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CONSTITUTIVE EQUATIONS FOR A CLASS OF NONLINEAR ELASTIC SOLIDS

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ABSTRACT

Constitutive equations are developed for elastic solids sustaining deformation for which displacement gradients are small but where complete physical nonlinearity is permitted. The constitutive equation includes as special cases forms considered by recent authors; at the same time, more general effects are considered, in particular, coupling between volumetric and deviatoric effects.

Some simple states of deformation are examined and the plane elastostatic problem is formulated together with the approximate solution of an example by perturbation techniques.

NOTATION

U	Strain energy density
z_i	Rectangular cartesian coordinates
u_i	Displacements
$u_{i,j} = \frac{\partial u_i}{\partial z_j}$	Displacement gradients
ϵ_{ij}	Strain tensor
τ_{ij}	Stress tensor
I_i	Invariants of strain
$\phi_i = \frac{\partial U}{\partial I_i}$	Material strain functions
C	Complementary energy density
Θ_i	Invariants of stress
$\alpha_i = \frac{\partial C}{\partial \Theta_i}$	Material stress functions
A_{AB}, B_{AB}	Elastic constants
σ_i	Principal stress components
e_i	Principal strain components
$H(\ )$	Hessian determinant
$\nabla^2(\ )$	Harmonic operator
$\nabla^4(\ )$	Biharmonic operator
x^i	Curvilinear coordinates

INTRODUCTION

There exist real materials which, even for small deformations, exhibit nonlinear mechanical effects.

Examples of interest are materials such as concrete, solid propellants and foamed elastomers whose tensile and compressive responses differ and whose behavior is strongly dependent on superimposed hydrostatic stress. Another example is sand which dilates when subjected to a state of simple shearing stress.

While some of these materials are not entirely elastic, they are often analyzed by classical elasticity theory and, in order to describe more adequately some aspects of their mechanical behavior under small deformations, one is led to examine a theory of elastic solids for which kinematic linearity is retained but where physical nonlinearity is permitted. In other words, for the class considered, nonlinearity in the stress-strain relations is postulated to be more important than in the strain-displacement equations.

Although this nonlinearity is but a special case of the general nonlinear theory of elasticity (1)[1], rather than simplify more general results which are generally complicated, it has been more convenient to introduce kinematic restrictions initially and to develop the theory from this viewpoint.

The first significant contribution to physically nonlinear elasticity theory appears to have been in 1894 by Voigt (2), who extended the stress-strain law to include quadratic terms in strain and thus developed

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[1] Numbers in parentheses refer to the bibliography at the end of the text.

a five constant elasticity theory, applying it to the solution of simple problems. The same form of law was considered in 1937 by Murnaghan (3) and in 1940 by Biot (4).

Novozhilov (5) pointed out the restrictive nature of Voigt's five constant theory in its application to the behavior of most real materials and discussed a constitutive law retaining linear and cubic terms in strain and having six elastic constants. He also referred to earlier work by Bulffinger in 1729.

Sternberg (6), in 1946, further examined the five constant theory and applied it to the extension of a rod and to torsion of a circular cylinder.

The first application to boundary value problems appears to have been by Kauderer (7) who used perturbation techniques to obtain approximate solutions to a number of problems. His form of constitutive law, as will be seen, is quite restrictive.

Most recently, Savin (8) has extended Kauderer's work to include formulation of approximate solutions for the extension of an infinite plate with a hole.

Mention should also be made of the work of Dillon (9) who has considered coupled thermoelastic theory where nonlinearity is present with respect to mechanical and thermal variables.

A special class of viscoelastic materials has been considered by Rivlin (10) and by Bergen, Messersmith and Rivlin (11) with a constitutive law that is equivalent, for a given class of deformations, to a sub-class of that considered here.

The first section of the present work is devoted to the development of the most general form of constitutive law for isotropic media and to consideration of the corresponding inverse constitutive law. Restrictions on these laws are considered and special classes of constitutive laws are examined.

In the second section some simple states of deformation are investigated and the third section is devoted to formulation of the plane elastostatic problem together with an approximate solution, by perturbation techniques, of the extension of a nonlinear elastic plate containing a circular hole.

I. THE CONSTITUTIVE LAW FOR PHYSICALLY NONLINEAR ELASTIC SOLIDS

1.1 Introduction

In the following development, for mathematical simplicity, attention is restricted to homogeneous isotropic media.

The constitutive law is derived and its inverse is considered. The requirement that the constitutive law have a unique inverse places certain restrictions on the form of the constitutive law. Further restrictions are obtained from physical reasoning.

The section concludes with consideration of special classes of constitutive laws and those laws considered by other authors are related to the general case.

1.2 Formulation of the constitutive law for homogeneous isotropic solids

An elastic solid is defined (12) as one for which the state of stress depends only on the current state of deformation. A Green elastic, or hyperelastic, solid is one for which, in addition, there exists a scalar potential function, U, dependent only on the current state of deformation from which the constitutive law may be derived. The latter sub-class of materials will be considered in this work.

If displacement gradients are small such that

$$u_{i,j} << 1 \quad [1][2]$$

[1]Latin indices take on values 1, 2 or 3.

[2]Tensor notation is used and, for convenience, quantities are referred to rectangular Cartesian coordinates, z_i, unless otherwise specified.

the material deformation measure,

$$E_{ij} = 1/2\,(u_{i,j} + u_{j,i} + u_{k,i}\,u_{k,j}) ,$$

may be approximated by

$$\epsilon_{ij} = 1/2\,(u_{i,j} + u_{j,i}) .$$

In order that U , the strain energy density function, be objective, its assumed functional form

$$U = U(z_k, u_{i,j})$$

must be replaced by

$$U = U(z_k, \epsilon_{ij}) \tag{1.1}$$

and stress is determined from

$$\tau_{ij} = \frac{\partial U}{\partial \epsilon_{ij}} . \tag{1.2}$$

For homogeneous media, (1.1) becomes

$$U = U(\epsilon_{ij}) .$$

If the medium is now restricted to be isotropic, U is a scalar invariant under proper orthogonal coordinate transformations and may be expressed as

$$U = U(I_1, I_2, I_3) \tag{1.3}$$

where I_i , independent invariants of the strain tensor, are here taken to be

$$I_1 = \epsilon_{kk} ,$$

$$I_2 = \tfrac{1}{2}\epsilon_{km}\epsilon_{km} , \qquad (1.4)$$

$$I_3 = \tfrac{1}{3}\epsilon_{km}\epsilon_{kn}\epsilon_{mn} .$$

From (1.2) and (1.3)

$$\tau_{ij} = \frac{\partial U}{\partial I_1}\frac{\partial I_1}{\partial \epsilon_{ij}} + \frac{\partial U}{\partial I_2}\frac{\partial I_2}{\partial \epsilon_{ij}} + \frac{\partial U}{\partial I_3}\frac{\partial I_3}{\partial \epsilon_{ij}}$$

or

$$\tau_{ij} = \phi_1 \delta_{ij} + \phi_2 \epsilon_{ij} + \phi_3 \epsilon_{ik}\epsilon_{jk} , \qquad (1.5)$$

where

$$\phi_i = \phi_i(I_j) = \frac{\partial U}{\partial I_j} . \qquad (1.6)$$

From (1.6), the functions ϕ_i (material strain functions) are related by three equations

$$\frac{\partial \phi_i}{\partial I_j} = \frac{\partial \phi_j}{\partial I_i} . \qquad (1.7)$$

The constitutive law (1.5) may also be derived from the theory of isotropic matrices (13) without assuming the existence of U. The material strain functions, no longer restricted by (1.7), define a Cauchy elastic solid.

As an example of the constitutive law, let U be a continuous function which is approximated by a polynomial in I_i. If terms in U from second to fourth order in strain are retained, the 'cubic' stress-strain law is obtained:

$$\tau_{ij} = A_{11}\epsilon_{kk}\delta_{ij} + A_{12}\epsilon_{ij} + A_{21}\epsilon_{kk}^{2}\delta_{ij} + A_{22}\epsilon_{km}\epsilon_{km}\delta_{ij}$$
$$+ 2A_{22}\epsilon_{kk}\epsilon_{ij} + A_{23}\epsilon_{ik}\epsilon_{jk} + A_{31}\epsilon_{kk}^{3}\delta_{ij} + A_{32}\epsilon_{km}\epsilon_{km}\epsilon_{ij}$$
$$+ A_{33}\epsilon_{km}\epsilon_{kn}\epsilon_{nn}\delta_{ij} + A_{33}\epsilon_{kk}^{2}\epsilon_{ij} + A_{34}\epsilon_{kk}\epsilon_{ip}\epsilon_{jp}$$
$$+ 3A_{34}\epsilon_{km}\epsilon_{kn}\epsilon_{mn}\delta_{ij} \,. \tag{1.8}$$

Terms are grouped in order in (1.8), and it is readily seen that

$$\Phi_1 = A_{11}I_1 + A_{21}I_1^{2} + 2A_{22}I_2 + A_{31}I_1^{3} + 2A_{33}I_1I_2 + A_{34}I_3 \,,$$
$$\Phi_2 = A_{12} + 2A_{22}I_1 + 2A_{32}I_2 + A_{33}I_1^{2} \,,$$
$$\Phi_3 = A_{23} + A_{34}I_1 \,. \tag{1.9}$$

It is to be noted that for a nonlinear solid at small strain, reduction to a linear law for $\epsilon_{ij} \to 0$ need not be required.

There is, furthermore, no a priori reason for requiring that terms of a prescribed order be present in all material functions. The only reason for doing so here is to examine the most general polynomial law of a given order.

1.3 <u>The incompressible case</u>

For incompressible media,

$$I_1 = 0$$

and

$$U = U(I_2, I_3) \,.$$

Introducing a Lagrangian multiplier, $-P$, (1.2) becomes

$$\tau_{ij} = \frac{\partial U}{\partial \epsilon_{ij}} + (-P)\frac{\partial I_1}{\partial \epsilon_{ij}} \tag{1.10}$$

and, from (1.10),

$$\tau_{ij} = -P\delta_{ij} + \Phi_2\epsilon_{ij} + \Phi_3\epsilon_{ik}\epsilon_{jk} \,. \tag{1.11}$$

Φ_2 and Φ_3 are as defined by (1.6), and (1.7) gives the single equation

$$\frac{\partial \Phi_2}{\partial I_3} = \frac{\partial \Phi_3}{\partial I_2} \quad .$$

p has dimensions of stress and in certain cases has physical significance as hydrostatic pressure.

