of $K_i, i \in I$, and K follows from Remark 4.4.3. For $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$, let

$$\begin{aligned}
&\mathrm{k}_i^{(1)}(\xi) = -\kappa_i \rho_i^{-\alpha} \phi_i(\langle \mathbb{1}, \xi \rangle)(\langle \mathbb{1}, \xi_i \rangle)^\alpha, \\
&\mathrm{k}_i^{(2)}(\xi) = -\kappa_i \rho_i^{-\alpha} \int_0^\infty \left(\frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle} \right)^\alpha \langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle \left(\frac{\alpha \phi_i(\langle \mathbb{1}, \xi \rangle)}{\langle \mathbb{1}, \xi_i \rangle} \frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle \langle \chi, \vartheta_i \rangle} \right) \mathbb{1}_{(0,\infty)}(\langle \mathbb{1}, \xi_i \rangle) dx, \\
&\mathrm{k}_i^{(3)}(\xi) = \kappa_i \rho_i^{-\alpha} \int_0^\infty \left(\frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle} \right)^\alpha \langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle \nu_i(\alpha+1) \mathbb{1}_{(0,\infty)}(\langle \mathbb{1}, \xi_i \rangle) dx.
\end{aligned}$$

Fix $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$ and let $z = \langle \mathbb{1}, \xi \rangle$. We first show that, for each $i \in I_+(z)$, K_i is continuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$ at ξ. Fix $i \in I_+(z)$. Suppose $\{\xi^n\}_{n \in \mathbb{N}}$ is a sequence in $\mathbf{M}_\upsilon^{\mathbf{I}}$ that converges to ξ (weakly). We want to show that $\lim_{n \to \infty} K_i(\xi^n) = K_i(\xi)$. For $\mathrm{k}_i^{(1)}$, by the continuity of $\phi_i(\cdot)$ at z when $z_i \neq 0$, and the fact that ξ^n converges to ξ implying $\langle \mathbb{1}, \xi^n \rangle \to \langle \mathbb{1}, \xi \rangle$, we have $\lim_{n \to \infty} \mathrm{k}_i^{(1)}(\xi^n) = \mathrm{k}_i^{(1)}(\xi)$. For $\mathrm{k}_i^{(2)}$, we have $\langle \mathbb{1}_{(x,\infty)}, \xi_i^n \rangle \to \langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle$ as $n \to \infty$ for almost every $x \in [0,\infty)$, $\langle \mathbb{1}, \xi_i^n \rangle \to z_i \neq 0$ and $\phi_i(\langle \mathbb{1}, \xi^n \rangle) \to \phi_i(\langle \mathbb{1}, \xi \rangle)$ as $n \to \infty$ (by the continuity of ϕ_i at z such that $z_i \neq 0$), $\frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i^n \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle} \leq \upsilon$ for all $n \in \mathbb{N}$ and x such that $\langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle > 0$, and $\int_0^\infty \langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle dx = \langle \chi, \vartheta_i \rangle < \infty$, and so using the fact that $\phi_i(\langle \mathbb{1}, \xi^n \rangle) \leq \max_{j \in \mathcal{J}} C_j$ for all $n \in \mathbb{N}$, we can apply the dominated convergence theorem to conclude that $\lim_{n \to \infty} \mathrm{k}_i^{(2)}(\xi^n) = \mathrm{k}_i^{(2)}(\xi)$. For $\mathrm{k}_i^{(3)}$, we can also apply the dominated convergence theorem to conclude that $\lim_{n \to \infty} \mathrm{k}_i^{(3)}(\xi^n) = \mathrm{k}_i^{(3)}(\xi)$. It follows that $K_i = \mathrm{k}_i^{(1)} + \mathrm{k}_i^{(2)} + \mathrm{k}_i^{(3)}$, is continuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$ at ξ for $i \in I_+(z)$. This proves the last statement of the lemma.

For $i \in I \setminus I_+(z)$, we will show that K_i is upper semicontinuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$ at ξ, where $\xi_i = 0$. For this it suffices to show for $\{\xi^n\}_{n \in \mathbb{N}}$, a sequence in $\mathbf{M}_\upsilon^{\mathbf{I}}$ that converges to ξ (weakly), we have $\limsup_{n \to \infty} K_i(\xi^n) \leq K_i(\xi)$. Notice that $\mathrm{k}_i^{(1)}(\xi^n) \leq 0$ and $\mathrm{k}_i^{(2)}(\xi^n) \leq 0$, while $\mathrm{k}_i^{(1)}(\xi) = 0$ and $\mathrm{k}_i^{(2)}(\xi) = 0$. It follows that $\limsup_{n \to \infty} (\mathrm{k}_i^{(1)}(\xi^n) + \mathrm{k}_i^{(2)}(\xi^n)) \leq \mathrm{k}_i^{(1)}(\xi) + \mathrm{k}_i^{(2)}(\xi)$. For $\mathrm{k}_i^{(3)}(\xi^n)$, the integrand is dominated by the integrable function $x \to \upsilon^\alpha \nu_i(\alpha+1) \langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle$ and tends to zero as $n \to \infty$, since $\langle \mathbb{1}_{(x,\infty)}, \xi_i^n \rangle \leq z_i^n \to z_i = 0$ as $n \to \infty$. It follows by the dominated convergence theorem that $\mathrm{k}_i^{(3)}(\xi^n) \to 0 = \mathrm{k}_i^{(3)}(\xi)$ as $n \to \infty$. Combining, we see that K_i is upper semicontinuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$ at ξ, for $i \notin I_+(z)$.

Since $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$ was arbitrary and any continuous function is upper semicontinuous, it follows that K_i is upper semicontinuous on $\mathbf{M}_\upsilon^\mathbf{I}$ for each $i \in I$. Furthermore, $K = \sum_{i \in I} K_i/(\alpha+1)$ is upper semicontinuous on $\mathbf{M}_\upsilon^\mathbf{I}$, being a linear combination, with positive coefficients, of such functions. $\square$

The following is a key lemma, proved in Section 4.9.1. For the statement of this, let δ_0 denote the probability measure on $\mathbb{R}_+$ that has unit mass at the origin and define

$$\mathcal{M}^* = \{\xi \in \mathbf{M}^\mathbf{I} : \text{ for each } i \in I,\ \xi_i = a_i\delta_0 + b_i\vartheta_i^e,\ \text{where } a \in \mathbb{R}_+^\mathbf{I} \text{ and } b \in \mathcal{P}\}. \tag{4.20}$$

Lemma 4.4.4. *Given* $\xi \in \bigcup_{\upsilon>0} \mathbf{M}_\upsilon^\mathbf{I}$, *with* $z = \langle \mathbb{1}, \xi\rangle$ *and* $z_i = \langle \mathbb{1}, \xi_i\rangle$ *for each* $i \in I$, *we have*

$$K_i(\xi) \le \kappa_i z_i^\alpha \left(\frac{-\phi_i(z)}{\rho_i^\alpha} + \frac{\rho_i}{\big(\phi_i(z)\big)^\alpha} \right) \mathbb{1}_{(0,\infty)}(z_i). \tag{4.21}$$

Moreover, if Assumption 4.1 is satisfied, we have

$$K(\xi) \le \sum_{i \in I_+(z)} \kappa_i \left(\frac{z_i}{\phi_i(z)} \right)^\alpha \big(\rho_i - \phi_i(z)\big) \le 0, \tag{4.22}$$

where equality holds everywhere in (4.22) if and only if $\xi \in \mathcal{M}^*$.

Definition 4.4.4. *Suppose that Assumption 4.2 holds. Given a fluid model solution* ζ *with* $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}_\upsilon^\mathbf{I}$, *for each* $t \ge 0$ *and* $i \in I$, *define*

$$\begin{aligned} \mathcal{K}_i^\zeta(t) = K_i\big(\zeta(t)\big) = \frac{\kappa_i}{\rho_i^\alpha} \Bigg(& -\Lambda_i(t)\big(z_i(t)\big)^\alpha \\ & + \int_0^\infty \left(\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \right)^\alpha \overline{N}_i(x) \left(-\frac{\alpha\Lambda_i(t)}{z_i(t)} \frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)\langle\chi,\vartheta_i\rangle} + \nu_i(\alpha+1) \right) \mathbb{1}_{(0,\infty)}\big(z_i(t)\big)dx \Bigg) \end{aligned} \tag{4.23}$$

and

$$\mathcal{K}^{\zeta}(t) = \frac{1}{\alpha+1} \sum_{i \in I_+(z(t))} \mathcal{K}_i^{\zeta}(t) \text{ for all } t \geq 0. \tag{4.24}$$

Lemma 4.4.5. *Suppose that Assumption 4.2 holds. Let ζ be a fluid model solution with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}_{\upsilon}^{\mathbf{I}}$. Then $\mathcal{K}_i^{\zeta}, i \in I$, and $\mathcal{K}^{\zeta}$ are real-valued, upper semicontinuous functions on $[0,\infty)$. Furthermore, for each $i \in I$, $\mathcal{K}_i^{\zeta}$ is continuous on $\{t \geq 0 : z_i(t) > 0\}$.*

Proof. This follows immediately on combining Lemma 4.3.1 with Lemma 4.4.3 and the continuity of $\zeta(\cdot)$ on $[0,\infty)$. □

4.4.3 Relationship between $\mathcal{H}^{\zeta}$ and $\mathcal{K}^{\zeta}$

Theorem 4.4.1. *Suppose that Assumptions 4.1 and 4.2 hold. Further suppose that ζ is a fluid model solution with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}_{\upsilon}^{\mathbf{I}}$. For each $i \in I$, $\mathcal{K}_i^{\zeta}(\cdot)$ is integrable over $[0,t]$ for each $t \geq 0$ and the function $\mathcal{H}_i^{\zeta}(\cdot)$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$, with density $\mathcal{K}_i^{\zeta}(\cdot)$, and so*

$$\mathcal{H}_i^{\zeta}(t) - \mathcal{H}_i^{\zeta}(0) = \int_0^t \mathcal{K}_i^{\zeta}(s)ds \text{ for each } t \geq 0. \tag{4.25}$$

Consequently, $\mathcal{H}^{\zeta}(\cdot)$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$ and $\mathcal{K}^{\zeta}(\cdot)$ is a density for $\mathcal{H}^{\zeta}(\cdot)$. Furthermore, for each $t \geq 0$,

$$\mathcal{K}^{\zeta}(t) \leq \sum_{i \in I_+(z(t))} \kappa_i \left(\frac{z_i(t)}{\Lambda_i(t)}\right)^{\alpha} \left(\rho_i - \Lambda_i(t)\right) \leq 0, \tag{4.26}$$

where equality holds everywhere in (4.26) if and only if $\zeta(t) \in \mathcal{M}$. Hence $\mathcal{H}^{\zeta}(\cdot)$ is non-increasing on $[0,\infty)$, and is strictly decreasing at times $t \in [0,\infty)$ where $\zeta(t) \notin \mathcal{M}$.

The proof of Theorem 4.4.1 is given in Section 4.9.

4.4.4 The Function $\underline{F}$

One characterization of the invariant states for the fluid model that we will give uses the following optimization problem. This optimization problem is similar to one used by Kelly and Williams [KW04], who studied properties of the fluid model when ϑ_i is exponentially distributed for each $i \in I$. The main difference in the form from [KW04] is that in two places (one in the function F and one in the constraint of the optimization problem (4.27)), $\frac{1}{\mu_i}$ from [KW04] is replaced by $\langle \chi, \vartheta_i^e \rangle$ for $i \in I$. We now describe the optimization problem.

For $z \in \mathbb{R}_+^{\mathbf{I}}$, let

$$F(z) = \frac{1}{\alpha+1} \sum_{i \in I} \frac{\kappa_i \langle \chi, \vartheta_i^e \rangle}{\rho_i^\alpha} z_i^{\alpha+1}.$$

For $\widetilde{w} \in \mathbb{R}_+^{\mathbf{J}_*}$, consider the optimization problem

$$\text{minimize } F(z) \text{ subject to } \sum_{i \in I} R_{ji} z_i \langle \chi, \vartheta_i^e \rangle \geq \widetilde{w}_j \text{ for all } j \in \mathcal{J}_* \text{ and } z \in \mathbb{R}_+^{\mathbf{I}}. \tag{4.27}$$

In Section 4.6, we give several different characterizations of the set $\mathcal{P}$, which features in the characterization (4.8) of invariant states for the fluid model. One of these uses the optimization problem (4.27). For $\widetilde{w} \in \mathbb{R}_+^{\mathbf{J}_*}$, let $\underline{F}(\widetilde{w})$ be the optimal value attained in the optimization problem (4.27) and let $\Delta(\widetilde{w})$ be the optimizing value of z. These exist and are unique. The following proposition gives properties of $\underline{F}$. Its proof is the same as that of Lemma 6.3 of [KW04] with $diag(\langle \chi, \vartheta_e \rangle)$ in place of $M^{-1} = diag(\mu_i^{-1} : i \in I)$, and we refer the reader to [KW04] for the details. We note that this proof uses the fact that R has full row rank.

Proposition 4.4.1. *The functions $\underline{F} : \mathbb{R}_+^{\mathbf{J}_*} \to \mathbb{R}_+$ and $\Delta : \mathbb{R}_+^{\mathbf{J}_*} \to \mathbb{R}_+^{\mathbf{I}}$ are continuous. In addition, $\underline{F}$ is a non-decreasing function, i.e., for $\widetilde{w}, \widetilde{w}^\dagger \in \mathbb{R}_+^{\mathbf{J}_*}$, if $\widetilde{w}_j \leq \widetilde{w}_j^\dagger$ for each $j \in \mathcal{J}_*$, then $\underline{F}(\widetilde{w}) \leq \underline{F}(\widetilde{w}^\dagger)$.*

The non-decreasing property of $\underline{F}$ will be a key property for proving that our Lyapunov function, when applied to a fluid model solution, yields a non-increasing function of time.

4.5 Characterization of Solutions for the Optimization Problems (2.2) and (4.27)

In this section, we characterize the optimizing solutions for the optimization problems used to define the bandwidth sharing policy and $\underline{F}$. The first optimization problem considered here is (2.2) and the second is (4.27). We characterize the optimal solutions for both problems below, so as to give an alternative characterization of the invariant states. The idea of using these two optimization problems to characterize invariant states was employed by Kelly and Williams [KW04] when the file sizes are exponentially distributed. Proposition 4.5.1, which characterizes the optimal solution for (2.2), is equivalent to Lemma A.4 in [KW04]. Proposition 4.5.2, which characterizes the optimal solution for (4.27), is similar to Lemma 6.4 in [KW04]. Both propositions are proved using Lagrange multipliers. For the proof of Proposition 4.5.2, in the proof of Lemma 6.4 in [KW04], substitute $\langle \chi, \vartheta_e^i \rangle$ for μ_i^{-1} in the constraints and in one place in F. The proof uses the fact that R has full row rank. We refer readers to [KW04] for details of the proofs of these two propositions.

Proposition 4.5.1. *Fix $z \in \mathbb{R}_+^{\mathbf{I}} \setminus \{0\}$, where 0 is the origin of $\mathbb{R}_+^{\mathbf{I}}$. A vector $\psi = (\psi_i : i \in I) \in O(z)$ is the unique optimal solution of (2.2), i.e. $\psi = \phi(z)$, if and only if there is $p \in \mathbb{R}_+^{\mathbf{J}}$ such that*

$$p_j\big(C_j - \sum_{i \in I_+(z)} R_{ji}\psi_i\big) = 0 \quad \textit{for all } j \in \mathcal{J}, \tag{4.28}$$

$$\sum_{j \in \mathcal{J}} p_j R_{ji} > 0 \quad \textit{for all } i \in I_+(z), \tag{4.29}$$

$$\psi_i = z_i \Big(\frac{\kappa_i}{\sum_{j \in \mathcal{J}} p_j R_{ji}} \Big)^{1/\alpha} \quad \textit{for all } i \in I_+(z) \quad \textit{and} \tag{4.30}$$

$$\sum_{i \in I_+(z)} R_{ji}\psi_i \le C_j \quad \textit{for all } j \in \mathcal{J}. \tag{4.31}$$

Proposition 4.5.2. *Suppose Assumption 4.1 holds. For each $\widetilde{w} \in \mathbb{R}_+^{\mathbf{J}_*}$, a vector $z \in \mathbb{R}_+^{\mathbf{I}}$ is the* *unique optimal solution of (4.27), i.e.* $z = \Delta(\widetilde{w})$*, if and only if there is* $p \in \mathbb{R}_+^{\mathbf{J}_*}$ *such that for each* $i \in I$*,*

$$z_i = \rho_i \left(\frac{\sum_{j \in \mathcal{J}_*} p_j R_{ji}}{\kappa_i} \right)^{1/\alpha}, \tag{4.32}$$

and for each $j \in \mathcal{J}_*$*,*

$$p_j \left(\sum_{i \in I} R_{ji} z_i \langle \chi, \vartheta_i^e \rangle - \widetilde{w}_j \right) = 0 \quad \textit{and} \quad \sum_{i \in I} R_{ji} z_i \langle \chi, \vartheta_i^e \rangle \geq \widetilde{w}_j.$$

4.6 Further Characterizations of Invariant States

In this section, we further characterize the set of invariant states. Under Assumption 4.1, recall the set of invariant states $\mathcal{M}$ is given by (4.8) and $\mathcal{P}$ is defined in (4.6). Here we characterize the set $\mathcal{P}$ in two further ways, similar to Lemma 6.4 of Gromoll and Williams [GW09], whose proof relies on those of Theorems 5.1 and 5.3 of Kelly and Williams [KW04].

Lemma 4.6.1. *Suppose Assumption 4.1 holds. The following three conditions are equivalent:*

(i) $z \in \mathcal{P}$*,*

(ii) for some $p \in \mathbb{R}_+^{\mathbf{J}_*}$, $z_i = \rho_i \left(\frac{1}{\kappa_i} \sum_{j \in \mathcal{J}_*} p_j R_{ji} \right)^{1/\alpha}$ *for all* $i \in I$*,*

(iii) $z = \Delta(\check{w}(z))$*, where* $\check{w}_j(z) = \sum_{i \in I} R_{ji} z_i \langle \chi, \vartheta_i^e \rangle$ *for all* $j \in \mathcal{J}_*$*.*

Proof. The proof is very similar to that of Theorems 5.1 and 5.3 of [KW04], with $\langle \chi, \vartheta_i^e \rangle$ replacing μ_i^{-1} in two places for each $i \in I$. For $(i) \Leftrightarrow (ii)$, one uses Proposition 4.5.1; and for $(ii) \Leftrightarrow (iii)$, one uses Proposition 4.5.2 in place of Lemma 6.4 of [KW04]. □

Remark 4.6.1. *The above characterization of* $\mathcal{P}$ *is slightly different from what is given by Gromoll and Williams [GW09]. The latter uses* $w(z)$ *and* μ_i^{-1} *in the constraints rather than* $\check{w}(z)$ *and* $\langle \chi, \vartheta_i^e \rangle$*. Both characterizations are correct and although the difference is subtle, we find that our form is more useful for our proofs.*

4.7 Lyapunov Function G and $\mathcal{G}^{\zeta}$

In this section, we define the Lyapunov function G on $\bigcup_{\upsilon>0} \mathbf{M}_{\upsilon}^{\mathbf{I}}$ and the function $\mathcal{G}^{\zeta}$ for any fluid model solution satisfying $\zeta(0) \in \bigcup_{\upsilon>0} \mathbf{K}_{\upsilon}^{\mathbf{I}}$.

