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Author

Lubliner, Jacob

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Structures and Materials Research
Department of Civil Engineering
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A STUDY OF THE THERMODYNAMICS OF ONE-DIMENSIONAL DEFORMATIONS
OF NONLINEAR MATERIALS CHARACTERIZED BY QUASILINEAR DIFFERENTIAL
CONSTITUTIVE EQUATIONS

by

J. Lubliner

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University of California
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Summary

The thermodynamics of one-dimensional deformations of materials of first-order quasilinear differential type is studied on the basis of the assumption that the strain is the sum of a "rate-independent" and a "rate-dependent" part, and that the material time derivative of the latter, but not of the former, is a kinematical variable of state. This assumption is shown to lead to the most general isothermal constitutive equation of such a material. Furthermore, non-isothermal behavior can be predicted from the isothermal characteristic functions at various temperatures. A distinction is made between viscoelastic and viscoplastic materials on the basis of the continuity (or lack of it) of the rate equation with respect to strain rate. For viscoelastic materials the functional form of the free-energy density is determined completely, and specific examples of such materials are discussed. Viscoplastic materials are discussed with regard to yield condition and relationship to plastic materials. The validity of the development for three-dimensional deformations is remarked upon.

1. Introduction

A one-dimensional constitutive equation of the form

$$\dot{\epsilon} = f(\sigma, \epsilon) \dot{\sigma} + g(\sigma, \epsilon) \quad (1)$$

where σ and ϵ denote stress and strain, respectively, while the superscript dot designates material derivatives with respect to time, has been used by many authors to represent viscoelastic and viscoplastic materials, in both quasi-static and dynamic treatments. Materials described by Eq. (1) may exhibit solid-like behavior both instantaneously and in equilibrium (though special forms of the functions $f(\quad)$ and $g(\quad)$ may correspond to fluid-like behavior), creep, relaxation, and, with a slight modification, plasticity.

The form of the equation, being a quasi-linear differential equation of the first order, lends itself particularly well to the treatment of problems of wave propagation, since it forms, with the equations of motion and continuity, a quasi-linear hyperbolic system (provided $f(\quad) > 0$).

When Eq. (1) is linear, i.e., when

$$\bar{f}(\sigma, \epsilon) = \frac{1}{E}, \quad g(\sigma, \epsilon) = \frac{1}{\mu}(\sigma - E'\epsilon)$$

with E , E' and μ constant, then it describes the well-known "standard linear solid", whose behavior is also represented by either of the two models shown in Fig. 1. The special cases $\frac{1}{E} = 0$ and $E' = 0$ characterize, respectively, the simpler Kelvin-Voigt and Maxwell materials. The standard linear solid - as, indeed, any linear viscoelastic

material described by a differential equation in which the orders of the highest derivatives of stress and strain are the same - is also a Boltzmann material with a special form of the relaxation function, and hence, in turn, a simple material with fading memory (see Coleman & Noll 1961, Gurtin & Sternberg 1962). Problems of wave propagation in bodies of standard-linear-solid material have been treated by authors too numerous to list.

A semilinear form of Eq. (1), i.e., one with $f(\cdot)$ constant but $g(\cdot)$ nonlinear, has long been used (with ϵ usually denoting logarithmic strain) by investigators of creep in metals, and was introduced into the theory of wave propagation by Malvern (1951) who was likewise followed by many others.

The general form of Eq. (1) has been used in wave-propagation problem by Simmons, Hauser & Dorn (1962), Cristescu (1964), Lubliner (1964), and Greenberg (1967). A three dimensional constitutive equation of which Eq. (1) would be the one-dimensional reduction would be of the form

$$\dot{E}_{ij} = F_{ijkl}(S, E) \dot{S}_{kl} + G_{ij}(S, E). \quad (2)$$

Here E_{ij} , S_{ij} are cartesian components of the Green strain tensor E and the Kirchhoff stress tensor S . A material described by Eq. (2) is, in accordance with the classification of Noll (1958), of differential type; in addition, it may be called quasilinear. A semilinear form of Eq. (2), restricted to small deformations, was used by Perzyna (1963) to describe elastic-viscoplastic materials; recently, Perzyna

(1967) has also studied the thermodynamics of such materials. However, there appears to exist no study of the thermodynamics of the more general, quasilinear, material of differential type.* The present paper represents the beginning of such a study.

