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The Core when Strategies are Restricted by Law^{1, 2}

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As described by Scarf [4], the α -core of a game consists of those feasible allocations with the property that no coalition is able to do better for its members by selecting alternative strategies independently of the strategies chosen by non-members. The properties of the α -core depend, of course, on which strategies are available to individuals and coalitions. It is argued here that it is useful to distinguish between those strategies which are physically possible and those strategies which are deemed admissible under a specified code of laws.

To illustrate this distinction, let us examine an exchange economy (with purely private commodities) where preferences display the properties of convexity and continuity requisite for the existence of competitive equilibrium. Suppose that there is a set I of all agents such that each agent $i \in I$ has a vector of initial holdings w^i . Where preferences are monotone in all goods, it is customary to suppose that any coalition $K \subset I$ is able to select any consumption vectors x^i for its members such that $\sum_{i \in K} x^i \leq \sum_{i \in K} w^i$. When strategy sets are thus specified, it is well known that all competitive equilibria belong to the α -core. Such a specification of available strategies may be interpreted in either of two ways. One might suppose that these and only these strategies are technically feasible. More reasonably one might suppose that though individuals are physically capable of taking goods away from others, the "law" prohibits any taking without consent of both parties to the transaction. With this legal restriction, the set of strategies available to coalition K without the consent of individuals not in K is precisely the set defined above.

Where some of the goods are unpleasant to some or all agents (though still purely private), one wishes not only to rule out theft but, for similar reasons, to outlaw the dumping of nasty things on others' doorsteps. An analytically workable procedure in this case is to restrict the set of strategies which members of any coalition K can pursue, in the absence of consent from non-members, to strategies which result in allocations such that $\sum_{i \in K} x^i = \sum_{i \in K} w^i$. Here again it can be shown that the α -core contains the set of competitive equilibria.

The problem of finding a reasonable system of property rights and legal restrictions when there are public goods is more subtle. In the first section of this paper a general model of social interaction is constructed. The second section introduces the notion of a restricted α -core. The final section explores the relation between Lindahl equilibrium and restricted α -cores.

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² This paper is based on research undertaken while the author was a visitor at CORE in Louvain, Belgium. The research was carried out within the project "Analyse économique de lutte contre la pollution des eaux" under contract between the Université Catholique de Louvain and the Belgian Ministry of Scientific Policy. The project is itself part of the "Premier programme nationale de recherche et de développement sur l'environnement physique et biologique", administered by Service de la Politique et de la Programmation Scientifique. I am grateful to Professor Henry Tulkens of CORE, who stimulated my interest in this problem.

1. A GENERAL MODEL OF SOCIAL INTERACTION

A *society* is a set I of n agents where each agent i has a non-empty activity set, $Y_i \subset E^m$. A vector $y_i \in Y_i$ is interpreted as representing an activity which it is possible for i to perform.¹ A *social state* is a vector $y = (y_1, \dots, y_n) \in \prod_i Y_i$. In the subsequent discussion it will occasionally be useful to single out one social state for special treatment. This social state will be called the *initial state*. There is no loss of generality and considerable gain in notational convenience if we choose the representation of activities in E^m in such a way that the initial state is represented as the vector $0 \in \prod_i Y_i$.²

Each agent i has a preference relation R_i defined on the set $Y_i \times \Sigma_I Y_i$. The statement $(y_i, y_I) R_i (y'_i, y'_I)$ is interpreted to mean that agent i likes every social state $(y_1, \dots, y_n)$ in which he performs y_i and in which $\Sigma_I y_i = y_I$, at least as well as every social state $(y'_1, \dots, y'_n)$ in which he performs y'_i and $\Sigma_I y'_i = y'_I$. This interpretation requires that the level of preference of any agent is determined by his own activity and the vector sum of the activities of all agents.

