Title

The Tropical Geometry of Nonequilibrium String Theory

ISBN
9798189279713

Author
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Publication Date
2026-05-01

Peer reviewed|Thesis/dissertation

The Tropical Geometry of Nonequilibrium String Theory

by

Andrés Antonio Franco Valiente

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Physics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Petr Horava, Co-chair
Professor Ori Ganor, Co-chair
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Spring 2026

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Abstract

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This dissertation investigates ingredients necessary for the worldsheet formulation of string theory in nonequilibrium regimes, moving beyond the traditional framework of static, equilibrium target spaces. While conventional perturbative string theory has yielded profound mathematical insights into differential geometry and algebraic topology, these successes are overwhelmingly predicated on equilibrium dynamics and simplifying assumptions like supersymmetry. However, modeling the complex, time dependent evolution of realistic quantum systems from the rapid expansion of the early universe to driven quantum materials requires a complete generalization of string theory. The aim of this thesis is primarily to introduce the scientific reader to the problem of nonequilibrium string theory.

To address this problem, we leverage the Schwinger Keldysh formalism alongside path integral localization. We suggest that from the perspective of the string worldsheet, the turnaround region of the Schwinger Keldysh closed time contour degenerates into a piecewise linear structure mathematically governed by tropical geometry. We explicitly show this in the simplified case of a topological sigma model by applying a tropical limit and through using the localization principle of cohomological path integrals, we formally construct *tropological* sigma models. These novel models do not require a nondegenerate metric or complex structure on the worldsheet, yet their path integrals successfully localize to tropical limits of pseudoholomorphic maps and yield a non relativistic string worldsheet foliation structure based on nilpotent worldsheet endomorphisms that we coin *Jordan structures.*

Finally, the dissertation explores the extended dynamics of these models, including the introduction of boundary conditions that give rise to tropical branes. It further demonstrates that tropological sigma models can admit unusual higher dimensional target spaces structured as filtered geometries which further admit exotic kinematical symmetries given by Nilmanifolds.

This forms an unusual connection between the geometric analysis of filtered geometries and Nilmanifolds with the real algebraic geometry given by tropical geometry.

Dedicated to the Knight of Immortal Fire walking through Early Summer Rain

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Acknowledgments

This thesis would not have been possible without the support of everybody (whom I've ever met directly or have been influenced by indirectly) who has pushed me to pursue study physics. I am deeply indebted to Petr Hořava for having been an incredibly supportive advisor, pedagogical collaborator,a kindred soul and a friend to such an unusual researcher. I now understand that the nonequilibrium string is not something that will be finished in a single doctorate and it will be set of questions that I will be thinking about for the rest of the time that I am alive. If mother nature shines upon us, I will seek you out in Czech for a hike.

I am also very deeply indebted to Ori Ganor for always being a very encouraging mentor and a friend when discussing life trajectories. I had decided to remain in Berkeley for graduate school primarily because of your outreach. Your ability to think in so many different dimensions is only outmatched by your kindness. Your support of Emil and I has drastically changed our trajectories. I believe that we'll be able to bring some impact to the world after the PhD.

The baffling amount of work that was done throughout this thesis would have simply been impossible to do without the collaboration of Emil Albrychiewicz. Together, it feels like we have done far more than a doctoral degree. Since our undergraduate years here at Berkeley, we learned and researched through so many diverse subject areas such as pure mathematics, physics, computer science, economics, finance, politics, healthcare and now seemingly, the business world. I do not know what it is like to have a brother but this might be as close as it gets. Thank you for picking me up from Fremont that night. You are definitely somebody who can not only outwork me but has a very vast intelligence across so many different subject areas. You are also somebody with very strong ethical values that I look up to and respect deeply. Although I do not yet know where our individual trajectories will take us, I am certain that we will continue working in some way. I wish us both luck in the endeavors we are about to undertake.

Louisa Weng Chi Man, you gave me a teaching job that changed my life. Prior to working for you at the Chemistry tutoring center, I was a butcher and although that line of work wasn't too physically demanding, it was quite limiting. You helped me discover that one of my passions was genuinely helping people. Through this job, I met so many people with so many different histories and the amount of extra empathy that I was able to obtain from these experiences is invaluable. Since then, I was able to use these experiences to try to help out as many transfer students as I could and in return, I have been supported by the transfers themselves. For each and every single student of Physics 153, you are each in my memory. I started the 1:1s precisely so that I could get to know as many of you.

I thank Max Dornfest for having an essential part in making my stochastic trajectory from chemical engineering to physics completely deterministic. When I was 18, you left such an enormous impression on me by demonstrating to me just how much knowledge a human could hold. It is not simply surface level knowledge.There are many statements of yours that

I remember and which have helped me develop into a better person over the years, I thank you.

I thank the Berkeley admissions officers who first allowed me to obtain an undergraduate education through the Regents & Chancellors scholarship. I thank the College of Chemistry for first approving me to study chemical engineering before I switched to theoretical physics. I would have not attended college without this scholarship. Furthermore, I thank whoever forwarded my PhD application to the graduate division and gave me the additional support of the Regents fellowship. This fellowship solidified my life purpose in trying to make sure that whatever I work in is primarily for the community around me. Upon arriving at the Berkeley Center for Theoretical Physics, I felt as though I had completed a very specific dream of mine. Once again, I must now thank again Petr Hořava, Ori Ganor and David T. Limmer for supporting me enough that the UC system has awarded me the President's postdoctoral fellowship. Quite interestingly, it seems as though I may be returning to my first academic love: Chemistry. I would also like to further extend my thanks to Aaron Fisher and Raul Briceno for having agreed to read my thesis at the last minute and specifically Professor Fisher for having also gone through two years of string theory within one hour during my doctoral qualifications. Although I do not really know how my career trajectory will evolve given external circumstances, be certain that I will somehow give back this support to the world.

Prior to undergraduate admissions, I had been somewhat emotionally impacted by the state of the world and genuinely believed that a path to education was not possible. I thank Lowell E. Hazelton for essential words during that time period. You very firmly predicted that I will make my way and this thesis is proof of that.

It feels a little strange to thank my family directly. I believe whenever I say "I", I really mean that my family and I thank these individuals but nonetheless, I want to explicitly thank my mother and my father for having given up so much to try to give my sisters and I a chance. I know that there has been a lot of uncertainty about whether or not this was ever the right decision but I believe that this dissertation is at least one good signal that we did everything we could and believed was the right thing to do. It wasn't a straight path but there aren't any regrets. Pamela, Gissella. I know that you two have done quite a lot and have been through your own stories, I am sorry for not being present. There's a lot of things that we have all missed and a lot of family that we have in some sense all lost. I will not state everybody's name but I leave this a record that I am constantly thinking about all of you. Abuela, gracias por saber siempre, de alguna manera, cuándo llamarme. Parece que, de algún modo, eres capaz de percibir mis emociones. Creo que heredé de ti mi capacidad de comunicación, mi resiliencia y mi visión estratégica.

Obtaining an endless amount of scientific and mathematical knowledge to spread was very much an escape at times. There were periods that besides quick eating, sleep and training, I would spend effectively 16 hour workdays going through as much as I could. This would not have been possible without music, memories and people in the background. As silly as it may sound, the music of several games and animations including Warcraft III, Ragnarok Online, Gunbound, Final Fantasy X, Devil May Cry, Ratchet and Clank, God of

War, Dark Cloud, Shadow of the Colossus, Minecraft, Ultrakill, Skyrim, Naruto, Dragonball, Full Metal Alchemist, Hellsing, Jojo, Evangelion, Korn, Rhapsody of Fire, Luca Turili and most importantly Stratovarius has somehow forged the core of my character. Familiar music brings a peace to my being that I cannot describe and it is what has allowed me to work for so long.

I will now resort to chronological order of thanking individuals and groups of people in order to respect the privacy of individuals. April, I still listen to Ragnarok on almost a daily basis; it reminds me often of when we were just children. It's somewhat amazing how we're still talking almost daily after two decades. I know that I am often extremely busy with all the work but I really appreciate your patience. I have been able to see what a nurse faces through your eyes and it is a reminder of all the work that I must do. James Verges, I really appreciate you having matured so early in life; we had a lot of good laughs when were kids. Erick Daza, I am quite happy to have reconnected with you briefly when you were doing your engineering degree; I appreciate you and your family giving me a copy of Peskin and Schroeder's Introduction to Quantum Field Theory. I still have the copy and read through it entirely during my senior undergraduate year. Erik Correia, I think it is clear that you were my best friend throughout high school. You showed me a lot of what is it like to have a friendly rivalry. I really appreciate all our training sessions and I apologize for not having been more supportive at certain critical times. I want to thank your family and you for taking me on that Reno trip for the water polo tournament. To this date, this is still one of the furthest distances I've ever gone outside of California. Those granola bars that you would give me during lunch time were real savers. In hindsight, you kinda kept me from starving during high school. Sarah Brown, I appreciate us spending time as kids and thank you for the Minecraft creeper cookies. I am wishing you, your new family and your new babies a wonderful life! Gordon Triano, we spent a lot of time when we were younger, I still remember our times playing motherload. I want to thank my first water polo team. I can't remember most of your names since many of you weren't the most friendly people but I very much enjoyed challenging you in all those practice sessions. I thank my "unnamed" military water polo coach, you specifically mentioned that I have a work ethic that nobody else has. I intend to make sure those statements stay true. James Drage, when I did not think I could attend college, although you did not say anything in particular, you were always very indirectly supportive of the small achievements that I made. It very much made me fall in love with chemistry. Evan Jenks, you have inspired me in such a deep way during high school. I try to reproduce the same passion that you had with my students. You were very genuine and you were really seeking for truth. I will continue to do that. Heeda Pathan, you gave me a lot of confidence in many different times in high school. Robseth Taas, you really showed me a lot of practical solutions to simple problems that I had not considered. Rendale Taas, your positivity was so refreshing. I really enjoyed just how much energy you put into things. Moises Mares, you are a very interesting fellow. I was always entertained by your comments. Qianhui Amiee Loo, I appreciate learning biology with you for a year. That year of biology has proven again and again to be extremely useful throughout my life and has formed the foundation for many later things in biochemistry and pharmacology. James

Macleod, I appreciate your support for the PharmD letter of recommendation. Brandon Chu, I was honestly so impressed by the accuracy of your words ever since we met in high school. I am very happy to see that you have completed your MD at UCSF and I wish you luck in your residency. We only had a few conversations but I hold a very deep respect for you. Inju Yi, you were an enormously friendly individual, I really appreciated every single time you popped up in high school. Patricia Chen, you are my first pyrefly. I know that there are a lot of things that I could have done better early on during our time and I believe that you saw the most important parts of me before you left to Taiwan. It was the first time that I knew I was a knight of immortal fire. Despite our disconnect, you were an enormously important motivational push for me to give back as much as possible. Chelsea Erling, I very much enjoyed our brief hangouts. Professor Leonard, I would have not learned mathematics without you. I was graduated high school without ever touching calculus and after 8 weeks learning from you, I had already become quite comfortable in vector calculus. You were essential in fixing my starting point. Jhevon Smith, I have never had you officially as one of my teachers but you were the person who taught me ordinary differential equations and introduced me to partial differential equations all while I was still enrolled in precalculus. I am deeply indebted to you. Ed Rye or the Kaizen effect, *clap*, I loved your videos as well. You were the first to introduce me to the calculus of variations. It's still a tool that I use today and occasionally scare economists with. Pavel Grinfeld, we also did not meet but I somehow heard additional stories about you from your student Johannes Wagner. I believe it was around my second semester of college that I began to learn tensor calculus from you. Shahnawaz Soroya and Masood M. Saaed, man. I swear I had so much fun with you guys, I don't think I ever reproduced that amount of fun. I'm glad to have met people like you, I apologize for being too much of a uber nerd. Peter Sohn, you really tried to save me at one of my darkest points in life. Although I did not find the specific faith you were looking for, I found a lot of faith in the good people who are present. Thank you. Matilda G. Lau, I know, I know. Sometimes the best policy is to not say anything, I'm sorry and thank you for your time. You had your own plan of supporting people, I will try to do it indirectly. Akshatha Muralidhar, thank you very much for the tea, that was a very thoughtful thing and unexpected. Kanisha Sha, you are a fireball but it was always very good spirited. Kinglsey, our conversations were actually pretty deep always. I appreciate it. Yash Kumar Johar, you were one of the most prominent people for me during our organic chemistry years. I could not believe that there was somebody that good at chemistry. I would try Chaos mode with you again. Malvee Vasan, you are a definitely another sister. We always have so much to say that I know it can get quite overwhelming. I know you yourself have a very interesting story and despite all the things, you have been very supportive all these years. You are in my thoughts. Sophie May, you as well. Siddharth Mukherjee, you are one of my most important students and now you are a very important person. You are in some sense a little brother. I learned quite a bit about mentoring from you and I hope that you enjoy this physics journey as much as I did. I want to thank your parents for also having supported me during the time, I was able to eat from our exchanges. Andrew Wise, you provided me the opportunity to go further into mathematics. You also brought an important person to my life which

was Cynthia. Cynthia Wang, I absolutely cannot believe the amount of things that we went through from nighttime mountain running to sleeping in the middle of nowhere. It was quite a free time, I hope you are doing well. Jarna Patel, thank you and your uncle for providing a place for me to study. I unironically love smelling like Suju's Coffee. Baotong Han, thank you for the rides. Meelood Rahimi, you were also one of my hardest working students, I appreciate your tenacity. Raza Bakr, it was incredibly fun to workout physics with you. I won't forget our relativistic rockets. Adam Liron, you are also a very interesting character. I was very entertained by some of your comments. Allin Chen, you really did scare me the first time we met with your crazy personalized problem sets. We both transferred to Berkeley and stayed here for our PhDs, although we didn't talk too many times, I appreciate the few times we hung out. Eric Soto, you are a very genuine friend and despite what others may say, I appreciate your mathematical insights. You are always trying and it is a private inspiration to me. Xiaoyu Niu, you are still around all these years and you are probably the most wild person I know. I think you will be there in the background for an extremely long time. Scout Xingzhi Zhou, I genuinely only meant to do a quantum mechanics problem set with you but we ended up getting far more than we bargained for. I appreciate the LA trip. That was the first time that I ever flew on a plane alone. Your have a very fierce way of caring and I appreciate it. I will try to get you out at some point. Andrew Chang, you have no clue how amazed I am at every single time that we speak. It's somewhat baffling that somebody has such a wide range of mathematical abilities and social abilities. It's rather refreshing that we can discuss on such a wide array of things on so many different levels for such a long time. I apologize that we don't talk too often, I believe that we can easily speak for dozens of hours but we have life to attend to. I hope to continue to have you around as a friend for years to come. Connor Duncan, I appreciate you for all your smart remarks. You really saved me with 111A. Annan Deng, I almost died running in the smoke. Katherine Ping Zhou, that was indeed a very crazy year. Mathias Reinsch, thank you so much for your letter of recommendation. I very much miss your big chalks that you used for lecture. You had a very unique aura when developing subjects and I appreciate your patience when I was asking you about Nakahara's Geometry and Topology. It was my first semester there so I did not know that it was unusual to ask. Christopher Mogni and Marie Hepfer, you two are one person. But Chris, I think you are the most influential GSI. I really appreciate all the different times we went to Asha to discuss string theory. I am also sorry for spamming you with an ungodly amount of questions during our first semester, I now realize how unusual that is. I also wish I had not been so spread so that we could've developed some of the more interesting ideas that you had about nonequilibrium strings and I also apologize for not having the bandwidth to also do the software work. You really care about people and I will repay back what you have tried to give me during my undergrad and during my PhD. You are also perhaps, one of the most intelligent friends that I have. I suspect that we may still cross paths in the future since there is only a very small set of skills left that I think I need to master before I have the ability to fully context switch to anything on a hourly basis. Tymothy Mangan, you are also a very caring person. I appreciate so many of our conversations throughout the years. In some sense, we have been playing out the same life story in our personal lives, I hope to eventually meet up with you. Dizhi Wang, you are my second pyrefly. Although I think we both disagree on what is uniqueness, I know that the whole experience was very much worth it. As you said, there is no regrets. There were many things that I did not express and you coming here to take care of me is something that I can never repay. Yasunori Nomura, thank you for your letter of recommendation. Your energy is quite contagious and your birthday is very amusing. I always remember it. Chao Ju, I think you taught me my first non stringy scattering amplitude through Feynman rules when we were taking field theory. I also really appreciate you taking me out to see the fireworks that night. We had a lot of deep rare conversations throughout the years and I will never forget the several hour long chess game we had. I think we both got somewhat tired and maybe just threw the game because we kept checking each other past a certain point so perhaps one day we will have a rematch. Xuyang Yu, you are also somebody whose presence I appreciated very much. I apologize for changing throughout the years. Hitoshi Murayama, your ability to communicate scientific topics is so refined. I try to reproduce it in all my lectures. Jessie Cara Zhao, you were there for one of the lowest points of my life. I know that I had and perhaps still have quite an unorthodox way of doing things. I am still somewhat shocked that I could spend so many days talking to somebody for topics that were not at all mathematical. I will respect our agreements and I hope to see you achieve everything you want. Johannes Wagner (and now Amari Cowan), you are one of the men that I respect the most, not because of how big your biceps are but precisely because of your calculated, true kindness. I am very excited to see you get married. I expect to be your lifelong friend until our time passes. Chad Harper, you have some of the most interesting conversations about social structures, I hope to have some more. Malte Schwarz, you have a very gentle nature, I appreciate it. Ivan Maurico Burbano Aldana, you were quite an inspiration since we first met. I really do appreciate your background and how you can full send theoretical physics despite everything that goes in life. I believe that in an alternative worldline where we have more time, we would have been very excellent collaborators. Kai Ellers, I apologize also not always being present but I have appreciated having you as a peaceful housemate for all these years. Edward Yam, we have a lot of work to do! Tehya and Iman, you two were very motivating throughout the years. I wanted to continue learning more research level topics so that I could teach you both as much as I could. I hope you reach your career goals. Michelle Dong, it is somewhat amazing how well everything turned out. I appreciate the mindfulness between us now. Being an artist is actually quite fun. Vi Hong, you brought back a lot of the playful nature of life to physics. I will be very excited for the day that you are makign a movie about noncommutative geometry. E. Noelle Blose, I really appreciate you helping me out with some legal documents. It was really a lifesaver. Ira Young, developing the SEEDs program with you was quite an important pillar of mine. Seeing a lot of these individuals grow up through my doctorate gave me several different layers of support. Yuan Feng, thank you for letting me be your mentor. I appreciate the life wisdom that you exchanged. Jesus Sanchez, you were an absolutely amazing collaborator. You really walked a path that I wish I had taken occasionally. You showed me the beauty of geometric analysis and you really showed me just how far the index theorems can be pushed. I really hope that you make the

path integral rigorous. Clara Hung, you as well. Those few study sessions helped me get out of a cycle. Aaron Fisher, it was around this time that I came to you for help on attending my doctoral qualifications. I have to say I was very surprised that you agreed but it started to become very apparent to me once I started reading a bit more into your works and how you support people. Thank you very much. Kathleen Cooney, thank you so much for all your support with the transfer course over these years. Ariana Castro, you really saved my thesis last minute. Thank you for the last minute support! Claudia Trujillo, you really felt like an additional auntie. Thank you so much for bringing me into the hispanic community at Berkeley. The few moments that we've discussed, I very much felt at home. Charles Sagun, you aren't going anywhere. I'm pretty sure we will intersect at some point in the future very strongly. You really kept the years of my PhD quite interesting. I've had so many laughs with you and it will be interesting to see how we build our life over the next decade. Benardo "Vegeta" Jr, you were an excellent person to compete against. I know you are so proud of your achievements, I don't think I've ever seen anybody so ridiculously prideful. What is fun is that I know you are real good at what you do and I know that it bothers you that I can get as strong as you. You are a very worthy challenger. I never once held negative feelings towards you, I just like messing with ya. I appreciate your students Kathleen Liang, Emily Lee and Jose Rivera. I appreciate the people that I have through the RSF gym since 2018. Many of you I have never spoken to but we've acknowledged eachothers existence by training often and I appreciate the atmosphere. For many of you, we had some very nice dinners and discussions, many that were just simply playful but still nonetheless appreciate. Thank you Takashi, Lynn, Grace He, Dior Chen, Garrett Louie (and Matthew Tao), Peter Jin, Alicia Zhang, Ian Miller, Nick Pham, Julie Lee, Amy Tran, Nao Ohashi and Nari Yum. Chien-I Chiang, thank you for showing me that there is a way back into teaching. Seeing you and your family and how you support all the students at Berkeley and at your other colleges is very inspiring. Ilan Roth, you were always a pleasure to talk to and walk with. I hope to keep talking to you whenever we have a moment. Thank you for your offerings. Juan Peres, you have the right attitude for so many things. I appreciate the reality checks you gave out. The year of 2024 was a specifically special one for me. In that year, I've met two people who I will be indebted to for an incredibly long time. Li-Ching Chen, you will be an incredibly important bedrock for the work that is to come. I believe that we will be able to setup the healthcare wedge. Viola Zixin Zhao, the last two years were incredibly profound. Although now it is time to separate ways as you go and do your biophysics PhD, I will be looking forward to verifying specific aspects of what you have said. Ten years is quite a long time. There are No Promises to Keep. Thank you Aditi Krishnapriyan for giving me a chance and introducing me to some of the graduate students at BAIR. In the last year, I had the pleasure of meeting and working Swapan Chaterji. I appreciate our dinners together. You have a very unique insight into so many different subject areas, it somewhat feels as though I am talking to my late grandfather at times. David T. Limmer, we have only met recently as well but I am very much indebted to you for supporting Viola's research career as well as my own. I know that we are settling into the group since we are making a transfer over from high energy physics however it seems that we are all kindred spirits in the sense that we have often been the type of people who want to research and unify as many fields as possible using nonequilibrium physics as the core guide. Sierra Elbert, it is very clear to me that we are slightly different copies of eachother. I hope to keep on supporting you for as long as I possibly can. I suspect that you'll be the third horsemen of Alignment. We'll be able to challenge those who are temporarily ahead. Raul A Briceno, thank you again for your support. You had offered me on several occasions many opportunities to do nuclear physics research and now I am learning quite a lot of useful techniques from you. You are a very chill guy. Lastly, I want to thank Julian Hong and Travis Zack. Ever since I originally left my trajectory into pharmacy, I've always wanted a way to help out in healthcare again. I believe that there is a create deal of overlap between the stochastic thermodynamics that I intend to learn and the RL algorithms that you are both interested in clinical reasoning models through Li-Ching's research. In the last year, I've spent way more time developing this part of the journey. There were a few individuals that I was particular moved by and I hope to continue in good relations with over the next few years. Oscar Simon, it looks like we resonate far beyond just business. I'm glad that you also care for people. Galym Imanbayev, you have really provided a lot of feedback when we focused in on the healthcare wedge. I look forward to our dinner. Jacob Vogelstein, I truly appreciate the support that you have given us to this point. I do not yet know what will be our ultimate business relation but I can really tell that we already match in terms of scientific thinking. I look forward to having good relations with you for the long run. Arvind Gupta; I will do everything I possibly can to make sure we can make an profound impact. I believe that there is truly a profound reason when you wrote your book. I will do my best to live up to the role.