2. Kinematics and the Clausius-Duhem Inequality

We are concerned with a one-dimensional state of deformation, i.e., one in which the state varies in space only along one cartesian material coordinate, X , and in which only one cartesian spatial coordinate, y (which may be parallel or perpendicular to X) is variable, so that the deformation is described by

$$y = y(X, t). \quad (3)$$

The deformation gradient F is given by

$$F(X, t) = \frac{\partial y}{\partial X}. \quad (4)$$

Let T denote the stress component corresponding to F , ρ_0 the density in the reference state, ψ the free-energy density, η the entropy density, θ the absolute temperature, and q the X -component of the heat-flow vector; all these variables are function of X and t , except ρ_0 , which is a function of X only. The "strain" ϵ may be defined as a function of F in any one of several conventional ways; for example,

* Witness the following from a recent paper by Laws (1967): "As far as the author is aware there has been no attempt to discuss the thermodynamics of simple materials which include any of the fluids of the differential type as special cases."

$$\varepsilon = F - 1, \quad (5a)$$

$$\varepsilon = F, \quad (5b)$$

$$\varepsilon = \frac{1}{2}(F^2 - 1). \quad (5c)$$

The "conjugate stress" σ (to be called simply "stress" henceforth) is then determined as a function of T and F (or T and ε) in such a way that

$$\sigma \dot{\varepsilon} = T \dot{F}$$

where $(\dot{}) = \frac{\partial}{\partial t} () \Big|_X$.

Hence, corresponding respectively to (5a), (5b), and (5c) we find

$$\sigma = T, \quad (6a)$$

$$\sigma = TF = Te^{\varepsilon}, \quad (6b)$$

$$\sigma = T/F = T(1 + 2\varepsilon)^{-\frac{1}{2}}. \quad (6c)$$

The Clausius-Duhem inequality now takes the form

$$\sigma \dot{\varepsilon} - \rho_0 (\dot{\psi} + \eta \dot{\theta}) - \frac{q}{\theta} \frac{\partial \theta}{\partial x} \geq 0, \quad (7)$$

which is valid for longitudinal or transverse deformations.

3. Constitutive Assumptions

We proceed to study the thermodynamics of materials which possess an isothermal stress-strain relation of the form (1), but in which

$f(\quad)$ and $g(\quad)$ depend on temperature. Consequently, we assume the constitutive equation to be

$$\dot{\epsilon} = f(\sigma, \epsilon, \theta) \dot{\sigma} + g(\sigma, \epsilon, \theta) \quad \text{when } \theta = \text{const.} \quad (8)$$

Consider the material characterized by the model in Fig. 1a. We set

$$\epsilon = \epsilon^0 + \epsilon^1 \quad (9)$$

where ϵ^0 and ϵ^1 are, respectively, the strains of the spring and the Kelvin-Voigt elements. The constitutive equation is derived from two independent relations: (a) Hooke's law linking σ and ϵ^0 , and (b) a "rate equation" linking σ , ϵ^1 , and $\dot{\epsilon}^1$. Both relations may be assumed to depend on temperature and temperature gradient as well.

We shall now assume that our non-linear material can likewise be modeled as in Fig. 1a, but with the elements non-linear and coupled. This means that Eq. (9) is still valid, and that ϵ^0 , ϵ^1 , and $\dot{\epsilon}^1$ are the kinematical variables of state. If the thermodynamical variables are θ and $\gamma = \partial\theta/\partial X$, then the constitutive hypothesis is that ψ , η , and σ are all functions of ϵ^0 , ϵ^1 , $\dot{\epsilon}^1$, θ , and γ . This hypothesis leads to the following form of the Clausius-Duhem inequality (7):

$$\begin{aligned} (\sigma - \rho_0 \frac{\partial\psi}{\partial\epsilon^0})\dot{\epsilon}^0 + (\sigma - \rho_0 \frac{\partial\psi}{\partial\epsilon^1})\dot{\epsilon}^1 - \rho_0 (\eta + \frac{\partial\psi}{\partial\theta}) \dot{\theta} \\ - \rho_0 \frac{\partial\psi}{\partial\dot{\epsilon}^1} \ddot{\epsilon}^1 - \rho_0 \frac{\partial\psi}{\partial\gamma} \dot{\gamma} - q\gamma/\theta \geq 0. \end{aligned} \quad (10)$$

Following Coleman & Mizel (1964) we note that the inequality must be obeyed independently of the signs or magnitudes of $\dot{\varepsilon}^0$, $\dot{\theta}$, $\dot{\varepsilon}^1$ and $\dot{\gamma}$, since these are, by hypothesis, not variables of state. Consequently, their coefficients must vanish, and we obtain

$$\frac{\partial \psi}{\partial \dot{\varepsilon}^1} = \frac{\partial \psi}{\partial \dot{\gamma}} = 0, \quad (11)$$

so that

$$\psi = \psi(\varepsilon^0, \varepsilon^1, \theta); \quad (12)$$

furthermore

$$\eta = - \frac{\partial \psi}{\partial \theta}, \quad (13)$$

and

$$\sigma = \rho_0 \frac{\partial \psi}{\partial \varepsilon^0}. \quad (14)$$

Hence the Clausius-Duhem inequality reduces to

$$(\sigma - \rho_0 \frac{\partial \psi}{\partial \varepsilon^0}) \dot{\varepsilon}^1 - q\gamma / \theta \geq 0 \quad (15)$$

Equation (13) is a caloric equation of state, and Eq. (14) is the analogue of the Hookean relation of the linear material. We still need, therefore, an independent rate equation, which we may assume in the most general form

$$\Phi(\sigma, \varepsilon^0, \varepsilon^1, \dot{\varepsilon}^1, \theta, \gamma) = 0. \quad (16)$$

It is easy to see, however, that, in order to be compatible with Eq.

