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Common Variance Fractional Factorial Designs for Model Comparisons

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Common Variance Fractional Factorial Designs for Model Comparisons

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by

Shrabanti Chowdhury

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ABSTRACT OF THE DISSERTATION

Common Variance Fractional Factorial Designs for Model Comparisons

by

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Doctor of Philosophy, Graduate Program in Applied Statistics
University of California, Riverside, June 2016
Dr. Subir Ghosh, Chairperson

In designing a fractional factorial experiment, a class of models with some common parameters is considered for describing the data to be obtained from the experiment. The uncommon parameters of these models are to be estimated with the same variance as best as possible. Fractional factorial designs are obtained with the various variance structures in terms of their equalities. A special variance structure having the equal variances of the estimators of all uncommon parameters is the main theme of this thesis. In particular the $2-$factor interaction effect is considered as the uncommon parameter in each model. Such plans with the ability of estimating the uncommon parameter with equal precision are called Common Variance (CV) designs. From the class of all CV designs for particular values of the number of factors m and the number of runs n designs giving smallest value of CV are obtained. Such designs are called Optimum CV designs. Both symmetric and asymmetric factorial experiments are considered with factors at two and three levels.

Two series of CV designs are obtained for general 3^m factorial experiment

with different number of runs. The common variance property is characterized for general fractional factorial designs. Several sufficient conditions are obtained using projection matrix and runs of the designs. The projection matrices of the series of CV designs for general m are investigated and a special structure of the projection matrix is presented for the CV designs including the optimum CV designs. Optimum CV designs are also presented for these two series for different m. CV designs are obtained with replicated runs. It is shown that a 3^2 CV design which is optimum in the class of all CV designs for $n = 6$ remains CV after replicating any of its six runs any number of times. Several other 3^2 CV designs for $n = 6$ are presented which satisfy this general replication property. Condition is derived for obtaining hierarchical CV designs for a general fractional factorial experiment. The determination of CV designs was also extended to a mixed level factorial experiment with factors at two and three levels. For a 2×3 factorial experiment CV designs exist only under a constraint of replications, for $2^m \times 3$ and $2^m \times 3^3$ factorial experiments designs are presented which give common variance within groups of similar structured interactions.

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Chapter 1

Introduction

1.1 Factorial Experiment

Many scientific investigations are carried out to study the effects of two or more factors simultaneously on the response variable. In such investigation factorial experiments are widely used as they provide a systematic and statistically valid strategy to find the best result. Factorial designs are used in these experiments which consider all the level combinations of the diferent factors and thus can study the factors simultaneously.

In factorial experiments the treatments are formed by the different level combinations of the factors. A general factorial experiment is of the form $s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_t^{m_t}$, where s_i's ($s_i \geq 2$) are all distinct and there are m_i number of factors each with level s_i, $i = 1(1)t$. In factorial experiments we want to express the factorial effects as a linear combination of the treatment effects. In particular we consider levels 2 and 3, i.e. we take $s_1 = 2$, $s_2 = 3$ and all other s_i's to

be zero. We denote the number of factors with two levels by m_a and that with three levels by m_b, i.e. we consider factorial experiments of the form $2^{m_a} \times 3^{m_b}$. We denote the factors with 2 levels by $A_1,\ A_2, \ldots, A_{m_a}$ and those with 3 levels by $B_1,\ B_2, \ldots, B_{m_b}$. Also we denote the levels of the factors of a 2^{m_a} factorial experiment by $(x_1,\ x_2, \ldots,\ x_{m_a})$ and the levels of the factors of a 3^{m_b} experiment by $(y_1,\ y_2, \ldots,\ y_{m_b})$ and thus a treatment of a $2^{m_a} \times 3^{m_b}$ experiment is of the form $(x_1,\ x_2, \ldots,\ x_{m_a},\ y_1,\ y_2, \ldots,\ y_{m_b})\,,\ x_i \in \{0,1\}\,,\ y_j \in \{0,1,2\}\,,\ i = 1\,(1)\,m_a,\ j = 1(1)m_b$. Any treatment and its effect for a 2^{m_a} factorial experiment is expressed as $(x_1, \ldots, x_u, \ldots, x_{m_a})$ where x_u is the level of the factor $A_u,\ x_u = 0, 1,\ u = 1, \ldots, m_a$. Similarly, for a 3^{m_b} experiment, any treatment and its effect is expressed as $(y_1, \ldots, y_v, \ldots, y_{m_b})$ where y_v is the level of factor $B_v,\ y_v = 0, 1, 2,\ v = 1, \ldots, m_b$. For a 2^{m_a} factorial experiment out of the 2^{m_a} factorial effects, there are $\binom{m_a}{1} = p_{1A}$ main effects and $\binom{m_a}{u} = p_{uA}$ u-factor interaction effects, $u = 2, \ldots, m_a$. Similarly, for a 3^{m_b} factorial experiment out of the 3^{m_b} factorial effects, there are $2\binom{m_b}{1} = p_{1B}$ main effects and $2^v\binom{m_b}{v} = p_{vB}$ v-factor interaction effects, $v = 2, \ldots, m_b$. For the 2^{m_a} factorial experiment all p_{1A} main effects are linear but for the 3^{m_b} factorial experiment each of the p_{1B} main effects has a linear and a quadratic component. Now any factorial effect of a $2^{m_a} \times 3^{m_b}$ factorial experiment can be represented as $A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_{m_a}^{\alpha_{m_a}} B_1^{\beta_1} B_2^{\beta_2} \ldots B_{m_b}^{\beta_{m_b}}$, $\alpha_i \in \{0,1\}\,,\ \beta_j \in \{0,1,2\}\,,\ i = 1\,(1)\,m_a,\ j = 1(1)m_b$. The factorial effects of the 2^{m_a} and 3^{m_b} factorial experiments are of the form $A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_{m_a}^{\alpha_{m_a}}$ and $B_1^{\beta_1} B_2^{\beta_2} \ldots B_{m_b}^{\beta_{m_b}}$ respectively, $\alpha_i \in \{0,1\}\,,\ \beta_j \in \{0,1,2\}\,,\ i = 1\,(1)\,m_a,\ \ j = 1(1)m_b$. When $\alpha_1 = \ldots = \alpha_{m_a} = \beta_1 = \ldots = \beta_{m_b} = 0$, the effect becomes the general mean. For

a 2^{m_a} factorial experiment the effect becomes the linear effect for the uth factor when $\alpha_u = 1$ and $\alpha_k = 0,\ k = 1, \ldots, u-1, u+1, \ldots, m_a$. For a 3^{m_b} factorial experiment the effect becomes the linear effect for the vth factor when $\beta_v = 1$ and $\beta_k = 0,\ k = 1, \ldots, v-1, v+1, \ldots, m_b$, it becomes the quadratic effect for the vth factor when $\beta_v = 2$ and $\beta_k = 0,\ k = 1, \ldots, v-1, v+1, \ldots, m_b$. For $\alpha_k = 0,\ k = 1, \ldots, u_1 - 1, u_1 + 1, \ldots, u_2 - 1, u_2 + 1, \ldots, m_a$, it becomes the two-factor interaction effect between the factors A_{u_1} and A_{u_2}. Similarly for $\beta_k = 0,\ k = 1, \ldots, v_1 - 1, v_1 + 1, \ldots, v_2 - 1, v_2 + 1, \ldots, m_b$, it becomes the two-factor interaction effect between the factors B_{v_1} and B_{v_2}: (i) linear x linear when $\beta_{v_1} = \beta_{v_2} = 1$, (ii) linear x quadratic when $\beta_{v_1} = 1,\ \beta_{v_2} = 2$, (iii) quadratic x linear when $\beta_{v_1} = 2,\ \beta_{v_2} = 1$ and (iv) quadratic x quadratic when $\beta_{v_1} = \beta_{v_2} = 2$. Define $\{x_{u_1} + x_{u_2} = c_a\},\ c_a \in \{0, 1\},\ u_1 < u_2 = 1, \ldots, m_a$ as the sum of all the treatment effects for (x_{u_1}, x_{u_2}) satisfying the equation $x_{u_1} + x_{u_2} = c_a$ over the finite field $GF(2)$ for a 2^{m_a} factorial experiment. Similarly for a 3^{m_b} factorial experiment define $\{y_{v_1} + b^* y_{v_2} = c_b\},\ b^* \in \{1, 2\},\ c_b \in \{0, 1, 2\},\ v_1 < v_2 = 1, \ldots, m_b$ as the sum of all the treatment effects for (y_{v1}, y_{v_2}) satisfying the equation $y_{v_1} + b^* y_{v_2} = c_b$ over the finite field $GF(3)$. For example in a 3^2 factorial experiment the set $\{x_1 + x_2 = 0\}$ corresponds to the sum of the treatment effects for $(x_1, x_2) = (0, 0),\ (2, 1),\ (1, 2)$, satisfying the equation $x_1 + x_2 = 0$ over $GF(3)$. Now for the 2^{m_a} factorial experiment the general mean, main effects and two-factor interaction effects can be expressed in terms of the treatment effects

as:

$$
\begin{aligned}
2^{m_a}\mu &= \{x_1 = 0\} + \{x_1 = 1\} \\
2^{m_a-1}A_u &= \{x_u = 1\} - \{x_u = 0\} \\
2^{m_a-1}A_{u_1}A_{u_2} &= \{x_{u_1} + x_{u_2} = 1\} - \{x_{u_1} + x_{u_2} = 0\},
\end{aligned}
$$

where $u_1 < u_2 = 1, \ldots, m_a$. Similarly for the 3^{m_b} factorial experiment the corresponding factorial effects are expressed as:

$$
\begin{aligned}
3^{m_b}\mu &= \{y_1 = 0\} + \{y_1 = 1\} + \{y_1 = 2\} \\
3^{m_b-1}B_v &= \{y_v = 2\} - \{y_v = 0\} \\
3^{m_b-1}B_v^2 &= \{y_v = 2\} - 2\{y_v = 1\} + \{y_v = 0\} \\
3^{m_b-1}B_{v_1}B_{v_2} &= \{y_{v_1} + y_{v_2} = 2\} - \{y_{v_1} + y_{v_2} = 0\} \\
3^{m_b-1}B_{v_1}^2B_{v_2}^2 &= \{y_{v_1} + y_{v_2} = 2\} - 2\{y_{v_1} + y_{v_2} = 1\} + \{y_{v_1} + y_{v_2} = 0\} \\
3^{m_b-1}B_{v_1}B_{v_2}^2 &= \{y_{v_1} + 2y_{v_2} = 2\} - \{y_{v_1} + 2y_{v_2} = 0\} \\
3^{m_b-1}B_{v_1}^2B_{v_2} &= \{y_{v_1} + 2y_{v_2} = 2\} - 2\{y_{v_1} + 2y_{v_2} = 1\} + \{y_{v_1} + 2y_{v_2} = 0\},
\end{aligned}
$$

where $v_1 < v_2 = 1, \ldots, m_b$. The higher order interaction effects can be expressed in the similar manner. The expressions of the factorial effects for $2^{m_a} \times 3^{m_b}$ factorial experiment are given in detail in chapter 7. In matrix notation the factorial effects can be expressed in terms of the treatment effects as:

$$
\boldsymbol{F} = \boldsymbol{R}\boldsymbol{t}, \tag{1.1}
$$

where $\boldsymbol{t}$ corresponds to the set of treatment effects and $\boldsymbol{F}$ corresponds to the set of factorial effects. From (1.1) we have $\boldsymbol{t} = \boldsymbol{R}^{-1}\boldsymbol{F}$. The rows of the matrix R are orthogonal to each other and therefore $\boldsymbol{R}\boldsymbol{R}'$ is a diagonal matrix with

non-zero diagonal elements. It can be seen that $(\boldsymbol{RR}')(\boldsymbol{RR}')^{-1} = \boldsymbol{I}$ and hence $\boldsymbol{R}'(\boldsymbol{RR}')^{-1} = \boldsymbol{R}^{-1}$. Define $\boldsymbol{F}^* = (\boldsymbol{RR}')^{-1}\boldsymbol{F}$. Then

$$\boldsymbol{t} = \boldsymbol{R}'\boldsymbol{F}^*. \tag{1.2}$$

Example. We present one example for 3^{m_b} factorial experiment for $m_b = 2$. We have $\boldsymbol{F}$, $\boldsymbol{R}$ and $\boldsymbol{t}$ as follows:

$$\boldsymbol{F} = \begin{pmatrix} 3\mu \\ A_1 \\ 3A_1^2 \\ A_2 \\ A_2^2 \\ A_1A_2 \\ A_1^2A_2^2 \\ A_1A_2^2 \\ A_1^2A_2 \end{pmatrix}, \quad \boldsymbol{R} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 & -2 & -2 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 \\ -1 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & -2 & 1 & -2 & 1 & 1 & 1 & 1 & -2 \\ -1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \\ 1 & 1 & -2 & -2 & 1 & 1 & 1 & -2 & 1 \end{pmatrix}$$

and

$$\boldsymbol{t} = \begin{pmatrix} (0,0) \\ (0,1) \\ (0,2) \\ (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \end{pmatrix}.$$

Here $(RR') = diag\,(9, 6, 18, 6, 18, 6, 18, 6, 18)$ and hence from (1.2) we have

$$\boldsymbol{R}' = \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & -2 & 0 & -2 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 & -2 \\ 1 & 0 & -2 & -1 & 1 & 0 & -2 & 0 & -2 \\ 1 & 0 & -2 & 0 & -2 & 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & -2 & -1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 1 & 1 & 0 & -2 & -1 & 1 \end{pmatrix}, \quad \boldsymbol{F}^* = \begin{pmatrix} \mu \\ \frac{A_1}{6} \\ \frac{A_1^2}{18} \\ \frac{A_2}{6} \\ \frac{A_2^2}{18} \\ \frac{A_1 A_2}{6} \\ \frac{A_1^2 A_2^2}{18} \\ \frac{A_1 A_2^2}{6} \\ \frac{A_1^2 A_2}{18} \end{pmatrix}.$$

1.2 Model for a Fractional Factorial Experiment Using CRD

From this section onwards we denote the number of main effects by p_1 and the number of two factor interaction effects by p_2 for any factorial experiment. Under a completely randomized design (CRD) we assume the general model as

$$E\left(y\left(\boldsymbol{t}\right)\right) = \boldsymbol{t}^*,\ Var\left(y\left(\boldsymbol{t}\right)\right) = \sigma^2\boldsymbol{I}, \tag{1.3}$$

where the vector $\boldsymbol{t}$ represents the set of treatments and $\boldsymbol{t}^*$ represents their effects. The $y\left(\boldsymbol{t}\right)$ is the vector of responses for treatments in $\boldsymbol{t}$. We now consider a fraction n_f of the treatments denoted by $\boldsymbol{t}_f$. Then (1.3) becomes

$$E\left(y\left(\boldsymbol{t}_f\right)\right) = \boldsymbol{t}_f^*,\ Var\left(y\left(\boldsymbol{t}_f\right)\right) = \sigma^2\boldsymbol{I}, \tag{1.4}$$

From (1.2)we write

$$E\left(y\left(\boldsymbol{t}_f\right)\right) = \boldsymbol{t}_f^* = \boldsymbol{R}_f'\boldsymbol{F}^* = \boldsymbol{j}\boldsymbol{\mu} + \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2, \tag{1.5}$$

where $\boldsymbol{j}$ is the vector of unity, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ corresponds to the main effects and two-factor interaction effects respectively and $\boldsymbol{X}_1\left(n_f \times p_1\right)$and $\boldsymbol{X}_2\left(n_f \times p_2\right)$ are the design matrices corresponding to $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ respectively.

Example (Contd.). In 3^2 factorial experiment we consider the fraction as

$$\boldsymbol{t}_f = \left((0,0)\,,\ (0,1),\ (0,2),\ (1,0),\ (1,1),\ (1,2),\ (2,0),\ \ (2,1)\,,\ \ (2,2)\right)'.$$

In 1.5 we have the following:

$$\boldsymbol{R}'_f = \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & -2 & 0 & -2 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 & -2 \\ 1 & 0 & -2 & -1 & 1 & 0 & -2 & 0 & -2 \\ 1 & 0 & -2 & 0 & -2 & 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 1 & 1 & -1 & 1 & 1 & 1 \end{pmatrix},$$

$$\boldsymbol{X}_1 = \begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & 0 & -2 \\ -1 & 1 & 1 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & 1 & 1 \end{pmatrix},$$

$$\boldsymbol{X}_2 = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & 0 & -2 \\ 0 & -2 & 0 & -2 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \ \boldsymbol{j} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\boldsymbol{\beta}_1 = \begin{pmatrix} \frac{A_1}{2} \\ \frac{A_1^2}{6} \\ \frac{A_2}{2} \\ \frac{A_2^2}{6} \end{pmatrix}, \ \boldsymbol{\beta}_2 = \begin{pmatrix} \frac{A_1 A_2}{2} \\ \frac{A_1^2 A_2^2}{6} \\ \frac{A_1 A_2^2}{2} \\ \frac{A_1^2 A_2}{6} \end{pmatrix}.$$

1.3 Class of Models

Consider the linear model

$$E(\boldsymbol{y}) = \boldsymbol{j}\boldsymbol{\mu} + \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2, \ Var(\boldsymbol{y}) = \sigma^2\boldsymbol{I},$$

where $\boldsymbol{y}\,(n \times 1)$ is a vector of responses, $\boldsymbol{\beta}_0$ is the general mean, $\boldsymbol{\beta}_1\,(p_1 \times 1)$ is the vector of parameters corresponding to the main effects, $\boldsymbol{\beta}_2\,(p_2 \times 1)$ is the vector of parameters corresponding to the interaction effects, $\boldsymbol{X}_1\,(n_f \times p_1)$and $\boldsymbol{X}_2\,(n_f \times p_2)$ are the design matrices corresponding to $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ respectively and σ^2 is a constant which may or may not be known. The general mean $(\boldsymbol{\beta}_0)$ and all p_1 main effects in $\boldsymbol{\beta}_1$ are important and they are estimated under the model anyway. But we are not sure about the importance of all the parameters in $\boldsymbol{\beta}_2$ except that only $k\,(\geq 1)$ out of its p_2 parameters are non-negligible, k is not known. In this

situation the number of possible models with k interaction effects is $\binom{p_2}{k}$. These models, each with general mean , main effects in $\boldsymbol{\beta}_1$ and k parameters from $\boldsymbol{\beta}_2$ are compared to identify k non-negligible parameters and then inferences are drawn on them. Following the hierarchical principle we consider the case where $\boldsymbol{\beta}_2$ is the vector of the two-factor interaction effects and all three factor and higher order interaction effects are assumed to be negligible. Thus each of the $\binom{p_2}{k}$ models contain the general mean $\boldsymbol{\beta}_0$, all p_1 main effects and k two factor interaction effects. We write the uth linear model as:

$$M_u: \; E(\boldsymbol{y}) = \boldsymbol{j}\boldsymbol{\mu} + \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_{2u}\boldsymbol{\beta}_{2u}, \; Var(\boldsymbol{y}) = \sigma^2\boldsymbol{I}, \; u = 1, 2, \ldots \binom{p_2}{k}, \quad (1.6)$$

where $\boldsymbol{\beta}_1$ is a $(p_1 \times 1)$ vector, $\boldsymbol{\beta}_2$ is a $(p_2 \times 1)$ vector and $\boldsymbol{\beta}_{2u}$ is the uth $(k \times 1)$ vector obtained from $\boldsymbol{\beta}_2$, $\boldsymbol{X}_1\,(n \times p_1)$and $\boldsymbol{X}_{2u}\,(n \times k)$ are the design matrices corresponding to $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_{2u}$ respectively. Define the following

$$\begin{aligned} \boldsymbol{X}^{(u)} &= \left[\boldsymbol{j}_n \vdots \boldsymbol{X}_1 \vdots \boldsymbol{X}_{2u}\right]', \; \boldsymbol{\beta}^{(u)} = \left[\boldsymbol{j}_n \vdots \boldsymbol{\beta}_1 \vdots \boldsymbol{\beta}_{2u}\right]', \\ \boldsymbol{X}_1^* &= \left[\boldsymbol{j}_n \vdots \boldsymbol{X}_1\right]. \end{aligned} \quad (1.7)$$

Then

$$\boldsymbol{X}^{(u)} = \left[\boldsymbol{X}_1^* \vdots \boldsymbol{X}_{2u}\right] \text{ and } \boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)} = \begin{bmatrix} \boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^* & \boldsymbol{X}_1^{*\prime}\boldsymbol{X}_{2u} \\ \boldsymbol{X}_{2u}'\boldsymbol{X}_1^* & \boldsymbol{X}_{2u}'\boldsymbol{X}_{2u} \end{bmatrix}. \quad (1.8)$$

Thus the model in (5.1) becomes

$$E(\boldsymbol{y}) = \boldsymbol{X}^{(u)}\boldsymbol{\beta}^{(u)},$$

We assume $|\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}| > 0$ holds for the design and hence all the parameters in the models can be unbiasedly estimated. For the u^{th}model, the least square

estimator of $\boldsymbol{\beta}^{(u)}$ is $\hat{\boldsymbol{\beta}^{(u)}} = \left(\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}\right)^{-1}\boldsymbol{X}^{(u)\prime}\boldsymbol{y}$ and its variance is given as

$$Var\left(\hat{\boldsymbol{\beta}}^{(u)}\right) = \sigma^2\left(\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}\right)^{-1}. \tag{1.9}$$

From Rao (1973) we have

$$\frac{Var\left(\hat{\boldsymbol{\beta}}_{2u}\right)}{\sigma^2} = \left(\boldsymbol{X}_{2u}'\boldsymbol{X}_{2u} - \boldsymbol{X}_{2u}'\boldsymbol{X}_1^*\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_{2u}\right)^{-1}.$$

For example if we consider a fraction of a 3^3 factorial experiment with $k = 1$ and $n < 27$, where $\boldsymbol{\beta}_1$ consists of 6 main effects and $\boldsymbol{\beta}_2$ of 12 two-factor interactions, there will be 12 possible models, each with the general mean, 6 main effects and 1 2-factor interaction effect. Similarly, if we consider a 2^3 factorial experiment with $k = 1$ and $n < 8$, the $\boldsymbol{\beta}_1$ consists of 3 main effects and $\boldsymbol{\beta}_2$ of 3 two-factor interactions and the possible number of models would be 3. Here we give the uth model for the 3^3 factorial experiment:

$$M_u: \; E\left(\boldsymbol{y}\right) = \boldsymbol{j}\boldsymbol{\mu} + \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_{2u}\beta_{2u}, \; Var\left(\boldsymbol{y}\right) = \sigma^2\boldsymbol{I}, \; u = 1, 2, \ldots 12, \tag{1.10}$$

The columns of $\boldsymbol{X}_1\left(n \times 6\right)$ correspond to the main effects and the column of $\boldsymbol{X}_{2u}\left(n \times 1\right)$ corresponds to the 2-factor interaction effect for the uth model. For these 12 models the common parameters are $\boldsymbol{\beta}_0$ and elements of $\boldsymbol{\beta}_1$ while the uncommon parameters in u^{th} and u'^{th} models are β_{2u} and $\beta_{2u'}$, $u \neq u'$. The variance of the estimator of 2-factor interaction effect in the uth model is the last diagonal element of $Var\left(\hat{\boldsymbol{\beta}}^{(u)}\right)$ which can be expressed as:

$$\frac{Var\left(\hat{\beta}_{2u}\right)}{\sigma^2} = \left(c - \boldsymbol{X}_{2u}'\boldsymbol{X}_1^*\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_{2u}\right)^{-1}, \; u = 1, 2, \ldots 12. \tag{1.11}$$

where c is the last diagonal element in $\left(\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}\right)$ as presented in Ghosh and Flores (2013).

1.4 Contributions of the Thesis

1.4.1 3^m Factorial Experiment

1. The orthogonal design ($d.1$) is compared with the CV design ($d.2$) for $n = 9$. These two designs are very similar with respect to their runs and the orthogonality property but $d.2$ is a resolution III plus one plan as it can estimate the general mean and all main effects in presence of any two factor interaction effect and at the same time gives equal precision to all the two factor interaction estimators. On the contrary $d.1$ is a resolution III plan since it can not even estimate all the main effects in presence of any two factor interaction from the set of two factor interactions that are aliased with the main effects.

2. CV designs are obtained for 3^m factorial experiment for $m = 3$ and $n = 8$, 9, 10 and 11 from complete computer search and then the search is extended to obtain CV designs for higher values of m. Also five 3^3 CV designs for $n = 10$ are compared with respect to the different CV values and other optimality criteria like AD, AT, AE, GD, GT and GE.

3. The two series of CV designs $d_m^{(1)}$ and $d_m^{(2)}$ are obtained for general 3^m factorial experiment. The design $d_m^{(1)}$ for $n = 2m+2$, $m \geq 2$ gives optimum CV design for $m = 2$ and the design $d_m^{(2)}$ for $n = 3m$, $m \geq 3$ gives optimum CV design for $m = 3$.

4. The projection matrices of the 3^m CV designs are completely analyzed and

are found to possess a particular structure in which the elements of the m rows and m columns are zeros corresponding to a particular set of m runs of the CV designs. Most of the CV as well as the optimal CV designs possess this particular structure of the projection matrix.

5. CV designs with $(n \pm r)$ runs are obtained from CV designs with n runs by deleting runs from or adding runs to the latter. The complete tree structure of the hierarchical CV designs for 3^3 factorial experiment is presented. Starting from $n = 8$ hierarchically CV designs are obtained for $n = 9,\ 10$ and 11 by adding one run at a time from the remaining runs at each step and then narrowing down the search from all possible designs to full rank designs some of which satisfy the CV property. Similarly starting from $n = 11$ CV designs are obtained for $n = 10,\ 9$ and 8 hierarchically by deleting the existing runs one at a time. Also the complete tree structure of the optimal CV designs is presented. The condition of obtaining a CV design for $(n \pm 1)$ from a CV design for n is derived in terms of the design matrix and the runs of the design.

6. A class of fractional factorial designs with n runs possessing the common variance property are characterized for general m. Several sufficient conditions are obtained by using pairs of interaction effects (null space and permutation matrix), independent columns of the projection matrix and runs of the designs. As the number of factors for a factorial experiment gets large it is not possible by computer check to search for CV designs from millions of

possible designs for different n which involves the tedious calculation of the inverse of the variance covariance matrix for each model in the class. The CV designs for factorial experiment with small m can be extended to designs for factorial experiment with higher m and the conditions can be checked for the CV property of the latter. These checkings can be done through simple calculations as the projection matrix needs to be calculated only once and the dimension of its independent columns is low for small n.

7. We derive the condition of obtaining a 3^3 CV design from a 3^2 CV design where every pair of columns of the 3^3 CV design consists of the same runs as that of the 3^2 CV design and the runs are replicated in the same way in both.

8. We prove that the optimal CV design $d_m^{(1)}$ for $m = 2$ always remains CV after replicating any of its six runs any number of times. We also obtain many more 3^2 designs for $n = 6$ which satisfy the CV property for any number of replication of the six runs. Some of these designs are balanced and isomorphic to each other w.r.t the runs. Replicated designs are also obtained for 3^3 factorial experiment for different number of runs.

1.4.2 $2^{m_a} \times 3^{m_b}$ Factorial Experiment

1. We also extend our search of CV designs to the mixed level factorial experiment. For the simplest 2×3 factorial experiment no CV design exists with distinct runs and hence we considered a very structured replication of

the six runs and under a particular condition of replications CV designs are obtained for different runs.

2. For higher values of m_a and m_b it is computationally challenging to obtain CV designs. We obtain designs that give common variance within each of the groups: (1) the pure interaction estimators between the factors with same levels, (2) the mixed interactions linear in both factors and (3) the mixed interactions quadratic in the factor with 3 levels.

3. The general replications of the runs for the 2×3 designs are presented giving the variances of the 2-factor interaction estimators almost identical to each other.

Chapter 2

Common Variance

2.1 Chapter Summary

In this chapter we discuss the common variance (CV) property of the designs and obtain CV designs for 3^3 factorial experiment by thorough computer check. Also we compare a 3^3 CV design with an orthogonal 3^3 design. Here is what we present in each section:

- (Section 2.2): In this secion we discuss the concept of common variance that was first introduced in the paper by Ghosh and Flores (2013). We present a 3^3 design with 10 runs which gives constant value to all the two-factor interaction variance and hence a CV design.

- (Section 2.3): In this section we present the CV designs for 3^3 factorial experiment for $n = 8, 9, 10$ and 11. Also we present one optimum CV design giving minimum value of CV for each of these values of n.

- (Section 2.4): In this section we compare a 3^3 CV design for $n = 9$ with an orthogonal one-third fraction of 3^3 factorial experiment. The CV design has the ability to identify a class models each with general mean, main effects and one two-factor interaction effect along with giving equal precision to all the two-factor interaction estimators while the one-third fraction can not even estimate the all main effects in presence any two-factor interactions aliased with the main effects.

2.2 Common Variance

The concept of common variance of the uncommon parameter in the models was first introduced in Ghosh and Flores (2013).

Definition 1. *A design is a common variance* (CV) *design if the variance of the uncommon parameter estimator is constant, i.e.,* $Var\left(\hat{\beta}_{2u}\right) = constant,\ \forall u.$

The statistical meaning of this notion is that in all the models the uncommon parameter is estimated with equal precision (precision is defined as the reciprocal of the variance of the parameter estimator). This is a desirable statistical property of the design about the estimation of the uncommon parameter. If instead one two factor interaction is estimated with greater precision than the other and it turns out that the latter is the true one then certainly this kind of a situation is not wanted. Since we do not have any *apriori* information about the true non-negligible two factor interaction and hence the true model is not known, so all the uncommon parameters in the models should be estimated with equal precision

Table 2.1: CV Design D_3^1 for $n = 10$

t_1	0	0	2	0	2	2	1	2	2	2
t_2	0	2	0	2	0	2	2	1	2	2
$t3$	2	0	0	2	2	0	2	2	1	2

or equivalently should have a common variance. No model should be preferred over the other while estimating the two factor interactions. This makes all the models stand on the same level of comparison to identify the true non-negligible component of $\boldsymbol{\beta}_2$. We give one example of a CV design with number of runs $n = 10$ and number of factors $m = 3$ each at three levels in Table 2.1. We consider the class of models M_u, $u = 1, \ldots, 12$ for 3^3 factorial experiment as presented in (1.3.5) in Chapter 1. For the design in Table 2.1 M_u satisfies the design condition $|\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}| > 0$, $\forall u$, i.e, this design can estimate the general mean, all main effects and one two factor interaction effect in each model. We find that $\frac{Var\left(\hat{\beta}_{2u}\right)}{\sigma^2} = 0.2963$, $\forall u$. Thus all the models are estimating their uncommon parameter with equal precision and hence the design D_3^1 is a CV design.

2.3 Common Variance Designs for $m = 3$

The number of all possible designs that could be formed with $n = 10$ and $m = 3$, all treatments being replicated only once, is $\binom{27}{10} = 84,36,285$. All of these designs are checked for CV property. It is found that $2,792,387$ (about 33%) designs can estimate all the 8 parameters in each model. Out of $27,92,387$ designs only $16,640$ designs are common variance designs that can estimate the 2-factor

Table 2.2: CV Designs for Different n

n	Possible designs=$\binom{27}{n}$	DC	# of Non CV designs	# of CV designs	Groups	
					#	CV
11	13,037,895	6,926,868	6,924,772	2,096	32	0.2151
					2,064	0.2222
10	8,436,285	2,792,387	2,775,747	16,640	0.2564	48
					0.2667	48
					0.2837	16
					0.2963	16,512
					0.4	16
9	4,686,825	636,348	588,348	48,000	0.3333	8,256
					0.381	32
					0.4167	13,056
					0.4444	26,640
					0.5	16
8	2,220,075	49,628	23,340	26,288	0.6667	9600
					0.8889	16,688

interaction effect with equal precision in all the 12 models. Similar results are obtained for $n = 11,\ 10,$ and 8. Table 2.2 shows the findings. From Table 2.2 we see that there are 5 different groups of CV value for $n = 9$ and $n = 10$ and there are two different groups for $n = 8$ and $n = 11$. We present the examples of CV designs with minimum CV value for $n = 8, 9, 10$ and 11 in Table 2.3. These designs are optimum CV designs for the 3^3 factorial experiment for the respective n. The five designs presented in Table 2.4 are selected from CV designs for $n = 10$, one from each category of common variance to study their treatment contents thoroughly and also to compare them with respect to some criterion functions. In chapter 9 we will compare these five CV designs with respect to the AD, AT, AE, GD, GT and GE optimality criteria.

Table 2.3: CV Designs with Minimum CV Value for Different n

n	CV designs											CV
8	0	0	0	0	1	1	2	2				0.6667
	0	0	1	2	1	2	1	2				
	0	1	2	0	2	1	1	2				
9	0	0	2	2	2	0	2	2	1			0.3333
	0	2	0	2	0	2	2	1	2			
	2	0	0	0	2	2	1	2	2			
10	0	0	0	0	1	1	1	2	2	2		0.2564
	0	1	2	2	1	1	2	0	0	2		
	1	1	0	1	1	2	1	0	1	2		
11	0	0	0	0	1	1	2	2	1	1	0	0.2151
	0	0	1	2	1	2	1	2	1	2	2	
	0	1	2	0	2	1	1	2	1	2	1	

2.4 Comparison of Two 3^3 Designs for $n = 9$

In this section we compare a 3^3 CV design $d.1$ for $n = 9$ with a standard one third fraction of 3^3 factorial design $d.2$. Both of these designs are resolution III plans and we will show that although the CV design is non orthogonal unlike the one-third fraction, the correlations among the estimates of the general mean and main effects are very week and also its variance-covariance matrix satisfies one important property of the diagonal matrix. So both the designs are very similar considering the main effects estimation only. But the CV design has the ability to estimate the additional $2-$factor interaction in each model with equal precision whereas the orthogonal one third fraction can not even estimate all the main effects in presence of any $2-$factor interaction effect from the alias set in the model.