At first glance it may appear that there is a substantial loss of generality in assuming that the environment is represented by the vector sum of individual activities rather than by their Cartesian product. At a purely formal level, this is not true. By making the dimension of the activity vectors suitably large, we can accommodate any non-additive interactions. To see how this is done, let us examine an extreme case. Suppose that the activities which any agent i can perform can be represented by points y_i in E^k . Activities, however similar, which are performed by different agents could be represented by different components of vectors in E^m where $m = kn$. Thus any vector $y_i \in Y_i$ would be of the form $y_i = (0, \dots, 0, y_{i1}, \dots, y_{ik}, 0, \dots, 0)$ with 0's appearing everywhere except in the components $k(i-1)+1$ through ki . Then the vector $y_I = \Sigma_I y_i$ contains an exact description of the activities of each agent in the society. In fact, the vector y_I is just the Cartesian product of the vectors y_i . In this way, our formal structure can be applied even to a society in which individual preference levels can be determined only from a complete specification of the activity of each agent. Of course, such an extreme disaggregation may result in an inconveniently large number of commodities. In fact, it will turn out that some of our results have much more interesting interpretations when it is possible to represent similar activities by different agents with the same component of E^m .

It is helpful to see how some familiar special cases can be treated with this formulation. Consider the usual exchange model with privately held commodities. Let there be r commodities and let each agent have a vector of initial holdings, $w_i \in E^r$, $w_i \geq 0$. Then $w = (w_1, \dots, w_n)$ is the allocation of commodities before trade. Feasible activities for any agent consist of transfers of commodities to other agents. A non-negative vector $y_{ij} \in E^r$ where $i \neq j$ is interpreted as the vector of commodities which agent i transfers to agent j . A non-positive vector y_{ii} is interpreted as the vector of commodities which i transfers away from himself. Thus for each $i \in I$, we let

$$Y_i = \{(y_{i1}, \dots, y_{in}) \in E^{rn} \mid y_{ij} \geq 0 \text{ for all } j \neq i \text{ and } y_{ii} = - \sum_{j \neq i} y_{ij} \geq -w_i\}.$$

If the social state is $(y_1, \dots, y_n)$, then the vector $\sum_{i \in I} y_i = \sum_{i \neq j} y_{ij} + y_{jj}$ is the vector of

¹ There is apparently some loss of generality in assuming that the set of activities possible for any agent can be defined independently of the activities chosen by others. This will not be of great consequence, however, because in the model treated here, the *utility* of an activity to an agent may depend on the activities of others. Thus even if the set of possible activities for an agent is unaffected by the activities of other agents, the model allows the interpretation that the effects of an activity may be radically altered by the activities of others.

² Since the vector $0 \in \prod_i Y_i$ does not necessarily represent a social state in which all agents are entirely inactive, we must of course allow the possibility that $\prod_i Y_i$ contains vectors with negative components. This will cause no difficulties in the subsequent discussion.

commodities received by j from other agents minus the vector of commodities transferred by j to other agents. Therefore the holdings of agent j after the exchanges represented by the social state $(y_1, \dots, y_n)$ are equal to $w_j + \sum_{i \in I} y_{ij}$ and the allocation of commodities

after exchange is $w + y_I$ where $y_I = \sum_I y_i$. Where preferences are concerned only with the allocation of resources after exchange, each agent's level of preference is determined by the vector y_I . Notice that this is true even if some agents are concerned about the consumption vectors allocated to other consumers.

We consider another example. Suppose that each agent cares about how much he uses a public facility (which, for simplicity, we suppose to be available in fixed supply) and about how congested the facility is. Each agent likes to use the facility but dislikes congestion. Suppose that each agent is able to use the facility for any number of hours between 0 and 1. Let the initial state represent a situation in which each agent i uses the facility for α_i hours. For agent i , a vector $y_i \in Y_i$ represents the difference between the amount which i uses the facility and α_i . Thus $Y_i = \{y_i \mid -\alpha_i \leq y_i \leq 1 - \alpha_i\}$. Where the social state is $(y_1, \dots, y_n)$ each agent i uses the facility for $\alpha_i + y_i$ hours and the total utilization of the facility is $\sum_I \alpha_i + \sum_I y_i$. Where the effects of congestion are impersonal in the sense that agents do not care who else is using the facility but only about how crowded it is, it is reasonable to suppose that preferences of each individual are determined by y_i and $\sum_I y_i$.

A formally similar model can be constructed where y_i represents an amount of a public good supplied by agent i . Here we might wish to assume that each individual finds it unpleasant to supply the public good but enjoys its presence regardless of who supplied it. More elaborate examples could include some commodities of each of the types discussed above. Public commodities could also be included which are pleasant to some agents and noxious to others.