1.4 The Inverse Constitutive Law

Consider the scalar function, C , the complementary energy density, dependent only on the current state of stress such that

$$\epsilon_{ij} = \frac{\partial C}{\partial \tau_{ij}} \quad .$$

Then, by arguments identical to those used for the strain energy density function, for a homogeneous, isotropic solid

$$C = C(\Theta_1, \Theta_2, \Theta_3) \quad ,$$

where Θ_i , the invariants of the stress tensor, are given by

$$\begin{aligned} \Theta_1 &= \tau_{kk} \quad , \\ \Theta_2 &= 1/2\, \tau_{km} \tau_{km} \quad , \\ \Theta_3 &= 1/3\, \tau_{km} \tau_{kn} \tau_{mn} \end{aligned} \tag{1.12}$$

The inverse constitutive law

$$\epsilon_{ij} = \alpha_1 \delta_{ij} + \alpha_2 \tau_{ij} + \alpha_3 \tau_{ik} \tau_{jk} \tag{1.13}$$

is thus obtained, where

$$\alpha_i = \alpha_i(\Theta_j) = \frac{\partial C}{\partial \Theta_i} \quad . \tag{1.14}$$

From (1.14) it follows that the material stress functions, α_i , are related by three equations

$$\frac{\partial \alpha_i}{\partial \Theta_j} = \frac{\partial \alpha_j}{\partial \Theta_i} \quad . \tag{1.15}$$

The existence of the complementary energy density function follows from the existence of the strain energy density function in that C is the Legendre transformation of U (14), the two functions being related by

$$C = \tau_{ij}\epsilon_{ij} - U \quad . \tag{1.16}$$

The condition that (1.5) has a unique inverse (1.13) is that the Hessian determinant of U is non vanishing, i.e.

$$H(U) \equiv \mathrm{DET}\left|\frac{\partial^2 U}{\partial \epsilon_{ij}\partial \epsilon_{km}}\right| \neq 0 \quad . \tag{1.17}$$

The condition (1.17), which is identical to the non vanishing of the Jacobian of $\tau_{ij}(\epsilon_{mn})$, is taken as a restriction on the material strain functions.

The restriction is, in fact, equivalent to Drucker's postulate of stability (15). A proof of this equivalence follows.

Drucker's stability postulate states that, if a unit volume of the body is subjected to a homogeneous state of stress, τ_{ij} , and corresponding strain, ϵ_{ij} , and that if, due to an additional small load, stress and strain increase by $\delta\tau_{ij}$ and $\delta\epsilon_{ij}$ respectively, then

$$\delta W = \delta\tau_{ij}\,\delta\epsilon_{ij} > 0 \tag{1.18}$$

for a stable material.

Since, for an elastic solid,

$$\tau_{ij} = \tau_{ij}(\epsilon_{km}) ,$$

it follows that

$$\delta \tau_{ij} = \frac{\partial \tau_{ij}}{\partial \epsilon_{mn}} \delta \epsilon_{mn} = \frac{\partial^2 U}{\partial \epsilon_{ij} \partial \epsilon_{mn}} \delta \epsilon_{mn}$$

and

$$\delta W = \frac{\partial^2 U}{\partial \epsilon_{ij} \partial \epsilon_{mn}} \delta \epsilon_{ij} \delta \epsilon_{mn} .$$

For convenience, ϵ_{ij} will now be denoted by ϵ_A where A can take on values from 1 to 6. Then

$$\delta W = \frac{\partial^2 U}{\partial \epsilon_A \partial \epsilon_B} \delta \epsilon_A \delta \epsilon_B . \qquad (1.19)$$

(1.19) is a quadratic form and, since $\frac{\partial^2 U}{\partial \epsilon_A \partial \epsilon_B}$ are the components of a real symmetric matrix, (1.19) may, by a change of coordinates, be expressed as

$$\delta W = \sum_{A=1}^{6} \lambda_A \delta \epsilon'_A \delta \epsilon'_A$$

where λ_A are the eigenvalues of the matrix $\frac{\partial^2 U}{\partial \epsilon_A \partial \epsilon_B}$ (16).

Since $\delta \epsilon'_A$ are arbitrary, the condition (1.18) requires that

$$\lambda_A > 0 , \quad (A = 1, \ldots, 6) \qquad (1.20)$$

Also

$$\mathrm{DET} \left| \frac{\partial^2 U}{\partial \epsilon_A \partial \epsilon_B} \right| \equiv H(U) = \prod_{A=1}^{6} \lambda_A .$$

Thus, if (1.20) is satisfied,

$$H(U) > 0$$

and (1.5) has a unique inverse (1.13).

Conversely, the condition

$$H(U) = 0$$

represents incipient instability.

Identical arguments show that stability infers

$$H(C) > 0 \tag{1.21}$$

with (1.13) having a unique inverse (1.5) and that the condition

$$H(C) = 0$$

represents incipient instability.

In figure 1, the conditions of stability are represented for a one-dimensional stress space.

(1.16) may be used as a starting point for obtaining inverse constitutive laws.

Suppose that the form of (1.5) and hence of U is known.

Since, from (1.5),

$$\tau_{ij}\epsilon_{ij} = \phi_1 I_1 + 2\phi_2 I_2 + 3\phi_3 I_3 , \tag{1.22}$$

(1.16) may be used to obtain $C(I_i)$.

From (1.5) and making use of the Cayley-Hamilton theorem, $\theta_i(I_j)$ are obtained, the relations being

$$\Theta_1 = 3\phi_1 + \phi_2 I_1 + 2\phi_3 I_2 \;,$$

$$\Theta_2 = 3/2\,\phi_1^2 + \phi_1\phi_2 I_1 + (\phi_2^2 + 2\phi_1\phi_3) I_2 + 3\phi_2\phi_3 I_3$$
$$+ \phi_3^2 \left(1/12\, I_1^4 - I_1^2 I_2 + I_2^2 + 2 I_1 I_3 \right) , \tag{1.23}$$

$$\Theta_3 = \phi_1^3 + \phi_1^2\phi_2 I_1 + 2(\phi_1\phi_2^2 + \phi_1^2\phi_3) I_2 + (\phi_2^3 + 6\phi_1\phi_2\phi_3) I_3$$
$$+ (\phi_1\phi_3^2 + \phi_2^2\phi_3)(1/6\, I_1^4 - 2 I_1^2 I_2 + 2 I_2^2 + 4 I_1 I_3)$$
$$+ \phi_2\phi_3^2 \left(1/6\, I_1^5 + 5/3\, I_1^3 I_2 + 5/2\, I_1^2 I_3 + 5 I_2 I_3 \right)$$
$$+ 1/3\,\phi_3^3 \left(1/12\, I_1^6 - 1/2\, I_1^4 I^2 - 3 I_1^2 I_2^2 + I_1^3 I_3 + 2 I_2^3 + 3 I_3^2 + 6 I_1 I_2 I_3 \right).$$

$C(\Theta_i)$ is then computed.

As Truesdell (ibid), who uses a different but equivalent approach based on isotropic functions, points out, the analysis in general will be purely formal.

If (1.5) is known in polynomial form, however, (1.13) may be obtained as a power series in Θ_i by using (1.23) and equating coefficients of the power series $I_1^L I_2^M I_3^N$.

As an example, suppose that (1.5) contains linear and quadratic terms, i.e.

$$\tau_{ij} = A_{11}\epsilon_{kk}\delta_{ij} + A_{12}\epsilon_{ij} + A_{21}\epsilon_{kk}^2\delta_{ij} + A_{22}\epsilon_{km}\epsilon_{km}\delta_{ij}$$
$$+ 2A_{22}\epsilon_{kk}\epsilon_{ij} + A_{23}\epsilon_{ik}\epsilon_{jk} \;. \tag{1.24}$$

This corresponds to

$$U = 1/2\, A_{11} I_1^2 + A_{12} I_2 + 1/3\, A_{21} I_1^3 + 2 A_{22} I_1 I_2 + A_{23} I_3 \,.$$

Using (1.22), (1.16) gives

$$C(I_i) = 1/2\, A_{11} I_1^2 + A_{12} I_2 + 2/3\, A_{21} I_1^3 + 4 A_{22} I_1 I_2 + 2 A_{23} I_3 \,. \tag{1.25}$$

C is now expressed as a power series in Θ_i . Retaining linear and quadratic terms only,

$$C = \tfrac{1}{2} B_{11} \Theta_1^2 + B_{12} \Theta_2 + \tfrac{1}{3} B_{21} \Theta_1^3 + 2 B_{22} \Theta_1 \Theta_2 + B_{23} \Theta_3 + \cdots , \tag{1.26}$$

corresponding to

$$\begin{aligned} \epsilon_{ij} = {} & B_{11} \tau_{kk} \delta_{ij} + B_{12} \tau_{ij} + B_{21} \tau^2_{kk} \delta_{ij} + B_{22} \tau_{km} \tau_{km} \delta_{ij} \\ & + 2 B_{22} \tau_{kk} \tau_{ij} + B_{23} \tau_{ik} \tau_{jk} + \cdots\cdots . \end{aligned} \tag{1.27}$$

Using (1.23) in (1.26) and comparing coefficients of terms in (1.25) and (1.26), the following results are obtained:

$$B_{12} = \frac{1}{A_{12}} ,$$

$$B_{11} = - \frac{A_{11}}{A_{12}(3A_{11} + A_{12})} ,$$

$$B_{23} = - \frac{A_{23}}{A_{12}^3} ,$$

$$B_{22} = \frac{A_{11}A_{23} - A_{12}A_{22}}{A_{12}^3 (3A_{11} + A_{12})} ,$$

$$B_{21} = \frac{3A_{11}\left[A_{12}A_{22}(3A_{11} + 2A_{12}) - A_{11}A_{23}(2A_{11} + A_{12})\right]}{A_{12}^3 (3A_{11} + A_{12})^3} .$$

The inverse law (1.27) containing only linear and quadratic terms in stress would be a close approximation to the actual inverse of the quadratic law (1.24) only for strains small enough such that the effect of nonlinearity in (1.24) is small. For larger strains where nonlinearity is significant, higher order terms would have to be retained in (1.26) in order to obtain a good approximation to the actual inverse law.

1.5 Further restrictions on the material functions

The material functions, ϕ_i , are restricted by (1.7) and (1.17). Further restrictions follow from considerations similar to those examined by Truesdell (ibid) and by Baker and Ericksen (17) for general nonlinear elastic solids:

1. A zero state of stress must correspond to a zero state of strain (for compressible media).

From (1.5), this requires

$$\Phi_1(0) = 0 \quad . \tag{1.28}$$

When a polynomial constitutive law is used, (1.28) is satisfied if U does not contain a term of the type $A_{01} I_1$

2. The greatest principal strain occurs in the direction of greatest principal stress.

Let σ_i be principal stresses, e_i be principal strains associated with a given deformation. From (1.5)

$$\sigma_1 = \Phi_1 + \Phi_2 e_1 + \Phi_3 e_1^2 \ ,$$

$$\sigma_2 = \Phi_1 + \Phi_2 e_2 + \Phi_3 e_2^2 \ ,$$

whence

$$(\sigma_1 - \sigma_2) = (e_1 - e_2)\left[\Phi_2 + \Phi_3(e_1 + e_2)\right] . \tag{1.29}$$

From (1.29) it is required that

$$\Phi_2 + \Phi_3(e_1 + e_2) > 0 \quad \text{if} \quad e_1 \neq e_2 \ ,$$

$$\Phi_2 + \Phi_3(e_1 + e_2) \geq 0 \quad \text{if} \quad e_1 = e_2 \ . \tag{1.30}$$

Arguments identical to those used above give restrictions on α_i in addition to (1.14) and (1.21), which are equivalent to (1.28) and (1.30), i.e.

$$\alpha_1(0) = 0$$

and

$$\alpha_2 + \alpha_3(\sigma_1 + \sigma_2) > 0 \quad \text{if} \quad \sigma_1 \neq \sigma_2 \ ,$$

$$\alpha_2 + \alpha_3(\sigma_1 + \sigma_2) \geqslant 0 \quad \text{if} \quad \sigma_1 = \sigma_2 \ .$$

For materials where the deviation from linearity is small, condition (1.30) is automatically satisfied. This follows from the dominance in Φ_2 of the positive constant A_{12} . In general, however, (1.30) is an independent restriction on the constitutive law and is not a consequence of (1.17).