Finally, I'm sorry that I could not take care of everybody simultaneously. I will continue to grow stronger and wiser over time despite the occasional dips. Everybody (and more more unnamed for privacy) who I've thought about during writing these acknowledgments forms a piece of my motivation to keep on working.

Chapter 1

Quantum Field Theory in and out of Equilibrium

Quantum field theory (QFT) stands as arguably the most profound conceptual synthesis in modern physics, providing a universal mathematical language that transcends traditional subdisciplinary boundaries [1]. Originally formulated to reconcile quantum mechanics with special relativity, QFT has evolved far beyond its triumphant genesis in the Standard Model of particle physics. Its profound interdisciplinary impact is perhaps best exemplified by the deep structural parallels it established between high energy physics and condensed matter physics. Through the revolutionary lens of the Wilsonian renormalization group, continuous phase transitions in macroscopic materials and the ultraviolet behavior of fundamental particles were revealed to be two sides of the same mathematical coin. This conceptual bridge allowed physicists to apply the exact same field theoretic machinery to phenomena ranging from the subatomic interactions of quarks and gluons to the collective excitations of macroscopic many-body systems, cementing QFT as the indispensable framework for understanding nature across vastly disparate length and energy scales.

This cross pollination of ideas has only accelerated in recent decades, particularly as the frontiers of both high energy and condensed matter physics have converged upon the formidable challenge of strongly coupled systems. In these regimes, where traditional perturbative approximations inevitably break down, physicists have been forced to seek out entirely new theoretical methodologies. Consequently, advanced non perturbative methods of QFT have transitioned from formal mathematical curiosities to indispensable analytical tools. Techniques such as localization [2] in supersymmetric and/or topological quantum field theories and bootstrap methods in conformal field theory [3] have become essential. Parallel to these analytical advancements, the explosive growth of high performance computing architectures has elevated numerical methodologies, enabling the direct, non perturbative evaluation of path integrals via lattice discretizations which is famously pioneered in lattice quantum chromodynamics [4] where physicists are now able to simulate strongly interacting matter from first principles. The advent of holographic dualities which map strongly correlated quantum many body dynamics onto weakly coupled gravitational systems have provided a revolutionary geometric dictionary for decoding the behavior of exotic phases of matter such as strange metals and high temperature superconductors.

Furthermore, the modern understanding of nature increasingly demands that we look beyond static, equilibrium configurations. The study of nonequilibrium dynamics arises commonly in investigations regarding the rapid expansion of the early universe, the thermalization of the quark gluon plasma in heavy ion collisions, or the real time evolution of driven quantum materials. Consequently, QFT is not merely a static lexicon of fundamental interactions, but also serves as a deeply dynamic framework necessary for modeling the complex, time dependent evolution of quantum systems driven far from equilibrium.

It is within this rich, interconnected landscape of quantum field theory that string theory naturally emerges, not as a departure from field theoretic principles, but as their most elegant and ambitious culmination. As a subarea, string theory has served as an unparalleled engine for formal discovery, yielding a profound corpus of distinct mathematical results. By requiring the internal quantum consistency of the string, the framework has uncovered deep, unexpected links to differential geometry, algebraic topology, and perhaps most famously manifesting in the rigorous classification of Calabi Yau manifolds parallel to the discovery of mirror symmetry. However, it is imperative to recognize that this formidable mathematical machinery, and the elegant geometric data it extracts, is overwhelmingly predicated on the assumption that the target space in which the string lives in is in a state of static equilibrium and with a great deal of simplifying assumptions, which is widely known as supersymmetry. Consequently, the primary objective of this thesis is to venture beyond this equilibrium paradigm and investigate how to formulate string theory far from equilibrium. However, it is not the intention of this thesis to have an overly technical tome of unreadable results due to just how vast and complicated the problem is but instead give the mathematically inclined scientific researcher an introduction to problem of nonequilibrium string theory and understand precisely why this problem is an exciting and important frontier to work on. The author apologizes for the lengthy exposition but believes that the introductory chapter is absolutely essential for the reader who comes without context. Experienced researchers in high energy theoretical physics may prefer to skip directly to Chapter 2 2.

In Chapter 1, we will give a wide and conceptual overview to the background context needed to understand the problem. The main tool that we will be utilizing to formulate the problem is the path integral. We will discuss how path integral localization is a key component of being able to formulate analytically tractable answers and how the Schwinger Keldysh formalism drives the underlying motivation for our starting path integral. We will then introduce the basic concepts of string theory and a simplified string theory known as a topological string theory which will be essential in our technical results in the subsequent chapters. We will then motivate their nonequilibrium extension through the Schwinger Keldysh formalism. The main technical statement being that the turnaround point of the Schwinger Keldysh closed time contour from the point of view of the string worldsheet degenerates and should be described by piecewise linear geometry known as tropical geometry.

In Chapter 2, we will shift over to a much more technical language and investigate the turnaround region of the Schwinger Keldysh contour in the simplified case of a topological sigma model. We will show that that one is able to take a "tropical limit" of the topological sigma mode by leveraging the localization principle of cohomological path integrals. This will result in what is known as a tropical topological sigma model or a *tropological* sigma model for short. We will go over the kinematics and dynamics surrounding the tropological sigma model and see very explicitly that they can be mathematically described in terms of a randomized extension of tropical geometry.

In Chapter 3, we will discuss further investigations of the dynamics surrounding tropological sigma models. In particular, we will see that tropological sigma models still admit boundary conditions that suggest the extension of *tropical branes.* We will also see that the tropological sigma model admits unusual higher dimensional target spaces which are themselves filtered geometries and have exotic kinematical symmetries related to nilpotent Lie algebras.

In Chapter 4, we will close with some discussions about the current results and where the community should be heading as a whole. The main lesson being that generalizing string theory into nonequilibrium string theory is not something that will be accomplished by a single researcher but will require a focused effort.

1.1 Infinite Dimensional Integration

Our story begins with some of the most complicated summations that humanity has had the pleasure of encountering, that is the Euclidean path integral. In order to make way, we will have to go back and review the normal distribution and their corresponding Gaussian integrals.

Consider a single, continuous random variable X and denote its realization by x, then the Gaussian integral is defined by

$$Z_0 \equiv \int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2}, \quad a > 0. \tag{1.1}$$

We will often call this object, the free/noninteracting partition function and denote all "noninteracting objects" with a $_0$ subscript. This seemingly simple integral is at the core of an endless amount of analytical results in quantum field theory. In quantum field theory, we will call x, the observable. A short calculation allows one to see that the partition function Z can be evaluated to be $\sqrt{\frac{2\pi}{a}}$. To match conventional statistics [5, 6], we often choose $a = \frac{1}{\sigma^2}$ and we shift by a constant amount μ. Using this fact, one can define a normalized shifted probability distribution function $p(x)$ known as the normal distribution to be

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \tag{1.2}$$

for later bookkeeping purposes, we will generically call the negative argument of the exponential, the Euclidean action S. In this case, it is $S(x) = \frac{(x-\mu)^2}{2\sigma^2}$, hence it is quadratic in

x. From this probability distribution function, one is then able to compute basic statistical observables such as moments of the random variable and/or correlation functions. These are computed through the standard definition

$$\langle \mathcal{O}(x) \rangle \equiv \frac{1}{Z} \int_{\mathbb{R}} dx \, \mathcal{O}(x) p(x) \tag{1.3}$$

A simple calculation shows

$$\begin{aligned} \langle x \rangle &= \mu, \\ \left\langle x^2 \right\rangle &= \mu^2 + \sigma^2, \\ \left\langle x^3 \right\rangle &= \mu^3 + 3\mu\sigma^2, \\ \left\langle x^4 \right\rangle &= \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4, \end{aligned} \tag{1.4}$$

where $\mu = \langle x \rangle$ is the mean and $\sigma^2 = \langle (x-\mu)^2 \rangle$ is the variance. From analyzing the above correlation functions, one can already begin to see a pattern that the first two moments form sufficient statistics in the sense that all higher order correlation functions may be explicitly written in terms of μ and σ^2. Notice that for the case when the mean vanishes, the odd correlation functions also vanish. The key lesson to obtain from this elementary example is that unlike standard statistics where we begin by claiming that a random variable is normally distributed from the start, we instead practically derive the probability law of the random variable by postulating a partition function through the corresponding Euclidean action.

Now let's consider the generalization of this case where we have N continuous random variables $x_i, \quad i = 1, \dots, N$. We will utilize Einstein summation convention, where repeated indices are summed over. Let the Euclidean action take the form

$$S_0(x_1, x_2, ..., x_N) = \frac{1}{2} x_i A_{ij} x_j \equiv \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i A_{ij} x_j, \tag{1.5}$$

where for demonstration purposes, we can restrict the matrix coefficients A_{ij} to arise from a real symmetric positive definite matrix $A_{ij} = A_{ji}, \quad v_i A_{ij} v_j > 0$ for all $v_i \neq 0$ to guarantee convergence and define the integration measure to be $d^N x \equiv \prod_{i=1}^{N} dx_i$, we then have a free N dimensional partition function given by

$$Z_0 = \int d^N x e^{-\frac{1}{2} x_i A_{ij} x_j}. \tag{1.6}$$

Due to the fact that we have restricted to symmetric matrices, we are able to utilize some elementary linear algebra to diagonalize the matrix coefficients A_{ij} by an orthogonal matrix O. This can be explicitly implemented by changing variables from x_i to a new set of variables y_i through an orthogonal transformation. The resulting integral is then factorizable into individual 1 dimensional Gaussian integrals given by the eigenvalues λ_k of A which can then be recombined to yield the determinant

$$Z_0 = \int d^N y e^{-\frac{1}{2} \lambda_k y_k^2} = (2\pi)^{N/2} (\det A)^{-1/2}. \tag{1.7}$$

Following the 1 dimensional example, we can see that the n point correlation functions $\langle x_{i_1} \cdots x_{i_n} \rangle$ where i_j is a multindex that can take any value in the set $\{1, \cdots, N\}$ can then be easily calculated through the evaluation of the 2 point correlation function

$$\langle x_i x_j \rangle_0 = \left(A^{-1}\right)_{ij}. \tag{1.8}$$

Quite remarkably, the consequence of this results states that the information present in any other correlation function of interest can be ultimately reduced down to the numerics present in the inverse of the matrix A which defined the original partition function. This aforementioned result is known as Wick's probability theorem. This has such profound consequences that in quantum field theory, we name the two point correlation function, the propagator. For example, it is easy to see that the 4 point correlation function can be decomposed into sums of products of two point correlators

$$\langle x_i x_j x_k x_\ell \rangle_0 = A^{-1}_{ij} A^{-1}_{k\ell} + A^{-1}_{ik} A^{-1}_{j\ell} + A^{-1}_{i\ell} A^{-1}_{jk}. \tag{1.9}$$

While these Gaussian results are mathematically elegant, they describe a universe of noninteracting/free particles that pass through one another without ever exchanging a single bit of information. To bridge the gap between this sterile toy model and a realistic model of the physical world, we must introduce interactions where we can actually begin to see collective behavior. The most general cubic interaction for N random variables that one can add in has the form

$$S^{(3)}_{\text{int}}(x) = \frac{g}{3!} T_{abc} x_a x_b x_c, \tag{1.10}$$

where T_{abc} can be taken to be a symmetric tensorial object and we have normalized by 3! to make the combinatorics work out nicely. However, to make the combinatorics maximally transparent, it is often convenient to choose the most explicit diagonal examples where $T_{abc} = g\delta_{ab}\delta_{ac}$ with g a real number, then

$$S^{(3)}_{\text{int}}(x) = \frac{g}{3!} \sum_{a=1}^{N} x_a^3. \tag{1.11}$$

Likewise, one can also consider quartic interaction terms of the form

$$S^{(4)}_{\text{int}}(x) = \frac{\lambda}{4!} \sum_{a=1}^{N} x_a^4. \tag{1.12}$$

As we have previously mentioned, in theoretical statistics, we often formally postulate that a random variable follows a particular probability law through language. Here, we instead tell you what the corresponding overall Euclidean action and the partition function to "derive" what the probability law is e.g.

$$S(x) = S_0(x) + \frac{g}{3!} \sum_{a=1}^{N} x_a^3, \tag{1.13}$$

we then have

$$Z = \int d^N x e^{-S_0(x) - \frac{g}{3!}\sum_a x_a^3}, \tag{1.14}$$

and can compute correlation functions from here as usual. In this particular model, we can see that the random variables x are still identically distributed but they are no longer independent due to the interaction term and hence in the large N limit, one can no longer expect a central limit theorem in its standard form. Now, there is an obvious technical issue which is that if you stare at this integral long enough, you can see that it is in fact divergent. The technical issue here is a matter of asymptotic competition. While the quadratic term S_0 ensures the integrand decays at large x, the cubic term $-\frac{g}{3!}x^3$ grows at a higher order. For sufficiently large negative values of x, the cubic term dominates the exponent, causing the integrand to blow up exponentially rather than vanish. Because the tail of the distribution never closes, the integral fails to converge. To resolve this, we have the choice of switching to a different theory where we introduce a quartic interaction term or doing a little trick and investigate the model perturbatively. By perturbatively, we mean that we will assume that g is a small number so that we may take a Taylor expansion of the interaction term exponential

$$e^{-\frac{g}{3!}\sum_a x_a^3} = 1 - \frac{g}{3!}\sum_a x_a^3 + \frac{1}{2}\left(\frac{g}{3!}\right)^2 \sum_{a,b} x_a^3 x_b^3 + \cdots, \tag{1.15}$$

hence, the full partition function can calculated order by order in g

$$Z = Z_0 \left[1 - \frac{g}{3!}\sum_a \left\langle x_a^3\right\rangle_0 + \frac{1}{2}\left(\frac{g}{3!}\right)^2 \sum_{a,b} \left\langle x_a^3 x_b^3\right\rangle_0 + \cdots\right]. \tag{1.16}$$

From the above asymptotic expansion, we can see that the perturbative correction terms are determined by correlation functions given by the free noninteracting non Gaussian partition function. The one point correlation function in the free noninteracting Gaussian theory vanishes but one can see that in the interacting case, the one point expectation function is in fact nonzero and is given by

$$\langle x_i \rangle = -\frac{g}{2}\sum_{a=1}^{N} A_{ia}^{-1} A_{aa}^{-1} + \cdots \tag{1.17}$$

and the two point correlation function in the interacting theory can now be computed to give

$$\langle x_i x_j \rangle = A_{ij}^{-1} + \frac{g^2}{4}\sum_{a,b} A_{ia}^{-1} A_{jb}^{-1} A_{aa}^{-1} A_{bb}^{-1} + \frac{g^2}{2}\sum_{a,b} A_{ia}^{-1} A_{jb}^{-1} A_{ab}^{-1} A_{ab}^{-1} + \cdots. \tag{1.18}$$

Once again, we still see that even in the interacting model, in the perturbative regime, we are still able to effectively compute everything in terms of the propagator A^{-1}.

Now having understood these finite dimensional cases, we are finally able to transition into a Euclidean field theory. Up to this point, we have treated $i = 1, \ldots, N$ as labeling N distinct random variables. For the path integral, we instead reinterpret the same label as labeling a sequence of closely spaced instants in Euclidean time. In this way, the collection of variables

$$x_1, x_2, \ldots, x_N \tag{1.19}$$

are no longer thought of as N unrelated realizations of a random variable, but rather as a discretized path trajectory. The basic idea is that a path is nothing more than a sufficiently long sequence of ordinary random variables, and the continuum limit is obtained by letting the spacing between neighboring labels go to zero. To make this precise, we can let the Euclidean time interval be $[0, T]$, and we divide it into N equal subintervals of size $\varepsilon \equiv \frac{T}{N}$, then the discrete time steps are $\tau_n = n\varepsilon, \quad n = 0, 1, \ldots, N$, and we identify $x_n \equiv x(\tau_n)$ where x is the path. In the continuum where we have infinitely many random variables, the discrete index i is formally replaced by a continuum index $t \in [0, T]$. The naive generalization of the Euclidean action is then

$$S[x(t)] = \frac{1}{2} \int_{[0,T]^2} dt dt' x(t) A\left(t, t'\right) x\left(t'\right). \tag{1.20}$$

One can see that such a Euclidean action depends on two distinct times t and t' signaling that we have a nonlocality in time. Although, this is fine for several applications outside of high energy physics [7, 8, 9, 10, 11], we are generically interested in case where $A(t, t')$ is a local differential operator [12] i.e.

$$A\left(t, t'\right) = -\delta\left(t - t'\right) \frac{d^2}{dt^2}, \tag{1.21}$$

where $\delta(t - t')$ is a Dirac delta distribution. With an appropriate set of boundary conditions, an integration by parts yields

$$S[x(t)] = \int_0^T dt \frac{1}{2} \dot{x}(t)^2, \tag{1.22}$$

which tells us that the Euclidean action is simply the nonrelativistic kinetic energy associated to a particle whose trajectory is along the 1 dimensional path $x(t)$ with velocity $\dot{x}(t)$ with a mass $m = 1$. This defines somewhat paradoxically what one would then call a one dimensional field theory despite being made out of infinitely many random variables x_i in the discretization. The reason for this due to the fact in the continuum limit, we are integrating over a 1 dimensional space in the action $[0, T]$, hence the previous discrete cases fall under the classification of zero dimensional field theories. The relevant integral for the case of infinitely many random variables is therefore an example of an infinite dimensional integration over the space of all paths and we denote the integral as

$$\mathcal{Z} = \int \mathcal{D}x(t) e^{-S_E[x]}. \tag{1.23}$$

This resulting object is known as the path integral. From the point of view of quantum physics, there is a standard derivation that is widely known which suggests that the path integral measure $\mathcal{D}x(t)$ is seemingly a divergent quantity

$$\mathcal{D}x(t) \sim \lim_{N\to\infty} \left(\frac{m}{2\pi\varepsilon}\right)^{N/2} \prod_{n=1}^{N-1} dx_n, \tag{1.24}$$

since in the naive limit as $\epsilon \to 0$ as $N \to \infty$, we seemingly have a factor of ∞^{∞}. To the mathematically rigorous reader, this is an enormous red flag [13, 14], however in the simple case of the free point particle, mathematicians have been able to rigorously construct the path integral measure through the stochastic analysis of the Wiener measure, namely through the rigorous formulation of Brownian motion on the space of continuous paths $C^0([0,\infty),\mathbb{R})$ [15, 16]. In this framework, one is able to provide rigorous probabilistic interpretation of the path integral in terms of geometric objects like the heat kernel [17]. However, despite making progress in this simplest case, there is no widely accepted rigorous construction of the path integral for the more complicated cases where the particle does not have simple diffusive type behavior. The moment that you introduce interactions, the underlying trajectories fail to have a naive pointwise definition and new ultraviolet divergences start to crop up. These problems lead to very technical waters involving renormalization and regularization to make sense out of these expressions. From a practical point of view, instead of trying to formulate a general theory of path integrals, it is the author's suggestion that unless we can classify the path integral to fall under a very specific class of path integrals[1], we should generically treat them to all be distinct problems with distinct methodologies in the most general case similar to how each nonlinear stochastic partial differential equation [18] generically requires its own research subprogram to truly map out its dynamics.