The following notational conventions will be used. For any set $K \subset I$, let $\sim K$ denote the complement of K in I . Where $K \subset I$, let $Y_K = \sum_K Y_i$ and let

$$Y^K = \{(y_1, \dots, y_n) \in \prod_I Y_i \mid y_i = 0 \text{ for all } i \in \sim K\}.$$

A vector $y^K \in Y^K$ is called a *strategy for coalition K*. Notice that $Y^I = \prod_I Y_i$, and for all $K \subset I$, $Y^K + Y^{\sim K} = Y^I$. Where $(y_1, \dots, y_n)$ is a specified vector in E^m , let y_K denote $\sum_K y_i$. Where $y = (y_1, \dots, y_n)$ and $y' = (y'_1, \dots, y'_n)$ are social states and $K \subset I$, we write $y R_K y'$ to signify that $(y_i, y_I) R_i(y'_i, y'_I)$ for all $i \in K$ and we write $y P_K y'$ to signify that $y R_K y'$ and $(y_i, y_I) P_i(y'_i, y'_I)$ for some $i \in K$.

2. THE α -CORE AND RESTRICTIONS ON STRATEGIES

A coalition $K \subset I$ is said to be *able to improve on social state* $\bar{y}$ if there exists $y^K \in Y^K$ such that for all $y^{\sim K} \in Y^{\sim K}$, $y^K + y^{\sim K} P_K \bar{y}$. A social state $\bar{y}$ is said to *belong to the α -core* if no coalition $K \subset I$ is able to improve on $\bar{y}$.

To show that for a large class of societies the α -core is not empty, we cite the following theorem for which Scarf [4] supplies an elegant proof.

Theorem 1. (Scarf) *For a society as described in Section 1, if for all $i \in I$, the preference relation R_i is a continuous, convex, pre-ordering and if Y_i is a compact, convex set then there exists a social state which belongs to the α -core.*

This is an interesting and rather surprising result since it applies even where activities of some agents may be harmful to others. On the other hand, in these cases the α -core has some unattractive features. To quote Scarf:

A coalition S in attempting to obtain an improved position for all its members must confront the entire range of strategic possibilities open to players not in S including

those which lead to disastrous consequences for the complementary coalition and in all probability would not be undertaken.

As an extreme example of the difficulty indicated by Scarf, consider a society in which each agent is able to kill any other agent. In this society, any Pareto optimal social state which each agent prefers to being dead would belong to the α -core. The α -core would therefore be of little interest since it is not significantly more restrictive than the set of Pareto optima.

It is tempting to deal with this problem by forbidding some feasible strategies by a general rule, much as theft is prohibited in the exchange model. Where strategies are restricted, a coalition may find that strategies with which it could improve on some social states are forbidden. On the other hand, the coalition needs to concern itself with fewer responses from the complementary coalition. It may therefore be able to improve on some social states by strategies which would have been rejected had the strategies of the complementary coalition been unrestricted. Thus, in general, neither the restricted α -core nor the unrestricted α -core need contain the other.

To further the discussion, we set up a formal vocabulary. A system of restrictions is a mapping A which specifies a set $A^K \subset Y^K$ for each $K \subset I$. The set A^K is called the *set of A-admissible strategies for K*. A coalition $K \subset I$ is *able to improve on the social state $\bar{y}$ by A-admissible strategies* if for some $y^K \in A^K$ and for all $y^{\sim K} \in A^{\sim K}$, $y^K + y^{\sim K} P_K \bar{y}$. A social state $\bar{y}$ is said to *belong to the A-restricted α -core* if no coalition $K \subset I$ is able to improve on $\bar{y}$ by A-admissible strategies.

A special class of systems of restrictions are those obtained by restricting the admissible activities of any individual agent i to a subset $\tilde{Y}_i$ of Y_i , and by restricting all coalitions other than the coalition of all members of society to strategies in which each member, i , of the coalition pursues an activity in $\tilde{Y}_i$. More formally, a *simple system of restriction* is a system of restrictions A such that $A^I = Y^I$ and such that for all $K \subsetneq I$, $A^K = Y^K \cap \prod_i \tilde{Y}_i$.

The existence of a non-empty A-restricted α -core for a broad class of simple systems of restrictions can be established as an easy corollary of Theorem 1.