Consider the one-third fraction with the defining relation as: $ABC = I$. We consider the design in Table 2.5 corresponding to the fraction: $x_1 + x_2 + x_3 = 1$

Table 2.4: 5 Designs Selected from CV Designs for $n = 10$

$CV = 0.2564$			$CV = 0.2667$			$CV = 0.2837$			$CV = 0.2963$			$CV = 0.4$		
0	0	1	0	0	1	0	0	2	2	0	0	0	0	2
0	1	1	0	2	0	0	1	1	0	2	0	0	1	2
0	2	0	0	2	1	0	2	0	0	0	2	1	0	1
0	2	1	1	1	1	0	2	1	2	2	0	1	0	2
1	1	1	1	1	2	1	1	1	2	0	2	2	0	0
1	1	2	1	2	2	1	2	1	0	2	2	2	0	1
1	2	1	2	0	0	2	0	0	2	2	1	2	0	2
2	0	0	2	0	1	2	0	1	2	1	2	2	1	2
2	0	1	2	1	2	2	2	0	1	2	2	2	2	0
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

Table 2.5: Design $d.1$

0	0	1	1	1	2	0	2	2
0	1	0	1	2	1	2	0	2
1	0	0	2	1	1	2	2	0

under mod (3) . This design $d.1$ is a resolution III plan which has the ability to estimate the general mean and main effects under the assumption that the two factor and higher order interactions are negligible. Also $d.1$ can estimate the general mean and all main effects orthogonally and hence its variance – covariance matrix is a diagonal matrix. We consider another 3^3 design for $n = 9$ in Table 2.6. This design $d.2$ also has the ability to estimate the general mean and all main effects but it is not an orthogonal design and hence its variance-covariance matrix is not a diagonal one. Table 2.7 gives the variances and the covariances of the main effects estimators for the two designs $d.1$ and $d.2$. From Table 2.7 we see that both $d.1$ and $d.2$ estimate all the linear main effects with equal precision as well as all the quadratic main effects with equal precision, i.e., we have $\frac{Var\left(\hat{\beta}_{2u}\right)}{\sigma^2}$ =constant, $u =$

Table 2.6: Design $d.2$

0	0	1	1	1	2	1	2	2
0	1	0	1	2	1	1	2	2
1	0	0	2	1	1	1	2	1

Table 2.7: Variance-Covariance of the Main Effects Estimators

	Main Effects	A	A^2	B	B^2	C	C^2
$d.1$	A	0.1667	0	0	0	0	0
	A^2	0	0.0556	0	0	0	0
	B	0	0	0.1667	0	0	0
	B^2	0	0	0	0.0556	0	0
	C	0	0	0	0	0.1667	0
	C^2	0	0	0	0	0	0.0556
$d.2$	A	0.3437	-0.0104	-0.1562	-0.0104	-0.0938	0.0104
	A^2	-0.0104	0.0521	-0.0104	-0.0035	0.0104	0.0035
	B	-0.1562	-0.0104	0.3437	-0.0104	-0.0938	0.0104
	B^2	-0.0104	-0.0035	-0.0104	0.0521	0.0104	0.0035
	C	-0.0938	0.0104	-0.0938	0.0104	0.3437	-0.0104
	C^2	0.0104	0.0035	0.0104	0.0035	-0.0104	0.0521

A, B, C and $\frac{Var(\hat{\beta}_{2u})}{\sigma^2}$ =constant, $u = A^2, B^2, C^2$ for both the designs. Comparing the variance-covariance structures of $d.1$ and $d.2$ we see that $d.1$ estimates the linear main effects with almost double precision as compared to $d.2$. But both the designs estimate the quadratic main effects with similar precision. Also since $d.1$ estimates the main effects orthogonally, we have $\frac{Cov(\hat{\beta}_{2u},\hat{\beta}_{2u'})}{\sigma^2} = 0, u \neq u'$. But for $d.2$, $\frac{Cov(\hat{\beta}_{2u},\hat{\beta}_{2u'})}{\sigma^2} \neq 0, u \neq u'$ since $d.2$ is not an orthogonal design. If $\boldsymbol{X}_2$ is the design matrix for $d.2$, the difference between $|\,(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\,|$ and the product of the diagonal elements of $(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}$ is 5.418699×10^{-7}. So we see that although the variance - covariance matrix of $d.2$ is not diagonal, it has one property of the diagonal matrix, its determinant being almost equal to the product of its diagonals. Moreover, $d.2$ has the ability to estimate one two-factor interaction effect along

Table 2.8: Aliased Two Factor Interactions with Main Effects for $d.1$

Main effect	Aliased 2− Factor Interaction Effects
A	B^2C^2
A^2	BC
B	A^2C^2
B^2	AC
C	A^2B^2
C^2	AB

with the general mean and all main effects in the class of models. But the one-third fraction $d.1$ can not even estimate all main effects in the presence of some two-factor interaction effects. This is because the main effects for a resolution III plan are aliased with some of the two-factor interaction effects which are shown in Table 2.8. From Table 2.8 we see that the main effects are aliased with six two-factor interactions and hence $d.1$ can not estimate the general mean and all main effects in presence of any one of these six interaction effects in the model. However, the general mean and all main effects with one of the interactions from the set $\{AB^2, A^2B, AC^2, A^2C, BC^2, B^2C\}$ can be estimated by $d.1$. The design $d.2$ can estimate all the two-factor interactions with equal variance, i.e., we have $\frac{Var(\hat{\beta}_{2u})}{\sigma^2} = 0.4444$, constant $\forall u$ and hence $d.2$ is a Common Variance (CV) design. But we can not compare the variances of the interaction estimators for the design $d.1$ since it does not even have the ability to estimate all the two-factor interaction effects. So we see that although the design $d.2$ is not an orthogonal design like $d.1$ but it has the advantage over $d.1$ in the sense of estimating the uncommon parameter in each model with equal precision.

Next consider a design $d.3$ for $n = 8$ by deleting one run 221 from $d.2$ which

Table 2.9: Design $d.3$

0	0	1	1	1	2	1	2
0	1	0	1	2	1	1	2
1	0	0	2	1	1	1	2

Table 2.10: Design $d.4$

1	1	2	0	2	0	0	0
1	2	1	0	2	0	1	2
2	1	1	0	2	1	2	0

is presented in Table 2.9. This design $d.3$ has the ability to estimate the general mean, all main effects and one two-factor interaction in each model. Moreover $d.3$ can estimate all the two-factor interactions with equal precision. Hence it is a CV design with $\frac{Var\left(\hat{\beta}_{2u}\right)}{\sigma^2} = 0.8889$, $\forall u$. However $d.3$ is not a CV design with minimum value of CV. So in Table 2.10 we present one CV design $d.4$ for $n = 8$ with $CV = 0.6667$ which is the minimum value of CV in the class of all 3^3 CV designs for $n = 8$. The design $d.4$ is one of the many optimum CV designs for $n = 8$. Adding the run 111 or 122 to the runs of $d.4$ produces a CV design for $n = 9$ with $CV = 0.3810$ which is the second best in the class of all 3^3 CV designs for $n = 9$. No optimum CV design ($CV = 0.3333$) for $n = 9$ can be obtained from $d.4$.

Chapter 3

Hierarchical CV Designs

3.1 Chapter Summary

In this chapter we present hierarchical CV designs for 3^3 factorial experiment starting from $n = 8$ to $n = 11$ and vice versa and derive condition for obtaining hierarchical CV designs. Here is what we present in each section:

- (Section 3.2): In this section we present our complete search of 3^3 CV designs that are obtained from CV designs for n runs by deleting runs from or adding runs to the latter. Also we present the condition derived for obtaining CV design with $(n \pm 1)$ runs from a CV design with n runs. So given a CV design for n this condition can be checked to determine the CV property of the design for $(n \pm 1)$ without calculating the variance of the two-factor interaction estimators for the latter.

- (Section 3.3): In this section we present the complete hierarchical structure of CV designs starting from $n = 8$ to $n = 11$ and the other way. Also we

present the hierarchical optimum CV designs from $n = 8$ to $n = 11$ and vice versa.

In the following we describe the notations that are used in Table 3.4 in Section 3.2 and Tables 3.7 and 3.8 in Section 3.3:

- CV_n: # of CV designs for n,
- $CV_n^{(n-1)}$: Subset of CV_n generating CV designs for $(n-1)$,
- $CV_{n(n-1)}$: # of CV designs for $(n-1)$ generated from CV designs for n,
- $CV_n^{(n+1)}$: Subset of CV_n generating CV designs for $(n+1)$,
- $CV_{n(n+1)}$: # of CV designs for $(n+1)$ generated from CV designs for n.
- CV_n^{n-r} : # of CV designs for $(n-r)$ obtained in the hierarchical order from CV designs for n.
- CV_n^{n+r} : # of CV designs for $(n+r)$ obtained in the hierarchical order from CV designs for n.
- *Opt* CV_n^{n-r} : # of optimum CV designs for $(n-r)$ obtained in the hierarchical order from optimum CV designs for n.
- *Opt* CV_n^{n+r} : # of optimum CV designs for $(n+r)$ obtained in the hierarchical order from optimum CV designs for n.

Table 3.1: Design D_3^1

0	0	2	0	2	2	1	2	2	2
0	2	0	2	0	2	2	1	2	2
2	0	0	2	2	0	2	2	1	2

Table 3.2: CV Designs for $n = 9$ from D_3^1

Delete	CV value	Designs								
0		0	0	2	2	2	1	2	2	2
2	0.4444	0	2	0	0	2	2	1	2	2
2		2	0	0	2	0	2	2	1	2
2		0	0	2	0	2	1	2	2	2
0	0.4444	0	2	0	2	2	2	1	2	2
2		2	0	0	2	0	2	2	1	2
2		0	0	2	0	2	1	2	2	2
2	0.4444	0	2	0	2	0	2	1	2	2
0		2	0	0	2	2	2	2	1	2
2		0	0	2	0	2	2	1	2	2
2	0.3333	0	2	0	2	0	2	2	1	2
2		2	0	0	2	2	0	2	2	1

3.2 CV Designs from n to $n \pm 1$

To check if CV designs could be obtained for $n = 9$ or $n = 11$ we deleted one run from and added one run to a CV design for $n = 10$ respectively. Consider the CV design D_3^1 in Table 3.1 for $n = 10$. The four CV designs presented in Table 3.2 for $n = 9$ are obtained from the design D_3^1 by deleting one run from it at a time. Naturally if we add the deleted runs one at a time to these four CV designs for $n = 9$ we get back D_3^1. The CV design for $n = 11$ in Table 3.3 is obtained by adding the run $(0, 0, 0)$ to D_3^1. Again deleting the run $(0, 0, 0)$ will give us back the design D_3^1. It follows from the above two tables how CV designs for $n \pm 1$ can

Table 3.3: CV Design for $n = 11$ from D_3^1

0	0	2	0	2	2	1	2	2	2	0
0	2	0	2	0	2	2	1	2	2	0
2	0	0	2	2	0	2	2	1	2	0

Table 3.4: CV Designs from n to $(n \pm 1)$

n	CV_n	$CV_n^{(n-1)}$	$CV_{n(n-1)}$	$CV_n^{(n+1)}$	$CV_{n(n+1)}$
8	$26,288$	-	-	$26,112$	$37,856$
9	$48,000$	$37,856$	$26,112$	$33,320$	$16,640$
10	$16,640$	$16,640$	$33,320$	$16,560$	$2,096$
11	$2,096$	$2,096$	$16,560$	-	-

be obtained from that of n. In chapter 2 Table 2.8 presents the number of 3^3 CV designs for $n = 8,\ 9,\ 10$ and 11. To the CV designs with n runs we add one run at a time from the remaining $(27 - n)$ runs to obtain designs with $(n + 1)$ runs. Similarly we delete one run at a time from the CV designs with n runs to obtain designs with$(n - 1)$ runs . The designs obtained for $(n \pm 1)$ are not all distinct and hence we ignore the repeated designs and only consider the distinct ones to check for their CV property. Out of all the distinct designs with $(n \pm 1)$ runs satisfying the design condition we determine the designs satisfying the condition of common variance. The CV designs with $(n \pm 1)$ runs obtained from the CV designs with n runs are infact a subset of the set of all CV designs for $(n \pm 1)$. Also all the CV designs for n do not generate the complete set of CV designs for $(n \pm 1)$. Only a subset of all CV designs for n generates CV designs for $(n \pm 1)$. Table 3.4 gives the result of the complete search for CV designs from n to $(n \pm 1)$.

In the following we derive the condition for obtaining a CV design for $(n + 1)$ number of runs from a CV design for n runs by adding one run to the latter.

The condition is also true for obtaining a CV design for n from a CV design for $(n+1)$. For a design with n runs we write the design matrix with the uth two factor interaction as:

$$\boldsymbol{X}_u^{(1)} = \left[\boldsymbol{X}_1 \vdots \boldsymbol{X}_{2u}\right], \tag{3.1}$$

where $\boldsymbol{X}_1$corresponds to the general mean and main effects and $\boldsymbol{X}_{2u}$ corresponds to the two factor interaction effect in the uth model. For a design with $(n+1)$ runs obtained by adding one run to the former design we write the design matrix as:

$$\boldsymbol{X}_u^{(2)} = \begin{bmatrix} \boldsymbol{X}_1 & \vdots & \boldsymbol{X}_{2u} \\ \boldsymbol{x}_2' & \vdots & x_{22u} \end{bmatrix}, \tag{3.2}$$

where $\boldsymbol{x}_2'$ and x_{22u} correspond to the new run added for the main effects and the two factor interaction effect respectively. Assuming that the design for n runs is a CV design the following condition holds true:

$$\frac{Var\left(\hat{\beta}_{2u}\right)}{\sigma^2} = \frac{1}{\left(\boldsymbol{X}_{2u}'\boldsymbol{X}_{2u} - \boldsymbol{X}_{2u}'\boldsymbol{X}_1\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_{2u}\right)} = constant,\ \forall u.$$

Let $v_u^{(1)} = \left(\boldsymbol{X}_{2u}'\boldsymbol{X}_{2u} - \boldsymbol{X}_{2u}'\boldsymbol{X}_1\left(\boldsymbol{X}_1'\boldsymbol{X}_{\mathbf{1}}\right)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_{2u}\right)$. A design with n runs is CV iff $v_u^{(1)}$ is constant,$\forall u$. From (3.2) we get

$$\boldsymbol{X}_u^{(2)\prime}\boldsymbol{X}_u^{(2)} = \begin{bmatrix} \boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{x}_2\boldsymbol{x}_2' & \vdots & \boldsymbol{X}_1'\boldsymbol{X}_{2u} + x_{22u}\boldsymbol{x}_2 \\ \boldsymbol{X}_{2u}'\boldsymbol{X}_1 + x_{22u}\boldsymbol{x}_2' & \vdots & \boldsymbol{X}_{2u}'\boldsymbol{X}_{2u} + x_{22u}^2 \end{bmatrix}. \tag{3.3}$$

Define $v_u^{(2)} = \boldsymbol{X}_{2u}'\boldsymbol{X}_{2u} + x_{22u}^2 - \left(\boldsymbol{X}_{2u}'\boldsymbol{X}_1 + x_{22u}\boldsymbol{x}_2'\right)\left(\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{x}_2\boldsymbol{x}_2'\right)^{-1}\left(\boldsymbol{X}_1'\boldsymbol{X}_{2u} + x_{22u}\boldsymbol{x}_2\right)$.

The variance of the uth two-factor interaction for the design with $(n+1)$ runs is proportional to the last diagonal element of $\left(\boldsymbol{X}_u^{(2)\prime}\boldsymbol{X}_u^{(2)}\right)^{-1}$ which is $\frac{1}{v_u^{(2)}}$. Hence the $(n+1)$ run design will be CV iff

$$v_u^{(2)} = constant,\ \forall u.$$

Table 3.5: CV Design (a) D^9 for $n = 9$ and (b) D^{10} for $n = 10$

(a)

D^9								
0	0	2	0	2	2	1	2	2
0	2	0	2	0	2	2	1	2
2	0	0	2	2	0	2	2	1

(b)

D^{10}									
0	0	2	0	2	2	1	2	2	2
0	2	0	2	0	2	2	1	2	2
2	0	0	2	2	0	2	2	1	2

After simplification of the expression of $v_u^{(2)}$ the following condition is obtained. The complete derivation is shown in the Appendix (Section 3.4).

$$v_u^{(2)} = v_u^{(1)} + \frac{(a_u - b_u)^2}{(1+k)}, \ \forall u, \tag{3.4}$$

where $a_u = x_{22u}$, $b_u = \boldsymbol{X}'_{2u}\boldsymbol{X}_1 \left(\boldsymbol{X}'_1\boldsymbol{X}_1\right)^{-1} \boldsymbol{x}_2$, $k = \boldsymbol{x}'_2 \left(\boldsymbol{X}'_1\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2$. The k does not depend on u and hence $v_u^{(2)}$ will be constant iff $(a_u - b_u)^2$ or equivalently $|a_u - b_u|$ is constant given that $v_u^{(1)}$ is constant. Again if we have $v_u^{(2)}$ constant then $v_u^{(1)}$ will be constant iff $|a_u - b_u|$ is constant independent of u. Thus given a CV design for n (or $n+1$) the design for $n+1$ (or n) will be CV iff $|a_u - b_u|$ is constant, $\forall u$. The (3.4) gives the condition to obtain a CV design for $(n \pm 1)$ from a CV design for n. We present one example of a CV design D^{10} with $(n+1) = 10$ runs obtained by adding one run to a CV design D^9 with $n = 9$ runs. Both D^9 and D^{10} are presented in Table 3.5. The design D^{10} is in fact the design D_3^1 in Table 3.1 which is already known to be CV. We demonstrate the CV property of D^{10} by

Table 3.6: Values of $|a_u - b_u|$

Interaction(u)	a_u	b_u	$a_u - b_u$	$(a_u - b_u)^2 = \|a_u - b_u\|$
AB	0	1	-1	1
A^2B^2	-2	-1	-1	1
AB^2	-1	0	-1	1
A^2B	1	0	1	1
AC	0	1	-1	1
A^2C^2	-2	-1	-1	1
AC^2	-1	0	-1	1
A^2C	1	0	1	1
BC	0	1	-1	1
B^2C^2	-2	-1	-1	1
BC^2	-1	0	-1	1
B^2C	1	0	1	1

using the condition in (3.4). The design D^9 is a CV design with $CV = 0.2963$ and hence $v_u^{(1)}$ is constant, $\forall u$. We add the run $(2, 2, 2)$ to the design D^9 to obtain design D^{10}. From (3.4) if we can show that $|a_u - b_u|$ is constant $\forall u$, then $v_u^{(2)}$ will be constant $\forall u$, and hence the design D^{10} will be a CV design. In Table 3.6 we present the values of $|a_u - b_u|$. From Table 3.6 wee see that $|a_u - b_u|$ is constant, $\forall u$ and this explains the CV property of the design D^{10}.

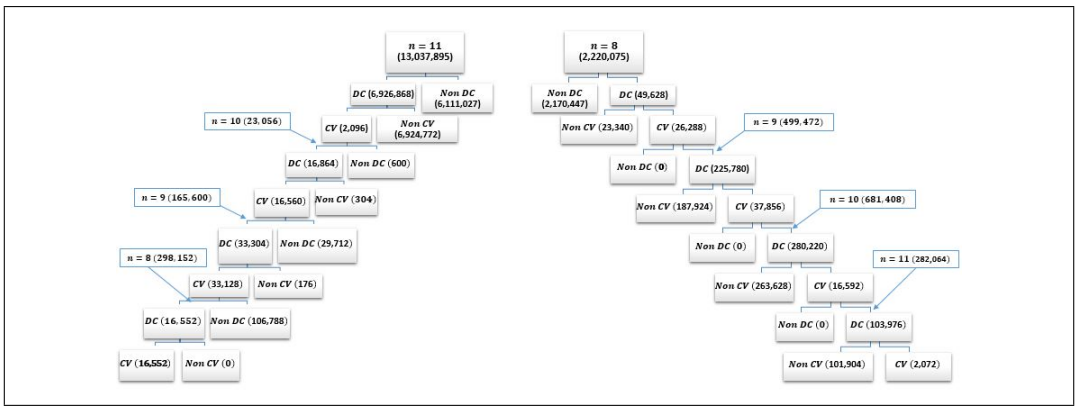
3.3 CV Designs from n to $(n \pm r)$

In this section we present some hierarchical CV designs starting from $n = 11$ going down to $n = 8$ and vice versa. Starting from the CV designs for $n = 11$ we delete one run at a time from the existing runs and construct designs for $n = 10$. Since all these designs are not distinct so we delete the repeated designs and work with the distinct ones only. A subset of these distinct designs are full rank and a

Table 3.7: CV Designs from n to $n \pm r$

$n,\ r$	$n-r$	$*CV_n^{n-r}$	$n,\ r$	$n+r$	$**CV_n^{n+r}$
11, 1	10	16,650	8, 1	9	37,856
11, 2	9	33,128	8, 2	10	16,592
11, 3	8	16,552	8, 3	11	2,072

Figure 3.1: Hierarchical CV Designs for $m = 3$



subset of these full rank designs satisfy the CV property. This is how we obtain CV designs for $n = 10$ from $n = 11$. From theseCV designs for $n = 10$ we obtain CV designs for $n = 9$ and finally from these CV designs for $n = 9$ we obtain CV designs for $n = 8$. Again we start from CV designs for $n = 8$ and add one run from the remaining runs to them and obtain CV designs for $n = 9$ and continue this way up to $n = 11$. Table 3.7 presents the number of hierarchical CV designs that could be obtained from $n = 11$ in hierarchical order through $n = 8$ and also the other way. This hierarchical setting is also displayed in Figure 3.1.

Table 3.8 presents the number of optimum CV designs (designs with smallest CV) in hierarchical order. We see that only 32 designs for $n = 11$ have minimum

Table 3.8: Optimum CV Designs from n to $n \pm r$

$n,\ r$	$n-r$	$Opt\ CV_n^{n-r}$	$Opt\ CV$	$n,\ r$	$n+r$	$Opt\ CV_n^{n+r}$	$Opt\ CV$
11, 0	11	32	0.2151	8, 0	8	2,096	0.6667
11, 1	10	32	0.2667	8, 1	9	32	0.3810
11, 2	9	16	0.3810	8, 2	10	16	0.2667
11, 3	8	40	0.6667	8, 3	11	8	0.2151

Table 3.9: One Optimum CV Design for $n = 11$

0	0	0	0	1	1	2	2	1	1	0
0	0	1	2	1	2	1	2	1	2	2
0	1	2	0	2	1	1	2	1	2	1

CV value and each of these 32 designs generates one optimum CV design for $n = 10$. Thus we have 32 optimum CV designs for $n = 10$ obtained from optimum CV designs for $n = 11$. From these 32 designs, only 8 designs generate 16 optimum CV designs for $n = 9$. And these 8 designs generate 40 optimum CV designs for $n = 8$. Similarly if we start from $n = 8$ there are $2,096$ optimum CV designs which give 32 optimum CV designs for $n = 9$ and 16 optimum CV designs for $n = 10$ and only 8 for $n = 11$. Thus, although there are 32 optimum CV designs for $n = 11$, only 8 of them are optimum in the hierarchical set up which give optimum CV designs for $n = 10$, 9 and 8 in the hierarchical order. Also for $n = 8$ and 11 the optimum CV values are the smallest in their class of all CV designs. But for $n = 9$ and 10 the optimum designs are only optimum in this hierarchical setting, these are the second best in their class of all CV designs. We present one example of the hierarchical CV design in Table 3.9. This is one of the 8 optimum CV designs for $n = 11$. Deleting $(0, 2, 1)$ gives optimum CV design for $n = 10$ in the hierarchical setting. Deleting both $(0, 2, 1)$ and $(1, 1, 1)$ gives optimum CV design for $n = 9$ in

the hierarchical setting and deleting $(0,2,1)$, $(1,1,1)$ and $(1,2,2)$ gives optimum CV design for $n = 8$.

3.4 Appendix

3.4.1 Proof of Equation (3.4)

The variance of the uth two-factor interaction for the design with n runs is

$$v_u^{(1)} = \boldsymbol{X}_{2u}'\boldsymbol{X}_{2u} - \boldsymbol{X}_{2u}'\boldsymbol{X}_1\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_{2u}.$$

The variance of the uth two-factor interaction for the design with $(n+1)$ runs is

$$v_u^{(2)} = \boldsymbol{X}_{2u}'\boldsymbol{X}_{2u} + x_{22u}^2 - \left(\boldsymbol{X}_{2u}'\boldsymbol{X}_1 + x_{22u}\boldsymbol{x}_2'\right)\left(\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{x}_2\boldsymbol{x}_2'\right)^{-1}\left(\boldsymbol{X}_1'\boldsymbol{X}_{2u} + x_{22u}\boldsymbol{x}_2\right).$$

From *Rao* (1973) we have

$$\left(\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{x}_2\boldsymbol{x}_2'\right)^{-1} = \left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1} - \frac{\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2\boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}}{1 + \boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2}.$$

Hence

$$\begin{aligned}
&\left(\boldsymbol{X}_{2u}'\boldsymbol{X}_1 + x_{22u}\boldsymbol{x}_2'\right)\left(\boldsymbol{X}_1'\boldsymbol{X}_1 + \boldsymbol{x}_2\boldsymbol{x}_2'\right)^{-1}\left(\boldsymbol{X}_1'\boldsymbol{X}_{2u} + x_{22u}\boldsymbol{x}_2\right)\\
=&\left(\boldsymbol{X}_{2u}'\boldsymbol{X}_1 + x_{22u}\boldsymbol{x}_2'\right)\left[\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1} - \frac{\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2\boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}}{1 + \boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2}\right]\left(\boldsymbol{X}_1'\boldsymbol{X}_{2u} + x_{22u}\boldsymbol{x}_2\right)\\
=&\boldsymbol{X}_{2u}'\boldsymbol{X}_1\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_{2u} - \frac{\boldsymbol{X}_{2u}'\boldsymbol{X}_1\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2\boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_{2u}}{1 + \boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2} +\\
&+x_{22u}^2\boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2 - \frac{x_{22u}^2\boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2 x_{22u}^2\boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2}{1 + x_{22u}^2\boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2}\\
&+2x_{22u}\boldsymbol{X}_{2u}'\boldsymbol{X}_1\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2 - \frac{2x_{22u}\boldsymbol{X}_{2u}'\boldsymbol{X}_1\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2\boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2}{1 + \boldsymbol{x}_2'\left(\boldsymbol{X}_1'\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2}.
\end{aligned}$$

Putting $a_u = x_{22u}$, $b_u = \boldsymbol{X}'_{2u}\boldsymbol{X}_1\left(\boldsymbol{X}'_1\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2$ and $k = \boldsymbol{x}'_2\left(\boldsymbol{X}'_1\boldsymbol{X}_1\right)^{-1}\boldsymbol{x}_2$, we have

$$\begin{aligned}
&\left(\boldsymbol{X}'_{2u}\boldsymbol{X}_1 + x_{22u}\boldsymbol{x}'_2\right)\left(\boldsymbol{X}'_1\boldsymbol{X}_1 + \boldsymbol{x}_2\boldsymbol{x}'_2\right)^{-1}\left(\boldsymbol{X}'_1\boldsymbol{X}_{2u} + x_{22u}\boldsymbol{x}_2\right)\\
=&\boldsymbol{X}'_{2u}\boldsymbol{X}_1\left(\boldsymbol{X}'_1\boldsymbol{X}_1\right)^{-1}\boldsymbol{X}'_1\boldsymbol{X}_{2u} - \frac{b_u^2}{1+k} + a_u^2 k - \frac{a_u^2 k^2}{1+k}\\
+&2a_u b_u - \frac{2a_u b_u k}{1+k}\\
=&\boldsymbol{X}'_{2u}\boldsymbol{X}_1\left(\boldsymbol{X}'_1\boldsymbol{X}_1\right)^{-1}\boldsymbol{X}'_1\boldsymbol{X}_{2u} - \frac{b_u^2}{1+k} + \frac{a_u^2 k}{1+k} + \frac{2a_u b_u}{1+k}.
\end{aligned}$$

Hence

$$\begin{aligned}
v_u^{(2)} &= \boldsymbol{X}'_{2u}\boldsymbol{X}_{2u} + x_{22u}^2 - \boldsymbol{X}'_{2u}\boldsymbol{X}_1\left(\boldsymbol{X}'_1\boldsymbol{X}_1\right)^{-1}\boldsymbol{X}'_1\boldsymbol{X}_{2u} + \frac{b_u^2}{1+k} - \frac{a_u^2 k}{1+k} - \frac{2a_u b_u}{1+k}\\
&= v_u^{(1)} + a_u^2 - \frac{a_u^2 k}{1+k} + \frac{b_u^2}{1+k} - \frac{2a_u b_u}{1+k}\\
&= v_u^{(1)} + \frac{\left(a_u - b_u\right)^2}{1+k}.
\end{aligned}$$

Chapter 4

Characterization of Common Variance Property

4.1 Chapter Summary

In this chapter we characterize a class of designs possessing the common variance property for general n and m. The characterizations lead to several conditions for a design to be CV. The characterizations are mostly obtained in terms of the projection matrix and are true for the general fractional factorial designs with n runs. Here is what we present in each section:

- (Section 4.2): In this section we present the form of the general factorial experiment with different factors at different levels which is already presented in Chapter 1. Fractional factorial designs with n runs is considered which can estimate general mean, main effects and one two-factor interaction in each model M_u presented in Chapter 1 and at the same time give constant

variance to all two-factor interaction estimators.

- (Section 4.3): This section presents several theorems stating conditions for a design to be CV. Several sufficient conditions are obtained by using pairs of interaction effects, independent columns of the projection matrix and runs of the designs. The conditions on the runs are true for 3^m factorial designs only but similar conditions can be obtained for general factorial designs. Computationally checking the conditions on the runs provide much faster ways to check for CV. Finally we illustrate these characterizations with examples. The conditions presented in Theorem 3 and Theorem 5 are checked using two 3^3 CV designs respectively.

4.2 Introduction

Consider a factorial experiment $s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_t^{m_t}$, where each of m_1 factors is at level s_1, each of m_2 factors at s_2 and so on. The total number of factors is $m_1 + m_2 + \ldots + m_t = m$. The total number of main effects is $\sum_{u=1}^{t} (s_u - 1) m_u$, the total number of k-factor interaction effects is $\sum_{u=1}^{t} (s_u - 1)^k m_u$. The total number of runs required to estimate all the factorial effects is at least $s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_t^{m_t} = d$. We consider the fractional factorial designs with $n\,(< d)$ runs. In Chapter 1 the class of models M_u, $\forall u$ with general mean (β_0) , p_1 main effects in $\boldsymbol{\beta}_1$ and k $2-$factor interactions in $\boldsymbol{\beta}_{2u}$ are given. In particular we consider the case for $k = 1$. The $3-$factor and higher order interaction effects are assumed to be negligible. We already know that a fractional factorial design with m factors and

n runs is a common variance (CV) design if $Var\left(\hat{\beta}_{2u}\right) = constant$, $\forall u$, where $\hat{\beta}_{2u}$ is the least square estimator of β_{2u}. We obtain conditions that would make $Var\left(\hat{\beta}_{2u}\right) = constant$, $\forall u$.

4.3 Characterization of CV Designs

In this section we characterize the CV designs in terms of the projection matrix and also the runs of the design. Section 4.3.1 presents five theorems on the conditions of finding CV designs. Section 4.3.2 illustrates the two conditions given in theorem 3 and the four conditions given in theorem 5 with examples. The designs satisfying these conditions given in the respective theorems possess the CV property.

4.3.1 Finding CV Designs

Consider the class of models M_u $\forall u$ for $k = 1$ in (1.3.1), the equations (1.3.2) and (1.3.3) in Chapter 1 and refer to the definition of CV design given in Chapter 2. Define the projection matrix $\boldsymbol{P}$ as $\boldsymbol{P} = \boldsymbol{I}_n - \boldsymbol{X}_1^* \left(\boldsymbol{X}_1^{*\prime} \boldsymbol{X}_1^*\right)^{-1} \boldsymbol{X}_1^{*\prime}$. The matrix $\boldsymbol{P}$ satisfies the properties $1-4$ below:

1. $\boldsymbol{P}$ is symmetric, i.e, $\boldsymbol{P} = \boldsymbol{P}'$.

2. $\boldsymbol{P}$ is idempotent, i.e, $\boldsymbol{P} = \boldsymbol{P}^2$.

3. $\boldsymbol{P}$ is orthogonal to $\boldsymbol{X}_1^*$, i.e, $\boldsymbol{P}\boldsymbol{X}_1^* = \boldsymbol{0}$.

4. $Null\left(\boldsymbol{P}\right) = \left\{\boldsymbol{x} \in \boldsymbol{R}^n : \boldsymbol{P}\boldsymbol{x} = \boldsymbol{0}\right\}$.

The $Var\left(\hat{\beta}_{2u}\right)$ is the last diagonal element of $Var\left(\hat{\boldsymbol{\beta}}^{(u)}\right)$ which is expressed as

$$Var\left(\hat{\beta}_{2u}\right) = \sigma^2 \frac{|\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*|}{|\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}|}. \tag{4.1}$$

From *Rao* (1973) we have

$$\begin{aligned}
|\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}| &= |\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*| \left(\boldsymbol{X}_{2u}'\boldsymbol{X}_{2u} - \boldsymbol{X}_{2u}'\boldsymbol{X}_1^* \left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1} \boldsymbol{X}_1^{*\prime}\boldsymbol{X}_{2u}\right) \\
&= |\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*| \left[\boldsymbol{X}_{2u}' \left(\boldsymbol{I} - \boldsymbol{X}_1^* \left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1} \boldsymbol{X}_1^{*\prime}\right) \boldsymbol{X}_{2u}\right] \\
&= |\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*| \left(\boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u}\right).
\end{aligned}$$

Hence

$$\frac{|\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*|}{|\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}|} = \frac{|\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*|}{|\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*| \left(\boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u}\right)} = \frac{1}{\boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u}}. \tag{4.2}$$

Now we state the following theorem.

Theorem 1. *A design is CV iff* $\boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u} = constant,\ \forall u.$

Proof. From (4.1) and (4.2) we get

$$Var\left(\hat{\beta}_{2u}\right) = constant \Leftrightarrow |\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}| = constant \Leftrightarrow \boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u} = constant, \forall u.$$

This proves the theorem. □

The $\boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u}$ can be expressed as

$$\boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u} = \boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{P}\boldsymbol{X}_{2u} = \boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{P}'\boldsymbol{X}_{2u} = \left(\boldsymbol{P}'\boldsymbol{X}_{2u}\right)' \left(\boldsymbol{P}'\boldsymbol{X}_{2u}\right). \tag{4.3}$$

Definition 2. *For a vector* $\boldsymbol{g} = (g_i)$*, where* g_i *is the* i^{th} *element of* $\boldsymbol{g}$*, we define the absolute* $\boldsymbol{g}$*,* $|\boldsymbol{g}|$*, as*

$$|\boldsymbol{g}| = \left(|g_i|\right), \tag{4.4}$$

where $|g_i|$ *is the absolute value of* g_i*.*

For two $(n \times 1)$ vectors $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ such that $\boldsymbol{g}_1 = \boldsymbol{P}_m(\boldsymbol{g}_2)$ where $\boldsymbol{P}_m(n \times n)$ is a permutation matrix, it can be seen that

$$\boldsymbol{g}_1'\boldsymbol{g}_1 = \boldsymbol{g}_2'\boldsymbol{g}_2. \tag{4.5}$$

Using (4.3) we have the following theorem.

