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DETERMINING CYCLICITY OF FINITE MODULES

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ABSTRACT. We present a deterministic polynomial-time algorithm that determines whether a finite module over a finite commutative ring is cyclic, and if it is, outputs a generator.

1. INTRODUCTION

If R is a commutative ring, then an R -module M is cyclic if there exists $y \in M$ such that $M = Ry$.

Theorem 1.1. *There is a deterministic polynomial-time algorithm that, given a finite commutative ring R and a finite R -module M , decides whether there exists $y \in M$ such that $M = Ry$, and if there is, finds such a y .*

We present the algorithm in Algorithm 4.1 below. The inputs are given as follows. The ring R is given as an abelian group by generators and relations, along with all the products of pairs of generators. The finite R -module M is given as an abelian group, and for all generators of the abelian groups R and all generators of the abelian group M we are given the module products in M .

Our algorithm depends on R being an Artin ring, and should generalize to finitely generated modules over any commutative Artin ring that is computationally accessible.

Theorem 1.1 is one of the ingredients of our work [4, 5] on lattices with symmetry, and a sketch of the proof is contained in [4]. Previously published algorithms of the same nature appear to restrict to rings that are algebras over fields. Subsequently to [4], I. Ciocănea-Teodorescu [2], using different and more elaborate techniques, greatly generalized our result, dropping the commutativity assumption on the finite ring R and finding, for any given finite R -module M , a set of generators for M of smallest possible size.

See Chapter 8 of [1] for commutative algebra background. For the purposes of this paper, commutative rings have an identity element 1, which may be 0.

2. LEMMAS ON COMMUTATIVE RINGS

If R is a commutative ring and $\mathfrak{a}$ is an ideal in R , let $\text{Ann}_R \mathfrak{a}$ denote the annihilator of $\mathfrak{a}$ in R . We will use that every finite commutative ring is an Artin ring,

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that every Artin ring is isomorphic to a finite direct product of local Artin rings, and that the maximal ideal in a local Artin ring is always nilpotent.

Lemma 2.1. *If A is a local Artin ring, $\mathfrak{a}$ is an ideal in A , and $\mathfrak{a}^2 = \mathfrak{a}$, then $\mathfrak{a}$ is 0 or A .*

Proof. If $\mathfrak{a}$ contains a unit, then $\mathfrak{a} = A$. Otherwise, $\mathfrak{a}$ is contained in the maximal ideal $\mathfrak{m}$, which is nilpotent. Thus there is an $r \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}^r = 0$. Now $\mathfrak{a} = \mathfrak{a}^2 = \dots = \mathfrak{a}^r \subset \mathfrak{m}^r = 0$. $\square$

Lemma 2.2. *Suppose that A is a finite commutative ring, $\mathfrak{a}$ is an ideal in A , $\mathfrak{b} = \text{Ann}_A \mathfrak{a}$, and $\mathfrak{a} \cap \mathfrak{b} = 0$. Then:*

- (i) $\mathfrak{a}^2 = \mathfrak{a}$;
- (ii) *there is an idempotent $e \in A$ such that $\mathfrak{a} = eA$, $\mathfrak{b} = (1 - e)A$, and $A = (1 - e)A \oplus eA = \mathfrak{b} \oplus \mathfrak{a}$;*
- (iii) *if $\mathfrak{b} = 0$ then $\mathfrak{a} = A$.*

Proof. Write A as a finite direct product of local Artin rings $A_1 \times \dots \times A_s$. Then $\mathfrak{a}$ is a direct product $\mathfrak{a}_1 \times \dots \times \mathfrak{a}_s$ of ideals $\mathfrak{a}_i \subset A_i$. Assume $\mathfrak{a}^2 \neq \mathfrak{a}$. Then there is an i such that $\mathfrak{a}_i^2 \neq \mathfrak{a}_i$. Let $\mathfrak{b}_i = \text{Ann}_{A_i} \mathfrak{a}_i$. Since $\mathfrak{a} \cap \mathfrak{b} = 0$, it follows that $\mathfrak{a}_i \cap \mathfrak{b}_i = 0$. Since A_i is a local ring, $\mathfrak{a}_i$ is contained in the maximal ideal of A_i , so $\mathfrak{a}_i$ is nilpotent. Let r denote the smallest positive integer such that $\mathfrak{a}_i^r = 0$. Since $\mathfrak{a}_i \neq 0$ we have $r \geq 2$. Then $\mathfrak{a}_i^{r-1}$ is contained in $\mathfrak{a}_i$ and kills $\mathfrak{a}_i$, so $0 \neq \mathfrak{a}_i^{r-1} \subset \mathfrak{a}_i \cap \mathfrak{b}_i = 0$, a contradiction. This gives (i).