Definition 4.7.1. *Given $\xi \in \bigcup_{\upsilon>0} \mathbf{M}_{\upsilon}^{\mathbf{I}}$, define*

$$G(\xi) = H(\xi) - \underline{F}\big(\widetilde{w}(\xi)\big) \tag{4.33}$$

where $\widetilde{w}_j(\xi) = \sum_{i\in I} R_{ji}\langle \chi, \xi_i\rangle$ for each $j \in \mathcal{J}_*$, and $\underline{F}\,(\widetilde{w}\,(\xi))$ is the optimal value for the optimization problem (4.27) with $\widetilde{w} = \widetilde{w}(\xi)$.

The following lemma is proved in Section 4.10.3.

Lemma 4.7.1. *For each $\upsilon > 0$,*

(i) $G : \mathbf{M}_{\upsilon}^{\mathbf{I}} \to [0,\infty)$ is continuous.

Moreover, if Assumption 4.1 holds, then for any $\xi \in \cup_{\upsilon>0}\mathbf{M}_{\upsilon}^{\mathbf{I}}$,

(ii) $G(\xi) = 0$ if and only if $\xi \in \mathcal{M}^*$, where $\mathcal{M}^*$ is given by (4.20).

Definition 4.7.2. *Suppose that Assumption 4.2 holds. Given a fluid model solution ζ with $\zeta(0) \in \bigcup_{\upsilon>0} \mathbf{K}_{\upsilon}^{\mathbf{I}}$, define*

$$\mathcal{G}^{\zeta}(t) = G(\zeta(t)) = \mathcal{H}^{\zeta}(t) - \underline{F}(\widetilde{w}(\zeta(t))) \text{ for all } t \geq 0. \tag{4.34}$$

Remark 4.7.1. *By Lemmas 4.3.1 and 4.7.1, $\mathcal{G}^{\zeta}(\cdot)$ is well defined and continuous on $[0,\infty)$.*

4.8 Main Results

The proofs of the next three theorems are given in Section 4.11.

Theorem 4.8.1. *Suppose that Assumptions 4.1 and 4.2 hold. Further suppose that ζ is a fluid model solution with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}^{\mathbf{I}}_{\upsilon}$. Then*

(i) $\mathcal{G}^{\zeta} : [0,\infty) \to [0,\infty)$ *is continuous,*

(ii) for any $t \geq 0$, $\mathcal{G}^{\zeta}(t) = 0$ if and only if $\zeta(t) \in \mathcal{M}$, and

(iii) $\mathcal{G}^{\zeta}$ *is a non-increasing function on $[0,\infty)$ and at times $t \in [0,\infty)$ where $\zeta(t) \notin \mathcal{M}$, $\mathcal{G}^{\zeta}$ is strictly decreasing.*

Theorem 4.8.2. *Suppose that Assumptions 4.1 and 4.2 hold. Fix $\upsilon > 0$. For any fluid model solution ζ with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$, $\mathcal{G}^{\zeta}(t)$ decreases monotonically to zero as $t \to \infty$. Furthermore, this convergence is uniform, i.e.,*

$$\lim_{t\to\infty} \sup\{\mathcal{G}^{\zeta}(t) : \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}\} = 0. \tag{4.35}$$

Theorem 4.8.3. *Suppose that Assumptions 4.1 and 4.2 hold. Fix $\upsilon > 0$. For any fluid model solution ζ satisfying $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$, $\zeta(t)$ converges towards $\mathcal{M}$ as $t \to \infty$, uniformly for all initial measures in $\mathbf{K}^{\mathbf{I}}_{\upsilon}$, i.e.,*

$$\lim_{t\to\infty} \sup\{\mathbf{d}_{\mathbf{I}}(\zeta(t), \mathcal{M}) : \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}\} = 0. \tag{4.36}$$

Furthermore, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\sup_{t\geq 0}\{\mathbf{d}_{\mathbf{I}}(\zeta(t), \mathcal{M}) : \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon} \text{ and } \mathbf{d}_{\mathbf{I}}(\zeta(0), \mathcal{M}) < \delta\} \leq \varepsilon. \tag{4.37}$$

4.9 Proofs of Lemma 4.4.4 and Theorem 4.4.1

4.9.1 Proof of Lemma 4.4.4

For our proof of Lemma 4.4.4, we need Lemmas 3.4.1, 3.4.2 and the following proposition. Proposition 4.9.1 is nearly the same as Lemma 3.4.3, with the condition for equality in the inequality specified. Here, we indicate the reasoning for that and leave the reader to consult Section 3.4 for the rest of the proof for Proposition 4.9.1.

Proposition 4.9.1. *For any strictly positive real numbers, a, b, q, we have*

$$-\frac{b}{q^a}+\frac{q}{b^a}\leq (a+1)\frac{q-b}{b^a}, \tag{4.38}$$

where equality holds if and only if $q=b$.

Proof of when equality holds in (4.38). The inequality (4.38) comes from the fact that the tangent line to the graph of $y=x^{a+1}$ at $x=q$ is a lower support line for the graph. It follows from the strict convexity of $x\to x^{a+1}$ that this support line touches the graph only at $x=q$ and hence the inequality in (4.38) is strict for all $b\neq q$. $\square$

Proof of Lemma 4.4.4. We first prove (4.21). Since both sides of the inequality are zero when $z_i=0$, it suffices to consider the case where $z_i>0$. In this case, we have

$$\begin{aligned}
\rho_i^\alpha K_i(\xi) &= -\kappa_i\phi_i(z)z_i^\alpha \\
&\quad +\kappa_i\int_0^\infty\left(\frac{\langle \mathbb{1}_{(x,\infty)},\xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)},\vartheta_i^e\rangle}\right)^\alpha\left((\alpha+1)\rho_i-\frac{\alpha\phi_i(z)}{z_i}\frac{\langle \mathbb{1}_{(x,\infty)},\xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)},\vartheta_i^e\rangle}\right)\mu_i\overline{N}_i(x)dx \\
&\leq -\kappa_i\phi_i(z)z_i^\alpha+\kappa_i\int_0^\infty\left(\frac{\rho_i z_i}{\phi_i(z)}\right)^\alpha\rho_i\mu_i\overline{N}_i(x)dx \qquad (4.39)\\
&= \kappa_i z_i^\alpha\left(-\phi_i(z)+\frac{\rho_i^{\alpha+1}}{(\phi_i(z))^\alpha}\right), \qquad (4.40)
\end{aligned}$$

where we used Lemma 3.4.2 with $a = \alpha$, $q = \rho_i$, $b = \frac{\alpha\phi_i(z)}{z_i}$ for $z_i > 0$ to obtain the inequality, and the fact that $\int_0^\infty \mu_i \overline{N}_i(x)dx = 1$ for the last equality. We note here that the inequality in (4.39) is strict unless $\langle \mathbb{1}_{(x,\infty)}, \xi_i\rangle = \frac{\rho_i z_i}{\phi_i(z)} \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e\rangle$ for all $x \in \mathbb{R}_+$; this follows for $x \in \mathbb{R}_+$ where $\overline{N}_i(x) > 0$ by the uniqueness of the maximum in Lemma 3.4.2, and the relation automatically holds for x where $\overline{N}_i(x) = 0$, since $\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e\rangle = 0$ for such x and $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$ for some $\upsilon > 0$. Thus,

$$K_i(\xi) \leq \kappa_i z_i^\alpha \left(\frac{-\phi_i(z)}{\rho_i^\alpha} + \frac{\rho_i}{(\phi_i(z))^\alpha} \right) \tag{4.41}$$

$$\leq \kappa_i z_i^\alpha (\alpha+1) \frac{\rho_i - \phi_i(z)}{(\phi_i(z))^\alpha}, \tag{4.42}$$

where the last step follows by Proposition 4.9.1 with $a = \alpha, b = \phi_i(z)$ and $q = \rho_i$. We note here that by Proposition 4.9.1, the last inequality is strict unless $\phi_i(z) = \rho_i$. The inequality (4.41) yields (4.21).

Assuming that Assumption 4.1 holds, we shall now use inequality (4.42) to prove (4.22), and we shall use the conditions for equality in (4.39) and (4.42) to determine conditions for equality in (4.22). For $i \in I_+(z)$, $U'\left(\frac{\phi_i(z)}{z_i}\right) = \left(\frac{z_i}{\phi_i(z)}\right)^\alpha$. Furthermore, ρ has positive components and satisfies $\sum_{i\in I} R_{ji}\rho_i \leq C_j$ for all $j \in \mathcal{J}$, by Assumption 4.1. Then, by (4.42) and replacing $z, \psi, \phi(z)$ by $z, \rho, \phi(z)$, respectively, in Lemma 3.4.1, we obtain

$$K(\xi) = \frac{1}{\alpha+1} \sum_{i \in I_+(z)} K_i(\xi) \leq \sum_{i \in I_+(z)} \kappa_i \left(\frac{z_i}{\phi_i(z)} \right)^\alpha \big(\rho_i - \phi_i(z)\big) \leq 0. \tag{4.43}$$

Hence (4.22) holds. By the conditions for equality in (4.39) and (4.42), the first inequality in (4.43) is an equality if and only if $\langle \mathbb{1}_{(x,\infty)}, \xi_i\rangle = \frac{\rho_i z_i}{\phi_i(z)} \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e\rangle$ for all $x \in \mathbb{R}_+$ and $\phi_i(z) = \rho_i$, for all $i \in I_+(z)$. Noting that $z_i = 0$ for all $i \notin I_+(z)$, it then follows that $K(\xi) = 0$ if and only if for all $i \in I$, $\langle \mathbb{1}_{(x,\infty)}, \xi_i\rangle = z_i \overline{N}_i^e(x)$ for all $x \in \mathbb{R}_+$, where z is such that $\phi_i(z) = \rho_i$ for all $i \in I_+(z)$. When the measures are restricted to $(0,\infty)$, this is the characteristic property of elements of the invariant manifold $\mathcal{M}$, as described in (4.8). Because K only captures the behavior of ξ on $(0,\infty)$, and a general $\xi \in \mathbf{M}^{\mathbf{I}}$ could be such that any of its components has an atom at zero, it follows that $K(\xi) = 0$ if and only if $\xi \in \mathcal{M}^*$, as defined in (4.20).

□

4.9.2 Smooth Approximation of Measures

We use an approximation argument to prove Theorem 4.4.1. An approximation argument was also used in Section 3.5.1. Consequently, some propositions and proofs are the same as in Section 3.5.1 and we record those results here without proof. We focus on the details that differ from those in Section 3.5.1. For each positive integer n, let $\varphi_n \in \mathbf{C}_c^\infty(\mathbb{R})$ be such that $\varphi_n \geq 0, \varphi_n(x) = 0$ for all $x \in (-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, \infty)$, $\varphi_n(x) = \varphi_n(-x)$ for all $x > 0$, and $\int_{\mathbb{R}} \varphi_n(x)dx = 1$. Given $\xi \in \mathbf{M}$ and $n \in \mathbb{N}$, let ξ^n be the non-negative, absolutely continuous Borel measure on $\mathbb{R}_+$ whose continuous density is given by

$$d_n(x) = \int_{\mathbb{R}_+} \varphi_n(x-y)\xi(dy) = \int_{\mathbb{R}_+} \varphi_n(y-x)\xi(dy) \quad \text{for } x \in \mathbb{R}_+, \tag{4.44}$$

where we have used the symmetry of φ_n for the last equality. Note that $d_n(\cdot)$ is in $\mathbf{C}_b^\infty(\mathbb{R}_+)$, since φ_n is infinitely differentiable with compact support and ξ is a finite measure on $\mathbb{R}_+$. For any bounded, Borel measurable function f defined on $\mathbb{R}_+$, let $(f * \varphi_n)(y) = \int_{\mathbb{R}_+} \varphi_n(y-x)f(x)dx$ for $y \in \mathbb{R}_+$. Then, by Fubini's theorem,

$$\langle f, \xi^n \rangle = \int_{\mathbb{R}_+} f(x) \int_{\mathbb{R}_+} \varphi_n(y-x)\xi(dy)dx = \langle f * \varphi_n, \xi \rangle. \tag{4.45}$$

The next two propositions are the same as Lemma 3.5.1 and Lemma 3.5.2, where the first of these is proved by an argument similar to that in Lemma 7.12 of [PW16]. We refer the reader to Section 3.5.1 and [PW16] for the proofs noting that they do not rely on whether the fluid model is in the strictly subcritical regime or not.

Proposition 4.9.2. *Let $\xi \in \mathbf{K} \cap \mathbf{M}_1$. For each $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$, we have*

$$\left\langle \mathbb{1}_{(x+\frac{1}{n},\infty)}, \xi \right\rangle \leq \langle \mathbb{1}_{(x,\infty)}, \xi^n \rangle \leq \left\langle \mathbb{1}_{\left((x-\frac{1}{n})^+,\infty\right)}, \xi \right\rangle, \tag{4.46}$$

$$\langle \chi, \xi \rangle - \frac{\langle \mathbb{1}, \xi \rangle}{n} \leq \langle \chi, \xi^n \rangle \leq \langle \chi, \xi \rangle + \frac{\langle \mathbb{1}, \xi \rangle}{n}. \tag{4.47}$$

Furthermore, we have $\xi^n \in \mathbf{A}$ for each $n \in \mathbb{N}$ and as $n \to \infty$,

$$\xi^n \xrightarrow{w} \xi \quad \textit{and} \quad \langle \chi, \xi^n \rangle \to \langle \chi, \xi \rangle. \tag{4.48}$$

Given a fluid model solution ζ, for each $t \geq 0$ and $i \in I$, let $\{\zeta_i^n(t)\}_{n=1}^{\infty}$ be the approximating sequence of measures for $\zeta_i(t)$, as defined above with $\zeta_i(t)$ in place of ξ. Similarly, define ϑ_i^n for each $i \in I$, $n \in \mathbb{N}$. For any positive integer ℓ, let $\mathcal{C}_{0,\ell} = \{g \in \mathbf{C}_b^1(\mathbb{R}_+) : g = 0 \text{ on } [0, \frac{1}{\ell}]\}$. For $g \in \mathcal{C}_{0,\ell}$ and all $n > \ell$, we have $(g * \varphi_n)(0) = 0$ and $(g * \varphi_n)'(0) = 0$. It follows that $g * \varphi_n \in \mathcal{C}$, where $\mathcal{C}$ is defined in (2.3). For each positive integer n, $i \in I$, $t \geq 0$ and $x \in \mathbb{R}_+$, let $\vartheta_i^{n,e}$ be the excess lifetime distribution for ϑ_i^n, and

$$\overline{M}_t^{i,n}(x) = \langle \mathbb{1}_{(x,\infty)}, \zeta_i^n(t) \rangle, \qquad \overline{N}_i^n(x) = \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^n \rangle, \qquad \overline{N}_i^{n,e}(x) = \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^{n,e} \rangle. \tag{4.49}$$

We note that $\vartheta_i^{n,e}$ has density $\overline{N}_i^n(\cdot)/\langle \chi, \vartheta_i^n \rangle$.

The following proposition shows that for all n sufficiently large, $(t,x) \to \overline{M}_t^{i,n}(x)$ satisfies a transport partial differential equation with nonlinear, nonlocal coefficients on intervals of time where $z_i(\cdot)$ is not zero and on intervals for x that are bounded away from zero.