Despite the amount of effort that we've used to get here, this is still one of the simplest path integrals. In order to make further conceptual progress, despite the fact that our trajectories are labeled by a one dimensional source space that denotes the mathematical time along the trajectory, the trajectory itself can live in a higher dimensional space that we call the target space. We typically denote the components in the target space by superscripts as $x^a(t), \quad a = 1, \dots, d$. For example, in three spatial dimensions, one writes $\mathbf{x}(t) = (x^1(t), x^2(t), x^3(t)) = (x(t), y(t), z(t))$ and hence our Euclidean action for a higher dimensional target space would be the sum of kinetic energies

$$S_E[x,y,z] = \int_0^T dt \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right). \tag{1.25}$$

From, here we can introduce typical interaction terms that one would encounter by elementary studies of electromagnetism like a constant electric field E along the x direction $S_E[x,y,z] = \int_0^T dt\,(-qEx)$ which would be nonzero if the point particle along the trajectory

[1] For example, the path integrals we have considered thus far, fall informally under a class of worldline path integrals.

has charge q. From here, there are many different integrals that you may consider for whatever purpose of interest e.g. you can still add in nonlinear interaction terms to have the path interact with itself

$$S_{\text{int}} = \int_0^T dt \left[\frac{g}{3!} T_{abc} x^a x^b x^c + \frac{\lambda}{4!} U_{abcd} x^a x^b x^c x^d \right] \tag{1.26}$$

You can consider distinct trajectories in the same target space and introduce a coupling between them

$$S_E \left[\mathbf{x}_1, \mathbf{x}_2 \right] = \int_0^T dt \left[\frac{m_1}{2} \dot{\mathbf{x}}_1^2 + \frac{m_2}{2} \dot{\mathbf{x}}_2^2 + \frac{k}{2} \left(\mathbf{x}_1 - \mathbf{x}_2 \right)^2 \right], \tag{1.27}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are two distinct trajectories each with their own x, y, z components and k is a spring constant. The introduction of a constant magnetic field B forces us to consider derivative interaction terms between different components of the path which would not be possible in zero dimensions where there no derivatives

$$S_E[x, y, z] = \int_0^T dt \left[\frac{m}{2} \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + \frac{iqB}{2} (x\dot{y} - y\dot{x}) \right], \tag{1.28}$$

A more careful analysis of the Euclidean action also forces us to add a factor of the imaginary number i which is suggestive that turning on background fields might generically require a complexification of the action. In fact, from the point of view of quantum mechanics, the Euclidean path integral formulation that we have developed thus far is not the one we should be using. Quantum mechanics instead starts from a real time formulation where the free partition function instead takes the form

$$Z_0 = \int \mathcal{D}x(t) \, e^{i \int_{t_i}^{t_f} dt \left(\frac{1}{2} m \dot{x}^2 \right)}, \tag{1.29}$$

in this simple case, we see that although the action is formally the same, we have now lost the notion of a positive definite probability measure since the integrand is now an oscillatory phase e^{iS} instead of the damped exponential e^{-S} in the Euclidean formulation. For the experienced reader, we note that we have temporarily ignored the discussion of boundary conditions. The convergence of this real time path integral is now also suspect due to the fact that we have an infinite dimensional sum of oscillatory phases. The standard physicist's remedy to tame this wildly oscillating integral is to introduce a small imaginary regulator, commonly known as the $i\epsilon$ prescription [19]. By deforming the action $S \to S + i\epsilon \int x^2 dt$, one essentially forces the integrand to possess a slight exponential damping, ensuring that trajectories far from the classical path destructively interfere and safely vanish at infinity. Deeply tied to this regulator is the concept of Wick rotation, an analytic continuation that maps the real time variable t to an imaginary time variable $\tau = it$. This is formal deformation that ones allow one to connect simple real time theories to Euclidean theories however it is known that the process of Wick rotating a field theory generically has profound consequences on the underlying dynamics e.g. the differential equations associated to real time quantum mechanical evolution are no longer diffusion processes given by elliptic operators but are instead wave processes given by hyperbolic differential operators [20]. It is expected that one cannot simply Wick rotate a theory far from equilibrium and it often requires a much more careful analytic continuation.

With the mathematical scaffolding of the one dimensional path integral seemingly secure, the natural progression is to escalate the dimensionality of our source space, thereby crossing the threshold from quantum mechanics into true quantum field theory. If quantum mechanics is the study of a zero dimensional point particle tracing out a one dimensional path $x(t)$ in a target space, then a standard scalar field theory (in the historical sense of the word) arises when we replace this point particle with an extended continuous medium. Our source space is no longer just a one dimensional time interval, but a d dimensional spacetime manifold, and our dynamical variable becomes a field $\phi(t, \mathbf{x})$ that assigns a value to every point in this extended geometry. The path integral measure $\mathcal{D}x(t)$ correspondingly bloats into an unimaginably vast functional measure $\mathcal{D}\phi(x)$, integrating over all possible configurations/realizations of the field across all of spacetime. This monumental leap in complexity brings with it the full bestiary of ultraviolet divergences and necessitates the full machinery of the renormalization group to extract finite, physical predictions [21]. However, both the Euclidean and the standard real time path integrals harbor a subtle but profound limitation in their typical formulation i.e. they are fundamentally anchored to the assumption of thermal equilibrium or pure state transitions. The traditional real time formalism, often formulated to compute S matrix scattering amplitudes [22, 23], implicitly assumes that the system begins in an asymptotic vacuum state in the infinite past and evolves into another asymptotic vacuum state in the infinite future. In a vast array of physical scenarios, this assumption spectacularly fails.

In these highly dynamic environments, we are not dealing with pure vacuum states, but rather evolving macroscopic ensembles described by density matrices that are driven far from thermal equilibrium.To capture this nonequilibrium dynamics, we must abandon the standard “in out” path integral and transition to the Schwinger Keldysh, or “in in”, formalism. Instead of calculating transition amplitudes between a past and a future state, the Schwinger Keldysh formalism [24] computes expectation values by evolving the system’s density matrix forward in time from a known initial state to the time of measurement, and then evolving it backward in time to the initial state. This necessitates a path integral defined over a complex time contour that doubles back on itself, known as the closed time contour. Consequently, our degrees of freedom are doubled and we must track a forward field and a time reversed backwards field. It is precisely the interactions between these two branches that encodes the intricate dissipative and statistical fluctuations of the nonequilibrium system.

1.2 The Schwinger Keldysh Formalism, or one thousand and one lessons in nonequilibrium quantum physics.

Before we continue to make the path integrals more complicated, we must clarify how one incorporates the ideas of thermal ensembles and characterizes near thermal equilibrium regimes or far away from equilibrium regimes. It is no coincidence that our path integrals were also simultaneously named partition functions which are a key tool in equilibrium statistical physics in order to extract thermodynamical observables. In Euclidean time τ, thermal equilibrium is represented by a compact Euclidean time circle $\tau \sim \tau + \beta$, where $\beta = \frac{1}{T}$ is the inverse temperature and the thermal partition function is then

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta \widehat{H}} = \int_{x(\beta)=x(0)} \mathcal{D}x(\tau) e^{-S_E[x]}. \tag{1.30}$$

Here, we have made explicit that the existence of the Euclidean path integral assumes that there is a corresponding Hamiltonian operator which generates the Euclidean time evolution and gives rise to a density matrix of the form

$$\widehat{\rho}_\beta = \frac{e^{-\beta \widehat{H}}}{Z(\beta)}, \tag{1.31}$$

consequently, thermal equilibrium should simply be viewed as a property of the Euclidean action but it is a property of the entire path integral since it encodes the density matrix through the boundary conditions on the path integral. In the above, we have denoted operators via hats. In order to truly go out of thermal equilibrium, the natural object is no longer the Euclidean circle but the Schwinger Keldysh closed time contour [24] which in the operator formalism can be formulated as follows. Define a time dependent Hamiltonian operator $\widehat{H}(t)$ and a corresponding density matrix $\widehat{\rho}(t)$. The density matrix can be shown to evolve in time according to the von Neumann equation

$$i\frac{\partial}{\partial t}\widehat{\rho}(t) = \widehat{H}(t)\widehat{\rho}(t) - \widehat{\rho}(t)\widehat{H}(t), \tag{1.32}$$

this can be formally solved by the introduction of a unitary time evolution operator

$$\widehat{U}(t, t_0) = \mathcal{T}\, e^{-i\int_{t_0}^{t} dt' \widehat{H}(t')} \tag{1.33}$$

which satisfies $\widehat{U}(t_0, t_0) = \mathbf{I}$ and $\mathcal{T}$ refers to the time ordering. Consequently, a short calculation yields

$$\widehat{\rho}(t) = \widehat{U}(t, t_0)\, \widehat{\rho}_0 \widehat{U}^\dagger(t, t_0). \tag{1.34}$$

Now consider an observable $\widehat{\mathcal{O}}$. Its expectation value at a time t is given by $\langle\widehat{\mathcal{O}}(t)\rangle = \mathrm{Tr}(\widehat{\rho}(t)\widehat{\mathcal{O}})$ and given the time evolution of the density matrix, we can see that

$$\langle\widehat{\mathcal{O}}(t)\rangle = \mathrm{Tr}\left(\widehat{U}(t, t_0)\, \widehat{\rho}_0 \widehat{U}^\dagger(t, t_0)\, \widehat{\mathcal{O}}(t)\right). \tag{1.35}$$

Where we have chosen to interpret t_0 as the time where the initial density matrix is prepared. If you choose a time $t^\wedge$ later than every operator insertion and insert a resolution of the identity in terms of the unitary time evolution operators $\mathbf{I} = \widehat{U}\left(t, t^\wedge\right)\widehat{U}\left(t^\wedge, t\right)$, then after some algebraic simplifications, one arrives at the form

$$\langle\widehat{\mathcal{O}}(t)\rangle = \mathrm{Tr}\left(\widehat{U}\left(t_0, t^\wedge\right)\widehat{U}\left(t^\wedge, t\right)\widehat{\mathcal{O}}(t)\widehat{U}\left(t, t_0\right)\widehat{\rho}_0\right). \tag{1.36}$$

From here, the physical (and mathematical interpretation) becomes transparent when the equation is read from right to left. The quantum system must be prepared in an initial density matrix $\widehat{\rho}_0$ and then evolved forward in time from t_0 to t where the observable is placed. Upon arriving at t, we further evolve the system to future time $t^\wedge$ which we will call the turnaround time. In the context of nonequilibrium string theory [25], we will call it the wedge time. And finally, we evolve the system backwards in time back to the initial time t_0. This closed time contour

$$\mathcal{C}_{SK}: \quad t_0 \longrightarrow t^\wedge \longrightarrow t_0, \tag{1.37}$$

is the Schwinger Keldysh contour that underlies quantum mechanical systems out of thermal equilibrium. The first arrow is known as the forward branch is often denoted $\mathcal{C}_+$ which begins at t_0 and ends at the wedge time $t^\wedge$, the second arrow refers to the time reversed backwards branch denoted $\mathcal{C}_-$ which begins at $t^\wedge$ and returns to t_0. In practice, we often introduce additional sources J_+ and J_- to the Hamiltonians on the forwards and backwards branch separately in order to generate nonequilibrium dynamics, and minimally, we end up with

$$\begin{aligned}\widehat{U}_{J_+}\left(t^\wedge, t_0\right) &= \mathcal{T}\, e^{-i\int_{t_0}^{t^\wedge} dt\left(\widehat{H}(t) - J_+(t)\widehat{x}\right)},\\ \widehat{U}^\dagger_{J_-}\left(t^\wedge, t_0\right) &= \widetilde{\mathcal{T}}\, e^{+i\int_{t_0}^{t^\wedge} dt\left(\widehat{H}(t) - J_-(t)\widehat{x}\right)}.\end{aligned} \tag{1.38}$$

where $\widetilde{\mathcal{T}}$ refers to anti time ordering. In order to connect this back to the worldline path integral of the previous subsection, we write the trace in the configuration basis and obtain the explicit expression

$$Z_{\rho_0}\left[J_+, J_-\right] = \int dx_* dx_0^+ dx_0^- \left\langle x^\wedge\right|\widehat{U}_{J_+}\left(t^\wedge, t_0\right)\left|x_0^+\right\rangle\left\langle x_0^+\right|\widehat{\rho}_0\left|x_0^-\right\rangle\left\langle x_0^-\right|\widehat{U}^\dagger_{J_-}\left(t^\wedge, t_0\right)\left|x^\wedge\right\rangle, \tag{1.39}$$

the individual inner products can be evaluated and they result in double path integrals of the form

$$\left\langle x_0^-\right|\widehat{U}^\dagger_{J_-}\left(t^\wedge, t_0\right)\left|x^\wedge\right\rangle = \int_{x_-(t_0)=x_0^-}^{x_-(t_*)=x^\wedge} \mathcal{D}x_-\,\mathcal{D}p_-\, e^{-i\int_{t_0}^{t^\wedge} dt\left(p_-\dot{x}_- - H(x_-,p_-,t) + J_-(t)x_-(t)\right)}. \tag{1.40}$$

These path integrals are known as Hamiltonian path integrals [26, 27] where we have an additional field $p_-(t)$ which is a momentum field along the reverse branch. If the Hamiltonian on the backwards branch has momentum dependency that is at almost quadratic, the momentum fields can be integrated out and this results in

$$\left\langle x_0^- \right| \widehat{U}_{J_-}^\dagger \left(t^\wedge, t_0\right) \left| x^\wedge \right\rangle = \int_{x_-(t_0)=x_0^-}^{x_-(t^\wedge)=x^\wedge} \mathcal{D}x_- \, e^{-iS_- - i\int_{t_0}^{t^\wedge} dt\, J_-(t) x_-(t)}. \tag{1.41}$$

Notice that on the backwards branch, the real time action obtains an overall minus sign. Altogether, the Schwinger Keldysh generating functional is then

$$\begin{aligned} &Z_{\rho_0}\left[J_+, J_-\right] = \\ &= \int_{x_+(t^\wedge)=x_-(t^\wedge)} \mathcal{D}x_+ \, \mathcal{D}x_- \, \rho_0\left(x_+\left(t_0\right), x_-\left(t_0\right)\right) e^{i(S_+-S_-)+i\int_{t_0}^{t^\wedge} dt (J_+x_+-J_-x_-)}. \end{aligned} \tag{1.42}$$

The above equation shows that formulating a path integral for the Schwinger Keldysh generating functional doubles our degrees of freedom. In fact, one can push this further and recover the Euclidean circle that characterizes thermal equilibrium by choosing the initial density matrix to be the canonical ensemble

$$\widehat{\rho}_0 = \widehat{\rho}_\beta = \frac{e^{-\beta \widehat{H}}}{Z(\beta)}, \quad Z(\beta) = \mathrm{Tr}\, e^{-\beta \widehat{H}}. \tag{1.43}$$

For the generating functional with no additional observables on the forward and backwards branch inserted, this now leads to a deformed contour known as the thermal Schwinger Keldysh contour

$$\mathcal{C}_\beta : \quad t_0 \to t^\wedge \to t_0 \to t_0 - i\beta, \tag{1.44}$$

where we acquire an additional Euclidean contour segment which goes in the direction of imaginary time along an extent β. Without those additional observables, this reduces down to the Euclidean path integral by the principle of unitarity. The takeaway is that the a Euclidean thermal field theory is the equilibrium subsector of the full Schwinger Keldysh nonequilibrium quantum field theory.

Once the forward and backward fields have been introduced, the next natural step is to reorganize the doubled degrees of freedom into variables which have a more direct physical interpretation. The most useful choice is the Keldysh, or c/q, rotation. For a bosonic degree of freedom $x_\pm(t)$ on the two branches, define

$$x_c(t) \equiv \frac{x_+(t) + x_-(t)}{2}, \qquad x_q(t) \equiv x_+(t) - x_-(t), \tag{1.45}$$

and likewise for the sources

$$J_c(t) \equiv \frac{J_+(t) + J_-(t)}{2}, \qquad J_q(t) \equiv J_+(t) - J_-(t). \tag{1.46}$$

This is merely a linear change of variables, but it is one of the most important changes of variables in nonequilibrium quantum theory. The inverse transformation is

$$x_+(t) = x_c(t) + \frac{1}{2} x_q(t), \qquad x_-(t) = x_c(t) - \frac{1}{2} x_q(t), \tag{1.47}$$

and similarly

$$J_+(t) = J_c(t) + \frac{1}{2}J_q(t), \qquad J_-(t) = J_c(t) - \frac{1}{2}J_q(t). \tag{1.48}$$

Substituting these into the source term gives

$$J_+x_+ - J_-x_- = \left(J_c + \frac{1}{2}J_q\right)\left(x_c + \frac{1}{2}x_q\right) - \left(J_c - \frac{1}{2}J_q\right)\left(x_c - \frac{1}{2}x_q\right) \tag{1.49}$$

$$= J_q x_c + J_c x_q. \tag{1.50}$$

Hence the Schwinger Keldysh generating functional may be rewritten as

$$Z_{\rho_0}[J_c, J_q] = \int \mathcal{D}x_c\, \mathcal{D}x_q\, \rho_0\, e^{i\left(S[x_c+\frac{1}{2}x_q]-S[x_c-\frac{1}{2}x_q]\right)+i\int dt\,(J_q x_c+J_c x_q)}. \tag{1.51}$$

The interpretation of these new variables is immediate. The field x_c is the "classical" or average field, while x_q is the "quantum" or difference field. Despite their names, they are both subject to field fluctuations and don't necessarily describe the classical or quantum dynamics. In the limit where you ignore higher order terms in x_q, you can show that the path integral localizes into the classical equations of motion. One can see this by noticing how the dynamical content reogranizes itself under a Keldysh rotation, if you expand the doubled action for small x_q, then

$$S\left[x_c + \frac{x_q}{2}\right] - S\left[x_c - \frac{x_q}{2}\right] = \int dt\, x_q(t)\frac{\delta S[x_c]}{\delta x_c(t)} + \cdots . \tag{1.52}$$

hence at leading order, the q field imposes the classical equations of motion. In this precise sense, x_q acts as a Lagrange multiplier enforcing real time dynamics. The latter x_q measures the failure of the two contour branches to coincide. In particular, the physical limit in which the same source is applied on both branches is $J_q = 0$, and unitarity implies the normalization condition

$$Z_{\rho_0}[J_c, 0] = 1. \tag{1.53}$$

This innocuous looking statement has strong consequences. It implies, for example, that correlation functions with only q-type insertions vanish in the physical limit, and it forces the effective action to vanish whenever $x_q = 0$. This is often a very useful check

For the two point functions, it is convenient to define

$$G^>(t,t') \equiv -i\langle\widehat{x}(t)\widehat{x}(t')\rangle, \qquad G^<(t,t') \equiv -i\langle\widehat{x}(t')\widehat{x}(t)\rangle. \tag{1.54}$$

From these, one constructs the retarded, advanced, and Keldysh correlators,

$$G^R(t,t') \equiv \Theta(t-t')\Big(G^>(t,t') - G^<(t,t')\Big), \tag{1.55}$$

$$G^A(t,t') \equiv -\Theta(t'-t)\Big(G^>(t,t') - G^<(t,t')\Big), \tag{1.56}$$

$$G^K(t,t') \equiv G^>(t,t') + G^<(t,t'). \tag{1.57}$$

In the c/q basis, the propagator matrix takes the triangular form

$$G_{\alpha\beta}(t,t') = -i \begin{pmatrix} \langle x_c(t)x_c(t')\rangle & \langle x_c(t)x_q(t')\rangle \\ \langle x_q(t)x_c(t')\rangle & \langle x_q(t)x_q(t')\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2}G^K & G^R \\ G^A & 0 \end{pmatrix}. \tag{1.58}$$

This matrix summarizes the philosophy of the formalism. The retarded and advanced entries encode causal response, whereas G^K encodes the statistical occupation data of the state. Nonequilibrium physics is precisely the problem of determining how these two sectors talk to one another when the system is not in a stationary Gibbs ensemble.

At thermal equilibrium, the doubled contour is not arbitrary, it inherits a hidden periodicity from the Euclidean thermal circle. This is the Kubo Martin Schwinger, or KMS condition [28, 29]. For bosonic operators in a stationary thermal state, it states

$$G^>(t) = G^<(t - i\beta), \tag{1.59}$$

which in frequency space becomes

$$G^>(\omega) = e^{\beta\omega} G^<(\omega). \tag{1.60}$$

This one relation is enough to derive the fluctuation dissipation theorem. One can check that

$$G^K(\omega) = G^>(\omega) + G^<(\omega) \tag{1.61}$$

$$= \left(e^{\beta\omega} + 1\right) G^<(\omega), \tag{1.62}$$

while

$$G^R(\omega) - G^A(\omega) = G^>(\omega) - G^<(\omega) \tag{1.63}$$

$$= \left(e^{\beta\omega} - 1\right) G^<(\omega). \tag{1.64}$$

Dividing the first equation by the second yields

$$G^K(\omega) = \frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\left(G^R(\omega) - G^A(\omega)\right) = \coth\left(\frac{\beta\omega}{2}\right)\left(G^R(\omega) - G^A(\omega)\right). \tag{1.65}$$

This is the bosonic fluctuation dissipation theorem. It states that at equilibrium the noisy, statistical correlator is not independent data. It is completely determined by the dissipative spectral response. Physically, it says that the same microscopic processes which permit the system to relax also force it to fluctuate. Dissipation without noise is inconsistent with a thermal steady state, and noise without dissipation would indefinitely heat the system.

In the classical high temperature limit $\beta\omega \ll 1$, one has

$$\coth\left(\frac{\beta\omega}{2}\right) \approx \frac{2}{\beta\omega}, \tag{1.66}$$

so that the fluctuation dissipation relation reduces to

$$G^K(\omega) \approx \frac{2T}{\omega}\left(G^R(\omega) - G^A(\omega)\right). \tag{1.67}$$

This is the form that survives in classical stochastic hydrodynamics and in Langevin descriptions of dissipative systems. Thus, the classical Einstein relation is not a separate phenomenon; it is the low frequency, high temperature limit of the full quantum Schwinger Keldysh statement.