Theorem 2. *A design is CV if $|\boldsymbol{P}'\boldsymbol{X}_{2u}|$ is constant independent of u, except for the permutation of its elements.*

Proof. Suppose that $\boldsymbol{g} = \boldsymbol{P}'\boldsymbol{X}_{2u}$. We write

$$\boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u} = \boldsymbol{g}'\boldsymbol{g} = \sum_i g_i^2 = \sum_i |g_i|^2.$$

Let $\boldsymbol{g}_1 = \boldsymbol{P}'\boldsymbol{X}_{2u_1}$ and $\boldsymbol{g}_2 = \boldsymbol{P}'\boldsymbol{X}_{2u_2}$ for $u_1 \neq u_2$ such that $\boldsymbol{g}_1 = \boldsymbol{P}_m(\boldsymbol{g}_2)$. The theorem is proved from (4.5). □

Now we want to characterize the CV property further using pairs of 2−factor interactions. For every pair (u, v), except for the permutation of the elements

$$\boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u} = \boldsymbol{X}_{2v}'\boldsymbol{P}\boldsymbol{X}_{2v} \Rightarrow \boldsymbol{X}_{2u^*}'\boldsymbol{P}\boldsymbol{X}_{2u^*} = constant,\ \forall u^*.$$

From Theorem 1 we already know that $\boldsymbol{X}_{2u^*}'\boldsymbol{P}\boldsymbol{X}_{2u^*} = constant,\ \forall u^*$ is an NSC for a design to be CV. The pairs of interactions may come from within a group like $G_1 : \left(A_jA_k, A_j^2A_k^2, A_jA_k^2, A_j^2A_k\right),\ j < k$, or from between the groups like $G_2 : \left(A_j^\alpha A_k^\beta, A_l^\alpha A_r^\beta\right),\ \alpha, \beta \in (1, 2)$. We find conditions that would make $\boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u} = \boldsymbol{X}_{2v}'\boldsymbol{P}\boldsymbol{X}_{2v}$ for any (u, v) belonging to G_1 or G_2. For any pair (u, v), $\boldsymbol{X}_{2u}$ and $\boldsymbol{X}_{2v}$ can be expressed as a linear combination of the columns of $\boldsymbol{X}_1^*(n \times p_1 + 1)$ and

the columns of $\boldsymbol{P}\,(n \times n)$ in the following way:

$$\begin{aligned} \boldsymbol{X}_{2u} &= \boldsymbol{X}_1^* \boldsymbol{w}_{1u} + \boldsymbol{P}\boldsymbol{w}_{2u} \\ \boldsymbol{X}_{2v} &= \boldsymbol{X}_1^* \boldsymbol{w}_{1v} + \boldsymbol{P}\boldsymbol{w}_{2v}. \end{aligned} \tag{4.6}$$

where $\boldsymbol{w}_{1u}\,(p_1 + 1 \times 1)\,,\ \boldsymbol{w}_{2u}\,(n \times 1)\,,\ \boldsymbol{w}_{1v}\,(p_1 + 1 \times 1)$ and $\boldsymbol{w}_{2v}\,(n \times 1)$ are vectors of linear combinations. From (4.6) we have

$$\begin{aligned} \boldsymbol{X}_{2u} \pm \boldsymbol{X}_{2v} &= \boldsymbol{X}_1^* \left(\boldsymbol{w}_{1u} \pm \boldsymbol{w}_{1v}\right) + \boldsymbol{P}\left(\boldsymbol{w}_{2u} \pm \boldsymbol{w}_{2v}\right) \\ \Rightarrow \boldsymbol{P}\left(\boldsymbol{X}_{2u} \pm \boldsymbol{X}_{2v}\right) &= \boldsymbol{P}\left(\boldsymbol{w}_{2u} \pm \boldsymbol{w}_{2v}\right). \end{aligned} \tag{4.7}$$

So for any pair (u, v), if $(\boldsymbol{X}_{2u} \pm \boldsymbol{X}_{2v})$ can be expressed as linear combination of the columns of $\boldsymbol{X}_1^*$ only, then from (4.7) we have

$$\boldsymbol{P}\left(\boldsymbol{w}_{2u} \pm \boldsymbol{w}_{2v}\right) = \boldsymbol{0}.$$

Again from (4.7) we have

$$\boldsymbol{P}\left(\boldsymbol{w}_{2u} \pm \boldsymbol{w}_{2v}\right) = \boldsymbol{0} \Leftrightarrow \boldsymbol{P}\left(\boldsymbol{X}_{2u} \pm \boldsymbol{X}_{2v}\right) = \boldsymbol{0} \Rightarrow \boldsymbol{X}_{2u}'\boldsymbol{P}\boldsymbol{X}_{2u} = \boldsymbol{X}_{2v}'\boldsymbol{P}\boldsymbol{X}_{2v}. \tag{4.8}$$

From the definition of $Null\,(\boldsymbol{P})\,,$

$$\boldsymbol{P}\left(\boldsymbol{X}_{2u} \pm \boldsymbol{X}_{2v}\right) = \boldsymbol{0} \Leftrightarrow \left(\boldsymbol{X}_{2u} \pm \boldsymbol{X}_{2v}\right) \in Null\,(\boldsymbol{P})\,. \tag{4.9}$$

Below we state the properties of any Permutation matrix $\boldsymbol{Q}$ obtained from the Identity matrix by interchanging its rows or columns:

1. $\boldsymbol{Q} = \boldsymbol{Q}'$
2. $\boldsymbol{Q}^2 = \boldsymbol{I}$
3. $\boldsymbol{Q}\boldsymbol{Q}' = \boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}$

If $\boldsymbol{Q'PQ} = \boldsymbol{P}$, we have

$$\boldsymbol{X}_{2u} = \boldsymbol{Q}\boldsymbol{X}_{2v} \Rightarrow \boldsymbol{X}'_{2u}\boldsymbol{P}\boldsymbol{X}_{2u} = \boldsymbol{X}'_{2v}\boldsymbol{Q'}\boldsymbol{P}\boldsymbol{Q}\boldsymbol{X}_{2v} = \boldsymbol{X}'_{2v}\boldsymbol{P}\boldsymbol{X}_{2v}. \tag{4.10}$$

Theorem 3. *For any pair* (u, v), $\boldsymbol{X}'_{2u}\boldsymbol{P}\boldsymbol{X}_{2u} = \boldsymbol{X}'_{2v}\boldsymbol{P}\boldsymbol{X}_{2v}$ *holds if at least one of the following two conditions hold:*

1. $\boldsymbol{X}_{2u} \pm \boldsymbol{X}_{2v} \in Null\,(\boldsymbol{P})$.

2. $\boldsymbol{X}_{2u} = \boldsymbol{Q}\boldsymbol{X}_{2v}$, where $\boldsymbol{Q}$ is a permutation matrix such that $\boldsymbol{Q'PQ} = \boldsymbol{P}$ holds.

Proof. (1) is proved from (4.8) and (4.9) and (2) is proved from (4.10). □

Corollary 3.1. *A design is CV iff* $\boldsymbol{X}'_{2u}\boldsymbol{P}\boldsymbol{X}_{2u} = \boldsymbol{X}'_{2v}\boldsymbol{P}\boldsymbol{X}_{2v}$ *holds for all pair* (u, v).

We already obtained conditions on $\boldsymbol{P}$ for a design to be CV. Now we want to see if instead of working with the whole $\boldsymbol{P}$ matrix we can work with only the independent columns of $\boldsymbol{P}$. Let the matrix $\boldsymbol{P}_s$ consist of the independent columns of $\boldsymbol{P}$. Working with $\boldsymbol{P}_s$ makes the calculation even faster because the dimension of $\boldsymbol{P}_s$ is small as compared to that of $\boldsymbol{P}$ for designs with small n. In the following we obtain the CV conditions on $\boldsymbol{P}_s$. Without any loss of generality we partition the projection matrix $\boldsymbol{P}\,(n \times n)$ as $\boldsymbol{P} = \left[\boldsymbol{P}_s \vdots \boldsymbol{P}_{\bar{s}}\right]$, where $Rank\,(\boldsymbol{P}_s\,(n \times r)) = Rank\,(\boldsymbol{P}) = r$. We have

$$\boldsymbol{P'}\boldsymbol{X}_{2u} = \begin{pmatrix} \boldsymbol{P}'_s \\ \boldsymbol{P}'_{\bar{s}} \end{pmatrix} \boldsymbol{X}_{2u} = \begin{pmatrix} \boldsymbol{P}'_s\boldsymbol{X}_{2u} \\ \boldsymbol{P}'_{\bar{s}}\boldsymbol{X}_{2u} \end{pmatrix}. \tag{4.11}$$

Let $\boldsymbol{g}_1 = \boldsymbol{P}'\boldsymbol{X}_{2u_1}$ and $\boldsymbol{g}_2 = \boldsymbol{P}'\boldsymbol{X}_{2u_2}$ for $u_1 \neq u_2$ such that the elements of $\boldsymbol{g}_1$ are permutations of the elements of $\boldsymbol{g}_2$, i.e, $\boldsymbol{g}_1 = \boldsymbol{P}_m(\boldsymbol{g}_2)$. So the elements of $\boldsymbol{P}'_s\boldsymbol{X}_{2u}$ may not appear in the same order in $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$. However by permuting the elements of $\boldsymbol{g}_2$ by $\boldsymbol{P}_m$ we can make the elements of $\boldsymbol{P}'_s\boldsymbol{X}_{2u}$ appear in the same order in both $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$. Hence from (4.11) we have

$$|\boldsymbol{P}'_s\boldsymbol{X}_{2u}| = constant \text{ and } |\boldsymbol{P}'_{\bar{s}}\boldsymbol{X}_{2u}| = constant \Rightarrow |\boldsymbol{P}'\boldsymbol{X}_{2u}| = constant, \forall u. \tag{4.12}$$

The matrix $\boldsymbol{P}_{\bar{s}}$ consists of the columns of $\boldsymbol{P}$ that are dependent on the columns of $\boldsymbol{P}_s$ and hence the columns in $\boldsymbol{P}_{\bar{s}}$ can be expressed as the linear combinations of the columns of $\boldsymbol{P}_s$ as:

$$\boldsymbol{P}_{\bar{s}} = \boldsymbol{P}_s\boldsymbol{W}, \tag{4.13}$$

where $\boldsymbol{W}$ is an $(r \times n - r)$ matrix of linear combinations. Hence from (4.13) we get

$$\boldsymbol{P}'_{\bar{s}} = \boldsymbol{W}'\boldsymbol{P}'_s \Rightarrow \boldsymbol{P}'_{\bar{s}}\boldsymbol{X}_{2u} = \boldsymbol{W}'\boldsymbol{P}'_s\boldsymbol{X}_{2u}. \tag{4.14}$$

Hence from (4.14) we have

$$|\boldsymbol{P}'_s\boldsymbol{X}_{2u}| = constant \Rightarrow |\boldsymbol{P}'_{\bar{s}}\boldsymbol{X}_{2u}| = constant, \forall u. \tag{4.15}$$

We state the following theorem.

Theorem 4. *A design is CV if $|\boldsymbol{P}'_s\boldsymbol{X}_{2u}| = constant$ independent of u, except for the permutation of its elements.*

Proof. From (4.12)and (4.15) we have

$$|\boldsymbol{P}'_s\boldsymbol{X}_{2u}| = constant \Rightarrow |\boldsymbol{P}'\boldsymbol{X}_{2u}| = constant, \forall u.$$

Table 4.1: 2−Factor Interaction Vector $\boldsymbol{X}_{2u}$

u	Expression
A_jA_k	$\left(x_1^{(jk)},\ x_2^{(jk)},\ \ldots\ ,x_n^{(jk)}\right)'$
$A_j^2A_k^2$	$\left(3x_1^{2(jk)}-2,\ 3x_2^{2(jk)}-2,\ \ldots\ ,3x_n^{2(jk)}-2\right)'$
$A_jA_k^2$	$\left(z_1^{(jk)},\ z_2^{(jk)},\ \ldots\ ,z_n^{(jk)}\right)'$
$A_j^2A_k$	$\left(3z_1^{2(jk)}-2,\ 3z_2^{2(jk)}-2,\ \ldots\ ,3z_n^{2(jk)}-2\right)'$
A_lA_r	$\left(x_1^{(lr)},\ x_2^{(lr)},\ \ldots\ ,x_n^{(lr)}\right)'$

This completes the proof. □

From the previous theorem we know that $|\boldsymbol{P}_s'\boldsymbol{X}_{2u}| = constant$, $\forall u$, except for the permutation of its elements, will make a design CV. Now instead of calculating $|\boldsymbol{P}_s'\boldsymbol{X}_{2u}|$, $\forall u$ we find faster ways to check for the CV property. We find conditions on the design runs that will make $|\boldsymbol{P}_s'\boldsymbol{X}_{2u}|$ constant $\forall u$ except for the permutation of its elements. Given a design, its $\boldsymbol{P}$ matrix and hence the $\boldsymbol{P}_s$ matrix can be easily calculated. We present the conditions in terms of the runs and the matrix $\boldsymbol{P}_s$. For a design with m factors and n runs denote the levels corresponding to the m factors by $s_{1i},\ s_{2i},\ldots s_{mi},\ i=1(1)n$ respectively, $s_{ji}\in\{0,1,2\}$, $j=1(1)m$. For $i=1(1)n,\ \ j<k=1(1)m$ and $l<r=1(1)m$ define the following

$$x_i^{(jk)} = (s_{ji}+\ s_{ki})_{mod(3)} - 1$$

$$z_i^{(jk)} = (s_{ji}+2s_{ki})_{mod(3)} - 1$$

$$x_i^{(lr)} = (s_{li}+s_{ri})_{mod(3)} - 1$$

In Table 4.1 we give the expression of $\boldsymbol{X}_{2u}$ corresponding to all the 2−factor interactions involving the factors A_j and A_k and the linear×linear interaction between A_l and A_r. Except for the permutation of the elements, for $u,u'\in$

$\{A_jA_k,\ A_j^2A_k^2,\ A_jA_k^2,\ A_j^2A_k\}$, $j < k = 1(1)m$ and for any $l < r = 1(1)m$ and $\delta = \pm 1$

$$\boldsymbol{P}'_s\boldsymbol{X}_{2u} = \delta\boldsymbol{P}'_s\boldsymbol{X}_{2u'} \text{ and } \boldsymbol{P}'_s\boldsymbol{X}_{2A_jA_k} = \delta\boldsymbol{P}'_s\boldsymbol{X}_{2A_lA_r} \Leftrightarrow |\boldsymbol{P}'_s\boldsymbol{X}_{2u}| = constant,\ \forall u. \tag{4.16}$$

$\begin{pmatrix} \boldsymbol{P}'_s \\ \boldsymbol{P}'_{\bar{s}} \end{pmatrix} \boldsymbol{X}_1 = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} \Rightarrow \begin{pmatrix} \boldsymbol{P}'_s \\ \boldsymbol{P}'_{\bar{s}} \end{pmatrix} \boldsymbol{j} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} \Rightarrow \boldsymbol{P}'_s\boldsymbol{j} = \boldsymbol{0}$, where $\boldsymbol{j}$ is the first column of $\boldsymbol{X}_1^*$. Hence

$$\begin{aligned} \boldsymbol{P}'_s\boldsymbol{X}_{2A_j^2A_k^2} &= 3\boldsymbol{P}'_s \begin{pmatrix} x_1^{2(jk)} \\ x_2^{2(jk)} \\ \vdots \\ x_n^{2(jk)} \end{pmatrix} - 2\boldsymbol{P}'_s\boldsymbol{j} \\ &= 3\boldsymbol{P}'_s \begin{pmatrix} x_1^{2(jk)} \\ x_2^{2(jk)} \\ \vdots \\ x_n^{2(jk)} \end{pmatrix}. \end{aligned}$$

Similarly

$$\boldsymbol{P}'_s\boldsymbol{X}_{2A_j^2A_k} = 3\boldsymbol{P}'_s \begin{pmatrix} z_1^{2(jk)} \\ z_2^{2(jk)} \\ \vdots \\ z_n^{2(jk)} \end{pmatrix}.$$

In Table 4.2 we give the necessary and sufficient conditions for $\boldsymbol{P}'_s\boldsymbol{X}_{2u} = \delta\boldsymbol{P}'_s\boldsymbol{X}_{2u'},\ u, u' \in \{A_jA_k,\ A_j^2A_k^2,\ A_jA_k^2,\ A_j^2A_k\}$ and $\boldsymbol{P}'_s\boldsymbol{X}_{2A_jA_k} = \delta\boldsymbol{P}'_s\boldsymbol{X}_{2A_lA_r}$ to hold in terms of runs of the designs.

Table 4.2: NSC for $\boldsymbol{P}'_s\boldsymbol{X}_{2u} = \delta\boldsymbol{P}'_s\boldsymbol{X}_{2u'}$

(u, u')	NSC
$\left(A_jA_k, A_j^2A_k^2\right)$	$\boldsymbol{P}'_s \begin{pmatrix} 3x_1^{2(jk)} - \delta x_1^{(jk)} \\ 3x_2^{2(jk)} - \delta x_2^{(jk)} \\ \vdots \\ 3x_n^{2(jk)} - \delta x_n^{(jk)} \end{pmatrix} = \mathbf{0}$
$(A_jA_k, A_jA_k^2)$	$\boldsymbol{P}'_s \begin{pmatrix} z_1^{(jk)} - \delta x_1^{(jk)} \\ z_2^{(jk)} - \delta x_2^{(jk)} \\ \vdots \\ z_n^{(jk)} - \delta x_n^{(jk)} \end{pmatrix} = \mathbf{0}$
$\left(A_jA_k, A_j^2A_k\right)$	$\boldsymbol{P}'_s \begin{pmatrix} 3z_1^{2(jk)} - \delta z_1^{(jk)} \\ 3z_2^{2(jk)} - \delta z_2^{(jk)} \\ \vdots \\ 3z_n^{2(jk)} - \delta z_n^{(jk)} \end{pmatrix} = \mathbf{0}$
(A_jA_k, A_lA_r)	$\boldsymbol{P}'_s \begin{pmatrix} x_1^{(jk)} - \delta x_1^{(lr)} \\ x_2^{(jk)} - \delta x_2^{(lr)} \\ \vdots \\ x_n^{(jk)} - \delta x_n^{(lr)} \end{pmatrix} = \mathbf{0}$

Theorem 5. *$|\boldsymbol{P}'_s\boldsymbol{X}_{2u}|$ is constant $\forall u$ iff the following conditions hold:*

(1) $\boldsymbol{P}'_s \begin{pmatrix} 3x_1^{2(jk)} - \delta x_1^{(jk)} \\ 3x_2^{2(jk)} - \delta x_2^{(jk)} \\ \vdots \\ 3x_n^{2(jk)} - \delta x_n^{(jk)} \end{pmatrix} = \mathbf{0}$

(2) $\boldsymbol{P}'_s \begin{pmatrix} z_1^{(jk)} - \delta x_1^{(jk)} \\ z_2^{(jk)} - \delta x_2^{(jk)} \\ \vdots \\ z_n^{(jk)} - \delta x_n^{(jk)} \end{pmatrix} = \mathbf{0}$

(3) $\boldsymbol{P}'_s \begin{pmatrix} 3z_1^{2(jk)} - \delta z_1^{(jk)} \\ 3z_2^{2(jk)} - \delta z_2^{(jk)} \\ \vdots \\ 3z_n^{2(jk)} - \delta z_n^{(jk)} \end{pmatrix} = \mathbf{0}$

(4) $\boldsymbol{P}'_s \begin{pmatrix} x_1^{(jk)} - \delta x_1^{(lr)} \\ x_2^{(jk)} - \delta x_2^{(lr)} \\ \vdots \\ x_n^{(jk)} - \delta x_n^{(lr)} \end{pmatrix} = \mathbf{0}$, *where* $x_i^{(jk)}$ *and* $z_i^{(jk)}$ *are the* i^{th} *components of* $\boldsymbol{X}_{2A_jA_k}$ *and* $\boldsymbol{X}_{2A_jA_k^2}$ *respectively,* $x_i^{(lr)}$ *is the* i^{th} *component of* $\boldsymbol{X}_{2A_lA_r}$, $j < k = 1(1)m$, $l < r = 1(1)m$, $i = 1(1)n$, $\delta = \pm 1$.

Proof. The proof of this theorem follows from Table 4.2 and (4.16). □

4.3.2 Illustration with Examples

4.3.2.1 Illustration of the Conditions of Theorem 3

To illustrate the conditions given in Theorem 3, we consider 6 structured fractional factorial designs for $m_1 = 3,\ s_1 = 3,\ m_t = 0,\ t > 1$ and $n = 10$. In all of these 6 designs seven runs are in common which are $(0,0,2)$, $(0,2,0)$, $(2,0,0)$, $(0,2,2)$, $(2,0,2)$, $(2,2,0)$, $(2,2,2)$ and the remaining set of three runs is from one of the six sets presented in Table 4.3. For 3^3 factorial experiment we have $\boldsymbol{\beta}_1 = (A, A^2, B, B^2, C, C^2)$ and $\boldsymbol{\beta}_2 =$(AB, A^2B^2, AB^2, A^2B, AC, A^2C^2, AC^2, A^2C, BC, B^2C^2, BC^2, B^2C). The number of pairs of 2-factor interaction effects is $\binom{12}{2} = 66$. The $\boldsymbol{P}$ matrices are identical for all the 6 designs although $\boldsymbol{X}^{(u)}, \forall u$

Table 4.3: 6 Sets of 3 Runs

I	II	III	IV	V	VI
$(0,0,1)$	$(0,1,1)$	$(1,1,2)$	$(1,2,2)$	$(0,1,2)$	$(0,2,1)$
$(0,1,0)$	$(1,0,1)$	$(1,2,1)$	$(2,1,2)$	$(1,2,0)$	$(2,1,0)$
$(1,0,0)$	$(1,1,0)$	$(2,1,1)$	$(2,2,1)$	$(2,0,1)$	$(1,0,2)$

are not identical for them. The common $\boldsymbol{P}$ matrix is given below.

$$\boldsymbol{P} = \begin{bmatrix} 3a & -a & -a & -2a & -2a & 2a & 0 & 0 & 0 & a \\ -a & 3a & -a & -2a & 2a & -2a & 0 & 0 & 0 & a \\ -a & -a & 3a & 2a & -2a & -2a & 0 & 0 & 0 & a \\ -2a & -2a & 2a & 4a & 0 & 0 & 0 & 0 & 0 & -2a \\ -2a & 2a & -2a & 0 & 4a & 0 & 0 & 0 & 0 & -2a \\ 2a & -2a & -2a & 0 & 0 & 4a & 0 & 0 & 0 & -2a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & a & a & -2a & -2a & -2a & 0 & 0 & 0 & 3a \end{bmatrix}, a = 0.125.$$

To verify the two conditions of Theorem 3 on design II we observe the following results:

1. The 18 pairs of 2−factor interactions satisfy the condition $\boldsymbol{P}\left(\boldsymbol{X}_{2u} \pm \boldsymbol{X}_{2v}\right) = \boldsymbol{0}$. Out of these 18 pairs 6 are from the group containing the factors A and B where the pairs are formed by taking all possible 2 out of 4 interaction effects (AB, A^2B^2, AB^2, A^2B). Similarly the remaining 12 pairs are 6 from the group containing the factors B and C and 6 from the group containing A and C respectively. We present Table 4.4 which shows that the coefficients

Table 4.4: $(\boldsymbol{X_{2u}} \pm \boldsymbol{X_{2v}})$ Expressed as Linear Combinations of Cols of $\boldsymbol{X}_1^*$ and $\boldsymbol{P}$ for Group AB.

$(\boldsymbol{X_{2u}} \pm \boldsymbol{X_{2v}})$	Coefficients of linear combinations of $\boldsymbol{X}_1^*$	Coefficients of linear combinations of $\boldsymbol{P}$
$\left(\boldsymbol{X}_{2AB} - \boldsymbol{X}_{2A^2B^2}\right)$	$(1, 1, 0, 1, 0, 0, -1)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
$\left(\boldsymbol{X}_{2AB} - \boldsymbol{X}_{2AB^2}\right)$	$(0.5, 0, -0.5, 0.5, 0, 0, 0.5)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
$\left(\boldsymbol{X}_{2AB} + \boldsymbol{X}_{2A^2B}\right)$	$(0.5, 1, -0.5, 0.5, -1, 0, 0.5)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
$\left(\boldsymbol{X}_{2A^2B^2} - \boldsymbol{X}_{2AB^2}\right)$	$(-0.5, -1, -0.5, -0.5, 0, 0, 1.5)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
$\left(\boldsymbol{X}_{2A^2B^2} + \boldsymbol{X}_{2A^2B}\right)$	$(-0.5, 0, 0.5, -1.5, -1, 0, 1.5)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
$\left(\boldsymbol{X}_{2AB^2} + \boldsymbol{X}_{2A^2B}\right)$	$(0, 1, 1, -1, -1, 0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

of the linear combinations corresponding to the columns of $\boldsymbol{P}$ are zero for all 6 pairs of interactions in the group of A and B. . The findings of Table 4.4 implies that $(\boldsymbol{X}_{2u} \pm \boldsymbol{X}_{2v})$ can be expressed by the columns of $\boldsymbol{X}_1^*$ only. Hence $(\boldsymbol{X}_{2u} \pm \boldsymbol{X}_{2v}) \in Null\,(\boldsymbol{P})$. Similar tables can be obtained for the pairs of interactions in the group of A and C and the group of B and C as well. We note that the condition (1) of Theorem 3 holds for all pairs of interaction effects belonging to G_1.

2. We find that pairs formed by similar interaction effects from 2 groups (e.g. $A^\alpha B^\beta$, $A^\alpha C^\beta$, $\alpha, \beta \in \{1, 2\}$) satisfy the conditions $\boldsymbol{X}_{2u} = \boldsymbol{Q}\boldsymbol{X}_{2v}$ and $\boldsymbol{Q'PQ} = \boldsymbol{P}$. Here is the result for the pair $(AB,\ AC)$. We present the interaction vectors $\boldsymbol{X}_{2u}$, for $u = AB$ and AC and the permutation matrix $\boldsymbol{Q}$ below.

$$\boldsymbol{X}_{2AB} = (-1, 1, 1, 1, 1, 0, 0, 0, 1, 0)',\ \boldsymbol{X}_{2AC} = (1, -1, 1, 1, 0, 1, 0, 1, 0, 0)',$$

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here $\boldsymbol{PQ} = \boldsymbol{QP}$ holds and this implies $\boldsymbol{Q'PQ} = \boldsymbol{P}$. Similar permutation matrix exists for other pairs like (A^2B^2, A^2C^2), (AB^2, AC^2), (A^2B, A^2C) etc. and $\boldsymbol{Q'PQ} = \boldsymbol{P}$ holds for these pairs. However, this condition holds for 2-factor interaction effects belonging to G_2 because no permutation matrix exists for any pair belonging to G_1.

In Table 4.5 we present the connections among the variances of the 2- factor interaction estimators for design II. Similar results are also obtained for the remaining 5 structured designs.

4.3.2.2 Illustration of the Conditions of Theorem 5

We consider the design $D^{(3)}$ for $m = 3$ and $n = 8$ in Table 4.6. Here we have $\boldsymbol{X}_{2AB} = (-1, -1,\ 0, -1, -1,\ 1,\ 1,\ 0)'$, $\boldsymbol{X}_{2A^2B^2} = (1,\ 1, -2,\ 1,\ 1,\ 1,\ 1, -2)'$, $\boldsymbol{X}_{2AB^2} =$

Table 4.5: Equality of the Variances by the Two Conditions

Equality of Variances	Conditions
$Var\,(AB) = Var\,(A^2B^2) = Var\,(AB^2) = Var\,(A^2B)$	1
$Var\,(AC) = Var\,(A^2C^2) = Var\,(AC^2) = Var\,(A^2C)$	1
$Var\,(BC) = Var\,(B^2C^2) = Var\,(BC^2) = Var\,(B^2C)$	1
$Var\,(AB) = Var\,(AC) = Var\,(BC)$	2
$Var\,(A^2B^2) = Var\,(A^2C^2) = Var\,(B^2C^2)$	2
$Var\,(AB^2) = Var\,(AC^2)$	2
$Var\,(A^2B) = Var\,(A^2C)$	2

Table 4.6: Design $D^{(3)}$

1	2	2	0	0	0	2	2
2	1	2	0	0	2	0	2
2	2	1	0	2	0	0	2

$(1,\ 0, -1, -1, -1,\ 0,\ 1, -1)'$, $\boldsymbol{X}_{2A^2B} = (1, -2,\ 1,\ 1,\ 1, -2,\ 1,\ 1)'$ and $\boldsymbol{X}_{2AC} =$

$(-1,\ 0, -1, -1,\ 1, -1,\ 1,\ 0)'$. Also we have

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & -.25 & -.25 & -.25 & .25 \\ 0 & 0 & 0 & -.25 & .125 & .125 & .125 & -.125 \\ 0 & 0 & 0 & -.25 & .125 & .125 & .125 & -.125 \\ 0 & 0 & 0 & -.25 & .125 & .125 & .125 & -.125 \\ 0 & 0 & 0 & .25 & -.125 & -.125 & -.125 & .125 \end{pmatrix},$$

$$\boldsymbol{X}_1^* = \begin{pmatrix} 1 & 0 & -2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & -2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Here $rank\,(\boldsymbol{P}) = r = 1$. Any column of P from 4 to 8 is independent, so we can take the 4th column of $\boldsymbol{P}$ as $\boldsymbol{P}_s$:

$$\boldsymbol{P}_s = \begin{pmatrix} 0 \\ 0 \\ 0 \\ .5 \\ -.25 \\ -.25 \\ -.25 \\ .25 \end{pmatrix} = 0.25 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Consider the pair $(\boldsymbol{X}_{2AB}, \boldsymbol{X}_{2A^2B^2})$. For $\delta = 1$ from condition (1) of Theorem 5 we have

$$\boldsymbol{P}'_s \begin{pmatrix} 3x_1^{2(AB)} - x_1^{(AB)} \\ 3x_2^{2(AB)} - x_2^{(AB)} \\ \vdots \\ 3x_8^{2(AB)} - x_8^{(AB)} \end{pmatrix} = .25 \begin{pmatrix} 0 & 0 & 0 & 2 & -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 0 \\ 4 \\ 4 \\ 2 \\ 2 \\ 0 \end{pmatrix} = 0.$$

Hence we have $\boldsymbol{P}_s'\boldsymbol{X}_{2AB} = \boldsymbol{P}_s'\boldsymbol{X}_{2A^2B^2}$. From condition (2) of Theorem 5, for $\delta = 1$, we have

$$\boldsymbol{P}_s' \begin{pmatrix} z_1^{(AB)} - x_1^{(AB)} \\ z_2^{(AB)} - x_2^{(AB)} \\ \vdots \\ z_8^{(AB)} - x_8^{(AB)} \end{pmatrix} = .25 \begin{pmatrix} 0 & 0 & 0 & 2 & -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} = 0$$

and hence $\boldsymbol{P}_s'\boldsymbol{X}_{2AB} = \boldsymbol{P}_s'\boldsymbol{X}_{2AB^2}$ is confirmed. Next considering the pair $(\boldsymbol{X}_{2AB}, \boldsymbol{X}_{2AB^2})$, from condition (3) of Theorem 5 we have

$$\boldsymbol{P}_s' \begin{pmatrix} 3z_1^{2(AB)} + z_1^{(AB)} \\ 3z_2^{2(AB)} + z_2^{(AB)} \\ \vdots \\ 3z_8^{2(AB)} + z_8^{(AB)} \end{pmatrix} = .25 \begin{pmatrix} 0 & 0 & 0 & 2 & -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 2 \\ 2 \\ 2 \\ 0 \\ 4 \\ 2 \end{pmatrix} = 0,$$

$\delta = -1$ and hence $\boldsymbol{P}_s'\boldsymbol{X}_{2AB^2} = -\boldsymbol{P}_s'\boldsymbol{X}_{2A^2B}$. Finally consider the pair $(\boldsymbol{X}_{2AB}, \boldsymbol{X}_{2AC})$. From the condition (4) of Theorem 5, for $\delta = 1$, we have

$$\boldsymbol{P}_s' \begin{pmatrix} x_1^{(AB)} - x_1^{(AC)} \\ x_2^{(AB)} - x_2^{(AC)} \\ \vdots \\ x_8^{(AB)} - x_8^{(AC)} \end{pmatrix}$$

$$= .25 \begin{pmatrix} 0 & 0 & 0 & 2 & -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -2 \\ 2 \\ 0 \\ 0 \end{pmatrix} = 0,$$

which shows $\boldsymbol{P}_s'\boldsymbol{X}_{2AB} = \boldsymbol{P}_s'\boldsymbol{X}_{2AC}$. Similarly it can be shown that $\boldsymbol{P}_s'\boldsymbol{X}_{2AC} = \boldsymbol{P}_s'\boldsymbol{X}_{2A^2C^2} = \boldsymbol{P}_s'\boldsymbol{X}_{2AC^2} = \boldsymbol{P}_s'\boldsymbol{X}_{2A^2C}$ and $\boldsymbol{P}_s'\boldsymbol{X}_{2BC} = \boldsymbol{P}_s'\boldsymbol{X}_{2B^2C^2} = \boldsymbol{P}_s'\boldsymbol{X}_{2BC^2} = \boldsymbol{P}_s'\boldsymbol{X}_{2B^2C}$ by using conditions (1)-(3) of Theorem 5. Again by using condition (4) it can be shown that $\boldsymbol{P}_s'\boldsymbol{X}_{2AC} = \boldsymbol{P}_s'\boldsymbol{X}_{2BC}$. So all four conditions of Theorem 5 hold for the design $D^{(3)}$ and hence it is a CV design.

Chapter 5

Two General and Other Special CV Designs and Their Characterization Using the P Matrix

5.1 Chapter Summary

In this chapter we present two designs for 3^m factorial experiment for general m which possess CV property. Also we analyze the structure of the projection matrix for these CV designs. Here is what we present in each section:

- (Section 5.2): In this section we present the design $d_m^{(1)}$ for $n = 2m + 2$ runs, $m \geq 2$ and demonstrate the condition of CV in terms of the design runs and the projection matrix of the design.

- (Section 5.3): In this section we present another 3^m CV design $d_m^{(2)}$ for $n = 3m$ runs, $m \geq 3$. Like $d_m^{(1)}$ the CV property of $d_m^{(2)}$ is also demonstrated in terms of the design runs and projection matrix.

- (Section 5.4): In this section we present the optimum CV designs. The design $d_m^{(1)}$ is optimum for $m = 2$ and the design $d_m^{(2)}$ is optimum for $m = 3$. This section also presents some more CV designs for different m that satisfy a particular structure of $\boldsymbol{P}$.

- (Section 5.5): In this section we analyze a particular structure of the projection matrix $\boldsymbol{P}$ which is satisfied by both the designs $d_m^{(1)}$ and $d_m^{(2)}$ as well as all the other CV designs presented in section 4.4.