Since A is a finite product of local Artin rings, $\mathfrak{a}$ is generated by an idempotent e , by Lemma 2.1. Then $\mathfrak{b} = (1 - e)A$ and $A = (1 - e)A \oplus eA = \mathfrak{b} \oplus \mathfrak{a}$. This gives (ii) and (iii). $\square$

3. PREPARATORY LEMMAS

If R is a commutative ring, then a commutative R -algebra is a commutative ring A equipped with a ring homomorphism from R to A . Whenever A is an R -algebra, we let M_A denote the A -module $A \otimes_R M$.

From now on, suppose R is finite commutative ring and M is a finite R -module. Let $\mathcal{S}$ denote the set of quadruples (A, B, y, N) such that:

- (i) A and B are finite commutative R -algebras for which the natural map $f : R \rightarrow A \times B$ is surjective and has nilpotent kernel,
- (ii) $y \in M$ is such that the map $B \rightarrow M_B = B \otimes_R M$ defined by $b \mapsto b \otimes y$ is an isomorphism and such that $1 \otimes y = 0$ in M_A ,
- (iii) and N is a submodule of M such that the natural map $N \rightarrow M_A$ defined by $z \mapsto 1 \otimes z$ is onto and such that the natural map $N \rightarrow M_B$ is the zero map.

In Algorithm 4.1 below, initially we take $(A, B, y, N) = (R, 0, 0, M)$. Clearly, $(R, 0, 0, M) \in \mathcal{S}$. Throughout that algorithm, we always have $(A, B, y, N) \in \mathcal{S}$. While A and B occur in the proof of correctness of Algorithm 4.1, the R -algebra B does not actually occur in the algorithm itself.

Lemma 3.1. *If $(A, B, y, N) \in \mathcal{S}$ and $M_A = 0$, then $M = Ry$.*

Proof. Let J denote the kernel of $f : R \rightarrow A \times B$, and let I_A (resp., I_B) denote the kernel of the composition of f with projection from $A \times B$ onto A (resp., B). Since J is nilpotent we have $J^r = 0$ for some $r \in \mathbb{Z}_{>0}$. Since $0 = M_A =$

$A \otimes_R M = (R/I_A) \otimes_R M \cong M/I_A M$ it follows that $I_A M = M$. Since $JM \subseteq I_B M = I_B I_A M \subseteq (I_B \cap I_A)M = JM$, it follows that $JM = I_B M$. Letting $y' = (y \bmod I_B M) \in M/I_B M$, then $M_B \cong M/I_B M = By'$. Thus,

$$\begin{aligned} M &= Ry + I_B M = Ry + JM = Ry + J(Ry + JM) \\ &= Ry + J^2 M = \dots = Ry + J^r M = Ry. \end{aligned}$$

□

Lemma 3.2. *Suppose $(A, B, y, N) \in \mathcal{S}$ and $M_A \neq 0$. Then there exists $x \in N$ such that $1 \otimes x \neq 0$ in M_A . Choosing x and letting $\mathbf{a} = \text{Ann}_A(1 \otimes x)$ and $\mathbf{b} = \text{Ann}_A \mathbf{a}$, we have:*

- (i) $(A/(\mathbf{a} \cap \mathbf{b}), B, y, N) \in \mathcal{S}$;
- (ii) If $\mathbf{a} \cap \mathbf{b} = 0$ and $(A/\mathbf{a}) \otimes x = M_{A/\mathbf{a}}$, then $(A/\mathbf{b}, (A/\mathbf{a}) \times B, x + y, \mathbf{a}N) \in \mathcal{S}$, where $\mathbf{a}N$ denotes $f^{-1}(\mathbf{a} \times B)N$.
- (iii) If $\mathbf{a} \cap \mathbf{b} = 0$ and $(A/\mathbf{a}) \otimes x \neq M_{A/\mathbf{a}}$, then M is not cyclic.

Proof. Since the map $N \rightarrow M_A, z \mapsto 1 \otimes z$ is onto, as long as $M_A \neq 0$ there exists $x \in N$ such that $1 \otimes x \neq 0$ in M_A .

Since $\mathbf{a}\mathbf{b} = 0$, we have $(\mathbf{a} \cap \mathbf{b})^2 = 0$, so $\mathbf{a} \cap \mathbf{b}$ is a nilpotent ideal in A . It follows that $(A/(\mathbf{a} \cap \mathbf{b}), B, y, N) \in \mathcal{S}$, giving (i).