(14) at all times, $\Phi(\quad)$ must be independent of γ . In addition, Eqs. (9) and (14) may be solved for ε^0 and ε^1 as functions of σ, ε and θ . When these are substituted in $\Phi(\quad)$ and Eq. (16) is solved for $\dot{\varepsilon}^1$, we obtain

$$\dot{\varepsilon}^1 = q(\sigma, \varepsilon, \theta). \quad (17)$$

On the other hand Eq. (16) may be solved for σ ; since this is independent of γ , substitution into the reduced Clausius-Duhem inequality (15) yields the two inequalities

$$(\sigma - \rho_0 \frac{\partial \psi}{\partial \varepsilon^1}) \dot{\varepsilon}^1 \geq 0 \quad (18)$$

and

$$- q\gamma/\theta \geq 0. \quad (19)$$

being, respectively, the mechanical-dissipation and the heat-conduction inequalities.* We shall henceforth not be concerned with heat conduction, so that inequality (18) is the only remaining thermodynamical restriction on material behavior.

4. Development of the Constitutive Equation

We shall now retrace the process of eliminating ε^0 and ε^1 between Eqs. (9), (14) and (16). We may define a Gibbs free-energy density ϕ as

* This result is analogous to the one obtained by Coleman (1964) for general simple materials with fading memory. It does not hold for materials which do not undergo any "rate-independent" strain ε^0 , i.e. for which Eq. (16) is replaced by $\varepsilon^0=0$. It is this type of material (the nonlinear Kelvin-Voigt material) whose thermodynamics was studied by Coleman & Mizel (1964); see the discussion at the end of the paper by Coleman (1964).

$$\phi = \frac{1}{\rho_0} \sigma \varepsilon^0 - \psi. \quad (20)$$

Equation (20) is a Legendre transformation defining ϕ as a function of σ , ε^1 , and θ , such that

$$\varepsilon^0 = H(\sigma, \varepsilon^1, \theta) \quad (21)$$

where $H(\quad) = \rho_0 \partial\phi/\partial\sigma$.

We consider next the equation

$$\varepsilon - \varepsilon^1 = H(\sigma, \varepsilon^1, \theta), \quad (22)$$

which we solve for ε^1 , obtaining

$$\varepsilon^1 = J(\sigma, \varepsilon, \theta) \quad (23)$$

and

$$\varepsilon^0 = \varepsilon - J(\sigma, \varepsilon, \theta). \quad (24)$$

The constitutive equation is obtained by differentiating Eqs. (9) and (21) with respect to time, eliminating $\dot{\varepsilon}^0$, and inserting Eq. (17), to wit:

$$\dot{\varepsilon} = \frac{\partial H}{\partial \sigma} \dot{\sigma} + \frac{\partial H}{\partial \theta} \dot{\theta} + \left(1 + \frac{\partial H}{\partial \varepsilon^1}\right) G(\sigma, \varepsilon, \theta) \quad (25)$$

When ε^1 is eliminated from the several partial derivatives of $H(\quad)$ by means of Eq. (23), we obtain an equation of the form

$$\dot{\varepsilon} = f(\sigma, \varepsilon, \theta) \dot{\sigma} + g(\sigma, \varepsilon, \theta) \dot{\theta} + h(\sigma, \varepsilon, \theta) \dot{\theta}, \quad (26)$$

which clearly reduces to Eq. (8) in the isothermal case.

The functions $f(\quad)$ and $g(\quad)$ are obtained from the two independent functions $H(\quad)$ and $G(\quad)$; hence they are, in turn, independent. The functions $f(\quad)$ and $h(\quad)$, however, are both derived from $H(\quad)$; hence there must exist a relation between them, which, after some analysis, turns out to be

$$\frac{\partial h}{\partial \sigma} + \bar{r} \frac{\partial h}{\partial \epsilon} = \frac{\partial f}{\partial \sigma} + h \frac{\partial f}{\partial \epsilon} . \quad (27)$$

In fact, we we shall see in the next section, knowledge of $f(\quad)$ leads to a unique solution for $H(\quad)$, and hence to $h(\quad)$. Consequently, we can determine the non-isothermal behavior of a material obeying the constitutive assumptions of Section 3 from its isothermal behavior over a range of temperatures.

5. Derivation of the Free-Energy Density for Viscoelastic Materials

We shall now consider the problem inverse to that of Section 4: Given a material obeying the constitutive equation (26) subject to the restriction (27), find the partial strains ϵ^0 and ϵ^1 , the free-energy density $\psi(\epsilon^0, \epsilon^1, \theta)$, and the rate equation (17).