5.2 Design $d_m^{(1)}$ and Its $\boldsymbol{P}$

In this section we present the 3^m fractional factorial design $d_m^{(1)}$ for $n = 2m+2$, $m \geq 2$ runs and obtain its variance covariance matrix and the projection matrix as a function of m. Using the necessary and sufficient condition for CV in terms of the projection matrix presented in Theorem 1 of Chapter 4 we demonstrate the CV property of this design. Below we present $d_m^{(1)}$:

$$d_m^{(1)} = \begin{bmatrix} 2\boldsymbol{I}_m \\ 2\boldsymbol{J}_m - \boldsymbol{I}_m \\ \boldsymbol{0}'_m \\ 2\boldsymbol{j}'_m \end{bmatrix},$$

where $\boldsymbol{I}_m$ is the identity matrix of order m, $\boldsymbol{J}_m$ is the matrix of unity, $\boldsymbol{0}'_m$ is the $(m \times 1)$ vector of 0's and $\boldsymbol{j}'_m$ is the $(m \times 1)$ vector of 1's. Consider the uth model M_u in (5.1)

$$M_u: \; E(\boldsymbol{y}) = \boldsymbol{j}\boldsymbol{\mu} + \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_{2u}\beta_{2u}, \; Var(\boldsymbol{y}) = \sigma^2\boldsymbol{I}, \tag{5.1}$$

Define $\boldsymbol{X}_1^* = \left[\boldsymbol{j}_n \vdots \boldsymbol{X}_1\right]$. The matrices $\boldsymbol{X}_1^*$ and $(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*)$ are given below:

$$\boldsymbol{X}_1^* = \begin{bmatrix} \boldsymbol{j}_m & 2\boldsymbol{J}_m - \boldsymbol{I}_m & \boldsymbol{J}_m \\ \boldsymbol{j}_m & \boldsymbol{J}_m - \boldsymbol{I}_m & -3\boldsymbol{I}_m + \boldsymbol{J}_m \\ 1 & -\boldsymbol{j}'_m & \boldsymbol{j}'_m \\ 1 & \boldsymbol{j}'_m & \boldsymbol{j}'_m \end{bmatrix},$$

$$\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^* = \begin{bmatrix} 2m+2 & \boldsymbol{j}'_m & (2m-1)\,\boldsymbol{j}'_m \\ \boldsymbol{j}_m & 5\boldsymbol{I}_m + (2m-4)\,\boldsymbol{J}_m & 3\boldsymbol{I}_m - 2\boldsymbol{J}_m \\ (2m-1)\,\boldsymbol{j}_m & 3\boldsymbol{I}_m - 2\boldsymbol{J}_m & 9\boldsymbol{I}_m + (2m-4)\,\boldsymbol{J}_m \end{bmatrix}.$$

From Theorem 1 of Chapter 4 we know that a design is CV iff $\boldsymbol{X}'_{2u}\boldsymbol{P}\boldsymbol{X}_{2u} = constant, \forall u$. And $\boldsymbol{X}'_{2u}\boldsymbol{P}\boldsymbol{X}_{2u} = constant$ $\Leftrightarrow$sum of square of the elements of $\boldsymbol{P}\boldsymbol{X}_{2u}$ is constant $, \forall u$. Using this condition we want to demonstrate that the design $d_m^{(1)}$ is a CV design for all m. We find the $\boldsymbol{P}$ matrix which is given as $\boldsymbol{I}_n - X_1^*\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}\boldsymbol{X}_1^{*\prime}$. We first calculate the matrix $\boldsymbol{P}$. A general representation of the matrix $\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}$ is

$$\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1} = \begin{bmatrix} a_1 & a_2\boldsymbol{j}'_m & a_3\boldsymbol{j}'_m \\ a_2\boldsymbol{j}_m & a_4\boldsymbol{I}_m + a_5\boldsymbol{J}_m & a_6\boldsymbol{I}_m + a_7\boldsymbol{J}_m \\ a_3\boldsymbol{j}_m & a_6\boldsymbol{I}_m + a_7\boldsymbol{J}_m & a_8\boldsymbol{I}_m + a_9\boldsymbol{J}_m \end{bmatrix},$$

where $a_1, \ldots, a_9$ are unknown quantities which are determined from

$$\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^{*}\right)\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^{*}\right)^{-1} = \boldsymbol{I}. \tag{5.2}$$

Equating the first column of both sides of (5.2) we get

$$\begin{aligned}
(2m+2)\,a_1 + ma_2 + m\,(2-m)\,a_3 &= 1. \\
a_1 + (5 + m\,(2m-4))\,a_2 + (3-2m)\,a_3 &= 0. \\
a_1 + (3-2m)\,a_2 + (9 + m\,(2m-4))\,a_2 &= 0. \qquad (5.3)
\end{aligned}$$

The (5.3) gives

$$\begin{aligned}
ca_1 &= 4\left(9 - 11m + 10m^2 - 4m^3 + m^4\right). \\
ca_2 &= -6\left(2 - 2m + m^2\right). \\
ca_3 &= 2\left(4 - 8m + 5m^2 - 2m^3\right).
\end{aligned}$$

where $c = 36\,(2 - m + m^2)$. Now equating the second column of both sides of (5.2) we get

$$\begin{aligned}
a_4 &= \frac{1}{4}. \\
ca_5 &= 9\,(2-m). \\
a_6 &= -\frac{1}{12}. \\
ca_7 &= 3\,(3-2m).
\end{aligned}$$

Again equating the third column of both sides of (5.2) we get

$$\begin{aligned}
a_8 &= \frac{5}{36}. \\
ca_9 &= \left(6 - 9m + 4m^2\right).
\end{aligned}$$

The matrices $\boldsymbol{X}_1^* \left(\boldsymbol{X}_1^{*\prime} \boldsymbol{X}_1^*\right)^{-1} \boldsymbol{X}_1^{*\prime}$ and $\boldsymbol{P}$ are

$$\boldsymbol{X}_1^* \left(\boldsymbol{X}_1^{*\prime} \boldsymbol{X}_1^*\right)^{-1} \boldsymbol{X}_1^{*\prime} = \begin{bmatrix} b_1 \boldsymbol{I}_m + b_2 \boldsymbol{J}_m & b_3 \boldsymbol{I}_m + b_4 \boldsymbol{J}_m & b_5 \boldsymbol{j}_m & b_6 \boldsymbol{j}_m \\ b_3 \boldsymbol{I}_m + b_4 \boldsymbol{J}_m & b_7 \boldsymbol{I}_m + b_8 \boldsymbol{J}_m & -b_9 \boldsymbol{j}_m' & -b_{10} \boldsymbol{j}_m' \\ b_5 \boldsymbol{j}_m' & b_9 \boldsymbol{j}_m' & b_{11} & b_{12} \\ b_6 \boldsymbol{j}_m' & b_{10} \boldsymbol{j}_m' & b_{12} & b_{13} \end{bmatrix},$$

$$\boldsymbol{P} = \begin{bmatrix} (1 - b_1) \boldsymbol{I}_m - b_2 \boldsymbol{J}_m & -\left(b_3 \boldsymbol{I}_m + b_4 \boldsymbol{J}_m\right) & -b_5 \boldsymbol{j}_m & -b_6 \boldsymbol{j}_m \\ -\left(b_3 \boldsymbol{I}_m + b_4 \boldsymbol{J}_m\right) & (1 - b_7) \boldsymbol{I}_m - b_8 \boldsymbol{J}_m & -b_9 \boldsymbol{j}_m & -b_{10} \boldsymbol{j}_m \\ -b_5 \boldsymbol{j}_m' & -b_9 \boldsymbol{j}_m & (1 - b_{11}) & -b_{!2} \\ -b_6 \boldsymbol{j}_m' & -b_{10} \boldsymbol{j}_m & -b_{12} & (1 - b_{13}) \end{bmatrix},$$

where $b_1 = 1$, $cb_2 = -36$, $b_3 = b_4 = 0$, $cb_5 = 36\,(m-1)$, $cb_6 = 36$, $b_7 = 1$, $b_8 = b_9 = b_{10} = 0$, $cb_{11} = 36\,(m+1)$, $cb_{12} = -36\,(m-1)$, $cb_{13} = 36\,(1 - m + m^2)$. Thus wee see that $b_1 - 1 = 0$, $b_3 = b_4 = 0$, $b_5 = -\,(m-1)\,b_2$, $b_6 = -b_2$, $b_7 - 1 = 0$, $b_8 = b_9 = b_{10} = 0$, $1 - b_{11} = (m-1)^2\,b_2$, $b_{12} = (m-1)\,b_2$, $1 - b_{13} = -b_2$. Hence $\boldsymbol{P}$ can be expressed as

$$\boldsymbol{P} = -b_2 \begin{bmatrix} 1 & \dots & 1 & 0 & \dots & 0 & -(m-1) & -1 \\ 1 & \dots & 1 & 0 & \dots & 0 & -(m-1) & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 1 & 0 & \dots & 0 & -(m-1) & -1 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ -(m-1) & \dots & -(m-1) & 0 & \dots & 0 & (m-1)^2 & 0 \\ -1 & \dots & -1 & 0 & \dots & 0 & (m-1) & 1 \end{bmatrix}.$$

For real vector $\boldsymbol{X}_{2u} = (x_{1,u}, x_{2,u}, \dots, x_{2m+1,u}, x_{2m+2,u})'$ corresponding to the uth interaction we get

$$\boldsymbol{P}\boldsymbol{X}_{2u} = -b_2 \begin{pmatrix} x_{1,u} + x_{2,u} + \dots + x_{m,u} - (m-1)\, x_{2m+1,u} - x_{2m+2,u} \\ \vdots \\ x_{1,u} + x_{2,u} + \dots + x_{m,u} - (m-1)\, x_{2m+1,u} - x_{2m+2,u} \\ 0 \\ \vdots \\ 0 \\ -(m-1)\,(x_{1,u} + x_{2,u} + \dots + x_{m,u} - (m-1)\, x_{2m+1,u} - x_{2m+2,u}) \\ -(x_{1,u} + x_{2,u} + \dots + x_{m,u} - (m-1)\, x_{2m+1,u} - x_{2m+2,u}) \end{pmatrix}. \tag{5.4}$$

Table 5.1: 2−Factor Interaction Vectors

Levels of B_1	Levels of B_2	B_1B_2	$B_1^2B_2^2$	$B_1B_2^2$	$B_1^2B_2$
2	0	1	1	1	1
0	2	1	1	0	-2
0	0	-1	1	-1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	-1	1	-1	1
0	0	-1	1	-1	1
2	2	0	-2	-1	1

Table 5.2: $(x_1 + x_2 + \ldots + x_m - (m-1)\, x_{2m+1} - x_{2m+2})$ for the 4 Interactions

	B_1B_2	$B_1^2B_2^2$	$B_1B_2^2$	$B_1^2B_2$
$(x_1 + x_2 + \ldots + x_m)$	$-m+4$	m	$-m+3$	$m-3$
$(-(m-1)\, x_{2m+1} - x_{2m+2})$	$m-1$	$-m+3$	m	$-m$
$(x_1 + x_2 + \ldots + x_m - (m-1)\, x_{2m+1} - x_{2m+2})$	3	3	3	-3

Hence

$$\boldsymbol{X}'_{2u}\boldsymbol{P}\boldsymbol{X}_{2u} = \frac{(x_{1,u} + x_{2,u} + \ldots + x_{m,u} + x_{2m+1,u} + x_{2m+2,u})^2}{(2 - m + m^2)}. \tag{5.5}$$

The $\boldsymbol{X}'_{2u}\boldsymbol{P}\boldsymbol{X}_{2u}$ will be constant independent of u iff $[x_{1,u} + x_{2,u} + \ldots + x_{m,u} - (m-1)\, x_{2m+1,u} - x_{2m+2,u}]^2$ or equivalently $|x_{1,u}+x_{2,u}+\ldots+x_{m,u}-(m-1)\, x_{2m+1,u} - x_{2m+2,u}|$ is constant for all 2−factor interaction vectors from (5.5). We present the 2−factor interaction vector corresponding to the factors B_1 & B_2 in Table 5.1. Because of the symmetric structure of $d_m^{(1)}$ the interaction vectors corresponding to any two factors B_i & B_j, $i < j = 1, \ldots, m$ are of similar form. Now for each interaction vector corresponding to B_1 & B_2 we calculate $[x_{1,u} + x_{2,u} + \ldots + x_{m,u} - (m-1)\, x_{2m+1,u} - x_{2m+2,u}]^2$ in Table 5.2. From Table 5.2 we see that $|x_{1,u} + x_{2,u} + \ldots + x_{m,u} - (m-1)\, x_{2m+1,u} - x_{2m+2,u}| = 3, \forall u$ corresponding to B_1 and B_2. Identical result holds for any B_i & B_j, $i < j = 1, \ldots, m$. Thus $\boldsymbol{X}'_{2u}\boldsymbol{P}\boldsymbol{X}_{2u}$

is constant independent of u and hence $d_m^{(1)}$ is CV from Theorem 1 of Chapter 4. The expression of CV becomes $\frac{(2-m+m^2)}{9}$.

5.3 Design $d_m^{(2)}$ and Its $\boldsymbol{P}$

In this section we present another 3^m design $d_m^{(2)}$ for $n = 3m$, $m \geq 3$ and like $d_m^{(1)}$ the CV property of $d_m^{(2)}$ is also demonstrated using Theorem 1 of Chapter 4. Below we present $d_m^{(2)}$:

$$d_m^{(2)} = \begin{bmatrix} 2\boldsymbol{I}_m \\ 2\boldsymbol{J}_m - 2\boldsymbol{I}_m \\ 2\boldsymbol{J}_m - \boldsymbol{I}_m \end{bmatrix},$$

where $\boldsymbol{I}_m$ is the identity matrix of order m, $\boldsymbol{J}_m$ is the matrix of unity, $\boldsymbol{0}'_m$ is the $(m \times 1)$ vector of 0's and $\boldsymbol{j}'_m$ is the $(m \times 1)$ vector of 1's. The CV property is now characterized by the projection matrix $\boldsymbol{P}$. Below we present the matrices $\boldsymbol{X}_1^*$ and $\boldsymbol{P}$:

$$\boldsymbol{X}_1^* = \begin{pmatrix} 2\boldsymbol{I}_m - \boldsymbol{J}_m \\ \boldsymbol{J}_m - 2\boldsymbol{I}_m \\ \boldsymbol{J}_m - \boldsymbol{I}_m \end{pmatrix},$$

$$
\boldsymbol{P} = \begin{bmatrix}
\frac{1}{2}-\frac{1}{2m} & -\frac{1}{2m} & \cdots & -\frac{1}{2m} & \frac{1}{2}-\frac{1}{2m} & -\frac{1}{2m} & \cdots & -\frac{1}{2m} & 0 & 0 & \cdots & 0 \\
-\frac{1}{2m} & \frac{1}{2}-\frac{1}{2m} & \cdots & -\frac{1}{2m} & -\frac{1}{2m} & \frac{1}{2}-\frac{1}{2m} & \cdots & -\frac{1}{2m} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\frac{1}{2m} & -\frac{1}{2m} & \cdots & \frac{1}{2}-\frac{1}{2m} & -\frac{1}{2m} & -\frac{1}{2m} & \cdots & \frac{1}{2}-\frac{1}{2m} & 0 & 0 & \cdots & 0 \\
\frac{1}{2}-\frac{1}{2m} & -\frac{1}{2m} & \cdots & -\frac{1}{2m} & \frac{1}{2}-\frac{1}{2m} & -\frac{1}{2m} & \cdots & -\frac{1}{2m} & 0 & 0 & \cdots & 0 \\
-\frac{1}{2m} & \frac{1}{2}-\frac{1}{2m} & \cdots & -\frac{1}{2m} & -\frac{1}{2m} & \frac{1}{2}-\frac{1}{2m} & \cdots & -\frac{1}{2m} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\frac{1}{2m} & -\frac{1}{2m} & \cdots & \frac{1}{2}-\frac{1}{2m} & -\frac{1}{2m} & -\frac{1}{2m} & \cdots & \frac{1}{2}-\frac{1}{2m} & 0 & 0 & \cdots & 0 \\
0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 \\
0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0
\end{bmatrix}.
$$

Table 5.3: $(x_i + x_{m+i})\,,\ i = 1(1)m$ and $\sum_{i=1}^{2m} x_i$ for the 4 Interactions

	B_1B_2	$B_1^2B_2^2$	$B_1B_2^2$	$B_1^2B_2$
$\sum_{i=1}^{2m} x_i$	$(-m+6)$	$(-m+6)$	$-2\,(m-3)$	$2\,(m-3)$
$x_1 + x_m$	2	2	1	-1
$x_2 + x_{m+1}$	2	2	1	-1
$x_3 + x_{m+3}$	-1	-1	-2	2
$x_4 + x_{m+4}$	-1	-1	-2	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_m + x_{m+m}$	-1	-1	-2	2

For real vector $\boldsymbol{X}_{2u} = (x_{1,u}, \ldots, x_{m,u}, x_{m+1,u}, \ldots, x_{2m,u}, x_{2m+1,u}, \ldots, x_{3m,u})'$ corresponding to the uth $2-$ factor interaction we get

$$\boldsymbol{PX}_{2u} = \begin{bmatrix} \frac{1}{2}\left(x_{1,u} + x_{m+1,u}\right) - \frac{1}{2m}\left(x_{1,u} + \ldots + x_{2m,u}\right) \\ \frac{1}{2}\left(x_{2,u} + x_{m+2,u}\right) - \frac{1}{2m}\left(x_{1,u} + \ldots + x_{2m,u}\right) \\ \vdots \\ \frac{1}{2}\left(x_{m,u} + x_{2m,u}\right) - \frac{1}{2m}\left(x_{1,u} + \ldots + x_{2m,u}\right) \\ \frac{1}{2}\left(x_{1,u} + x_{m+1,u}\right) - \frac{1}{2m}\left(x_{1,u} + \ldots + x_{2m,u}\right) \\ \frac{1}{2}\left(x_{2,u} + x_{m+2,u}\right) - \frac{1}{2m}\left(x_{1,u} + \ldots + x_{2m,u}\right) \\ \vdots \\ \frac{1}{2}\left(x_{m,u} + x_{2m,u}\right) - \frac{1}{2m}\left(x_{1,u} + \ldots + x_{2m,u}\right) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We present $(x_{i,u} + x_{m+i,u})\,,\ i = 1(1)m$ and $\sum_{i=1}^{2m} x_i$ for the 4 interaction vectors corresponding to the factors $B_1$and B_2 in Table 5.3. Because of the symmetric

Table 5.4: 3^2 Design: $d_2^{(1)}$

2	0	1	2	0	2
0	2	2	1	0	2

structure of $d_m^{(2)}$ the interaction vectors corresponding to any two factors B_i & B_j and hence the vector $\boldsymbol{PX}_{2u}$, $u = B_iB_j,\ B_i^2B_j^2,\ B_iB_j^2,\ B_i^2B_j$, $i < j = 1, \ldots, m$ are the same except for a permutation. From Table 5.3 we compute the vector $\boldsymbol{PX}_{2u}$ which is found to be identical $\forall u$ corresponding to B_1 and B_2. The $\boldsymbol{PX}_{2u}$ is given below

$$\boldsymbol{PX}_{2u} = \begin{pmatrix} \frac{(3m-6)}{2m}\boldsymbol{j}'_2 \\ -\frac{3}{m}\boldsymbol{j}'_{m-2} \\ \frac{(3m-6)}{2m}\boldsymbol{j}'_2 \\ -\frac{3}{m}\boldsymbol{j}'_{m-2} \\ 0\boldsymbol{j}'_m \end{pmatrix},\ u = B_1B_2,\ B_1^2B_2^2,\ B_1B_2^2,\ B_1^2B_2. \tag{5.6}$$

Identical result is obtained for any B_i & B_j, $i < j = 1, \ldots, m$. From (5.6) we get $\boldsymbol{X}'_{2u}\boldsymbol{PX}_{2u} = 9\left(\frac{m-2}{m}\right)$ which is constant independent of u and hence from Theorem 1 of chapter 4 $d_m^{(2)}$ is CV. The expression of CV becomes $\frac{m}{9(m-2)}$.

5.4 Optimal CV Designs

The design $d_m^{(1)}$ for $n = 2m+2$ is an optimal CV design for $m = 2$ with the CV value 0.4444 as this is the minimum value of CV in the class of all CV designs for $m = 2$ and $n = 6$. Similarly the design $d_m^{(2)}$ for $n = 3m$ is an optimal CV design for $m = 3$ with the CV value of 0.3333. We present the two optimum CV designs

Table 5.5: 3^3 Design

$d_3^{(2)}$								
2	0	0	0	2	2	1	2	2
0	2	0	2	0	2	2	1	2
0	0	2	2	2	0	2	2	1

$d_m^{(1)}$ for $m = 2$ and $d_m^{(2)}$ for $m = 3$ in Table 5.4 and Table 5.5 respectively.

Now we present some more CV designs for 3^m, $4 \leq m \leq 7$ with $n = 2m + 2$ in Table 5.6. Some of these designs are optimum CV designs. These designs have a clear pattern in them, the first $2m - 1$ runs are identical. All of these designs have the similar structure of $\boldsymbol{P}$ as discussed in the next section.

5.5 The structure of $\boldsymbol{P}$

In this section we analyze the projection matrix of the CV designs presented in the earlier two sections. Both the designs are found to possess a particular structure of $\boldsymbol{P}$. The design matrix for the uth model is $\boldsymbol{X}^{(u)} = \left[\boldsymbol{X}_1^* \vdots \boldsymbol{X}_{2u}\right]$. The projection matrix $\boldsymbol{P}$ is defined as $\boldsymbol{P} = \boldsymbol{I}_n - \boldsymbol{X}_1^* \left(\boldsymbol{X}_1^{*\prime} \boldsymbol{X}_1^*\right)^{-1} \boldsymbol{X}_1^{*\prime} = \boldsymbol{I}_n - \boldsymbol{Q}$. After rearranging the runs it can be seen that both the designs $d_m^{(1)}$ and $d_m^{(2)}$ have the following structure for$\boldsymbol{P}$:

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^* \end{bmatrix}, \ \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}, \ \boldsymbol{A} = \boldsymbol{I}_{n-m} - \boldsymbol{A}^*.$$

Partition the matrix $\boldsymbol{X}_1^*$ as $\boldsymbol{X}_1^* = \begin{bmatrix} \boldsymbol{X}_{11} \\ \boldsymbol{X}_{12} \end{bmatrix}$, where $\boldsymbol{X}_{11}$ corresponds to the m runs giving $0's$ in $\boldsymbol{P}$ and $\boldsymbol{X}_{12}$ corresponds to the remaining $(n - m)$ runs. Again

Table 5.6: Some CV Designs: for (a) $m = 4$, (b) $m = 5$, (c) $m = 6$ and (d) $m = 7$

(a)

1	1	1	2	0	2	0	0	0	2
1	1	2	1	0	2	0	1	1	1
1	2	1	1	0	2	1	2	0	0
2	1	1	1	0	2	1	2	1	2

(b)

1	1	1	1	2	0	2	0	0	0	2	1
1	1	1	2	1	0	2	0	1	1	1	1
1	1	2	1	1	0	2	1	2	0	0	2
1	2	1	1	1	0	2	1	2	1	2	2
2	1	1	1	1	0	2	1	2	0	1	2

(c)

1	1	1	1	1	2	0	2	0	0	0	2	1	1
1	1	1	1	2	1	0	2	0	1	1	1	1	1
1	1	1	2	1	1	0	2	1	2	0	0	2	1
1	1	2	1	1	1	0	2	1	2	1	2	2	1
1	2	1	1	1	1	0	2	1	2	0	1	2	1
2	1	1	1	1	1	0	2	1	2	0	1	0	1

(d)

1	1	1	1	1	1	2	0	2	0	0	0	2	1	1	0
1	1	1	1	1	2	1	0	2	0	1	1	1	1	1	2
1	1	1	1	2	1	1	0	2	1	2	0	0	2	1	1
1	1	1	2	1	1	1	0	2	1	2	1	2	2	1	0
1	1	2	1	1	1	1	0	2	1	2	0	1	2	1	1
1	2	1	1	1	1	1	0	2	1	2	0	1	0	1	0
2	1	1	1	1	1	1	0	2	1	2	0	1	0	1	2

partition $\boldsymbol{X}_{12}$ as $\boldsymbol{X}_{12} = \begin{bmatrix} \boldsymbol{X}_{12}^{(1)} \\ \\ \boldsymbol{X}_{12}^{(2)} \end{bmatrix}$, where $\boldsymbol{X}_{12}^{(1)}$ corresponds to the independent

runs of $\boldsymbol{X}_{12}$. Therefore

$$\begin{aligned}
\boldsymbol{P} &= \boldsymbol{I}_n - \boldsymbol{Q} \\
\Rightarrow \boldsymbol{Q} &= \boldsymbol{X}_1^* \left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1} \boldsymbol{X}_1^{*\prime} \\
&= \begin{bmatrix} \boldsymbol{X}_{11} \\ \boldsymbol{X}_{12} \end{bmatrix} \left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1} \begin{bmatrix} \boldsymbol{X}_{11}' & \boldsymbol{X}_{12}' \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}. \qquad (5.7)
\end{aligned}$$

Define $\boldsymbol{V} = \boldsymbol{X}_{11}'\boldsymbol{X}_{11} + \boldsymbol{X}_{12}^{\prime(1)}\boldsymbol{X}_{12}^{(1)}$.

Theorem 6. *The following characterizations hold true:*

1. $\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}$ is a generalized inverse of $\boldsymbol{X}_{11}'\boldsymbol{X}_{11}$.

2. $\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}$ is a generalized inverse of $\boldsymbol{X}_{12}'\boldsymbol{X}_{12}$.

3. $\boldsymbol{X}_{11}\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}\boldsymbol{X}_{12}' = \boldsymbol{0} \Leftrightarrow \boldsymbol{X}_{11}\boldsymbol{V}^{-1}\boldsymbol{X}_{12}' = \boldsymbol{0}$.

Proof. The proof follows from the structure of the projection matrix presented in (5.7).

1. It follows from (5.7) by equating the first diagonal component of $\boldsymbol{Q}$ with $\boldsymbol{I}_m$:

$$\begin{aligned}
\boldsymbol{X}_{11}\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}\boldsymbol{X}_{11}' &= \boldsymbol{I}_m \\
\Rightarrow \boldsymbol{X}_{11}'\boldsymbol{X}_{11}\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}\boldsymbol{X}_{11}'\boldsymbol{X}_{11} &= \boldsymbol{X}_{11}'\boldsymbol{X}_{11}.
\end{aligned}$$

2. It follows from (5.7) by equating the last diagonal component of $\boldsymbol{Q}$ with $\boldsymbol{A}$:

$$\begin{aligned}
\boldsymbol{X}_{12}\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}\boldsymbol{X}_{12}' &= \boldsymbol{A} \\
\Rightarrow \boldsymbol{X}_{12}'\boldsymbol{X}_{12}\left(\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^*\right)^{-1}\boldsymbol{X}_{12}'\boldsymbol{X}_{12} &= \boldsymbol{X}_{12}'\boldsymbol{A}\boldsymbol{X}_{12},
\end{aligned}$$

where $\boldsymbol{A}$ satisfies $\boldsymbol{A}\boldsymbol{X}_{12} = \boldsymbol{X}_{12}$ and $\boldsymbol{A}^2 = \boldsymbol{A}$.

3. Assume

$$\boldsymbol{X}_{11}\boldsymbol{V}^{-1}\boldsymbol{X}'_{12} = \boldsymbol{0}$$

which is equivalent to

$$\boldsymbol{X}_{11}\boldsymbol{V}^{-1}\boldsymbol{X}'^{(1)}_{12} = \boldsymbol{0} \text{ and } \boldsymbol{X}_{11}\boldsymbol{V}^{-1}\boldsymbol{X}'^{(2)}_{12} = \boldsymbol{0}. \tag{5.8}$$

From *Rao* (1973) we have

$$\begin{aligned}
\boldsymbol{X}_{11}\left(\boldsymbol{X}^{*\prime}_{1}\boldsymbol{X}^{*}_{1}\right)^{-1}\boldsymbol{X}'_{12} &= \boldsymbol{X}_{11}\left(\boldsymbol{V} + \boldsymbol{X}'^{(2)}_{12}\boldsymbol{X}^{(2)}_{12}\right)^{-1}\boldsymbol{X}'_{12} \\
&= \boldsymbol{X}_{11}\left(\boldsymbol{V}^{-1} - \frac{\boldsymbol{V}^{-1}\boldsymbol{X}'^{(2)}_{12}\boldsymbol{X}^{(2)}_{12}\boldsymbol{V}^{-1}}{1 + \boldsymbol{X}^{(2)}_{12}\boldsymbol{V}^{-1}\boldsymbol{X}'^{(2)}_{12}}\right)^{-1}\boldsymbol{X}'_{12} \\
&= \boldsymbol{X}_{11}\boldsymbol{V}^{-1}\boldsymbol{X}'_{12} - \frac{\boldsymbol{X}_{11}\boldsymbol{V}^{-1}\boldsymbol{X}'^{(2)}_{12}\boldsymbol{X}^{(2)}_{12}\boldsymbol{V}^{-1}\boldsymbol{X}'_{12}}{1 + \boldsymbol{X}^{(2)}_{12}\boldsymbol{V}^{-1}\boldsymbol{X}'^{(2)}_{12}} \\
&= \boldsymbol{0} \text{ from (5.8).}
\end{aligned}$$

Again assume

$$\boldsymbol{X}_{11}\left(\boldsymbol{X}^{*\prime}_{1}\boldsymbol{X}^{*}_{1}\right)^{-1}\boldsymbol{X}'_{12} = \boldsymbol{0}$$

which is equivalent to

$$\boldsymbol{X}_{11}\left(\boldsymbol{X}^{*\prime}_{1}\boldsymbol{X}^{*}_{1}\right)^{-1}\boldsymbol{X}'^{(1)}_{12} = \boldsymbol{0} \text{ and } \boldsymbol{X}_{11}\left(\boldsymbol{X}^{*\prime}_{1}\boldsymbol{X}^{*}_{1}\right)^{-1}\boldsymbol{X}'^{(2)}_{12} = \boldsymbol{0} \tag{5.9}$$

From *Rao* (1973) we have

$$
\begin{aligned}
\boldsymbol{X}_{11}\boldsymbol{V}^{-1}\boldsymbol{X}'_{12} &= \boldsymbol{X}_{11}\left(\boldsymbol{X}'_1\boldsymbol{X}_1 - \boldsymbol{X}'^{(2)}_{12}\boldsymbol{X}^{(2)}_{12}\right)^{-1}\boldsymbol{X}'_{12} \\
&= \boldsymbol{X}_{11}\left[\left(\boldsymbol{X}^{*\prime}_1\boldsymbol{X}^{*}_1\right)^{-1}\right. \\
&\quad \left.+\frac{\left(\boldsymbol{X}^{*\prime}_1\boldsymbol{X}^{*}_1\right)^{-1}\boldsymbol{X}'^{(2)}_{12}\boldsymbol{X}^{(2)}_{12}\left(\boldsymbol{X}^{*\prime}_1\boldsymbol{X}^{*}_1\right)^{-1}}{1+\boldsymbol{X}^{(2)}_{12}\left(\boldsymbol{X}^{*\prime}_1\boldsymbol{X}^{*}_1\right)^{-1}\boldsymbol{X}'^{(2)}_{12}}\right]^{-1}\boldsymbol{X}'_{12} \\
&= \boldsymbol{X}_{11}\left(\boldsymbol{X}^{*\prime}_1\boldsymbol{X}^{*}_1\right)^{-1}\boldsymbol{X}'_{12} \\
&\quad +\frac{\boldsymbol{X}_{11}\left(\boldsymbol{X}^{*\prime}_1\boldsymbol{X}^{*}_1\right)^{-1}\boldsymbol{X}'^{(2)}_{12}\boldsymbol{X}^{(2)}_{12}\left(\boldsymbol{X}^{*\prime}_1\boldsymbol{X}^{*}_1\right)^{-1}\boldsymbol{X}'_{12}}{1+\boldsymbol{X}^{(2)}_{12}\left(\boldsymbol{X}^{*\prime}_1\boldsymbol{X}^{*}_1\right)^{-1}\boldsymbol{X}'^{(2)}_{12}} \\
&= \mathbf{0} \text{ from (5.9).}
\end{aligned}
$$

□

Chapter 6

Special Properties of the Design $d_m^{(1)}$ when $m = 2$

6.1 Chapter summary

In this chapter we present some properties of the optimal CV design with six runs for a factorial experiment with two factors each at three levels we presented in Chapter 5 when all the runs are replicated a number of times. We characterize the CV property in terms of the determinant of the inverse of the variance-covariance matrix of the parameter estimators for each model. Also we obtain the condition of CV for a design with three factors from a CV design with two factors. Here is the summary of what we present in each section:

- (Section 6.2): Upto and including Chapter 5 we presented and obtained CV conditions for the designs with distinct runs. In this section we consider designs for factorial experiment with two factors each at three levels with

replicated runs. We prove that for any number of replications of the six runs of the optimal design, the replicated design satisfies the CV property.

- (Section 6.3): In this section we present some more CV designs with six runs for factorial experiment with two factors each at three levels which also satisfy the CV property w.r.t the general replication presented in section 6.2. Among these CV designs some are balanced and isomorphic to the optimal CV design discussed in the previous section. We also characterize the CV designs in terms of the determinant of their variance-covariance matrices. The constant determinant of the inverse of the variance-covariance matrix of the parameter estimators for each model gives NSC for a design to be CV.

- (Section 6.4): In this section we demonstrate how the CV property can be extended from a design for factorial experiment with two factors to a design for factorial experiment with three factors. We obtain the condition of the CV for a design with three factors whose every pair of columns contains the runs of the CV design with two factors. The runs for both the designs are replicated in the same way. So for a CV design with two factors these conditions can be checked to see if it can be extended to a CV design with three factors.

6.2 Replications of the Runs of $d_2^{(1)}$

In Chapter 5 we presented the design $d_m^{(1)}$ for $n = 2m + 2$ which is an optimal CV design for $m = 2$. In this section we consider the general replication of the

Table 6.1: Replicated Design $d_{2R}^{(1)}$

Runs	Replications
$(2,0)$	r_1
$(0,2)$	r_2
$(2,1)$	r_3
$(1,2)$	r_4
$(0,0)$	r_5
$(2,2)$	r_6

Table 6.2: Design Matrices

Design	With u^{th} interaction	Without u^{th} interaction
$d_2^{(1)}$	$\boldsymbol{X}^{u(1)}(6 \times 6)$	$\boldsymbol{X}_1^{*(1)}(6 \times 5)$
$d_{2R}^{(1)}$	$\boldsymbol{X}^{u(2)}\left(\sum_{i=1}^{6} r_i \times 6\right)$	$\boldsymbol{X}_1^{*(2)}\left(\sum_{i=1}^{6} r_i \times 5\right)$

design $d_m^{(1)}$ for $m = 2$ and show mathematically that it remains CV irrespective of the number of replications of any of its runs. Also from the CV expression we will see that it does not depend on the replications of the 2 runs which give columns of zeroes in the projection matrix as presented in Chapter 5.