1.6 Special classes of constitutive laws

There are sub-classes of the constitutive laws (1.5) and (1.13) which have been considered previously or which may describe the behavior of special classes of materials. Two particular sub-classes are now considered.

1. Materials for which the strain energy function does not depend on I_3

If

$$U = U(I_1, I_2) \ , \tag{1.31}$$

then

$$\Phi_3 = 0 \tag{1.32}$$

and (1.5) reduces to

$$\tau_{ij} = \Phi_1 \delta_{ij} + \Phi_2 \epsilon_{ij} \quad . \tag{1.33}$$

Such a constitutive law was considered by Wainwright (18) in application to thin shell theory and by Dong (19) for viscoelastic solids.

For incompressible solids, (1.33) has the form

$$\tau_{ij} = -p\delta_{ij} + \Phi_2 \epsilon_{ij}$$

This form of law was considered by Berger, Messersmith and Rivlin (ibid) in connection with work on viscoelastic solids.

Reduction of the constitutive law to the form (1.33) is valid only where experimental evidence shows the condition (1.32) to be true. The form of (1.5) cannot be simplified by geometric arguments.

When (1.32) holds, then the inverse law (1.13) may be similarly simplified; i.e.

$$\alpha_3 = 0 \tag{1.34}$$

and

$$\epsilon_{ij} = \alpha_0 \delta_{ij} + \alpha_1 \tau_{ij} \quad . \tag{1.35}$$

A proof of this is as follows:

When condition (1.32) holds, it is seen from (1.22) and (1.23) that

$$\tau_{ij}\epsilon_{ij} = \tau_{ij}\epsilon_{ij}(I_1, I_2) \tag{1.36}$$

and that

$$\Theta_1 = \Theta_1(I_1, I_2),$$

$$\Theta_2 = \Theta_2(I_1, I_2), \qquad (1.37)$$

$$\Theta_3 = \Theta_3(I_1, I_2, I_3).$$

From (1.16), making use of (1.31) and (1.36),

$$C(I_i) = C(I_1, I_2) \quad . \qquad (1.38)$$

If the complementary energy density is now expressed in terms of stress invariants, it follows from (1.37) and (1.38) that C cannot depend on Θ_3 since Θ_3, in turn, depends on I_3.

Hence,

$$C(\Theta_i) = C(\Theta_1, \Theta_2)$$

and (1.34) and (1.35) follow.

2. Materials for which hydrostatic stress depends only on volumetric strain

A class of materials which may be of interest is that for which the hydrostatic stress, τ_{kk}, is a function of the volumetric strain, ϵ_{kk}, i.e.

$$\Theta_1 = \Theta_1(I_1) \quad . \qquad (1.39)$$

The form taken by (1.5) to satisfy this condition is obtained as follows:

Rewriting the first of (1.23),

$$\Theta_1 = 3\Phi_1 + I_1\Phi_2 + 2I_2\Phi_3 \quad . \qquad (1.40)$$

With a view to satisfying (1.39), let

$$\phi_1 = f_1(I_1) + f_2(I_1, I_2, I_3) \tag{1.41}$$

From (1.39), (1.40) and (1.7), f_2 , ϕ_2 and ϕ_3 must satisfy the four conditions

$$\begin{aligned} &3f_2 + I_1\phi_2 + 2I_2\phi_3 = 0 \,, \\ &\frac{\partial f_2}{\partial I_2} = \frac{\partial \phi_2}{\partial I_1} \,, \\ &\frac{\partial f_2}{\partial I_3} = \frac{\partial \phi_3}{\partial I_1} \,, \\ &\frac{\partial \phi_2}{\partial I_3} = \frac{\partial \phi_3}{\partial I_2} \,. \end{aligned} \tag{1.42}$$

Conditions (1.42) can be satisfied only if f_2 , ϕ_2 and ϕ_3 are of the form

$$\begin{aligned} f_2 &= 1/3\, I_1 f_3(I_2 - 1/6\, I_1^2) \,, \\ \phi_2 &= f_3(I_2 - 1/6\, I_1^2) \,, \\ \phi_3 &= 0 \,. \end{aligned} \tag{1.43}$$

Thus, from (1.41) and (1.43), it follows that, if the constitutive law satisfies the condition (1.39), it has the form

$$\begin{aligned} \tau_{ij} = &\left[f_1(\epsilon_{kk}) - 1/3\,\epsilon_{kk} f_3(\epsilon_{km}\epsilon_{km} - 1/3\,\epsilon_{kk}^2)\right]\delta_{ij} \\ &+ f_3(\epsilon_{km}\epsilon_{km} - 1/3\,\epsilon_{kk}^2)\,\epsilon_{ij} \,. \end{aligned} \tag{1.44}$$

Equation (1.44) is the form of constitutive law used by Kauderer (ibid) who obtained it by assuming, in addition to (1.39), that the deviatoric stress was related to deviatoric strain through the second invariant of deviatoric strain. From (1.44) it is readily seen that

$$\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij} = \left[\epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij}\right] f_3\left(\epsilon_{km}\epsilon_{km} - \tfrac{1}{3}\epsilon_{kk}^2\right) \tag{1.45}$$

If f_1 and f_3 are analytic, (1.44) may be expressed as

$$\tau_{ij} = \left[\sum_{n=1}^{N} a_n \epsilon_{kk}^n - \tfrac{1}{3}\epsilon_{kk}\sum_{n=0}^{M} b_n\left(\epsilon_{km}\epsilon_{km} - \tfrac{1}{3}\epsilon_{kk}^2\right)^n\right]\delta_{ij}$$
$$+ \sum_{n=0}^{M} b_n\left(\epsilon_{km}\epsilon_{km} - \tfrac{1}{3}\epsilon_{kk}^2\right)^n \epsilon_{ij} \; . \tag{1.46}$$

If (1.5) is further restricted so that deviatoric stress is a function of deviatoric strain only, i.e.

$$\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij} = f\left(\epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij}\right) ,$$

then from (1.45), f_3 must be a constant and the coefficients in (1.46) are restricted to

$$b_0 = b \, ,$$

$$b_n = 0 \quad \text{if } n \geq 1 \; .$$

Then (1.5) becomes

$$\tau_{ij} = \sum_{n=1}^{M} a_n'(\epsilon_{kk})^n \delta_{ij} + b\epsilon_{ij} \, , \tag{1.47}$$

where

$$a'_1 = a_1 - \tfrac{1}{3}b ,$$

$$a'_n = a_n \text{ if } n \geqslant 2 .$$

Equation (1.47) describes behavior which is linear in shear and non-linear in bulk response. It describes, for instance, the behavior of certain fibrous composites which are linear in shear but whose response in simple compression differs from that in simple tension even for very small strains.

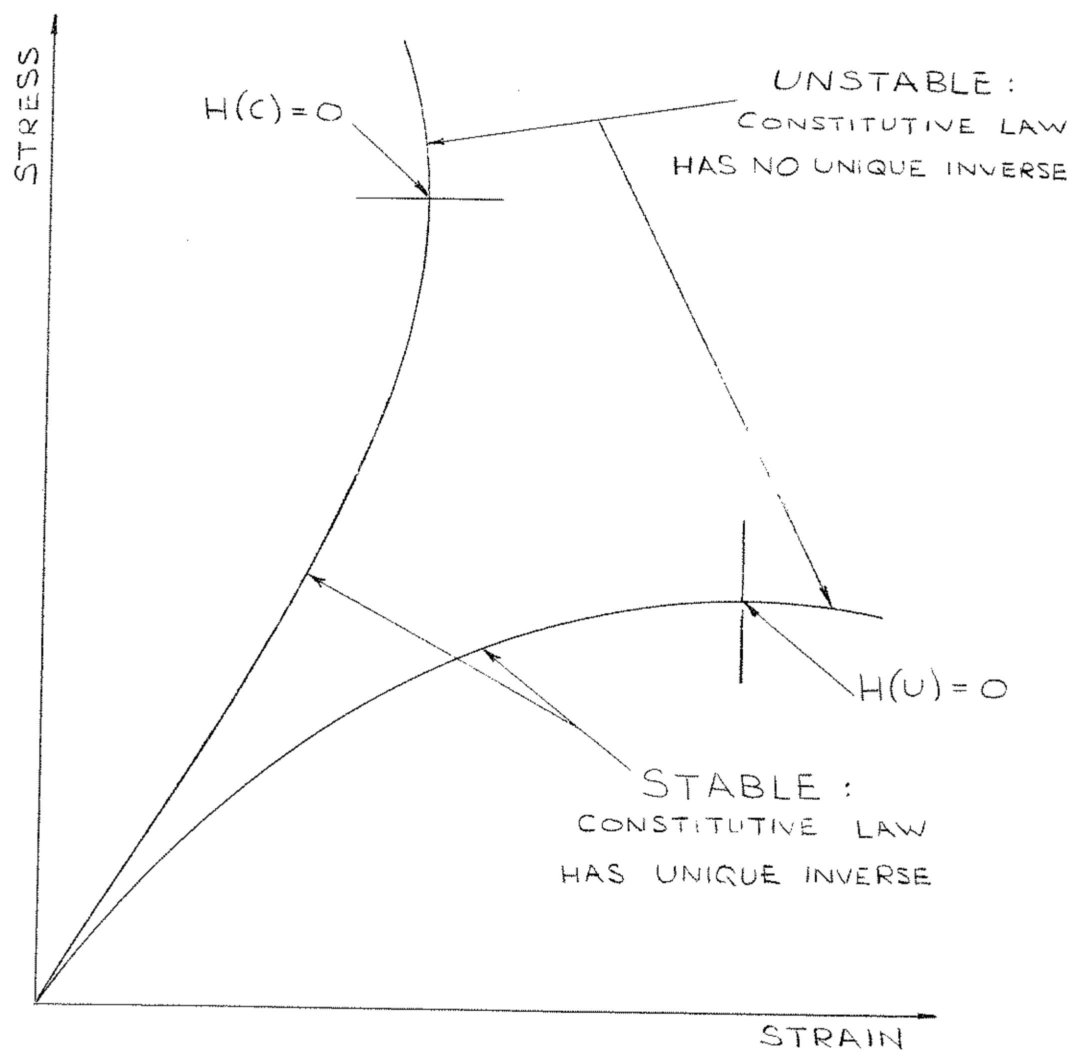

STRESS
H(C)=0
UNSTABLE :
CONSTITUTIVE LAW
HAS NO UNIQUE INVERSE
H(U)=0
STABLE :
CONSTITUTIVE LAW
HAS UNIQUE INVERSE
STRAIN

Figure 1

II. SIMPLE STATES OF DEFORMATION

2.1 Introduction

The examination of simple states of deformation gives a clear illustration of effects resulting from nonlinear constitutive laws.

When left in general form involving material functions of strain or stress, however, the resulting expressions may be misleading since material functions are, in certain cases, isolated. The material functions, however, are themselves functions of the state of strain (or stress) and cannot be determined from a single test.