Corollary 1. *For a society which satisfies the conditions of Theorem 1, if A is a simple system of restrictions such that for all $i \in I$, $\tilde{Y}_i$ is a closed convex set, then the A-restricted α -core is non-empty.*

Proof. Consider the society obtained by replacing Y_i with $\tilde{Y}_i$ for each $i \in I$. According to Theorem 1 there exists a social state $\tilde{y}$ which belongs to the α -core of the society so constructed. Since by assumption Y^I is compact and preference relations are continuous pre-orders, there exists a social state $\bar{y}$ such that $\bar{y} R_I \tilde{y}$ and such that $\bar{y}$ is Pareto optimal. Since $\bar{y}$ is Pareto optimal, the coalition I cannot improve on $\bar{y}$ by A-admissible methods. Since $\tilde{y}$ belongs to the α -core for the constructed society, it is clear that no coalition $K \subsetneq I$ can improve on $\tilde{y}$ by A-admissible methods. Since $\bar{y} R_I \tilde{y}$, it follows that no coalition K can improve on $\bar{y}$ by A-admissible methods. QED

Of special interest are those systems of restrictions, A , which ensure that all social states in the A-restricted α -core are at least as good for each agent as the initial state. Such restrictions can be viewed as protecting "property rights" which are embodied in the initial state. A set S of social states is said to *defend the initial state* if $y R_I 0$ for all $y \in S$.

Remark. If the A-restricted α -core is non-empty and defends the initial state, then if the initial state is Pareto optimal, the initial state belongs to the A-restricted α -core.

Proof. Suppose the A-restricted α -core defends the initial state and contains the social state y . Then $y R_I 0$. But if 0 is Pareto optimal, $y R_I 0$ implies $0 R_I y$.

Since $0R_I y$ and y belongs to the A -restricted α -core, 0 must also belong to the A -restricted α -core. QED

As an example of a simple system of restrictions which defends the initial state and for which the A -restricted α -core is non-empty, consider the following. Let A be the simple system of restrictions constructed from the sets, $\tilde{Y}_i$, where for all $i \in I$,

$$\tilde{Y}_i = Y_i \cap \{y_i \mid (0, (n-1)y_i)R_i(0, 0) \text{ for all } j \neq i\}.$$

We call this the system of restrictions derived from the *schoolmarm's rule* because the restrictions embodied are close to those implicit in the schoolteacher's rebuke: "You may not do that because if everyone did it, someone would be harmed." Thus, the activity y_i belongs to $\tilde{Y}_i$ only if, for any $j \neq i$, when j chooses the activity 0 and all agents other than j choose the activity y_i , the resulting social state is at least as good for j as the initial state.

Theorem 2. *For a society which satisfies the conditions of Theorem 1, where A is the simple system of restrictions derived from the schoolmarm's rule, the A -restricted α -core is non-empty and defends the initial state.*

Proof. From continuity and convexity of preferences it follows that the sets $\tilde{Y}_i$ are closed and convex. Therefore, according to Corollary 1, the A -restricted α -core is non-empty.

Let $\bar{y} \in Y^I$ and suppose that $(0, 0)P_i(\bar{y}_i, \bar{y}_I)$. We show that $\{i\}$ can improve on y by admissible strategies. Clearly $0 \in A^{(i)}$. Let $y^{(i)} \in A^{(i)}$. Then $y_j \in \tilde{Y}_j$ for all $j \neq i$. Hence $(0, (n-1)y_j)R_i(0, 0)$ for all $j \neq i$. Convexity of preferences requires that

$$\sum_{j \neq i} \frac{1}{n-1} (0, (n-1)y_j)R_i(0, 0),$$

or equivalently that $(0, \sum_{j \neq i} y_j)R_i(0, 0)$. But since $(0, 0)P_i(\bar{y}_i, \bar{y}_I)$, it must be that $(0, \sum_{j \neq i} y_j)P_i(\bar{y}_i, \bar{y}_I)$. Therefore $0 + y^{(i)}P_{\{i\}}\bar{y}$ for all $y^{(i)} \in A^{(i)}$. Thus, if for some $i \in I$,

$$(0, 0)P_i(\bar{y}_i, \bar{y}_I),$$

the coalition $\{i\}$ is able to improve on $\bar{y}$ by A -admissible strategies. It follows that the A -restricted α -core defends the initial state. QED