Proposition 4.9.3. *Assume that ζ is a fluid model solution. Suppose that $i \in I$ and $0 \leq a < b < \infty$ are such that $z_i(t) \neq 0$ for all $t \in [a,b]$. Then, for each positive integer ℓ and all $n > \ell$, $t \to \overline{M}_t^{i,n}(x)$ is continuously differentiable on $[a,b]$ for each fixed $x \in \mathbb{R}_+$, and $x \to \overline{M}_t^{i,n}(x)$ is continuously* *differentiable on* $[\frac{1}{\ell},\infty)$ *for each fixed* $t \in [a,b]$*, and furthermore,*

$$\frac{\partial \overline{M}_t^{i,n}(x)}{\partial t} = \frac{\Lambda_i(t)}{z_i(t)} \frac{\partial \overline{M}_t^{i,n}(x)}{\partial x} + \nu_i \overline{N}_i^n(x), \tag{4.50}$$

for $t \in [a,b],\ x \geq \frac{1}{\ell}$*, where the partial derivatives with respect to time at* $t = a,b$ *are from the right, left, respectively, and the partial derivative with respect to x at* $x = 1/\ell$ *is from the right.*

Remark 4.9.1. *From (4.44), for each fixed* $t \in [0,\infty)$*, the measure* $\zeta_i^n(t)$ *on* $\mathbb{R}_+$ *has a continuous density function given by*

$$m_t^{i,n}(x) = \int_{\mathbb{R}_+} \varphi_n(y-x)\zeta_i(t)(dy) \ \text{ for all } x \in \mathbb{R}_+. \tag{4.51}$$

For any fixed $x \in \mathbb{R}_+$*,* $t \to \frac{\partial \overline{M}_t^{i,n}(x)}{\partial x} = -m_t^{i,n}(x)$ *(where the derivative at* $x = 0$ *is from the right) is continuous on* $[0,\infty)$*, because the fluid model solution* ζ_i *is continuous as a function of time. It follows that* $(t,x) \to \frac{\partial \overline{M}_t^{i,n}(x)}{\partial x}$ *is separately continuous in t and x and hence is jointly measurable on* $[0,\infty) \times \mathbb{R}_+$*. Via (4.50), this implies joint measurability of* $(t,x) \to \frac{\partial \overline{M}_t^{i,n}(x)}{\partial t}$ *on* $[a,b] \times [\frac{1}{\ell},\infty)$ *for any* $n > \ell \geq 1$ *when* $z_i(t) \neq 0$ *for all* $t \in [a,b]$*. Furthermore, from (4.50) and (4.51), we have for any* $n > \ell \geq 1$*,* $t \in [a,b]$ *and* $x \in \mathbb{R}_+$*,*

$$\begin{aligned} \left| \frac{\partial \overline{M}_t^{i,n}(x)}{\partial t} \right| &\leq \frac{\Lambda_i(t)}{z_i(t)} \left| -m_t^{i,n}(x) \right| + \nu_i \overline{N}_i^n(x) &(4.52)\\ &\leq \Lambda_i(t) \sup_{y \in \mathbb{R}} \varphi_n(y) + \nu_i \overline{N}_i^n(x). &(4.53) \end{aligned}$$

It follows that $(t,x) \to \frac{\partial \overline{M}_t^{i,n}(x)}{\partial t}$ *is measurable and integrable over the interval* $[a,b] \times [\frac{1}{\ell},\ell]$ *for each fixed* $n > \ell \geq 1$*. These measurability and integrability properties will be needed for a use of Fubini's theorem in the proof of Theorem 4.4.1 below.*

Lemma 4.9.1. *Suppose that* $\vartheta \in \mathbf{K^I}$ *and* ζ *is a fluid model solution with* $\zeta(0) \in \mathbf{K^I} \cap \mathbf{M_1^I}$*. For* *any $0 \le a < b < \infty$, for each $i \in I$, we have the following uniform bounds:*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[a,b]} \sup_{x\in\mathbb{R}_+} \overline{M}_t^{i,n}(x) \le \sup_{t\in[a,b]} z_i(t) < \infty, \tag{4.54}$$

$$\sup_{n\in\mathbb{N}} \sup_{t\in[a,b]} \langle \chi, \zeta_i^n(t) \rangle \le \sup_{t\in[a,b]} (w_i(t) + z_i(t)) < \infty. \tag{4.55}$$

In addition, for each $i \in I$, as $n \to \infty$, $\zeta_i^n(t) \xrightarrow{w} \zeta_i(t)$, $\vartheta_i^n \xrightarrow{w} \vartheta_i$, $\langle \chi, \vartheta_i^n \rangle \to \langle \chi, \vartheta_i \rangle$, $\vartheta_i^{n,e} \xrightarrow{w} \vartheta_i^e$, $\overline{M}_t^{i,n}(x) \to \overline{M}_t^i(x)$, $\overline{N}_i^n(x) \to \overline{N}_i(x)$ and $\overline{N}_i^{n,e}(x) \to \overline{N}_i^e(x)$ for each $x \in [0,\infty)$.

Proof. By Proposition 4.2.2 and Remark 2.3.1, $\zeta(t) \in \mathbf{K}^{\mathbf{I}} \cap \mathbf{M}_1^{\mathbf{I}}$ for each $t \ge 0$, and by Proposition 4.9.2, for each $i \in I, n \in \mathbb{N}, t \ge 0$ and $x \in \mathbb{R}_+$, we have $\overline{M}_t^{i,n}(x) := \langle \mathbb{1}_{(x,\infty)}, \zeta_i^n(t) \rangle \le \overline{M}_t^i\big((x - \frac{1}{n})^+\big) \le z_i(t)$ and $\langle \chi, \zeta_i^n(t) \rangle \le w_i(t) + z_i(t)$. Since $z_i(\cdot)$ and $w_i(\cdot)$ are continuous, it follows that for any $0 \le a < b < \infty$, $\overline{M}_t^{i,n}(x)$ and $\langle \chi, \zeta_i^n(t) \rangle$ have uniform bounds for all $t \in [a,b]$, $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$. Furthermore, by Proposition 4.9.2, $\zeta_i^n(t) \xrightarrow{w} \zeta_i(t)$, $\vartheta_i^n \xrightarrow{w} \vartheta_i$ and $\langle \chi, \vartheta_i^n \rangle \to \langle \chi, \vartheta_i \rangle$ as $n \to \infty$. It follows, since $\zeta_i(t) \in \mathbf{K}$ and $\vartheta_i \in \mathbf{K}$, that $\overline{M}_t^{i,n}(x) \to \overline{M}_t^i(x)$ and $\overline{N}_i^n(x) \to \overline{N}_i(x)$ for each $x \in \mathbb{R}_+$ as $n \to \infty$, and the density $\overline{N}_i^n(\cdot)/\langle \chi, \vartheta_i^n \rangle$ for $\vartheta_i^{n,e}$ converges everywhere on $\mathbb{R}_+$ to $\overline{N}_i(\cdot)/\langle \chi, \vartheta_i \rangle$, the density for ϑ_i^e, as $n \to \infty$. Since the last sequence of densities is eventually dominated by $2\overline{N}_i((\cdot - 1)^+)/\langle \chi, \vartheta_i \rangle$, which is integrable on $\mathbb{R}_+$, it follows by dominated convergence that $\vartheta_i^{n,e} \xrightarrow{w} \vartheta_i^e$ as $n \to \infty$, which implies, since $\overline{N}_i^e(\cdot)$ is continuous, that $\overline{N}_i^{n,e}(x) \to \overline{N}_i^e(x)$ as $n \to \infty$ for all $x \in [0,\infty)$. $\square$

The following lemma is used to control $x \to \frac{\overline{M}_t^{i,n}(x)}{\overline{N}_i^{n,e}(x)}$ uniformly in n.

Lemma 4.9.2. *Suppose Assumption 4.2 holds and that ζ is a fluid model solution with $\zeta(0) \in \mathbf{K}^{\mathbf{I}} \cap \mathbf{M}_1^{\mathbf{I}}$. Let $0 \le a < b < \infty$ and $i \in I$ be such that $\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \le \upsilon_{a,b}$ for all $x \in \mathbb{R}_+$ and $t \in [a,b]$, for some $\upsilon_{a,b} \in (0,\infty)$. Then there is $n_{a,b} \in \mathbb{N}$ (depending only on a, b, C_ϑ and ϑ_i) such that for all $n \ge n_{a,b}$, we have $\frac{\overline{M}_t^{i,n}(x)}{\overline{N}_i^{n,e}(x)} \le 2\upsilon_{a,b}$ for all $x \in \mathbb{R}_+$ and $t \in [a,b]$.*

Proof. By Proposition 4.9.2, we have for $t \in [a,b]$, $x \in \mathbb{R}_+$,

$$\begin{aligned}
\frac{\overline{M}_t^{i,n}(x)}{\overline{N}_i^{n,e}(x)} = \frac{\langle \mathbb{1}_{(x,\infty)}, \zeta_i^n(t)\rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^{n,e}\rangle} &\le \frac{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \zeta_i(t)\rangle \langle \chi, \vartheta_i^n\rangle}{\int_x^\infty \langle \mathbb{1}_{(y,\infty)}, \vartheta_i^n\rangle dy} \\
&\le \frac{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \zeta_i(t)\rangle \langle \chi, \vartheta_i^n\rangle}{\int_{x+\frac{1}{n}}^\infty \langle \mathbb{1}_{(y,\infty)}, \vartheta_i\rangle dy} \\
&= \frac{\langle \chi, \vartheta_i^n\rangle}{\langle \chi, \vartheta_i\rangle} \frac{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \zeta_i(t)\rangle}{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle - \langle \mathbb{1}_{((x-\frac{1}{n})^+,x+\frac{1}{n})}, \vartheta_i^e\rangle} \\
&= \frac{\langle \chi, \vartheta_i^n\rangle}{\langle \chi, \vartheta_i\rangle} \frac{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \zeta_i(t)\rangle / \langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle}{1 - (\langle \mathbb{1}_{((x-\frac{1}{n})^+,x+\frac{1}{n})}, \vartheta_i^e\rangle / \langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle)}. \qquad (4.56)
\end{aligned}$$

Note that $\lim_{n\to\infty} \frac{\langle \chi, \vartheta_i^n\rangle}{\langle \chi, \vartheta_i\rangle} = 1$. For the other term in (4.56), the numerator is bounded above by $\upsilon_{a,b}$ and for the denominator, by Assumption 4.2, $\frac{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i\rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e\rangle} \le C_\vartheta$ for all $x \ge 0$, which implies that

$$\begin{aligned}
\frac{\langle \mathbb{1}_{((x-\frac{1}{n})^+,x+\frac{1}{n})}, \vartheta_i^e\rangle}{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle} &= \frac{\int_{(x-\frac{1}{n})^+}^{x+\frac{1}{n}} \frac{\langle \mathbb{1}_{(y,\infty)}, \vartheta_i\rangle}{\langle \chi, \vartheta_i\rangle} dy}{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle} \\
&\le \frac{2\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i\rangle}{n\langle \chi, \vartheta_i\rangle \langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle} \\
&\le \frac{2C_\vartheta}{n\langle \chi, \vartheta_i\rangle}.
\end{aligned}$$

Thus for all sufficiently large n (not depending on x), the denominator of the second fraction in the right hand side of (4.56) is greater than $1/2$. It follows that for all sufficiently large n (depending only on C_ϑ and ϑ_i),

$$\frac{\overline{M}_t^{i,n}(x)}{\overline{N}_i^{n,e}(x)} \le 2\upsilon_{a,b} \quad \text{for all } t \in [a,b], x \in \mathbb{R}_+.$$

□

4.9.3 Proof of Theorem 4.4.1

The following proof is similar to a combination of the proofs in Section 3.5.2 and Section 3.5.3 with $(\rho, 0)$ in place of $(\widetilde{\rho}, \delta)$ there. However, since we are now in the critical case, rather than the strictly subcritical case, some aspects in our proof are more delicate and require different justifications due to the more singular form of the Lyapunov function considered here. In addition, our development of conditions for equality to hold everywhere in (4.26) is new.

Proof of Theorem 4.4.1. Assume that the hypotheses of Theorem 4.4.1 hold. Since $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$ for some $\upsilon > 0$, we have by Lemma 4.3.1 that $\zeta(t) \in \mathbf{K}^{\mathbf{I}}_{\upsilon_t^*}$ for all $t \geq 0$, where υ_t^* is given in Lemma 4.3.1. Hence, $\zeta(t) \in \mathbf{K}^{\mathbf{I}} \cap \mathbf{M}^{\mathbf{I}}_1$ for all $t \geq 0$. It follows that for each $i \in I$ and $t \geq 0$, $x \to \overline{M}^i_t(x)$ is continuous and integrable with respect to Lebesgue measure (with integral equal to $\langle \chi, \zeta_i(t) \rangle < \infty$) on $\mathbb{R}_+$. Also, under the assumptions on ϑ_i, including Assumption 4.2, we have that $\vartheta_i \in \mathbf{K}$ and $x \to \overline{N}_i(x)$ is continuous and integrable with respect to Lebesgue measure (with integral equal to $\langle \chi, \vartheta_i \rangle < \infty$) on $\mathbb{R}_+$.

Fix $i \in I$. By the upper semicontinuity of $\mathcal{K}^{\zeta}_i(\cdot)$ (see Lemma 4.4.5), this function is Borel measurable on $[0,t]$ for each $t \geq 0$. To prove the absolute continuity of $\mathcal{H}^{\zeta}_i(\cdot)$, it suffices to prove that $\mathcal{K}^{\zeta}_i(\cdot)$ is integrable over $[0,t]$ and that (4.25) holds, for each $t \geq 0$.

We first prove that if $0 \leq a < b < \infty$ such that $z_i(s) \neq 0$ for all $s \in [a,b]$, then $\mathcal{K}^{\zeta}_i(\cdot)$ is integrable on $[a,b]$ and

$$\mathcal{H}^{\zeta}_i(b) - \mathcal{H}^{\zeta}_i(a) = \int_a^b \mathcal{K}^{\zeta}_i(s) ds. \tag{4.57}$$

Assuming we have such $a < b$, note that by the last part of Lemma 4.4.5, $\mathcal{K}^{\zeta}_i$ is continuous on $[a,b]$ and hence integrable there. To prove that (4.57) holds, recall the form of $\mathcal{K}^{\zeta}_i(\cdot)$ from (4.23). Using the facts that $\Lambda_i(\cdot) \leq \max_j C_j$; $z_i(\cdot)$ is bounded on $[a,b]$, being continuous there; $\frac{\overline{M}^i_s(\cdot)}{\overline{N}^e_i(\cdot)} \leq \upsilon_b^*$ for all $s \in [a,b]$, by Lemma 4.3.1; $\left|\frac{\Lambda_i(\cdot)}{z_i(\cdot)}\right|$ is bounded on $[a,b]$ since $z_i(\cdot)$ is continuous and strictly positive there; and $\int_0^\infty \overline{N}_i(x)dx = \langle \chi, \vartheta_i \rangle = \mu_i^{-1} < \infty$; we see that by dominated convergence,

$$\int_a^b \mathcal{K}_i^{\zeta}(s)ds = -\frac{\kappa_i}{\rho_i^{\alpha}}\int_a^b \Lambda_i(s)(z_i(s))^{\alpha}ds \tag{4.58}$$
$$+\lim_{\ell\to\infty}\frac{\kappa_i}{\rho_i^{\alpha}}\int_a^b\int_{\frac{1}{\ell}}^{\ell}\left(\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\right)^{\alpha}\left(\frac{-\Lambda_i(s)}{z_i(s)}\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\frac{\alpha\overline{N}_i(x)}{\langle\chi,\vartheta_i\rangle}+\nu_i(\alpha+1)\overline{N}_i(x)\right)dxds.$$

Now, for positive integers ℓ and $n>\ell$, by Assumption 4.2 and since $\zeta(s)\in\mathbf{K}^{\mathbf{I}}_{\upsilon_b^*}$, we have $\vartheta_i\in\mathbf{K}\cap\mathbf{M}_1$ and $\zeta_i(s)\in\mathbf{K}\cap\mathbf{M}_1$ for all $s\in[a,b]$. Then by Lemma 4.9.1, we have that as $n\to\infty$, $\langle\chi,\vartheta_i^n\rangle\to\langle\chi,\vartheta_i\rangle>0$, $\overline{N}_i^n(x)\to\overline{N}_i(x)$, $\overline{N}_i^{n,e}(x)\to\overline{N}_i^e(x)$ and $\overline{M}_s^{i,n}(x)\to\overline{M}_s^i(x)$ for each $x\in\mathbb{R}_+$, $s\in[a,b]$. Furthermore, for $s\in[a,b]$, since $\zeta(s)\in\mathbf{K}^{\mathbf{I}}_{\upsilon_b^*}$, we have $\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\le\upsilon_b^*$ for all $x\in\mathbb{R}_+$, and then by Lemma 4.9.2 there is a positive integer $n_{a,b}$ such that for all $n\ge n_{a,b}$, $\frac{\overline{M}_s^{i,n}(x)}{\overline{N}_{i,n}^e(x)}\le 2\upsilon_b^*$ for all $x\in\mathbb{R}_+$ and $s\in[a,b]$. It then follows by the dominated convergence theorem (using the fact from Proposition 4.9.2 that $\overline{N}_i^n(x)\le\overline{N}_i((x-1)^+$, where the latter is integrable over $x\in[\frac{1}{\ell},\ell]$), that for each fixed positive integer ℓ,

$$\int_a^b\int_{\frac{1}{\ell}}^{\ell}\left(\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\right)^{\alpha}\left(\frac{-\Lambda_i(s)}{z_i(s)}\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\frac{\alpha\overline{N}_i(x)}{\langle\chi,\vartheta_i\rangle}+\nu_i(\alpha+1)\overline{N}_i(x)\right)dxds$$
$$=\lim_{n\to\infty}\int_a^b\int_{\frac{1}{\ell}}^{\ell}\left(\frac{\overline{M}_s^{i,n}(x)}{\overline{N}_i^{n,e}(x)}\right)^{\alpha}\left(\frac{-\Lambda_i(s)}{z_i(s)}\frac{\overline{M}_s^{i.n}(x)}{\overline{N}_i^{n,e}(x)}\frac{\alpha\overline{N}_i^n(x)}{\langle\chi,\vartheta_i^n\rangle}+\nu_i(\alpha+1)\overline{N}_i^n(x)\right)dxds. \tag{4.59}$$

Using integration by parts on the first term in the integral in (4.59), and the fact that

$$\frac{\partial}{\partial x}\left(\overline{N}_i^{n,e}(x)\right)^{-\alpha}=\frac{\alpha\overline{N}_i^n(x)}{\left(\overline{N}_i^{n,e}(x)\right)^{\alpha+1}\langle\chi,\vartheta_i^n\rangle},$$

the last line in (4.59) is equal to

$$\begin{aligned}
&\lim_{n\to\infty}\Bigg(\int_a^b\left(\frac{\Lambda_i(s)}{z_i(s)}\right)[-(\overline{M}_s^{i,n}(\cdot))^{\alpha+1}(\overline{N}_i^{n,e}(\cdot))^{-\alpha}]_{\frac{1}{\ell}}^{\ell}ds\\
&\quad+(\alpha+1)\int_a^b\int_{\frac{1}{\ell}}^{\ell}\left(\frac{\overline{M}_s^{i,n}(x)}{\overline{N}_i^{n,e}(x)}\right)^{\alpha}\left(\frac{\Lambda_i(s)}{z_i(s)}\frac{\partial\overline{M}_s^{i,n}(x)}{\partial x}+\nu_i\overline{N}_i^n(x)\right)dxds\Bigg)\\
&=\lim_{n\to\infty}\Bigg(\int_a^b\left(\frac{\Lambda_i(s)}{z_i(s)}\right)[-(\overline{M}_s^{i,n}(\cdot))^{\alpha+1}(\overline{N}_i^{n,e}(\cdot))^{-\alpha}]_{\frac{1}{\ell}}^{\ell}ds\\
&\quad+(\alpha+1)\int_a^b\int_{\frac{1}{\ell}}^{\ell}\left(\frac{\overline{M}_s^{i,n}(x)}{\overline{N}_i^{n,e}(x)}\right)^{\alpha}\left(\frac{\partial\overline{M}_s^{i,n}(x)}{\partial s}\right)dxds\Bigg),
\end{aligned}\tag{4.60}$$

where we have used Proposition 4.9.3 for the last equality. By Fubini's theorem, the above is equal to the expression immediately below. For this use of Fubini's theorem, the joint measurability and absolute integrability of the integrand for each fixed $n\geq\max(n_{a,b},\ell+1)$ follow from Remark 4.9.1 and the fact that $\frac{\overline{M}_s^{i,n}(x)}{\overline{N}_i^{n,e}(x)}\leq\upsilon_b^*$ for all $x\in[\frac{1}{\ell},\ell],s\in[a,b]$.