We present the different replications of the 6 runs of the design $d_2^{(1)}$ in Table 6.1 and the replicated design is denoted by $d_{2R}^{(1)}$. The replications $r_1, r_2, \ldots, r_6$ can take positive integer values. When $r_i = 1, \forall i$ then $d_{2R}^{(1)}$ becomes $d_2^{(1)}$. The total number of runs for the replicated design $d_{2R}^{(1)}$ is $\sum_{i=1}^{6} r_i \, (\geq 6)$. We present the the notation of the design matrices for the two designs $d_2^{(1)}$ and $d_{2R}^{(1)}$ in Table 6.2. The rows of $\boldsymbol{X}_1^{*(2)}$ are formed by replicating the rows of $\boldsymbol{X}_1^{*(1)}$. For example if the ith row of $\boldsymbol{X}_1^{*(1)}$ is replicated r_i times it gives r_i rows of $\boldsymbol{X}_1^{*(2)}$, $i = 1(1)6$. So $\boldsymbol{X}_1^{*(2)}$ can be obtained from $\boldsymbol{X}_1^{*(1)}$ by pre-multiplying the latter by the matrix $\boldsymbol{R}$

of order $\left(\sum_{i=1}^{6} r_i \times 6\right)$ obtained from an identity matrix of order 6. So we have the following:

$$\boldsymbol{X}_1^{*(2)} = \boldsymbol{R}\boldsymbol{X}_1^{*(1)},$$

where

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \rightarrow replicated\ r_1\ times \\ \rightarrow replicated\ r_2\ times \\ \rightarrow replicated\ r_3\ times \\ \rightarrow replicated\ r_4\ times \\ \rightarrow replicated\ r_5\ times \\ \rightarrow replicated\ r_6\ times \end{array},$$

and

$$\boldsymbol{X}_1^{*(1)} = \begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

This implies

$$\boldsymbol{X}_1^{*(2)\prime}\boldsymbol{X}_1^{*(2)} = \boldsymbol{X}_1^{*(1)\prime}\boldsymbol{R}'\boldsymbol{R}\boldsymbol{X}_1^{*(1)}, \tag{6.1}$$

where

$$\boldsymbol{R}'\boldsymbol{R} = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_6 \end{bmatrix}.$$

Now we find the sufficient condition of CV for the replicated design $d_{2R}^{(1)}$ in terms of its projection matrix and the interaction vectors. Let the projection matrix corresponding to $d_{2R}^{(1)}$ be $\boldsymbol{P}_R$. We write $\boldsymbol{P}_R$ in terms of $\boldsymbol{X}_1^{*(2)}$. The condition of CV for the replicated design turns out to be same as that of the unreplicated design $d_2^{(1)}$ which is known to be CV. Hence the CV property of the replicated design follows. The $\boldsymbol{P}_R$ is given as

$$\boldsymbol{P}_R \;=\; \boldsymbol{I}_n - \boldsymbol{X}_1^{*(2)} \left(\boldsymbol{X}_1^{*(2)\prime}\boldsymbol{X}_1^{*(2)}\right)^{-1} \boldsymbol{X}_1^{*(2)\prime}.$$

We express $\boldsymbol{P}_R$ in terms of the design matrix $\boldsymbol{X}_1^{*(1)}$of $d_2^{(1)}$ because we want to obtain the condition of CV for $d_{2R}^{(1)}$ in terms of the CV design $d_2^{(1)}$. From (6.1) $\boldsymbol{P}_R$ can be expressed in terms of $\boldsymbol{X}_1^{*(1)}$ as

$$\boldsymbol{P}_R = \boldsymbol{I}_n - \boldsymbol{R}\boldsymbol{X}_1^{*(1)} \left(\boldsymbol{X}_1^{*(1)\prime}\boldsymbol{R}'\boldsymbol{R}\boldsymbol{X}_1^{*(1)}\right)^{-1} \boldsymbol{X}_1^{*(1)\prime}\boldsymbol{R}'.$$

Defining $\boldsymbol{W} = \boldsymbol{X}_1^{*(1)} \left(\boldsymbol{X}_1^{*(1)\prime}\boldsymbol{R}'\boldsymbol{R}\boldsymbol{X}_1^{*(1)}\right)^{-1} \boldsymbol{X}_1^{*(1)\prime}$, $\boldsymbol{P}_R$ can be simplified as $\boldsymbol{P}_R = \boldsymbol{I}_n - \boldsymbol{R}\boldsymbol{W}\boldsymbol{R}'$, where all the elements of $\boldsymbol{P}_R$ are in terms of the replications $r_1, \ldots, r_6$ and the elements of the design matrix of $d_2^{(1)}$. Given the replications and the design matrix of $d_2^{(1)}$, $\boldsymbol{P}_R$ can be easily obtained. The matrices $\boldsymbol{W}$ and $\boldsymbol{R}\boldsymbol{W}\boldsymbol{R}'$ are given

below:

$$\boldsymbol{W} = \begin{bmatrix} w_1 & w_2 & 0 & 0 & w_3 & w_4 \\ w_2 & w_5 & 0 & 0 & w_6 & w_7 \\ 0 & 0 & w_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_9 & 0 & 0 \\ w_3 & w_6 & 0 & 0 & w_{10} & w_{11} \\ w_4 & w_7 & 0 & 0 & w_{11} & w_{12} \end{bmatrix},$$

$$\boldsymbol{RWR}' = \begin{bmatrix} w_1\boldsymbol{J}_{r_1} & w_2\boldsymbol{J}_{r_1r_2} & \boldsymbol{0}_{r_1r_3} & \boldsymbol{0}_{r_1r_4} & w_3\boldsymbol{J}_{r_1r_5} & w_4\boldsymbol{J}_{r_1r_6} \\ w_2\boldsymbol{J}_{r_2r_1} & w_5\boldsymbol{J}_{r_2} & \boldsymbol{0}_{r_2r_3} & \boldsymbol{0}_{r_2r_4} & w_6\boldsymbol{J}_{r_2r_5} & w_7\boldsymbol{J}_{r_2r_6} \\ \boldsymbol{0}_{r_3r_1} & \boldsymbol{0}_{r_3r_2} & w_8\boldsymbol{J}_{r_3} & \boldsymbol{0}_{r_3r_4} & \boldsymbol{0}_{r_3r_5} & \boldsymbol{0}_{r_3r_6} \\ \boldsymbol{0}_{r_4r_1} & \boldsymbol{0}_{r_4r_2} & \boldsymbol{0}_{r_4r_3} & w_9\boldsymbol{J}_{r_4} & \boldsymbol{0}_{r_4r_5} & \boldsymbol{0}_{r_4r_6} \\ w_3\boldsymbol{J}_{r_5r_1} & w_6\boldsymbol{J}_{r_5r_2} & \boldsymbol{0}_{r_5r_3} & \boldsymbol{0}_{r_5r_4} & w_{10}\boldsymbol{J}_{r_5} & w_{11}\boldsymbol{J}_{r_5r_6} \\ w_4\boldsymbol{J}_{r_6r_1} & w_7\boldsymbol{J}_{r_6r_2} & \boldsymbol{0}_{r_6r_3} & \boldsymbol{0}_{r_6r_4} & w_{11}\boldsymbol{J}_{r_6r_5} & w_{12}\boldsymbol{J}_{r_6} \end{bmatrix},$$

where $w_i's$ are functions of $r_1, \ldots, r_6$ given at the end of this section. Hence the matrix $\boldsymbol{P}_R$ becomes

$$\boldsymbol{P}_R = \begin{bmatrix} \boldsymbol{I}_{r_1} - w_1\boldsymbol{J}_{r_1} & -w_2\boldsymbol{J}_{r_1r_2} & \boldsymbol{0}_{r_1r_3} & \boldsymbol{0}_{r_1r_4} & -w_3\boldsymbol{J}_{r_1r_5} & -w_4\boldsymbol{J}_{r_1r_6} \\ -w_2\boldsymbol{J}_{r_2r_1} & \boldsymbol{I}_{r_2} - w_5\boldsymbol{J}_{r_2} & \boldsymbol{0}_{r_2r_3} & \boldsymbol{0}_{r_2r_4} & -w_6\boldsymbol{J}_{r_2r_5} & -w_7\boldsymbol{J}_{r_2r_6} \\ \boldsymbol{0}_{r_3r_1} & \boldsymbol{0}_{r_3r_2} & \boldsymbol{I}_{r_3} - w_8\boldsymbol{J}_{r_3} & \boldsymbol{0}_{r_3r_4} & \boldsymbol{0}_{r_3r_5} & \boldsymbol{0}_{r_3r_6} \\ \boldsymbol{0}_{r_4r_1} & \boldsymbol{0}_{r_4r_2} & \boldsymbol{0}_{r_4r_3} & \boldsymbol{I}_{r_4} - w_9\boldsymbol{J}_{r_4} & \boldsymbol{0}_{r_4r_5} & \boldsymbol{0}_{r_4r_6} \\ -w_3\boldsymbol{J}_{r_5r_1} & -w_6\boldsymbol{J}_{r_5r_2} & \boldsymbol{0}_{r_5r_3} & \boldsymbol{0}_{r_5r_4} & \boldsymbol{I}_{r_5} - w_{10}\boldsymbol{J}_{r_5} & -w_{11}\boldsymbol{J}_{r_5r_6} \\ -w_4\boldsymbol{J}_{r_6r_1} & -w_7\boldsymbol{J}_{r_6r_2} & \boldsymbol{0}_{r_6r_3} & \boldsymbol{0}_{r_6r_4} & -w_{11}\boldsymbol{J}_{r_6r_5} & \boldsymbol{I}_{r_6} - w_{12}\boldsymbol{J}_{r_6} \end{bmatrix}.$$

Now we express the uth interaction vector of $\boldsymbol{X}^{u(2)}$ in terms of the uth interaction vector of $\boldsymbol{X}^{u(1)}$ and obtain the sufficient condition of CV for $d_{2R}^{(1)}$ in terms of its projection matrix and the interaction vector of $d_2^{(1)}$. For real valued vector $\boldsymbol{X}_{2u} = (x_{1,u}, x_{2,u}, \ldots, x_{6,u})'$ corresponding to the column of interaction vector of

$\boldsymbol{X}^{u(1)}$ the interaction vector of $\boldsymbol{X}^{u(2)}$ will be $\boldsymbol{X}^*_{2u} = \boldsymbol{R}\boldsymbol{X}_{2u}$ since the rows of $\boldsymbol{X}^*_{2u}$ are formed by replicating the rows of $\boldsymbol{X}_{2u}$. Thus $\boldsymbol{X}^*_{2u}$ can be expressed as

$$\boldsymbol{X}^*_{2u} = \begin{bmatrix} x_{1,u}\boldsymbol{j}_{r_1} \\ x_{2,u}\boldsymbol{j}_{r_2} \\ x_{3,u}\boldsymbol{j}_{r_3} \\ x_{4,u}\boldsymbol{j}_{r_4} \\ x_{5,u}\boldsymbol{j}_{r_5} \\ x_{6,u}\boldsymbol{j}_{r_6} \end{bmatrix}.$$

From Theorem 2 of Chapter 4 we know that the sufficient condition for a design to be CV is $|\boldsymbol{P}\boldsymbol{X}_{2u}|$ =constant,$\forall u$, $\boldsymbol{X}_{2u}$ being the uth interaction vector. Below we find the sufficient condition of CV for $d^{(1)}_{2R}$:

$$\boldsymbol{P}_R\boldsymbol{X}^*_{2u} = (x_{1,u} + x_{2,u} - x_{5,u} - x_{6,u}) \begin{bmatrix} \frac{r_2r_5r_6}{k}\boldsymbol{j}_{r_1} \\ \frac{r_1r_5r_6}{k}\boldsymbol{j}_{r_2} \\ 0\boldsymbol{j}_{r_3} \\ 0\boldsymbol{j}_{r_4} \\ -\frac{r_1r_2r_6}{k}\boldsymbol{j}_{r_5} \\ -\frac{r_1r_2r_5}{k}\boldsymbol{j}_{r_6} \end{bmatrix}, \; k = \frac{1}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)}. \tag{6.2}$$

So from the above expression we see that $\boldsymbol{P}_R\boldsymbol{X}^*_{2u}$ will be constant iff

$$|x_{1,u} + x_{2,u} - x_{5,u} - x_{6,u}| = \text{constant}, \forall u. \tag{6.3}$$

So (6.3) is a sufficient condition for the replicated design $d^{(1)}_{2R}$ to be CV. In Chapter 5 we found the sufficient condition for the 3^m design $d^{(1)}_m$ to be CV as

$$|x_{1,u} + x_{2,u} + \ldots + x_{m,u} - (m-1)\, x_{2m+1,u} - x_{2m+2,u}| = \text{constant}, \forall u \tag{6.4}$$

Table 6.3: Interaction Columns of $d_2^{(1)}$ and the Condition

t_1	t_2	$\boldsymbol{X}_{2B_1B_2}$	$\boldsymbol{X}_{2B_1^2B_2^2}$	$\boldsymbol{X}_{2B_1B_2^2}$	$\boldsymbol{X}_{2B_1^2B_2}$
0	2	1	1	0	-2
2	0	1	1	1	1
1	2	-1	1	1	1
2	1	-1	1	0	-2
0	0	-1	1	-1	1
2	2	0	-2	-1	1
Condition		3	3	3	3

where $\boldsymbol{X}_{2u} = (x_{1,u}, x_{2,u}, \ldots, x_{m,u}, x_{2m+1,u}, x_{2m+2,u})'$ is the uth interaction vector. For $m = 2$ condition (6.4) becomes identical to (6.3). So the condition of CV for both the designs $d_2^{(1)}$ and $d_{2R}^{(1)}$ are identical. In Table 6.3 we give the four interaction vectors corresponding to the four interaction effects along with the value of $|x_{1,u} + x_{2,u} - x_{5,u} - x_{6,u}|, \forall u$ for the design $d_2^{(1)}$. So from Table 6.3 we see that for all the 2-factor interaction vectors of $d_2^{(1)}$ we have

$$|x_{1u} + x_{2u} - x_{5u} - x_{6u}| = 3, \forall u \tag{6.5}$$

which is constant independent of u. Hence given this 3^2 CV design we have shown mathematically that if any of its run is replicated any number of times, the replicated design also satisfies the CV property. The CV expression for the replicated design is given as

$$\frac{Var\left(\hat{\beta}_{2u}\right)}{\sigma^2} = \frac{1}{\boldsymbol{X}_{2u}^{*}\boldsymbol{P}_R\boldsymbol{X}_{2u}^{*}} = \frac{r_1r_2r_5 + r_1r_2r_6 + r_1r_5r_6 + r_2r_5r_6}{9r_1r_2r_5r_6}. \tag{6.6}$$

From the variance expression in (6.6) we see that it does not depend on the replication of the runs $(1, 2)$ and $(2, 1)$ which gives the columns of zeroes in the projection matrix of $d_2^{(1)}$. Here are the expressions of $w_i's$ in terms of the $r_i's$, $i =$

1(1)6:

$$
\begin{aligned}
w_1 &= \frac{r_5r_6 + r_2\left(r_5 + r_6\right)}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)} \\
w_2 &= -\frac{r_5r_6}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)} \\
w_3 &= \frac{r_2r_6}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)} \\
w_4 &= \frac{r_2r_5}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)} \\
w_5 &= \frac{r_5r_6 + r_1\left(r_5 + r_6\right)}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)} \\
w_6 &= \frac{r_1r_6}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)} \\
w_7 &= \frac{r_1r_5}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)} \\
w_8 &= \frac{1}{r_3} \\
w_9 &= \frac{1}{r_4} \\
w_{10} &= \frac{r_2r_6 + r_1\left(r_2 + r_6\right)}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)} \\
w_{11} &= -\frac{r_1r_2}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)} \\
w_{12} &= \frac{r_2r_5 + r_1\left(r_2 + r_5\right)}{r_2r_5r_6 + r_1\left(r_5r_6 + r_2\left(r_5 + r_6\right)\right)}.
\end{aligned}
$$

Theorem 7. *The design $d_2^{(1)}$ remains CV after replicating any of its runs any number of times with the CV value $\frac{r_1r_2r_5+r_1r_2r_6+r_1r_5r_6+r_2r_5r_6}{9r_1r_2r_5r_6}$.*

Proof. The Theorem follows from (6.2), (6.5) and (6.6). □

6.3 Some More Balanced Designs Connected to $d_2^{(1)}$

In this section we present some more designs which have similar CV property as that of the design $d_2^{(1)}$ w.r.t the general replications of the runs presented in the previous section. Some of these designs are balanced and are isomorphic to $d_2^{(1)}$ w.r.t the runs. Also we characterize the CV property of these designs in terms of the determinant of the inverse of the respective variance-covariance matrices of the parameter estimators. So instead of calculating the variance of the $2-$factor interaction estimators only the determinant condition can be checked to identify the CV design. We consider the balanced 3^2 designs for $n = 6$ and find out how many of them satisfy the CV property by checking the determinant condition.

In Table 6.4 we present 30 more 3^2 designs all of which remain CV after replicating any of their runs any number of times. So all of these 30 designs have identical property as $d_2^{(1)}$ w.r.t the replication of the runs. From Table 6.4 we see that the design # 30 can be obtained from the design $d_2^{(1)}$ by renaming "0" as "1" and "1" as "0". The design # 2 can be obtained from $d_2^{(1)}$ by renaming "0" as "2" and "2" as "0". The 27th design in the set is obtained from the design # 30 by renaming "2" as "1" and "1" as "2". So all these four balanced designs are isomorphic to one another w.r.t the replications and the runs as well. In 3^2 factorial experiment the three pairs of balanced runs are $\{(0,2),(2,0)\}$, $\{(0,1),(1,0)\}$ and $\{(1,2),(2,1)\}$. From computer check we found that 3^2 designs with distinct runs are CV only for $n = 6$. We find out the balanced designs which are CV for $n = 6$.

Table 6.4: 3^2 CV Designs Which Remain CV for Any Replication

0	0
0	1
0	2
1	0
2	0
2	1

0	0
0	1
0	2
1	0
2	0
2	2

0	0
0	1
0	2
1	1
2	0
2	1

0	0
0	1
0	2
1	2
2	0
2	1

0	0
0	1
0	2
1	0
1	1
2	2

0	0
0	1
1	1
1	2
2	0
2	1

0	0
0	1
1	1
1	2
2	0
2	1

0	0
0	1
1	1
1	2
2	0
2	2

0	0
0	1
0	2
1	2
2	0
2	1

0	0
0	1
0	2
1	2
2	0
2	2

0	0
0	2
1	0
2	0
2	1
2	2

0	0
0	2
1	0
1	1
1	2
2	2

0	0
0	2
1	0
1	1
1	2
2	2

0	0
0	2
1	0
1	1
1	2
2	2

0	0
0	2
1	0
1	1
1	2
2	2

0	0
1	1
1	2
2	0
2	1
2	2

0	1
0	2
1	0
1	1
1	2
2	0

0	1
0	2
1	0
1	1
1	2
2	1

0	1
0	2
1	0
1	1
1	2
2	2

0	1
0	2
1	0
1	1
1	2
2	2

0	1
0	2
1	0
1	1
1	2
2	1

0	1
0	2
1	1
1	2
2	0
2	2

0	1
0	2
1	1
1	2
2	0
2	1

0	2
1	0
1	1
1	2
2	0
2	2

0	2
1	1
1	2
2	0
2	1
2	2

Table 6.5: Balanced 3^2 CV Designs

I	
0	2
2	0
1	2
2	1
2	2
x	x

II	
0	2
2	0
1	0
0	1
2	2
x	x

III	
0	1
1	0
1	2
2	1
2	2
x	x

We fix the run$(2,2)$ and choose any two pairs from the three balanced pairs and add one run from the remaining set $\{(0,0),(1,1)\}$. In Table 6.5 we present the balanced designs with fixed $(2,2)$ and the added run (x,x), $x \in \{0,1\}$. We have

$$\frac{Var\left(\hat{\beta}_{2u}\right)}{\sigma^2} = \frac{|\boldsymbol{X}_1^{*\prime}\boldsymbol{X}_1^{*}|}{|\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}|}, \tag{6.7}$$

where $\boldsymbol{X}_1^*$ is the design matrix corresponding to the general mean and main effects and the design matrix $\boldsymbol{X}^{(u)}$ corresponds to the general mean, main effects and the uth 2−factor interaction effect, $u = 1(1)4$. From (6.7) we see that a design is CV iff

$$|\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}| = constant, \forall u.$$

We present the $|\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}|, \forall u$ in terms of x for the three balanced designs in Table 6.6. In Table 6.7 we present the determinant value of $\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}, \forall u$ for both $x = 0$ and $x = 1$. From Table 6.7 we see that for design I all the determinants are equal to 11664. So both $(1,1)$ and $(0,0)$ gives CV design. For design II for both $x = 0$ and $x = 1$ all the determinants are equal to 11664. So both $(1,1)$ and $(0,0)$ gives CV design. For the design II for $x = 0$ all determinants are equal to 11664 and hence $(0,0)$ gives a CV design. But for $x = 1$, $|\boldsymbol{X}^{(B_1B_2^2)\prime}\boldsymbol{X}^{(B_1B_2^2)}| =$

Table 6.6: $|\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}|$ for the Balanced Designs

I	
$\lvert\boldsymbol{X}'_{B_1B_2}\boldsymbol{X}_{B_1B_2}\rvert$	$11664 - 69984x + 128304x^2 - 69984x^3 + 11664x^4$
$\lvert\boldsymbol{X}'_{B_1^2B_2^2}\boldsymbol{X}_{B_1^2B_2^2}\rvert$	$11664 + 116640x + 174960x^2 - 583200x^3 + 291600x^4$
$\lvert\boldsymbol{X}'_{B_1B_2^2}\boldsymbol{X}_{B_1B_2^2}\rvert$	$11664 + 11664x - 8748x^2 - 5832x^3 + 2916x^4$
$\lvert\boldsymbol{X}'_{B_1^2B_2}\boldsymbol{X}_{B_1^2B_2}\rvert$	$11664 + 11664x - 8748x^2 - 5832x^3 + 2916x^4$

II	
$\lvert\boldsymbol{X}'_{B_1B_2}\boldsymbol{X}_{B_1B_2}\rvert$	$11664 - 69984x + 128304x^2 - 69984x^3 + 11664x^4$
$\lvert\boldsymbol{X}'_{B_1^2B_2^2}\boldsymbol{X}_{B_1^2B_2^2}\rvert$	$11664 - 69984x + 81648x^2 + 69984x^3 + 11664x^4$
$\lvert\boldsymbol{X}'_{B_1B_2^2}\boldsymbol{X}_{B_1B_2^2}\rvert$	$11664 - 34992x + 37908x^2 - 17496x^3 + 2916x^4$
$\lvert\boldsymbol{X}'_{B_1^2B_2}\boldsymbol{X}_{B_1^2B_2}\rvert$	$11664 - 34992x + 37908x^2 - 17496x^3 + 2916x^4$

III	
$\lvert\boldsymbol{X}'_{B_1B_2}\boldsymbol{X}_{B_1B_2}\rvert$	$11664 - 69984x + 128304x^2 - 69984x^3 + 11664x^4$
$\lvert\boldsymbol{X}'_{B_1^2B_2^2}\boldsymbol{X}_{B_1^2B_2^2}\rvert$	$104976 - 769824x + 1901232x^2 - 1796256x^3 + 571536x^4$
$\lvert\boldsymbol{X}'_{B_1B_2^2}\boldsymbol{X}_{B_1B_2^2}\rvert$	$46656x^2 - 46656x^2 + 11664x^4$
$\lvert\boldsymbol{X}'_{B_1^2B_2}\boldsymbol{X}_{B_1^2B_2}\rvert$	$46656x^2 - 46656x^2 + 11664x^4$

Table 6.7: Value of $|\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}|, \forall u$ for $x = 0$ and $x = 1$

Design I		
u	$x = 0$	$x = 1$
B_1B_2	11664	11664
$B_1^2B_2^2$	11664	11664
$B_1B_2^2$	11664	11664
$B_1^2B_2$	11664	11664

Design II		
u	$x = 0$	$x = 1$
B_1B_2	11664	11664
$B_1^2B_2^2$	11664	104976
$B_1B_2^2$	11664	0
$B_1^2B_2$	11664	0

Design III		
u	$x = 0$	$x = 1$
B_1B_2	11664	11664
$B_1^2B_2^2$	104976	11664
$B_1B_2^2$	0	11664
$B_1^2B_2$	0	11664

$|\boldsymbol{X}^{(B_1^2B_2)\prime}\boldsymbol{X}^{(B_1^2B_2)}| = 0$ and hence the design II with $(1,1)$ does not satisfy the design condition. Hence $(1,1)$ does not give a CV design. For design III for $x = 0$, $|\boldsymbol{X}^{(B_1B_2^2)\prime}\boldsymbol{X}^{(B_1B_2^2)}| = |\boldsymbol{X}^{(B_1^2B_2)\prime}\boldsymbol{X}^{(B_1^2B_2)}| = 0$ and hence $(0,0)$ does not work. But $x = 1$ gives all the determinants equal to 11664 and hence this design with $(1,1)$ is a CV design. Thus we get four balanced designs which are CV and these four designs are the ones isomorphic to one another already discussed. The design I with $x = 0$ is the design $d_2^{(1)}$, design I with $x = 1$ is the design # 30 in Table 6.4, design II with $x = 0$ is design # 2 in Table 6.4 and the design III with $x = 1$ is the design # 27 in Table 6.4. From Table 6.6 we see that $|\boldsymbol{X}^{(B_1B_2^2)\prime}\boldsymbol{X}^{(B_1B_2^2)}| = |\boldsymbol{X}^{(B_1^2B_2)\prime}\boldsymbol{X}^{(B_1^2B_2)}|$ for all the three designs. This can be seen from the fact that the columns of $\boldsymbol{X}^{(B_1^2B_2)}$ are linear combinations of the columns of $\boldsymbol{X}^{(B_1B_2^2)}$ or vice versa and determinant does not change for elementary operations on the columns of a matrix. For example the 6th column of $\boldsymbol{X}^{(B_1^2B_2)}$ can be obtained by $\left(2^{nd} - 3^{rd} - 4^{th} + 5^{th} - 6^{th}\right)$ columns of $\boldsymbol{X}^{(B_1B_2^2)}$ and this representation is unique since these matrices are full rank matrices. For any $u_1 \neq u_2$ and $n \times n$ design matrices $\boldsymbol{X}^{(u_1)}$ and $\boldsymbol{X}^{(u_2)}$

$$\boldsymbol{X}^{(u_2)} = \boldsymbol{D}\boldsymbol{X}^{(u_1)} \Rightarrow |\boldsymbol{X}^{(u_2)\prime}\boldsymbol{X}^{(u_2)}| = |\boldsymbol{X}^{(u_1)\prime}\boldsymbol{D}'\boldsymbol{D}\boldsymbol{X}^{(u_1)}| = |\boldsymbol{X}^{(u_1)\prime}||\boldsymbol{D}'\boldsymbol{D}||\boldsymbol{X}^{(u_1)}|,$$

where $\boldsymbol{D}\,(n \times n)$ is the matrix of elementary operations. Hence $|\boldsymbol{D}'\boldsymbol{D}| = 1 \Rightarrow Var(\hat{\beta}_{2u_1}) = Var(\hat{\beta}_{2u_2})$. Thus for every pair (u_1, u_2) if there exists a $\boldsymbol{D}$ such that $\boldsymbol{X}^{(u_2)} = \boldsymbol{D}\boldsymbol{X}^{(u_1)}$, both $\boldsymbol{X}^{(u_1)}$ and $\boldsymbol{X}^{(u_2)}$ being square matrices, then the design is CV.

Table 6.8: 3^2 Design Extended to 3^3 Design

D_1		D_2		
B_1	B_2	B_1	B_2	B_3
2	0	2	0	0
0	2	0	2	0
2	1	2	1	2
1	2	1	2	2
0	0	0	0	2
0	0	0	0	0
2	2	2	2	1
2	2	2	2	2

6.4 $3^2 \to 3^3$ Common Variance Design

In this section we present the conditions of obtaining a 3^3 CV design whose every pair of columns is formed of the runs of a 3^2 CV design. So without even calculating the variance of the $2-$factor interaction estimators of the 3^3 design, we can check its CV property by verifying these conditions obtained from the 3^2 CV design. Conditions are obtained by taking the example of the design $d_2^{(1)}$ as the 3^2 CV design and the design $d_m^{(1)}$ for $m = 3$ as the 3^3 design.

In a 3^3 factorial experiment denote the 3 factors by B_1, B_2 and B_3. There are one general mean, 6 main effects and 12 $2-$factor interaction effects. Consider the model M_u $\forall u$ in (1.3.1) for $k = 1$ in Chapter 1 for 3^3 experiment. In each model there are 8 parameters and hence we need designs with at least 8 runs in order to estimate all of them. Consider the 3^2 design D_1 with $n = 8$ runs by replicating the runs $(0, 0)$ and $(2, 2)$ twice which is extended to the 3^3 design D_2 with $n = 8$ runs as presented in Table 6.8. From Table 6.8 we see that all the runs of the 3^2 design are present in every pair of columns of the 3^3 design and also the

runs are replicated in the same way in both the designs. We already know that both of these designs are CV since D_1 is the replicated design obtained from $d_2^{(1)}$ and D_2 is the CV design $d_m^{(1)}$ for $m = 3$ presented in Chapter 5. Consider the columns corresponding to the factors B_1 and B_2 of the 3^3 design D_2. The runs corresponding to B_1 and B_2 are identical to the runs of the 3^2 design. For the uth model let the design matrix of D_1 be $\boldsymbol{X}_2^{(u)}$ whose columns correspond to μ, B_1, B_1^2, B_2, B_2^2 and $u = B_1^\alpha B_2^\beta$, $\alpha, \beta \in \{1, 2\}$ and the design matrix of D_2 be $\boldsymbol{X}_3^{(u)}$ whose columns correspond to μ, B_1, B_1^2, B_2, B_2^2, B_3, B_3^2 and u for the interactions corresponding to B_1 and B_2. Now we make some re-arrangements in the columns of $\boldsymbol{X}_3^{(u)}$ and write them in the order: μ, B_1, B_1^2, B_2, B_2^2, u, B_3, B_3^2. Then $\boldsymbol{X}_3^{(u)}$ can be written as

$$\boldsymbol{X}_3^{(u)} = \left(\boldsymbol{X}_2^{(u)} \vdots \boldsymbol{X}_1\right),$$

where $\boldsymbol{X}_1$ consists of the columns corresponding to B_3 and B_3^2. So we have

$$\begin{aligned}
\boldsymbol{X}_3^{(u)\prime}\boldsymbol{X}_3^{(u)} &= \begin{pmatrix} \boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_2^{(u)} & \boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_1 \\ \boldsymbol{X}_1'\boldsymbol{X}_2^{(u)} & \boldsymbol{X}_1'\boldsymbol{X}_1 \end{pmatrix} \\
\Rightarrow \left(\boldsymbol{X}_3^{(u)\prime}\boldsymbol{X}_3^{(u)}\right)^{-1} &= \begin{pmatrix} \boldsymbol{A}_u & \boldsymbol{B}_u \\ \boldsymbol{C}_u & \boldsymbol{D}_u \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{A}_u &= \left(\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_2^{(u)}\right)^{-1} + \left(\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_2^{(u)}\right)^{-1}\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_1\left[\boldsymbol{X}_1'\boldsymbol{X}_1\right. \\
&\quad \left. -\boldsymbol{X}_1'\boldsymbol{X}_2^{(u)}\left(\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_2^{(u)}\right)^{-1}\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_1\right]^{-1}\boldsymbol{X}_1'\boldsymbol{X}_2^{(u)}\left(\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_2^{(u)}\right)^{-1}. \qquad (6.8)
\end{aligned}$$

Writing $\boldsymbol{W}_u = \boldsymbol{X}_2^{(u)}\left(\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_2^{(u)}\right)^{-1}\boldsymbol{X}_2^{(u)\prime}$, $\boldsymbol{M}_u = \left[\boldsymbol{X}_1'\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{W}_u\boldsymbol{X}_1\right]^{-1}$ and $\boldsymbol{Z}_u = \left(\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_2^{(u)}\right)^{-1}\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_1$, (6.8) becomes

$$\boldsymbol{A}_u = \left(\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_2^{(u)}\right)^{-1} + \boldsymbol{Z}_u\boldsymbol{M}_u\boldsymbol{Z}_u'. \qquad (6.9)$$

We are only interested in the last diagonal element of the matrix $\boldsymbol{A}_u$ which is proportional to $Var\left(\hat{\beta}_{2u}\right)$. The last diagonal element of $\left(\boldsymbol{X}_2^{(u)\prime}\boldsymbol{X}_2^{(u)}\right)^{-1}$ is independent of u since the design D_1 is CV. So we need to show that the last diagonal element of the matrix $\boldsymbol{Z}_u\boldsymbol{M}_u\boldsymbol{Z}_u'$ in (6.8) is constant independent of u. The matrix $\boldsymbol{X}_1$ and hence $(\boldsymbol{X}_1'\boldsymbol{X}_1)$ are independent of u since $\boldsymbol{X}_1$ does not contain any interaction vector. For D_1 we have

$$\boldsymbol{W}_u = same, \forall u.$$

From the expression of $\boldsymbol{M}_u$ we have

$$\boldsymbol{W}_u = same \Rightarrow \boldsymbol{M}_u = same, \forall u. \tag{6.10}$$

Now the last diagonal element of $\boldsymbol{Z}_u\boldsymbol{M}_u\boldsymbol{Z}_u'$ is the last row of $\boldsymbol{Z}_u$ multiplied by $\boldsymbol{M}_u$ multiplied by the last column of $\boldsymbol{Z}_u'$ which is same as the last row of $\boldsymbol{Z}_u$. Hence

$$\begin{aligned} &\text{Last row of } \boldsymbol{Z}_u \quad \text{is same}, \forall u \\ \Rightarrow &\text{Last diagonal element of } \boldsymbol{Z}_u\boldsymbol{M}_u\boldsymbol{Z}_u' \quad = \quad constant, \forall u. \end{aligned} \tag{6.11}$$

For D_1 we observe that the last row of $\boldsymbol{Z}_u$ is same in magnitude of its elements, $\forall u$. Hence we have

$$\begin{aligned} \text{Last row of } \boldsymbol{Z}_u &= same, \forall u \text{ in absolute value of the elements and} \\ \boldsymbol{W}_u &= same, \forall u \\ \Rightarrow Var\left(\hat{\beta}_{2u}\right) &= constant, \forall u. \end{aligned} \tag{6.12}$$

This result can be shown by taking any pair of columns of the design D_2 and rearranging the runs according to that of the design D_1 and forming the matrices

$\boldsymbol{X}_2^{(u)}$ and $\boldsymbol{X}_1$ appropriately. In the following we present the matrix $\boldsymbol{W}_u$ which is same for all u:

$$\boldsymbol{W}_u = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}, \ \forall u.$$

We present the matrix $\boldsymbol{M}_u$ and the vector $\boldsymbol{Z}_u$ whose last row is constant in magnitude of its elements in Table 6.9.

Theorem 8. *A 3^3 design whose every pair of columns contains a 3^2 CV design, the runs being replicated in the same way, is CV if $\boldsymbol{W}_u = \boldsymbol{X}_2^{(u)} \left(\boldsymbol{X}_2^{(u)\prime} \boldsymbol{X}_2^{(u)}\right)^{-1} \boldsymbol{X}_2^{(u)\prime}$ is same, $\forall u$ and the last row of $\boldsymbol{Z}_u = \left(\boldsymbol{X}_2^{(u)\prime} \boldsymbol{X}_2^{(u)}\right)^{-1} \boldsymbol{X}_2^{(u)\prime} \boldsymbol{X}_1$ is same in absolute value of its elements, $\forall u$, where $\boldsymbol{X}_{2u}$ corresponds to the uth interaction of the 3^2 design and $\boldsymbol{X}_1$ corresponds to the main effects of the factor not present in the 3^2 design.*

Proof. The Theorem follows from (6.9), (6.10), (6.11) and (6.12). □

Table 6.9: $\boldsymbol{M}_u$ and $\boldsymbol{Z}_u$

Interaction	$\boldsymbol{M}_u$	$\boldsymbol{Z}_u$
B_1B_2	$\begin{pmatrix} 0.5 & -1.667 \\ -1.667 & 2.778 \end{pmatrix}$	$\begin{pmatrix} -0.1667 & 1.5 \\ 0.3333 & -0.5 \\ 0 & -0.5 \\ 0.3333 & -0.5 \\ 0 & -0.5 \\ -0.8333 & 0.5 \end{pmatrix}$
$B_1^2B_2^2$	$\begin{pmatrix} 0.5 & -1.667 \\ -1.667 & 2.778 \end{pmatrix}$	$\begin{pmatrix} 1.5 & 0.5 \\ -0.5 & 0 \\ -0.8333 & 0 \\ -0.5 & 0 \\ -0.8333 & 0 \\ -0.8333 & 0.5 \end{pmatrix}$
$B_1B_2^2$	$\begin{pmatrix} 0.5 & -1.667 \\ -1.667 & 2.778 \end{pmatrix}$	$\begin{pmatrix} 0.6667 & 1 \\ 0.3333 & -0.5 \\ -0.8333 & 0 \\ -0.8333 & -0.25 \\ -0.4167 & -0.25 \\ -0.8333 & 0.5 \end{pmatrix}$
$B_1^2B_2$	$\begin{pmatrix} 0.5 & -1.667 \\ -1.667 & 2.778 \end{pmatrix}$	$\begin{pmatrix} 0.6667 & 1 \\ -0.5 & 0 \\ 0 & -0.5 \\ 0.75 & -0.75 \\ -1.25 & 0.25 \\ 0.8333 & -0.5 \end{pmatrix}$

Chapter 7

$2^{m_a} \times 3^{m_b}$ Factorial Experiment

7.1 Chapter Summary

In this chapter we consider mixed level factorial experiments where different factors are at different levels. In the first two sections we express the treatment effects in terms of the factorial effects for such factorial experiments. In the latter sections we check for the CV property of the mixed level designs and obtain conditions of CV on the replications of the design runs when unreplicated design does not possess CV property. Also we obtain designs which satisfy the CV property within groups of similar interactions. Here is the summary of what we present in each section:

- (Section 7.2): Up to and including Chapter 6 we only presented designs for factorial experiment with factors each at three levels. In many scientific experiments it is necessary to consider designs with combinations of different factors at different levels. A more general setting is an asymmetrical factorial

experiment where some factors are each at two levels, some are each at three levels and so on. In particular we consider a factorial experiment where some factors are each at two levels and some are each at three levels only. In this section we present the factorial effects of the mixed experiment and express them in terms of the treatment effects.

- (Section 7.3): In this section we illustrate the relations between the factorial effects and the treatment effects with different examples. In particular we present examples for factorial experiments with one and two factors each at two or three levels.

- (Section 7.4): The unreplicated full factorial design for factorial experiment with one factor at two levels and the other factor at three levels gives different values to the variance of its two-factor interaction estimators. Therefore to obtain CV designs we consider different replications of the six runs of this design. In this section we present some structured replications of this full factorial design and for one particular type of replication we found condition on the replications for the replicated design to be CV. If this condition is satisfied by the replicated full factorial designs then variance calculation is not needed to check for the CV property. For all other types of structured replications it is found that the variances of the $2-$factor interaction estimators can never be equal. Also for each type of replications we compare the variance of the $2-$factor interaction estimators obtained from the separate models with the ones obtained from the full model.

- (Section 7.5): In this section we obtain the variances of the two-factor interaction estimators expressesd in terms of the general replication of the six runs of the full factorial simplest mixed design. But finding condition of CV on the general replications is computationally very tedious. Hence we consider different replications within a range and replicated designs are obtained which are not CV but the variances of the interaction estimators are close to one another with very small differences. For the general replication also we compare the variances of the $2-$factor interaction estimators obtained from the separate models with the ones obtained from the full model.

- (Section 7.6): In this section we consider mixed designs for factorial experiment with some factors at two levels each and some at three levels each which are not CV but they give equal variance within the different groups of $2-$factor interaction estimators. The search of CV designs for different number of runs for the mixed level experiment is computationally challenging as the number of factors becomes large. So instead of finding designs giving equal variance to all $2-$factor interaction estimators, we present designs which give equal variance within groups of similar interaction effects.