From now on, suppose that $\mathbf{a} \cap \mathbf{b} = 0$. By Lemma 2.2, there is an idempotent $e \in A$ such that $\mathbf{a} = eA$, $\mathbf{b} = (1 - e)A$, and $A = (1 - e)A \oplus eA = \mathbf{b} \oplus \mathbf{a}$. It follows that $A \xrightarrow{\sim} A/\mathbf{a} \times A/\mathbf{b}$, so $M_A \xrightarrow{\sim} M_{A/\mathbf{a}} \times M_{A/\mathbf{b}}$. If (x', x'') is the image of $1 \otimes x$ under the latter map, then $x'' = 0$ (we have $\mathbf{b}x'' = 0$ since $x'' \in (A/\mathbf{b}) \otimes_R M$, and $\mathbf{a}x'' = 0$ since $\mathbf{a}(1 \otimes x) = 0$; thus $Ax'' = (\mathbf{a} + \mathbf{b})x'' = 0$, so $x'' = 0$). The map $i_{\mathbf{a}} : A/\mathbf{a} \rightarrow M_{A/\mathbf{a}}$ defined by $i_{\mathbf{a}}(t) = tx' = t \otimes x$ is injective since $\text{Ann}_{A/\mathbf{a}} x' = 0$.

First suppose $(A/\mathbf{a}) \otimes x = M_{A/\mathbf{a}}$. Then the injective map $i_{\mathbf{a}}$ is an isomorphism. Since $0 = x'' = 1_{A/\mathbf{b}} \otimes x$, we have $1 \otimes (x + y) = 0$ in $M_{A/\mathbf{b}}$. It is now easy to check that $(A/\mathbf{b}, (A/\mathbf{a}) \times B, x + y, \mathbf{a}N) \in \mathcal{S}$, giving (ii). Note that $\mathbf{b} \neq 0$ (if $\mathbf{b} = 0$, then $\mathbf{a} = A$ by Lemma 2.2, contradicting that $1 \otimes x \neq 0$ in M_A).

Now suppose that $(A/\mathbf{a}) \otimes x \neq M_{A/\mathbf{a}}$. By way of contradiction, suppose M is a cyclic R -module. Then $M_{A/\mathbf{a}}$ is a cyclic $A/\mathbf{a}$ -module. Since the domain and codomain of $i_{\mathbf{a}} : A/\mathbf{a} \hookrightarrow M_{A/\mathbf{a}}$ are both finite, it now follows that $i_{\mathbf{a}}$ is surjective, so $(A/\mathbf{a}) \otimes x = M_{A/\mathbf{a}}$. This contradiction gives (iii). □

The intuition behind Algorithm 4.1 is that throughout the algorithm, y generates the “non- A part” of M , and the goal is to shrink the “ A -part” of M , namely N .

4. MAIN ALGORITHM

Algorithm 4.1. Input a finite commutative ring R and a finite R -module M . Decide whether there exists $y \in M$ such that $M = Ry$, and if there is, find such a y .

- (i) Initially, take $A = R$, $y = 0$, and $N = M$.
- (ii) If $M_A = 0$, stop and output “yes” with generator y .
- (iii) Otherwise, pick $x \in N$ such that $1 \otimes x \neq 0$ in M_A , and compute $\mathbf{a} = \text{Ann}_A(1 \otimes x)$, $\mathbf{b} = \text{Ann}_A \mathbf{a}$, and $\mathbf{a} \cap \mathbf{b}$.
- (iv) If $\mathbf{a} \cap \mathbf{b} \neq 0$, replace A by $A/(\mathbf{a} \cap \mathbf{b})$ and go back to step (ii).

- (v) If $\mathbf{a} \cap \mathbf{b} = 0$, then if $(A/\mathbf{a}) \otimes x \neq M_{A/\mathbf{a}}$ terminate with “no”, and if $(A/\mathbf{a}) \otimes x = M_{A/\mathbf{a}}$ replace A , y , and N by $A/\mathbf{b}$, $x+y$, and $\mathbf{a}N$, respectively, and go back to step (ii).

Proposition 4.2. *Algorithm 4.1 runs in polynomial time, and on input a finite commutative ring R and a finite R -module M , decides whether there exists $y \in M$ such that $M = Ry$, and if there is, finds such a y .*

Proof. Since A is a finite ring, if the algorithm does not stop with “no” then eventually $A = 0$ and $M_A = 0$. Step (ii) of the algorithm is justified by Lemma 3.1, while steps (iii), (iv), and (v) are justified by Lemma 3.2.

The computations of annihilators and of the decompositions $A \xrightarrow{\sim} A/\mathbf{a} \times A/\mathbf{b}$ can be done in polynomial time using linear algebra (see §14 of [3]); in particular, $\mathbf{a}$ is the kernel of the map $A \rightarrow M_A$ defined by $t \mapsto t(1 \otimes x)$. For any B , compute M_B by computing $M/I_B M$ (and analogously for M_A). Each new A is at most half the size of the A it replaces. This implies that the number of steps is at most linear in the length of the input. $\square$

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