$$\begin{aligned}
&\lim_{n\to\infty}\left(\int_a^b\left(\frac{\Lambda_i(s)}{z_i(s)}\right)\left[-\frac{(\overline{M}_s^{i,n}(\ell))^{\alpha+1}}{(\overline{N}_i^{n,e}(\ell))^{\alpha}}+\frac{\left(\overline{M}_s^{i,n}\left(\frac{1}{\ell}\right)\right)^{\alpha+1}}{\left(\overline{N}_i^{n,e}\left(\frac{1}{\ell}\right)\right)^{\alpha}}\right]ds\right.\\
&\quad\left.+\int_{\frac{1}{\ell}}^{\ell}\left((\overline{M}_b^{i,n}(x))^{\alpha+1}-(\overline{M}_a^{i,n}(x))^{\alpha+1}\right)\left(\overline{N}_i^{n,e}(x)\right)^{-\alpha}dx\right)\\
&=\int_a^b\left(\frac{\Lambda_i(s)}{z_i(s)}\right)\left[-\frac{(\overline{M}_s^{i}(\ell))^{\alpha+1}}{\left(\overline{N}_i^{e}(\ell)\right)^{\alpha}}+\frac{\left(\overline{M}_s^{i}\left(\frac{1}{\ell}\right)\right)^{\alpha+1}}{\left(\overline{N}_i^{e}\left(\frac{1}{\ell}\right)\right)^{\alpha}}\right]ds\\
&\quad+\int_{\frac{1}{\ell}}^{\ell}\left((\overline{M}_b^{i}(x))^{\alpha+1}-(\overline{M}_a^{i}(x))^{\alpha+1}\right)\left(\overline{N}_i^{e}(x)\right)^{-\alpha}dx,
\end{aligned}\tag{4.61}$$

where we have used dominated convergence, provided by Lemmas 4.9.1 and 4.9.2, to pass to the limit for the last equality. Note that as $\ell\to\infty$, $\overline{M}_s^i(\ell)\to 0$, $\overline{M}_s^i\left(\frac{1}{\ell}\right)\to z_i(s)$, $\overline{N}_i^e\left(\frac{1}{\ell}\right)\to 1$ and $\overline{M}_s^i(x)\leq z_i(s)$, $\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\leq\upsilon_b^*$ for all $s\in[a,b],x\in\mathbb{R}_+$. Combining this with the fact that for $s=a,b$, $\left|\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\right|\leq\upsilon_b^*$ for all $x\in\mathbb{R}_+$ and $x\to\overline{M}_s^i(x)$ is integrable on $\mathbb{R}_+$, we see by dominated convergence that as $\ell \to \infty$, the above expression converges to

$$\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)}\right)(z_i(s))^{\alpha+1}ds + \int_0^\infty \left((\overline{M}_b^i(x))^{\alpha+1} - (\overline{M}_a^i(x))^{\alpha+1}\right)\left(\overline{N}_i^e(x)\right)^{-\alpha}dx. \quad (4.62)$$

On substituting the above into (4.58), we obtain

$$\begin{aligned}\int_a^b \mathcal{K}_i^\zeta(s)ds &= -\frac{\kappa_i}{\rho_i^\alpha}\int_a^b \Lambda_i(s)(z_i(s))^\alpha ds + \frac{\kappa_i}{\rho_i^\alpha}\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)}\right)(z_i(s))^{\alpha+1}ds \\ &\quad + \frac{\kappa_i}{\rho_i^\alpha}\int_0^\infty \left((\overline{M}_b^i(x))^{\alpha+1} - (\overline{M}_a^i(x))^{\alpha+1}\right)\left(\overline{N}_i^e(x)\right)^{-\alpha}dx \\ &= \mathcal{H}_i^\zeta(b) - \mathcal{H}_i^\zeta(a), \end{aligned} \quad (4.63)$$

as desired.

We now turn to proving that $\mathcal{K}_i^\zeta(\cdot)$ is integrable over $[0,t]$ and (4.25) holds for each $t \geq 0$. This clearly holds for $t = 0$, so we consider $t > 0$ fixed. If $z_i(s) \neq 0$ for all $s \in [0,t]$, then the result follows immediately from what we proved for (4.57) with $a = 0$ and $b = t$. So we only need to treat the case where $z_i(s) = 0$ for some $s \in [0,t]$. Assuming this, let $s^* = \inf\{s \in [0,t] : z_i(s) = 0\}$ and $t^* = \sup\{s \in [0,t] : z_i(s) = 0\}$. Then, $0 \leq s^* \leq t^* \leq t$, $z_i(s^*) = z_i(t^*) = 0$ and $z_i(s) > 0$ for $s \in (0,s^*) \cup (t^*,t)$. (Note that the interval $(0,s^*)$ is empty if $z_i(0) = 0$ and (t^*,t) is empty if $z_i(t) = 0$.) In any event, we can write the open set $\mathcal{T}_t^i = \{s \in (0,t) : z_i(s) > 0\}$ as a (finite or countable) union of disjoint open intervals:

$$\mathcal{T}_t^i = (0,s^*) \cup \left(\bigcup_n (s_n,t_n)\right) \cup (t^*,t), \quad (4.64)$$

where $\bigcup_n (s_n,t_n) \subset (s^*,t^*)$ and $z_i(s_n) = z_i(t_n) = 0$ for each n.

Recall the definitions of $\mathrm{k}_i^{(1)}, \mathrm{k}_i^{(2)}, \mathrm{k}_i^{(3)}$ from the proof of Lemma 4.4.3. For all $s \geq 0$, let

$$\begin{aligned}
k_i^{(1)}(s) &= \mathrm{k}_i^{(1)}(\zeta(s)) = -\kappa_i \rho_i^{-\alpha} \Lambda_i(s) \big(z_i(s)\big)^{\alpha}, \\
k_i^{(2)}(s) &= \mathrm{k}_i^{(2)}(\zeta(s)) = -\kappa_i \rho_i^{-\alpha} \int_0^\infty \left(\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)} \right)^{\alpha+1} \overline{N}_i(x) \frac{\alpha \Lambda_i(s)}{z_i(s)\langle \chi, \vartheta_i \rangle} \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) dx, \\
k_i^{(3)}(s) &= \mathrm{k}_i^{(3)}(\zeta(s)) = \kappa_i \rho_i^{-\alpha} \nu_i (\alpha+1) \int_0^\infty \left(\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)} \right)^{\alpha} \overline{N}_i(x) \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) dx. \qquad (4.65)
\end{aligned}$$

Then we have $|k_i^{(1)}(s)| \leq \kappa_i \rho_i^{-\alpha} (\max_{j \in \mathcal{J}} C_j) \sup_{s \in [0,t]} (z_i(s))^{\alpha}$ for $s \in [0,t]$, which implies that $\int_{[0,t]} |k_i^{(1)}(s)| ds < \infty$. By Lemma 4.3.1, $|k_i^{(3)}(s)| \leq \kappa_i \rho_i^{-\alpha} (\alpha+1) \nu_i (\upsilon_t^*)^{\alpha} \langle \chi, \vartheta_i \rangle$ for $s \in [0,t]$, which implies that $\int_{[0,t]} |k_i^{(3)}(s)| ds < \infty$.

For each fixed n, equation (4.57) gives that for any $[a,b] \subset (s_n, t_n)$

$$\int_{[a,b]} k_i^{(1)}(s) ds + \int_{[a,b]} k_i^{(2)}(s) ds + \int_{[a,b]} k_i^{(3)}(s) ds = \mathcal{H}_i^{\zeta}(b) - \mathcal{H}_i^{\zeta}(a).$$

Thus,

$$-\int_{[a,b]} k_i^{(2)}(s) ds = \int_{[a,b]} k_i^{(1)}(s) ds + \int_{[a,b]} k_i^{(3)}(s) ds + \mathcal{H}_i^{\zeta}(a) - \mathcal{H}_i^{\zeta}(b).$$

By the continuity of $\mathcal{H}_i^{\zeta}(\cdot)$ established in Lemma 4.4.2, as $a \to s_n$ and $b \to t_n$, $\mathcal{H}_i^{\zeta}(b) \to \mathcal{H}_i^{\zeta}(t_n) = 0$ and $\mathcal{H}_i^{\zeta}(a) \to \mathcal{H}_i^{\zeta}(s_n) = 0$. It follows from the above and since $k_2(s) \leq 0$ for all $s \geq 0$, that

$$\begin{aligned}
\int_{(s_n,t_n)} |k_2(s)| ds &= -\int_{(s_n,t_n)} k_i^{(2)}(s) ds && (4.66) \\
&= \int_{(s_n,t_n)} k_i^{(1)}(s) ds + \int_{(s_n,t_n)} k_i^{(3)}(s) ds + \mathcal{H}_i^{\zeta}(s_n) - \mathcal{H}_i^{\zeta}(t_n) && (4.67) \\
&= \int_{(s_n,t_n)} k_i^{(1)}(s) ds + \int_{(s_n,t_n)} k_i^{(3)}(s) ds. && (4.68)
\end{aligned}$$

In a similar manner, we can obtain

$$\int_{(0,s^*)} |k_2(s)| ds = -\int_{(0,s^*)} k_i^{(2)}(s) ds = \int_{(0,s^*)} k_i^{(1)}(s) ds + \int_{(0,s^*)} k_i^{(3)}(s) ds + \mathcal{H}_i^{\zeta}(0), \qquad (4.69)$$

since $\mathcal{H}_i^{\zeta}(s^*) = 0$, and

$$\int_{(t^*,t)} |k_2(s)|ds = -\int_{(t^*,t)} k_i^{(2)}(s)ds = \int_{(t^*,t)} k_i^{(1)}(s)ds + \int_{(t^*,t)} k_i^{(3)}(s)ds - \mathcal{H}_i^{\zeta}(t), \qquad (4.70)$$

since $\mathcal{H}_i^{\zeta}(t^*) = 0$. Hence using the integrability of $k_i^{(1)}$ and $k_i^{(3)}$ on $[0,t]$, the fact that $k_i^{(2)}$ is zero on $(0,t) \setminus \mathcal{T}_t^i$ and non-positive on $\mathcal{T}_t^i$, together with the disjointness of the intervals in the representation (4.64) for $\mathcal{T}_t^i$, we have $\int_{(0,t)} |k_i^{(2)}(s)|ds < \infty$. Thus, $\mathcal{K}_i^{\zeta} = k_i^{(1)} + k_i^{(2)} + k_i^{(3)}$ is integrable on $(0,t)$, and by (4.68)–(4.70), we have

$$\int_{(s_n,t_n)} \mathcal{K}_i^{\zeta}(s)ds = 0 \text{ for each } n, \qquad (4.71)$$

$$\int_{(0,s^*)} \mathcal{K}_i^{\zeta}(s)ds = -\mathcal{H}_i^{\zeta}(0) \text{ and } \int_{(t^*,t)} \mathcal{K}_i^{\zeta}(s)ds = \mathcal{H}_i^{\zeta}(t). \qquad (4.72)$$

Combining all of the above, and using the integrability of $\mathcal{K}_i^{\zeta}$ on $[0,t]$, the fact that $\mathcal{K}_i^{\zeta}(\cdot)$ is zero on $(0,t)\setminus\mathcal{T}_t^i$ and the disjointness of the intervals in the representation (4.64), we have

$$\begin{aligned} \int_0^t \mathcal{K}_i^{\zeta}(s)ds &= \int_{(0,s^*)} \mathcal{K}_i^{\zeta}(s)ds + \sum_n \int_{(s_n,t_n)} \mathcal{K}_i^{\zeta}(s)ds + \int_{(t^*,t)} \mathcal{K}_i^{\zeta}(s)ds \\ &= -\mathcal{H}_i^{\zeta}(0) + 0 + \mathcal{H}_i^{\zeta}(t), \end{aligned}$$

which is the desired result (3.16).

The inequality (4.26) follows immediately from Lemma 4.4.4 with $\xi = \zeta(t)$ where $\zeta(t) \in \mathbf{K}^{\mathbf{I}}_{\upsilon_t^*} \subset \mathbf{M}^{\mathbf{I}}_{\upsilon_t^*}$. By Lemma 4.4.4, equality holds everywhere in (4.26) if and only if $\zeta(t) \in \mathcal{M}^*$. Furthermore, since $\zeta(t) \in \mathbf{K}^{\mathbf{I}}_{\upsilon_t^*}$, its components cannot have atoms at zero and so the $\mathcal{M}^*$ can be replaced by $\mathcal{M}$ in this "if and only if statement" just stated. The non-positivity of $\mathcal{K}^{\zeta}(\cdot)$ yields the non-increasing property of $\mathcal{H}^{\zeta}(\cdot)$, and the fact from that $\mathcal{K}^{\zeta}(t) < 0$ at times $t \in [0,\infty)$ where $\zeta(t) \notin \mathcal{M}$ yields that $\mathcal{H}^{\zeta}(\cdot)$ is strictly decreasing at such times. $\square$

4.10 Properties of Workload, H, $\underline{F}$, G, and Total Mass for Fluid Model Solutions

In this section, we develop some properties of fluid model solutions and the relationship between H and $\underline{F}$ that will be needed for the proofs of our main results.

4.10.1 Properties of Workload

Lemma 4.10.1. *Suppose that Assumption 4.1 holds and ζ is a fluid model solution satisfying $w_i(0) < \infty$ for all $i \in I$. Then $t \to \widetilde{w}_j(\zeta(t))$ is a non-decreasing function on $[0,\infty)$ for each $j \in \mathcal{J}_*$.*

Proof. For $j \in \mathcal{J}_*$, by (2.5), Definition 2.3.1 and Assumption 4.1, we have for each $t \geq 0$,

$$\begin{aligned}
\widetilde{w}_j(\zeta(t)) &= \sum_{i\in I} R_{ji} w_i(t) \\
&= \sum_{i\in I} R_{ji}\Big(w_i(0) + \int_0^t \big(\rho_i - \Lambda_i(s)\big)\mathbb{1}_{(0,\infty)}\big(z_i(s)\big)ds\Big) \\
&= \widetilde{w}_j(\zeta(0)) + \sum_{i\in I} R_{ji}\rho_i t - \sum_{i\in I} R_{ji}\tau_i(t) \\
&= \widetilde{w}_j(\zeta(0)) + u_j(t).
\end{aligned}$$

The desired result follows from the fact that $u_j(\cdot)$ is non-decreasing, by Definition 2.3.2(ii) for a fluid model solution. □

Lemma 4.10.2. *Suppose Assumptions 4.1 and 4.2 hold. Let $\upsilon > 0$. Then, for any fluid model solution ζ satisfying $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$, we have*

$$\sup_{t\geq 0}\max_{i\in I} w_i(t) \leq B_\upsilon, \tag{4.73}$$

where B_υ is a finite, positive constant depending only on $\upsilon, \alpha, \rho, \kappa, \langle \chi, \vartheta^e\rangle$.

Proof. Fix $i \in I$ and $t \geq 0$. Then

$$\begin{aligned} w_i(t) &= \int_0^\infty \overline{M}_t^i(x)dx \\ &= \langle \chi, \vartheta_i^e \rangle \int_0^\infty \frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \frac{\overline{N}_i^e(x)}{\langle \chi, \vartheta_i^e \rangle} dx \\ &\leq \langle \chi, \vartheta_i^e \rangle \left(\int_0^\infty \left(\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \right)^{\alpha+1} \frac{\overline{N}_i^e(x)}{\langle \chi, \vartheta_i^e \rangle} dx \right)^{\frac{1}{\alpha+1}}, \end{aligned} \tag{4.74}$$

where the last inequality follows from Jensen's inequality, since $\frac{\overline{N}_i^e(\cdot)}{\langle \chi, \vartheta_i^e \rangle}$ is a probability density (for the probability measure $(\vartheta_i^e)^e$). We observe that the last line in (4.74) equals

$$\left(\frac{\rho_i^\alpha \mathcal{H}_i^\zeta(t) \langle \chi, \vartheta_i^e \rangle^\alpha}{\kappa_i} \right)^{\frac{1}{\alpha+1}}. \tag{4.75}$$

By the definition of $\mathcal{H}^\zeta(\cdot)$, $\mathcal{H}_i^\zeta(t) \leq (\alpha+1)\mathcal{H}^\zeta(t)$. By Theorem 4.4.1, $\mathcal{H}^\zeta(\cdot)$ is non-increasing and so $\mathcal{H}^\zeta(t)$ is bounded above by $\mathcal{H}^\zeta(0)$ and hence (4.75) is bounded above by

$$\left(\frac{\rho_i^\alpha (\alpha+1) \mathcal{H}^\zeta(0) \langle \chi, \vartheta_i^e \rangle^\alpha}{\kappa_i} \right)^{\frac{1}{\alpha+1}} \leq \left(\frac{\rho_i^\alpha}{\kappa_i} \langle \chi, \vartheta_i^e \rangle^\alpha \upsilon^{\alpha+1} \sum_{k \in I} \frac{\kappa_k \langle \chi, \vartheta_k^e \rangle}{\rho_k^\alpha} \right)^{\frac{1}{\alpha+1}},$$

where we have used the fact that $\zeta(0) \in \mathbf{K}_\upsilon^\mathbf{I}$ for the last inequality. The desired result follows because $i \in I$ and $t \geq 0$ were arbitrary and by taking the maximum over $i \in I$. $\square$

4.10.2 Relationship between H and $\underline{F}$

Lemma 4.10.3. *Let* $\xi \in \cup_{\upsilon > 0} \mathbf{M}_\upsilon^\mathbf{I}$. *Then*

$$\underline{F}\left(\widetilde{w}(\xi)\right) \leq F(\check{z}) \leq H(\xi), \tag{4.76}$$

where $\check{z}_i = \frac{\langle \chi, \xi_i \rangle}{\langle \chi, \vartheta_i^e \rangle}$ *for* $i \in I$ *and* $\widetilde{w}_j(\xi) = \sum_{i \in I} R_{ji} \langle \chi, \xi_i \rangle$ *for* $j \in \mathcal{J}_*$*. If, in addition, Assumption 4.1 holds, then the inequalities in (4.76) are all equalities if and only if* $\xi \in \mathcal{M}^*$*.*

Proof. We have

$$\begin{aligned}
H(\xi) &= \sum_{i \in I} \frac{\kappa_i \langle \chi, \vartheta_i^e \rangle}{(\alpha+1)\rho_i^\alpha} \int_0^\infty \left(\frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle} \right)^{\alpha+1} \frac{\overline{N}_i^e(x)}{\langle \chi, \vartheta_i^e \rangle} dx \\
&\geq \sum_{i \in I} \frac{\kappa_i \langle \chi, \vartheta_i^e \rangle}{(\alpha+1)\rho_i^\alpha} \left(\int_0^\infty \frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle} \frac{\overline{N}_i^e(x)}{\langle \chi, \vartheta_i^e \rangle} dx \right)^{\alpha+1} \quad \text{by Jensen's Inequality} \quad (4.77) \\
&= \sum_{i \in I} \frac{\kappa_i \langle \chi, \vartheta_i^e \rangle}{(\alpha+1)\rho_i^\alpha} \left(\frac{\langle \chi, \xi_i \rangle}{\langle \chi, \vartheta_i^e \rangle} \right)^{\alpha+1} \\
&= F(\check{z}) \\
&\geq \underline{F}(\widetilde{w}(\xi)), \qquad (4.78)
\end{aligned}$$

where the last inequality follows because $\check{w}(\check{z}) = \widetilde{w}(\xi)$ for $\check{w}(\cdot)$ defined as in Lemma 4.6.1, and so $\check{z}$ is feasible for the optimization problem (4.27) for which $\underline{F}(\widetilde{w}(\xi))$ is the optimal value. The stream of inequalities above establishes (4.76).