The equivalence between Eqs. (25) and (26) furnishes a partial differential equation for $H(\sigma, \epsilon^1, \theta)$:

$$f(\sigma, \epsilon^1 + H, \theta) = \frac{\partial H}{\partial \sigma} \quad (28)$$

with the initial condition

$$H(0, \epsilon^1, \theta) = 0. \quad (29)$$

Equation (29) connotes the simultaneous vanishing of σ and ϵ^0 , so that

ε^0 is viewed as an "elastic" strain. This view is consistent with the linear analogue.

We may treat ε^1 and θ in Eqs. (28-29) as parameters, so that on recalling

$$\varepsilon = \varepsilon^1 + H(\quad) \quad (30)$$

we find the ordinary differential equation

$$f(\sigma, \varepsilon, \theta) = \frac{d\varepsilon}{d\sigma} \quad (31)$$

which we solve for

$$\varepsilon = \bar{\varepsilon}(\sigma; \varepsilon^1, \theta) \quad (32)$$

subject to the initial condition

$$\bar{\varepsilon}(0; \varepsilon^1, \theta) = \varepsilon^1. \quad (33)$$

Hence

$$H(\sigma, \varepsilon^1, \theta) = \bar{\varepsilon}(\sigma; \varepsilon^1, \theta) - \varepsilon^1. \quad (34)$$

The existence and uniqueness of $H(\quad)$ are subject only to the classical conditions for first-order ordinary differential equations: single-valuedness, boundedness, and Lipschitz condition of $f(\quad)$.

We may also solve Eqs. (31) for

$$\sigma = \bar{\sigma}(\varepsilon, \varepsilon^1; \theta) \quad (35)$$

subject to the initial condition

$$\bar{\sigma}(\varepsilon^1, \varepsilon^1; \theta) = 0 \quad (36)$$

(For a reason to be evident later, we treat ε^1 in $\bar{\sigma}(\quad)$ as a variable and not a parameter). On comparing Eqs. (14) and (35), we find

$$\rho_0 \frac{\partial \psi}{\partial \varepsilon^0} = \bar{\sigma}(\varepsilon^0 + \varepsilon^1, \varepsilon^1; \theta). \quad (37)$$

Hence, by quadrature,

$$\psi(\varepsilon^0, \varepsilon^1, \theta) = \frac{1}{\rho_0} \int_0^{\varepsilon^0 + \varepsilon^1} \bar{\sigma}(u, \varepsilon^1; \theta) du + \psi'(\varepsilon^1, \theta), \quad (38)$$

with $\psi'(\quad)$ undetermined at this point.

Once we know $H(\quad)$ we can also, in principle, determine $J(\quad)$, so that we can make $\partial H / \partial \varepsilon^1$ a function of σ , ε , and θ , whence we find

$$G(\sigma, \varepsilon, \theta) = g(\sigma, \varepsilon, \theta) / \partial H / \partial \varepsilon^1. \quad (39)$$

We shall proceed to determine $\psi'(\quad)$ for materials for which Eq. (17) possesses a solution for σ that is continuous in $\dot{\varepsilon}^1$ at $\dot{\varepsilon}^1 = 0$, and that therefore has the form

$$\sigma = K(\varepsilon, \theta) + o(\dot{\varepsilon}^1) \quad \text{as } \dot{\varepsilon}^1 \rightarrow 0, \quad (40)$$

with $K(\quad)$ being the "static stress". (We shall call such materials viscoelastic, in contradistinction to viscoplastic materials which will be discussed later). Following Coleman & Mizel (1964) we insert Eq. (40) into the inequality (18) and let $\dot{\varepsilon}^1$ vanish. Since the inequality must be satisfied whether $\dot{\varepsilon}^1$ approached zero through positive

or negative values, we must have

$$K(\varepsilon, \theta) = \rho_0 \frac{\partial \psi}{\partial \varepsilon^1}.$$

We note, however, that the left-hand and right-hand sides of this equation are not functions of the same variables. This is because it is valid only when $\dot{\varepsilon}^1 = 0$, i.e. when

$$\sigma = K(\varepsilon, \theta). \quad (41)$$

Combining Eqs. (35) and (41), we obtain

$$K(\varepsilon, \theta) = \bar{\sigma}(\varepsilon, \varepsilon^1; \theta) \quad (42)$$

which we solve for ε :

$$\varepsilon = \lambda(\varepsilon^1, \theta) \quad (43)$$

We see that at equilibrium the total strain (and hence the "rate-independent" strain ε^0) is determined by the "rate-dependent" strain ε^1 .