No attempt is made here to describe general methods for experimental determination of the material functions, this, in general, not being possible unless they are expressed in polynomial form.

Three homogeneous states of stress are first examined using either form of constitutive law (1.5) or (1.13) depending on which is more convenient.

The problem of combined tension and torsion of a circular bar is then formulated and, for incompressible media, it is shown that the resulting equations simplify and may be solved in closed form.

2.2 Homogeneous states of stress

1. Simple tension: $\tau_{11} = T$, $\tau_{ij} = 0$ $i \neq 1, j \neq 1$

It is convenient to use (1.13) as the constitutive law.

$$\Theta_1 = T ,$$

$$\Theta_2 = 1/2\, T^2 ,$$

$$\Theta_3 = 1/3\, T^3 ,$$

and strains are given by

$$\epsilon_{11} = \alpha_1 + \alpha_2 T + \alpha_3 T^2,$$

$$\epsilon_{22} = \epsilon_{33} = \alpha_1 ,$$

$$\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0 .$$

For a quadratic law (1.27), for instance,

$$\alpha_1 = B_{11} T + B_{21} T^2 + B_{22} T^2,$$

$$\alpha_2 = B_{12} + 2 B_{22} T ,$$

$$\alpha_3 = B_{23} ,$$

and

$$\epsilon_{11} = (B_{11} + B_{12}) T + (B_{21} + 3 B_{22} + B_{23}) T^2$$

$$\epsilon_{22} = \epsilon_{33} = B_{11} T + (B_{21} + B_{22}) T^2 .$$

It is to be noted that by using (1.13) instead of (1.5), the problem of existence and uniqueness of solution, such as occurs in general nonlinear theory (1), is avoided.

2. Simple shear: Let $\epsilon_{12} = \epsilon$ be the only non-vanishing component of strain.

Using the constitutive law (1.5),

$$I_1 = 0 ,$$

$$I_2 = \epsilon^2 ,$$

$$I_3 = 0 ,$$

and

$$\tau_{12} = \Phi_2 \epsilon ,$$

$$\tau_{11} = \tau_{22} = \Phi_1 + \Phi_3 \epsilon^2 ,$$

$$\tau_{33} = \Phi_1 ,$$

$$\tau_{13} = \tau_{23} = 0 .$$

In general, for this state of strain, Φ_1 and Φ_3 will not be zero and consequently, in order to maintain simple shear deformation, normal stresses must be applied. This is the Poynting effect.

Furthermore, since

$$\tau_{kk} = 3\Phi_1 + 2\Phi_3 \epsilon^2 ,$$

in general, a hydrostatic stress must be applied in order to maintain the deformation. The requirement of such stress to maintain simple shear is known as the Kelvin effect.

This state of stress illustrates the difficulty, mentioned in the introduction to this chapter, regarding experimental determination of the material functions. Although the relationship between shear stress and shear strain involves only the material function Φ_2 , this does not mean that it would be possible to determine Φ_2 from a shear test. Since, for simple shear,

$$I_1 = I_3 = 0 ,$$

$\Phi_2 (I_1, I_2, I_3)$ could only be determined to the extent of its dependence on I_2 .

3. <u>Combined hydrostatic stress and shear stress</u>

$$\tau_{12} = \tau, \quad \tau_{11} = \tau_{22} = \tau_{33} = 1/3\,P, \quad \tau_{13} = \tau_{23} = 0 .$$

Using (1.13)

$$\Theta_1 = P ,$$

$$\Theta_2 = 1/6\,P^2 + \tau^2 ,$$

$$\Theta_3 = P(1/27\,P^2 + 2/3\,\tau^2) ,$$

and strains are given by

$$\epsilon_{11} = \epsilon_{22} = \alpha_1 + 1/3\,\alpha_2 P + \alpha_3 (1/9\,P^2 + \tau^2) ,$$

$$\epsilon_{33} = \alpha_1 + 1/3\,\alpha_2 P + 1/9\,\alpha_3 P^2 ,$$

$$\epsilon_{12} = \alpha_2 \tau + 1/3\,\alpha_3 P \tau ,$$

$$\epsilon_{13} = \epsilon_{23} = 0 .$$

Using the quadratic law (1.27) ,

$$\alpha_1 = B_{11} P + B_{21} P^2 + B_{22} (1/6\,P^2 + \tau^2) ,$$

$$\alpha_2 = B_{12} + 2 B_{22} P ,$$

$$\alpha_3 = B_{23} ,$$

and

$$\epsilon_{11} = \epsilon_{22} = (B_{11} + 1/3\,B_{12}) P + (B_{21} + 5/6\,B_{22} + 1/9\,B_{23}) P^2 + (B_{22} + B_{23}) \tau^2 ,$$

$$\epsilon_{33} = (B_{11} + \tfrac{1}{3}B_{12})P + (B_{21} + \tfrac{5}{6}B_{22} + \tfrac{1}{9}B_{23})P^2 + B_{22}\tau^2 ,$$

$$\epsilon_{12} = B_{12}\tau + (2B_{22} + \tfrac{1}{3}B_{23})P\tau .$$

This state of stress illustrates the coupling which, in general, exists between bulk and deviatoric effects and is particularly significant for filled heterogeneous materials.

2.3 Combined tension and torsion of a circular rod

Let a circular rod of outer radius R_o and inner radius R_I be subjected to an axial force, N , and a twisting moment, T .

The problem is formulated by assuming the form of the displacement field and calculating the corresponding stresses required to support this field.

Using polar coordinates $r, \theta, z\ (x^1, x^2, x^3)$ as shown in figure 2, the assumed displacement field is

$$\begin{aligned} u^{(1)} &= u(r) , \\ u^{(2)} &= \psi z r , \\ u^{(3)} &= \lambda z . \end{aligned} \tag{2.1}$$

where ψ is the angle of twist per unit length, λ is the uniform axial extension and $u(r)$ the unknown radial displacement.

To use the constitutive law (1.5) in curvilinear coordinates, it is first written, using mixed tensor components, as

$$\tau^i{}_j = \Phi_1 \delta^i{}_j + \Phi_2 \epsilon^i{}_j + \Phi_3 \epsilon^i{}_k \epsilon^k{}_j \tag{2.2}$$

where

$$\epsilon^i{}_j = g^{ik}\epsilon_{kj} = 1/2\, g^{ik}(u_{k|j} + u_{j|k}) ,$$

$$u_{i|j} = u_{i,j} - \left\{ {m \atop ij} \right\} u_m ,$$

and

$$u_i = \sqrt{g_{ii}}\, u^{(i)} . \quad (\text{no sum})$$

In expressing the above quantities with respect to curvilinear coordinates, the usual notation of tensor analysis (20) is employed.

For polar coordinates

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\left\{ {i \atop jk} \right\} = 0 \qquad \text{except} \qquad \left\{ {1 \atop 22} \right\} = -r , \left\{ {2 \atop 12} \right\} = \left\{ {2 \atop 21} \right\} = \frac{1}{r} .$$

Then from (2.1)

$$u_i = \begin{bmatrix} u \\ \psi z r^2 \\ \lambda z \end{bmatrix}$$

and the mixed strain components are

$$\epsilon^i{}_j = \begin{bmatrix} \frac{du}{dr} & 0 & 0 \\ 0 & \frac{u}{r} & \frac{1}{2}\psi \\ 0 & \frac{1}{2}\psi r^2 & \lambda \end{bmatrix}$$

The strain invariants are given by

$$I_1 = \epsilon^k_k = \frac{du}{dr} + \frac{u}{r} + \lambda \quad , \tag{2.3}$$

$$2\,I_2 = \epsilon^j_k \epsilon^k_j = \left(\frac{du}{dr}\right)^2 + \left(\frac{u}{r}\right)^2 + \lambda^2 + \tfrac{1}{2}\psi^2 r^2 ,$$

$$3\,I_3 = \epsilon^i_j \epsilon^j_k \epsilon^k_i = \left(\frac{du}{dr}\right)^3 + \left(\frac{u}{r}\right)^3 + \lambda^3 + \tfrac{1}{2}\lambda\psi^2 r^2 + \tfrac{1}{2}u\psi^2 r ,$$

and the mixed stress components by

$$\tau^1_1 = \Phi_1 + \Phi_2 \frac{du}{dr} + \Phi_3 \left(\frac{du}{dr}\right)^2 ,$$

$$\tau^2_2 = \Phi_1 + \Phi_2 \frac{u}{r} + \Phi_3 \left[\left(\frac{u}{r}\right)^2 + \tfrac{1}{4}\psi^2 r^2\right] , \tag{2.4}$$

$$\tau^3_3 = \Phi_1 + \Phi_2 \lambda + \Phi_3 \left[\lambda^2 + \tfrac{1}{4}\psi^2 r^2\right] ,$$

$$\tau^2_3 = \tfrac{1}{2}\psi\left[\Phi_2 + \Phi_3\left(\frac{u}{r} + \lambda\right)\right] ,$$

$$\tau^1_2 = \tau^1_3 = 0 \quad .$$

The equilibrium equations ,

$$\tau^i_{j|i} = 0 \quad ,$$

are identically satisfied except for the first which yields

$$\frac{d\tau^1_1}{dr} + \frac{1}{r}\left(\tau^1_1 - \tau^2_2\right) = 0 \quad . \tag{2.5}$$

The boundary conditions are

$$\tau'_1(r = R_I) = 0 \ ,$$

$$\tau'_1(r = R_o) = 0 \ , \qquad (2.6)$$

$$2\pi \int_{R_I}^{R_o} \tau_3^3 \, r \, dr = N \ ,$$

$$2\pi \int_{R_I}^{R_o} \tau_3^2 \, r^2 dr = T \ ,$$

Substitution from (2.4) into (2.5) leads to a nonlinear homogeneous differential equation of second order in u and first order in ψ and λ. This, together with the boundary conditions (2.6) formulates the problem. The solution will, in general, not be obtainable in closed form and, rather than examine approximate solution schemes, the particular case of incompressibility is considered, for which an exact solution is obtainable.

The constitutive law is now (1.11) instead of (1.5) and, by replacing ϕ_o by $-p$ in (2.4) and setting

$$I_1 = 0 \qquad (2.7)$$

in the first of (2.3), the above formulation carries over to the incompressible case.

Making use of (2.7), (2.3) yields

$$\frac{du}{dr} + \frac{u}{r} - \lambda = 0 \ ,$$

from which u is given by

$$u = \frac{A}{r} - \tfrac{1}{2}\lambda r \ .$$

and the incompressible problem is reduced to quadratures.