The true power of the closed time contour becomes visible when the system of interest is not isolated. Suppose we split the degrees of freedom into a system variable x and an environmental variable q, with total action

$$S_{\text{tot}}[x, q] = S_{\text{sys}}[x] + S_{\text{env}}[q] + S_{\text{int}}[x, q]. \tag{1.68}$$

The object of physical interest is usually not the full density matrix of system plus bath, but the reduced density matrix of the system alone. This is obtained by tracing over the environmental degrees of freedom. In the path integral language, one finds

$$Z[J_+, J_-] = \int \mathcal{D}x_+\mathcal{D}x_-\, e^{iS_{\text{sys}}[x_+]-iS_{\text{sys}}[x_-]}\, \mathcal{F}[x_+, x_-]\, e^{i\int dt(J_+x_+-J_-x_-)}, \tag{1.69}$$

where the entire effect of the environment is captured by the Feynman Vernon influence [30]

$$\mathcal{F}[x_+, x_-] \equiv \int \mathcal{D}q_+\,\mathcal{D}q_-\,\rho_{\text{env}}\, e^{iS_{\text{env}}[q_+]-iS_{\text{env}}[q_-]+iS_{\text{int}}[x_+,q_+]-iS_{\text{int}}[x_-,q_-]}. \tag{1.70}$$

By definition, one writes

$$\mathcal{F}[x_+, x_-] = e^{iS_{\text{IF}}[x_+,x_-]}, \tag{1.71}$$

where S_{IF} is the influence action. Although, it looks as though we're simply defining additional objects and increasing the complexity of the problem, this is actually helping us organize the system of interest by integrating out the environment whose details we cannot fully describe and remember the nonlocal imprint that it leaves on the system.

For a Gaussian environment linearly coupled to the system [31], the influence action in the c/q basis takes the schematic nonlocal form

$$S_{\text{IF}}[x_c, x_q] = -\int dtdt'\, x_c(t)D^R(t,t')x_q(t') + \frac{i}{2}\int dtdt'\, x_q(t)N(t,t')x_q(t'). \tag{1.72}$$

Here D^R is a retarded dissipation kernel and N is a symmetric noise kernel. The first term is real and modifies the equations of motion. The second term is purely imaginary and suppresses off diagonal configurations with large x_q, thereby encoding decoherence and fluctuations. This second term is precisely where the nonlocality and memory effects arise. It also hints that upon creating effective actions, we inherently deal with complexified systems whose path integral weight is no longer a pure phase but has a dampened exponential component. The reduced effective action therefore becomes

$$S_{\text{eff}}[x_c, x_q] = S_{\text{sys}}\left[x_c + \frac{x_q}{2}\right] - S_{\text{sys}}\left[x_c - \frac{x_q}{2}\right] - \int dtdt'\, x_c D^R x_q + \frac{i}{2}\int dtdt'\, x_q N x_q. \quad (1.73)$$

At this stage one may perform a Hubbard Stratonovich transformation,

$$e^{-\frac{1}{2}\int dt\,dt'\,x_q(t)\,N(t,t')\,x_q(t')} = \int \mathcal{D}\xi\, e^{-\frac{1}{2}\int dt\,dt'\,\xi(t)\,N^{-1}(t,t')\,\xi(t') + i\int dt\,\xi(t)\,x_q(t)}. \quad (1.74)$$

which introduces an explicit stochastic noise field ξ. Integrating over x_q then yields an effective Langevin type equation

$$\frac{\delta S_{\text{sys}}}{\delta x_c(t)} + \int dt'\, D^R(t,t')x_c(t') = \xi(t), \qquad \langle \xi(t)\xi(t')\rangle = N(t,t'). \quad (1.75)$$

Thus, we can see that the Schwinger Keldysh path integral creates a direct branch from real time quantum dynamics to stochastic dynamics. The noise field doesn't arise by hand but it is instead arises through the Hubbard Stratonovich transformation as a result of the degrees of freedoms that we are integrating out. It is worth noting that the noise field ξ is not simply a white noise but can be other forms of noise in general due to the specific form of the noise kernel $N(t, t')$.

The specific form of the noise kernel $N(t, t')$ depends heavily on the spectral density of the environment and its temperature. In the simplest high temperature limit with an Ohmic bath, the environment retains no memory, yielding a local, delta correlated white noise kernel, $N(t, t') = 2\gamma T\delta(t - t')$, which recovers standard Markovian Brownian motion with the familiar local form given by stochastic differential equation driven by white noise

$$m\ddot{x} + \gamma\dot{x} + V'(x) = \xi(t), \qquad \langle \xi(t)\xi(t')\rangle = 2\gamma T\,\delta(t - t'), \quad (1.76)$$

However, if the environment possesses a characteristic finite response time such as in a bath with a Lorentz-Drude high frequency cutoff, the noise becomes colored and exhibits an exponentially decaying memory, $N(t, t') \propto \exp(-\Gamma|t - t'|)$. Furthermore, in the strict zero temperature quantum limit where classical thermal kicks freeze out, the fluctuation dissipation theorem dictates that the kernel decays algebraically as $N(t, t') \propto -1/(t - t')^2$. From this these results, it becomes clear that many common questions of nonequilibrium statistical physics can be explored through the Schwinger Keldysh formalism.

If one now coarse grains further and focuses on conserved densities (or mesoscopic observables) rather than microscopic coordinates, the same formalism transmutes into the language of fluctuating hydrodynamics and, in suitable diffusive limits, macroscopic fluctuation theory [32]. Consider a conserved density $\rho(t, x_k)$ with current $j_i(t, x_k)$ where the indices i denote spatial directions, satisfying the continuity equation

$$\partial_t\rho + \partial_i j_i = 0. \quad (1.77)$$

A generic constitutive relation for a driven diffusive medium takes the form

$$j_i = -D(\rho)\partial_i\rho + \chi(\rho)E_i + \eta_i, \tag{1.78}$$

where $D(\rho)$ is a diffusivity, $\chi(\rho)$ is a mobility, E_i is an driving background field, and η_i is a stochastic noise. If the noise is Gaussian with covariance

$$\langle \eta_i(t, x_k)\eta_j(t', x'_k)\rangle = \sigma(\rho)\delta_{ij}\delta(t-t')\delta^{(d)}(x_k - x'_k), \tag{1.79}$$

then the trajectory probability of the hydrodynamic fields is weighted by

$$\mathbb{P}[\rho, j_i] \propto e^{\left[-\frac{1}{2}\int dt\, d^dx\, (j_i + D(\rho)\partial_i\rho - \chi(\rho)E_i)\sigma(\rho)^{-1}(j_i + D(\rho)\partial_i\rho - \chi(\rho)E_i)\right]}, \tag{1.80}$$

subject to the continuity equation. Introducing a response field $\widehat{\rho}$ to enforce conservation gives the Martin Siggia Rose Janssen de Dominicis functional or MSRJD functional [33, 34, 35]for short,

$$Z = \int \mathcal{D}\rho\, \mathcal{D}\widehat{\rho}\, e^{-\int dt\, d^dx\, \left(\widehat{\rho}\,\partial_t\rho + D(\rho)\,(\partial_i\rho)(\partial_i\widehat{\rho}) - \frac{1}{2}\sigma(\rho)\,(\partial_i\widehat{\rho})(\partial_i\widehat{\rho}) - \chi(\rho)\,E_i\partial_i\widehat{\rho}\right)}. \tag{1.81}$$

This has the same structural content as the classical limit of the Schwinger Keldysh action. A hydrodynamic "classical" field ρ coupled to a response "quantum" field $\widehat{\rho}$. In this sense, macroscopic fluctation theory encoded within the Schwinger Keldysh formalism as well. The rate functionals studied in nonequilibrium statistical mechanics, particularly in stochastic thermodynamics [36, 37], are then naturally recovered through the standard large deviation principles of rare events evaluated on this path integral.

Just as classical rare events are captured by large deviation minimizers, real time quantum tunneling transitions are dominated by saddle point trajectories known as instantons. Within the Schwinger Keldysh action, these instantons manifest as complex valued solutions to the equations of motion that explicitly mix the forward and backward time contours. By evaluating the Keldysh effective action along these instanton paths, one can rigorously compute the decay rates of metastable states and nonperturbative tunneling amplitudes directly in real time, bypassing the need for analytic continuation from purely imaginary time Matsubara formalisms [38].

The same contour technology also underlies quantum stochastic thermodynamics, whose goal is to define work, heat, entropy production, and irreversibility at the level of individual quantum trajectories or conditioned histories. A particularly important effective description is the Lindblad equation [39].

$$\frac{d\widehat{\rho}}{dt} = -i[\widehat{H}, \widehat{\rho}] + \sum_\alpha \left(\widehat{L}_\alpha\widehat{\rho}\widehat{L}_\alpha^\dagger - \frac{1}{2}\{\widehat{L}_\alpha^\dagger\widehat{L}_\alpha, \widehat{\rho}\}\right), \tag{1.82}$$

where the jump operators $\widehat{L}_\alpha$ encode exchanges with an environment. In a trajectory unraveling, each realization consists of smooth nonunitary evolution interrupted by quantum jumps. One then defines a stochastic entropy production

$$\Delta s_{\mathrm{tot}}[\gamma] = \log\frac{\mathbb{P}[\gamma]}{\mathbb{P}[\widetilde{\gamma}]}, \tag{1.83}$$

where γ is a forward trajectory and $\widetilde{\gamma}$ is its time reversed partner. The integral fluctuation theorem [40, 41] follows immediately,

$$\left\langle e^{-\Delta s_{\text{tot}}} \right\rangle = 1, \tag{1.84}$$

and by Jensen's inequality this implies $\langle \Delta s_{\text{tot}} \rangle \geq 0$, which is the statistical form of the second law.

There is one final lesson hidden here. Although ordinary Schwinger Keldysh contours are designed for time ordered observables, some of the most interesting probes of many body quantum dynamics are out of time order. A canonical example is the thermal out of time ordered correlator [42]

$$F(t) = \left\langle \widehat{V}^{\dagger}(t)\widehat{W}^{\dagger}(0)\widehat{V}(t)\widehat{W}(0) \right\rangle_{\beta}, \tag{1.85}$$

from which one constructs the squared commutator

$$C(t) = -\left\langle [\widehat{V}(t), \widehat{W}(0)]^2 \right\rangle_{\beta}. \tag{1.86}$$

Here, $\widehat{V}$ and $\widehat{W}$ are generic, local quantum operators chosen to probe how information spreads and scrambles. Typically, $\widehat{W}(0)$ acts as a highly localized initial perturbation, while $\widehat{V}(t)$ acts as a delayed measurement at a distant spatial location. At early times, the influence of the initial kick has not yet reached the distant measurement; the two operators act on different Hilbert subspaces and commute, yielding $C(t) \approx 0$ and $F(t) \approx 1$. However, as the Heisenberg operator $\widehat{W}(t) = e^{i\widehat{H}t}\widehat{W}(0)e^{-i\widehat{H}t}$ evolves in a strongly interacting system, it continuously entangles with its neighbors, growing into a massively complex, nonlocal operator that eventually overlaps with $\widehat{V}$.

When this overlap occurs and $C(t)$ grows rapidly, initially simple operators spread across many degrees of freedom and the system exhibits quantum chaotic behavior. In large N systems, one often finds an early regime of the form

$$C(t) \sim \frac{1}{N} e^{\lambda_L t}, \tag{1.87}$$

where λ_L is the many body Lyapunov exponent. Because the operator ordering in $F(t)$ is not capturable by a single forward/backward contour, one must enlarge the contour, often to a four leg contour or to two coupled Schwinger Keldysh contours. Thus, even the diagnosis of quantum chaos is ultimately a statement about how far we are willing to generalize the closed time path.

We may now summarize the moral of the story. The Euclidean circle is in some sense the geometry of equilibrium subsector and the full closed time contour is the geometry of real time dynamics. The Keldysh rotation separates response from fluctuations. The fluctuation dissipation theorem tells us that at thermal equilibrium these two are locked together by the KMS condition. Influence functionals show that when we coarse grain over inaccessible variables, dissipation and noise are generated automatically. In the hydrodynamic limit this becomes macroscopic fluctuation theory. In weakly coupled settings it becomes quantum stochastic thermodynamics; and when one asks how operators spread and information scrambles, one is naturally led to enlarged Schwinger Keldysh contours adapted to out of time ordering.

The reason this formalism is so powerful is that it is not merely a trick for doing perturbation theory. It is a bookkeeping device for causality, unitarity, decoherence, dissipation, and information flow all at once. Put differently, the Schwinger Keldysh contour is the minimal geometric arena in which one can ask genuinely nonequilibrium questions. The Euclidean partition function tells us what states are allowed at equilibrium, but the Schwinger Keldysh generating functional tells us how states fluctuate, respond, thermalize, remember, forget, and sometimes chaotically scramble the information that was initially put into them. For nonequilibrium quantum field theory, statistical mechanics, and eventually nonequilibrium string theoretic constructions, this is not an optional embellishment. It is the native language.

1.3 Localization, or one thousand and one lessons in nonperturbative physics.

If the Schwinger Keldysh formalism teaches us how badly behaved nonequilibrium quantum field theory can become, localization teaches us that under sufficiently rigid symmetry conditions even an infinite dimensional path integral can sometimes be forced to reveal exact information. This is one of the deepest lessons in modern high energy theory. Although the generic path integral is not something one evaluates in closed form. There exist remarkable circumstances in which symmetry is so constraining that the answer is already encoded in a much smaller subset of configurations e.g. fixed points of an action, critical points of a Morse functional, instanton moduli spaces, flat connections, or even a finite dimensional integral over a Cartan subalgebra [2]. The miracle of localization is that it turns this slogan into a systematic computational principle. In its most primitive form, it is the statement that a supersymmetric or equivariantly closed deformation of the action may be scaled to infinity without changing the value of suitable observables, so that the path integral collapses onto the bosonic zeros of that deformation. Beyond their role in formal mathematical physics, these exact results provide theoretical guidelines for what to expect in nonperturbative quantum field theory. By reducing infinite dimensional path integrals to finite dimensional random matrix models or localized sums, researchers can probe the strong coupling limits of theories that are otherwise inaccessible to standard perturbation theory. This has direct utility in verifying dualities, such as the AdS/CFT correspondence, where localization allows for a precise matching of gravitational observables with gauge theory predictions. Furthermore, the techniques used in instanton counting and Chern Simons matrix models have migrated into the realm of condensed matter physics and quantum information, offering exact templates for understanding topological phases of matter, the fractional quantum Hall effect, and the construction of error correcting codes in quantum computing. In essence, localization serves as a bridge, transforming abstract symmetry constraints into a precision laboratory for the most complex systems in modern physics.

There is a second lesson, equally important for the broader ambitions of this thesis. Localization is not merely a computational technology for special supersymmetric models. It is also a conceptual guide to which geometric structures matter in a quantum theory. Every successful localization computation exposes a hidden rigidity i.e. a nilpotent charge, a cohomological complex, a moment map, an equivariant action encoding symmetries, a set of localization equations expressed in terms of partial differential equations, and a distinguished moduli problem. Once one accepts that the raw functional integral is too large to understand directly, the problem becomes one of searching for symmetry principles strong enough to collapse it without destroying the physics one cares about. Supersymmetry has historically been the most effective such principle, but the deeper point is cohomological leading to topological field theory. Once a cohomological path integral can be organized around a nilpotent operator and its fixed locus, exact nonperturbative statements become possible far beyond ordinary perturbation theory.

The most elementary setting in which all of these ideas already appear is in the quantum mechanics of a particle moving on a curved manifold. In flat space our Euclidean action for a particle trajectory was simply the nonrelativistic kinetic energy. The natural generalization is to replace the flat target space with a Riemannian manifold M with local coordinates $x^i(\tau)$, $i = 1, \dots, d$, and metric tensor components $g_{ij}(x)$. Then the particle no longer moves in a vector space but in a curved target whose geometry is encoded in the metric tensor, and the corresponding path integral becomes what is known as a one dimensional nonlinear sigma model [43, 44]. The degrees of freedom is the embedding map

$$x : \Sigma_1 \to M, \tag{1.88}$$

where Σ_1 is the worldline, usually a line interval or a Euclidean circle, and M is the target space. The basic Euclidean action is

$$S_E[x] = \frac{1}{2} \int d\tau \, g_{ij}(x) \, \dot{x}^i \dot{x}^j. \tag{1.89}$$

Already at this stage, the worldline formulation is much more than a toy model. It is the universal local model for heat kernels, spectral asymptotics, semiclassical tunneling, supersymmetric quantum mechanics, and the worldline representation of one loop effective actions in quantum field theory. Once the target space is curved, one must introduce the standard language of differential geometry to write down the equations of motion, they are

$$\ddot{x}^i + \Gamma^i{}_{jk} \dot{x}^j \dot{x}^k = 0. \tag{1.90}$$

This is known as the geodesic equation and is the coordinate expression of the geometric statement that a free non interacting particle must move on a straight line within a non Euclidean space. In a local coordinate system, we can ensure that the particle is influenced by its local geometry through the derivatives of the metric tensor i.e. the connection coefficients given by the Koszul formula

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{i\ell}\big(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk}\big). \tag{1.91}$$

If one tries to compute naively the semiclassical expansions of the worldline path integral, one naturally begins to encounter second derivatives of the metric tensor which are encoded in the the Riemann curvature tensor $R^i{}_{jkl}$, Ricci tensor R_{ij}, and scalar curvature R. Even the short time asymptotics of the propagator already knows about curvature. Thus the curved worldline sigma model is not a decorative refinement of the flat particle; it is the minimal system in which one sees that geometry and quantization are inseparable.

If one wants a reparametrization invariant formulation, it is better to introduce a one dimensional worldline metric in addition to the target space metric tensor. Since a one dimensional metric has only one independent component, one usually writes

$$h_{\tau\tau}(\tau) = e(\tau)^2, \tag{1.92}$$

where $e(\tau)$ is the einbein. Then the classical action becomes

$$S[x,e] = \frac{1}{2} \int d\tau\, e(\tau) \Big(e(\tau)^{-2} g_{ij}(x)\dot{x}^i \dot{x}^j + m^2 \Big). \tag{1.93}$$

Varying with respect to e imposes the worldline Hamiltonian constraint, while gauge fixing $e(\tau) = 1$ returns the more familiar form of the action. In one dimension this metric looks trivial, but conceptually it is indispensable since we will see that it is the one dimensional analog of the worldsheet metric in string theory, and it makes clear that the true dynamical object is not simply a function of time but a map modulo reparametrizations. This is exactly the perspective one wants before generalizing from worldlines to worldsheets.

A particularly useful way to think about the worldline path integral is through its Euclidean transition kernel

$$K(\beta; x_f, x_i) = \langle x_f | e^{-\beta H} | x_i \rangle = \int_{x(0)=x_i}^{x(\beta)=x_f} \mathcal{D}x\, e^{-S_E[x]}. \tag{1.94}$$

For small β, this is dominated by short paths and hence by local geometry near $x_i \approx x_f$. For closed loops one obtains traces,

$$\mathrm{Tr}(e^{-\beta H}) = \int_M d\mu(x)\, K(\beta; x, x). \tag{1.95}$$

This is the bridge to the heat kernel. When we later say that localization teaches one to compute a partition function from a small set of configurations, the heat kernel is one of the oldest and cleanest prototypes of that party line.

The heat kernel of an elliptic operator D is the kernel of e^{-tD}. For Laplace type operators one has the diagonal asymptotic expansion

$$K(t;x,x) \sim \frac{1}{(4\pi t)^{d/2}} \sum_{n=0}^{\infty} a_n(x)\, t^n, \qquad t \to 0^+. \tag{1.96}$$

The coefficients $a_n(x)$ are the Seeley DeWitt coefficients. They are local curvature invariants built from the background geometry and any bundle data appearing in the operator. From the worldline point of view, these coefficients emerge from a semiclassical small t expansion of the one dimensional sigma model around short loops.

In order to perform supersymmetric localization, the bosonic worldline sigma model is not enough. One must enlarge it by fermionic degrees of freedom. Geometrically, this forces us to introduce frames and spinors. Let $e_i{}^a(x)$ be a vielbein on the target space so that

$$g_{ij}(x) = e_i{}^a e_j{}^b \delta_{ab}, \tag{1.97}$$

with flat tangent space indices a, b. The corresponding spin connection $\omega_{i\,ab}$ is determined by metric compatibility and vanishing torsion. The reason this language is useful is that fermions transform most naturally not as coordinate vectors on the manifold but as objects carrying tangent space indices and acted upon by the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}. \tag{1.98}$$

This is the starting point for Dirac operators, spin bundles, and the supersymmetric quantum mechanics that underlies index theorems. The index theorems are very deep results in mathematics that proves the number of solutions to a local differential equation is fixed and protected by the topology of the solution space itself.

In one dimensional supersymmetric quantum mechanics, the simplest fermions are Grassmann fields $\psi^i(\tau)$ valued in the pullback of the tangent bundle $x^*(TM)$. Their covariant derivative along the worldline is

$$D_\tau \psi^i = \dot{\psi}^i + \Gamma^i{}_{jk}(x)\dot{x}^j \psi^k. \tag{1.99}$$

A standard $\mathcal{N} = 1$ Euclidean supersymmetric sigma model action is

$$S_E[x,\psi] = \int d\tau \left[\frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j + \frac{1}{2}\psi_i D_\tau \psi^i \right], \tag{1.100}$$

one is able to obtain this action either through inspection or through standard superspace methods. While extended supersymmetry or couplings to bundles can generate additional curvature terms such as quartic fermion interactions proportional to $R_{ijkl}\psi^i\psi^j\psi^k\psi^l$. What matters physically is that the bosons x^i and fermions ψ^i assemble into a supermultiplet which encodes supersymmetry, and the nilpotent supercharge acts by

$$Qx^i = \psi^i, \qquad Q\psi^i = \dot{x}^i. \tag{1.101}$$

This is the first real appearance of the cohomological structure that localization needs.