7.2 Factorial Effects in Terms of Treatment Effects

Consider mixed level factorial experiment of the form $s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_t^{m_t}$, where s_i $(s_i \geq 2)$ is the level of the ith factor m_i, $i = 1(1)t$ and s_i's are all dis-

tinct. In particular we take $s_1 = 2$ and $s_2 = 3$ and all $s_i = 0,\ i = 3(1)t$, i.e, some factors are at 2 levels and some at 3 levels only. Denote the m_a factors each with 2 levels by $A_1, A_2, \ldots, A_{m_a}$ and m_b factors each with 3 levels by $B_1, B_2, \ldots B_{m_b}$. Denote the levels of the factors of the 2^{m_a} factorial experiment by $(x_1,\ x_2, \ldots,\ x_{m_a})$ and the levels of the factors of the 3^{m_b} experiment by $(y_1,\ y_2, \ldots,\ y_{m_b})$ and thus a treatment of $2^{m_a} \times 3^{m_b}$ experiment is of the form $(x_1,\ x_2, \ldots,\ x_{m_a},\ y_1,\ y_2, \ldots,\ y_{m_b})$, $x_i \in \{0,1\}$, $y_j \in \{0,1,2\}$, $i = 1\,(1)\,m_a,\ j = 1(1)m_b$. Any factorial effect of $2^{m_a} \times 3^{m_b}$ factorial design can be represented by $A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_{m_a}^{\alpha_{m_a}} B_1^{\beta_1} B_2^{\beta_2} \ldots B_{m_b}^{\beta_{m_b}}$, $\alpha_i \in \{0,1\}$, $\beta_j \in \{0,1,2\}$, $i = 1\,(1)\,m_a,\ j = 1(1)m_b$. When $m_b = 0$ we have 2^{m_a} factorial experiment and the factorial effects are represented as $A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_{m_a}^{\alpha_{m_a}}$, $\alpha_i \in \{0,1\}$, $i = 1\,(1)\,m_a$. In the following we define the factorial effects in terms of the treatment effects for a 2^{m_a} factorial experiment:

$$\begin{aligned} 2^{m_a-\delta_a} A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_{m_a}^{\alpha_{m_a}} &= \{a_1x_1 + \ldots + a_{m_a}x_{m_a} = 1\} \\ &\quad + (-1)^{\delta_a} \{a_1x_1 + \ldots + a_{m_a}x_{m_a} = 0\}, \end{aligned}$$

where

$$\delta_a = \begin{cases} 0, & \alpha_1 + \alpha_2 + \ldots + \alpha_{m_a} = 0 \\ 1, & 1 \le \alpha_1 + \alpha_2 + \ldots + \alpha_{m_a} \le m_a \end{cases}, \tag{7.1}$$

$$(a_1, a_{2,} \ldots, a_{m_a}) = \begin{cases} (\alpha_1, \alpha_2, \ldots, \alpha_{m_a}), & \delta_a = 1 \\ (1-\alpha_1, 1-\alpha_2, \ldots, 1-\alpha_{m_a}), & \delta_a = 0 \end{cases}.$$

The expression $\{a_1x_1 + \ldots + a_{m_a}x_{m_a} = c_a\}$, $c_a \in \{0,1\}$ in (7.1) represents the number of treatments satisfying the condition $a_1x_1 + \ldots + a_{m_a}x_{m_a} = c_a$ under

$mod\,(2)$. If all α_i's are zero then the factorial effect becomes the general mean denoted by μ. Similarly, when $m_a = 0$ we have 3^{m_b} factorial experiment and the factorial effects are represented as $B_1^{\beta_1} B_2^{\beta_2} \ldots B_{m_b}^{\beta_{m_b}}$, $\beta_j \in \{0, 1, 2\}$,. In the following we define the linear and the quadratic factorial effects for a 3^{m_b} factorial experiment respectively in terms of the treatment effects:

$$\begin{aligned}\text{Linear: } 3^{m_b-\delta_b} B_1^{\beta_1} B_2^{\beta_2} \ldots B_{m_b}^{\beta_{m_b}} &= \{b_1 y_1 + \ldots + b_{m_b} y_{m_b} = 2\} \\ &+ (1-\delta_b)\{b_1 y_1 + \ldots + b_{m_b} y_{m_b} = 1\} \\ &+ (-1)^{\delta_b}\{b_1 y_1 + \ldots + b_{m_b} y_{m_b} = 0\}, \end{aligned} \tag{7.2}$$

$$\begin{aligned}\text{Quadratic: } 3^{m_b-\delta_b} B_1^{\beta_1} B_2^{\beta_2} \ldots B_{m_b}^{\beta_{m_b}} &= \{b_1 y_1 + \ldots + b_{m_b} y_{m_b} = 2\} \\ &+ (-2)^{\delta_b}\{b_1 y_1 + \ldots + b_{m_b} y_{m_b} = 1\} \\ &+ \{b_1 y_1 + \ldots + b_{m_b} y_{m_b} = 0\}, \end{aligned} \tag{7.3}$$

where

$$\delta_b = \begin{cases} 0, & \beta_1 + \beta_2 + \ldots + \beta_{m_b} = 0 \\ 1, & 1 \le \beta_1 + \beta_2 + \ldots + \beta_{m_b} \le 2m_b \end{cases},$$

$$(b_1, b_{2,} \ldots, b_{m_b}) = \begin{cases} (\beta_1, \beta_2, \ldots, \beta_{m_b}), & \delta_b = 1 \\ (1-\beta_1, 1-\beta_2, \ldots, 1-\beta_{m_b}), & \delta_b = 0 \end{cases}.$$

The expression $\{b_1 y_1 + \ldots + b_{m_b} y_{m_b} = c_b\}$, $c_b \in \{0, 1, 2\}$ in (7.2) and (7.3) represents the number of treatments satisfying the condition $b_1 y_1 + \ldots + b_{m_b} y_{m_b} = c_b$ under $mod\,(3)$. For the linear effect the first non zero β_u is 1, i.e, $\beta_1 = \beta_2 = \ldots = \beta_{u-1} = 0$, $\beta_u = 1$. For the quadratic effect the first non zero β_u is 2, i.e,

$\beta_1 = \beta_2 = \ldots = \beta_{u-1} = 0,\ \beta_u = 2$. If all β_j's are zero then both the linear and the quadratic factorial effects become the general mean denoted by μ. We define the factorial effects of $2^{m_a} \times 3^{m_b}$ factorial experiment as follows:

Linear in B :

$$
\begin{aligned}
&2^{m_a-\delta_a} 3^{m_b-\delta_b} A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_{m_a}^{\alpha_{m_a}} B_1^{\beta_1} B_2^{\beta_2} \ldots B_{m_b}^{\beta_{m_b}} \\
&= [\{a_1x_1 + \ldots + a_{m_a}x_{m_a} = 1\} \\
&+ (-1)^{\delta_a} \{a_1x_1 + \ldots + a_{m_a}x_{m_a} = 0\}\Big] \\
&\otimes [\{b_1y_1 + \ldots + b_{m_b}y_{m_b} = 2\} \\
&+ (1-\delta_b) \{b_1y_1 + \ldots + b_{m_b}y_{m_b} = 1\} \\
&+ (-1)^{\delta_b} \{b_1y_1 + \ldots + b_{m_b}y_{m_b} = 0\}\Big],
\end{aligned}
$$

Quadratic in B :

$$
\begin{aligned}
&2^{m_a-\delta_a} 3^{m_b-\delta_b} A_1^{\alpha_1} A_2^{\alpha_2}, \ldots, A_{m_a}^{\alpha_{m_a}} B_1^{\beta_1} B_2^{\beta_2} \ldots B_{m_b}^{\beta_{m_b}} \\
&= [\{a_1x_1 + \ldots + a_{m_a}x_{m_a} = 1\} \\
&+ (-1)^{\delta_a} \{a_1x_1 + \ldots + a_{m_a}x_{m_a} = 0\}\Big] \\
&\otimes [\{b_1y_1 + \ldots + b_{m_b}y_{m_b} = 2\} \\
&+ (-2) \{b_1y_1 + \ldots + b_{m_b}y_{m_b} = 1\} \\
&+ \{b_1y_1 + \ldots + b_{m_b}y_{m_b} = 0\}],
\end{aligned}
$$

where $\{a_1x_1 + \ldots + a_{m_a}x_{m_a} = c_a\} \otimes \{b_1y_1 + \ldots + b_{m_b}y_{m_b} = c_b\}$ represents the number of treatments of the form $(x_1,\ x_2, \ldots,\ x_{m_a},\ y_1,\ y_2, \ldots,\ y_{m_b})$, $x_i \in \{0, 1\}$, $y_j \in$

$\{0,1,2\}$, $i = 1(1)m_a$, $j = 1(1)m_b$ for a $2^{m_a} \times 3^{m_b}$ factorial design satisfying the conditions $a_1x_1 + \ldots + a_{m_a}x_{m_a} = c_a, c_a \in \{0,1\}$ under $mod(2)$ and $b_1y_1 + \ldots + b_{m_b}y_{m_b} = c_b, c_b \in \{0,1,2\}$ under $mod(3)$ simultaneously.

7.3 Illustration

For the illustration of the expression of factorial effects in terms of the treatment effects we consider four choices of (m_a, m_b) : $(1,1), (1,2), (2,1), (2,2)$. Table 7.1 considers the case $m_a = 1$ and $m_b = 1$.

Table 7.1: Treatment Effects and Factorial Effects for 2×3 Factorial Experiment

2 Factors: A_1 at 2 levels and B_1 at 3 levels	
Treatment/Treatment Effects	$(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$
Factorial Effects	μ, A_1, B_1, B_1^2, A_1B_1, $A_1B_1^2$

The factorial effects for 2×3 experiment are represented in terms of the treatment effects in the following:

$$6\mu = [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_1 = 2\} + \{y_1 = 1\} + \{y_1 = 0\}],$$

$$3A_1 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 = 2\} + \{y_1 = 1\} + \{y_1 = 0\}],$$

$$2B_1 = [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}],$$

$$2B_1^2 = [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}],$$

$$A_1B_1 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}],$$

$$A_1B_1^2 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}].$$

In matrix notation the above expressions can be written as:

$$\begin{pmatrix} 6\mu \\ 3A_1 \\ 2B_1 \\ 2B_1^2 \\ A_1B_1 \\ A_1B_1^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ -1 & 2 & -1 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} (0,0) \\ (0,1) \\ (0,2) \\ (1,0) \\ (1,1) \\ (1,2) \end{pmatrix}.$$

Table 7.2 considers the case $m_a = 1$ and $m_b = 2$.

Table 7.2: Treatment Effects and Factorial Effects for 2×3^2 Factorial Experiment

3 Factors: A_1at 2 levels, B_1 and B_2 at 3 levels	
Treatment Effects	$(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,1,0)$, $(0,1,1)$, $(0,1,2)$,
	$(0,2,0)$, $(0,2,1)$, $(0,2,2)$,$(1,0,0)$, $(1,0,1)$, $(1,0,2)$,
	$(0,1,0)$, $(0,1,1)$, $(0,1,2)$, $(1,2,0)$, $(1,2,1)$, $(1,2,2)$
Factorial Effects	μ, A_1, B_1, B_1^2, B_2, B_2^2, B_1B_2, $B_1^2B_2^2$, $B_1B_2^2$, $B_1^2B_2$, A_1B_1,
	$A_1B_1^2$, A_1B_2, $A_1B_2^2$, $A_1B_1B_2$, $A_1B_1^2B_2^2$, $A_1B_1B_2^2$, $A_1B_1^2B_2$

The factorial effects are represented in terms of the treatment effects in the

following:

$$
\begin{aligned}
18\mu &= [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_1 + y_2 = 2\} + \{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}]\,, \\
9A_1 &= [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 + y_2 = 2\} + \{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}]\,, \\
6B_1 &= [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}]\,, \\
6B_1^2 &= [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}]\,, \\
6B_2 &= [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_2 = 2\} - \{y_2 = 0\}]\,, \\
6B_2^2 &= [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_2 = 2\} - 2\{y_2 = 1\} + \{y_2 = 0\}]\,, \\
3A_1B_1 &= [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}]\,, \\
3A_1B_1^2 &= [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}]\,, \\
3A_1B_2 &= [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_2 = 2\} - \{y_2 = 0\}]\,, \\
3A_1B_2^2 &= [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_2 = 2\} - 2\{y_2 = 1\} + \{y_2 = 0\}]\,, \\
6B_1B_2 &= [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_1 + y_2 = 2\} - \{y_1 + y_2 = 0\}]\,, \\
6B_1^2B_2^2 &= [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_1 + y_2 = 2\} - 2\{y_1 + y_2 = 1\} \\
&\quad + \{y_1 + y_2 = 0\}]\,, \\
6B_1B_2^2 &= [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_1 + 2y_2 = 2\} - \{y_1 + 2y_2 = 0\}]\,, \\
6B_1^2B_2 &= [\{x_1 = 1\} + \{x_1 = 0\}] \otimes [\{y_1 + 2y_2 = 2\} - 2\{y_1 + 2y_2 = 1\}]\,, \\
3A_1B_1B_2 &= [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 + y_2 = 2\} - \{y_1 + y_2 = 0\}]\,, \\
3A_1B_1^2B_2^2 &= [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 + y_2 = 2\} - 2\{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}]\,, \\
3A_1B_1B_2^2 &= [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 + 2y_2 = 2\} - \{y_1 + 2y_2 = 0\}]\,, \\
3A_1B_1^2B_2 &= [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 + 2y_2 = 2\} - 2\{y_1 + 2y_2 = 1\} \\
&\quad + \{y_1 + 2y_2 = 0\}]\,.
\end{aligned}
$$

In matrix notation the above expressions can be written as:

$$
\begin{pmatrix}
18\mu \\
9A_1 \\
6B_1 \\
6B_1^2 \\
6B_2 \\
6B_2^2 \\
3A_1B_1 \\
3A_1B_1^2 \\
3A_1B_2 \\
3A_1B_2^2 \\
6B_1B_2 \\
6B_1^2B_2^2 \\
6B_1B_2^2 \\
6B_1^2B_2 \\
3A_1B_1B_2 \\
3A_1B_1^2B_2^2 \\
3A_1B_1B_2^2 \\
3A_1B_1^2B_2
\end{pmatrix} =
$$

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & -2 & -2 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 & 1 & 1 & 1 \\
-1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\
1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\
-1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 1 & 1 & 1 & -2 & -2 & -2 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\
-1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 \\
-1 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & 0 \\
1 & -2 & 1 & -2 & 1 & 1 & 1 & 1 & -2 & 1 & -2 & 1 & -2 & 1 & 1 & 1 & 1 & -2 \\
-1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \\
1 & 1 & -2 & -2 & 1 & 1 & 1 & -2 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & -2 & 1 \\
1 & 0 & -1 & 0 & -1 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & 0 \\
-1 & 2 & -1 & 2 & -1 & -1 & -1 & -1 & 2 & 1 & -2 & 1 & -2 & 1 & 1 & 1 & 1 & -2 \\
1 & -1 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \\
-1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & -2 & 1
\end{pmatrix}
\begin{pmatrix}
(0,0,0) \\
(0,0,1) \\
(0,0,2) \\
(0,1,0) \\
(0,1,1) \\
(0,1,2) \\
(0,2,0) \\
(0,2,1) \\
(0,2,2) \\
(1,0,0) \\
(1,0,1) \\
(1,0,2) \\
(1,1,0) \\
(1,1,1) \\
(1,1,2) \\
(1,2,0) \\
(1,2,1) \\
(1,2,2)
\end{pmatrix}.
$$

Table 7.3 considers the case $m_a = 2$ and $m_b = 1$.

Table 7.3: Treatment Effects and Factorial Effects for $2^2 \times 3$ Factorial Experiment

3 Factors: A_1 and A_2 at 2 levels, B_1at 3 levels	
Treatment/Treatment Effects	$(0,0,0)$, $(0,1,0)$, $(1,0,0)$, $(1,1,0)$, $(0,0,1)$, $(0,1,1)$,
	$(1,0,1)$, $(1,1,1)$, $(0,0,2)$, $(0,1,2)$, $(1,0,2)$, $(1,1,2)$
Factorial Effects	μ, A_1, A_2, A_1A_2, B_1, B_1^2, A_1B_1, $A_1B_1^2$,
	A_2B_1, $A_2B_1^2$, $A_1A_2B_1$, $A_1A_2B_1^2$

The factorial effects are represented in terms of the treatment effects in the following:

$$12\mu = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_1 = 2\} + \{y_1 = 1\} + \{y_1 = 0\}] ,$$

$$6A_1 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 = 2\} + \{y_1 = 1\} + \{y_1 = 0\}] ,$$

$$6A_2 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_1 = 2\} + \{y_1 = 1\} + \{y_1 = 0\}] ,$$

$$6A_1A_2 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_1 = 2\} + \{y_1 = 1\} + \{y_1 = 0\}] ,$$

$$4B_1 = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}] ,$$

$$4B_1^2 = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}] ,$$

$$2A_1B_1 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}] ,$$

$$2A_1B_1^2 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}] ,$$

$$2A_2B_1 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}] ,$$

$$2A_2B_1^2 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}] ,$$

$$2A_1A_2B_1 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}] ,$$

$$2A_1A_2B_1^2 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}] .$$

In matrix notation the above expressions can be written as:

$$\begin{pmatrix} 12\mu \\ 6A_1 \\ 6A_2 \\ 6A_1A_2 \\ 4B_1 \\ 4B_1^2 \\ 2A_1B_1 \\ 2A_1B_1^2 \\ 2A_2B_1 \\ 2A_2B_1^2 \\ 2A_1A_2B_1 \\ 2A_1A_2B_1^2 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 2 & 2 & -2 & -2 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 2 & -2 & 2 & -2 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -2 & 2 & 2 & -2 & 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} (0,0,0) \\ (0,1,0) \\ (1,0,0) \\ (1,1,0) \\ (0,0,1) \\ (0,1,1) \\ (1,0,1) \\ (1,1,1) \\ (0,0,2) \\ (0,1,2) \\ (1,0,2) \\ (1,1,2) \end{pmatrix}$$

Table 7.4 considers the case $m_a = 2$ and $m_b = 2$.

Table 7.4: Treatment Effects and Factorial Effects for $2^2 \times 3^2$ Factorial Experiment

4 Factors: A_1 and A_2 at 2 levels, B_1 and B_2 at 3 levels	
Treatment Effects	$(0,0,0,0)$, $(0,1,0,0)$, $(1,0,0,0)$, $(1,1,0,0)$, $(0,0,1,0)$, $(0,1,1,0)$,
	$(1,0,1,0)$, $(1,1,1,0)$, $(0,0,2,0)$, $(0,1,2,0)$, $(1,0,2,0)$, $(1,1,2,0)$,
	$(0,0,0,1)$, $(0,1,0,1)$, $(1,0,0,1)$, $(1,1,0,1)$, $(0,0,1,1)$, $(0,1,1,1)$,
	$(1,0,1,1)$, $(1,1,1,1)$, $(0,0,2,1)$, $(0,1,2,1)$, $(1,0,2,1)$, $(1,1,2,1)$,
	$(0,0,0,2)$, $(0,1,0,2)$, $(1,0,0,2)$, $(1,1,0,2)$, $(0,0,1,2)$, $(0,1,1,2)$,
	$(1,0,1,2)$, $(1,1,1,2)$, $(0,0,2,2)$, $(0,1,2,2)$, $(1,0,2,2)$, $(1,1,2,2)$
Factorial Effects	μ, A_1, A_2, A_1A_2, B_1, B_1^2, B_2, B_2^2, B_1B_2, $B_1^2B_2^2$, $B_1B_2^2$, $B_1^2B_2$, A_1B_1,
	$A_1B_1^2$, A_2B_1, $A_2B_1^2$, A_1B_2, $A_1B_2^2$, A_2B_2, $A_2B_2^2$, $A_1A_2B_1$, $A_1A_2B_1^2$
	$A_1A_2B_2$, $A_1A_2B_2^2$, $A_1B_1B_2$, $A_1B_1^2B_2^2$, $A_1B_1B_2^2$, $A_1B_1^2B_2$, $A_2B_1B_2$,
	$A_2B_1^2B_2^2$, $A_2B_1B_2^2$, $A_2B_1^2B_2$, $A_1A_2B_1B_2$, $A_1A_2B_1^2B_2^2$, $A_1A_2B_1B_2^2$, $A_1A_2B_1^2B_2$

The factorial effects are represented in terms of the treatment effects in the following:

$$36\mu = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} + \{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}],$$

$$18A_1 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 + y_2 = 2\} + \{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}],$$

$$18A_2 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} + \{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}],$$

$$18A_1A_2 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} + \{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}],$$

$$12B_1 = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}],$$

$$12B_1^2 = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}],$$

$$12B_2 = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_2 = 2\} - \{y_2 = 0\}],$$

$$12B_2^2 = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_2 = 2\} - 2\{y_2 = 1\} + \{y_2 = 0\}],$$

$$6A_1B_1 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}],$$

$$6A_1B_1^2 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}],$$

$$6A_2B_1 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}],$$

$$6A_2B_1^2 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}],$$

$$6A_1B_2 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_2 = 2\} - \{y_2 = 0\}],$$

$$6A_1B_2^2 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_2 = 2\} - 2\{y_2 = 1\} + \{y_2 = 0\}],$$

$$6A_2B_2 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_2 = 2\} - \{y_2 = 0\}],$$

$$6A_2B_2^2 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_2 = 2\} - 2\{y_2 = 1\} + \{y_2 = 0\}],$$

$$6A_1A_2B_1 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_1 = 2\} - \{y_1 = 0\}],$$

$$6A_1A_2B_1^2 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_1 = 2\} - 2\{y_1 = 1\} + \{y_1 = 0\}],$$

$$6A_1A_2B_2 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_2 = 2\} - \{y_2 = 0\}],$$

$$6A_1A_2B_2^2 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_2 = 2\} - 2\{y_2 = 1\} + \{y_2 = 0\}],$$

$$12B_1B_2 = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} - \{y_1 + y_2 = 0\}],$$

$$12B_1^2B_2^2 = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} - 2\{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}],$$

$$12B_1B_2^2 = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_1 + 2y_2 = 2\} - \{y_1 + 2y_2 = 0\}],$$

$$12B_1^2B_2 = [\{x_1 + x_2 = 1\} + \{x_1 + x_2 = 0\}] \otimes [\{y_1 + 2y_2 = 2\} - 2\{y_1 + 2y_2 = 1\} + \{y_1 + 2y_2 = 0\}],$$

$$6A_1B_1B_2 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 + y_2 = 2\} - \{y_1 + y_2 = 0\}],$$

$$6A_1B_1^2B_2^2 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 + y_2 = 2\} - 2\{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}],$$

$$6A_1B_1B_2^2 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 + 2y_2 = 2\} - \{y_1 + 2y_2 = 0\}],$$

$$6A_1B_1^2B_2 = [\{x_1 = 1\} - \{x_1 = 0\}] \otimes [\{y_1 + 2y_2 = 2\} - 2\{y_1 + 2y_2 = 1\} + \{y_1 + 2y_2 = 0\}],$$

$$6A_2B_1B_2 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} - \{y_1 + y_2 = 0\}],$$

$$6A_2B_1^2B_2^2 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} - 2\{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}],$$

$$6A_2B_1B_2^2 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_1 + 2y_2 = 2\} - \{y_1 + 2y_2 = 0\}],$$

$$6A_2B_1^2B_2 = [\{x_2 = 1\} - \{x_2 = 0\}] \otimes [\{y_1 + 2y_2 = 2\} - 2\{y_1 + 2y_2 = 1\} + \{y_1 + 2y_2 = 0\}],$$

$$6A_1A_2B_1B_2 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} - \{y_1 + y_2 = 0\}],$$

$$6A_1A_2B_1^2B_2^2 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} - 2\{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}],$$

$$6A_1A_2B_1B_2^2 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} - \{y_1 + y_2 = 0\}],$$

$$6A_1A_2B_1^2B_2 = [\{x_1 + x_2 = 1\} - \{x_1 + x_2 = 0\}] \otimes [\{y_1 + y_2 = 2\} - 2\{y_1 + y_2 = 1\} + \{y_1 + y_2 = 0\}].$$

The above expressions can be written in matrix notation like the previous cases.

7.4 2×3 Factorial Experiment with Structured Replication

We consider mixed factorial experiment with one factor A_1 at two levels and another factor B_1 at three levels. All possible treatments of this experiment are given in Table 7.5.

Table 7.5: 2×3 Full Factorial Design

0	0	0	1	1	1
0	1	2	0	1	2

In this section we consider various structured replications of the runs of the 2×3 design. Without replication we find $\frac{Var\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} = 0.25$ and $\frac{Var\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} = 0.0833$. Thus we see that the 2×3 full factorial design is not CV. We are interested in finding replicated 2×3 designs that would satisfy the CV property in this mixed factorial set up. For 2×3 factorial experiment there are one general mean (μ),

three main effects $(A_1,\ B_1,\ B_1^2)$ and two 2-factor interaction effects $(A_1B_1,\ A_1B_1^2)$. Here we consider two models each with the general mean, main effects and one 2-factor interaction effect. Our objective is to construct designs for which the variances of the two 2-factor interaction estimators are equal. The uth model is given below:

$$M_u:\ E(\boldsymbol{y}) = \boldsymbol{j}\boldsymbol{\mu} + \boldsymbol{X}_1\beta_1 + \boldsymbol{X}_{2u}\boldsymbol{\beta}_{2u},\ Var(\boldsymbol{y}) = \sigma^2\boldsymbol{I},\ u = A_1B_1,\ A_1B_1^2, \quad (7.4)$$

where$\boldsymbol{\beta}_1$ is the vector corresponding to the main effects and $\boldsymbol{\beta}_{2u}$ corresponds to the uth $2-$factor interaction effect. We construct the design by replicating all six treatments as presented in Table 7.6.

Table 7.6: One Kind of Stuctured Replication

A/B	0	1	2
0	r_1	r_2	r_3
1	r_1	r_2	r_3

The replications r_1, r_2 and r_3 in Table 7.6 are positive integers and we see that those runs where B_1 is fixed are replicated equal number of times. The total number of runs in the design is $n = 2(r_1 + r_2 + r_3)$. The design matrix of the uth model can be expressed as $\boldsymbol{X}^{(u)} = \left[\boldsymbol{j}_n \vdots \boldsymbol{X}_1 \vdots \boldsymbol{X}_{2u}\right]'$. The order of $\boldsymbol{X}^{(u)}$ is $(n \times 5)$.

The two design matrices are as follows:

$$\boldsymbol{X}^{(A_1B_1)} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & -2 & 0 \\ 1 & 1 & 0 & -2 & 0 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \rightarrow \text{replicated } r_1 \text{ times} \\ \rightarrow \text{replicated } r_1 \text{ times} \\ \rightarrow \text{replicated } r_2 \text{ times} \\ \rightarrow \text{replicated } r_2 \text{ times} \\ \rightarrow \text{replicated } r_3 \text{ times} \\ \rightarrow \text{replicated } r_3 \text{ times} \end{matrix},$$

$$\boldsymbol{X}^{(A_1B_1^2)} = \begin{bmatrix} 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & -2 & 2 \\ 1 & 1 & 0 & -2 & -2 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \rightarrow \text{replicated } r_1 \text{ times} \\ \rightarrow \text{replicated } r_1 \text{ times} \\ \rightarrow \text{replicated } r_2 \text{ times} \\ \rightarrow \text{replicated } r_2 \text{ times} \\ \rightarrow \text{replicated } r_3 \text{ times} \\ \rightarrow \text{replicated } r_3 \text{ times} \end{matrix},$$

where the first two rows of the matrices are replicated r_1 times each, the third and fourth rows are replicated r_2 times each and the fifth and sixth rows are replicated r_3 times each. To obtain the variance of the two-factor interaction estimators we need to obtain $\left(\boldsymbol{X}'^{(u)\prime}\boldsymbol{X}^{(u)}\right)^{-1}$, $u = A_1B_1,\ A_1B_1^2$. To present the matrices

$\left(\boldsymbol{X}^{(A_1B_1)\prime}\boldsymbol{X}^{(A_1B_1)}\right)$ and $\left(\boldsymbol{X}^{(A_1B_1^2)\prime}\boldsymbol{X}^{(A_1B_1^2)}\right)$ define the following:

$$\begin{aligned}
p &= 2\left(r_1+r_2+r_3\right), \\
q &= 2\left(r_3-r_1\right), \\
r &= 2\left(r_1+r_3\right)-4r_2, \\
s &= 2\left(r_1+r_3\right)+8r_2, \\
u &= 2\left(r_1+r_3\right).
\end{aligned}$$

Hence we have

$$\boldsymbol{X}^{(A_1B_1)\prime}\boldsymbol{X}^{(A_1B_1)} = \begin{bmatrix} p & 0 & q & r & 0 \\ 0 & p & 0 & 0 & q \\ q & 0 & u & q & 0 \\ r & 0 & q & s & 0 \\ 0 & q & 0 & 0 & u \end{bmatrix},$$

$$\boldsymbol{X}^{(A_1B_1^2)\prime}\boldsymbol{X}^{(A_1B_1^2)} = \begin{bmatrix} p & 0 & q & r & 0 \\ 0 & p & 0 & 0 & r \\ q & 0 & u & q & 0 \\ r & 0 & q & s & 0 \\ 0 & q & 0 & 0 & s \end{bmatrix}.$$

Now, the variance of the uth $2-$factor interaction estimator is proportional to the last diagonal element of the variance-covariance matrix $\left(\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}\right)^{-1}$,$u = A_1B_1,\ A_1B_1^2$. We have

$$\begin{aligned}
\frac{Var\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} &= \frac{\left(r_1+r_2+r_3\right)}{2\left(r_1r_2+r_2r_3+4r_1r_3\right)}, \\
\frac{Var\left(\hat{\beta}_{2A_1B_1^2}\right)}{\sigma^2} &= \frac{\left(r_1+r_2+r_3\right)}{18r_2\left(r_1+r_3\right)}.
\end{aligned} \tag{7.5}$$

We want to find condition on $r_1,\ r_2$ and r_3 that will give CV designs. The two variances presented in (7.5) will be equal iff the following condition holds:

$$\begin{aligned} \frac{(r_1+r_2+r_3)}{2\left(r_1r_2+r_2r_3+4r_1r_3\right)} &= \frac{(r_1+r_2+r_3)}{18r_2\left(r_1+r_3\right)} \\ \Leftrightarrow 2r_2 &= \frac{r_1r_3}{(r_1+r_3)}. \end{aligned} \tag{7.6}$$

Any $r_1,\ r_2$ and r_3 satisfying (7.6) gives CV design. We obtain solutions of (7.6). The r_2 is a positive integer equal to k (say), which implies $\frac{r_1r_3}{(r_1+r_3)} = 2k$ from (7.6). However taking $r_3 = \alpha r_1$, where $\alpha\,(>0)$ is any number satisfying $\alpha = \frac{r_3}{r_1}$, we get

$$\begin{aligned} \frac{r_1r_3}{(r_1+r_3)} &= \frac{r_1\alpha}{(1+\alpha)} = 2k \\ \Rightarrow r_1 &= 2k\frac{(1+\alpha)}{\alpha}. \end{aligned}$$

Since r_1 is a positive integer, $2k\frac{(1+\alpha)}{\alpha}$ should also be a positive integer. The Table 7.7 shows some possible solutions of (7.6) along with the values of α and the number of runs in each case.

Table 7.7: Possible Replications to have Equal Variance

α	r_1	r_2	r_3	$n = 2\left(r_1+r_2+r_3\right)$
1	4	1	4	18
1	8	2	8	36
1	12	3	12	54
1	16	4	16	72
1	20	5	20	90
2	3	1	6	20
2	6	2	12	40
2	9	3	18	60
4	5	2	20	54
1/4	20	2	5	54
1/2	6	1	3	20
1/2	12	2	6	40
1/2	18	3	9	60
2/3	15	3	10	56
3/2	10	3	15	56

Now if we consider equal replication of all the runs, i.e, $r_1 = r_2 = r_3 = r$, then

$$\begin{aligned} \frac{Var\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} &= \frac{1}{4r}, \\ \frac{Var\left(\hat{\beta}_{2A_1B_1^2}\right)}{\sigma^2} &= \frac{1}{12r}. \end{aligned} \tag{7.7}$$

(7.7) implies that equal replications cannot make the two variances equal. The equal replication case is presented later in detail. The design with smallest number of runs satisfying (7.6) has $r_1 = 4,\ r_2 = 1$ and $r_3 = 4$ and is presented in Table 7.8.

Table 7.8: Replicated CV Design with $n = 18$

0	0	0	0	1	1	1	1	0	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	1	1	2	2	2	2	2	2	2	2

The total number of runs in this design is $n = 18$ and we get $\frac{Var\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} = \frac{Var\left(\hat{\beta}_{2A_1B_1^2}\right)}{\sigma^2} = 0.0625$. Next we compare the variances of the interaction estimators already obtained from the separate models with the ones obtained from full model and check if the condition of CV on the replications remain same in both the cases. We consider the full model of the 2×3 experiment as follows:

$$M:\ E(\boldsymbol{y}) = \boldsymbol{j}_n\boldsymbol{\mu} + \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2,\ Var(\boldsymbol{y}) = \sigma^2\boldsymbol{I}, \tag{7.8}$$

where $\boldsymbol{\beta}_1$ is the vector corresponding to the three main effects and $\boldsymbol{\beta}_2$ is the vector corresponding to the two 2−factor interaction effects. From the model in (7.8) we write $\boldsymbol{X} = \left[\boldsymbol{j}_n \vdots \boldsymbol{X}_1 \vdots \boldsymbol{X}_2\right]$. Using the replications given in the Table 7.6 we want to obtain $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ to get the variances of the interaction estimators. We present

the matrix $(\boldsymbol{X}'\boldsymbol{X})$ in the following:

$$\boldsymbol{X}'\boldsymbol{X} = \begin{bmatrix} 2(r_1+r_2+r_3) & 0 & 2(r_3-r_1) & 2(r_1+r_3)-4r_2 & 0 & 0 \\ 0 & 2(r_1+r_2+r_3) & 0 & 0 & 2(r_3-r_1) & 2(r_1+r_3)-4r_2 \\ 2(r_3-r_1) & 0 & 2(r_1+r_3) & 2(r_3-r_1) & 0 & 0 \\ 2(r_1+r_3)-4r_2 & 0 & 2(r_3-r_1) & 2(r_1+r_3)+8r_2 & 0 & 0 \\ 0 & 2(r_3-r_1) & 0 & 0 & 2(r_1+r_3) & 2(r_3-r_1) \\ 0 & 2(r_1+r_3)-4r_2 & 0 & 0 & 2(r_3-r_1) & 2(r_1+r_3)+8r_2 \end{bmatrix}.$$

The variance-covariance matrix of the estimators of the two interaction effects is the last 2×2 block diagonal matrix of $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ which is $\begin{bmatrix} \frac{r_1r_2+r_2r_3}{8r_1r_2r_3} & 0 \\ 0 & \frac{r_1r_2+r_2r_3+4r_1r_3}{24r_1r_2r_3} \end{bmatrix}$.

Thus we have

$$\begin{aligned} \frac{Var^{Full}\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} &= \frac{r_1r_2+r_2r_3}{8r_1r_2r_3}, \\ \frac{Var^{Full}\left(\hat{\beta}_{2A_1B_1^2}\right)}{\sigma^2} &= \frac{r_1r_2+r_2r_3+4r_1r_3}{24r_1r_2r_3}. \end{aligned} \tag{7.9}$$

From (7.5) and (7.9) we see that the variances of the interaction effects estimators obtained from the full model are different from those obtained from the separate models. But if we equate the two full model variances we end up getting the same relation among the $r_i's$ which is $2r_2 = \frac{r_1r_3}{(r_1+r_3)}$ that makes the replicated design CV. So the CV condition remains same in both the cases. Next we consider the structured replication presented in Table 7.9.