3. LINDAHL EQUILIBRIUM AND THE RESTRICTED α -CORE

In the Lindahl allocation system, unanimous agreement on a social state is secured subject to appropriately chosen budget constraints for each individual. Discussions of this notion can be found in the writings of Lindahl [3], Foley [2], and Bergstrom [1]. Lindahl equilibrium is shown by Bergstrom to exist under very weak assumptions. For the model studied in this paper, Lindahl equilibrium can be defined as follows. A price vector $\bar{p} = (\bar{p}_1, \dots, \bar{p}_n) \in E^{mn}$ (where each $\bar{p}_i \in E^m$) and a social state $(\bar{y}_1, \dots, \bar{y}_n)$ are said to *constitute a Lindahl equilibrium* if for all $i \in I$, $(\bar{y}_i, \bar{y}_I)$ maximizes R_i on $Y_i \times Y_I \cap \{y_i, y_I \mid \bar{p}_i y_I \leq \bar{p}_i y_i\}$. (Recalling our notational conventions, $y_I = \sum_I y_i$ and $\bar{p}_I = \sum_I \bar{p}_i$.)

Lindahl equilibrium may be viewed as a social exchange in which the value to any coalition K of the strategy of its complementary coalition must equal the value to the complementary coalition of the strategy of K . Theorem 3 makes this interpretation explicit.

Theorem 3. *If $(\bar{p}_1, \dots, \bar{p}_n)$ and $(\bar{y}_1, \dots, \bar{y}_n)$ constitute a Lindahl equilibrium and if preferences of all agents are locally non-satiated, then for every $K \subsetneq I$, $\bar{p}_K \bar{y}_{\sim K} = \bar{p}_{\sim K} \bar{y}_K$. Furthermore, if $y R_K \bar{y}$, then $\bar{p}_K y_{\sim K} \geq \bar{p}_{\sim K} y_K$ and if $y P_K \bar{y}$, then $\bar{p}_K y_{\sim K} > \bar{p}_{\sim K} y_K$.*

Proof. Since preferences are locally non-satiated, it must be that for all $i \in I$, $\bar{p}_i \bar{y}_I = \bar{p}_i \bar{y}_i$.

Therefore for $K \subset I$, $\sum_K \bar{p}_i \bar{y}_I = \sum_K \bar{p}_I \bar{y}_i$. But this implies that $\bar{p}_K \bar{y}_I = \bar{p}_I \bar{y}_K$ or, equivalently, $\bar{p}_K (\bar{y}_K + \bar{y}_{\sim K}) = (\bar{p}_K + \bar{p}_{\sim K}) \bar{y}_K$. Subtracting $\bar{p}_K \bar{y}_K$ from both sides of the equation leaves $\bar{p}_K \bar{y}_{\sim K} = \bar{p}_{\sim K} \bar{y}_K$.

From the definition of Lindahl equilibrium it follows that if $(y_i, y_I) P_i(\bar{y}_i, \bar{y}_I)$, then $\bar{p}_i y_I > \bar{p}_I y_i$. From local non-satiation it then follows that if $(y_i, y_I) R_i(\bar{y}_i, \bar{y}_I)$, then

$$\bar{p}_i y_I \geq \bar{p}_I y_i.$$

Therefore if $y R_K \bar{y}$, it must be that $\bar{p}_i y_I \geq \bar{p}_I y_i$ for all $i \in K$. Adding these inequalities, we have $\bar{p}_K y_I \geq \bar{p}_I y_K$. Subtracting $\bar{p}_K y_K$ from both sides of the inequality leaves

$$\bar{p}_K y_{\sim K} \geq \bar{p}_{\sim K} y_K.$$

We conclude that if $y R_K \bar{y}$, then $\bar{p}_K y_{\sim K} \geq \bar{p}_{\sim K} y_K$. A similar line of reasoning leads to the conclusion that if $y P_K \bar{y}$, then $\bar{p}_K y_{\sim K} > \bar{p}_{\sim K} y_K$. QED

Theorem 4. *If preferences are locally non-satiated, then the set of Lindahl equilibrium social states defends the initial state and is contained in the set of Pareto optimal social states.*

Proof. Clearly $\bar{p}_i 0 \leq \bar{p}_I 0$ for all $i \in I$. Therefore if $\bar{y}$ is a Lindahl equilibrium social state, then $(\bar{y}_i, \bar{y}_I) R_i(0, 0)$ for all $i \in I$. It follows that the set of Lindahl equilibrium social states defends the initial state.