We now differentiate Eq. (38) with respect to ε^1 :

$$\begin{aligned} \rho_0 \frac{\partial \psi}{\partial \varepsilon^1} &= \bar{\sigma}(\varepsilon^0 + \varepsilon^1, \varepsilon^1; \theta) + \int_0^{\varepsilon^0 + \varepsilon^1} \frac{\partial \bar{\sigma}}{\partial \varepsilon^1}(u, \varepsilon^1; \theta) du \\ &+ \rho_0 \frac{\partial \psi'}{\partial \varepsilon^1}. \end{aligned} \quad (44)$$

This, however, must also equal

$$\bar{\sigma}(\varepsilon^0 + \varepsilon^1, \varepsilon^1; \theta)$$

when

$$\varepsilon^0 + \varepsilon^1 = \lambda(\varepsilon^1, \theta).$$

Consequently

$$\rho_0 \frac{\partial \psi^1}{\partial \varepsilon^1} = - \int_0^{\lambda(\varepsilon^1, \theta)} \frac{\partial \bar{\sigma}}{\partial \varepsilon^1}(u, \varepsilon^1; \theta) du, \quad (45)$$

whence $\psi^1(\quad)$ may be obtained by quadrature, subject to an arbitrary initial condition. If we let this be

$$\psi^1(0, \theta) = 0 \quad (46)$$

then we obtain, finally,

$$\rho_0 \psi(\varepsilon^0, \varepsilon^1, \theta) = \int_0^{\varepsilon^0 + \varepsilon^1} \bar{\sigma}(u, \varepsilon^1; \theta) du - \int_0^{\varepsilon^1} \int_0^{\lambda(v, \theta)} \frac{\partial \bar{\sigma}}{\partial v}(u, v; \theta) du dv. \quad (47)$$

6. Examples of Viscoelastic Materials

(a) "Separable-free-energy" materials. The analysis of Sections 4 and 5 is greatly simplified if the free-energy density of the material is of the form

$$\psi(\varepsilon^0, \varepsilon^1, \theta) = \psi^0(\varepsilon^0, \theta) + \psi^1(\varepsilon^1, \theta). \quad (48)$$

A necessary and sufficient condition for this form is

$$\frac{\partial^2 \psi}{\partial \varepsilon^0 \partial \varepsilon^1} = 0, \quad (49)$$

From Eq. (38) it is clear that Eq. (49) is equivalent to

$$\frac{\partial \bar{\sigma}}{\partial \varepsilon} + \frac{\partial \bar{\sigma}}{\partial \varepsilon^1} = 0, \quad (50)$$

which in turn implies

$$\bar{\sigma}(\varepsilon, \varepsilon^1, \theta) = \bar{\bar{\sigma}}(\varepsilon - \varepsilon^1, \theta). \quad (51)$$

Since Eqs. (32) and (35) are equivalent, we have

$$\bar{\varepsilon}(\sigma; \varepsilon^1, \theta) = \varepsilon^1 + \bar{\bar{\varepsilon}}(\sigma, \theta), \quad (52)$$

so that, from Eq. (34)

$$H(\sigma, \varepsilon^1, \theta) = \bar{\bar{H}}(\sigma, \theta). \quad (53)$$

Hence,

$$f(\sigma, \varepsilon, \theta) = \bar{f}(\sigma, \theta) \quad (54)$$

and

$$h(\sigma, \varepsilon, \theta) = \bar{h}(\sigma, \theta), \quad (55)$$

with $\bar{f}(\quad) = \partial \bar{\varepsilon} / \partial \sigma$ and $\bar{h}(\quad) = \partial \bar{\varepsilon} / \partial \theta$,

so that

$$\frac{\partial \bar{f}}{\partial \theta} = \frac{\partial \bar{h}}{\partial \sigma}. \quad (56)$$

Furthermore

$$g(\sigma, \varepsilon, \theta) = \bar{g}(\sigma, \varepsilon, \theta). \quad (57)$$

The special case

$$g(\sigma, \varepsilon, \theta) = \bar{g}(\sigma, \theta), \quad (58)$$

with

$$\bar{g}(0, \theta) = 0, \quad (59)$$

characterizes the nonlinear Maxwell material.

If $\bar{f}(\quad)$ is given, we determine $\bar{\sigma}(\quad)$ by solving

$$u = \int_0^\sigma \bar{f}(v, \theta) dv \quad (60)$$

for

$$\sigma = \bar{\sigma}(u, \theta). \quad (61)$$

For the partial free-energy densities we have

$$\rho_0 \psi^0 = \int_0^\varepsilon \bar{\sigma}(u, \theta) du, \quad (62)$$

$$\rho_0 \psi^1 = \int_0^\varepsilon \bar{\sigma}(\lambda(v, \theta) - v, \theta) dv. \quad (63)$$

For the nonlinear Maxwell material $\lambda(\varepsilon^1, \theta) = \varepsilon^1$, so that $\psi^1 = 0$.