For simplicity, the rod is taken to be solid, so that

$$R_I = 0 \;, \quad R_0 = a \;.$$

Then

$$A = 0$$

and

$$u = -\tfrac{1}{2}\lambda r \,.$$

The strain invariants from (2.3) become

$$I_1 = \tfrac{1}{4}\left(3\lambda^2 + \psi^2 r^2\right) \,,$$

$$I_2 = \tfrac{1}{12}\lambda\left(3\lambda^2 + \psi^2 r^2\right) , \tag{2.8}$$

and the stress components

$$\tau^1_1 = -p - \tfrac{1}{2}\lambda\Phi_2 + \tfrac{1}{4}\lambda^2\Phi_3 \;,$$

$$\tau^2_2 = -p - \tfrac{1}{2}\lambda\Phi_2 + \tfrac{1}{4}\left(\lambda^2 + \psi^2 r^2\right)\Phi_3 \;,$$

$$\tau^3_3 = -p + \lambda\Phi_2 + \left(\lambda^2 + \tfrac{1}{4}\psi^2 r^2\right)\Phi_3 \;,$$

$$\tau^2_3 = \tfrac{1}{2}\psi\left(\Phi_2 + \tfrac{1}{2}\lambda\Phi_3\right) \;.$$

p is determined from the equilibrium equation (2.5) which gives

$$p = -\tfrac{1}{2}\lambda\Phi_2 + \tfrac{1}{4}\lambda^2\Phi_3 - \psi^2\int_0^r \Phi_3 r' dr' + \text{Const} \,.$$

The constant of integration is determined from the second of (2.6) whence

$$p = -\tfrac{1}{2}\lambda\Phi_2 + \tfrac{1}{4}\lambda^2\Phi_3 + \psi^2\int_r^a \Phi_3 r' dr'$$

Thus the stresses become

$$\tau^1_1 = -\psi^2\int_r^a \Phi_3 r' dr' ,$$

$$\tau^2_2 = \psi^2\left[\tfrac{1}{4}r^2\Phi_3 - \int_r^a \Phi_3 r' dr'\right], \tag{2.9}$$

$$\tau^3_3 = \tfrac{3}{2}\lambda\Phi_2 + \tfrac{3}{4}\lambda^2\Phi_3 + \psi^2\left[\tfrac{1}{4}r^2\Phi_3 - \int_r^a \Phi_3 r' dr'\right].$$

$$\tau^2_3 = \tfrac{1}{2}\psi\left(\Phi_2 + \tfrac{1}{2}\lambda\Phi_3\right) ,$$

and to relate ψ and λ with N and T

$$N = 2\pi\int_0^a\left[\tfrac{3}{2}\lambda\Phi_2 + \tfrac{3}{4}\lambda^2\Phi_3 + \psi^2\left(\tfrac{1}{4}r^2\Phi_3 - \int_r^a \Phi_3 r' dr'\right)\right] r\,dr ,$$

$$T = \pi\psi\int_0^a\left[\Phi_2 + \tfrac{1}{2}\lambda\Phi_3\right] r^3 dr . \tag{2.10}$$

In order to evaluate the integrals involved in (2.9) and (2.10), a specific form of constitutive law must be used.

As an example, for a general polynomial law retaining terms up to fourth order in strain

$$\Phi_2 = A_{12} + A_{32} I_2 + A_{45} I_3 ,$$

$$\Phi_3 = A_{23} + A_{45} I_2 ,$$

and, making use of (2.8), (2.10) becomes

$$T = \tfrac{1}{4}\pi a^4 \psi \left[A_{12} + \tfrac{1}{2} A_{23}\lambda + \left(A_{32} + \tfrac{5}{6}\lambda A_{45}\right)\left(\tfrac{3}{4}\lambda^2 + \tfrac{1}{6} a^2 \psi^2\right), \right.$$

$$, \quad (2.11)$$

$$N = \pi a^2 \left[\tfrac{3}{2} A_{12}\lambda + \tfrac{1}{4} A_{23}\left(3\lambda^2 - \tfrac{1}{2} a^2\psi^2\right) + \tfrac{3}{8} A_{32}\lambda\left(3\lambda^2 + \tfrac{1}{2} a^2\psi^2\right) \right.$$
$$\left. + \tfrac{1}{16} A_{45}\left(60\lambda^4 - \tfrac{1}{3} a^4\psi^4 + \tfrac{17}{2}\lambda^2\right) \right.$$

Equations (2.10) and (2.11) further illustrate the coupling that exists, in general, between volumetric and deviatoric effects.

If, however, the material is such that

$$\phi_3 = 0 ,$$

it follows from (2.9) that

$$\tau^1_1 = \tau^2_2 = 0 .$$

Also, the first of (2.10) then gives

$$N = 3\pi\lambda \int_0^a \phi_2 \, r \, dr$$

and, thus, in the absence of axial force, there is no axial extension. This will not be the case if ϕ_3 is non-zero.

The preceding development is very close to that for finite torsion of an elastic rod of circular section (21) of which the problem considered here is a special case.

The results are also equivalent to those obtained by Rivlin (22) for a special class of time-dependent materials.

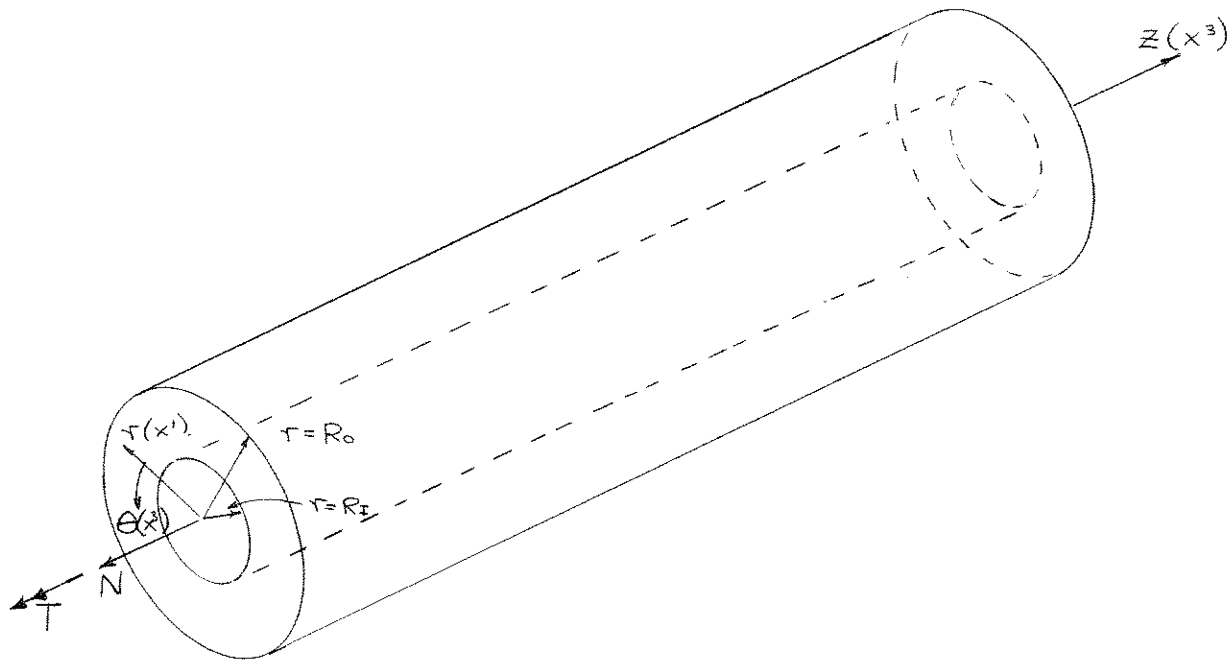


Figure 2

III. PLANE ELASTOSTATIC BOUNDARY VALUE PROBLEMS

3.1 Introduction

In this section two classes of two-dimensional problems are considered, plane strain and generalized plane stress.

By use of Airy's stress function, it is possible to formulate the generalized plane stress boundary value problem in a manner identical to that for classical elasticity, the governing equation being a fourth order nonlinear homogeneous partial differential equation. The generalized plane stress problem is thus formulated and an approximate solution method is described. An alternative approximate formulation is then shown which is also applicable to the plane strain problem.

The section concludes with the solution of an example problem.

3.2 Simplification of the constitutive law for the plane problem

In what follows it will be assumed that, for the generalized plane stress problem, quantities have been averaged by integration over the small thickness of the solid plate.

By redefining the material functions ϕ_i and α_i, the constitutive law for plane problems may be simplified from the forms (1.5) or (1.13) to forms similar to those of (1.33) and (1.35).

For the case of plane strain, the only non-vanishing components of strain are ϵ_{11}, ϵ_{12} and ϵ_{22}.

From (1.4), the strain invariants become

$$\begin{aligned} I_1 &= \epsilon_{11} + \epsilon_{22} \ , \\ I_2 &= \tfrac{1}{2}\left(\epsilon_{11}^2 + \epsilon_{22}^2 + 2\epsilon_{12}^2\right) \ , \\ I_3 &= \tfrac{1}{3}\left[\epsilon_{11}^3 + \epsilon_{22}^3 + 3\epsilon_{12}^2\left(\epsilon_{11} + \epsilon_{22}\right)\right] \ , \end{aligned} \tag{3.1}$$

and from (3.1)

$$I_3 = I_1(I_2 - 1/6\, I_1^2) \, . \tag{3.2}$$

Thus, if (3.2) is used in (1.3), the strain energy density function may be written as

$$U = U(I_1, I_2)$$

and hence

$$\tau_{ij} = \phi_1' \delta_{ij} + \phi_2' \epsilon_{ij} \tag{3.3}$$

where the prime $(\,'\,)$ indicates that ϕ_i' is not the same as ϕ_i.

As an example, if the 'cubic' constitutive law as given in (1.8) were used in a plane strain problem, the material functions ϕ_i given by (1.9) could be replaced by ϕ_1' and ϕ_2' in (3.3) where

$$\phi_1' = A_{11} I_1 + (A_{21} - \tfrac{1}{2} A_{23}) I_1^2 + (2A_{22} + A_{21}) I_2 + (A_{31} - \tfrac{2}{3} A_{34}) I_1^3 + 2(A_{33} + A_{34}) I_1 I_2$$

$$\phi_2' = A_{12} + (2A_{22} + A_{23}) I_1 I_2 + 2A_{32} I_2 + (A_{33} + A_{34}) I_1^2$$

As shown in section 1.6, a constitutive law of the form (3.3) has an inverse

$$\epsilon_{ij} = \alpha_1' \delta_{ij} + \alpha_2' \tau_{ij} \tag{3.4}$$

For generalized plane stress

$$\Theta_3 = \Theta_1(\Theta_2 - 1/6\,\Theta_1^2)$$

and by the same arguments as used above, the constitutive law may be simplified to the form (3.3) or (3.4).

Throughout the remainder of this chapter, the primes on the material functions in (3.3) and (3.4) will be omitted with the understanding that the constitutive laws (1.5) and (1.13) have been reduced to the simplified form (3.3) and (3.4).