Let $\bar{y}$ be a Lindahl equilibrium social state and let y be a social state such that $y P_I \bar{y}$. Then for all $i \in I$, $\bar{p}_i y_I \geq \bar{p}_I y_i$ with strict inequality for some $i \in I$. Therefore

$$\sum_{i \in I} \bar{p}_i y_I > \sum_{i \in I} \bar{p}_I y_i$$

or, equivalently, $\bar{p}_I y_I > \bar{p}_I y_I$. But this is impossible. We must therefore conclude that $\bar{y}$ is Pareto optimal. QED

In general, a Lindahl equilibrium need not belong to the α -core. This is demonstrated by example in the Appendix. However, certain systems of restrictions do ensure that the set of Lindahl equilibria is contained in the restricted α -core.

A system of restrictions A is said to *limit coalitions to harmless strategies* if for all $K \subset I$, $0 \in A^K$ and $A^K \subset Y^K \cap \{y^K \mid y^K + y R_{\sim K} y \text{ for all } y \in Y^I\}$.

Systems of restrictions which limit coalitions to harmless strategies may leave considerable latitude for decentralized bargaining. In the exchange model discussed in Section 1, if agents are concerned only with their own consumption, a system of restrictions which limited each coalition to just those strategies which involve exchange of commodities among its members would limit coalitions to harmless strategies. A more elaborate example is the following. Consider a society where there are some private commodities concerning which preferences are selfish, some public goods whose presence is desirable to all consumers and some pollutants which individuals may enjoy producing but whose total supply is noxious to all agents. Let there be an initial situation in which there are specified quantities of private goods held by each agent and specified quantities of each public good and of each pollutant produced by each agent. A system of restrictions will limit coalitions to harmless activities if, for example, it restricts each coalition to strategies in which the total consumption of private goods by its members is the same as in the initial state, total production of desirable public goods by its members is undiminished and total production of pollution by its members is not increased from the initial state.

Theorem 5. *Let A be a system of restrictions which limits coalitions to harmless activities. Then if preferences are locally non-satiated, the set of Lindahl equilibria is contained in the A -restricted α -core.*

Proof. Let $(\bar{p}_1, \dots, \bar{p}_n)$ and $(\bar{y}_1, \dots, \bar{y}_n)$ constitute a Lindahl equilibrium. Suppose that coalition K is able to improve on $\bar{y}$ by A -admissible strategies. Then there exists

$y^K \in A^K$ such that $y^K + y^{\sim K} P_K \bar{y}$ for all $y^{\sim K} \in A^{\sim K}$. According to Theorem 3, it must be that $\bar{p}_K y_{\sim K} > \bar{p}_{\sim K} y_K$ for all $y^{\sim K} \in A^{\sim K}$. Since $0 \in A^{\sim K}$, it follows that $\bar{p}_{\sim K} y_K < 0$.

But if $y^K \in A^K$, then $y^K + \bar{y} R_{\sim K} \bar{y}$ or, equivalently $(y^K + \bar{y}^K) + \bar{y}^{\sim K} R_{\sim K} \bar{y}^K + \bar{y}^{\sim K}$. From Theorem 3 it follows that $\bar{p}_{\sim K}(y_K + \bar{y}_K) \geq \bar{p}_K \bar{y}_{\sim K}$ and that $\bar{p}_{\sim K} \bar{y}_K = \bar{p}_K \bar{y}_{\sim K}$. Therefore $\bar{p}_{\sim K} y_K \geq 0$. But this contradicts the result of the previous paragraph. Therefore no coalition is able to improve on $\bar{y}$ by A -admissible strategies. QED

A special application of Theorem 5 is to the case where the sets Y_i are constructed in such a way that for every i , every activity in Y_i is harmless to all agents other than i . This is the case of "external economies" discussed by Shapley and Shubik [5]. An example is a society in which activities available to any agent consist only of transferring desirable private goods to other agents or adding to the supply of a universally desired public good. This is the case discussed by Foley [2].