We now assume that Assumption 4.1 holds and characterize when equality holds everywhere in (4.76). By the sharp version of Jensen's inequality, equality holds in (4.77) if and only if $x \to \frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle}$ is a constant for $x \in \mathbb{R}_+$ such that $\overline{N}_i^e(x) \neq 0$. It follows that equality holds in (4.77) if and only if for each $i \in I$, $\xi_i = a_i \delta_0 + b_i \vartheta_i^e$ for some $a_i, b_i \in [0, \infty)$. For ξ of this form, $\check{z} = b = (b_1, \ldots, b_{\mathbf{I}})$. The inequality in (4.78) is an equality if and only if $\check{z}$ is the optimal solution for the optimization problem (4.27) with $\widetilde{w} = \widetilde{w}(\xi) = \check{w}(\check{z})$, i.e., $\check{z} = \Delta(\check{w}(\check{z}))$. It then follows from Lemma 4.6.1, which requires Assumption 4.1, that the inequality in (4.78) is an equality if and only if $\check{z} \in \mathcal{P}$. Hence, by the definition of $\mathcal{M}^*$, both inequalities in (4.76) are equalities if and only if $\xi \in \mathcal{M}^*$, as defined in (4.20). □

4.10.3 Properties of G: Proof of Lemma 4.7.1

Proof of Lemma 4.7.1. For (i), fix $\upsilon > 0$. If $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$, then $H(\xi)$ is finite and $w_i(\xi) \leq \upsilon\langle \chi, \vartheta_i^e \rangle < \infty$ for all $i \in I$, and so $\widetilde{w}_j(\xi) < \infty$ for each $j \in \mathcal{J}_*$, since $|I|$ is finite. It follows that $G(\xi)$ is well defined and finite. By Lemma 4.10.3, $G(\xi) \geq 0$. For the continuity, suppose $\{\xi_n\}_{n\in\mathbb{N}}, \xi$ are in $\mathbf{M}_\upsilon^{\mathbf{I}}$ and $\xi_n \xrightarrow{w} \xi$ as $n \to \infty$. Then $\langle \mathbb{1}_{(x,\infty)}, \xi_n \rangle \to \langle \mathbb{1}_{(x,\infty)}, \xi \rangle$ for almost every $x \in \mathbb{R}_+$ where $\langle \mathbb{1}_{(x,\infty)}, \xi_n \rangle \leq \upsilon\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle$ for each n and x, and so it follows by dominated convergence that $w_i(\xi_n) = \int_0^\infty \langle \mathbb{1}_{(x,\infty)}, \xi_n \rangle dx \to w_i(\xi) = \int_0^\infty \langle \mathbb{1}_{(x,\infty)}, \xi \rangle dx$ and $\widetilde{w}_j(\xi_n) \to \widetilde{w}_j(\xi)$ as $n \to \infty$ for each $i \in I, j \in \mathcal{J}_*$. It then follows from the continuity of H on $\mathbf{M}_\upsilon^{\mathbf{I}}$ (see Lemma 4.4.1) and of $\underline{F}$ on $\mathbb{R}_+^{\mathbf{J}_*}$ (see Proposition 4.4.1) that $G(\xi_n) \to G(\xi)$ as $n \to \infty$. Hence G is continuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$.

For (ii), assume that Assumption 4.1 holds and suppose that $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$ for some $\upsilon > 0$. Noting that $G(\xi) = 0$ if and only if equality holds everywhere in (4.76), we conclude from the last part of Lemma 4.10.3 (which assumes that Assumption 4.1 holds) that $G(\xi) = 0$ if and only if $\xi \in \mathcal{M}^*$. $\square$

4.10.4 Property of Total Mass

The next lemma is an important element in our proof of the convergence of fluid model solutions to the invariant manifold. The proof, in part, uses some ideas from the proof of Lemma 5.1 in the paper of Puha and Williams [PW16] for a critical fluid model of a single class processor sharing queue. However, the proof given here also has new elements needed to treat general bandwidth sharing policies, which allocate bandwidth to routes in a utility-based, state-dependent manner, whereas for the single class processor sharing queue situation treated in [PW16], the bandwidth allocated to the class is always one.

Lemma 4.10.4. *Suppose Assumptions 4.1 and 4.2 hold. Fix $\upsilon > 0$. Then, for any fluid model* *solution* ζ *with* $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$*, we have*

$$\sup_{t \geq 0} \max_{i \in I} z_i(t) \leq \widetilde{B}_{\upsilon}, \tag{4.79}$$

where $z_i(t) = \langle \mathbb{1}, \zeta_i(t) \rangle$*,* $i \in I$*,* $t \geq 0$*, and* $\widetilde{B}_{\upsilon}$ *is a finite, positive constant depending only on* $\upsilon, \alpha, \nu, \rho, C, \kappa, \langle \chi, \vartheta^e \rangle$.

Proof. It is apparent from the form of the objective function in the optimization problem (2.2) that we have the scaling property:

$$\phi_i(rz) = \phi_i(z) \quad \text{for all } i \in I, z \in \mathbb{R}^{\mathbf{I}}_{+} \text{ and } r > 0. \tag{4.80}$$

Fix $\upsilon > 0$. Consider a fluid model solution ζ with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$. By Lemma 4.10.2, we know that

$$\langle \chi, \zeta_i(t) \rangle \leq B_{\upsilon} \quad \text{for all } t \geq 0,\ i \in I. \tag{4.81}$$

Let $\nu^* = \max_{i \in I} \nu_i$ and

$$\gamma = \min_{i \in I} \min \left\{ \phi_i(z) : z \in \mathbb{R}^{\mathbf{I}}_{+}, z_i \geq \frac{1}{4}, z_k \leq \frac{3}{2} \text{ for all } k \in I \right\}. \tag{4.82}$$

We note from the properties of ϕ described in Remark 2.2.1 that, for each $i \in I$, ϕ_i is continuous and strictly positive on the compact set

$$\left\{ z \in \mathbb{R}^{\mathbf{I}}_{+} : z_i \geq \frac{1}{4}, z_k \leq \frac{3}{2} \text{ for all } k \in I \right\},$$

and so $\gamma > 0$. Furthermore, from the scaling property (4.80) of ϕ, we have that for each $a > 0$,

$$\gamma = \min_{i \in I} \min \left\{ \phi_i(z) : z \in \mathbb{R}^{\mathbf{I}}_{+}, z_i \geq \frac{a}{4}, z_k \leq \frac{3a}{2} \text{ for all } k \in I \right\}. \tag{4.83}$$

Let $\beta = \frac{B_\upsilon \nu^*}{\gamma}$ and $f(x) = x^2 - 6\beta x + \beta^2$. The quadratic function f has two roots, the largest of which is $x^* = \beta(3+2\sqrt{2})$, and so $f(x) \geq 0$ for $x \geq x^*$. Let $a^* = \max(\upsilon, x^*)$, $\ell = (a^* - \beta)/2\nu^*$, and $b^* = a^* + \nu^* \ell$. Then $f(a^*) \geq 0$, $\ell > 0$, $\nu^* \ell \leq \frac{a^*}{2}$, and $b^* \leq \frac{3a^*}{2}$.

We shall prove the following: for $n = 0, 1, 2, \ldots$, for each $i \in I$,

$$
\begin{aligned}
z_i(n\ell) &\leq a^* \text{ and} && (4.84)\\
z_i(t) &\leq b^* \text{ for all } t \in [n\ell, (n+1)\ell]. && (4.85)
\end{aligned}
$$

Once this is proved, we obtain that

$$
\sup_{t \in [0,\infty)} \sup_{i \in I} z_i(t) \leq b^*, \tag{4.86}
$$

and the desired result holds with $\tilde{B}_\upsilon = b^*$.

We shall prove (4.84)–(4.85) by induction. Before commencing that proof, we first prove some preliminary estimates that hold for all $n = 0, 1, 2, \ldots$. For this, fix $n \in \{0, 1, 2, \ldots\}$ and $i \in I$. We consider two cases:

(I) $z_i(s) \neq 0$ for all $s \in [n\ell, (n+1)\ell]$,

(II) $z_i(s) = 0$ for some $s \in [n\ell, (n+1)\ell]$.

In case (I), by Proposition 4.2.3, on setting $x = 0$ in (4.9), we have for $t \in [n\ell, (n+1)\ell]$,

$$
\begin{aligned}
z_i(t) = \overline{M}^i_t(0) &= \overline{M}^i_{n\ell}(S^i_{n\ell,t}) + \nu_i \int_{n\ell}^t \overline{N}_i(S^i_{u,t}) du \\
&\leq \overline{M}^i_{n\ell}(S^i_{n\ell,t}) + \nu^* \ell && (4.87)\\
&\leq \frac{\int_0^{S^i_{n\ell,t}} \overline{M}^i_{n\ell}(x) dx}{S^i_{n\ell,t}} + \nu^* \ell, && (4.88)\\
&\leq \frac{w_i(n\ell)}{S^i_{n\ell,t}} + \nu^* \ell, && (4.89)
\end{aligned}
$$

where we used the non-increasing property of $\overline{M}^i_{n\ell}(\cdot)$ for the inequality in (4.88). Setting $t=(n+1)\ell$ in (4.89), we obtain

$$z_i((n+1)\ell) \leq \frac{w_i(n\ell)}{S^i_{n\ell,(n+1)\ell}} + \nu^*\ell \leq \frac{B_\upsilon}{S^i_{n\ell,(n+1)\ell}} + \nu^*\ell, \tag{4.90}$$

where we used Lemma 4.10.2 for the last inequality.

Thus, in case (I), if $z_i(n\ell) \leq a^*$, then by (4.87), since $\overline{M}^i_{n\ell}(\cdot)$ is non-increasing, we have for all $t \in [n\ell,(n+1)\ell]$:

$$z_i(t) \leq \overline{M}^i_{n\ell}(0) + \nu^*\ell \leq a^* + \nu^*\ell = b^*. \tag{4.91}$$

Hence, we see that in case (I), (4.85) follows once (4.84) is proved.

In case (II), by Proposition 4.2.3, for $t \in [n\ell, s_0)$ where $s_0 = \inf\{s \geq n\ell : z_i(s) = 0\}$, if $z_i(n\ell) \leq a^*$, then

$$\begin{aligned} z_i(t) &\leq z_i(n\ell) + \nu^*\ell && (4.92)\\ &\leq a^* + \nu^*\ell = b^*, && (4.93) \end{aligned}$$

and for any $t \in [s_0,(n+1)\ell]$, either $z_i(t) = 0$ or $z_i(t) > 0$ and by Remark 4.2.3, for $s_t = \sup\{s \in [n\ell,t) : z_i(s) = 0\}$, we have

$$\begin{aligned} z_i(t) &= \nu_i \int_{s_t}^t \overline{N}_i(x + S^i_{u,t})du \\ &\leq \nu^*\ell \\ &\leq b^*. && (4.94) \end{aligned}$$

Thus, in case (II), if $z_i(n\ell) \leq a^*$, then $z_i(t) \leq b^*$ for all $t \in [n\ell,(n+1)\ell]$.

Combining all of the above, we see that in either case (I) or case (II), (4.85) follows once

(4.84) is proved. Also, in case (II),

$$
\begin{aligned}
z_i((n+1)\ell) &\leq \nu_i \int_{s_{(n+1)\ell}}^{(n+1)\ell} \overline{N}_i(x+S^i_{u,t})du \\
&\leq \nu^* \ell \\
&\leq \frac{a^*}{2}.
\end{aligned} \tag{4.95}
$$

We now proceed to the induction proof. Consider first the case of $n=0$. Fix $i \in I$. Then by the definition of a^*, $z_i(0) \leq \upsilon \leq a^*$, and from the consideration of cases (I) and (II) above, it follows that $z_i(t) \leq b^*$ for all $t \in [0,\ell]$. Thus, (4.84) and (4.85) hold for $n=0$ and since $i \in I$ was arbitrary, they hold for all $i \in I$ for $n=0$.

Suppose now for the induction step that (4.84) and (4.85) hold for some $n \geq 0$ for all $i \in I$. We desire to prove that these inequalities hold with $n+1$ in place of n for all $i \in I$. For this, fix $i \in I$. By the consideration of cases (I) and (II) above, we know that it suffices to prove (4.84) holds with $n+1$ in place of n, since (4.85) follows once (4.84) is proved with $n+1$ in place of n. We consider two cases:

(i) $z_i(s) < \frac{a^*}{4}$ for some $s \in [n\ell,(n+1)\ell]$,

(ii) $z_i(s) \geq \frac{a^*}{4}$ for all $s \in [n\ell,(n+1)\ell]$.

Consider case (i) first. If $z_i(s)=0$ for some $s \in [n\ell,(n+1)\ell]$, then we are in case (II) and by (4.95), we have that $z_i((n+1)\ell) \leq a^*/2 < a^*$ and then (4.84) holds. On the other hand, if $z_i(s) \neq 0$ for all $s \in [n\ell,(n+1)\ell]$, then by Proposition 4.2.3 we have

$$
\begin{aligned}
z_i((n+1)\ell) &\leq \overline{M}^i_{(n+1)\ell}(0) = \overline{M}^i_{t_n}(S^i_{t_n,(n+1)\ell}) + \nu_i \int_{t_n}^{(n+1)\ell} \overline{N}_i(S^i_{u,(n+1)\ell})du \\
&\leq z_i(t_n) + \nu^* \ell \\
&\leq \frac{a^*}{4} + \frac{a^*}{2} \\
&< a^*,
\end{aligned} \tag{4.96}
$$

where $t_n = \inf\{s \geq n\ell : z_i(s) \leq \frac{a^*}{4}\}$. Thus, (4.84) holds with $n+1$ in place of n in case (i).

Now we suppose that we are in case (ii). Then, we are also in case (I), and by (4.90) we have that

$$z_i((n+1)\ell) \quad \leq \quad \frac{w_i(n\ell)}{S^i_{n\ell,(n+1)\ell}} + \nu^*\ell \leq \frac{B_\upsilon}{S^i_{n\ell,(n+1)\ell}} + \nu^*\ell, \tag{4.97}$$

where

$$S^i_{n\ell,(n+1)\ell} \quad = \quad \int_{n\ell}^{(n+1)\ell} \frac{\phi_i(z(s))}{z_i(s)} ds \geq \frac{\gamma\ell}{b^*}, \tag{4.98}$$

and we have used the property (4.83) of γ with $a = a^*$, and the facts that for all $s \in [n\ell,(n+1)\ell]$, $z_i(s) \geq \frac{a^*}{4}$ (since we are in case (ii)), and $z_k(s) \leq b^* \leq \frac{3a^*}{2}$ for all $k \in I$ (since (4.85) holds with arbitrary k in place of i, by the induction assumption). Combining (4.97) with (4.98), we obtain

$$\begin{aligned} z_i((n+1)\ell) \quad &\leq \quad \frac{B_\upsilon b^*}{\gamma\ell} + \nu^*\ell \\ &= \quad \frac{1}{4\nu^*\ell}\left(4\beta(a^* + \nu^*\ell) + (2\nu^*\ell)^2\right) \\ &= \quad \frac{1}{2(a^*-\beta)}\left(4\beta a^* + 2\beta(a^*-\beta) + (a^*-\beta)^2\right) \\ &= \quad \frac{1}{2(a^*-\beta)}\left((a^*)^2 + 4\beta a^* - \beta^2\right) \end{aligned} \tag{4.99}$$

where we substituted for b^* and used the definition of β for the second line, substituted for ℓ for the third line, and simplified the expression for the last line. The expression on the right hand side of the inequality in (4.99) is less than or equal to a^* if and only if

$$(a^*)^2 - 6\beta a^* + \beta^2 \geq 0. \tag{4.100}$$

The left hand side of (4.100) is $f(a^*)$ and it follows from the fact $a^* \geq x^*$, the largest root of the quadratic f, that (4.100) holds. It follows that we must have $z_i((n+1)\ell) \leq a^*$. This concludes the proof that (4.84) holds with $n+1$ in place of n in case (ii).

Combining all of the preceding arguments, and using the fact that $i \in I$ was arbitrary, we see that (4.84) (and hence (4.85)) holds with $n+1$ in place of n for all $i \in I$. This completes the induction step and hence (4.84) and (4.85) hold for all $i \in I$ and $n = 0, 1, 2, \ldots$. $\square$

4.11 Proofs of Main Results: Theorems 4.8.1, 4.8.2 and 4.8.3

4.11.1 Proof of Theorem 4.8.1

Proof of Theorem 4.8.1. Property (i) follows by combining Lemma 4.3.1 with Lemma 4.7.1 and the continuity of $\zeta(\cdot)$ on $[0, \infty)$. (We note that this part uses Assumption 4.2, but does not need Assumption 4.1.) For property (ii), for $t \geq 0$, by Lemma 4.3.1 and (ii) of Lemma 4.7.1, $\mathcal{G}^{\zeta}(t) = 0$ if and only $\zeta(t) \in \mathcal{M}^*$. Furthermore, by Lemma 4.3.1, $\zeta(t) \in \mathbf{K}^{\mathbf{I}}_{\upsilon_t^*}$ and so it has no atoms (including no atom at zero). It follows that $\mathcal{M}^*$ can be replaced by $\mathcal{M}$ in the "if and only if" statement. Hence, property (ii) holds. For property (iii), by Theorem 4.4.1, $\mathcal{H}^{\zeta}(\cdot)$ is non-increasing. Furthermore, $\underline{F}(\widetilde{w}(\zeta(\cdot)))$ is non-decreasing by Proposition 4.4.1 and Lemma 4.10.1. Hence $\mathcal{G}^{\zeta}(\cdot)$ is non-increasing. In addition, by Theorem 4.4.1, $\mathcal{H}^{\zeta}(\cdot)$ is strictly decreasing at all $t \geq 0$ such that $\zeta(t) \notin \mathcal{M}$, which implies $\mathcal{G}^{\zeta}(\cdot)$ is strictly decreasing at all $t \geq 0$ such that $\zeta(t) \notin \mathcal{M}$. $\square$

4.11.2 Fluid Model Solutions Stay in Relatively Compact Sets

The next lemma provides a key step in the proof that fluid model solutions stay in certain relatively compact sets.