A technical remark that is often skipped but is conceptually important is that the worldline spinors are almost trivial, whereas spinors on the target space carry the real geometric content. In one Euclidean dimension, the only real distinction in the fermionic sector is the spin structure around the worldline circle i.e. periodic boundary conditions for supersymmetry protected quantities and antiperiodic ones for thermal quantities. This simple difference becomes decisive. Periodic fermions compute supertraces and index data. Antiperiodic fermions compute ordinary traces and thermodynamics. Thus even before one reaches quantum field theory, the choice of spin structure already determines whether the path integral behaves like a thermal partition function or a topological invariant.

The index theorem enters when one considers a supertrace rather than an ordinary trace. The McKean Singer idea is that for a suitable supersymmetric system,

$$\mathrm{ind}(\not{D}) = \mathrm{Tr}\left((-1)^F e^{-\beta H}\right), \tag{1.102}$$

and the right hand side is independent of β. The same quantity may therefore be evaluated in two radically different limits. In the large β limit it counts zero modes, hence the analytical index. In the small β limit it is governed by the local short time asymptotics of the heat kernel, hence by curvature densities. Equating the two yields the index theorem. In its most famous form,

$$\mathrm{ind}(\not{D}_E) = \int_M \widehat{A}(M) \wedge \mathrm{ch}(E). \tag{1.103}$$

From the physics side, this is not a coincidence but a localization phenomenon where the supersymmetric path integral reduces down an infinite dimensional path integral into a finite dimensional integral over a manifold M. Witten's work on supersymmetry and Morse theory, together with the supersymmetric path integral proofs of index theorems, made this perspective canonical in modern high energy theory.

This viewpoint is worth emphasizing because it teaches several general lessons that persist in higher dimensional localization. First, the exact answer is often independent of a continuous deformation parameter. Second, one is free to compute it in the most convenient limit. Third, one loop fluctuations around the fixed locus are not a nuisance but the source of the characteristic classes and determinants that appear in the final answer. These are precisely the principles that later reappear in topological field theory and supersymmetric gauge theory.

In finite dimensions this philosophy is closely related to equivariant localization. The physics translation is that a Q exact deformation

$$S \longrightarrow S + t\,QV \tag{1.104}$$

does not change the value of a Q closed observable, provided the measure and contour behave properly. Sending $t \to \infty$ then reduces the integral to the zeros of QV.

The field theoretic localization principle can be stated schematically as follows. Suppose the theory has a fermionic symmetry Q such that

$$Q^2 = \mathcal{L}_v, \tag{1.105}$$

where $\mathcal{L}_v$ is a Lie derivative along a Killing vector field that generates the symmetry action. Then let V be a fermionic functional chosen so that the bosonic part of QV is nonnegative. If we denote a general field configuration by Φ, then for any Q closed observable $\mathcal{O}$,

$$\langle \mathcal{O} \rangle_t = \int \mathcal{D}\Phi \, \mathcal{O} \, e^{-S[\Phi]-tQV[\Phi]} \tag{1.106}$$

is independent of t. In the limit $t \to \infty$, the integral localizes onto the solutions of the BPS equations obtained from the vanishing of the bosonic part of QV. The full result takes the schematic form

$$Z = \sum_{\Phi_\star} e^{-S_{\mathrm{cl}}[\Phi_\star]} \, Z_{1-\mathrm{loop}}[\Phi_\star] \, Z_{\mathrm{nonpert}}[\Phi_\star], \tag{1.107}$$

where $\Phi_\star$ is a fixed point configuration. This formula is intentionally schematic because its detailed realization varies enormously from theory to theory. Sometimes the fixed locus consists only of constant scalar zero modes, sometimes it is an instanton moduli space, sometimes it includes nontrivial vortex or monopole sectors. But the overall structure of the localization procedure is the same in that it contains classical fixed points, one loop determinants, and possibly additional localized nonperturbative sectors.

Equivariance is the geometric language that explains why Q^2 need not vanish strictly. In a rigid supersymmetric theory on a curved manifold, the square of a supercharge is typically not zero but a compact bosonic symmetry. One then computes not ordinary cohomology but equivariant cohomology. From this angle, path integral localization is never purely about supersymmetry, it is about supersymmetry combined with the bosonic symmetries of the background geometry. This is why rotating backgrounds, Ω deformation backgrounds [45, 46], toric manifolds, and spheres with isometry groups are so natural in localization problems. They supply the equivariant parameters that both regulate and organize the exact answer. For this purposes of this thesis, we will begin with the case with cases that are non equivariant before dealing with their equivariant extensions.

Despite the rich nature of one dimensional quantum field theories, nonequilibrium string theory lies in the realm of two dimensional quantum field theories since their source space is a two dimensional worldsheet. The obvious question to ask is whether or not path integral localization extends beyond one dimensional field theories and it turns out that there is a resounding affirmative answer. Two dimensions are the natural home of sigma models which form the backbone of string theories, conformal field theories, and the first historically decisive topological twists. In Witten's topological sigma models [47], the A and B models arise from twists of $\mathcal{N} = (2, 2)$ supersymmetric sigma models. The point is that after twisting, a scalar nilpotent charge appears, the stress tensor becomes Q exact up to topological terms, and the path integral localizes onto geometrically distinguished maps. In the A model, this leads to pseudo holomorphic maps and ultimately to Gromov Witten theory. In the B model, localization favors constant maps and brings complex structure data to the theoretical analysis.

Two dimensional supersymmetric gauge theories furnish a complementary story. Partition functions on S^2, hemisphere amplitudes, the elliptic genus on T^2, and equivariant A

twisted correlators can all be computed by localization. Depending on the supercharge and deformation chosen, one may obtain Coulomb branch formulas, Higgs branch formulas, or Jeffrey Kirwan residue expressions.

Three dimensional supersymmetric gauge theories made localization dramatically concrete for many physicists because the answers very visibly collapsed to ordinary matrix integrals [48, 49]. For $\mathcal{N} = 2$ Chern Simons matter theories on S^3, the full path integral reduces to a finite-dimensional integral over the Cartan algebra. The complicated functional integral over gauge fields, matter multiplets, ghosts, and fermions becomes a matrix model with a calculable one loop measure and classical Chern Simons phase. This was conceptually explosive because it turned strongly interacting three dimensional quantum field theories into an standard finite dimensional integral that one could actually analyze at finite N and in large N limits. In the ABJM case, this random matrix model description opened the door to exact tests of AdS_4/CFT_3.

This is also where one begins to appreciate the independent life of matrix models after localization. Once the theory has been reduced to a finite dimensional eigenvalue integral, one can bring to bear the entire machinery of matrix model analysis. One studies eigenvalue distributions $\rho(a)$ through their resolvents

$$\omega(z) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z - a_i}, \tag{1.108}$$

large N saddles, and genus expansions. In favorable cases, the free energy admits a $1/N$ expansion that behaves like a topological expansion of a string theory. In still more specialized limits, one may take a double scaling limit and obtain continuum theories from critical matrix ensembles.

Pushing the dimensions even higher, four dimensional gauge theories are precisely the sort of models that describe our universe through the Standard model of particle physics. The electromagnetic field, electroweak and Yang Mills field of quantum chromodynamics can all be described in this language. However, extracting exact physical predictions from these non Abelian Yang-Mills theories is notoriously difficult due to their strongly coupled nature at low energies. Because traditional perturbative techniques completely break down in this regime, we often often turn to simplified toy models to uncover underlying mathematical structures and extract exact results. This leads to topological Yang Mills theory and supersymmetric localization for four dimensional field theories [50, 51].

Topological Yang Mills theory was historically approached via Witten's topological twist of $\mathcal{N} = 2$ supersymmetric Yang Mills theory, which reinterpreted Donaldson invariants as correlation functions in a topological quantum field theory. The localization equations in that setting are the anti self duality equations

$$F^+ = 0, \tag{1.109}$$

where F^+ is the self dual part of the electromagnetic curvature two form. This forces by construction that path integral localizes on instanton moduli spaces. This was a watershed moment because it showed that quantum field theory could compute previously inaccessible topological invariants of smooth four manifolds.

Beyond four dimensions, rigid supersymmetric localization becomes increasingly selective. One can formulate supersymmetric gauge theories on spheres and other special manifolds in dimensions above four, but preserving enough supersymmetry generally demands very special background structures, and the resulting theories may involve much more complicated geometric datum and higher derivative terms. The moral is not that localization stops, but that the geometric prerequisites become more rigid. As dimension increases, the amount of ultraviolet singular behavior also grows, renormalization becomes harsher, and the existence of a clean cohomological complex becomes correspondingly more precious.

Another important road into localization, especially relevant for nonequilibrium physics, is stochastic quantization [52, 53]. Instead of quantizing a Euclidean field theory directly, one introduces a fictitious stochastic time s and evolves the fields by a Langevin equation,

$$\partial_s \phi(x,s) = -\frac{\delta S[\phi]}{\delta \phi(x,s)} + \eta(x,s), \tag{1.110}$$

with Gaussian noise

$$\langle \eta(x,s)\eta(x',s')\rangle = 2\,\delta(x-x')\delta(s-s'). \tag{1.111}$$

For large stochastic times, the distribution of ϕ is supposed to approach the Euclidean Gibbs weight $e^{-S[\phi]}$. Formally this gives a new representation of the quantum field theory in terms of stochastic dynamics.

What makes this relevant here is that stochastic quantization naturally comes with its own supersymmetry, the Parisi Sourlas supersymmetry. In favorable settings this stochastic supersymmetry can be interpreted cohomologically. That means some of the conceptual machinery of localization reappears in a seemingly very different context i.e. through noise, Langevin dynamics, and Fokker Planck equations. This is extremely suggestive for any project concerned with nonequilibrium extensions of quantum field theory, because it hints that stochasticity and cohomological control are not opposites. If localization were tied only to pristine Euclidean supersymmetry, it would seem too far removed from that world. But stochastic quantization shows that supersymmetric and cohomological structures can also govern noisy systems

In summary, the first general lesson is that exact results are usually not accidents, they are hints of some underlying symmetric structure. Whenever a partition function, protected correlator, or supersymmetric loop operator can be computed exactly, one should immediately ask what nilpotent symmetry, equivariant structure, or topological twist is doing the work. Localization teaches us to look for entire complexes, not just Lagrangians.

The second lesson is that geometry is not just extra mathematics for the sake of mathematics. Metrics, spin structures, almost complex structures, contact structures, torus actions, and moment maps are often the very ingredients out of which the localizing supercharge is built. A localization problem is as much a problem in background geometry as in quantum field theory. This is why the transition from flat space physics to curved manifolds in exact

QFT is so pedagogical, it exposed geometric structures that can be hard to identify in flat space perturbation theory.

The third lesson is that one loop determinants are conceptually central. In ordinary semiclassical approximations, the quadratic fluctuations are corrections to a classical story. In localization they are often half the answer. Characteristic classes, anomalies, and dependence on background parameters are encoded precisely in the determinant of the fluctuation complex. The heat kernel and index theorem viewpoint already foreshadowed this: local curvature data and zero mode counting meet in the one loop sector.

The fourth lesson is that nonperturbative sectors become computable once symmetry isolates them. One can check that these nonperturbative sectors are governed by a bestiary of distinguished field configurations: instantons, monopoles, vortices, domain walls, kinks, and branes. These objects represent the dominant semiclassical saddles in sectors completely inaccessible to ordinary perturbation theory. Instantons are Euclidean finite action saddles and govern tunneling amplitudes, vacuum splitting, and anomalies. Solitons are finite energy classical field configurations in Lorentzian signature and encode stable extended objects, topological sectors, and BPS states. Localization repeatedly gravitates toward such sectors because supersymmetry pairs fluctuations so efficiently around BPS configurations. The very same symmetry that collapses the path integral often singles out precisely these nontrivial saddles as fixed points.

While we have only scratched the surface of what localization can achieve, an exhaustive accounting of the subject falls beyond the scope of this thesis. However, before moving forward, we must temper this optimism. Despite its mathematical elegance and power in isolating nonperturbative sectors, the localization program is not without its shortcomings, and it is worth detailing a few of its primary weaknesses.

First, localization does not automatically compute all observables. It computes protected observables such as supersymmetric indices, certain loop operators, twisted correlators, and other operators annihilated by the chosen supercharge or lying in its cohomology. Generic time ordered correlators, scattering amplitudes, and chaotic real time observables are usually not localizable in the standard sense. This boundary is conceptually important.

There is a philosophical lesson about formal path integrals themselves. Localization repeatedly succeeds in situations where the original functional integral is mathematically ill defined in any naive sense. One should therefore distinguish between full path integral and the protected quantity extracted from a supersymmetric localization of a supersymmetric subsector of the theory. The latter may be exact and robust even when the former is formally delicate. This does not excuse mathematical sloppiness, but it does explain why physicists have repeatedly been able to obtain correct nontrivial answers from formal manipulations once enough symmetry is present. In this sense, localization teaches humility. The theories it solves exactly are not generic. They are very special field theories.

Second, localization does not remove the need to choose integration contours carefully. In Euclidean supersymmetric theories, positivity of the bosonic part of QV is usually essential for the standard argument. When complex saddles, noncompact directions, or indefinite kinetic terms are present, the contour question can become subtle. In many modern applications one must implicitly or explicitly choose a middle dimensional contour, regularize noncompact zero modes, or interpret the final answer through analytic continuation. Thus the slogan "add a Q exact term tQV and send $t \to \infty$" always hides a contour problem in the background. As we discussed in the first part of the introduction, the Schwinger Keldysh formalism teaches us that this analytic continuation portion may be quite difficult to do.

Finally, the relation between localization and renormalization is subtler than it first appears. Localization does not bypass renormalization, rather, it often computes quantities whose scheme dependence is highly constrained or absent. In the case of supersymmetric field theories, this often manifests as nonrenormalization theorems. As a second example, in nonconformal theories, exact answers arising from localization may still depend on contact terms, counterterms, or subtle regularization choices. One should therefore resist the temptation to say that localization completely bypasses quantum field theory's usual subtleties.

1.4 The Problem of Nonequilibrium String Theory

Finally, we arrive at our problem of interest i.e. the nonequilibrium string. The question of how covariant string perturbation theory can be defined on a Schwinger Keldysh time contour was first explored in [54, 25, 55] with the use of expected holographic dualities of gauge theories in the large N expansion. The key result was that Schwinger Keldysh formulation of nonequilibrium string theory requires not just a doubling of the fields but has an extra sector known as the wedge region which is needed for its complete formulation. In order to fully digest this, we will begin by introducing the main lessons of one of the most primitive form of string theory i.e. the bosonic string.

We previously mentioned that a point particle sweeps out a one dimensional manifold known as a worldline, but a string sweeps out a two dimensional worldsheet Σ, coordinatized by worldsheet coordinates $\sigma^a = (\tau, \sigma)$. The dynamical variables are no longer one dimensional trajectories $x^\mu(\tau)$, but target space embedding fields

$$X^\mu : \Sigma \to \mathcal{M}, \qquad \mu = 0, 1, \ldots, D-1, \tag{1.112}$$

which specify how the worldsheet sits inside a D dimensional target spacetime $\mathcal{M}$. The first lesson of string theory is therefore that the analog of a worldline sigma model is now a two dimensional nonlinear sigma model [56, 57, 58].

In order to build up an intuition for string dynamics, we will begin in a target space where we simply have a flat metric tensor. The corresponding action for the string worldsheet is then given by the Polyakov action which may be written down as

$$S_P[X, h] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h}\, h^{ab}\, \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \tag{1.113}$$

where h_{ab} is the intrinsic worldsheet metric and α' is the Regge slope. The string tension is therefore

$$T = \frac{1}{2\pi\alpha'}. \tag{1.114}$$

This formulation is classically equivalent to the Nambu Goto action which is simply the area of the worldsheet,

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma\,\sqrt{-\det(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu})}, \tag{1.115}$$

but the Polyakov form is much more useful because the worldsheet metric appears as an independent field, making the gauge symmetries manifest and the quantization much cleaner. Although, a complete technical discussion would usually go over its basic symmetries as quantum field theory i.e worldsheet diffeomorphisms, Weyl rescaling and rigid invarianec under target space Poincare transformations. The main conceptual result is that the diffeomorphism and Weyl symmetries imply that the worldsheet metric does not carry local propagating degrees of freedom. In two dimensions this is an enormous simplification and is the basic reason the Polyakov formulation is so powerful.

Because of diffeomorphism and Weyl invariance, one may choose conformal gauge where the worldsheet metric tensor drasticially simplifies,

$$h_{ab}(\tau,\sigma) = e^{2\varphi(\tau,\sigma)}\eta_{ab}, \qquad \eta_{ab} = \mathrm{diag}(-1,+1). \tag{1.116}$$

And the action reduces to

$$S_P = -\frac{1}{4\pi\alpha'}\int d\tau\, d\sigma\, \eta^{ab}\,\partial_a X^\mu \partial_b X_\mu = \frac{1}{4\pi\alpha'}\int d\tau\, d\sigma\, \left(\dot{X}^\mu \dot{X}_\mu - X'^\mu X'_\mu\right), \tag{1.117}$$

where

$$\dot{X}^\mu \equiv \partial_\tau X^\mu, \qquad X'^\mu \equiv \partial_\sigma X^\mu. \tag{1.118}$$

In this gauge, the equations of motion become the two-dimensional wave equation

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0. \tag{1.119}$$

For a closed string, one imposes periodicity in the spatial worldsheet coordinate,

$$X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma). \tag{1.120}$$

The general mode expansion then takes the form

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in(\tau-\sigma)} + \widetilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}\right). \tag{1.121}$$

The constants x^μ and p^μ are the center of mass position and momentum of the string, while α_n^μ and $\widetilde{\alpha}_n^\mu$ are the oscillator modes of the left moving and right moving sectors. From here, the canonical quantization of the string theory proceeds as in any quantum mechanical theory. The result being that the quantized string is a relativistic system with infinitely many harmonic oscillators, one for each Fourier mode and each target space direction. One is able to show that these string states have masses which are explicitly computable and scale as

$$M^2 = \frac{4}{\alpha'}\left(N + \widetilde{N} - 2\right), \qquad N - \widetilde{N} = 0, \tag{1.122}$$

so that at large excitation level the string spectrum behaves as

$$M^2 \sim \frac{N}{\alpha'}, \qquad M \sim \frac{\sqrt{N}}{\sqrt{\alpha'}}. \tag{1.123}$$

The first nontrivial massless level occurs when

$$N = \widetilde{N} = 1, \qquad M^2 = 0. \tag{1.124}$$

The corresponding states take the form

$$|\epsilon; k\rangle = \epsilon_{\mu\nu}\, \alpha^{\mu}_{-1}\widetilde{\alpha}^{\nu}_{-1}|0; k\rangle, \qquad k^2 = 0. \tag{1.125}$$

The polarization tensor $\epsilon_{\mu\nu}$ decomposes into irreducible parts under the Lorentz group

$$\epsilon_{\mu\nu} = \epsilon^{\text{traceless}}_{(\mu\nu)} + \epsilon_{[\mu\nu]} + \frac{1}{D}\eta_{\mu\nu}\, \epsilon^{\rho}{}_{\rho}. \tag{1.126}$$

These three pieces are interpreted as distinct target space background fields. The traceless symmetric tensor becomes a graviton fluctuation $g_{\mu\nu}$, the antisymmetric part becomes an antisymmetric tensor $B_{\mu\nu}$ which acts as a higher dimensional generalization of the electromagnetic field and the trace is the dilation Φ.

Although we began with a completely flat target space background and no background fields turned on, the quantum spectrum of the closed string necessarily contains massless excitations that transform exactly like fluctuations of a target space metric, a two form gauge field, and a scalar dilaton. In other words, these fields are not put in by hand. They are forced upon us by the quantum theory itself.

Ensuring that the gauge symmetry of Weyl invariance holds at the quantum mechanical level gives the key result that the number of target space dimensions isn't arbitrary but is in fact

$$D = 26 \tag{1.127}$$

for the bosonic string. Thus even the supposedly simple flat space Polyakov theory is highly constrained at the quantum level.

One may express this point more concretely by expanding around flat spacetime. Once the massless closed string modes are turned on, the Polyakov action with background fields turned on becomes

$$\begin{aligned} S_{\text{ws}} = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \sqrt{h}\, h^{ab}\, \partial_a X^{\mu}\partial_b X^{\nu}\, G_{\mu\nu}(X) + \frac{i}{4\pi\alpha'}\int d^2\sigma\, \epsilon^{ab}\, \partial_a X^{\mu}\partial_b X^{\nu}\, B_{\mu\nu}(X) \\ + \frac{\alpha'}{4\pi}\int d^2\sigma\, \sqrt{h}\, \Phi(X)\, R^{(2)}\,. \end{aligned} \tag{1.128}$$

Here $R^{(2)}$ is the two dimensional Ricci scalar on the worldsheet. This is the precise sense in which flat space string quantization already knows about curved target space geometry, antisymmetric flux backgrounds, and the dilaton.