3.3 Direct formulation of the generalized plane stress problem

In plane problems the strain compatibility equation,

$$\epsilon_{ij,km} + \epsilon_{km,ij} - \epsilon_{ik,jm} - \epsilon_{jm,ik} = 0 \,, \tag{3.5}$$

is identically satisfied except when

$$i = j = 1 \,, \quad k = m = 2 \,.$$

Substituting from (3.4) into (3.5) and setting

$$i = j = 1 \,, \quad k = m = 2 \,,$$

$$\alpha_{1,11} + \alpha_{1,22} + (\alpha_2 \tau_{11})_{,22} + (\alpha_2 \tau_{22})_{,11} - 2(\alpha_2 \tau_{12})_{,12} = 0. \tag{3.6}$$

In order that stresses satisfy equilibrium in the absence of body forces, i.e.

$$\tau_{ij,j} = 0 \,,$$

the Airy stress function, Φ ,is introduced such that

$$\tau_{\alpha\beta} = \nabla^2\Phi\delta_{\alpha\beta} - \Phi_{,\alpha\beta} \, .^{(1)} \tag{3.7}$$

Substituting from (3.7) into (3.6), the compatibility equation becomes

$$\alpha_{1,\beta\beta} + (\alpha_2\Phi_{,22})_{,22} + (\alpha_2\Phi_{,11})_{,11} + 2(\alpha_2\Phi_{,12})_{,12} = 0,$$

i.e.

$$\alpha_{1,\beta\beta} + (\alpha_2\Phi_{,\alpha\beta})_{,\alpha\beta} = 0 \, . \tag{3.8}$$

Thus, for given material stress functions, α_1 and α_2 , (3.8) may be expressed explicitly as a nonlinear fourth order homogeneous partial differential equation.

The invariants of stress in terms of Φ are

$$\Theta_1 = \Phi_{,\alpha\alpha} \, , \tag{3.9}$$

$$\Theta_2 = \tfrac{1}{2}\Phi_{,\alpha\beta}\Phi_{,\alpha\beta} \, .$$

As an example of (3.8), for the 'cubic' strain-stress law,

$$\begin{aligned}\epsilon_{ij} = {} & B_{11}\tau_{kk}\delta_{ij} + B_{12}\tau_{ij} + B_{21}\tau_{kk}^2\delta_{ij} + B_{22}\tau_{km}\tau_{km}\delta_{ij} \\ & + 2B_{22}\tau_{kk}\tau_{ij} + B_{31}\tau_{kk}^3\delta_{ij} + B_{32}\tau_{km}\tau_{km}\tau_{ij} \\ & + B_{33}\tau_{km}\tau_{km}\tau_{nn}\delta_{ij} + B_{33}\tau_{kk}{}^2\tau_{ij} \, ,\end{aligned} \tag{3.10}$$

for which

---

[1] Greek indices take on values 1, 2 i.e. $\Phi_{,\alpha\alpha} = \Phi_{,11} + \Phi_{,22}$

$$\alpha_1 = B_{11}\Theta_1 + B_{21}\Theta_1^2 + 2B_{22}\Theta_2 + B_{31}\Theta_1^3 + 2B_{33}\Theta_1\Theta_2 ,$$

$$\alpha_2 = B_{12} + 2B_{22}\Theta_1 + 2B_{32}\Theta_2 + B_{33}\Theta_1^2 , \tag{3.11}$$

(3.8) becomes

$$\begin{aligned}(B_{11} + B_{12})\nabla^4\Phi &+ B_{21}\nabla^2[(\nabla^2\Phi)^2] \\ &+ B_{22}[\nabla^2(\Phi_{,\alpha\beta}\Phi_{,\alpha\beta}) + 2(\nabla^2\Phi\,\Phi_{,\alpha\beta})_{,\alpha\beta}] \\ &+ B_{31}\nabla^2[(\nabla^2\Phi)^3] + B_{32}(\Phi_{,\alpha\beta}\Phi_{,\alpha\beta}\Phi_{,\gamma\delta})_{,\gamma\delta} \\ &+ B_{33}[\nabla^2(\nabla^2\Phi\,\Phi_{,\alpha\beta}\Phi_{,\alpha\beta}) + ((\nabla^2\Phi)^2\Phi_{,\alpha\beta})_{,\alpha\beta}] = 0 .\end{aligned} \tag{3.12}$$

3.4 Perturbation solution scheme for the compatibility equation

It is not possible, in general, to obtain a closed form solution for (3.8). If, however, the constitutive law is "close" to the linear law and is in a polynomial form, then an approximate solution scheme may be generated by perturbing the linear solution. This solution scheme has been used by Kauderer (ibid) and by Savin (ibid) for the particular class of nonlinearity referred to in section 1.6.

Φ is expanded in terms of a characteristic parameter, , such that

$$\Phi = \alpha\Phi^{(1)} + \alpha^2\Phi^{(2)} + \alpha^3\Phi^{(3)} + \cdots\cdots . \tag{3.13}$$

Substituting from (3.13) into (3.8) and requiring the coefficient of each power of α to vanish, a succession of linear differential equations is obtained. The coefficient of α^1 gives the compatibility equation associated with the linear problem while, for each successive power of α, there is obtained a differential equation which is the biharmonic equation with a forcing function dependent on the preceding solutions, i.e.

$$
\begin{aligned}
&\nabla^4 \Phi^{(1)} = 0 \, , \\
&(B_{11} + B_{12}) \nabla^4 \Phi^{(2)} + F_1(\Phi^{(1)}) = 0 \, , \\
&\quad \vdots \\
&(B_{11} + B_{12}) \nabla^4 \Phi^{(n)} + F_n(\Phi^{(1)}, \ldots\ldots, \Phi^{(n-1)}) = 0 \, .
\end{aligned}
\tag{3.14}
$$

It is convenient to take the solution for $\alpha \Phi^{(1)}$ to satisfy the actual boundary conditions whilst $\Phi^{(n)}$ $(n > 1)$ satisfy homogeneous boundary conditions.

To illustrate (3.14), for a material with constitutive law (3.10), the first three terms of (3.14) become

$$\nabla^4 \Phi^{(1)} = 0 \, ,$$

$$
\begin{aligned}
&(B_{11} + B_{12}) \nabla^4 \Phi^{(2)} + B_{21} \nabla^2 [(\nabla^2 \Phi^{(1)})^2] \\
&+ B_{22} [\nabla^2 (\Phi^{(1)}_{,\alpha\beta} \Phi^{(1)}_{,\alpha\beta}) + 2(\nabla^2 \Phi^{(1)} \Phi^{(1)}_{,\alpha\beta})_{,\alpha\beta}] = 0 \, ,
\end{aligned}
$$

$$
\begin{aligned}
&(B_{11} + B_{12}) \nabla^4 \Phi^{(3)} + 2 B_{21} \nabla^2 (\nabla^2 \Phi^{(1)} \nabla^2 \Phi^{(2)}) \\
&+ B_{22} [2 \nabla^2 (\Phi^{(1)}_{,\alpha\beta} \Phi^{(2)}_{,\alpha\beta}) + (\nabla^2 \Phi^{(1)} \Phi^{(2)}_{;\alpha\beta} + \Phi^{(1)}_{,\alpha\beta} \nabla^2 \Phi^{(2)})_{,\alpha\beta}] \\
&+ B_{31} \nabla^2 [(\nabla^2 \Phi^{(1)})^3] + B_{32} (\Phi^{(1)}_{,\alpha\beta} \Phi^{(1)}_{,\alpha\beta} \Phi^{(1)}_{,\gamma\delta})_{,\gamma\delta} \\
&+ B_{33} [\nabla^2 (\nabla^2 \Phi^{(1)} \Phi^{(1)}_{,\alpha\beta} \Phi^{(1)}_{,\alpha\beta}) + ((\nabla^2 \Phi^{(1)})^2 \Phi^{(1)}_{,\alpha\beta})_{,\alpha\beta}] = 0 \, .
\end{aligned}
\tag{3.15}
$$

The problem of uniqueness and convergence of perturbation series has received slight attention (23). The convergence of (3.13) in general will not be considered here but is qualitatively considered in section 3.6 for a specific problem.

3.5 <u>An alternative approximate formulation of the plane problem</u>

Although, for generalized plane stress, the formulation (3.8) is exact for a given constitutive law (3.4), it is not valid for plane strain since τ_{33}, which occurs in (3.4), is not zero. Furthermore, in the nonlinear case, the condition

$$\epsilon_{33} = 0 \tag{3.16}$$

will not enable τ_{33} to be expressed, in closed form, in terms of $\tau_{\alpha\beta}$.

An alternative approximate formulation for the problem is developed by expanding both strain and stress in powers of a characteristic parameter, α, i.e.

$$\begin{aligned} \epsilon_{ij} &= \alpha\epsilon_{ij}^{(1)} + \alpha^2\epsilon_{ij}^{(2)} + \alpha^3\epsilon_{ij}^{(3)} + \cdots\cdots , \\ \tau_{ij} &= \alpha\tau_{ij}^{(1)} + \alpha^2\tau_{ij}^{(2)} + \alpha^3\tau_{ij}^{(3)} + \cdots\cdots . \end{aligned} \tag{3.17}$$

For polynomial law (3.4), substituting from (3.17) and requiring the coefficient of each power of α to be zero, a series of constitutive laws is obtained, each having the form

$$
\begin{aligned}
\epsilon_{ij}^{(1)} &= B_{11}\tau_{kk}^{(1)}\delta_{ij} + B_{12}\tau_{ij}^{(1)} , \\
\epsilon_{ij}^{(2)} &= B_{11}\tau_{kk}^{(2)}\delta_{ij} + B_{12}\tau_{ij}^{(2)} + F(\tau_{ij}^{(1)}) , \\
&\vdots \\
\epsilon_{ij}^{(n)} &= B_{11}\tau_{kk}^{(n)}\delta_{ij} + B_{12}\tau_{ij}^{(n)} + F'(\tau_{ij}^{(n-1)} \ldots\ldots \tau_{ij}^{(1)}) .
\end{aligned}
\tag{3.18}
$$

The coefficients of the first three powers of α in (3.10), for example, give

$$
\epsilon_{ij}^{(1)} = B_{11}\tau_{kk}^{(1)}\delta_{ij} + B_{12}\tau_{ij}^{(1)} ,
$$

$$
\begin{aligned}
\epsilon_{ij}^{(2)} &= B_{11}\tau_{kk}^{(2)}\delta_{ij} + B_{12}\tau_{ij}^{(2)} + B_{21}\tau_{kk}^{(1)2}\delta_{ij} \\
&\quad + B_{22}\tau_{km}^{(1)}\tau_{km}^{(1)}\delta_{ij} + 2B_{22}\tau_{kk}^{(1)}\tau_{ij}^{(1)} ,
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
\epsilon_{ij}^{(3)} &= B_{11}\tau_{kk}^{(3)}\delta_{ij} + B_{12}\tau_{ij}^{(3)} + 2B_{21}\tau_{kk}^{(1)}\tau_{mm}^{(2)}\delta_{ij} \\
&\quad + 2B_{22}(\tau_{km}^{(1)}\tau_{km}^{(2)}\delta_{ij} + \tau_{kk}^{(1)}\tau_{ij}^{(2)} + \tau_{ij}^{(1)}\tau_{kk}^{(2)}) \\
&\quad + B_{31}\tau_{kk}^{(1)3}\delta_{ij} + B_{32}\tau_{km}^{(1)}\tau_{km}^{(1)}\tau_{ij}^{(1)} \\
&\quad + B_{33}(\tau_{km}^{(1)}\tau_{km}^{(1)}\tau_{nn}^{(1)}\delta_{ij} + \tau_{kk}^{(1)2}\tau_{ij}) ,
\end{aligned}
$$

To formulate the plane stress problem, the Airy stress function (3.7) with (3.13) is used in the constitutive laws (3.18) which are then substituted into the compatibility equations (3.5). The resulting differential equations are, of course, precisely those given by (3.14).