The linear material has

$$\bar{f}(\sigma, \theta) = \frac{1}{E} \quad (64)$$

and

$$K(\varepsilon, \theta) = E' \varepsilon, \quad (65)$$

with E and E' functions of θ , and $E > E'$. We find that

$$\bar{\sigma}(u, \theta) = Eu \quad (66)$$

and

$$\lambda(\epsilon^1, \theta) = \frac{E}{E-E'} \epsilon^1, \quad (67)$$

so that

$$\rho_o \psi = \frac{1}{2} \left[E(\epsilon^o)^2 + \frac{EE'}{E-E'} (\epsilon^1)^2 \right]. \quad (68)$$

(b) Nonlinear materials corresponding to Fig. 1b. We shall now derive the constitutive equation for a material represented by the model of Fig. 1b with nonlinear but uncoupled elements. For simplicity we shall treat isothermal behavior only and omit the dependence on θ . We then have

$$\sigma = \sigma_a + \sigma_b, \quad (69)$$

with

$$\sigma_a = k(\epsilon) \quad (70)$$

describing the nonlinear spring, and

$$\dot{\epsilon} = m'(\sigma_b) \dot{\sigma}_b + n(\sigma_b) \quad (71)$$

describing the nonlinear Maxwell element in accord with the preceding subsection. On eliminating σ_a and σ_b we obtain

$$\dot{\varepsilon} = \frac{m'(\cdot)}{1 + k'(\varepsilon)m'(\cdot)} \dot{\sigma} + \frac{n(\cdot)}{1 + k'(\varepsilon)m'(\cdot)}, \quad (72)$$

the argument of $m'(\cdot)$ and $n(\cdot)$ being $\sigma - k(\varepsilon)$. Equation (72) is clearly of the form (1).

It is easy to show that

$$\bar{\sigma}(\varepsilon, \varepsilon^1) = k(\varepsilon) + m^{-1}(\varepsilon - \lambda(\varepsilon^1)) \quad (73)$$

and

$$\lambda(\varepsilon^1) = \varepsilon^1 - m(-k(\varepsilon^1)). \quad (74)$$

Hence, if we define partial free-energy densities $\psi_a(\cdot)$, $\psi_b(\cdot)$ by

$$k(z) = \rho \int_0^z \psi_a(z), \quad m^{-1}(z) = \rho \int_0^z \psi_b(z) \quad (75)$$

with

$$\psi_a(0) = \psi_b(0) = 0, \quad (76)$$

then

$$\psi(\varepsilon^0, \varepsilon^1) = \psi_a(\varepsilon^0 + \varepsilon^1) + \psi_b(\varepsilon^0 + m(-k(\varepsilon^1))) \quad (77)$$

A material of this type will also be of the "separable-free-energy" type if

$$\frac{\partial}{\partial \varepsilon} \frac{m'(\cdot)}{1 + k'(\varepsilon)m'(\cdot)} = 0, \quad (78)$$

which leads to

$$k'(\varepsilon) m''(\cdot) + k''(\varepsilon) [m'(\cdot)]^2 = 0; \quad (79)$$

hence, writing z for $\sigma - k(\epsilon)$, we find

$$\frac{k''(\epsilon)}{k'(\epsilon)} = - \frac{m''(z)}{[m'(z)]^2} = c \quad (80)$$

with c constant. Equation (80) yields ordinary differential equations for $k(\epsilon)$ and $m(z)$, which may be solved subject to

$$k(0) = m(0) = 0 \quad (81)$$

to give

$$k(\epsilon) = \frac{E'}{c} (e^{c\epsilon} - 1), \quad (82)$$

$$m(z) = \frac{1}{c} \ln \left(1 + \frac{cz}{E-E'} \right), \quad (83)$$

with E and E' constant, $E > E' > 0$. We thus have

$$\bar{f}(\sigma) = \frac{1}{E + c\sigma}, \quad (84)$$

$$g(\sigma, \epsilon) = \left(1 - \frac{E'e^{c\epsilon}}{E+c\sigma} \right) n \left(\sigma - \frac{E'}{c} (e^{c\epsilon} - 1) \right), \quad (85)$$

and

$$\begin{aligned} \rho_0 \psi = \frac{1}{c^2} & \left[E(e^{c\epsilon^0} - c\epsilon^0 - 1) \right. \\ & \left. + (E - E') \ln \left(\frac{E-E'}{E-E'e^{c\epsilon^1}} \right) - E'c\epsilon^1 \right]. \end{aligned} \quad (86)$$

7. Viscoplastic Materials

We shall now consider materials for which Eq. (17) does not possess a solution for σ whose behavior near $\dot{\epsilon}^1 = 0$ is described by

Eq. (40), but for which, instead, the solution of

$$G(\sigma, \varepsilon, \theta) = 0 \quad (87)$$

takes the form of an inequality:

$$-K_-(\varepsilon, \theta) \leq \sigma \leq K_+(\varepsilon, \theta). \quad (88)$$

Furthermore, for $\sigma > K_+(\quad)$ and $\sigma < -K_-(\quad)$ we have $G(\quad) > 0$ and $G(\quad) < 0$, respectively. Such materials will be called viscoplastic.