Having identified the string spectrum with states of the worldsheet conformal field theory, one next requires a mechanism for inserting such external string states into correlation functions. This is achieved by means of vertex operators. In the operator state correspondence of two dimensional conformal field theory, to every physical string state $|\Psi\rangle$ there is associated a local worldsheet operator $V_\Psi(z,\bar{z})$ where z refers to a complexified coordinate $z=\tau+i\sigma, \bar{z}=\tau-i\sigma$. For the bosonic string in flat target space, the simplest example is the tachyon vertex operator

$$V_k(z,\bar{z}) = g_c\, :e^{ik\cdot X(z,\bar{z})}:,$$

where g_c is the closed string coupling and k^μ is the spacetime momentum and the dots refer to normal ordering operation in order to avoid creating divergent contributions. It is precisely these vertex operators that we insert into the worldsheet path integral.

More generally, excited string states are represented by vertex operators containing derivatives of the embedding fields. For example, the massless graviton, Kalb Ramond field, and dilaton arise from the operator

$$V_\epsilon(z,\bar{z}) = g_c\,\epsilon_{\mu\nu}\, :\partial X^\mu(z)\,\bar{\partial} X^\nu(\bar{z})\, e^{ik\cdot X(z,\bar{z})}:,$$

where $\epsilon_{\mu\nu}$ is the polarization tensor. Quantum consistency requires imposes several conditions including a massless condition

$$k^2=0,$$

together with transversality constraints

$$k^\mu\epsilon_{\mu\nu}=0, \qquad k^\nu\epsilon_{\mu\nu}=0.$$

Then as before, we are able to decompose the polarization $\epsilon_{\mu\nu}$ into irreducible tensors that we can identify with specific field excitations. In this way, spacetime particle states are encoded as local insertions on the worldsheet, and string scattering amplitudes are obtained by integrating appropriate correlation functions of these vertex operators over the insertion points and over the moduli of the worldsheet geometry. Schematically, the worldsheet path integral without vertex insertions is written as

$$Z = \frac{1}{\mathrm{Vol}(\mathrm{Diff}\times\mathrm{Weyl})}\int \mathcal{D}g_{ab}\,\mathcal{D}X^\mu\, e^{-S_{\mathrm{Polyakov}}[X,g]}.$$

Thus one integrates over all worldsheet geometries and all embeddings of the worldsheet into spacetime, while dividing by the gauge redundancy coming from worldsheet diffeomorphisms and Weyl rescalings.

However, if one wishes to include vertex operator insertions in order to compute scattering amplitudes which encodes stringy interactions, one writes

$$\mathcal{A}_n = \frac{1}{\mathrm{Vol}(\mathrm{Diff}\times\mathrm{Weyl})}\int \mathcal{D}g_{ab}\,\mathcal{D}X^\mu\, e^{-S_{\mathrm{Polyakov}}[X,g]}\prod_{i=1}^{n}\int d^2\sigma_i\,\sqrt{g(\sigma_i)}\,V_i(\sigma_i).$$

After gauge fixing, this formal expression reduces to an integral over the moduli space of punctured Riemann surfaces. One begins by decomposing this in topologically distinct subsectors distinguished by the genus g. At genus g, the n point scattering amplitude of closed bosonic strings then takes the form

$$\mathcal{A}_{g,n} = g_s^{2g-2+n} \int_{\mathcal{M}_{g,n}} d\mu_{g,n} \left\langle \prod_{i=1}^{n} V_i \right\rangle_g ,$$

where $\mathcal{M}_{g,n}$ denotes the moduli space of genus g Riemann surfaces with n marked points, $d\mu_{g,n}$ is the appropriate measure including ghost contributions, and $\langle \cdots \rangle_g$ denotes the worldsheet correlation function evaluated on a genus-g surface. The factor

$$g_s^{2g-2+n}$$

arises from the dilaton coupling to the worldsheet Euler characteristic and from the normalization of external insertions. Summing over genera then produces the full perturbative string S-matrix.

$$\mathcal{A}_n = \sum_{g=0}^{\infty} \mathcal{A}_{g,n},$$

For a genus$g$ surface with n punctures, the complex dimension of moduli space is

$$\dim_{\mathbb{C}} \mathcal{M}_{g,n} = 3g - 3 + n, \qquad \text{for } 2g - 2 + n > 0.$$

Thus, at tree level for the sphere one has

$$\dim_{\mathbb{C}} \mathcal{M}_{0,n} = n - 3,$$

while for the torus with n punctures one has

$$\dim_{\mathbb{C}} \mathcal{M}_{1,n} = n.$$

These moduli parametrize inequivalent conformal structures that cannot be removed by diffeomorphisms and Weyl transformations. Equivalently, they are the true geometric parameters over which the gauge fixed Polyakov path integral must still be integrated and hence the worldsheet path integral reduces to a finite dimensional integral through symmetries but naively through different mechanism than is done in path integral localization.

As a standard example, the four point tree level amplitude on the sphere depends on a single complex modulus. Fixing three insertion points through a residual symmetry group as

$$z_1 = 0, \qquad z_2 = 1, \qquad z_3 = \infty, \qquad z_4 = z,$$

one obtains an amplitude of the form

$$\mathcal{A}_{0,4} \sim g_s^2 \int_{\mathbb{C}} d^2 z \left\langle V_1(0,0)\, V_2(1,1)\, V_3(\infty,\infty)\, V_4(z,\bar{z}) \right\rangle.$$

For tachyon vertex operators this correlation function evaluates, up to normalization and momentum-conserving delta functions, to

$$\mathcal{A}_{0,4} \sim g_s^2 \int_{\mathbb{C}} d^2 z \, |z|^{-\alpha' s/2-4} |1-z|^{-\alpha' t/2-4},$$

where the Mandelstam invariants are

$$s = -(k_1+k_2)^2, \qquad t = -(k_1+k_3)^2, \qquad u = -(k_1+k_4)^2.$$

This integral yields the Virasoro Shapiro amplitude, whose analytic structure exhibits the infinite tower of intermediate string states. The poles of the resulting Gamma functions occur precisely at the masses of string excitations, demonstrating that a single smooth worldsheet amplitude unifies what would appear in field theory as infinitely many exchange diagrams.

At higher genus, the same essential structure persists, but now the integration extends over the full moduli space $\mathcal{M}_{g,n}$. In the gauge fixed Polyakov formulation, the ghost system provides the measure on moduli space through insertions of the b ghost paired with Beltrami differentials. Schematically, one writes

$$d\mu_{g,n} \sim \prod_{a=1}^{3g-3+n} dm^a \, d\bar{m}^a \Big\langle \prod_{a=1}^{3g-3+n} (b, \mu_a) \, (\bar{b}, \bar{\mu}_a) \prod_{i=1}^{n} V_i \Big\rangle,$$

where m^a are local complex coordinates on $\mathcal{M}_{g,n}$, and μ_a are the associated Beltrami differentials. This expression makes explicit that the moduli are the remaining gauge inequivalent degrees of freedom of the worldsheet metric after quotienting by diffeomorphisms and Weyl rescalings. In this sense, string scattering amplitudes are integrals over the geometry of punctured Riemann surfaces, and the structure of perturbative string theory is inseparable from the geometry of moduli space itself. Importantly, we mention that ensuring that these expressions encounter no practical pathologies or singularities will be precisely one of the obstacles that we will encounter in constructing the nonequilibrium string.

In the setting where the background fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ are turned on, they are constrained by quantum Weyl invariance of the sigma model. At leading order in α', the corresponding beta functions can computed via standard renormalization group flow methods to be

$$\beta^G_{\mu\nu} = \alpha' \left(R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + 2\nabla_\mu \nabla_\nu \Phi \right) + O(\alpha'^2), \tag{1.129}$$

$$\beta^B_{\mu\nu} = \alpha' \left(-\frac{1}{2} \nabla^\rho H_{\rho\mu\nu} + \nabla^\rho \Phi \, H_{\rho\mu\nu} \right) + O(\alpha'^2), \tag{1.130}$$

$$\beta^\Phi = \frac{D-26}{6} + \alpha' \left(4(\nabla\Phi)^2 - 4\nabla^2\Phi - R + \frac{1}{12} H^2 \right) + O(\alpha'^2), \tag{1.131}$$

where

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \tag{1.132}$$

The conformal invariance conditions

$$\beta^G_{\mu\nu} = 0, \qquad \beta^B_{\mu\nu} = 0, \qquad \beta^\Phi = 0 \tag{1.133}$$

are then interpreted as the target space equations of motion for the background fields. It is incredibly interesting to note here that the beta equation for the metric tensor is precisely the Einstein field equations that describes gravitation but sourced by the antisymmetric B field and the dilaton Φ

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}, \tag{1.134}$$

with

$$T_{\mu\nu} = \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma}_{\nu} - 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \left(\frac{1}{4} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} - 2\nabla^2 \Phi \right). \tag{1.135}$$

This is the correct point of departure for discussing nonequilibrium string theory. In equilibrium perturbative string theory, one typically studies strings in static or stationary backgrounds for which the worldsheet theory is a conformal field theory. The nonequilibrium problem begins once these background fields become genuinely time dependent and break time translation invariance,

$$G_{\mu\nu} = G_{\mu\nu}(X^0, \mathbf{X}), \qquad B_{\mu\nu} = B_{\mu\nu}(X^0, \mathbf{X}), \qquad \Phi = \Phi(X^0, \mathbf{X}), \tag{1.136}$$

where we have singled out the timelike component X^0. Now the couplings of the worldsheet sigma model themselves depend on the target space time coordinate. In general, we can expect there to be no global timelike Killing vector, no preferred notion of positive frequency, and no canonical asymptotic vacuum with respect to which one may define an ordinary S matrix. The worldsheet theory ceases to be a simple static conformal field theory and instead becomes a genuinely time dependent two dimensional quantum field theory. This is the basic origin of the difficulty of nonequilibrium string theory.

At a formal level, one expects that a real time treatment of such backgrounds should require an "in in" rather than an "in out" formalism as suggested by the Schwinger Keldysh formalism

$$\begin{aligned} &Z_{\mathrm{SK}}[J_+, J_-] = \\ &= \int \mathcal{D}X_+ \, \mathcal{D}X_- \, \mathcal{D}h_+ \, \mathcal{D}h_- \; \rho_0[X_+, X_-] \, \exp\left\{ iS_{\mathrm{ws}}[X_+, h_+] - iS_{\mathrm{ws}}[X_-, h_-] \right\}. \end{aligned} \tag{1.137}$$

Here, we would naively expect that the worldsheet path integral would obtain a doubling of not just the target space embedding coordinates into X_+ and X_- but the worldsheet metric tensors would also double into a pair h_+ and h_-.

However, unlike in ordinary quantum field theory, such a formulation is far from straightforward because the string path integral also involves dealing with the worldsheet diffeomorphisms and Weyl symmetry, their associated ghost systems, and boundary terms arising from the turnaround point of the Schwinger Keldysh contour. Consequently, while the flat space

Polyakov action is one of the cleanest and most beautiful starting points in all of theoretical physics, extending it to genuinely nonequilibrium settings remains one of the central open structural problems in string theory.

In order to begin probing the nonequilibrium string, we can leverage the large N limit of random matrix models where, the large N limit organizes perturbation theory in a way that is naturally interpreted as a sum over discretized two dimensional geometries. Consider, for example, a Hermitian $N \times N$ matrix integral with action of the schematic form

$$Z = \int dM \exp[-N \operatorname{Tr} V(M)] \, ,$$

where V is a polynomial potential. When one expands this partition function perturbatively around the Gaussian theory, each interaction vertex contributes a factor of N, each propagator contracts a pair of matrix indices, and each closed index loop contributes an additional factor of N. Using the double line notation of 't Hooft, every Feynman diagram becomes a ribbon graph, and such ribbon graphs are in 1:1 correspondence with triangulated two dimensional surfaces. If a given graph has V vertices, E edges, and F faces, then its overall scaling is

$$N^{V-E+F} = N^{\chi},$$

where χ is the Euler characteristic of the associated surface. Writing $\chi = 2-2g$ for a closed orientable surface of genus g, one finds that the perturbative expansion is weighted as

$$Z \sim \sum_{g=0}^{\infty} N^{2-2g} Z_g.$$

This is precisely the same topological organization that appears in closed string perturbation theory, where each genus g worldsheet contributes with a factor of g_s^{2g-2}, so that one identifies the string coupling as $g_s \sim N^{-1}$. The planar sector, corresponding to $g = 0$, dominates as $N \to \infty$, while higher genus corrections are suppressed by powers of $1/N^2$, exactly as higher loop string amplitudes are suppressed by powers of the string coupling. The lesson here being that the large N expansion of random matrix models yields qualitative hints about the dynamics of a corresponding string theory.

If, however, one places the matrix degrees of freedom on a Schwinger Keldysh time contour, then each propagator and vertex acquires an additional contour label indicating whether it lies on the forward branch, the backward branch, or at the turn around segment where the contour closes. Repeating the same large N ribbon graph analysis therefore no longer produces a single undifferentiated closed worldsheet, but instead yields a worldsheet endowed with a natural real time decomposition [25, 55, 54],

$$\Sigma = \Sigma^{+} \cup \Sigma^{\wedge} \cup \Sigma^{-}.$$

where Σ^{+} corresponds to the part of the worldsheet on the forward branch of the time contour, Σ^{-} corresponds to the backward part of the time contour, and the "wedge region"

$\Sigma^\wedge$ is the portion of the worldsheet that connects them, and corresponds to the turn around late time where the forward and backward branch of the time contour meet. Surprisingly, the wedge region is itself a topologically two dimensional region with arbitrarily complicated topology and its own genus expansion. Geometrically, however, it is expected to be highly anisotropic on the worldsheet, in such a way that the worldsheet develops a nonrelativistic foliation structure where the leaves of foliation connect points on the boundary with the Σ^+ region to a point on the boundary with the Σ^- region. The immediate question one might ask is: "How can this wedge region be explicitly constructed in a manner suitable for worldsheet calculations?"

To make progress in this direction, it is necessary to provide a prescription for addressing the fact that string perturbation theory would need to be defined on analytically continued worldsheets, which are permitted to evolve both forward and backward in time. It is well known that dealing with a covariant and strictly Lorentzian string perturbation theory can lead to uncontrollable singularities upon integrating over all worldsheet geometries and hence it was argued in [59], that one can minimally modify the string perturbation theory by allowing the worldsheets to be generically described by a Riemann surface and only analytically continued into Lorentzian signature when the Riemann surface develops a nodal singularity, in other words, as we approach the boundary of the moduli space of punctured Riemann surfaces. The physical nature of these singularities depends on the type of degeneration, but the singularity of interest here arises from a non-separating degeneration. This node can be conformally mapped to an infinitely long tube, which is interpreted as a free Lorentzian string going on-shell.

The real issue then becomes how to connect the two Lorentzian worldsheets with oppositely oriented analytic continuations. From the arguments presented above, we know that the worldsheet must degenerate in the wedge region, and this degeneration suggests that we are working near the boundary of the moduli space of Riemann surfaces. A useful language for describing such boundary regions is provided by tropical geometry. Very schematically, as a family of punctured Riemann surfaces degenerates, thin necks and long tubes develop on the worldsheet, and in the limiting regime these are replaced by edges of a combinatorial graph whose lengths record the degeneration parameters. In this way, the boundary strata of moduli space are described by metric graphs, or tropical curves, which capture how the smooth complex surface breaks apart into simpler components joined at nodes. Consequently, one is led to the conclusion that in order to construct the wedge region, one must investigate the behavior of analytically continued tropicalized worldsheets. In the next chapter, we will go in the deep technical details behind the construction of these tropicalized worldsheets in the simplified scenario of a topological sigma model where we can leverage path integral localization methods. These tropical topological sigma models are coined *tropological* sigma models.

Chapter 2

Tropological Sigma Models

2.1 Introduction

These analytically continued tropical worldsheets were constructed in [60] and were shown to have a non relativistic worldsheet foliation structure as suggested by the large N arguments. They are known as tropical topological sigma models or *tropological* sigma models for short.

The work presented in this paper is at least partially motivated by a remarkable result of Mikhalkin, established twenty years ago [61]: Certain Gromov-Witten invariants of pseudoholomorphic maps [62, 63] can be computed by first going to the tropical limit [64, 65, 66, 67] of the maps involved, effectively reducing the problem to counting certain types of piecewise-linear maps between tropicalized manifolds. The original proofs of this perhaps surprising and potentially computationally powerful result were rooted in abstract mathematical arguments. Since Gromov-Witten invariants emerged from quantum field theory (more precisely, a topological sigma model [63] coupled to worldsheet topological gravity [68, 69]), we would like to understand this relation between the counting of complex and tropical maps directly in physics terms, from the first principles of quantum field theory, and in particular to find its direct path-integral realization. In the process, we are hoping to learn new things about topological strings, worldsheet quantum field theories, and more generally about the path integral method extended to the tropical regime.

Over the past twenty years (or much longer, since there were many significant precursors of tropical mathematics long before the name "tropical" was coined), tropical geometry has developed into a thriving field, connected to surprisingly many areas of mathematics, physics, computer science, and more (see [64, 65, 66, 67, 70] for an introduction to tropical geometry and its applications). Much like real and complex geometry deal with geometric objects defined over the field of real or complex numbers $\mathbb{R}$ and $\mathbb{C}$, the natural objects of tropical geometry are defined over a certain semifield $\mathbb{T}$, known as the "tropical semifield": Roughly speaking, the field operation of multiplication has been replaced in $\mathbb{T}$ by addition, and the field operation of addition has been replaced by maximization. For example, given a polynomial in real variable x with real coefficients,

$$p(x) = a_1 x^{k_1} + a_2 x^{k_2} + \ldots + a_n x^{k_n}, \tag{2.1}$$

with $k_1 < \ldots < k_n$ a sequence of positive integers, its tropicalized version would be given in terms of standard mathematical operations by

$$p_{\text{trop}}(x) = \max\left\{a_1 + k_1 x, a_2 + k_2 x, \ldots, a_n + k_n x\right\}. \tag{2.2}$$

Hence, a nonlinear structure in traditional mathematics has turned into a piece-wise linear structure in tropical mathematics. Moreover, the coefficients of the linear terms in x are integers. Such a radical departure from more traditional geometry over fields has profound consequences: Objects of tropical geometry are typically locally modeled by piece-wise linear polyhedra, and tropical maps between two such geometries respect this structure, including the integrality of the coefficients of the linear terms. Calculations in the tropical setting therefore take on a very different nature compared to those in more traditional areas of geometry, often taking on a purely combinatorial form. They bring in new and unexpected connections to other areas of mathematics and computer science, such as combinatorial polyhedral geometry, dynamical programming, and optimization algorithms. If one can rewrite traditional problems of enumerative geometry into their equivalent tropical form, one might gain a computational and conceptual advantage, and solve problems that would otherwise be inaccessible by more traditional methods. We would like to take this remarkable relation between classical geometry and tropical geometry and look for its direct manifestations in quantum field theory, with the hope that such a direct reformulation in the physics language can teach us new and perhaps unexpected facts about path integrals and quantization.

In their original quantum field theory formulation, the standard Gromov-Witten invariants first appeared in topological string theory, as certain correlation functions of gauge-invariant observables in a theory whose fields are maps from the worldsheet Σ to the target space M (and referred to as the "topological sigma model" for short)[1] coupled to topological gravity. Such quantum theories with topological symmetries can be given a very natural path-integral representation, using the methods of BRST quantization. Since the maps that the path integrals of this theory localize to are (pseudo)holomorphic maps

$$\Phi : \Sigma \to M, \tag{2.3}$$

the worldsheet must be equipped with a dynamical complex structure – or, equivalently, a conformal class of dynamical metrics – and the target space must carry an almost complex structure, which may or may not be integrable. The underlying worldsheet quantum theory is then an example of a relativistic topological theory of the cohomological type [72], and

[1] More precisely, they correspond to the observables in the topological sigma model known as the A-model. Throughout this paper, we focus entirely on the tropicalization of the A-model, leaving the analogous consideration of the B-model (see [71] for a review) outside of our scope.

the Gromov-Witten invariants appear among the gauge-invariant (or BRST invariant) observables of this topological theory. How do we formulate the analog of this topological field theory framework, when the pseudoholomorphic maps are replaced by tropical maps?

This question has many ramifications: Do the topological worldsheet theories that define the Gromov-Witten invariants have a tropical analog which would be accessible to the traditional methods of quantum field theory? Is there a tropical analog of the path integral formulation of such field theories? Do the techniques of the BRST quantization of gauge theories naturally extend to the tropical case? These questions need to be answered before one can offer a purely path-integral-based proof of Mikhalkin's results.

We will see that constructive answers to these questions will take us outside of the limits of traditional worldsheet theories with relativistic invariance. The worldsheets will no longer carry complex structures, nor will they be equipped with nondegenerate metrics. Yet, the resulting theories are consistent, and correlation functions of their physical observables are calculable. The worldsheet theories that we will encounter turn out to belong to the class of theories with Lifshitz-like behavior [73, 74], sensitive to a worldsheet foliation structure. Clearly, explorations of any such extension of string theory beyond its traditional scope could be valuable even outside the realm of topological theories.