Table 7.9: Another Kind of Structured Replication

A/B	0	1	2
0	r_1	r_1	r_1
1	r_2	r_2	r_2

From this table we see that the replications for the runs are same where level of A_1 is fixed. The total number of runs is $n = 3(r_1+r_2)$. Again our objective

is to obtain condition of CV for this particular replication considering both the separate models and the full model. From the separate models in (7.4) we get the following matrices:

$$\boldsymbol{X}^{(A_1B_1)\prime}\boldsymbol{X}^{(A_1B_1)} = \begin{bmatrix} 3\left(r_1+r_2\right) & 0 & 0 & 0 & 0 \\ 0 & 3\left(r_1+r_2\right) & 0 & 0 & 0 \\ 0 & 0 & 2\left(r_1+r_2\right) & 0 & 2\left(r_2-r_1\right) \\ 0 & 0 & 0 & 6\left(r_1+r_2\right) & 0 \\ 0 & 0 & 2\left(r_2-r_1\right) & 0 & 2\left(r_1+r_2\right) \end{bmatrix},$$

$$\boldsymbol{X}^{(A_1B_1^2)\prime}\boldsymbol{X}^{(A_1B_1^2)} = \begin{bmatrix} 3\left(r_1+r_2\right) & 0 & 0 & 0 & 0 \\ 0 & 3\left(r_1+r_2\right) & 0 & 0 & 0 \\ 0 & 0 & 2\left(r_1+r_2\right) & 0 & 0 \\ 0 & 0 & 0 & 6\left(r_1+r_2\right) & 6\left(r_2-r_1\right) \\ 0 & 0 & 0 & 6\left(r_2-r_1\right) & 6\left(r_1+r_2\right) \end{bmatrix}.$$

The variances of the interaction estimators are proportional to the last diagonal element of the inverse of the matrices presented above. These are given below:

$$\begin{aligned} \frac{Var\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} &= \frac{(r_1+r_2)}{8r_1r_2}, \\ \frac{Var\left(\hat{\beta}_{2A_1B_1^2}\right)}{\sigma^2} &= \frac{(r_1+r_2)}{24r_1r_2}. \end{aligned} \tag{7.10}$$

From the variance expressions in (7.10) we see that they can never be equal for any values of $r_1\,(>0)$ and $r_2\,(>0)$. We want to compare these CV expressions with the ones from the full model. Considering the full model M for the replications

given in Table 7.9 we get the following matrix:

$$\boldsymbol{X}'\boldsymbol{X} = \begin{bmatrix} 3(r_1+r_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 3(r_1+r_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 2(r_1+r_2) & 0 & 2(r_2-r_1) & 0 \\ 0 & 0 & 0 & 6(r_1+r_2) & 0 & 6(r_2-r_1) \\ 0 & 0 & 2(r_2-r_1) & 0 & 2(r_1+r_2) & 0 \\ 0 & 0 & 0 & 6(r_2-r_1) & 0 & 6(r_1+r_2) \end{bmatrix}.$$

The variance-covariance matrix of the estimators of the two interaction effects is proportional to the last 2×2 block diagonal matrix of $(\boldsymbol{X}'\boldsymbol{X})^{-1}$which is $\begin{bmatrix} \frac{r_1+r_2}{8r_1r_2} & 0 \\ 0 & \frac{r_1+r_2}{24r_1r_2} \end{bmatrix}$.

Hence we have

$$\begin{aligned} \frac{Var^{Full}\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} &= \frac{r_1+r_2}{8r_1r_2}, \\ \frac{Var^{Full}\left(\hat{\beta}_{2A_1B_1^2}\right)}{\sigma^2} &= \frac{r_1+r_2}{24r_1r_2}. \end{aligned} \tag{7.11}$$

From (7.10) and (7.11) we see that the variance expressions are exactly identical. Also from (7.11) we see that for the particular replication presented in Table 7.9 the replicated design can never be CV. Next we consider the equal replication of all the six runs as presented in Table 7.10.

Table 7.10: Equal Replication for All Treatments

A/B	0	1	2
0	r	r	r
1	r	r	r

Considering the separate model we already had $\frac{Var\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} = \frac{1}{4r}$ and $\frac{Var\left(\hat{\beta}_{2A_1B_1^2}\right)}{\sigma^2} =$

$\frac{1}{12r}$ in (7.7). Considering the full model we have

$$\boldsymbol{X}'\boldsymbol{X} = \begin{bmatrix} 6r & 0 & 0 & 0 & 0 & 0 \\ 0 & 6r & 0 & 0 & 0 & 0 \\ 0 & 0 & 4r & 0 & 0 & 0 \\ 0 & 0 & 0 & 12r & 0 & 0 \\ 0 & 0 & 0 & 0 & 4r & 0 \\ 0 & 0 & 0 & 0 & 0 & 12r \end{bmatrix}.$$

From $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ we have $\frac{Var^{Full}\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} = \frac{1}{4r}$ and $\frac{Var^{Full}\left(\hat{\beta}_{2A_1B_1^2}\right)}{\sigma^2} = \frac{1}{12r}$ which exactly coincide with the variances obtained from the separate model. So for equal replication of the six runs the replicated design can never be CV.

Again we consider the replication structure given in Table 7.6. Without imposing any condition on the replication of the runs we consider r_1, r_2, $r_3 \in [1, 5]$ and search for the designs which may not be CV but the difference between the variance of the 2−factor interaction estimators is as minimum as possible. The different replications along with the variances of the 2−factor interaction estimators are given in Table 7.11.

Table 7.11: Replicated Designs

Difference	r_1	r_2	r_3	n	$\frac{Var\left(\hat{AB}\right)}{\sigma^2}$	$\frac{Var\left(\hat{AB^2}\right)}{\sigma^2}$
0	4	1	4	18	0.0625	0.0625
$(0, 0.004)$	3	1	5	18	0.0662	0.0625
	5	1	3	18	0.0662	0.0625
$(0.004, 0.006)$	4	1	5	20	0.0562	0.0617
	5	1	4	20	0.0562	0.0617
$(0.006, 0.01)$	3	1	4	16	0.0727	0.0635
	4	1	3	16	0.0727	0.0635

From Table 7.11 we see that the replication in the first row satisfies the condition in (7.6) and hence the variances are equal. For all other replications the condition is not satisfied and hence they do not give equal variance but the difference of the variances are very small.

7.5 2×3 Factorial Experiment with General Replication

In the previous section we considered structured replication of 2×3 full factorial design. In this section we consider general replication for the runs of the design without any condition on the levels of any factor. The replications are given in Table 7.12.

Table 7.12: General Replication

A/B	0	1	2
0	r_1	r_2	r_3
1	r_4	r_5	r_6

We want to find condition for the replicated 2×3 design to be CV for this general replication. Considering the models in (7.4) we get the following two

design matrices respectively:

$$\boldsymbol{X}^{(A_1B_1)} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 0 & -2 & 0 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \rightarrow \text{replicated } r_1 \text{ times} \\ \rightarrow \text{replicated } r_2 \text{ times} \\ \rightarrow \text{replicated } r_3 \text{ times} \\ \rightarrow \text{replicated } r_4 \text{ times} \\ \rightarrow \text{replicated } r_5 \text{ times} \\ \rightarrow \text{replicated } r_6 \text{ times} \end{matrix},$$

$$\boldsymbol{X}^{(A_1B_1^2)} = \begin{bmatrix} 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & -1 & 0 & -2 & -1 \\ 1 & 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 & -2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \rightarrow \text{replicated } r_1 \text{ times} \\ \rightarrow \text{replicated } r_2 \text{ times} \\ \rightarrow \text{replicated } r_3 \text{ times,} \\ \rightarrow \text{replicated } r_4 \text{ times} \\ \rightarrow \text{replicated } r_5 \text{ times} \\ \rightarrow \text{replicated } r_6 \text{ times} \end{matrix}$$

where the ith row of each of the two matrices is replicated r_i times, $i = 1(1)6$. To obtain the variance of the interaction estimators we need to compute $\left(\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}\right)^{-1}$.

To present $\left(\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}\right)$, $u = A_1B_1,\ A_1B_1^2$. we define the following:

$$
\begin{aligned}
a &= (r_1 + r_2 + r_3 + r_4 + r_5 + r_6)\,, \\
b &= (r_4 + r_5 + r_6 - r_1 - r_2 - r_3)\,, \\
c &= (r_3 + r_6 - r_1 - r_4)\,, \\
d &= \{(r_1 + r_3 + r_4 + r_6) - 2r_2 - 2r_5\}\,, \\
e &= (r_1 + r_6 - r_3 - r_4)\,, \\
f &= \{(r_4 + r_6 - r_1 - r_3) + 2r_2 - 2r_5\}\,, \\
g &= (r_1 + r_3 + r_4 + r_6)\,, \\
h &= (r_4 + r_6 - r_1 - r_3)\,, \\
k &= (r_1 + 4r_2 + r_3 + r_4 + 4r_5 + r_6)\,, \\
l &= \{(r_4 + r_6 - r_1 - r_3) + 4r_5 - 4r_2\}\,.
\end{aligned}
$$

Thus $\left(\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}\right)$, $u = A_1B_1,\ A_1B_1^2$ becomes

$$
\boldsymbol{X}^{(A_1B_1)\prime}\boldsymbol{X}^{(A_1B_1)} = \begin{bmatrix} a & b & c & d & e \\ b & a & e & f & c \\ c & e & g & c & h \\ d & f & c & k & e \\ e & c & h & e & g \end{bmatrix},
$$

$$\boldsymbol{X}^{(A_1B_1^2)\prime}\boldsymbol{X}^{(A_1B_1^2)} = \begin{bmatrix} a & b & c & d & f \\ b & a & e & f & d \\ c & e & g & c & e \\ d & f & c & k & l \\ f & d & e & l & k \end{bmatrix}.$$

The variances are obtained from the last diagonal element of $\left(\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}\right)^{-1}$ which are presented below:

$$\begin{aligned}\frac{Var\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} &= \Bigg[\frac{r_1r_2r_3\left(r_4+r_5+r_6\right)+r_1r_3r_5\left(r_4+r_6\right)}{4\left\{\left(r_3r_4r_5+r_3r_5r_6+r_4r_5r_6\right)+r_2r_3r_4r_5r_6\right\}+192\left(r_1r_2r_3r_4r_5+r_1r_3r_4r_5r_6\right)} \\ &\quad \frac{+r_1r_2r_6\left(r_4+r_5\right)+r_1r_4r_5r_6}{4\left\{\left(r_3r_4r_5+r_3r_5r_6+r_4r_5r_6\right)+r_2r_3r_4r_5r_6\right\}+192\left(r_1r_2r_3r_4r_5+r_1r_3r_4r_5r_6\right)} \\ &\quad \frac{+r_2r_3r_4\left(r_5+r_6\right)+r_4r_5r_6\left(r_2+r_3\right)}{4\left\{\left(r_3r_4r_5+r_3r_5r_6+r_4r_5r_6\right)+r_2r_3r_4r_5r_6\right\}+192\left(r_1r_2r_3r_4r_5+r_1r_3r_4r_5r_6\right)}\Bigg],\end{aligned}$$

$$\begin{aligned}\frac{Var\left(\hat{\beta}_{2A_1B_1^2}\right)}{\sigma^2} &= \Bigg[\frac{\left\{r_4\left(r_3r_5r_6+r_2\left(r_5r_6+r_3\left(r_5+r_6\right)\right)\right)\right\}+r_1\left[r_5\left(r_4r_6+r_3\left(r_4+r_6\right)\right)\right]}{36r_2r_5\left\{r_3r_4r_6+r_1r_4r_6+r_3\left(r_4+r_6\right)\right\}} \\ &\quad \frac{+r_2\left\{\left(r_4+r_5\right)r_6+r_3\left(r_4+r_5+r_6\right)\right\}}{36r_2r_5\left\{r_3r_4r_6+r_1r_4r_6+r_3\left(r_4+r_6\right)\right\}}\Bigg].\end{aligned}$$

The design with general replication of the six runs is CV iff

$$\frac{Var\left(\hat{\beta}_{2A_1B_1}\right)}{\sigma^2} = \frac{Var\left(\hat{\beta}_{2A_1B_1^2}\right)}{\sigma^2}. \tag{7.12}$$

But (7.12) does not simplify to a descent expression and thus finding the values of r_i's satisfying (7.12) would be a tedious task. Hence computationally it is challenging to obtain the condition of CV for the general replication of the 2×3 design. Instead of solving for r_i's we consider $r_1, \ldots, r_6 \in [1, 5]$. In this range we search for the replicated 2×3 designs which may not be CV but gives very small difference among the variances of the two $2-$factor interaction estimators. The replicated designs with the variances are given in Table 7.13 with the smallest possible difference among the variances which is less than 0.004.

Table 7.13: 2×3 Full Factorial Design with General Replication in the range $[1, 5]$ with Variance Difference< 0.004

r_1	r_2	r_3	r_4	r_5	r_6	V_1	V_2
3	3	1	1	5	5	0.0662	0.0625
3	4	1	1	4	5	0.0645	0.0626
3	4	1	1	5	4	0.0645	0.0626
3	4	1	1	5	5	0.0612	0.0621
3	5	1	1	3	5	0.0667	0.0630
3	5	1	1	4	4	0.0646	0.0627
3	5	1	1	4	5	0.0614	0.0623
3	5	1	1	5	3	0.0667	0.0630
3	5	1	1	5	4	0.0614	0.0623
3	5	1	1	5	5	0.0582	0.0619
4	3	1	1	4	5	0.0645	0.0626
4	3	1	1	5	4	0.0645	0.0626
4	3	1	1	5	5	0.0612	0.0621
4	4	1	1	3	5	0.0646	0.0627
4	4	1	1	4	4	0.0625	0.0625
4	4	1	1	4	5	0.0594	0.0621
4	4	1	1	5	3	0.0646	0.0627
4	4	1	1	5	4	0.0594	0.0621
4	5	1	1	3	4	0.0645	0.0626
4	5	1	1	3	5	0.0614	0.0623
4	5	1	1	4	3	0.0645	0.0626

r_1	r_2	r_3	r_4	r_5	r_6	V_1	V_2
4	5	1	1	4	4	0.0594	0.0621
4	5	1	1	5	3	0.0614	0.0623
5	3	1	1	3	5	0.0667	0.0630
5	3	1	1	4	4	0.0646	0.0627
5	3	1	1	4	5	0.0614	0.0623
5	3	1	1	5	3	0.0667	0.0630
5	3	1	1	5	4	0.0614	0.0623
5	3	1	1	5	5	0.0582	0.0619
5	4	1	1	3	4	0.0645	0.0626
5	4	1	1	3	5	0.0614	0.0623
5	4	1	1	4	3	0.0645	0.0626
5	4	1	1	4	4	0.0594	0.0621
5	4	1	1	5	3	0.0614	0.0623
5	5	1	1	3	3	0.0662	0.0625
5	5	1	1	3	4	0.0612	0.0621
5	5	1	1	3	5	0.0582	0.0619
5	5	1	1	4	3	0.0612	0.0621
5	5	1	1	5	3	0.0582	0.0619
5	5	1	2	5	5	0.05	0.0472
5	5	2	1	5	5	0.05	0.0472

In table 7.13 by V_1 we denote $\frac{Var(\hat{\boldsymbol{\beta}}_{2A_1B_1})}{\sigma^2}$ and by V_2 we denote $\frac{Var(\hat{\boldsymbol{\beta}}_{2A_1B_1^2})}{\sigma^2}$. Also we replicate 5 out of 6 runs of the 2×3 design in the range $[1, 5]$ and obtain replicated designs which give small difference among the variances of the $2-$factor interaction estimators. We present the 2×3 designs with 5 replicated runs with smallest possible difference among the variances in the range $[1, 5]$ in Table 7.14.

Table 7.14: Replicated 2×3 Designs with 5 Distinct Runs

Delete	r_1	r_2	r_3	r_4	r_5	r_6	$\frac{Var(\hat{\boldsymbol{\beta}}_{2A_1B_1})}{\sigma^2}$	$\frac{Var(\hat{\boldsymbol{\beta}}_{2A_1B_1^2})}{\sigma^2}$	Difference
$0,0$		1	5	5	5	5	0.2	0.2222	0.1778
		2	5	5	5	5	0.2	0.2222	0.1778
		3	5	5	5	5	0.2	0.2222	0.1778
		4	5	5	5	5	0.2	0.2222	0.1778
		5	5	5	5	5	0.2	0.2222	0.1778
$0,1$	4		5	1	4	5	0.0563	0.0384	0.0179
$0,2$	5	5		5	5	1	0.2	0.2222	0.1778
	5	5		5	5	2	0.2	0.2222	0.1778
	5	5		5	5	3	0.2	0.2222	0.1778
	5	5		5	5	4	0.2	0.2222	0.1778
	5	5		5	5	5	0.2	0.2222	0.1778
$1,0$	1	5	5		5	5	0.2	0.2222	0.1778
	2	5	5		5	5	0.2	0.2222	0.1778
	3	5	5		5	5	0.2	0.2222	0.1778
	4	5	5		5	5	0.2	0.2222	0.1778
	5	5	5		5	5	0.2	0.2222	0.1778
$1,1$	5	4	1	5		4	0.0562	0.0384	0.0179
$1,2$	5	5	5	5	1		0.2	0.2222	0.1778
	5	5	5	5	2		0.2	0.2222	0.1778
	5	5	5	5	3		0.2	0.2222	0.1778
	5	5	5	5	4		0.2	0.2222	0.1778
	5	5	5	5	5		0.2	0.2222	0.1778

7.6 Other Mixed Designs

In this section we consider different mixed designs which do not satisfy the CV property but they possess a particular structure of the variance of their $2-$factor interaction estimators. For a general $2^{m_a} \times 3^{m_b}$ factorial experiment there are four different kinds of $2-$factor interactions which are presented in Table 7.15

Table 7.15: Different Types of 2−Factor Interactions

Type	Notation
Pure in $A's$	A_iA_j
Pure in $B's$	B_iB_j, $B_i^2B_j^2$, $B_iB_j^2$, $B_i^2B_j$
Mixed linear in $B's$	A_iB_j
Mixed quadratic in $B's$	$A_iB_j^2$

. We did computer search to obtain CV designs for $2^{m_a} \times 3^{m_b}$ factorial experiment for small values of m_a and m_b. We did not find any CV design with distinct runs. Searching for higher values of m_a and m_b was beyond the scope as computationally it is very challenging. Hence we start searching for designs whose 2−factor interaction estimators possess common variance within each group of interactions as presented in Table 7.15.

We consider mixed designs for $2^m \times 3$ and $2^m \times 3^3$ factorial experiments, $m \geq 2$. Consider the following CV design for 2^m factorial experiment with $(m+2)$ runs:

$$d_{4A} = \begin{bmatrix} \mathbf{0}' \\ \boldsymbol{j}'_m \\ \boldsymbol{J}_m - \boldsymbol{I}_m \end{bmatrix}.$$

Consider the following:

$$d_{4B} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Each row of d_{4A} is mixed with each row of d_{4B} to form the design for $2^m \times 3$ experiment. This design satisfies the CV property within each type of its 2−factor interaction. In Table 7.16 we give the variance of the estimators of different types of 2−factor interactions for $m = 2$, 3 and 4.

Table 7.16: Variance of 2−Factor Interaction Estimators for Different m for $2^m \times 3$

m	Interaction type	Variance
2	A_1A_2	0.0833
	A_iB	0.125
	A_iB^2	0.0417
3	A_1A_2	0.1667
	A_iB	0.1042
	A_iB^2	0.0347
4	A_1A_2	0.2917
	A_iB	0.0938
	A_iB^2	0.0313

Again consider the following:

$$d_{5B} = \begin{bmatrix} 2\boldsymbol{J}_3 - \boldsymbol{I}_3 \\ 2\boldsymbol{I}_3 \\ 2\boldsymbol{J}_3 - 2\boldsymbol{I}_3 \end{bmatrix}.$$

Now each row of d_{4A} is mixed with each row of d_{5B} to form the design for $2^m \times 3^3$ experiment. This design also satisfies the CV property within each type of its 2−factor interaction. In Table 7.17 we give the variance of the estimators of different types of 2−factor interactions for this design for different m.

Table 7.17: Variance of 2−Factor Interaction Estimators for Different m for $2^m \times 3^3$

m	Interaction type	Variance
2	A_1A_2	0.0278
	A_iB	0.0331
	A_iB^2	0.0313
	$B_1B_2,\ B_1^2B_2^2,\ B_1B_2^2,\ B_1^2B_2$	0.0833
3	A_1A_2	0.0556
	A_iB	0.0276
	A_iB^2	0.0260
	$B_1B_2,\ B_1^2B_2^2,\ B_1B_2^2,\ B_1^2B_2$	0.0667
4	A_1A_2	0.0972
	A_iB	0.0248
	A_iB^2	0.0234
	$B_1B_2,\ B_1^2B_2^2,\ B_1B_2^2,\ B_1^2B_2$	0.0556

From both Tables 7.16 and 7.17 we see that as the number of factors m gets large the difference between the variances of the mixed interaction linear in B and the mixed interaction quadratic in B is getting smaller.

Chapter 8

Replicated 3^3 CV Designs and Comparisons

8.1 Chapter Summary

In this chapter we present replicated CV designs for 3^3 factorial experiment for different n and make comparisons among few 3^3 CV designs w.r.t some optimality criteria. Here is what we present in each section:

- (Section 8.2): In this section we present replicated 3^3 designs which are CV for $n \geq 12$. In Chapter 2 we obtained CV designs for 3^3 factorial experiment through complete computer check for $n = 8,\ 9,\ 10$ and 11. We did not find any CV design beyond $n = 11$ with distinct runs for 3^3 factorial experiment. Thus to obtain CV designs for $n > 11$ we replicate one or more runs of the 3^3 designs which are already found to possess the CV property. Also we compare the CV values of the mixed and pure replications.

Table 8.1: 3^3 CV Design for $n = 9$

t_1	t_2	t_3
0	0	2
0	2	0
2	0	0
0	2	2
2	0	2
2	2	0
1	2	2
2	1	2
2	2	1

- (Section 8.3): In this section we compare the five 3^3 CV designs for $n = 10$ presented in Chapter 2 w.r.t different optimality criteria like AD, AT, AE, GD, GT and GE and their CV values.

8.2 Replicated 3^3 CV Designs

Consider the 3^3 CV design for $n = 9$ in Table 8.1 which is the design $d_m^{(2)}$ for $m = 3$ presented in Chapter 5. We add the runs $(0, 0, 0)$ twice and $(2, 2, 2)$ twice separately to this design and obtain the respective CV designs for $n = 11$. Also if both the runs are added simultaneously once the design is CV for $n = 11$. Since the two runs worked for giving CV designs we replicate them a couple more times and obtain 3^3 CV designs for $n \geq 12$. The replications along with the variances are presented in Table 8.2. From Table 8.2 we see that the mixed replications (where both the runs are replicated) give smaller variance as compared to the pure replications (where one run is replicated a couple of times) for different n.

Table 8.2: Replicated 3^3 CV Designs

n	Runs added to $d_3^{(2)}$	CV
11	$(0,0,0)$ twice	0.2889
	$(2,2,2)$ twice	0.2889
	$(0,0,0)$ once and $(2,2,2)$ once	0.2222
12	$(0,0,0)$ thrice	0.2857
	$(2,2,2)$ thrice	0.2857
	$(0,0,0)$ once and $(2,2,2)$ twice	0.2051
	$(0,0,0)$ twice and $(2,2,2)$ once	0.2051
13	$(0,0,0)$ 4 times	0.284
	$(2,2,2)$ 4 times	0.284
	$(0,0,0)$ once and $(2,2,2)$ thrice	0.1975
	$(0,0,0)$ thrice and $(2,2,2)$ once	0.1975
	$(0,0,0)$ twice and $(2,2,2)$ twice	0.1852

Next we consider the 3^3 CV design for $n = 11$ with all distinct runs as presented in Table 8.3. We want to see if replicating the existing runs of this design give CV for $n \geq 12$. Hence we replicate the existing runs one at a time and obtain CV designs with pure replications which are presented in Table 8.4. From Table 8.4 we see that by replicating any of the first 9 runs of $D_3^{(11)}$ CV designs for $n \geq 12$ are obtained. The replication of $(0,0,0)$ and $(2,2,2)$ gives CV that is already presented in Table 8.2. Interestingly we see that the replication of any of the runs from the set $\{(1,2,2), (2,1,2), (2,2,1)\}$ always gives $CV = 0.2222$ irrespective of the number of replication. Although in Table 8.4 we have presented the replicated designs for $n = 12, 13$ and 14 only but through computer check we found that these replications can be extended to any number $r > 1$. So given the CV design $D_3^{(11)}$ for $n = 11$ any of the existing runs can be replicated any number of times (pure replications) and in all the cases respective CV designs are obtained for $n > 11$.

Table 8.3: 3^3 CV Design $D_3^{(11)}$ for $n = 11$

t_1	t_2	t_3
0	0	2
0	2	0
2	0	0
0	2	2
2	0	2
2	2	0
1	2	2
2	1	2
2	2	1
0	0	0
2	2	2

Table 8.4: Replicated 3^3 CV designs for $n \geq 12$

n	Runs added	CV
12	$(0,0,2)/(0,2,0)/(2,0,0)$ once	0.2051
	$(0,2,2)/(2,0,2)/(2,2,0)$ once	0.2051
	$(1,2,2)/(2,1,2)/(2,2,1)$ once	0.2222
13	$(0,0,2)/(0,2,0)/(2,0,0)$ twice	0.1975
	$(0,2,2)/(2,0,2)/(2,2,0)$ twice	0.1975
	$(1,2,2)/(2,1,2)/(2,2,1)$ twice	0.2222
14	$(0,0,2)/(0,2,0)/(2,0,0)$ thrice	0.1932
	$(0,2,2)/(2,0,2)/(2,2,0)$ thrice	0.1932
	$(1,2,2)/(2,1,2)/(2,2,1)$ thrice	0.2222

We presented one CV design for $n = 11$ from which several replicated designs are obtained. Similar replicated designs can be obtained from several other 3^3 CV designs.

8.3 Comparison of the Five 3^3 CV Designs for $n = 10$

In Chapter 2 we presented five 3^3 CV designs for $n = 10$ from five groups of different CV value in Table 2.4. We make comparisons among them w.r.t different optimality criteria like the arithmatic and geometric average of the determinant, trace and the maximum eigen value of the variance-covariance matrices, average being taken over all the models. We define these optimality criteria in the following:

1. Determinant: For a model having full rank design matrix, the variance-covariance matrix of the estimators of the parameters is given by

$$\frac{Var(\hat{\boldsymbol{\beta}})}{\sigma^2} = (\boldsymbol{X}'\boldsymbol{X})^{-1}. \tag{8.1}$$

   The $D-$Optimal designs are those obtained by maximizing the determinant of the information matrix, i.e, maximizing $|\boldsymbol{X}'\boldsymbol{X}|$ or equivalently by minimizing the determinant of the variance-covariance matrix given in (8.1) among the possible designs in a particular class of m and n. Lowering the $|(\boldsymbol{X}'\boldsymbol{X})^{-1}|$ is an optimality criteria because it is directly proportional to the volume of the confidence region of the parameters. Hence designs giving smaller value of $|(\boldsymbol{X}'\boldsymbol{X})^{-1}|$ are better.

2. Trace: Trace of any matrix is defined as the sum of all the diagonal elements of the matrix. Optimal designs can be obtained by minimizing the trace of $(\boldsymbol{X}'\boldsymbol{X})^{-1}$. This is an optimality criteria because smaller the value of the

average variance of the parameter estimators better is the esimation.

3. Eigenvalue: Eigenvalues of a matrix are obtained from solving $\boldsymbol{Ax} = \lambda \boldsymbol{x}$, λ is the Eigenvalue and $\boldsymbol{x}$ is the Eigenvector. Maximum Eigenvalue of $(\boldsymbol{X'X})^{-1}$is proportional to $Max_{\boldsymbol{a} \in \boldsymbol{R}^p} \frac{\boldsymbol{a}' Var(\hat{\boldsymbol{\beta}}) \boldsymbol{a}}{\boldsymbol{a}' \boldsymbol{a}}$. Hence minimizing the maximum Eigenvalue is an optimality criteria since it minimizes the maximum value of $Var(\boldsymbol{a}' \hat{\boldsymbol{\beta}})$ over all real vector $\boldsymbol{a} \in \boldsymbol{R}^p$.

Since we consider a class of models with general mean and main effects as the common parameters and the $2-$factor interaction as the uncommon parameter, the design matrices vary from one model to the other and hence the values of the criterion functions are different for different models. But optimality criteria should not depend on any model as all the models are being treated equally. So we take the average of all the criterion functions over all the models to get one value of each of the functions from one design. The average is taken by arithmatic mean as well as geometric mean. The average arithmatic mean of the three criterion functions are denoted by AD, AT and AE whereas the geometric mean of the functions are denoted by GD, GT and GE respectively. In Table 8.5 we present the values of the different criterion functions for the five CV designs along with their CV values. So from the table we see that although design I is an optimal 3^3 CV design for $n = 10$ because of the the minimum CV value but it is not optimal w.r.t the other criterion functions. Design III which is the third best design w.r.t the CV, is optimal w.r.t all other criterion functions. Also we see that there is not much difference between the optimal CV design (I) and the second best CV design

Table 8.5: Criterion Functions

Design	CV	Determinant	Trace		Max Eigen Value	
		AM=GM($\times 10^8$)	AM	GM	AM	GM
(I)	0.2564	6.10	1.957	1.944	1.007	0.98
(II)	0.2667	5.95	1.959	1.938	1.03	0.988
(III)	0.2837	5.07	1.742	1.717	0.897	0.838
(IV)	0.2963	19.85	2.5	2.462	1.467	1.402
(V)	0.4	15.88	2.619	2.516	1.603	1.413

(II) w.r.t the CV value as well as all the criterion functions since the values of the functions are very similar for these two designs. Design IV followed by design V are the worst among all w.r.t any of the optimality criteria as well as the CV value.

Chapter 9

Dose Finding Experiment and Simulation

9.1 Chapter Summary

In this chapter we present an example of a dose finding experiment where factorial designs are widely used to identify the optimal potential drug dosage combinations to treat any disease. Also we do a simulation study using two CV designs to compare the class of models to identify the true interaction. Here is what we present in each section:

- (Section 9.2): In this section we present the dose finding experiment where both two and three level experiments are carried out sequentially to identify the optimal drug dosage combinations for treating the Herpes Simplex Virus. For the three level experiment an 81-run resolution IV design is used which can estimate the general mean, all main effects and some of the two-factor

interactions along with the block factors. From the significance test of the parameters only one two-factor interaction is found to be significant along with general mean, some of the main effects and the block parameters.

- (Section 9.3): In this section for the three level factorial experiment a class of models is considered each with general mean, all main effects and the block factors as the common parameters and the two-factor interaction as the uncommon parameter. All the models are fitted and the sum of squares due to error is calculated for them. The model with the minimum error sum of squares contains the possible non negligible two-factor interaction. The result exactly matches with the one obtained in section 9.2 for the three level experiment.

- (Section 9.4): In this section we do a simulation study using two CV designs for 3^3 factorial experiment. We generate artificial data from an assumed true model and compare the error sum of square values for all the fitted models in the class to identify the true interaction.

9.2 Dose Finding Experiment

9.2.1 Introduction

In USA, UK and other western countries one of the very common virus called the Herpes Simplex Virus type 1 (HSV-1) causes various severe diseases like mucocu-taneous diseases, neonatal herpes and herpes encephalitis and it can even

lead to the increasing risk of HIV infection. Many therapeutics have been developed to treat HSV infections but the drug resistance and toxicity have always been concerns. Often times instead of using the individual drugs the combination of different anti viral drugs are preferred as their low dosage combinations are found to be more effective as well as reduce the cytotoxity. But it is huge time, cost and labor consuming to test for every possible combination when various dosages of multiple drugs are considered. Factorial designs have been widely used to find the optimal drugs and their interactions and also predict the optimal combination by building statistical models. Since often times in most of the scientific experiments the three factor and higher order interactions are found to be non important so using a full factorial design is just waste of most of the degrees of freedom to estimate the non important higher order interactions. Hence a more practical and economical approach is to use fractional factorial designs which use much smaller number of runs that allows estimation of the lower order interactions. Different combinations of the six different anti viral drugs: (1) Interferon-alpha (A), (2) Interferon-beta (B), (3) Interferon-gamma (C), (4) Ribavirin, (5) Acyclovir and (6) TNF-alpha are used to treat HSV-1 and then two level and three level experiments are carried out sequentially.

9.2.2 $2-$level Experiment

9.2.2.1 Design and Model

In the two level experiment a half fraction of 2^6 design is used which is a resolution VI design and hence can estimate the general mean, all main effects

Table 9.1: Dosages for 2^6 Experiment

Factors	Levels (ng/mL)	
	Low	High
A	3.12	50
B	3.12	50
C	3.12	50
D	1560	25000
E	31	5m
F	0.31	5

and all $2-$factor interaction effects under the assumption that the four factor and higher order interactions are negligible. The half fraction design with 32 runs is obtained from the generator $F = ABCDE$. Along with these 32 runs three center points are also added to estimate the pure error and carry out the lack of fit test to check for the model adequacy. From the pilot study the minimum response dosage and the plateau dosage of each drug are determined. In the study the plateau dosage is chosen as the high level (coded as 1) and the minimum dosage which is 16 times diluted than the plateau dosage is chosen as the low level (coded as -1). The high and low dosage levels of different drugs are given in Table 9.1. The different combinations of the six drugs are added to the host cells simultaneously with HSV-1. The virus are engineered to carry the green fluorescent protein (GFP) gene which serves as a biomarker to measure the percentage of infected cells. The readout of the percentage of infected cells along with the design with 35 runs are given in Table 9.2. The distribution of the readouts is positively skewed and hence the logarithm of the readouts with base 10 are considered as the response. The model with general mean, main effects and the two and three factor interactions

Table 9.2: 2^6 Resolution VI Design

A	B	C	D	E	F	Read out
-1	-1	-1	-1	-1	-1	31.6
-1	-1	-1	-1	1	1	32.6
-1	-1	-1	1	-1	1	13.4
-1	-1	-1	1	1	-1	13.2
-1	-1	1	-1	-1	1	27.5
-1	-1	1	-1	1	-1	32.5
-1	-1	1	1	-1	-1	11.6
-1	-1	1	1	1	1	20.8
-1	1	-1	-1	-1	1	37.2
-1	1	-1	-1	1	-1	51.6
-1	1	-1	1	-1	-1	14.1
-1	1	-1	1	1	1	19.9
-1	1	1	-1	-1	-1	27.3
-1	1	1	-1	1	1	40.2
-1	1	1	1	-1	1	19.3
-1	1	1	1	1	-1	23.3
1	-1	-1	-1	-1	1	31.2
1	-1	-1	-1	1	-1	32.6
1	-1	-1	1	-1	-1	14.2
1	-1	-1	1	1	1	22.4
1	-1	1	-1	-1	-1	32.7
1	-1	1	-1	1	1	41.0
1	-1	1	1	-1	1	20.1
1	-1	1	1	1	-1	18.7
1	1	-1	-1	-1	-1	29.6
1	1	-1	-1	1	1	42.3
1	1	-1	1	-1	1	18.5
1	1	-1	1	1	-1	20.0
1	1	1	-1	-1	1	30.9
1	1	1	-1	1	-1	34.3
1	1	1	1	-1	-1	19.4
1	1	1	1	1	1	23.4
0	0	0	0	0	0	16.8
0	0	0	0	0	0	17.5
0	0	0	0	0	0	16.2

are given below:

$$
\begin{aligned}
E\left(y_i\right) &= \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + \beta_6 x_{6i} + \sum_{j<k} \beta_{jk} x_{ji} x_{ki} \\
&+ \sum_{j<k<l} \beta_{ijk} x_{ji} x_{ki} x_{li}, \\
Var\left(y_i\right) &= \sigma^2, \forall i,
\end{aligned}
$$

where y_i is the response variable corresponding to the ith run of the design, β_0 is the general mean, β_i, $i = 1(1)6$ are the main effects, β_{jk}, $j < k = 1(1)6$ are the $2-$factor interaction effects, β_{jkl}, $j < k < l = 1(1)6$ are the $3-$factor interaction effects, x_{si}, $s = 1(1)6$, $i = 1(1)35$ is the ith level of the sth factor, the levels are coded as -1 and 1. The least square estimates of the factorial effects are twice as that of the corresponding β's.