We shall write

$$G(\sigma, \varepsilon, \theta) = \bar{G}(\tau, \varepsilon, \theta), \quad (89)$$

where $\tau = \sigma - K_+(\quad)$ if $\sigma > K_+(\quad)$, $\tau = \sigma + K_-(\quad)$ if $\sigma < -K_-(\quad)$, and $\tau = 0$ whenever inequality (88) holds. Hence the equation

$$\bar{G}(\tau, \varepsilon, \theta) = \dot{\varepsilon}^1 \quad (90)$$

has a solution

$$\tau = M(\varepsilon, \dot{\varepsilon}^1, \theta) \quad (91)$$

which is continuous at $\dot{\varepsilon}^1 = 0$, with

$$M(\varepsilon, 0, \theta) = 0. \quad (92)$$

The solution for σ therefore has the form

$$\sigma = K(\varepsilon, \theta) + L(\varepsilon, \theta) \operatorname{sgn} \dot{\varepsilon}^1 + M(\varepsilon, \dot{\varepsilon}^1, \theta), \quad (93)$$

where

$$K(\quad) = \frac{1}{2} [K_+(\quad) - K_-(\quad)], \quad L(\quad) = \frac{1}{2} [K_+(\quad) + K_-(\quad)]. \quad (94)$$

A viscoplastic material for which $L(\quad)$ vanishes identically is consequently viscoelastic.

A plastic material may be defined as a viscoplastic one whose function $M(\quad)$ becomes vanishingly small. In the limit, however, the function $G(\quad)$ ceases to exist, so that a plastic material thus defined is not represented by the class of constitutive equations treated here. An alternative definition of plasticity may be obtained from a generalization of this class which takes instantaneous or rate-independent plastic deformation into effect; that is, the "rate-independent" strain ϵ^0 is assumed to consist of an elastic and a plastic part.

If a material obeying this assumption can undergo, in addition, a "rate-dependent" strain ϵ^1 governed by a rate equation, then it will be called plastic-viscoplastic. If there is no ϵ^1 , then the material is elastic-plastic; this is the type of material which, with some restrictive assumptions, was treated by Green & Naghdi (1965), who also studied its thermodynamics.

For the plastic-viscoplastic material the constitutive equation is no longer strictly quasilinear but piecewise quasilinear. That is, the term

$$f(\sigma, \epsilon, \theta) \dot{\sigma}$$

in Eq. (8) or (26) must be replaced by

$$f^e(\sigma, \epsilon, \theta) \dot{\sigma} + f^p(\sigma, \epsilon, \theta) \langle \dot{\sigma} \operatorname{sgn} \sigma \rangle,$$

where

$$\langle z \rangle = \frac{1}{2}(z + |z|).$$

The isothermal constitutive equation of the plastic-viscoplastic material was used as the basis of studies of wave propagation by Cristescu (1963) and Lubliner (1964). A thermodynamical study which would lead to a non-isothermal constitutive equation is in progress.

8. Materials of n th order

If instead of Eq. (9) we postulated

$$\epsilon = \epsilon^0 + \epsilon^1 + \dots + \epsilon^n \quad (95)$$

with ψ , η , and σ assumed as functions of $\epsilon^0, \epsilon^1, \dots, \epsilon^n, \dot{\epsilon}^1, \dots, \dot{\epsilon}^n$, θ , and γ , the Clausius-Duhem inequality would tell us that

$$\psi = \psi(\epsilon^0, \epsilon^1, \dots, \epsilon^n, \theta) \quad (96)$$

and that Eqs. (13) and (14) are valid. If, in addition, we had n rate equations

$$\Phi_i(\sigma, \epsilon^0, \epsilon^1, \dots, \epsilon^n, \dot{\epsilon}^1, \dots, \dot{\epsilon}^n, \theta) = 0, \quad i = 1, \dots, n, \quad (97)$$

then we would obtain, in principle, a single constitutive equation by eliminating the $(n + 1)^2$ quantities

$$\begin{array}{c} \epsilon^0, \epsilon^1, \dots, \epsilon^n \\ \dot{\epsilon}^0, \dot{\epsilon}^1, \dots, \dot{\epsilon}^n \\ \vdots \\ \binom{(n)}{\epsilon^0} \dots \dots \dots \binom{(n)}{\epsilon^n} \end{array}$$

from the set $(n + 1)^2 + 1$ equation consisting of the following:

Eq. (95)	1
Eq. (95) differentiated 1, ..., n times	n
Eq. (14)	1
Eq. (14) differentiated 1, ..., n times	n
Eq. (97)	n
Eq. (97) differentiated 1, ..., n-1 times	n(n-1)
	n ² +2n+2=(n+1) ² +1

Moreover, the equation thus obtained would be quasilinear.