Lemma 4.11.1. *Suppose Assumptions 4.1 and 4.2 hold. Fix $\upsilon > 0$. For any fluid model solution*

ζ with $\zeta(0) \in \mathbf{K}_\upsilon^\mathbf{I}$ and any $t \geq 0$, let $z^\zeta(t) = \langle \mathbb{1}, \zeta(t) \rangle$. Define

$$M_\upsilon = \sup \left\{ \frac{z_i^\zeta(t)}{\phi_i(z^\zeta(t))} \mathbb{1}_{\{z_i^\zeta(t) \neq 0\}} : i \in I, t \geq 0, \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}_\upsilon^\mathbf{I} \right\}. \tag{4.101}$$

Then $M_\upsilon < \infty$.

Proof. For any fluid model solution ζ with $\zeta(0) \in \mathbf{K}_\upsilon^\mathbf{I}$, for any $t \geq 0$ and $i \in I$ such that $z_i^\zeta(t) > 0$, by Proposition 4.5.1, we have

$$\frac{z_i^\zeta(t)}{\phi_i(z^\zeta(t))} = \left(\frac{\sum_{j \in \mathcal{J}} p_j^\zeta(t) R_{ji}}{\kappa_i} \right)^{\frac{1}{\alpha}} \tag{4.102}$$

where $p^\zeta(t) \in \mathbb{R}_+^\mathbf{J}$ satisfies conditions (4.28)–(4.31) with $p = p^\zeta(t), z = z^\zeta(t)$ and $\psi = \phi(z^\zeta(t))$.

Suppose, for a proof by contradiction, that there is $i \in I$, a sequence of fluid model solutions $\{\zeta^n\}_{n \in \mathbb{N}}$ with $\zeta^n(0) \in \mathbf{K}_\upsilon^\mathbf{I}$, and an associated sequence of times $\{t_n\}_{n \in \mathbb{N}}$, such that $z_i^n(t_n) \neq 0$ and $\{\frac{z_i^n(t_n)}{\phi_i(z^n(t_n))}\}_{n \in \mathbb{N}}$ is unbounded. Here we use $z^n(\cdot)$ to represent $z^{\zeta^n}(\cdot)$ for simplicity. (Note also that ζ^n here is not the smoothed version of ζ used in Section 4.9.) Since $|\mathcal{J}| = \mathbf{J}$ is finite, R is a matrix of zeros and ones, and κ_i and α are fixed positive constants, by (4.102), we have that there exists $\{j_n^*\}_{n \in \mathbb{N}} \subset \mathcal{J}$ such that $R_{j_n^* i} = 1$ for each n and such that $\{p_{j_n^*}^{\zeta^n}(t_n)\}_{n \in \mathbb{N}}$ is an unbounded sequence of positive real numbers. By (4.28), for each n, since $p_{j_n^*}^{\zeta^n}(t_n) > 0$, we have

$$\sum_{k \in I_+(z^n(t_n))} R_{j_n^* k} \phi_k(z^n(t_n)) = C_{j_n^*}. \tag{4.103}$$

Let $C_{min} = \min\{C_j : j \in \mathcal{J}\}$ and $\delta = \frac{C_{min}}{2\mathbf{I}} > 0$. Then for each n, there is $i_n^* \in I_+(z^n(t_n))$ such that $R_{j_n^* i_n^*} = 1$ and $\phi_{i_n^*}(z^n(t_n)) > \delta$. Combining this with Lemma 4.10.4, we have

$$\frac{z_{i_n^*}^n(t_n)}{\phi_{i_n^*}(z^n(t_n))} < \frac{\widetilde{B}_\upsilon}{\delta} \tag{4.104}$$

for each n. Now, by (4.30) with i_n^* in place of i, we have

$$\frac{z_{i_n^*}^n(t_n)}{\phi_{i_n^*}(z^n(t_n))} = \left(\frac{\sum_{j\in\mathcal{J}} p_j^{\zeta^n}(t_n)R_{ji_n^*}}{\kappa_{i_n^*}}\right)^{\frac{1}{\alpha}}. \tag{4.105}$$

Since $R_{j_n^*i_n^*}=1$ and $\{p_{j_n^*}^{\zeta^n}(t_n)\}_{n\in\mathbb{N}}$ is unbounded, it follows that $\frac{z_{i_n^*}^n(t_n)}{\phi_{i_n^*}(z^n(t_n))}$ diverges as $n\to\infty$, which contradicts (4.104). Because of this contradiction, it follows that M_υ is finite. □

With Lemma 4.11.1, we can prove the following strengthened form of Lemma 4.3.1, under the added assumption that the fluid model is critical, i.e., Assumption 4.1 holds.

Lemma 4.11.2. *Suppose Assumptions 4.1 and 4.2 hold. Fix* $\upsilon>0$. *For any fluid model solution* ζ *with* $\zeta(0)\in\mathbf{K}_\upsilon^{\mathbf{I}}$, *we have* $\zeta(t)\in\mathbf{K}_{\upsilon^*}^{\mathbf{I}}$ *for all* $t\geq 0$, *where*

$$\upsilon^* = \upsilon + M_\upsilon \max_{i\in I}\rho_i. \tag{4.106}$$

Proof. Fix $i\in I$ and a fluid model solution ζ with $\zeta(0)\in\mathbf{K}_\upsilon^{\mathbf{I}}$. For any $t\geq 0$, if $\zeta_i(t)=0$, then the result holds for any $\upsilon^*>0$. If $\zeta_i(t)\neq 0$, let $t_0=\sup\{0\leq s<t:\zeta_i(s)=0\}$, where $\sup(\emptyset)=0$. Then $\zeta_i(\cdot)$ is nonzero on $(t_0,t]$ and $\zeta_i(t_0)=0$ if $t_0>0$. For $s\in(t_0,t]$ and $x\in[0,\infty)$, by (4.9),

$$\begin{aligned}
\overline{M}_t^i(x) &= \overline{M}_s^i(x+S_{s,t}^i) + \nu_i\int_s^t \overline{N}_i(x+S_{u,t}^i)du \\
&\leq \overline{M}_s^i(x) + \int_s^t \nu_i\overline{N}_i(x+S_{u,t}^i)\frac{z_i(u)}{\Lambda_i(u)}\frac{\Lambda_i(u)}{z_i(u)}du \\
&\leq \overline{M}_s^i(x) + M_\upsilon\nu_i\int_s^t \overline{N}_i(x+S_{u,t}^i)\frac{d(-S_{u,t}^i)}{du}du \\
&= \overline{M}_s^i(x) + M_\upsilon\nu_i\int_x^{x+S_{s,t}^i}\overline{N}_i(y)dy \quad \text{with } y=x+S_{u,t}^i \qquad (4.107)\\
&= \overline{M}_s^i(x) + M_\upsilon\nu_i\mu_i^{-1}\left(\overline{N}_i^e(x)-\overline{N}_i^e(x+S_{s,t}^i)\right) \\
&\leq \overline{M}_s^i(x) + M_\upsilon\rho_i\overline{N}_i^e(x), \qquad (4.108)
\end{aligned}$$

where we used Lemma 4.11.1 for the second inequality. Now let $s \downarrow t_0$ in (4.108) to obtain

$$\overline{M}_t^i(x) \leq \overline{M}_{t_0}^i(x) + M_\upsilon \rho_i \overline{N}_i^e(x),$$

where $\overline{M}_{t_0}^i(x) \leq z_i(t_0) = 0$ if $t_0 > 0$ and $\overline{M}_{t_0}^i(x) = \overline{M}_0^i(x) \leq \upsilon \overline{N}_i^e(x)$ if $t_0 = 0$. Then for all $t \geq 0$, $i \in I$,

$$\overline{M}_t^i(x) \leq \upsilon^* \overline{N}_i^e(x) \text{ for all } x \in [0, \infty), \tag{4.109}$$

where υ^* is given by (4.106). Combining with Proposition 4.2.2 yields the desired result. □

Remark 4.11.1. *The substitution step in (4.107) is similar to one used in the proof of Corollary 5.1 in [PW16]. However, the new crucial step here is to use the uniform bound on* $\frac{z_i(\cdot)}{\Lambda_i(\cdot)}$ *from Lemma 4.11.1.*

4.11.3 Proofs of Theorems 4.8.2 and 4.8.3

Our proofs of Theorems 4.8.2 and 4.8.3 draw on some arguments in the proofs of Theorems 3.2 and 3.1, respectively, given in [PW16] for the case of a single class processor sharing queue. However, multiple details are more complicated in our more general setting. In particular, our Lyapunov function G is different, our fluid model solutions can have components that reach zero and we also have a less restrictive precompact set $\mathbf{K}_\upsilon^{\mathbf{I}}$ than in [PW16].

Proof of Theorem 4.8.2 (Monotone convergence of $\mathcal{G}^\zeta(\cdot)$ to zero). Fix $\upsilon > 0$. The monotonic decreasing property is an immediate consequence of Theorem 4.8.1. So it suffices to prove the uniform convergence to zero. Note that by Lemma 4.11.2, for υ^* as given there, and all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, we have $\zeta(t) \in \mathbf{K}_{\upsilon^*}^{\mathbf{I}}$ for all $t \geq 0$. Given $\varepsilon > 0$, let

$$\mathbf{G}_\varepsilon = \{\xi \in \mathbf{M}_{\upsilon^*}^{\mathbf{I}} : G(\xi) < \varepsilon\}. \tag{4.110}$$

It suffices to show that there exists $T_\varepsilon > 0$ such that for all ζ with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$, we have $\zeta(t) \in \mathbf{G}_\varepsilon$ for all $t \geq T_\varepsilon$.

By Lemma 4.7.1, G is continuous on $\mathbf{M}^{\mathbf{I}}_{\upsilon^*}$. Then

$$\mathbf{G}^c_\varepsilon = \mathbf{M}^{\mathbf{I}}_{\upsilon^*} \backslash \mathbf{G}_\varepsilon = \{\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon^*} : G(\xi) \geq \varepsilon\} \tag{4.111}$$

is a closed set in the compact set $\mathbf{M}^{\mathbf{I}}_{\upsilon^*}$ and hence is compact. By Lemma 4.7.1(ii), we have $\mathbf{G}^c_\varepsilon \cap \mathcal{M}^* = \emptyset$. Then by Lemma 4.4.4, $K(\xi) < 0$ for all $\xi \in \mathbf{G}^c_\varepsilon$. By Lemma 4.4.3, K is upper semicontinuous on the compact set $\mathbf{G}^c_\varepsilon$, and so it achieves its maximum there, which will be strictly negative. Let $\delta > 0$ be such that $K(\xi) \leq -\delta$ for all $\xi \in \mathbf{G}^c_\varepsilon$. Then for any $t \geq 0$ and fluid model solution ζ with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$, since $\underline{F}(\cdot) \geq 0$ and using Theorem 4.4.1, we have for any $t \geq 0$,

$$\begin{aligned} 0 \leq \mathcal{G}^{\zeta}(t) &= \mathcal{H}^{\zeta}(t) - \underline{F}\left(\widetilde{w}(\zeta(t))\right) \\ &\leq \mathcal{H}^{\zeta}(t) \\ &= \mathcal{H}^{\zeta}(0) + \int_0^t \mathcal{K}^{\zeta}(s)ds. \end{aligned} \tag{4.112}$$

Let $\tau^{\zeta}_\varepsilon = \inf\{t \geq 0 : \zeta(t) \in \mathbf{G}_\varepsilon\}$. Then by (4.112), since $\mathcal{K}^{\zeta}(s) = K(\zeta(s))$ where K has a maximum of $-\delta$ on $\mathbf{G}^c_\varepsilon$, we have

$$\tau^{\zeta}_\varepsilon \leq \frac{\mathcal{H}^{\zeta}(0)}{\delta} \leq \frac{1}{\delta(\alpha+1)} \sum_{i \in I} \frac{\kappa_i \langle \chi, \vartheta^e_i \rangle (\upsilon^*)^{\alpha+1}}{\rho_i^{\alpha}} := T_\varepsilon.$$

Since $t \to \mathcal{G}^{\zeta}(t)$ is non-increasing, by Theorem 4.8.1, it follows that $\zeta(t) \in \mathbf{G}_\varepsilon$ for all $t \geq T_\varepsilon$. Since T_ε does not depend on the particular ζ chosen, the desired result follows. □

Before proving Theorem 4.8.3, we first prove the following two lemmas. The first lemma is like Theorem 4.8.3, but with $\mathcal{M}^*$ in place of $\mathcal{M}$. The second lemma will be used to derive Theorem 4.8.3 from the first lemma.

Lemma 4.11.3. *Suppose that Assumptions 4.1 and 4.2 hold. Fix $\upsilon > 0$. For any fluid model solution ζ satisfying $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$, $\zeta(t)$ converges towards $\mathcal{M}^*$ as $t \to \infty$, uniformly for all initial measures in $\mathbf{K}^{\mathbf{I}}_{\upsilon}$, i.e.,*

$$\lim_{t\to\infty} \sup\{\mathbf{d_I}(\zeta(t), \mathcal{M}^*) : \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}\} = 0. \tag{4.113}$$

Furthermore, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\sup_{t\geq 0}\{\mathbf{d_I}(\zeta(t), \mathcal{M}^*) : \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon} \text{ and } \mathbf{d_I}(\zeta(0), \mathcal{M}^*) < \delta\} \leq \varepsilon. \tag{4.114}$$

Proof. Fix $\upsilon > 0$. By Lemma 4.11.2, with υ^* as given there, for any fluid model solution with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$, we have $\zeta(t) \in \mathbf{K}^{\mathbf{I}}_{\upsilon^*}$ for all $t \geq 0$. For each $a > 0$, let

$$\mathbf{D}_a := \{\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon^*} : \mathbf{d_I}(\xi, \mathcal{M}^*) \geq a\} \quad \text{and} \quad \mathbf{G}_a := \{\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon^*} : G(\xi) < a\}.$$

For the proof of (4.113), consider $\varepsilon > 0$ fixed. Since $\xi \to d_{\mathbf{I}}(\xi, \mathcal{M}^*)$ is a continuous function on $\mathbf{M}^{\mathbf{I}}_{\upsilon^*}$, $\mathbf{D}_\varepsilon$ is a closed subset of the compact set $\mathbf{M}^{\mathbf{I}}_{\upsilon^*}$ and hence is compact. By Lemma 4.7.1, G is strictly positive on $\mathbf{D}_\varepsilon$. Then by the compactness of $\mathbf{D}_\varepsilon$, there is $\delta_1(\varepsilon) > 0$ such that $G(\xi) \geq \delta_1(\varepsilon)$ for all $\xi \in \mathbf{D}_\varepsilon$. Hence $\mathbf{D}_\varepsilon \subset \mathbf{G}^c_{\delta_1(\varepsilon)} = \mathbf{M}^{\mathbf{I}}_{\upsilon^*} \backslash \mathbf{G}_{\delta_1(\varepsilon)}$ and so $\mathbf{G}_{\delta_1(\varepsilon)} \subset \mathbf{D}^c_\varepsilon = \mathbf{M}^{\mathbf{I}}_{\upsilon^*} \backslash \mathbf{D}_\varepsilon$. By Theorem 4.8.2, there is $T_{\delta_1(\varepsilon)} < \infty$ such that $\zeta(t) \in \mathbf{G}_{\delta_1(\varepsilon)}$ for all $t \geq T_{\delta_1(\varepsilon)}$, for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$. It follows that $d_{\mathbf{I}}(\zeta(t), \mathcal{M}^*) < \varepsilon$ for all $t \geq T_{\delta_1(\varepsilon)}$ and all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$. The result (4.113) follows since $\varepsilon > 0$ was arbitrary.

For the proof of (4.114), fix $\varepsilon > 0$ and let $\delta_1(\varepsilon)$ be as defined above. Since G is a continuous function on the compact set $\mathbf{M}^{\mathbf{I}}_{\upsilon+1}$, it is uniformly continuous there. Also, G is zero on $\mathcal{M}^*_{\upsilon+1} = \mathcal{M}^* \cap \mathbf{M}^{\mathbf{I}}_{\upsilon+1}$. It follows that there is $\delta \in (0,1)$ such that $G(\xi) < \delta_1(\varepsilon)$ whenever $\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon+1}$ and $\mathbf{d_I}(\xi, \mathcal{M}^*_{\upsilon+1}) < \delta$. If ζ is a fluid model solution with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$ satisfying $\mathbf{d_I}(\zeta(0), \mathcal{M}^*) < \delta$, then there is $\eta \in \mathcal{M}^*$ such that $\mathbf{d_I}(\zeta(0), \eta) < \delta$. By the form of the elements of $\mathcal{M}^*$, we have for all $i \in I$, $\langle \mathbb{1}_{[x,\infty)}, \eta_i \rangle \le \langle \mathbb{1}, \eta_i \rangle \langle \mathbb{1}_{[x,\infty)}, \vartheta_i^e \rangle$ where $\langle \mathbb{1}, \eta_i \rangle \le \langle \mathbb{1}, \zeta_i(0) \rangle + \delta \le \upsilon + 1$, and so $\eta \in \mathbf{M}_{\upsilon+1}^{\mathbf{I}}$. It follows that $\mathbf{d_I}(\zeta(0), \mathcal{M}_{\upsilon+1}^*) < \delta$ and hence, by the choice of δ, $G(\zeta(0)) < \delta_1(\varepsilon)$. By Theorem 4.8.1, $t \to \mathcal{G}(t) = G(\zeta(t))$ is a non-increasing function and so $G(\zeta(t)) < \delta_1(\varepsilon)$ for all $t \ge 0$. By Lemma 4.11.2, we also have that $\zeta(t) \in \mathbf{K}_{\upsilon^*}^{\mathbf{I}}$. Thus, $\zeta(t) \in \mathbf{G}_{\delta_1(\varepsilon)} \subset \mathbf{M}_{\upsilon^*}^{\mathbf{I}} \setminus \mathbf{D}_\varepsilon$, from the first part of this proof. It follows that $\mathbf{d_I}(\zeta(t), \mathcal{M}^*) < \varepsilon$ for all $t \ge 0$. The desired result (4.114) follows. □

The following lemma is a vector measure analogue of Lemma 4.4 in [PW16].