9.2.2.2 Results

The resolution VI design used in the experiment can estimate the general mean, all six main effects, all fifteen $2-$factor interaction effects and ten pairs of $3-$factor interaction effects. Table 9.3 presents the scaled estimates (estimates/SE) along with the sum of squares and the p-values for each of the parameter in the model. From this table we see that the overall sum of squares for the main effects is maximum followed by that of the $2-$factor interactions followed by the ten pairs of $3-$factor interactions. Also we see that the sum of square due to the main effect of the drug D is maximum followed by that of the drug E. The significance test also gives the minimum p-value for the coefficient of the drug D which shows that the drug D is highly significant as compared to the other drugs used in treating

Table 9.3: Estimates and p-values for 2^6 Experiment

Effect	Estimates	% Sum of Squares	$p-$value
A	0.02	1	~ 1
B	0.04	3.1	~ 1
C	0.01	0.2	~ 1
D	-0.20	68	~ 1
E	0.06	7.3	~ 1
F	0.03	1.9	~ 1
AB	-0.03	1.6	~ 1
AC	0.008	0.1	~ 1
AD	0.03	1.2	~ 1
AE	-0.01	0.3	~ 1
AF	0.007	0.1	~ 1
BC	-0.01	0.3	~ 1
BD	0.01	0.2	~ 1
BE	0.01	0.2	~ 1
BF	-0.01	0.2	~ 1
CD	0.03	1.9	~ 1
CE	0.003	0	~ 1
CF	0.005	0	~ 1
DE	0.002	0	~ 1
DF	0.02	0.7	~ 1
EF	-0.002	0	~ 1
$ABC + DEF$	-0.003	0	~ 1
$ABD + CEF$	0.002	0	~ 1
$ABE + CDF$	-0.008	0.1	~ 1
$ABF + CDE$	-0.002	0	~ 1
$ACE + BDF$	-0.02	0.9	~ 1
$ACF + BDE$	-0.02	0.8	~ 1
$ACD + BEF$	-0.02	0.5	~ 1
$ADE + BCF$	-0.005	0	~ 1
$ADF + BCE$	-0.01	0.2	~ 1
$AEF + BCD$	0.02	0.7	~ 1
Residuals		8.3	
Total		100	

Table 9.4: Lack of Fit Test

Source	DF	SS	MS	F	p
Model	31	0.858	0.028	1.08	> 0.5
Error	3	0.077	0.026		
lack of Fit	1	0.0766	0.0766	272.46	0.0037
Pure Error	2	0.00056	0.00028		
Corrected Total	34	0.935			

HSV-1. Moreover the negative estimate of the coefficient associated with drug D suggests that high dosage of this drug has the capability of lowering the viral infection. All other drugs A through F except the drug D have positive coefficients which implies that if the dosages of all the drugs except drug D are lowered and the dosage level of drug D is increased then the minimum viral infection can be achieved. But increasing the dosage level of a drug can simultaneously bring toxicity to the subjects. So in the follow up experiment all drugs are set at lower dosage level to avoid unacceptable toxicity.

Using the three independent center points in the design the lack of fit test is carried out to check for the non linearity in the response. The result is presented in Table 9.4. From this table we see that the lack of fit is very significant with a p-value of 0.0037. This clearly shows that the relationship between the response and the drug dosages is non linear. Hence to study this non linear relationship additional levels and runs are required and this is the motivation for the follow up $3-$level experiment.

Table 9.5: Dosages for 3^6 Experiment

Factors	Levels (ng/mL)		
	Low	Intermediate	High
A	0	0.78	12.5
B	0	0.78	12.5
C	0	0.78	12.5
D	0	390	6250
E	0	80	1250
F	0	0.08	1.25

9.2.3 3−level Experiment

9.2.3.1 Design and Model

In the 3−level experiment the drug dosage levels are lowered from the previous 2−level experiment to screen for less toxic drug combinations. The high dosage for the 3−level experiment is the intermediate dosage level for the 2−level experiment, the intermediate dosage for the 3−level experiment is 16 times diluted than the high dosage and the low dosage is set at no drug. The dosage levels of the six drugs are given in Table 9.5. The design used in this experiment is a one-ninth fraction of 3^6 factorial experiment and is obtained from the generators: $ABCD = E$ and $AB^2C = F$. This is a resolution IV design which can estimate the general mean, all main effects and some of the 2−factor interactions assuming that the 3−factor and higher order interactions are negligible. In practice it is not feasible to carry out the experiment with 81 runs in a single batch and hence they are divided into three batches of homogeneous experimental runs. Blocking factor is incorporated in to the model to reduce systematic sources of variation. Blocking was done using the generator $block = AC^2D$. The response is again the logarithm of the

percentage of infected cells. We denote the factors A through F by 1 through 6. Consider the following linear model:

$$E\left(y_i\right) = \beta_0 + \sum_s \beta_s x_{si} + \sum_i \beta_{ss} x_{si}^2 + \sum_{j<k} \beta_{jk} x_{jki} + \gamma_1 z_{1i} + \gamma_2 z_{2i},\ Var\left(y_i\right) = \sigma^2, \forall i, \tag{9.1}$$

where y_i is the response variable corresponding to the ith run, $i = 1(1)81$, β_0 is the general mean, β_s, $s = 1(1)6$ are the linear main effects, β_{ss}, $s = 1(1)6$ are the quadratic main effects β_{jk}, $j < k = 1(1)6$ are the linear$\times$linear $2-$factor interaction effects, x_{si}, $s = 1(1)6$ is the ith level of the sth factor, the levels being coded as -1, 0 1, x_{si}^2 is the quadratic term corresponding to the ith level of the sth factor, x_{jki} is the ith component of the two-factor interaction corresponding to the jth and the kth factor. For $i = 1(1)81$, the z_{1i} and z_{2i} are defined as

$$\begin{aligned} \boldsymbol{z}_{1i} &= 1,\ \text{if the } i^{th} \text{ run is in block 1} \\ &= 0,\ \text{otherwise.} \end{aligned}$$

$$\begin{aligned} \boldsymbol{z}_{2i} &= 1,\ \text{if the } i^{th} \text{ run is in block 2} \\ &= 0,\ \text{otherwise.} \end{aligned}$$

The γ_1 and γ_2 are the coefficients corresponding to the two blocks respectively. The block 0 is taken as the reference.

9.2.3.2 Results

The estimates of the coefficients of the model (9.1) depend on the type of coding of the quadratic and the interaction terms. We consider two types of coding: (1)

used in the paper (2) used in my research. The two types of coding and the results are illustrated in the following:

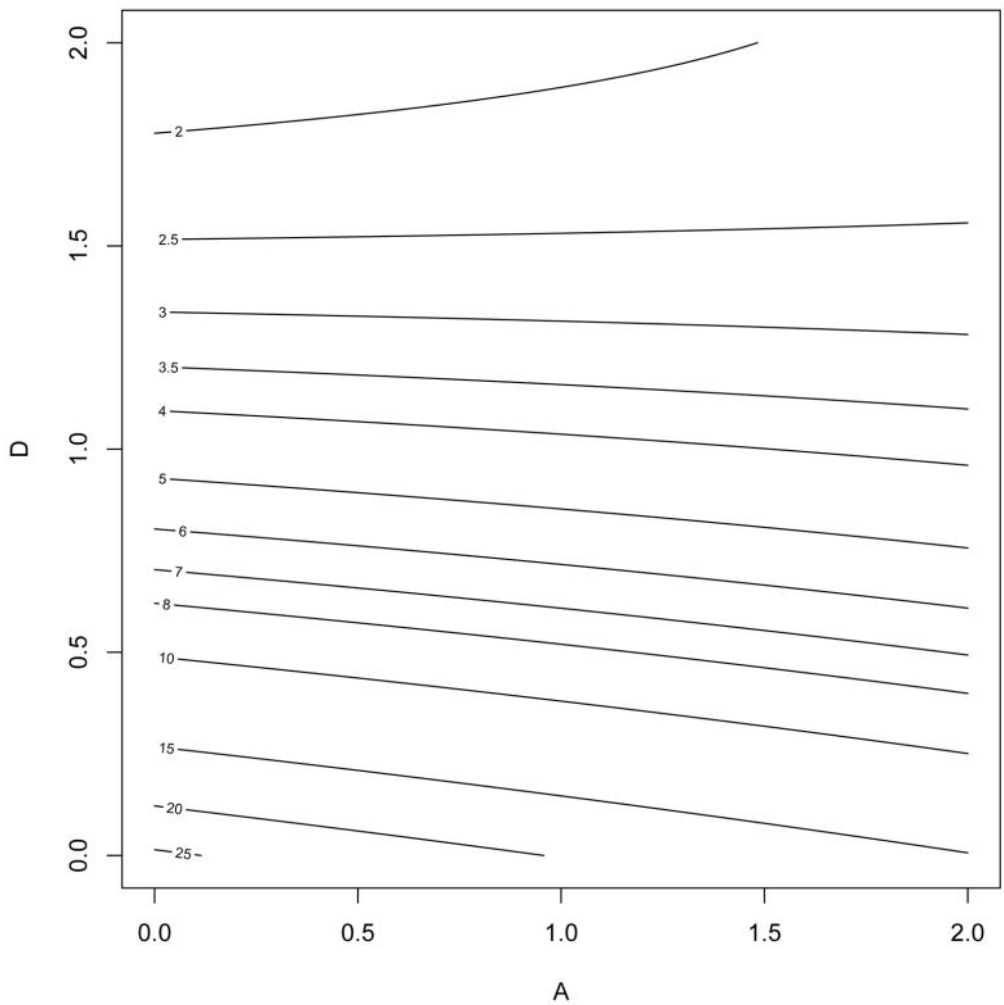
- Coding Used in the Paper

The ith level of the sth factor is $x_{si} \in (0, 1, 2)$, $s = 1(1)6$, $i = 1(1)81$ which is coded as -1, 0, 1 respectively. Now the quadratic term x_{si}^2 is calculated by squaring the coded x_{si}'s, i.e., the coded $x_{si}^2 \in (0, 1)$, $s = 1(1)6$. The ith component of the linear$\times$linear interaction corresponding to the jth and the kth factor, x_{jki}, is computed by multiplying the coded x_{ji} with x_{ki}, i.e, $x_{ji}x_{ki} \in (-1, 0, 1)$. In this setting the design matrix $\boldsymbol{X}$ (81×30) is formed with the first column as the vector of unity and the remaining columns correspond to the 6 linear main effects, 6 quadratic main effects, 15 linear$\times$linear 2$-$factor interactions and the two block variables. The initial analysis identifies the 80th run as an outlier and thus it is deleted and the model is fitted again without the outlier. The scaled estimates (estimates/SE) of the coefficients along with the p values for their significance tests are given in Table 9.6. From the p values we see that the linear main effects of the drugs B, C, D and E are significant at 5% level of significance and the linear main effect of drug A is significant at 10% level. But the linear main effect of the drug F is not significant implying that it is inert in minimization of the viral infection. Also the coefficients of the drugs A through E are negative indicating that the high dosage of these drugs have the potential to lower the viral infection. The quadratic main effect of drug D is very significant but that of the other drugs are not. Among the fifteen 2$-$factor interactions only AD is found to be significant.

Table 9.6: Estimates and $p-$values for 3^6 Experiment Following the Coding in the Paper

Effects	Estimates	$p-$values
β_0	12.45	0
γ_1	-8.45	0
γ_2	-4.47	0
A	-1.87	0.067
B	-2.82	0.007
C	-2.32	0.024
D	-25.94	0
E	-6.17	0
F	0.58	0.563
A^2	0.83	0.409
B^2	0.31	0.758
C^2	-0.77	0.446
D^2	4.95	0
E^2	1.58	0.119
F^2	1.52	0.135
AB	0.37	0.715
AC	0.18	0.858
AD	3.21	0.002
AE	1.52	0.134
AF	-0.46	0.646
BC	-0.22	0.824
BD	-0.74	0.46
BE	0.84	0.404
BF	-0.73	0.468
CD	1.14	0.261
CE	-1.22	0.229
CF	0.64	0.525
DE	0.23	0.815
DF	1.29	0.202
EF	-1.40	0.166

Figure 9.1: Contour Plot of the Drugs A and D



Although the drug A is not significant at 5% level we still keep it in the model as the interaction AD is significant. So the final fitted model is given as:

$$\begin{aligned}\hat{y}_i &= 0.761 - 0.037x_{1i} - 0.054x_{2i} - 0.046x_{3i} - 0.509x_{4i} - 0.119x_{5i} + 0.167x_{4i}^2 \\ &- 0.327z_{1i} - 0.176z_{2i} + 0.078x_{14i}. \qquad (9.2)\end{aligned}$$

To identify the potential drug dosage levels of A and D the contour plot of these two drugs are drawn for the predicted response from model (9.2) given the drugs B, C and E are set at high dosage level. The plot is shown in figure 9.1. The

Table 9.7: Coding of Linear and Quadratic Terms

Level	x_i	x_i^2
0	-1	1
1	0	-2
2	1	1

coordinates of the contour plot has the values of the two drugs respectively. The contour plot indicates that the high dosage of drug D combined with the low dosage (no drug) of drug A would produce the maximum viral infection minimization. So the final optimal potential drug dosage combination would be to set the drugs B, C, D and E at high dosage level and the drug A at low dosage level which is no drug.

- Coding Used in my Research

The levels $(0, 1, 2)$ of the factors are still coded as -1, 0, 1 respectively. The quadratic terms are coded as given in Table 9.7. The linear×linear interaction terms are calculated as $s_{jki} = (x_{ji} + x_{ki})_{mod(3)}$, $x_{ji}, x_{ki} \in (0, 1, 2)$. This will make $s_{jki} \in (0, 1, 2)$ and then x_{jki}'s are obtained by following the coding in Table 9.7. The z_{1i} and z_{2i} corresponding to the blocks remain the same. In this setting the model is again fitted without the outlier and the scaled estimates (estimates/SE) of the parameters along with the p values are given in Table 9.8. Because of this coding and the generators defined earlier the interaction BF and AC are aliased with each other and hence both can not be estimated separately. Again BE and DF are aliased with each other. The Table 9.8 shows the estimates of BE and AC but the estimates of BF and DF can not be obtained. The estimates get

Table 9.8: Estimates and $p-$values for 3^6 Experiment Using the Coding in my Research

Effects	Estimates	$p-$values
β_0	32.27	0
γ_1	-7.85	0
γ_2	-4.27	0
A	-1.85	0.0697
B	-2.61	0.0116
C	-2.27	0.0272
D	-24.21	0
E	-5.73	0
F	0.66	0.5150
A^2	0.71	0.4830
B^2	0.42	0.6766
C^2	-0.78	0.4386
D^2	4.53	0
E^2	1.6	0.1147
F^2	1.34	0.1854
AB	0.26	0.7993
AC	0.41	0.6865
AD	-1.6	0.1147
AE	-1.24	0.2204
AF	0.86	0.3912
BC	0.03	0.9796
BD	0.83	0.4130
BE	-2.12	0.0388
BF	NA	NA
CD	-0.97	0.3387
CE	1.02	0.3136
CF	0.72	0.4718
DE	0.034	0.7376
DF	NA	NA
EF	1.32	0.1941

slightly changed along with the p values but the overall significance remain the same as compared to the previous coding. The linear main effect of the drugs B through D are very significant followed by that of the drug A and drug F is again insignificant. The quadratic effect of only drug D is significant. Because of the negative coefficients of drugs B through E they are set at the high dosage level to minimize the infection. The interaction BE is significant at 5% level while the interaction AD is insignificant unlike the previous coding. The intercept and the blocks are very significant and hence are kept in the final model. Since the drug A is not significant at 5% level and no interaction is significant where drug A is present so it can be considered an inert in minimizing the viral infection like the drug F and hence can be removed from the model. Again since both drugs B and E are significant with negative coefficients and their interaction is also significant so the optimal potential drug dosage combination would be to set the drugs B through E at high level to minimize viral infection. Here is the final fitted model:

$$\begin{aligned} \hat{y}_i &= 0.951 - 0.054x_{2i} - 0.048x_{3i} - 0.511x_{4i} - 0.119x_{5i} + 0.055x_{4i}^2 \\ &- 0.327z_{1i} - 0.178z_{2i} - 0.045x_{25i}. \end{aligned} \tag{9.3}$$

9.3 Class of Models to Identify the True Interaction

For a 3^6 factorial experiment there are one general mean, twelve main effects and sixty 2−factor interaction effects. We consider the class of models M_u $\forall u$ in (1.3.1) in Chapter 1 for $k = 1$ for a 3^6 experiment. Thus there are 60 models

in the class. However in the 3−level experiment of the dose finding example only fifteen linear×linear 2−factor interactions are considered and hence there are 15 models in the class each with one linear×linear 2−factor interaction along with the general mean, all main effects and the two block parameters which are the common parameters. Thus each of these fifteen models in the class has 16 parameters. In this case the 81− run design without the outlier can estimate all the parameters in all the models since no 2−factor interaction is aliased with any main effect or general mean. Using the 3^6 design these fifteen models are fitted and the parameter estimates for all models are found to be identical with those of the bigger model. Also the significance of the common parameters for each of the models remain the same as that of the previous model. To identify the true model containing the true 2−factor interaction we compare the class of models w.r.t their SSE values. Write the uth model as

$$E(\boldsymbol{y}) = \boldsymbol{X}^{(u)}\boldsymbol{\beta}^{(u)}, \tag{9.4}$$

where $\boldsymbol{X}^{(u)} = \left[\boldsymbol{j}_n \vdots \boldsymbol{X}_1 \vdots \boldsymbol{X}_{2u}\right]'$, $\boldsymbol{\beta}^{(u)} = \left[\boldsymbol{j}_n \vdots \boldsymbol{\beta}_1 \vdots \beta_{2u}\right]'$. Under (9.4) the least square estimator of $\boldsymbol{\beta}^{(u)}$ and the sum of squares due to error $s_e^{2(u)}$ for the uth model are given as

$$\begin{aligned}
\hat{\boldsymbol{\beta}}^{(u)} &= \left(\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}\right)^{-1}\boldsymbol{X}^{(u)\prime}\boldsymbol{y}, \\
s_e^{2(u)} &= \boldsymbol{y}'\left[\boldsymbol{I}_n - \boldsymbol{X}^{(u)}\left(\boldsymbol{X}^{(u)\prime}\boldsymbol{X}^{(u)}\right)^{-1}\boldsymbol{X}^{(u)\prime}\right]\boldsymbol{y}.
\end{aligned}$$

We calculate $s_e^{2(u)}, \forall u$. Let β_{2u^*} be the 2−factor interaction such that $s_e^{2(u^*)}$ is minimum for some u^*. Then β_{2u^*} is the possible non-negligible 2−factor interaction effect. In the following we present the results for the 3^6 experiment using both

Table 9.9: $s_e^{2(u)}$ Following the Coding in Research

u	$s_e^{2(u)}$
AB	1.56
AC	1.56
AD	1.50
AE	1.52
AF	1.54
BC	1.56
BD	1.55
BE	1.46
BF	1.56
CD	1.54
CE	1.54
CF	1.55
DE	1.56
DF	1.46
EF	1.52

types of coding.

9.3.1 Coding Used in the Research

Following the coding used in my research we get $s_e^{2(u)}$ for all the fifteen models. We present the values of $s_e^{2(u)}$ in Table 9.9. From the table we see that the values of $s_e^{2(u)}$ are almost identical for all the models. Comparing the values we get minimum $s_e^{2(u)}$ for $u = BE$ and DF. This result is consistent with that of the bigger model which yielded BE (aliased with DF) as the most significant interaction.

9.3.2 Coding Used in the Paper

Following the coding used in the paper we get the values of $s_e^{2(u)}$ for all u which are presented in Table 9.10. From this table we see that $s_e^{2(u)}$ is minimum

Table 9.10: $s_e^{2(u)}$ Using the Coding in the Paper

u	$s_e^{2(u)}$
AB	1.55
AC	1.56
AD	1.35
AE	1.52
AF	1.56
BC	1.56
BD	1.51
BE	1.50
BF	1.55
CD	1.54
CE	1.53
CF	1.54
DE	1.56
DF	1.47
EF	1.48

for $u = AD$. From the bigger model also we obtained AD as the most significant interaction and hence the results are consistent.

We also note that if we would have considered all sixty $2-$factor interactions then the $80-$run 3^6 design can estimate all the parameters in all sixty models. Using the bigger model the design can not estimate all $2-$factor interactions because of the aliases and hence if any interaction from the set of $2-$factor interactions that can not be estimated is the true interaction then it can not be identified. On the contrary using the class of models all sixty $2-$factor interactions can be estimated and hence by comparing the models the true interaction can be identified.

9.4 Simulation Study

In this section we do a simulation study using two CV designs to compare the class of models for the identification of the true interaction. Consider the two 3^3 CV designs for $n = 10$ with $CV = 0.2564$ ($design1$) and $CV = 0.2667$ ($design2$) presented in Table 2.4 in Chapter 2. For a 3^3 factorial experiment there are one general mean, six main effects and twelve $2-$factor interactions. We consider the class of models each with the general mean, six main effects and one $2-$factor interaction effect. Thus there are twelve models in the class. We assume that the three factor and higher order interactions are negligible and out of the twelve $2-$factor interactions AB is the true non negligible one. Hence the true model becomes

$$\boldsymbol{y} = \boldsymbol{j}_n\beta_0 + \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_{2AB}\beta_{2AB} + \epsilon,\ Var\left(\boldsymbol{y}\right) = \sigma^2\boldsymbol{I}_n, \tag{9.5}$$

where $\boldsymbol{y}\,(10 \times 1)$ is the vector responses, $\boldsymbol{j}_n$ is (10×1) vector of unity, β_0 is the general mean, $\boldsymbol{\beta}_1\,(6 \times 1)$ is the vector corresponding to the 6 main effects, β_{2AB} corresponds to the $2-$factor interaction AB and $\boldsymbol{X}_1\,(10 \times 6)$ and $\boldsymbol{X}_{2AB}\,(10 \times 1)$ are the corresponding design matrices. We simulate the artificial data $\boldsymbol{y}$ under the model (9.5), generating error from Normal distribution with $\sigma^2 = 0.5, 1.0, 1.5$ and 2 one at a time for the two designs. The parameter values for the true model are taken as

$$\beta_0 = 3.2,\ \boldsymbol{\beta}_1 = (5, 2, 2.5, 1.3, 2.8, 3.5, 1.8, 1.7)\,,\ \beta_{2AB} = 6.7. \tag{9.6}$$

The design matrix $\boldsymbol{X}^{(AB)} = \left[\boldsymbol{j}_n \vdots \boldsymbol{X}_1 \vdots \boldsymbol{X}_{2AB}\right]$ for the two designs are given below:

$$\boldsymbol{X}^{(AB)}_{design1} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 & -2 & -1 \\ 1 & -1 & 1 & 0 & -2 & 0 & -2 & 0 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 0 & -2 & 1 \\ 1 & 0 & -2 & 0 & -2 & 0 & -2 & 1 \\ 1 & 0 & -2 & 0 & -2 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 & 0 & -2 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0 & -2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{X}^{(AB)}_{design2} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 & -2 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 0 & -2 & 1 \\ 1 & 0 & 1 & 0 & -2 & 0 & -2 & 1 \\ 1 & 0 & -2 & 0 & -2 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 & 1 & 1 & -1 \\ 1 & 0 & -2 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0 & -2 & 1 \\ 1 & 1 & 1 & 0 & -2 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The data $\boldsymbol{y}$ is used to fit all twelve models in the class. After fitting the models $s_e^{2(u)}$ is calculated for all u. The $s_e^{2(u)}$ values are compared to identify the true

Table 9.11: Average Proportion of Times the Correct Model is Identified

Design	σ^2	Average Proportion	No. of Iterations for one proportion	No. of Iterations for average proportion
1	0.5	0.988	1000	$100,000$
	1.0	0.912	1000	$100,000$
	1.5	0.827	1000	$100,000$
	2	0.747	1000	$100,000$
2	0.5	0.986	1000	$100,000$
	1.0	0.903	1000	$100,000$
	1.5	0.826	1000	$100,000$
	2	0.752	1000	$100,000$

model. The model with minimum $s_e^{2(u)}$ is the true model containing the possible non negligible parameter among all $2-$factor interactions. We repeat this process of identifying the true model 1000 times, i.e, we generate generate the error vector from Normal distribution 1000 times and using the error vector, design matrix $\boldsymbol{X}^{(AB)}$ and the true parameter values in (9.6) we generate the data vector $\boldsymbol{y}$ 1000 times and after fitting the models calculate $s_e^{2(u)}, \forall u$ for 1000 simulations. Out of 1000 times we calculate the proportion of times the true model is identified. We repeat this whole process $100,000$ times, i.e. such proportion of identification of the correct model is calculated $100,000$ times. We report the average of these proportions in Table 9.11. From Table 9.11 we see that the proportions are very similar for both the designs. For $\sigma^2 = 0.5$ the correct model is identified almost all the times since the proportion is close to 1. The proportion is decreasing as σ^2 is increased. In the following we give the detailed algorithm for obtaining the average proportion:

1. Given a design calculate the design matrix and fix the parameter values

assuming a true model.

2. Generate sample of size 10 from Normal distribution with mean 0 and variance σ^2, $\sigma^2 \in \{0.5, 1.0, 1.5, 2\}$.

3. Generate the data ($\boldsymbol{y}$) under the true model.

4. Fit all the models in the class using $\boldsymbol{y}$ and calculate $s_e^{2(u)}$ for all u. Find the model with minimum $s_e^{2(u)}$. If $s_e^{2(u)}$ turns out to be minimum for the true model assumed in the beginning then the correct model is identified.

5. Repeat (2)-(4) 1000 times and calculate the proportion of times the correct model is identified.

6. Repeat (2)-(5) $100,000$ times and calculate the average proportion.

Chapter 10

Conclusions

In this dissertation we obtain series of CV designs for 3^m factorial experiment and characterize the CV property in terms of the projection matrix and the design runs for general fractional factorial designs. Also we obtain designs satisfying a particular structure of the variance of the interaction estimators for $2^{m_a} \times 3^{m_b}$ factorial experiment. We conclude by presenting the most important contributions of this thesis.

10.1 3^m Factorial Experiment

1. The two series of CV designs $d_m^{(1)}$ and $d_m^{(2)}$ are obtained for general 3^m factorial experiment. The design $d_m^{(1)}$ for $n = 2m+2$, $m \geq 2$ gives optimum CV design for $m = 2$ and the design $d_m^{(2)}$ for $n = 3m$, $m \geq 3$ gives optimum CV design for $m = 3$. The projection matrices of these designs are found to possess a particular structure giving columns and rows of zeros corresponding to a particular set of m runs of the respective CV designs. Most of the CV as

well as optimal CV designs satisfy this structure of the projection matrix.

2. A class of fractional factorial designs with n runs possessing the common variance property are characterized for general m. Several sufficient conditions are obtained by using pairs of interaction effects (null space and permutation matrix), independent columns of the projection matrix and runs of the designs.

3. The condition of obtaining a CV design for $(n \pm 1)$ from a CV design for n is derived in terms of the design matrix and the runs of the design.

4. Replicated designs give smaller CV as compared to the designs with distinct runs. The optimal CV design $d_m^{(1)}$ for $m = 2$ always remains CV after replicating any of its six runs any number of times. Many more such designs exist for 3^2 experiment for $n = 6$.

5. The condition of obtaining a 3^3 CV design from a 3^2 CV design is derived where every pair of columns of the 3^3 CV design consists of the same runs as that of the 3^2 CV design and the runs are replicated in the same way in both.

10.2 $2^{m_a} \times 3^{m_b}$ Factorial Experiment

1. For the simplest 2×3 factorial experiment no CV design exists with distinct runs and hence we consider a very structured replication of the six runs and under a particular condition of replications CV designs are obtained for

different runs.

2. For higher values of m_a and m_b it is computationally challenging to obtain CV designs. We obtain designs that give common variance within each of the groups: (1) the pure interaction estimators between the factors with same levels, (2) the mixed interactions linear in both factors and (3) the mixed interactions quadratic in the factor with 3 levels.

Bibliography

[1] Addelman, S., 1962. Orthogonal Main-effect Plans for Asymmetrical Factorial Experiments. Technometrics, 4, 21-46.

[2] Addleman, S., 1964. Techniques for Constructing Fractional Factorial Replicate Plans. Journal of American Statistical Association 58, 45-71.

[3] Anderson, D.A. and Srivastava, J.N., 1972. Resolution IV Designs of $2^m \times 3$ series. J. Roy. Statist. Soc. Ser., B 34, 377-384.

[4] Anderson, D. A. and Thomas, A. M., 1980. Weakly Resolvable IV. 3 Search Designs for the p^n Factorial Experiment. Journal of Statistical Planning and Inference 4, 299-312.

[5] Atkinson, A. C., Donev, A. N., Tobias, R. D., 2007. Optimum Experimental Designs, with SAS. Oxford University Press, New York.

[6] Bose, R.C. and Iyer, H.K., 1984. Balanced E-Optimal Designs of Resolution III for the $2^m \times 3^n$ Series. Journal of Statistical Planning and Inference, 10, 345-364.

[7] Box, G. E. P. and Hunter, J.S., 1961. The 2^{k-p} Fractional Factorial Designs, Parts I and II. Technometrics 3, 311-352, 449-458.

[8] Chatterjee, K., 1989. Search Designs for General Symmetric Factorials. Comm. Statist. Theory Methods 42 (19), 2189-2200.

[9] Chatterjee, K., Deng, L. Y., Lin, D. K. J., 2001. Two-Level Search Design for Main-Effect Plus Two Plan. Metrika 54, 233-245.

[10] Chatterjee, K. and Mukherjee, R., 1986. Some Search Designs for Symmetric and Asymmetric Factorials. Journal of Statistical Planning and Inference 13, 357-363.

[11] Chatterjee, K. and Mukherjee, R., 1993. Search Designs for Estimating Main effects and Searching Several Two-Factor Interactions in General Factorials. Journal of Statistical Planning and Inference 35, 131-138.

[12] Connor, W. S. and Zelen, M., 1959. Fractional Factorial Experimental Designs for Factors at Three Levels. National Bureau of Standards, Washington, DC. Applied Mathematics Series, No. 54.

[13] Dey, A., Mukherjee, R., 1999. Fractional Factorial Plans. Wiley, New York.

[14] Doughlas C. Montgomery., 2001. Design and Analysis of Experiments: Fifth Edition. Wiley.

[15] Federov, V. V., 1972. Theory of Optimal Experiments. Academic Press, New York.

[16] Finney, D. J., 1945. The Fractional Replication of Factorial Arrangements. Ann. Eugenics 12, 291-301.

[17] Ghosh, S., 1980. On Main Effect Plus Two Plans for 2^m Factorials. Ann. Statist. 8, 922-930.

[18] Ghosh, S., 1981. On Some New Search Designs for 2^m Factorial Experiments. Journal of Statistical Planning and Inference 5, 381-389.

[19] Ghosh, S., 1985. Resolution V Plus One Plan for 3^m Factorial Experiments. Unpublished Technical Report No. 135, University of California Riverside.

[20] Ghosh, S., 1987. Influential Nonnegligible parameters under the Search Linear Model. Comm. Statist. Theory Methods 16, 1013-1025.

[21] Ghosh, S. and Avila, D., 1985. Some New Factor Screening Designs using the Search Linear Model. Journal of Statistical Planning and Inference 11, 259-266.

[22] Ghosh, S. and Burns, C., 2002. Comparison of Four New General Classes of Search Designs. Australian & New Zealand Journal of Statistics, 44, 357-366.

[23] Ghosh, S. and Dekker, M., 1990. Statistical Design and Analysis of Industrial Experiments, New York.

[24] Ghosh, S., Deng, H., Luan, L., 2007. Error Sum of Squares and Power Comparisons for Model Search, Identification, and Discrimination. Journal of Statistical Planning and Inference 137, 3475-3482.

[25] Ghosh, S. and Flores, A., 2013. Common Variance Fractional Factorial Designs and Their Optimality to Identify a Class of Models. Journal of Statistical Planning and Inference.

[26] Ghosh, S. and Huang, M., 2016. Full Estimation Capacity Hierarchical Fractional Factorial Designs for a Class of Models. Communications in Statistics - Simulation and Computation. Volume 45, Issue 5.

[27] Ghosh, S. and Rao, C.R., 1996. Design and Analysis of Experiments. North-Holland.

[28] Ghosh, S., Teschmacher, L., 2002. Comparisons of Search Designs Using Search Probabilities. Journal of Statistical Planning and Inference 104 (2), 439-458.

[29] Ghosh, S. and Tian, Y., 2006. Optimum Two Level Fractional Factorial Plans for Model Identification, and Discrimination. Journal of Multivariate Analysis, 97, 1437-1450.

[30] Ghosh, S. and Zhang, X., 1987. Two New Series of Search Designs for 3^m Factorial Experiments. Utilitas Mathematica, 32, 245-254.

[31] Hinkelmann, K. and Kempthorne, O., 2005. Design and Analysis of Experiments: Advanced Experimental Design, Vol-2. Wiley.

[32] Jaynes, Jessica et al., 2013. An Application of Fractional Factorial Designs to Study Drug Combinations. Statistics in Medicine, 32(2), 307-318.

[33] Kiefer, J., 1959. Optimum Experimental Designs (with discussion). Journal of Royal Statistical Society, Ser. B 21, 272-319.

[34] Lauter, E., 1974. Experimental Design in a Class of Models. Math. Oper. Stat. Ser. Stat., 5, 379-398.

[35] Margolin, B.H., 1967. Systematic Methods for Analyzing 2^n3^m Factorial Experiments with Applications. Technometrics, 9(2), 245-259.

[36] Margolin, B.H., 1968. Orthogonal Main-effect 2^n3^m Designs and Two-factor Interaction Aliasing. Technometrics, 10(3), 559-573.

[37] Margolin, B.H., 1969a. Results on Factorial Designs of Resolution IV for the 2^n3^m Series. Technometrics, 11(3), 431-444.

[38] Margolin, B.H., 1969b. Orthogonal Main-effect Plans Permitting Estimation of All Two- factor Interactions for the 2^n3^m Factorial Series of Designs.

[39] Ohnishi, T. and Shirakura, T., 1985. Search Designs for 2^m Factorial Experiments. Journal of Statistical Planning and Inference 11, 241-245.

[40] Patel, M. S., (Ed.), 1987. Experiments in Factor Screening. Comm. Statist. Theory Methods 10 (Special Edition).

[41] Pukelsheim, F., 1993. Optimum Design of Experiments. Wiley, New York.

[42] Rao, C.R., 1973. Linear Statistical Inference and Its Applications. Wiley, New York.

[43] Sarkar, A. and Chatterjee, K., 2010. An MEP.2 Plan in 3^n Factorial Experiment and Its Probability of Correct Identification. Journal of Statistical Planning and Inference, 140, 3531-3539.

[44] Shirakura, T., Tazawa, S., 1991. Series of Main effect Plus One or Two Plans for 2^m Factorials when Three-Factor and Higher Order Interactions are Negligible. J. Japan Statist. Soc. 21, 211-219.

[45] Srivastava, J.N., 1975. Designs for Searching Non-negligible Effects. In: Srivastava, J.N. (Ed.), In a Survey of Statistical Design and Linear Models, North-Holland, Amsterdam, 507-520.

[46] Srivastava, J. N., 1977. Optimal Search Designs, in: Gupta, S. S., Moore, D. S. (Eds.), Proceedings of the International Symposium on Decision Theory, Academic Press, New York, pp. 375-409.

[47] Srivastava, J.N. and Anderson, D.A., 1969. Fractional Factorial Designs for Estimating Main-effects Orthogonal to Two-factor Interactions: 3^n and $2^m \times 3^n$ series. ARL, 60-0123, Aerospace Res. Labs., Wright Patterson Air Force Base.

[48] Srivastava, J.N. and Ariyaratna, W.M., 1979. Theory of Optimal Balanced 3^m Factorial Designs. In: Rustagi, J.S. (Ed.), Optimization Methods in Statistics, Academic Press, New York, 443-460.

[49] Srivastava, J.N. and Ariyaratna, W.M., 1982. Inversion of Information Matrices of Balanced 3^m Factorial Designs of Resolution V, and Optimal Designs. In: Krishnaiah P.R. and Ghosh J.K. (Eds.), Statistics and Probability, Essays in Honor of C.R. Rao, North-Holland, Amsterdam, 671-688.

[50] Tamhane, A.C., 2009. Statistical Analysis of Designed Experiments: Theory and Applications. John Wiley & Sons, Inc., Hoboken, New Jersey.