9. Remarks

(a) It is easy to show that, had the entropy density η been used as a state variable in place of the temperature θ , the treatment would have proceeded analogously, leading to a non-isentropic constitutive equation:

$$\dot{\varepsilon} = \tilde{f}(\sigma, \varepsilon, \eta) \dot{\sigma} + \tilde{g}(\sigma, \varepsilon, \eta) + \tilde{h}(\sigma, \varepsilon, \eta) \dot{\eta} \tag{98}$$

with

$$\frac{\partial \tilde{h}}{\partial \sigma} + \tilde{f} \frac{\partial \tilde{h}}{\partial \varepsilon} = \frac{\partial \tilde{f}}{\partial \eta} + \tilde{h} \frac{\partial \tilde{f}}{\partial \varepsilon} . \tag{99}$$

This form would be of particular use in the study of shock waves.

In a "separable-free-energy" material the entropy density η is also separable: $\eta = \eta^0 + \eta^1$. For such a material η^0 would probably be a more convenient state variable than η .

(b) It would be a simple matter to extend the analysis to three-dimensional deformations if these were infinitesimal; Eq. (9) would be

replaced by

$$\epsilon_{ij} = \epsilon^0_{ij} + \epsilon^1_{ij} \quad (100)$$

and a consistent development would lead to

$$\dot{\epsilon}_{ij} = f_{ijkl} \dot{\sigma}_{kl} + g_{ij} + h_{ij} \dot{\theta}, \quad (101)$$

with f_{ijkl} , g_{ij} and h_{ij} symmetric tensor function of σ_{ij} , ϵ_{ij} , and θ , and with the restriction

$$\frac{\partial h_{ij}}{\partial \sigma_{kl}} + \frac{\partial h_{ij}}{\partial \epsilon_{mn}} f_{mnlk} = \frac{\partial f_{ijkl}}{\partial \theta} + \frac{\partial f_{ijkl}}{\partial \epsilon_{mn}} h_{mn}. \quad (102)$$

Viscoplastic materials could be included by defining a yield surface.

In principle there would be no obstacle to performing the same analysis with finite deformations if, for example, we replaced the infinitesimal strain ϵ_{ij} by the Green strain E_{ij} , and the stress σ_{ij} by the Kirchhoff-Piola stress S_{ij} . However, a hypothesis such as

$$E_{ij} = E^0_{ij} + E^1_{ij}, \quad (103)$$

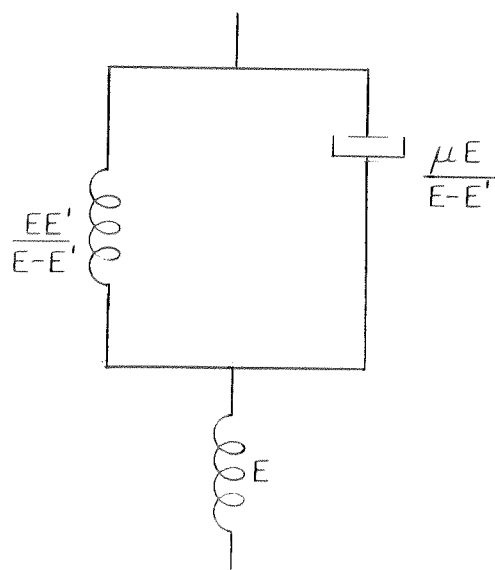
with superscripts interpreted as before, seems highly questionable on physical grounds. In particular, a preliminary study of thermodynamics based on Eq. (103) showed that an isotropic material of the "separable-free-energy" type could not, at the same time, be a fluid as defined by Noll (1958). For this and other reasons I believe that for finite three-dimensional deformations a better hypothesis would be

$$E_{ij} = E_{ij}(Z_{kl}^0, Z_{kl}^1) \quad (104)$$

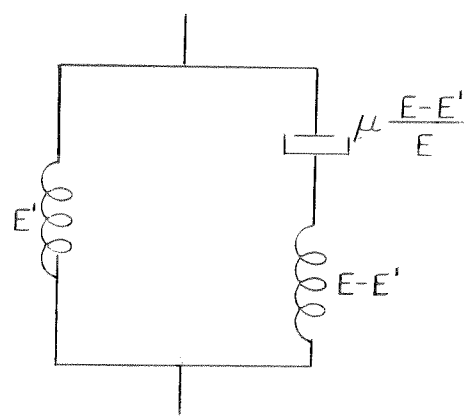
where Z_{kl}^0 and Z_{kl}^1 are hidden variables which describe, respectively, "rate-independent" and "rate-dependent" deformations, and whose choice is determined by additional physical requirements.

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(a)



(b)

FIG 1 "STANDARD LINEAR SOLID" MODELS