In fact, our second motivation for the present work originates from questions about string theory in the broader context, beyond topological, with propagating degrees of freedom. Historically, the formulation of fundamental string theory was deeply rooted in the theory of the S-matrix in Minkowski spacetime, and therefore attached to the implicit assumption of the existence of a stable, static, eternal vacuum. In order to study non-equilibrium systems using string theory, it would be highly desirable to relax this assumption, and formulate the string-theory analog of the Schwinger Keldysh formalism, which is suitable for the study of systems far from equilibrium. This formalism is based on a doubled time contour, in which the system is first evolved from the remote past into the far future, and then to the past again. But what would string perturbation theory look like, when extended far from equilibrium? This question was addressed using the techniques of large-N dualities [54, 25, 55], leading to a perhaps surprising prediction: The genus expansion of string perturbation theory, familiar from the study of equilibrium states, should undergo a refinement, whereby the string worldsheets Σ contributing to the such should decompose into three parts: One, Σ^+, at least roughly corresponds to the evolution forward in time, another – Σ^- – corresponds to the evolution backwards in time, and finally the "wedge region" $\Sigma^\wedge$ connecting Σ^+ to Σ^-, corresponds to the time in the far future where the two branches of the Schwinger Keldysh time contour meet. It is this wedge region $\Sigma^\wedge$ whose hypothetical worldsheet description remains quite enigmatic. The large-N arguments have shown that $\Sigma^\wedge$ does not represent simple Cutkosky-like cuts of the worldsheets, but it has its own topological expansion, with higher-genus $\Sigma^\wedge$ contributing to string perturbation theory. Thus, $\Sigma^\wedge$ is at least topologically two-dimensional, exhibiting the full topological complexity of two-dimensional surfaces. On the other hand, the structure of the ribbon diagrams in the large-N analysis indicates that the *geometry* of $\Sigma^\wedge$ is highly anisotropic [25]: The worldsheet distances in the directions connecting the boundaries with Σ^+ to the boundaries with Σ^- are effectively scaled to zero (when the distances in the transverse worldsheet direction are held fixed). As we will see below, very similar anisotropic features in the worldsheet path integral will emerge in the study of tropical topological theories, as well as for their non-topological cousins.

In this paper, and its sequel [75], we will construct – at least in the controlled setting of a topological theory – an example of a string theory which does not require the existence of a nondegenerate metric or a complex structure on the worldsheet. We will see how such a construction naturally emerges when we try to make sense of path integrals for topological theories of tropical maps.[2] We will formulate and study the topological quantum field theories describing the matter sector, defined on a surface Σ, whose path integrals localize to the solutions of the appropriately defined tropical limit of pseudoholomorphic maps from Σ to a target space M. For the lack of a better term, we refer to such tropical topological sigma models as *tropological sigma models*.[3] Throughout this paper, we will treat the appropriate version of worldsheet gravity as a fixed, nondynamical background. The construction of the appropriate worldsheet tropicalized topological quantum gravity that our tropological sigma models can naturally couple to will be presented in the sequel paper [75]. In the final parts of this paper, we will apply the lessons learned from the topological sigma-model case in a broader context, and explore some aspects of this type of theories without the restriction that the worldsheet theory be topological.

The present paper is organized as follows. In the rest of §1, we provide a lightning overview of some central features of tropical geometry, focusing on those directly relevant for this paper. Then we discuss some of the first obstacles and potential pitfalls that we are facing in an attempt to give a path-integral representation to a tropicalized sigma model, and propose our candidate for the tropical version of the localization equations. In §2, we study the worldsheet and target-space geometric structures associated with these tropicalized localization equations, clarifying their symmetries and highlighting the differences from the standard relativistic case. In §3 we construct the tropical version of the topological sigma model, using the traditional method of BRST quantization in the path integral formulation. We specifically address some of the novelties compared to the relativistic case, in particular in the structure of antighost and auxiliary BRST multiplets, and explain how to deal with a residual gauge symmetry that did not appear in the relativistic case. In §4, we consider the example with the tropicalized $\mathbb{C}P^1$ as the target space, solve for all its topological correlation functions of point-like observables at any genus, and confirm that the results

[2] There are other known examples of string theories whose worldsheet path integral description involves more exotic mathematics compared to the conventional critical (super)strings; perhaps most notably, the cases of p-adic strings and non-Archimedean strings have been studied in considerable detail [76, 77, 78, 79, 80].

[3] To the uninitiated, the word "tropological" may appear to be a random amalgam of the words "tropical" and "topological". However, a closer inspection reveals that the word *tropological* has an esteemed history going back almost two thousand years, having referred to "the use of a Scriptural text so as to give it a moral interpretation or significance apart from its direct meaning" [81] (see also [82]). Coincidentally, this word has already been used in mathematical physics recently, in a different context for the tropical version of topological theory constructions, in [83]; and it also appeared previously in [84]. (We thank Yoav Len for bringing Ref. [84] to our attention.)

match the correlation functions of the relativistic $\mathbb{C}P^1$ topological A-model. In §5 we depart from the limitations of topological theories, and study the simplest bosonic sector of the tropical sigma models as a theory with propagating degrees of freedom. We focus on the question of a proper analytic continuation of the theory to real worldsheet time. In §6 we conclude with some remarks about possible generalizations.

Tropical mathematics and Gromov-Witten invariants

For two real numbers, $a, b \in \mathbb{R}$, define a one-parameter family of two operations, labeled by $\hbar \geq 0$: the product $\odot_\hbar$ and the addition $\oplus_\hbar$, by

$$e^{a\odot_\hbar b/\hbar} = e^{a/\hbar} \cdot e^{b/\hbar}, \tag{2.4}$$

$$e^{a\oplus_\hbar b/\hbar} = e^{a/\hbar} + e^{b/\hbar}. \tag{2.5}$$

For $\hbar$ real and positive, these operations equip the real numbers with the structure of a field $\mathbb{R}_\hbar$, which is canonically isomorphic to the field $\mathbb{R}$. However, as $\hbar \to 0$ (and denoting $\odot_0$ and $\oplus_0$ simply by $\odot$ and $\oplus$), the rules contract to

$$a \odot b = \lim_{\hbar\to 0} \hbar \log\left\{e^{(a+b)/\hbar}\right\} = a + b, \tag{2.6}$$

$$a \oplus b = \lim_{\hbar\to 0} \hbar \log\left\{e^{a/\hbar} + e^{b/\hbar}\right\} = \begin{cases} a, & \text{if } a > b; \\ b, & \text{if } b > a. \end{cases} \tag{2.7}$$

Since $\oplus$ is now idempotent, $\mathbb{R}_\hbar$ itself contracts to the tropical semifield, $\mathbb{R}_{\hbar=0} \equiv \mathbb{T}$, with $a \odot b = a + b$ and $a \oplus b = \max(a, b)$. (More accurately, $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$, with the role of tropical unity played by 0, and the role of tropical zero played by $-\infty$.) This procedure has been known as the Maslov (or sometimes Litvinov-Maslov) dequantization [67, 70, 85, 64].

With this tool at hand, tropicalizations of complex manifolds can be generated, roughly speaking, as follows. Consider a complex manifold M, of complex dimension n. Choose a suitable system of complex coordinates Z^I on M, with $I = 1, \ldots, n$. Then take the absolute values $|Z^I|$, and apply the $\hbar \to 0$ tropicalization to each of these absolute values individually. This yields a piece-wise linear object M_{trop}, locally in generic points of M_{trop} of real dimension n. Such objects are naturally parametrized by tropical coordinates, consisting of $X^I \sim \log |Z^I|$ for $I = 1, \ldots, n$ in the $\hbar \to 0$ limit (and with the phases of all Z^I forgotten). The tropical coordinates X^I then satisfy the tropical algebraic rules in $\mathbb{T}$. This procedure works particularly nicely for simplest and most symmetric complex manifolds which are essentially covered by one privileged coordinate system. $\mathbb{C}P^n$ would be an example, leading to tropical projective spaces $\mathbb{T}P^n$, which carries the linear structure of a real n-dimensional polygon.

Reminder: Origins of tropical geometry in superstring theory and M-theory

Historically, one of the stronger streams that contributed to the formation of the field of tropical geometry came from string theory and M-theory. Tropicalizations of complex curves have naturally emerged in several different corners, mostly in the context of considering dualities of various extended BPS objects. In this § 2.1, we remind the reader briefly of some of the historical context, as an additional motivation for our own treatment of the tropicalization of topological sigma models below. Although this historical string-theory perspective reviewed in § 2.1 will provide heuristic insights that we will find useful for our further approach, it is not strictly necessary for the rest of this paper, and the reader not interested in the superstring/M-theory context can skip this section.

One of the first instances where it was understood how piece-wise linear supersymmetric BPS configurations of various branes ending on other branes can be lifted to a smooth holomorphic brane in M-theory was the case of Type IIA D4-branes ending on infinite NS5-branes (and possibly also with D6-branes present) [86]. Consider Type IIA superstring theory on the flat Minkowski spacetime $\mathbb{R}^{10}$, with coordinates $X^0, \ldots X^9$, first at very weak string coupling. We will place parallel NS5-branes at $X^7 = X^8 = X^9 = 0$ and at various fixed values of X^6. The worldvolume of the NS5-branes is thus parametrized by coordinates $X^0, X^1, \ldots, X^5$. We connect the NS5-branes by D4-branes which are at some fixed values of X^4 and X^5, stretching along X^6 between two NS5-branes. The D4-brane worldvolume is thus parametrized by $X^0, \ldots, X^3$ and X^6, with X^6 a compact interval. This piece-wise linear network is a 1/4 BPS state in Type IIA superstring theory.

A useful vantage point can be gained by lifting this configuration of intersecting D4-branes and NS5-branes to strong coupling, described by M-theory on $\mathbb{R}^{10} \times S^1$. The radius R of the extra dimension of M-theory plays the role of the Type IIA string coupling. We will denote by X^{10} the coordinate on this M-theory S^1. In this picture, D4-branes and NS5-branes have the common origin, a single smooth M-theory M5-brane; depending on whether the M5-brane wraps the S^1 or not, it gives rise to the D4-brane or the NS5-brane at weak string coupling. Such an M5-brane can again be viewed as located at $X^7 = X^8 = X^9 = 0$. Its worldvolume will again be parametrized by $X^0, \ldots, X^5$. For such an M5-brane, the condition of 1/4 BPS supersymmetry simply requires that the worldvolume should be embedded into $X^6 + iX^{10}$ holomorphically, with the corresponding boundary conditions at infinity. In particular, the configuration corresponding to the M-theory lift of our D4- NS5-system is described by the holomorphic map

$$X^6 + iX^{10} = R \sum \log(U - a_i) - R \sum \log(U - b_i); \tag{2.8}$$

in terms of the complex coordinate

$$U = X^4 + iX^5 \tag{2.9}$$

parametrizing in the directions transverse to the D4-branes. (The first sum is over all the D4-branes that end on the NS5-brane from the left, and the second sum over those D4-branes that end on the NS5-brane from the right; a_i and b_i are the locations of the D4-branes inside the NS5; see [86] for details). In the limit of $R \to 0$, which corresponds to the weak string coupling, this smooth single holomorphic curve degenerates into the network of piece-wise linear flat branes that we started with in the Type IIA theory. Mathematically, this relation is very reminiscent of tropicalization; indeed, the relation between U and $X^6 + iX^{10}$ that follows from (2.8) can be characterized as

$$U \sim \exp\left\{\frac{X^6 + iX^{10}}{R}\right\}, \tag{2.10}$$

and we see the limit of zero radius $R \to 0$ of the M-theory circle corresponds to the Litvinov-Maslov dequantization limit. The role of the Litvinov-Maslov $\hbar$ is thus played by the Type IIA string coupling, the tropical limit corresponding to the limit of zero string coupling.

Yet another prime example of tropicalization emerging from string and M-theory is given by BPS objects in Type IIB superstring theory in $\mathbb{R}^{10}$. There are two antisymmetric 2-form gauge fields $B^{(1)}$ and $B^{(2)}$ in the spacetime supergravity description of this system. Consequently, this theory contains half-BPS strings, labeled by two co-prime integers (p, q) and referred to as (p, q)-strings. p and q are the charges under the two B-fields. At weak coupling, one can interpret say the $(1, 0)$-string as the fundamental string, and the $(0, 1)$-string as the corresponding D-string.

Naively, one expects that the fundamental string is allowed to end on a stretched D-string; however, the rules for joining strings in this case turn out to be more sophisticated [87, 88, 89, 90, 91], as they require that when a string juncture is formed, the total p and q charges must be separately conserved,

$$\sum p_i = 0, \qquad \sum q_i = 0, \tag{2.11}$$

at each junction. Thus, a (1,0)-string can join a (0,1)-string, but the third leg in this junction must be a (1,1)-string, with the appropriate orientation. For the configuration to be 1/4 BPS supersymmetric, the strings must lie in a two-plane, and their directions in the plane are correlated with their (p, q) charges. Thus, not only must the charges be conserved at the junction, the strings must also come in under specific fixed angles. This procedure can be iterated, with many individual junctions conforming to the same rules, and leading to BPS objects known as *string networks*. The geometry of this network consists of piecewise-linear segments connected at vertices subjected to the charge and angle conservation rules. As it turns out, the resulting geometric object is mathematically just a tropical curve in the tropicalized $\mathbb{C}P^2$ (which itself is essentially the two-dimensional flat plane, modulo the compactification locus).

The lift of this Type IIB string network configuration to M-theory is again very illuminating [92]: All the different (p, q) strings originate from a single object, the M-theory M2-brane. Type IIB superstring theory on $\mathbb{R}^{10}$ is dual to M-theory compactified on a two-torus T^2. The complex structure of the T^2 determines the complexified Type IIB coupling constant; and we denote this modulus by

$$\tau = \tau_1 + i\tau_2. \tag{2.12}$$

Our coordinates X^9 and X^{10} on the T^2 will satisfy the periodicity conditions

$$(X^9, X^{10}) \sim (X^9 + 2\pi R, X^{10}) \sim (X^9 + 2\pi R\tau_1, X^{10} + 2\pi R\tau_2), \tag{2.13}$$

with R indicating the overall size of the T^2. It is this radius that will be taken to zero to recover the Type IIB superstring limit. The Type IIB S-duality symmetry group $SL(2,\mathbb{Z})$ acts naturally on this modulus by modular transformations, giving a geometric explanation of S-duality via M-theory.

In this picture, the various (p, q)-strings simply correspond to the unique M2-brane wrapping various corresponding (p, q) cycles inside the T^2. On the M-theory side of this duality, the M2-branes which preserve the same degree of supersymmetry as the Type IIB string networks if their worldvolume is again embedded into the spacetime holomorphically. Take the (p, q)-string network to lie in the X^1, X^2 plane. It is then useful to pair up these two coordinates with the two compact coordinates on the T^2 and use the complex coordinates

$$Z^1 = X^1 + iX^9, \qquad Z^2 = X^2 + iX^{10}. \tag{2.14}$$

Further changing the variables to

$$U = \exp\left\{\frac{Z^1}{R} - \frac{\tau_1}{\tau_2}\frac{Z^2}{R}\right\}, \qquad V = \exp\left\{\frac{1}{\tau_2}\frac{Z^2}{R}\right\}, \tag{2.15}$$

one finds that the condition for the M2-brane to satisfy the same 1/4 BPS condition as the Type IIB (p, q)-string network simply reduces to the single condition of a vanishing holomorphic function

$$f(U, V) = 0, \tag{2.16}$$

defining a smooth holomorphic embedding of the M2-brane into the compactified spacetime of M-theory, with boundary conditions at infinity set by the choice of the (p, q)-strings in the corresponding Type IIB network. Importantly, the Type IIB superstring limit results from taking $R \to 0$ in (2.15), again essentially reproducing the steps of the Litvinov-Maslov dequantization. This connection between the string networks and tropicalization was later highlighted again in [93].

In the limit of the small radius of the M-theory circle, the holomorphic surface Σ projects onto an amoeba [94] in the Type IIB dimensions, which in the strict zero-radius limit becomes the piece-wise linear tropical curve describing the geometry of the string network. Thus, the limit of small radius is equivalent to the Litvinov-Maslov dequantization, the role of $\hbar$ is being played by the radius of the M-theory compactification torus.

In the descent from M-theory to Type IIB theory, the size of the T^2 shrinks to zero; however, one of the main lessons from string/M-theory dualities has been that it is very beneficial to keep the shape of the T^2 (or the M-theory S^1 in our Type IIA example above) as a part of the spacetime geometry, instead of dropping it altogether. In the case of Type IIB string theory, such an extended formalism is known as *F-theory* [95], and it has lead to many original insights. We will keep these string-theory lessons in mind, when we ask how we should treat the tropicalized manifolds in our proposed path integral for tropical topological sigma models.

Tropical topological sigma models: Search for the localization equation

In order to specify a topological field theory of the cohomological type [72], we need three basic ingredients: A choice of the primary quantum fields, a list of symmetries that act on them (which will typically include some class of topological deformations of the fields), and the equations that we can use to gauge-fix the underlying symmetries. The path integral for the theory is then constructed using the methods of BRST quantization, which adds to the list of our primary fields the corresponding ghosts, antighosts and auxiliaries. The action is designed to be BRST exact modulo possible topological invariants, and the BRST symmetry can be used to show that the path integral localizes to the moduli space of the solutions of those chosen equations.

In our tropical case, none of the three choices that we need to make to specify our path integral are quite obvious. First, let us consider the choice of fields in the path integral. Should our fields be maps from the worldsheet to the tropicalized manifold? And, more importantly, should the two-dimensional worldsheet on which these quantum fields live be replaced by its real-one-dimensional tropical limit? With such tropical maps as fundamental integration variables, the hypothetical path integral would then likely have to be interpreted as a tropical integral. While we are agnostic as to whether such notions of tropicalized quantum field theory and tropical path integrals can be made sense of, in this paper we will choose a more traditional way, and will find a conventional path-integral definition of the tropical topological sigma models in which the fundamental fields are still the same maps

$$\Phi : \Sigma \to M \tag{2.17}$$

from a two-dimensional worldsheet Σ to a $2n$ dimensional target space manifold M. We interpret the tropicalization of M not as a reduction to a real n dimensional tropical variety M_{trop}, as would be common in the mathematical literature on tropical geometry; instead, we keep the same topology of the original complex n dimensional manifold M as a differentiable manifold of real dimension $2n$. The tropical limit will then come from taking a singular limit of various geometric structures on M.

We use local real coordinates Y^i on M, $i = 1, \ldots, 2n$. A suitable choice for Y^i would be, for example, the real and imaginary parts of the n complex coordinates on M,

$$Z^I = x^I + iy^I, \qquad I = 1, \ldots, n, \tag{2.18}$$

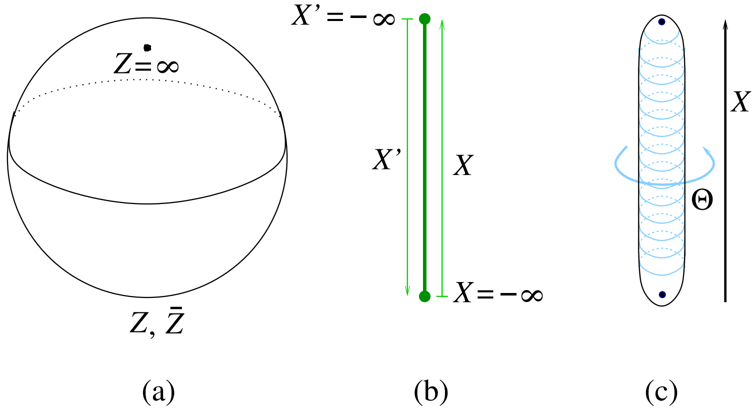

Figure 2.1: The tropicalization of the complex projective space $\mathbb{C}P^1$. **(a):** The original complex manifold $\mathbb{C}P^1$. The standard coordinate $Z \in \mathbb{C}$ covers the $\mathbb{C}P^1$ except at the point at infinity, and the coordinate $W = 1/Z$ covers all $\mathbb{C}P^1$ except the point at $Z = 0$. **(b):** The standard tropicalization $\mathbb{T}P^1$, with two coordinate systems indicated: $X \in \mathbb{T}$ and $X' \in \mathbb{T}$. The X coordinate system covers the entire $\mathbb{T}P^1$ except the point at $X' = -\infty$, while the X' coordinate system covers $\mathbb{T}P^1$ except the point at $X = -\infty$. They overlap along their open subset $\mathbb{R}$, and on this overlap the transition function must be a *linear function with integer coefficients*, in this case $X' = -X$. **(c):** The representation of $\mathbb{T}P^1$ that we find suitable for our construction of the topological sigma model with $\mathbb{T}P^1$ as the target space. The underlying differential topology is the same as in $\mathbb{C}P^1$, while the anisotropy between X and the angular dimension parametrized by Θ results from the deformation of the geometric structures on this manifold.

setting

$$Y^i = \begin{cases} x^I, & \text{for } i = 2I - 1, \\ y^I, & \text{for } i = 2I. \end{cases} \tag{2.19}$$

On the worldsheet, we will use general real coordinates denoted by σ^α, with $\alpha = 1, 2$. Using this local parametrization, the map Φ is then represented by specifying Y^i as functions of the worldsheet coordinates, $Y^i(\sigma^\alpha)$. These will be our quantum fields on Σ.

Next, we must address the symmetries. We will allow the theory to be gauge invariant under the standard topological deformations of Φ. In our local coordinates, these are represented by arbitrary gauge transformation functions $f^i(\sigma^\beta)$, acting simply via

$$\delta Y^i(\sigma^\alpha) = f^i(\sigma^\alpha). \tag{2.20}$$

Our goal is to gauge-fix this topological symmetry by choosing the appropriate localization equations as gauge-fixing conditions, and applying the standard BRST formalism.