For plane strain, $\tau_{33}^{(n)}$ $(n = 1, 2, \ldots)$ must be eliminated from (3.18). To do this, the condition (3.16) is used which, from the first of (3.17), requires that

$$\epsilon_{33}^{(n)} = 0 \quad , \quad n = 1, 2 \ldots \ldots . \qquad (3.20)$$

Using (3.20) in (3.18), the equivalent plane strain constitutive laws may be obtained.

The equations (3.19), for example, become

$$\epsilon_{\alpha\beta}^{(1)} = \bar{B}_{11} \tau_{\gamma\gamma}^{(1)} \delta_{\alpha\beta} + \bar{B}_{12} \tau_{\alpha\beta}^{(1)} ,$$

$$\epsilon_{\alpha\beta}^{(2)} = \bar{B}_{11} \tau_{\gamma\gamma}^{(2)} \delta_{\alpha\beta} + \bar{B}_{12} \tau_{\alpha\beta}^{(2)} + \bar{B}_{21} \tau_{\gamma\gamma}^{(1)\,2} \delta_{\alpha\beta}$$
$$+ \bar{B}_{22} \tau_{\gamma\delta}^{(1)} \tau_{\gamma\delta}^{(1)} \delta_{\alpha\beta} + 2 \bar{B}_{22} \tau_{\gamma\gamma}^{(1)} \tau_{\alpha\beta}^{(1)} ,$$

$$\begin{aligned} \epsilon_{\alpha\beta}^{(3)} = {} & \bar{B}_{11} \tau_{\gamma\gamma}^{(3)} \delta_{\alpha\beta} + \bar{B}_{12} \tau_{\alpha\beta}^{(3)} + 2 \bar{B}_{21} \tau_{\gamma\gamma}^{(1)} \tau_{\delta\delta}^{(2)} \delta_{\alpha\beta} \\ & + 2 \bar{B}_{22} \left(\tau_{\gamma\delta}^{(1)} \tau_{\gamma\delta}^{(2)} \delta_{\alpha\beta} + \tau_{\delta\delta}^{(1)} \tau_{\alpha\beta}^{(2)} + \tau_{\alpha\beta}^{(1)} \tau_{\delta\delta}^{(2)} \right) \\ & + \bar{B}_{31} \tau_{\delta\delta}^{(1)\,3} \delta_{\alpha\beta} + \bar{B}_{32} \tau_{\gamma\delta}^{(1)} \tau_{\gamma\delta}^{(1)} \tau_{\alpha\beta}^{(1)} \\ & + \bar{B}_{33} \left(\tau_{\gamma\delta}^{(1)} \tau_{\gamma\delta}^{(1)} \tau_{\rho\rho}^{(1)} \delta_{\alpha\beta} + \tau_{\gamma\gamma}^{(1)\,2} \tau_{\alpha\beta}^{(1)} \right) . \end{aligned} \qquad (3.21)$$

where

$$\bar{B}_{11} = \frac{B_{12}}{B_{11} + B_{12}} B_{11} ,$$

$$\begin{aligned}
\bar{B}_{12} &= B_{12} \quad , \\
\bar{B}_{21} &= \frac{B_{12}}{(B_{11}+B_{12})^3}\left[B_{12}^2 B_{21} - 3 B_{11}^2 B_{22} \right] \quad , \\
\bar{B}_{22} &= \frac{B_{12}}{(B_{11}+B_{12})} B_{22} \quad , \qquad\qquad (3.22) \\
\bar{B}_{31} &= \frac{1}{(B_{11}+B_{12})^5}\Big\{ (B_{11}+B_{12})\left[B_{12}^4 B_{31} - B_{11}^4 B_{32} + 2 B_{11}^2 B_{12}^2 B_{33} \right] \\
&\qquad - 2\left[B_{12}^2 B_{21} + B_{11}(B_{11}-2B_{12}) B_{22} \right]^2 \Big\} \quad , \\
\bar{B}_{32} &= B_{32} - \frac{2}{(B_{11}+B_{12})} B_{22}^2 \quad , \\
\bar{B}_{33} &= \frac{1}{(B_{11}+B_{12})^3}\Big\{ (B_{11}+B_{12})\left[B_{11}^2 B_{32} + B_{12}^2 B_{33} \right] \\
&\qquad - 2\left[B_{12}^2 B_{21} + B_{11}(B_{11}-2B_{12}) B_{22} \right] B_{22} \Big\} .
\end{aligned}$$

Thus, by replacing the constants B_{AB} in the constitutive law by equivalent constants $\bar{B}_{AB}$, the plane strain problem may be considered as if it were plane stress and, as in the linear case, a given solution may be adopted to plane strain or plane stress provided the elastic constants are properly interpreted.

It is to be noted that, for nonlinear solids, the solution of plane problems, in general, will not be independent of the elastic constants.

From (3.22) it is seen that the cubic constants in plane strain will not be zero even if the cubic constants in the constitutive law are zero. The second perturbation of quadratic terms is thus effected both through the coefficients $\bar{B}_{2A}$ and $\bar{B}_{3A}$

3.6 Example -- The extension of an infinite plate containing a circular hole

As mentioned in section 3.4, Kauderer obtained numerous approximate solutions to plane stress problems for a special class of nonlinearity. A single perturbation of quadratic terms was considered.

Using complex variables, Savin (ibid) formulated the problem of the tension of an infinite plate containing a hole, again using a single perturbation of Kauderer's quadratic law.

In the solution described below, the first and second perturbations are carried out for the 'cubic' strain-stress law as given by equation (3.10).

The equations solved are (3.15). These take into account the most general nonlinearity up to third order in the strain-stress law and, as shown above, may be used for plane stress or plane strain depending on the interpretation of the elastic constants.

Figure 3 shows part of the plate. The radius of the hole is a and a uniform tension, S, is applied at infinity.

It is convenient to use polar coordinates r and θ located as shown.

Since the equations (3.15) are in invariant form, they may be applied in curvilinear coordinates if the partial differentiation is replaced by appropriate invariant differentiation (20). The term $\bar{\Phi}^{(1)}_{,\alpha\beta}\,\bar{\Phi}^{(1)}_{,\alpha\beta}$, for instance, becomes $g^{\alpha\delta} g^{\beta\gamma} \bar{\Phi}^{(1)}_{|\delta\gamma}\,\bar{\Phi}^{(1)}_{|\alpha\beta}$.

Physical components of stress are required and, in polar coordinates, become

$$\tau^{(n)}_{\theta\theta} = \Phi^{(n)}_{|(1)(1)} = \frac{\partial^2 \Phi^{(n)}}{\partial r^2} ,$$

$$\tau^{(n)}_{r\theta} = -\Phi^{(n)}_{|(1)(2)} = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial \Phi^{(n)}}{\partial \theta}\right] , \qquad (3.23)$$

$$\tau^{(n)}_{rr} = \Phi^{(n)}_{|(2)(2)} = \frac{1}{r}\frac{\partial \Phi^{(n)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi^{(n)}}{\partial r^2} .$$

The method of solution of (3.15) is straightforward and is only briefly described.

The linear solution is well known (24) and, in order that $\alpha \Phi^{(1)}$ satisfy the actual boundary conditions

$$\Phi^{(1)} = \frac{S}{\alpha}\frac{a^2}{4}\left[\left(\frac{r^2}{a^2} - 2\log\frac{r}{a}\right) + \left(-\frac{r^2}{a^2} + 2 - \frac{a^2}{r^2}\right)\right] \qquad (3.24)$$

(3.24) is now used in the second of (3.15). Since the coefficients of B_{21} and B_{22} are mutually independent, they must separately satisfy the homogeneous boundary conditions which are:

i) $r = a$ $\qquad \tau_{rr} = \tau_{r\theta} = 0$,

ii) $r \to \infty$ $\qquad \tau_{\alpha\beta} \to 0$

A particular integral, $\Phi^{(2)}_{P}$, is first obtained, and the complementary function, $\Phi^{(2)}_{c}$, is determined such that:

i) $r = a$ $\qquad \Phi^{(2)}_{P|22} = -\Phi^{(2)}_{c|22}$,

$\qquad \Phi^{(2)}_{P|12} = -\Phi^{(2)}_{c|12}$,

ii) $r \to \infty$ $\qquad \Phi^{(2)}_{P|\alpha\beta} = -\Phi^{(2)}_{c|\alpha\beta}$.

Carrying out the above steps,

$$\Phi^{(2)} = \frac{S^2}{\alpha^2}\frac{a^2}{(B_{11}+B_{12})}\Big\{-B_{21}\Big[\log\frac{r}{a} + \frac{1}{2}\frac{a^2}{r^2}\Big] + \frac{B_{22}}{8}\Big[\Big(-16\log\frac{r}{a} - 9\frac{a^2}{r^2}$$

$$+2\frac{a^4}{r^4} - \frac{a^6}{r^6}\Big) + 2\Big(1 - 2\frac{a^2}{r^2} + \frac{a^4}{r^4}\Big)\cos 2\theta + 2\Big(-1 + \frac{a^2}{r^2} - \frac{a^4}{r^4}\Big)\cos 4\theta\Big]\Big\}. \tag{3.25}$$

(3.24) and (3.25) are now used in the third of (3.15) to obtain, by the same procedure as above,

$$\Phi^{(3)} = \frac{S^3}{\alpha^3}\frac{a^2}{(B_{11}+B_{12})}\Big\{\frac{B_{21}^2}{B_{11}+B_{12}}\Big[\Big(2\log\frac{r}{a} + \frac{a^2}{r^2}\Big) + \frac{2}{3}\Big(-1 + 2\frac{a^2}{r^2}$$

$$-\frac{a^4}{r^4}\Big)\cos 2\theta\Big] + \frac{B_{21}B_{22}}{B_{11}+B_{12}}\Big[\frac{1}{12}\Big(100\log\frac{r}{a} + 39\frac{a^2}{r^2} - 2\frac{a^4}{r^4} + 5\frac{a^6}{r^6}\Big)$$

$$+\frac{1}{60}\Big(-248 + 463\frac{a^2}{r^2} - 200\frac{a^4}{r^4} + 10\frac{a^6}{r^6} - 18\frac{a^8}{r^8}\Big)\cos 2\theta$$

$$+\frac{1}{10}\Big(5 - 7\frac{a^2}{r^2} - \frac{a^4}{r^4} + 3\frac{a^6}{r^6}\Big)\cos 4\theta + \frac{1}{4}\Big(\frac{a^2}{r^2} + 2\frac{a^4}{r^4} - \frac{a^6}{r^6}\Big)\cos 6\theta\Big]$$

$$-\frac{B_{22}^2}{B_{11}+B_{12}}\Big[\frac{1}{80}\Big(-752\log\frac{r}{a} - 200\frac{a^2}{r^2} - 40\frac{a^4}{r^4} + 20\frac{a^6}{r^6} - 39\frac{a^8}{r^8}\Big)$$

$$+\frac{1}{560}\Big(3484 - \Big(5405 + 1400\log\frac{r}{a}\Big)\frac{a^2}{r^2} + 1610\frac{a^4}{r^4} - 161\frac{a^6}{r^6} + 252\frac{a^8}{r^8}$$

$$+150\frac{a^{10}}{r^{10}}\Big)\cos 2\theta + \frac{1}{56}\Big(-49 + \Big(-38 + 56\log\frac{r}{a}\Big)\frac{a^2}{r^2} + \Big(148$$