Lemma 4.11.4. *Suppose that* $\xi \in \mathbf{M}^{\mathbf{I}}$ *and* $\theta > 0$ *such that* $\mathbf{d_I}(\xi, \mathcal{M}^*) < \theta$. *Then*

$$\mathbf{d_I}(\xi, \mathcal{M}) \le \max_{i \in I} \xi_i([0, \theta)) + 2\theta. \tag{4.115}$$

Proof. There is $\eta \in \mathcal{M}^*$ such that $\mathbf{d_I}(\xi, \eta) < \theta$, and there is $a \in \mathbb{R}_+^{\mathbf{I}}$ and $b \in \mathcal{P}$ such that for each $i \in I$, $\eta_i = a_i \delta_0 + b_i \vartheta_i^e$. Let $\vartheta_i^{e,b} = b_i \vartheta_i^e$ for $i \in I$. Note that $\vartheta^{e,b} \in \mathcal{M}$. Then,

$$\mathbf{d_I}(\xi, \mathcal{M}) \le \mathbf{d_I}(\xi, \vartheta^{e,b}) \le \mathbf{d_I}(\xi, \eta) + \mathbf{d_I}(\eta, \vartheta^{e,b}) \le \theta + \max_{i \in I} a_i, \tag{4.116}$$

where by the definition of the metric $\mathbf{d_I}$,

$$a_i = \eta_i(\{0\}) \le \xi_i([0, \theta)) + \theta \quad \text{for each } i \in I. \tag{4.117}$$

Combining the above, yields the desired result (4.115). □

Proof of Theorem 4.8.3 (Convergence to the invariant manifold). We first note that for any $\theta > 0$ and any fluid model solution ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, for $t \ge 0$, $i \in I$, and $t_i = \sup\{s \le t : z_i(s) = 0\}$ where $z_i(s) = \langle \mathbb{1}, \zeta_i(s) \rangle$, using the fact that $\zeta_i(t)$ has no atom at $\{0\}$ and letting $s \downarrow t_i$ in (4.9) (when $t_i \neq t$), we have that

$$
\begin{aligned}
\zeta_i(t)([0,\theta)) &= \overline{M}_t^i(0) - \overline{M}_t^i(\theta) \\
&= \mathbb{1}_{\{t_i=0\}}\left(\overline{M}_0^i(S_{0,t}^i) - \overline{M}_0^i(\theta + S_{0,t}^i)\right) + \nu_i \int_{t_i}^{t} (\overline{N}_i(S_{u,t}^i) - \overline{N}_i(\theta + S_{u,t}^i))du \\
&\leq \mathbb{1}_{\{t_i=0\}}\left(\overline{M}_0^i(S_{0,t}^i) - \overline{M}_0^i(\theta + S_{0,t}^i)\right) + \nu_i M_\upsilon \int_0^{S_{t_i,t}^i} (\overline{N}_i(y) - \overline{N}_i(\theta + y))dy \\
&\leq \mathbb{1}_{\{t_i=0\}}\left(\overline{M}_0^i(S_{0,t}^i) - \overline{M}_0^i(\theta + S_{0,t}^i)\right) + \nu_i M_\upsilon \int_0^{\theta} \overline{N}_i(y)dy \\
&\leq \mathbb{1}_{\{t_i=0\}}\left(\overline{M}_0^i(S_{0,t}^i) - \overline{M}_0^i(\theta + S_{0,t}^i)\right) + \nu_i M_\upsilon \theta, \qquad (4.118)
\end{aligned}
$$

where for the third inequality, we used the change of variable $y = S_{u,t}^i$ and the upper bound of M_υ on $z_i(u)/\Lambda_i(u)$ for $u \in (t_i, t)$ afforded by Lemma 4.11.1, and for the last inequality we used the fact that $\overline{N}_i(\cdot)$ is bounded by one.

We first prove (4.36). For this, let $\varepsilon > 0$ and

$$
\theta = \frac{\varepsilon}{3(1 + \upsilon + M_\upsilon \max_{i \in I} \nu_i)} \in \left(0, \frac{\varepsilon}{3}\right).
$$

By Lemma 4.11.3, there is $T_\theta^{(1)} > 0$ such that

$$
\mathbf{d_I}(\zeta(t), \mathcal{M}^*) < \theta \quad \text{for all } t \geq T_\theta^{(1)}, \qquad (4.119)
$$

for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$. Then for each such fluid model solution, by Lemma 4.11.4, we have

$$
\mathbf{d_I}(\zeta(t), \mathcal{M}) \leq \max_{i \in I} \zeta_i(t)([0,\theta)) + 2\theta \quad \text{for all } \; t \geq T_\theta^{(1)}, \qquad (4.120)
$$

and by (4.118), the fact that $\zeta(0) \in \mathbf{K}_{\upsilon}^{\mathbf{I}}$, and since by Lemma 4.11.1,

$$S_{0,t}^{i} = \int_{0}^{t} \frac{\phi_i(z(u))}{z_i(u)} du \geq \frac{t}{M_{\upsilon}} \quad \text{when } t_i = 0,$$

we have for each $i \in I$,

$$\begin{aligned} \zeta_i(t)([0,\theta)) &\leq \mathbb{1}_{\{t_i=0\}} \overline{M}_0^i(S_{0,t}^i) + \nu_i M_{\upsilon} \theta \\ &\leq \upsilon \overline{N}_i^e(t/M_{\upsilon}) + \nu_i M_{\upsilon} \theta. \end{aligned} \tag{4.121}$$

Let $T_{\theta}^{(2)}$ be such that for each $i \in I$, $\overline{N}_i^e(t/M_{\upsilon}) < \theta$ for all $t \geq T_{\theta}^{(2)}$. Combining this with (4.120), (4.121), and the definition of θ, we see that for all $t \geq T_{\theta}^{(1)} \vee T_{\theta}^{(2)}$, for any fluid model solution ζ satisfying $\zeta(0) \in \mathbf{K}_{\upsilon}^{\mathbf{I}}$, we have

$$\mathbf{d_I}(\zeta(t), \mathcal{M}) \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon. \tag{4.122}$$

Since $\varepsilon > 0$ was arbitrary, it follows that (4.36) holds.

We now turn to proving (4.37). For this, fix $\varepsilon \in (0,1)$. It suffices to consider such an ε, since a δ that works for such an ε also works for all larger ε. Because $\overline{N}_i^e(\cdot)$ is uniformly continuous on $[0,\infty)$ and $i \in I$ takes finitely many values, there is $h_{\varepsilon} > 0$ such that for all $i \in I$ and $0 \leq h \leq h_{\varepsilon}$, we have

$$\sup_{x \in [0,\infty)} (\overline{N}_i^e(x) - \overline{N}_i^e(x+h)) < \frac{\varepsilon}{4(\upsilon+1)}. \tag{4.123}$$

Let

$$\theta = \min\left(\frac{h_{\varepsilon}}{3}, \frac{\varepsilon}{4(1 + M_{\upsilon} \max_{i \in I} \nu_i)}\right). \tag{4.124}$$

By the last part of Lemma 4.11.3, with θ in place of ε there, we can find $\delta \in (0, \theta \wedge 1)$ (not depending on ζ) such that

$$\mathbf{d_I}(\zeta(t),\mathcal{M}^*) \le \theta \quad \text{for all } t \ge 0, \tag{4.125}$$

for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$ and $\mathbf{d_I}(\zeta(0),\mathcal{M}) < \delta$. It follows from Lemma 4.11.4 that for all such fluid model solutions ζ,

$$\mathbf{d_I}(\zeta(t),\mathcal{M}) \le \max_{i\in I} \zeta_i(t)([0,\theta)) + 2\theta \quad \text{for all } t \ge 0. \tag{4.126}$$

Since $\mathbf{d_I}(\zeta(0),\mathcal{M}) < \delta$, there is $b \in \mathcal{P}$ such that $\mathbf{d_I}(\zeta(0),\vartheta^{e,b}) < \delta$ where $\vartheta_i^{e,b} = b_i\vartheta_i^e$ and $b_i \le \upsilon+\delta$ for each $i \in I$. It follows from this and (4.118) that for any $t \ge 0$ and $i \in I$,

$$\begin{aligned}
\zeta_i(t)([0,\theta)) &\le \mathbb{1}_{\{t_i=0\}}(\langle \mathbb{1}_{(S_{0,t}^i,\theta+S_{0,t}^i]}, \zeta_i(0)\rangle) + \nu_i M_\upsilon \theta \\
&\le \mathbb{1}_{\{t_i=0\}}(b_i\langle \mathbb{1}_{((S_{0,t}^i-\delta)^+,\theta+S_{0,t}^i+\delta)}, \vartheta_i^e\rangle + \delta) + \nu_i M_\upsilon \theta \\
&= \mathbb{1}_{\{t_i=0\}}(b_i(\overline{N}_i^e((S_{0,t}^i-\delta)^+) - \overline{N}_i^e(\theta+S_{0,t}^i+\delta)) + \delta) + \nu_i M_\upsilon \theta \\
&\le (\upsilon+\delta)(\sup_{x\in[0,\infty)} (\overline{N}_i^e(x) - \overline{N}_i^e(x+\theta+2\delta)) + \delta + \nu_i M_\upsilon \theta \\
&\le (\upsilon+1)\frac{\varepsilon}{4(\upsilon+1)} + \theta + \nu_i M_\upsilon \theta \\
&\le \frac{\varepsilon}{2},
\end{aligned} \tag{4.127}$$

where we used (4.123), the facts that $\delta < \theta \wedge 1$ and $\theta + 2\delta \le 3\theta \le h_\varepsilon$ for the second last inequality, and we used the definition of θ for the last inequality. Combining (4.126) with (4.127) and the fact that $\theta \le \frac{\varepsilon}{4}$, we find that

$$\mathbf{d_I}(\zeta(t),\mathcal{M}) \le \varepsilon \quad \text{for all } t \ge 0, \tag{4.128}$$

for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$ and $\mathbf{d_I}(\zeta(0),\mathcal{M}) < \delta$. Hence (4.37) holds. $\square$

4.12 Acknowledgement

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Appendix A

Supplementary Lemmas

Lemma A.1. *For each $z^* \in \mathbb{R}_+^{\mathbf{I}}$, $\phi_i(\cdot)$ is continuous at z^* for each $i \in I_+(z^*)$.*

We first observe that $\kappa_i z_i U_i(\psi_i/z_i) = \kappa_i z_i^{\alpha_i} U_i(\psi_i)$ for all $\psi_i \geq 0, i \in I_+(z), z \in \mathbb{R}_+^{\mathbf{I}}$. So the objective function in the utility maximization problem (2.2) is the same as $\sum_{i \in I_+(z)} \kappa_i z_i^{\alpha_i} U_i(\psi_i)$.

Fix $z^* \in \mathbb{R}_+^{\mathbf{I}}$. We want to show that for each $i \in I_+(z^*)$, $z \to \phi_i(z)$ is continuous at $z = z^*$, where $\phi(z) = (\phi_1(z), \ldots, \phi_{\mathbf{I}}(z))$ is the optimal solution of (2.2). For convenience, for $z \in \mathbb{R}_+^{\mathbf{I}}$, let

$$G_z(\psi^+) = \sum_{i \in I_+(z)} \kappa_i z_i^{\alpha_i} U_i(\psi_i),$$

where $\psi^+ = (\psi_i : i \in I_+(z))$ will be regarded as a vector in $\mathbb{R}_+^{|I_+(z)|}$ (this vector contains all of the positive entries of any feasible vector $\psi \in \mathbb{R}_+^{\mathbf{I}}$ for the optimization problem (2.2)).

Let $\varepsilon > 0$ be sufficiently small that the open ball B_ε in $\mathbb{R}_+^{|I_+(z^*)|}$, that is centered at $\phi^*(z^*) = \{\phi_i(z^*) : i \in I_+(z^*)\}$ and has radius $\varepsilon > 0$, is a strictly positive distance from the boundary of the orthant $\mathbb{R}_+^{|I_+(z^*)|}$. Let D_ε denote the compact set of $\psi^\dagger = (\psi_i : i \in I_+(z^*))$ in $\mathbb{R}_+^{|I_+(z^*)|}$ that satisfy the constraints:

$$\sum_{i \in I_+(z^*)} R_{ji}\psi_i \leq C_j \text{ for all } j \in \mathcal{J}, \quad \psi_i \geq 0 \text{ for all } i \in I_+(z^*), \quad \psi^\dagger \notin B_\varepsilon.$$

We claim that there is $\eta > 0$ and $\delta_1 > 0$ such that for all $z \in \mathbb{R}_+^{\mathbf{I}}$ satisfying $|z - z^*| < \delta_1$ and $\psi^\dagger = (\psi_i : i \in I_+(z^*))$ in D_ε, we have

$$\sum_{i \in I_+(z^*)} \kappa_i z_i^{\alpha_i} U_i(\psi_i) < G_{z^*}(\phi^*(z^*)) - \eta. \tag{A.1}$$

Here $|\cdot|$ denotes the usual Euclidean norm. Note that the sum in (A.1) is only over $i \in I_+(z^*)$, even though the functions being summed have z_i not z_i^* in them. The claim can be proved using an argument by contradiction as follows. Suppose that for each positive integer n there is $z^n \in \mathbb{R}_+^{\mathbf{I}}$ such that $|z^n - z^*| < 1/n$ and $\psi^{\dagger,n} = (\psi_i^n, i \in I_+(z^*)) \in D_\varepsilon$ such that

$$\sum_{i \in I_+(z^*)} \kappa_i (z_i^n)^{\alpha_i} U_i(\psi_i^n) \geq G_{z^*}(\phi^*(z^*)) - \frac{1}{n}. \tag{A.2}$$

Then $z^n \to z^*$ as $n \to \infty$ and, since D_ε is compact, by passing to a suitable subsequence, denoted by $\{n_k\}_{k=1}^\infty$, we may assume that $\psi^{\dagger,n_k} \to \psi^*$ for some $\psi^* \in D_\varepsilon$ as $k \to \infty$. For any $i \in I_+(z^*)$ such that $\alpha_i \in (0,1)$, the term $\kappa_i (z_i^n)^{\alpha_i} U_i(\psi_i^n)$ in the left member of (A.2) is jointly continuous in z_i^n and ψ_i^n and so with n replaced by n_k, this term tends to the finite value $\kappa_i (z_i^*)^{\alpha_i} U_i(\psi_i^*)$ as $k \to \infty$. For any $i \in I_+(z^*)$ such that $\alpha_i \in [1,\infty)$, if $\psi_i^* > 0$, then $U_i(\psi_i^{n_k})$ tends to $U_i(\psi_i^*)$ as $k \to \infty$; on the other hand, if $\psi_i^* = 0$ then $U_i(\psi_i^{n_k})$ tends to $-\infty$. In fact, the latter cannot occur; because, if it did, taking the $\liminf$ as $k \to \infty$ in (A.2), with n_k in place of n, and using the fact that $z_i^{n_k} \to z_i^* > 0$ as $k \to \infty$ for $i \in I_+(z^*)$, would yield a contradiction to the finiteness of the right member of (A.2). Consequently, we can pass to the limit as $k \to \infty$ in (A.2), with n_k in place of n, to conclude that $\psi^* \in D_\varepsilon$ and

$$\sum_{i \in I_+(z^*)} \kappa_i (z_i^*)^{\alpha_i} U_i(\psi_i^*) \geq G_{z^*}(\phi^*(z^*)). \tag{A.3}$$

Recognizing the left member of (A.3) as $G_{z^*}(\psi^*)$, this implies that $\psi^* \in D_\varepsilon$ and $\phi^*(z^*) \in B_\varepsilon$ are two distinct maximizers for the optimization of $G_{z^*}(\cdot)$ over the set

$$\left\{\psi^\dagger \in \mathbb{R}_+^{|I_+(z^*)|} : \sum_{i\in I_+(z^*)} R_{ji}\psi_i \le C_j \text{ for all } j \in \mathcal{J}\right\}.$$

This contradicts the uniqueness of such a maximizer (see Remark 2.2.1). This last contradiction implies that the claim associated with (A.1) is true.