Corollary 2. *If for all $i \in I$ and for all $y \in Y^I$, $y + y^i R_{\sim \{i\}} y$ for every $y^i \in Y^I$, and if preferences are locally non-satiated, then every Lindahl equilibrium social state is contained in the α -core.*

Proof. The conditions of Corollary 2 imply that for all $K \subset I$,

$$Y^K = Y^K \cap \{y^K \mid y^K + y R_{\sim K} y \text{ for all } y \in Y^I\}.$$

Therefore, the system of restrictions, A , where $A^K = Y^K$ for all $K \subset I$ limits coalitions to harmless strategies. But in this case the A -restricted α -core is just the α -core. The result of the corollary is then immediate from Theorem 5. QED

The system of restrictions which set $A^K = Y^K \cap \{y^K \mid y_K = 0\}$ obviously limits coalitions to harmless strategies. Thus we can assert the following.

Corollary 3. *Where A is a system of restrictions such that $A^K = Y^K \cap \{y^K \mid y_K = 0\}$ for all $K \subset I$, if preferences are locally non-satiated every Lindahl equilibrium social state is in the A -restricted α -core.*

A result very similar to that of Corollary 3 is offered by Starrett [6].

It might be argued that a system of restrictions which limits coalitions to harmless strategies is overly restrictive since it allows coalitions access only to strategies which are harmless when taken from *any* starting point. Thus strategy y^K may be excluded from A^K even if $y^K + 0 R_{\sim K} y^K$ because for some $y \in Y^I$ it is not true that $y^K + y R_{\sim K} y$. On the other hand, the procedure suggested here has the advantage of providing a relatively simple general principle for imposing restrictions which does not require detailed knowledge of the general equilibrium consequences of each restriction.

A question which has not been discussed in this paper is whether it is in the interest of many or all agents to adopt a particular set of restrictions. Also, in keeping with the neo-classical tradition, we have left the question of the selection of the initial state as a problem distinct from the study of movements away from this state. A provocative discussion of these problems is conducted by the writers in Tullock [7].

APPENDIX

Here is a simple example of a society in which Lindahl equilibrium allocations do not belong to the α -core. There are two agents. There is a private commodity, bread, and a public commodity, smoke. Bread, which both agents like, is available in fixed supply. Agent 1 likes to produce smoke and is able to produce any amount of smoke between 0 and 1 units. No other resources are exhausted in the production of smoke. Agent 2 is unable to directly affect the quantity of smoke and dislikes smoke produced by agent 1. Possible social states can be represented in the rather ordinary looking Edgeworth box of Figure 1.

FIGURE 1

Any point in the box represents a social state in the following way. The distance of a point from the left side of the box is the amount of bread consumed by agent 1. The distance from the point to the right side of the box is the amount of bread consumed by agent 2. The distance from the point to the bottom of the box is the total amount of smoke produced by agent 1 (as perceived by both agent 1 and agent 2). Points on the top of the box represent social states where agent 1 produces as much smoke as he possibly can. Where preferences are convex and the direction of preference of each agent is as described above, indifference curves of each agent display the customary shapes. That is, indifference curves of agent 1 bow toward the lower left and those of agent 2 bow toward the upper right. The contract curve containing all Pareto optimal social states is represented by the curve OP . The initial state is represented by the point W .

Suppose that feasible activities for the agents are such that each agent is able to give bread to the other (but not to steal bread) and such that agent 1 is able to reduce or increase his smoke production to any amount between 0 and 1. Then in the diagram, the set $W + Y_1$ is the shaded area to the left of W and the set $W + Y_2$ is the line WA .

By choosing the activity in which he produces one unit of smoke and keeps his initial holding of bread, agent 1 can be assured of a social state which he likes at least as well as that represented by the point B . By choosing the activity in which he keeps his initial holding of bread, agent 2 can be assured of a social state which he likes as well as B . Neither agent can improve on B by himself. The α -core therefore consists of the points on the segment CD of the contract curve. On the other hand, the point E of Figure 1 is a Lindahl equilibrium social state. Clearly E does not belong to the α -core.

If coalitions are restricted to harmless activities, then without the consent of agent 2, agent 1 is not allowed to increase the amount of smoke which he produces beyond that which he produces in the initial state. The core would then consist of the segment of the contract curve which is Pareto superior to the initial situation which, of course, includes E . Thus the Lindahl equilibrium belongs to the restricted α -core.

Notice also that if the initial situation were represented by the point B , the α -core would still be the segment CD . With this initial state, Lindahl equilibrium must also lie on CD and hence belong to the α -core. This illustrates Corollary 2, since in this case all feasible activities for either agent are harmless to the other agent.

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