$$+84\log\frac{r}{a}\Big)\frac{a^4}{r^4} - 56\frac{a^6}{r^6} - 5\frac{a^8}{r^8}\Big)\cos 4\theta + \frac{1}{30}\Big(-5 + 42\frac{a^2}{r^2}$$

$$-69\frac{a^4}{r^4} + 32\frac{a^6}{r^6}\Big)\cos 6\theta\Big] + \frac{B_{31}}{2}\Big[3\Big(-2\log\frac{r}{a} - \frac{a^2}{r^2}\Big) + \Big(1 - 2\frac{a^2}{r^2} + \frac{a^4}{r^4}\Big)$$

$$\cos 2\theta\Big] - 6B_{32}\Big[\frac{1}{32}\Big(\frac{338}{45}\log\frac{r}{a} + 3\frac{a^2}{r^2} - \frac{1}{18}\frac{a^4}{r^4} - \frac{1}{9}\frac{a^6}{r^6} + \frac{3}{10}\frac{a^8}{r^8}\Big)$$

$$+\frac{1}{16}\left(-\frac{27}{14}+\left(\frac{631}{420}+\frac{10}{3}\log\frac{r}{a}\right)\frac{a^2}{r^2}+\frac{a^4}{r^4}-\frac{17}{30}\frac{a^6}{r^6}+\frac{1}{10}\frac{a^8}{r^8}-\frac{3}{28}\frac{a^{10}}{r^{10}}\right)\cos 2\theta$$

$$+\frac{1}{4}\left(\frac{5}{48}+\left(\frac{109}{420}-\frac{1}{3}\log\frac{r}{a}\right)\frac{a^2}{r^2}-\left(\frac{341}{840}+\frac{1}{2}\log\frac{r}{a}\right)\frac{a^4}{r^4}+\frac{1}{30}\frac{a^6}{r^6}+\frac{1}{112}\frac{a^8}{r^8}\right)\cos 4\theta$$

$$+\frac{1}{32}\left(\frac{1}{9}-\frac{9}{10}\frac{a^2}{r^2}+\frac{22}{15}\frac{a^4}{r^4}-\frac{61}{90}\frac{a^6}{r^6}\right)\cos 6\theta\Big]-B_{33}\Big[\Big(\frac{25}{6}\log\frac{r}{a}+\frac{17}{8}\frac{a^2}{r^2}$$

$$-\frac{1}{3}\frac{a^4}{r^4}+\frac{5}{24}\frac{a^6}{r^6}\Big)+\frac{1}{4}\left(-\frac{23}{5}+\frac{91}{10}\frac{a^2}{r^2}-5\frac{a^4}{r^4}+\frac{11}{10}\frac{a^6}{r^6}-\frac{3}{5}\frac{a^8}{r^8}\right)\cos 2\theta$$

$$+\frac{1}{4}\left(1-\frac{7}{5}\frac{a^2}{r^2}-\frac{1}{5}\frac{a^4}{r^4}+\frac{3}{5}\frac{a^6}{r^6}\right)\cos 4\theta\;+\frac{1}{8}\left(-\frac{a^2}{r^2}+2\frac{a^4}{r^4}+\frac{a^6}{r^6}\right)\cos 6\theta\Big]\Big\}. \tag{3.26}$$

From (3.23), (3.24), (3.25) and (3.26) the first three terms of the perturbation series for stresses may be obtained. Of particular interest is the effect of nonlinearity on the stress concentration obtained by linear theory, i.e. $\tau_{\theta\theta}(r = a, \theta = \frac{\pi}{2})$.

Computing this quantity one obtains

$$\tau_{\theta\theta}\Big|_{\substack{r=a,\\ \theta=\frac{\pi}{2}}} = S\Big[3+\frac{S}{B}\left(-2B_{21}-9B_{22}\right)+\frac{S^2}{B^2}\left(9.3B_{21}^2\right.$$
$$\left.+66B_{21}B_{22}+133B_{22}^2\right)+\frac{S^2}{B}\left(-10B_{31}-14.3B_{32}-25.5B_{33}\right)\Big],$$

where

$$B = B_{11}+B_{12}\ .$$

The general effect is as expected. As shown in figure 4, positive elastic constants, B_{AB} $(A > 1)$, indicate that the material softens under increasing uniaxial tensile load, and a corresponding reduction in stress concentration would be expected.

A feature brought out by the stress concentration is that the second perturbation for quadratic terms is not small compared with the first perturbation unless the deviation from linearity is very small.

As seen from figure 4a, the ratios $\frac{B_{21}S}{B}$ and $\frac{3B_{22}S}{B}$ are measures of nonlinearity with respect to the uniaxial test, and each ratio has a coefficient in the second perturbation of between four and five times its coefficient in the first perturbation. Thus, the magnitude of the stress concentration due to the second perturbation would be approximately the same as that due to the first perturbation for a nonlinearity of 20%.

In order to consider more fully the convergence of the perturbation series, two further perturbations were carried out for the coefficient B_{21}.

Omitting computational details, for a constitutive law

$$\epsilon_{ij} = B_{11}\tau_{kk}\delta_{ij} + B_{12}\tau_{ij} + B_{21}\tau_{kk}^{2}\delta_{ij} ,$$

the stress concentration factor is

$$\tau_{\theta\theta}\Big|_{\substack{r=a\\ \theta=\frac{\pi}{2}}} = S\left[3 - 2k + 9.33k^2 - 50.4k^3 + 297k^4 - \cdots\right] , \qquad (3.27)$$

where

$$k = \frac{B_{21}S}{B} .$$

is the nonlinearity ratio with respect to uniaxial tension.

If an approximation of 10% to the correction of stress concentration due to nonlinearity is taken as an acceptable criterion to terminate the perturbation series, one might infer from (3.27):

1. for one perturbation to be satisfactory, k must not exceed 2 %,

2. for $k = 5\%$, at least two perturbations are required.

It also appears doubtful that the alternating series (3.27) would converge for values of k greater than 15%.

It must be emphasized that general conclusions regarding convergence and accuracy cannot be based on qualitative conclusions for a particular stress state and a particular type of nonlinearity.

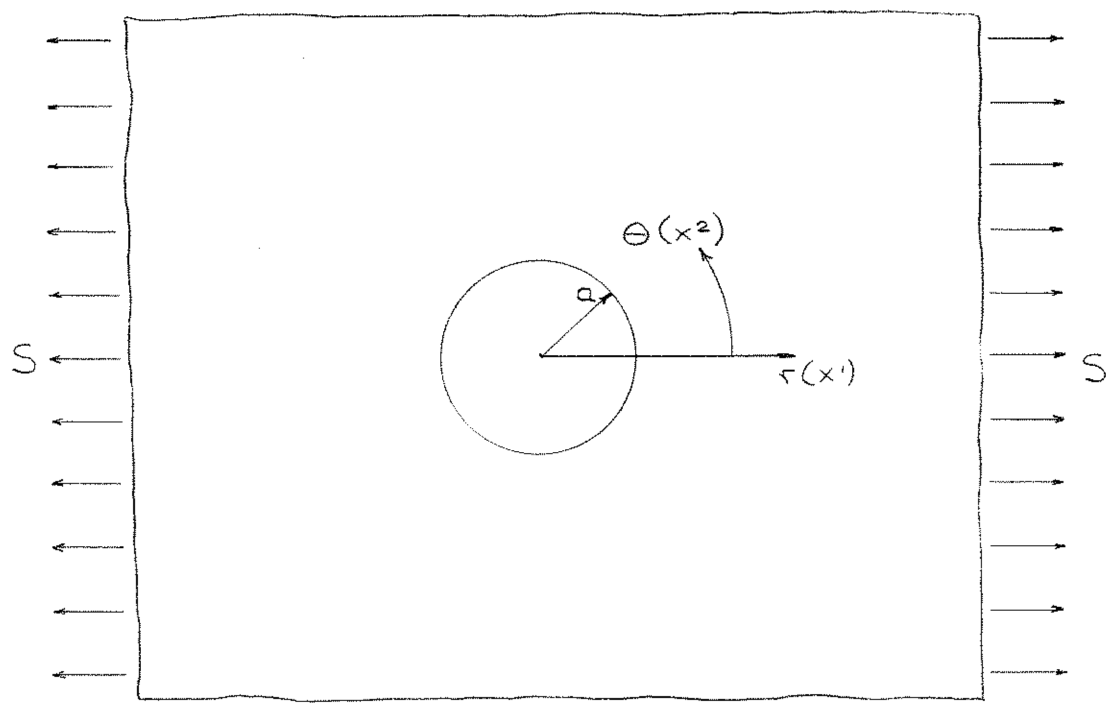
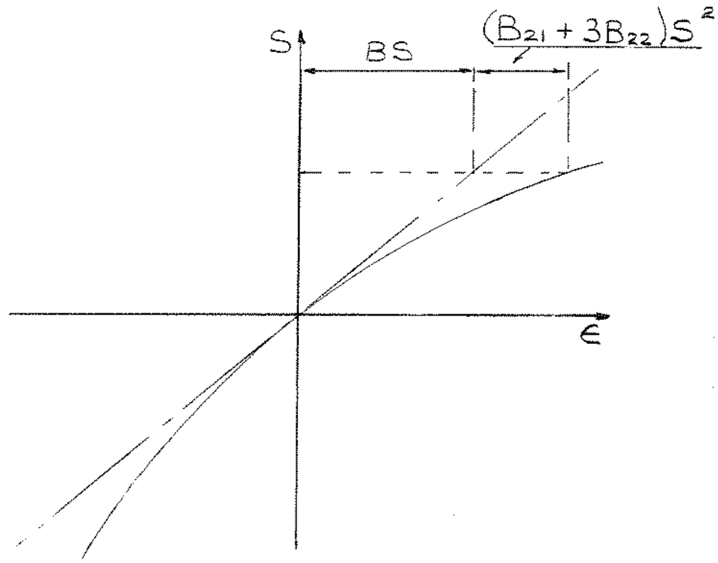
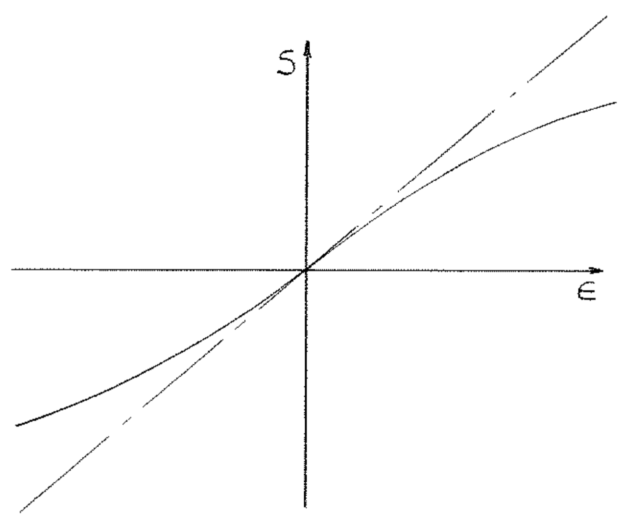


Figure 3



(a) "Quadratic" material , $B_{2A} > 0$.



(b) "Cubic" material , $B_{3A} > 0$.

Figure 4

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