Without loss of generality, we can assume that the $\delta_1 > 0$ in the claim proved above is small enough that for all $z \in \mathbb{R}_+^{\mathbf{I}}$ such that $|z - z^*| < \delta_1$ we have $z_i > 0$ for all $i \in I_+(z^*)$, which implies that $I_+(z^*) \subset I_+(z)$ and for $\psi^+ = (\psi_i : i \in I_+(z)) \in \mathbb{R}_+^{|I_+(z)|}$,

$$G_z(\psi^+) = \sum_{i\in I_+(z^*)} \kappa_i z_i^{\alpha_i} U_i(\psi_i) + \sum_{i\in I_+(z)\setminus I_+(z^*)} \kappa_i z_i^{\alpha_i} U_i(\psi_i). \tag{A.4}$$

Note $z_i^* = 0$ for all $i \in I_+(z) \setminus I_+(z^*)$, and for all ψ^+ satisfying

$$\sum_{i\in I_+(z)} R_{ji}\psi_i \le C_j, \quad \text{for all } j \in \mathcal{J}, \tag{A.5}$$

we have $U_i(\psi_i) \le U_i(C^*)$ for all $i \in I_+(z)$ where $C^* = \max_{j\in\mathcal{J}} C_j$. It follows that there is $\delta_2 \in (0, \delta_1)$ such that the last sum in (A.4) is less than $\eta/4$ for all $z \in \mathbb{R}_+^{\mathbf{I}}$ satisfying $|z - z^*| < \delta_2$ and $\psi^+ \in \mathbb{R}_+^{|I_+(z)|}$ satisfying (A.5). Combining this with (A.1) and (A.4), we see that for such z and ψ^+, if in addition, $\psi^\dagger = (\psi_i : i \in I_+(z^*)) \notin B_\varepsilon$, then

$$G_z(\psi^+) \le G_{z^*}(\phi^*(z^*)) - \frac{3\eta}{4}. \tag{A.6}$$

On the other hand, consider the first sum on the right side of (A.4). By the continuity of the $U_i(\cdot)$ on $(0, \infty)$ and since $\phi_i(z^*) > 0$ for all $i \in I_+(z^*)$, there is $\delta_3 \in (0, \delta_2)$ such that for all $z \in \mathbb{R}_+^{\mathbf{I}}$ satisfying $|z - z^*| < \delta_3$ and all $\psi^\dagger = (\psi_i : i \in I_+(z^*))$ satisfying $|\psi^\dagger - \phi^*(z^*)| < \delta_3$, we have

$$\psi_i > 0 \text{ for all } i \in I_+(z^*) \quad \text{ and } \quad \sum_{i\in I_+(z^*)} \kappa_i z_i^{\alpha_i} U_i(\psi_i) \geq G_{z^*}(\phi^*(z^*)) - \frac{\eta}{4}. \tag{A.7}$$

In particular, an allowed value of such a $\psi^\dagger$ is $\psi^\ddagger = (\phi_i^*(z^*) - \frac{\delta_3}{2\mathbf{I}} : i \in I_+(z^*))$. For this $\psi^\ddagger$, if $j \in \mathcal{J}$ such that $\sum_{i\in I_+(z^*)} R_{ji}\phi_i^*(z^*) = C_j$, we must have that $R_{ji} = 1$ for some $i \in I_+(z^*)$ and then $\sum_{i\in I_+(z^*)} R_{ji}\psi_i \leq C_j - \frac{\delta_3}{2\mathbf{I}}$. Furthermore, there is $\delta_4 \in (0, \delta_3/2\mathbf{I})$ such that for those $j \in \mathcal{J}$ satisfying $\sum_{i\in I_+(z^*)} R_{ji}\phi_i^*(z^*) < C_j$, we have $\sum_{i\in I_+(z^*)} R_{ji}\phi^*(z^*) < C_j - \delta_4$. Then, $\psi^\ddagger$ satisfies $\sum_{i\in I_+(z^*)} R_{ji}\psi_i < C_j - \delta_4$ for all $j \in \mathcal{J}$. Then, for any $z \in \mathbb{R}_+^{\mathbf{I}}$ satisfying $|z - z^*| < \delta_4$, we can define a vector $\psi^\flat(z)$ in $\mathbb{R}_+^{|I_+(z)|}$ such that $\psi_i^\flat(z) = \psi_i^\ddagger$ for $i \in I_+(z^*)$ and $\psi_i^\flat(z) = \frac{\delta_4}{\mathbf{I}}$ for all $i \in I_+(z) \setminus I_+(z^*)$. Then $\sum_{i\in I_+(z)} R_{ji}\psi_i^\flat(z) \leq C_j$ for all $j \in \mathcal{J}$, and by (A.4) and (A.7),

$$G_z(\psi^\flat(z)) \geq G_{z^*}(\phi^*(z^*)) - \frac{\eta}{4} + \sum_{i\in I_+(z)\setminus I_+(z^*)} \kappa_i z_i^{\alpha_i} U_i\left(\frac{\delta_4}{\mathbf{I}}\right). \tag{A.8}$$

For $i \in I_+(z) \setminus I_+(z^*)$, we have that $z_i^* = 0$, and so there is $\delta_5 \in (0, \delta_4)$ such that the sum that is the last term in the above is smaller in magnitude than $\eta/4$ for all $z \in \mathbb{R}_+^{\mathbf{I}}$ satisfying $|z - z^*| < \delta_5$. Then, for all $z \in \mathbb{R}_+^{\mathbf{I}}$ such that $|z - z^*| < \delta_5$, we have that $\psi^\flat(z)$ (expanded to a vector in $\mathbb{R}_+^{\mathbf{I}}$ that has zeros for the components indexed by $i \notin I_+(z)$) is feasible for the optimization problem (2.2) and by (A.8) and (A.6),

$$\begin{aligned} G_z(\psi^\flat(z)) &\geq G_{z^*}(\phi^*(z^*)) - \frac{\eta}{2} \\ &\geq G_z(\psi^+) + \frac{\eta}{4} \end{aligned}$$

for all $\psi^+ = (\psi_i : i \in I_+(z)) \in \mathbb{R}_+^{|I_+(z)|}$ that satisfy (A.5) and are such that $\psi^\dagger = (\psi_i : i \in I_+(z^*))$ is not in B_ε. It follows that the optimal solution $\phi(z)$ must be such that $(\phi_i(z) : i \in I_+(z^*))$ is in B_ε. Hence $\sum_{i\in I_+(z^*)} |\phi_i(z) - \phi_i(z^*)|^2 < \varepsilon^2$ whenever $|z - z^*| < \delta_5$. This proves the desired continuity. □

Lemma A.2. *Let $\widetilde{\mathcal{C}} = \{f \in \mathbf{C}_b^1(\mathbb{R}_+) : f(0) = 0\}$. If $\zeta(\cdot)$ is a solution for the fluid model, then for each $f \in \widetilde{\mathcal{C}}$, property (iii) in Definition 2.3.2 still holds.*

Proof. Fix $f \in \widetilde{\mathcal{C}}$. We first consider the case where f has compact support contained in $[0, M]$ for some $M > 1$. Let $\{g_n\}_{n=0}^{\infty}$ be a uniformly bounded sequence of continuous functions on $\mathbb{R}_+$ such that each g_n has support in $[0, M]$, $g_n(0) = 0$, and $g_n(x) \to f'(x)$ pointwise for each $x \in (0, \infty)$ as $n \to \infty$. For each n, let $f_n(x) = \int_0^x g_n(t)dt$, $x \in [0, \infty)$. Then $f_n \in \mathcal{C}$ for each n, $f_n' = g_n$ converges to f' pointwise and boundedly on $(0, \infty)$, and by bounded convergence, f_n converges pointwise to f on $[0, M]$ and also on $[M, \infty)$ since $f_n(x) = f_n(M) \to f(M) = f(x)$ for all $x \geq M$.

The property (2.4) holds with f_n, g_n in place of f, f', respectively. Hence,

$$\langle f_n, \zeta_i(t)\rangle = \langle f_n, \zeta_i(0)\rangle - \int_0^t \langle g_n, \zeta_i(s)\rangle \frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}(z_i(s))ds + \nu_i \langle f_n, \vartheta_i\rangle \int_0^t \mathbb{1}_{(0,\infty)}(z_i(s))ds. \tag{A.9}$$

By the bounded convergence theorem, since $\zeta_i(t)$, $\zeta_i(0)$, $\zeta_i(s)$, ϑ_i are finite measures on $\mathbb{R}_+$ that do not charge the origin, as $n \to \infty$, we have $\langle f_n, \zeta_i(t)\rangle \to \langle f, \zeta_i(t)\rangle$, $\langle f_n, \zeta_i(0)\rangle \to \langle f, \zeta_i(0)\rangle$, $\langle g_n, \zeta_i(s)\rangle \to \langle f', \zeta_i(s)\rangle$ for each $s \geq 0$, and $\langle f_n, \vartheta_i\rangle \to \langle f, \vartheta_i\rangle$. Furthermore,

$$\sup_{s \in [0,t]} \left| \langle g_n, \zeta_i(s)\rangle \frac{\Lambda_i(s)}{z_i(s)} \right| \leq \sup_n \|g_n\|_\infty (\max_{j \in \mathcal{J}} C_j) < \infty.$$

Combining the above and using bounded convergence again for the second term in the right side of (A.9), we can let $n \to \infty$ in (A.9) to show that (2.4) holds for f. Thus, (2.4) holds for $f \in \widetilde{\mathcal{C}}$ that has compact support in $[0, M]$ for any $M > 1$. In particular, for an arbitrary $f \in \widetilde{\mathcal{C}}$, it holds with $f\chi_M$ in place of f and $(f\chi_M)' = f'\chi_M + f\chi_M'$ in place of f', where χ_M is a function in $\mathbf{C}_b^1(\mathbb{R}_+)$ that equals 1 on $[0, M-1]$, is zero on $[M, \infty)$, and is monotonically decreasing on $[M-1, M]$ with first derivative bounded in absolute value by 2. Then using the facts that $f\chi_M$ and $(f\chi_M)'$ converge pointwise and boundedly on $\mathbb{R}_+$ to f and f', respectively, as $M \to \infty$, using bounded convergence again, we conclude that (2.4) holds for all $f \in \widetilde{\mathcal{C}}$. □

Appendix A in full is extracted from the Appendix of “Stability of a Subcritical Fluid Model for Fair Bandwidth Sharing with General File Size Distributions”, Stochastic Systems, Yingjia Fu and Ruth J. Williams, Volume 10, Number 3, 2020. The dissertation author was the co-author of this paper.

Appendix B

Hazard Rate

Definition B.1. *Assume that ξ is a probability measure on $\mathbb{R}_+$ defining the distribution of an absolutely continuous, non-negative random variable with probability density function $j(\cdot)$ and cumulative distribution function $J(\cdot)$. The hazard rate function for ξ is defined by*

$$q(x) = \frac{j(x)}{1 - J(x)} \quad \textit{for } \; 0 < x < x^*,$$

where $x^* = \inf\{x \geq 0 : J(x) = 1\}$. The distribution is said to have bounded hazard rate if there is a finite constant L such that

$$q(x) \leq L \quad \textit{for all } 0 < x < x^*.$$

It turns out that in order to have a bounded hazard rate, the support of the distribution must be unbounded and so, in this case, $x^* = \infty$ and

$$q(x) \leq L \quad \text{for all } x \in (0, \infty).$$

Under Assumption 4.2, we have $\langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle \leq C_\vartheta \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle$ for all $x \in [0, \infty)$, which is equivalent to ϑ_i^e having bounded hazard rate, noticing $\frac{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle}{\langle \chi, \vartheta_i \rangle}$ is the density of ϑ_i^e. A sufficient condition for ϑ_i^e to have bounded hazard rate is that ϑ_i is absolutely continuous with bounded hazard rate. To see this, note that if q_i, j_i, J_i are the hazard rate, probability density and cumulative distribution function, respectively, for an absolutely continuous ϑ_i for some $i \in I$ and L_i is a bound for q_i, then for all $x \geq 0$, we have

$$\int_x^\infty \big(1 - J_i(y)\big)dy \geq \frac{1}{L_i}\int_x^\infty j_i(y)dy = \frac{1 - J_i(x)}{L_i},$$

and consequently,

$$\frac{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i\rangle}{\langle \chi, \vartheta_i\rangle\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e\rangle} = \frac{1 - J_i(x)}{\int_x^\infty \big(1 - J_i(y)\big)dy} \leq L_i.$$

We now give some examples of common distributions with bounded hazard rates.

- *Gamma Distribution.* The probability density function has the form

$$j(x) = \frac{a^b x^{b-1}}{\Gamma(b)} e^{-ax} \quad \text{for } x > 0,$$

where $a, b > 0$ are parameters. The hazard rate corresponding to this Gamma distribution is

$$q(x) = \frac{x^{b-1}e^{-ax}}{\int_x^\infty y^{b-1}e^{-ay}dy} \quad \text{for } x > 0.$$

If $0 < b < 1$, the hazard rate function is decreasing, but it is unbounded on $(0, \infty)$. If $b = 1$, the Gamma distribution is the exponential distribution with constant hazard rate function equal to a, which is clearly bounded. If $b > 1$, the hazard rate function is increasing and using the asymptotic behavior of the incomplete gamma function at infinity, we see that $\lim_{x\to\infty} q(x) = a$ and so q is bounded.

- *Pareto Distribution.* The probability density function has form

$$j(x) = \frac{a x_m^a}{x^{a+1}} \quad \text{for } x \geq x_m,$$

where $a > 0$ and $x_m > 0$ are parameters. The corresponding hazard rate function is $q(x) = \frac{a}{x}$ for $x \geq x_m$. This function is decreasing and bounded on $[x_m, \infty)$. Note that we must have $a > 2$ in order for such a distribution to have finite first and second moments.

- *Lognormal Distribution.* A random variable X follows the lognormal distribution if $Y = \log(X)$ is normally distributed. The probability density function is therefore given by

$$j(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - a)^2}{2\sigma^2}\right) \quad \text{for } x > 0,$$

where $a \in (0, \infty)$ and $\sigma > 0$ are parameters. The hazard rate function is given by

$$q(x) = \frac{\exp\left(-\frac{(\ln x - a)^2}{2\sigma^2}\right)}{x\sigma\sqrt{2\pi}\left(1 - \Phi\left(\frac{\ln x - a}{\sigma}\right)\right)}, \quad \text{for } x > 0.$$

where Φ is the cumulative distribution function for the standard normal distribution. It can be shown, see e.g., [Swe90], that the hazard rate function tends to zero at zero and infinity and has a maximum in between. Thus it has an inverted bathtub shape and is bounded.

Appendix B in full is extracted from the Appendix of "Asymptotic Behavior of a Critical Fluid Model for Bandwidth Sharing with General File Size Distributions", preprint, Yingjia Fu and Ruth J. Williams, and is slightly rewritten. The manuscript has been submitted to a major applied probability journal. The dissertation author was the co-author of this paper.

Bibliography

[BEZ14] S. Borst, R. Egorova, and B. Zwart. Fluid limits for bandwidth-sharing networks in overload. *Mathematics of Operations Research*, 39(2):533–560, 2014.

[BM01] T. Bonald and L. Massoulié. Impact of fairness on internet performance. *SIGMETRICS Perform. Eval. Rev.*, 29(1):82–91, June 2001.

[Bra98] M. Bramson. State space collapse with application to heavy traffic limits for multiclass queueing networks. *Queueing Systems*, 30(1):89–140, 1998.

[Bra08] M. Bramson. Stability of queueing networks. *Probab. Surveys*, 5:169–345, 2008.

[Bra10] M. Bramson. Network stability under max–min fair bandwidth sharing. *Annals of Applied Probability*, 20(3):1126–1176, 06 2010.

[Csi67] I. Csiszár. Information-type measures of difference of probability distributions and indirect observations. *Studia Sci. Math. Hungar.*, 2:299–318, 1967.

[CST06] M. Chiang, D. Shah, and A. Tang. Stochastic stability under network utility maximization: General file size distribution. In *Proceedings of 44th Allerton Conference on Communication, Control and Computing*, volume 5, pages 49–77. IEEE, 2006.

[Dai95] J. G. Dai. On positive harris recurrence of multiclass queueing networks: a unified approach via fluid limit models. *Annals of Applied Probability*, pages 49–77, 1995.

[DVLK01] G. De Veciana, T. Lee, and T. Konstantopoulos. Stability and performance analysis of networks supporting elastic services. *IEEE/ACM Transactions on Networking*, 9(1):2–14, 2001.

[FW20] Y. Fu and R. J. Williams. Stability of a subcritical fluid model for fair bandwidth sharing with general file size distributions. *Stochastic Systems*, 10:251–273, 2020.

[GPW02] H. C. Gromoll, A. L. Puha, and R. J. Williams. The fluid limit of a heavily loaded processor sharing queue. *Annals of Applied Probability*, 12(3):797–859, 08 2002.

[Gro04] H. C. Gromoll. Diffusion approximation for a processor sharing queue in heavy traffic. *Annals of Applied Probability*, 14:555–611, 2004.

[GW08] H. C. Gromoll and R. J. Williams. *Fluid Model for a Data Network with α-Fair Bandwidth Sharing and General Document Size Distributions: Two Examples of Stability*, volume 4 of *Collections*, pages 253–265. Institute of Mathematical Statistics, Beachwood, Ohio, USA, 2008.

[GW09] H. C. Gromoll and R. J. Williams. Fluid limits for networks with bandwidth sharing and general document size distributions. *Annals of Applied Probability*, 19(1):243 – 280, 2009.

[Kal83] O. Kallenberg. *Random Measures*. Akademie-Verlag, 1983.

[KKLW09] W. N. Kang, F. P. Kelly, N. H. Lee, and R. J. Williams. State space collapse and diffusion approximation for a network operating under a fair bandwidth sharing policy. *Annals of Applied Probability*, 19:1719–1780, 2009.

[KL93] F. P. Kelly and C. N. Laws. Dynamic routing in open queueing networks: Brownian models, cut constraints and resource pooling. *Queueing Systems*, 13:47–86, 1993.

[KW04] F. P. Kelly and R. J. Williams. Fluid model for a network operating under a fair bandwidth-sharing policy. *Annals of Applied Probability*, 14(3):1055–1083, 2004.

[KW07] W. Kang and R. J. Williams. An invariance principle for semimartingale reflecting brownian motions in domains with piecewise smooth boundaries. *Annals of Applied Probability*, 17:741–779, 2007.

[Mas07] L. Massoulié. Structural properties of proportional fairness: Stability and insensitivity. *Annals of Applied Probability*, 17:809–839, 2007.

[MPW19] J. A. Mulvany, A. L. Puha, and R. J. Williams. Asymptotic behavior of a critical fluid model for a multiclass processor sharing queue via relative entropy. *Queueing Systems*, 93(3-4):351–397, 2019.

[MR00] L. Massoulié and J. W. Roberts. Bandwidth sharing and admission control for elastic traffic. *Telecommunication Systems*, 15(1):185–201, Nov 2000.

[MW00] J. Mo and J. Walrand. Fair end-to-end window-based congestion control. *IEEE/ACM Transactions on networking*, 8(5):556–567, 2000.

[Pro56] Y. V. Prokhorov. Convergence of random processes and limit theorems in probability theory. *Theory of Probability & Its Applications*, 1(2):157–214, 1956.

[PTFA12] F. Paganini, A. Tang, A. Ferragut, and L. L. H. Andrew. Network stability under alpha fair bandwidth allocation with general file size distribution. *IEEE Transactions on Automatic Control*, 57(3):579–591, March 2012.

[PW04] A. L. Puha and R. J. Williams. Invariant states and rates of convergence for a critical fluid model of a processor sharing queue. *Annals of Applied Probability*, 14(2):517–554, 05 2004.

[PW16] A. L. Puha and R. J. Williams. Asymptotic behavior of a critical fluid model for a processor sharing queue via relative entropy. *Stochastic Systems*, 6(2):251–300, 2016.

[Swe90] A. L. Sweet. On the hazard rate of the lognormal distribtuion. *IEEE Transactions on Reliability*, 39(3):325–328, 1990.

[VZZ14] M. Vlasiou, J. Zhang, and B. Zwart. Insensitivity of proportional fairness in critically loaded bandwidth sharing networks. *arXiv preprint:1411.4841*, 2014.

[Wil98] R. J. Williams. Diffusion approximations for open multiclass queueing networks: Sufficient conditions involving state space collapse. *Queueing Systems*, 30:27–88, 1998.

[Wil16] R. J. Williams. Stochastic processing networks. *Annual Review of Statistics and Its Application*, 3(1):323–345, 2016.

[YOY05] H. Q. Ye, J. Ou, and X. M. Yuan. Stability of data networks: Stationary and bursty models. *Operations Research*, 53(1):107–125, 2005.

[YY12] H. Q. Ye and D. D. Yao. A stochastic network under proportional fair resource control—diffusion limit with multiple bottlenecks. *Operations Research*, 60(3):716–738, 2012.