In many topological theories (such as Yang-Mills or gravity), there is also a secondary gauge symmetry, which then requires the introduction of ghost-for-ghosts in the BRST formalism. This secondary gauge symmetry does not occur in the standard relativistic topological sigma models, and as we will see below, no such symmetries will be required in our tropical case either.

Finally, the central subtle point is the choice of the localization equations themselves. Consider maps from a worldsheet Σ to a target space M, first in the well-studied case of a relativistic topological sigma model, with M which we will assume to be a complex manifold. In this paper, we focus on the tropical version of the relativistic A-model. In this case, the worldsheet carries a complex structure, and it is convenient use local complex coordinates $z, \bar{z}$ on Σ, and n complex coordinates $Z^I, \bar{Z}^{\bar{I}}$ on M. (A convenient choice for the $2n$ real coordinates Y^i used above would then be the real and imaginary parts of Z^I.)

Localization equation in topological sigma models: Holomorphic maps, i.e., locally satisfying the Cauchy-Riemann equations

$$\partial_{\bar{z}} Z^I = 0. \tag{2.21}$$

In traditional algebraic geometry over the complex numbers $\mathbb{C}$, one can get many examples of holomorphic maps into suitable target spaces by satisfying polynomial equations, i.e., identifying the zero loci of polynomials over $\mathbb{C}$ (or over some other field $\mathbb{F}$) in some ambient space (such as the projective space $\mathbb{C}P^n$). In contrast, tropical curves are not easily defined as "tropical zeros" of tropical polynomials over the tropical semifield: Indeed, in our example of a tropical polynomial in (2.2), we see by inspection that setting it equal to the tropical zero (whose role is played by $-\infty$)

$$p_{\text{trop}}(x) = -\infty \tag{2.22}$$

does not have any meaningful solutions. Instead, tropical geometers use $p_{\text{trop}}(x)$ to define the corresponding tropical variety in a rather more indirect (and somewhat cumbersome) way: It is the subspace consisting of all points where the polynomial is nondifferentiable; or alternatively, all points where the value of the polynomial is achieved by *at least two* distinct linear terms appearing under the maximization operation.

If we wish to apply the algorithmic construction of a cohomological field theory, it would seem absolutely vital to be able to define tropical curves as objects that satisfy a certain equation. The apparent deficiency of the tropical semifield $\mathbb{T}$ to allow a definition of tropical varieties in terms of solutions of an associated tropical equation was particularly stressed by Oleg Viro [85, 70], who suggested an intriguing resolution: The tropical semifield $\mathbb{T}$ (and its cousins in similar tropical constructions) should be replaced by an object that satisfies all the standard axioms of the field, except the operations in this field are sometimes multivalued. Such generalized fields are known in the literature as "hyperfields".

Let us refine the discussion by considering the complex case of the appropriate hyperfield, as introduced and studied in [85, 70]. Following Viro, we define the following *subtropical deformation* of the field of the complex numbers. First, introduce a map from $\mathbb{C}$ to $\mathbb{C}$,

$$S_\hbar(z) = \begin{cases} |z|^{1/\hbar} \dfrac{z}{|z|}, & \text{if } z \neq 0; \\ 0 & \text{if } z = 0. \end{cases} \tag{2.23}$$

Then define

$$z \oplus_{\mathbb{C},\hbar} w = S_\hbar^{-1}\left(S_\hbar(z) + S_\hbar(w)\right). \tag{2.24}$$

(As pointed out by Viro [70], a similar deformation of the complex torus $(\mathbb{C}\setminus\{0\})^n$ was also originally proposed by Mikhalkin in [61].) If we choose to parametrize the complex numbers as

$$z = e^{r+i\theta}, \tag{2.25}$$

the $S_\hbar(z)$ map takes a particularly intuitive form in the new real variables r, θ:

$$S_\hbar(e^{r+i\theta}) = e^{r/\hbar+i\theta}. \tag{2.26}$$

The limit of this operation as $\hbar \to 0$ then defines the complex tropical limit of the addition of complex numbers, $\oplus_{\mathbb{C},0}$. The tropical multiplication $\odot_{\mathbb{C},0}$ will again be the standard addition.

There are two distinct ways how to interpret the $\hbar \to 0$ limit of $\oplus_{\mathbb{C},\hbar}$. The first one follows the traditional strategy used in tropical geometry, which in the real case has lead to the semifield $\mathbb{T}$, and interprets this limit as yielding univalued operation of addition, given by

$$z_1 \oplus_{\mathbb{C},0} z_2 = \begin{cases} z_1, & \text{if } |z_1| > |z_2|; \\ z_2, & \text{if } |z_2| > |z_1|; \\ |z_1|\dfrac{z_1+z_2}{|z_1+z_2|}, & \text{if } |z_1| = |z_2| \text{ and } z_1 \neq -z_2; \\ 0, & \text{if } z_1 = -z_2. \end{cases} \tag{2.27}$$

These addition rules again become more intuitive if we use the parametrization of complex numbers (2.25). In particular, the meaning of the third line is as follows: If $r_1 = r_2$ and $\theta_1 \neq \theta_2 + \pi (\mathrm{mod}\, 2\pi)$, the result of their addition is a complex number with $r = r_1$ and the phase θ which is at the midpoint between θ_1 and θ_2 along the shortest arc connecting them along the circle of constant r. The meaning of the remaining three lines is self-explanatory.

While this single-valued operation has many good properties, it does not satisfy the axioms required of a field addition; in fact, it is not even associative. This has lead Viro to the second interpretation of the $\hbar \to 0$ limit, in terms of multivalued operations leading to a hyperfield. The result of such a multivalued tropical addition of two complex numbers z_1 and z_2 will now be given by specifying a set of values, as follows:

$$z_1 \oplus_{\mathbb{C},0} z_2 = \begin{cases} \{z_1\}, & \text{if } |z_1| > |z_2|; \\ \{z_2\}, & \text{if } |z_2| > |z_1|; \\ \{|z_1|e^{i\theta}, \theta \in [\theta_1, \theta_2]\}, & \text{if } |z_1| = |z_2| \text{ and } \theta_2 - \theta_1 < \pi; \\ \{z \in \mathbb{C}, |z| \leq |z_1|\}, & \text{if } z_1 = -z_2. \end{cases} \tag{2.28}$$

Thus, if two complex numbers have equal magnitude but are not opposites of each other, their addition is the shorter closed arc segment connecting them along the circle of constant magnitude; and the addition of z and $-z$ is the entire closed disk of radius $|z|$. With these rules, the addition together with the tropical multiplication satisfy the axioms of a hyperfield.

Over this hyperfield, the defining relations of a tropical variety can take the form of satisfying an equation, analogous to the polynomial equations known in classical complex geometry.

The use of hyperfields may have solved the mathematical problem of finding candidate equations whose solutions are the tropical curves, but it may have created a much greater difficulty for our formulation of the path integral: Are we now supposed to define path integrals for quantum fields that are multivalued, and satisfy multivalued algebraic relations under addition or multiplication, and invent a hypothetical new discipline of "quantum hyperfield theory"? Fortunately, the answer turns out to be more prosaic, at least in the present context of the topological sigma models: We will be able to avoid any use of hyperfields (such as Viro's hyperfield shown in 2.28) altogether, and will find a path-integral realization of our theory using the conventional theory of univalued functions, and with the appropriate localization equations. After this construction is completed, can one still find some signs of multivalued operations in our path integral? As we will see throughout this paper, the resulting theory will exhibit a new kind of residual gauge invariance, absent in the standard relativistic topological sigma models for complex target manifolds. We suspect that the apparent need for the multivalued operations in Viro's mathematical treatment of this problem is related to the existence of this gauge symmetry in our physical formulation.

Having been educated by the two sources – F-theory on one hand, and the Mikhalkin-Viro treatment of the tropicalization of the complex numbers on the other – we will keep the angular variables while performing the tropicalization of the classical geometric structures, both on the worldsheet and in the target space. Since we found the tropicalization to be most intuitive in the parametrization of complex numbers given in (2.25), our starting point will be to define similar new variables both on Σ and on M,

$$z = \exp\left\{\frac{r}{\hbar} + i\theta\right\}, \qquad Z^I = \exp\left\{\frac{X^I}{\hbar} + i\Theta^I\right\}. \tag{2.29}$$

From now on, we will suppress the index I on Z, since as we mentioned above, the standard tropicalization of complex manifolds is applied individually on each Z^i, index value by index value. In fact, we can simply consider this equation as describing a map from the punctured complex plane z (with $z = 0$ removed) to the punctured complex plane Z (with $Z = 0$ removed). In the new variables, the holomorphicity condition becomes

$$\partial_{\bar{z}} Z = \frac{Z}{2\bar{z}}\left\{\partial_r X - \partial_\theta\Theta + \frac{i}{\hbar}\left(\partial_\theta X + \hbar^2\partial_r\Theta\right)\right\} = 0. \tag{2.30}$$

Since we are not interested in the overall normalization while looking for a suitable localization equation, we will drop the overall normalization of the left-hand side, and keeping the leading terms in the $\hbar$ expansion both for the real and for the imaginary part. Thus, we are led to propose

$$\begin{aligned} \partial_r X - \partial_\theta\Theta &= 0, & (2.31)\\ \partial_\theta X &= 0, & (2.32) \end{aligned}$$

as our tropical localization equations.

This proposed form of the localization equations also suggests the natural use of adapted coordinates, both in the target space and on the worldsheet. From now on, we will often use the adapted worldsheet coordinates (r, θ), and the adapted target-space coordinates

$$Y^i = \begin{cases} X^I & \text{for } i = 2I-1, \\ \Theta^I & \text{for } i = 2I, \end{cases} \tag{2.33}$$

instead of the coordinates (2.19) that would have been more suitable for the standard relativistic case. Since the tropicalization treats each X, Θ pair independently of all the others, we will often consider the simplest case of just one such pair, described by the coordinates X, Θ which we will often collectively refer to as Y^i, with $i = 1, 2$, or $Y^1 = X$ and $Y^2 = \Theta$.

Let us first examine the proposed localization equations, to see if they deliver the desired solutions. Locally, the general solution of the system (2.31), (2.32) is given by

$$X(r,\theta) = X_0(r), \tag{2.34}$$
$$\Theta(r,\theta) = \Theta_0(r) + \theta\partial_r X_0(r), \tag{2.35}$$

where $X_0(r)$ and $\Theta_0(r)$ are arbitrary (differentiable) functions of their argument. However, globally, both θ and Θ are periodic with periodicity 2π, which restricts $\partial_r X_0(r)$ to be an integer, restricting the global solutions to

$$X(r,\theta) = x_0 + nr, \qquad n \in \mathbb{Z}, \tag{2.36}$$
$$\Theta(r,\theta) = \Theta_0(r) + n\theta, \tag{2.37}$$

with $\Theta_0(r)$ still an arbitrary function. Recall now that in the standard description of its local neighborhood, a tropical curve is described by a piece-wise linear map $X = x_0 + nr$ whose slope is an integer, $n \in \mathbb{Z}$. Thus, the solutions of our localization equations describe correctly the ingredients from which tropical curves are built, when we simply apply the forgetful map $(X, \Theta) \to X$ to our pair of fields (X, Θ). Thus, we see that at least locally in generic coordinate neighborhoods, the proposed equations indeed contain the information about the local structure of tropical maps. We will return to the matching conditions at junctions of several local neighborhoods once we establish a covariant formulation of the localization equations, applicable to more general maps from more general surfaces Σ to the tropical limit of the target space M. While the mathematical investigations in tropical geometry are often described in the language of solely X, we find it useful to keep both X and Θ as fundamental fields in our field theory formulation: This will not only allow a more-or-less conventional path-integral representation of the tropical theory using ordinary quantum fields, but also lead to various clarifications, such as the origin of the quantization of n (which in the reduced definition using X needs to be postulated axiomatically, but which in the (X, Θ) language is simply explained as the topological winding number around Θ).

Note that there is an apparent hierarchy between the two equations in (2.31), (2.32): $\partial_\theta X$ appears at one lower order in the $\hbar$ expansion than $\partial_r X - \partial_\theta \Theta$. We will encounter various intriguing consequences of this hierarchy in our investigations below.

Symmetries of the tropicalized target spaces

Here we make a few additional clarifying remarks about the structure and symmetries of the target spaces, which will be useful during the rest of this paper.

While many complex manifolds have large continuous Lie groups of symmetries (for example those constructed as homogeneous spaces, such as $\mathbb{C}P^n$), tropical manifolds exhibit only discrete symmetries [64, 66]. This is related to the fact that if they are constructed by the tropicalization limit of a complex manifold M, a preferred coordinate system Z^I is chosen on M, and tropicalization is applied to each component of Z^I individually, leading to a tropical coordinate system with real X^I coordinates, which is adapted to the piece-wise linear and integral structure of the resulting tropical manifold. Transition functions to another such local tropical coordinate system $\widetilde{X}^I$ is then restricted to be piece-wise linear, with the strictly linear terms having integer coefficients n^{IJ},

$$\widetilde{X}^I = n^{IJ}X^J + a^I. \tag{2.38}$$

As a result, those geometric symmetries of tropical manifolds that mix two or more X^I coordinates can be at most elements of the discrete symmetry group

$$\mathcal{G} \subset GL(n, \mathbb{Z}). \tag{2.39}$$

This factorization thus provides another, symmetry-based reason why we focus in the bulk of this paper on just one X, and its associated angle dimension Θ. Each such pair enters the description of the tropical manifold essentially independently of the others, and they are mixed together only by a subgroup of the discrete symmetry (2.39).

These simple observations will have important consequences in our construction of the path integral for tropological sigma models and its symmetries. In particular, one can question whether it is strictly necessary to formulate the theory in a fully covariant form in the target space, allowing arbitrary coordinate transformations of the Y^i coordinates, or restrict only to those that respect the actual geometric symmetries of the tropical target space. For now, we will attempt a construction which is fully covariant, and return to this issue as needed below.

Besides reducing rotational symmetries to discrete subgroups, the piece-wise linear structure of tropical geometry has another important consequence for our construction: The reader will find that throughout this paper, the worldsheet theories we deal with are mostly free-field theories, at least when described in local adapted coordinates. In contrast, most interesting relativistic topological sigma models correspond to targets with nonlinear geometric structures, described by highly nonlinear Lagrangians. This is not an essential simplification on our part: Rather, this is a feature of the tropical universe, in which nonlinear geometric structures of classical geometry have been replaced with the combinatorics of piece-wise linear structures [64, 65, 66], and are therefore amenable to a description by free quantum fields.

2.2 Geometry of the tropical limit of pseudoholomorphic maps

We wish to be able to write our localization equations (2.31) and (2.32) in a covariant form, in arbitrary coordinates. In traditional relativistic topological sigma models, the localization equation is usually written in the covariant form as follows. Consider maps from Σ to M, at first viewed as real differential manifolds, with arbitrary real coordinates σ^α on Σ, and Y^i on M. First, one introduces a complex structure $\widehat{\varepsilon}_\alpha{}^\beta$ on Σ and an almost complex structure $\widehat{J}_i^j$ on M. (In this paper, we denote complex and almost complex structures $\widehat{\varepsilon}$ and $\widehat{J}$ with a hat, reserving the symbols ε and J for their tropicalized limits that will be introduced below.) Then one demands

$$\partial_\alpha Y^i + \widehat{\varepsilon}_\alpha{}^\beta \widehat{J}_j{}^i \partial_\beta Y^j = 0. \tag{2.40}$$

We would like to rewrite our proposed localization equations in a similarly covariant form. In order to do so, we must first understand what happens to the complex structures $\widehat{\varepsilon}$ and $\widehat{J}$ in the tropical limit.

Tropicalized complex structures and Jordan structures

Consider first the standard complex structure $\widehat{\varepsilon}_\alpha{}^\beta$ on a relativistic Σ, which is a section of the tensor product of the tangent and cotangent bundle, $T\Sigma \otimes T^\star\Sigma$. Starting with the complex coordinate $z \equiv u + iv$ on Σ, $\widehat{\varepsilon}$ is simply given by

$$\widehat{\varepsilon}_\alpha{}^\beta d\sigma^\alpha \frac{\partial}{\partial\sigma^\beta} = du\frac{\partial}{\partial v} - dv\frac{\partial}{\partial u}. \tag{2.41}$$

Next, we substitute our tropicalization change of variables (2.29), and express $\widehat{\varepsilon}$ in terms of (r, θ) coordinates,

$$\widehat{\varepsilon}_\alpha{}^\beta d\sigma^\alpha \frac{\partial}{\partial\sigma^\beta} = \frac{1}{\hbar}dr\frac{\partial}{\partial \theta} - \hbar\, d\theta\frac{\partial}{\partial r}. \tag{2.42}$$

As we perform the tropical contraction, $\hbar \to 0$, in order for the complex structure $\widehat{\varepsilon}_\alpha{}^\beta$ to have a finite limit in our favorite coordinates (r, θ), we need to multiplicatively renormalize it by one power of $\hbar$, holding fixed

$$\varepsilon_\alpha{}^\beta d\sigma^\alpha \frac{\partial}{\partial\sigma^\beta} = dr\frac{\partial}{\partial \theta}. \tag{2.43}$$

Similarly, we perform the tropical contraction of the target-space almost complex structure $\widehat{J}$, leading to

$$J_i^j dY^i \frac{\partial}{\partial Y^j} = dX\frac{\partial}{\partial \Theta}. \tag{2.44}$$

Note that these renormalized limits of the original complex structure do not themselves represent almost complex structures; instead, they satisfy

$$\varepsilon^2 = 0, \qquad J^2 = 0. \tag{2.45}$$

As we will see in the rest of this paper, the worldsheets and target spaces in topological quantum field theories that correctly represent localization to tropical maps will carry such structures ε and J that square to zero (without vanishing, except perhaps at point-like singularities). Note also that in the natural coordinates r, θ or X^I, Θ^I, the resulting matrices representing ε and J take their Jordan normal form, with zero eigenvalues; therefore, from now on, we will refer to such structures ε on the worldsheet and J on the target space as *Jordan structures*.

Next, we need to write our localization equations covariantly, in terms of ε and J. Simply using (2.40) with the tropicalized ε and J will not work: These tensors are now degenerate, and imposing (2.40) would lead to four independent equations, not two. Instead, we first rewrite (2.40) in a form which would be equivalent to (2.40) in the relativistic case,

$$\widehat{E}_\alpha{}^i \equiv \widehat{\varepsilon}_\alpha{}^\beta \partial_\beta Y^i - \widehat{J}_j{}^i \partial_\alpha Y^j = 0. \tag{2.46}$$

We can now replace the relativistic almost complex structures in the relativistic localization equation (2.46) with the Jordan structures ε and J, proposing the localization equations in the covariant form

$$E_\alpha{}^i \equiv \varepsilon_\alpha{}^\beta \partial_\beta Y^i - J_j{}^i \partial_\alpha Y^j = 0. \tag{2.47}$$

We observe that in coordinates (r, θ) and (X, Θ), these covariant equations reduce to our desired tropical equations (2.31), (2.32):

$$E_r{}^X = \partial_\theta X, \qquad E_r{}^\Theta = \partial_\theta \Theta - \partial_r X, \qquad E_\theta{}^X = 0, \qquad E_\theta{}^\Theta = -\partial_\theta X. \tag{2.48}$$

Self-duality and Jordan structures

In the relativistic case, the holomorphicity condition (2.21) can be viewed as a self-duality relation in two dimensions. Note that, similarly, the expression $E_\alpha{}^i$ that we intend to use to define our localization equation also satisfies a condition that reduces its number of independent components from four to two. In adapted coordinates, we find

$$E_\theta{}^X = 0, \qquad E_\theta{}^\Theta = -E_r{}^X, \tag{2.49}$$

with $E_r{}^\Theta$ unconstrained. These relations are the Jordan-structure analogs of the self-duality equations. Can they be written in a covariant form? The answer is yes, they are equivalent to

$$\varepsilon_\alpha{}^\beta E_\beta{}^i = -J_k{}^i E_\alpha{}^k. \tag{2.50}$$

Similarly, as in the relativistic case with the complex structure, one could define an anti-self-duality condition for generic sections $\mathcal{E}_\alpha{}^i$ of $T^*\Sigma \otimes \Phi^\star(TM)$ by

$$\varepsilon_\alpha{}^\beta \mathcal{E}_\beta{}^i = J_k{}^i \mathcal{E}_\alpha{}^k. \tag{2.51}$$

While there are some similarities with the standard notion of self-duality on Σ with a complex structure, there are also significant differences. For example, while in the case of the complex structure the self-duality and anti-self-duality conditions decompose the corresponding tensors into the sum of their self-dual and anti-self-dual parts, in the case of the Jordan structure such a decomposition does not occur. To understand this phenomenon better, let us first investigate more carefully the structures induced on a tensor algebra by the existence of a Jordan structure on a vector space.

Jordan structures on vector spaces

Consider a two-dimensional vector space V, such as the tangent space $T_p\Sigma$ of the two-dimensional worldsheet surface Σ at some general point p. We define the *Jordan structure* on V as a nonzero element ε of $V^*\otimes V$ which satisfies, when interpreted as an endomorphism of V, the condition

$$\varepsilon^2 = 0. \tag{2.52}$$

A choice of a Jordan structure ε induces various unique structures on the tensor algebra associated with V.

It is natural to represent such a Jordan structure in an adapted basis, in which it takes the following canonical form,

$$\varepsilon_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.53}$$

Vectors $v \in V$ annihilated by ε form a one-dimensional vector subspace, $F^1V \subset V$. It is natural to view this subspace as defining a natural filtration structure on V, consisting of subspaces

$$F^0V \equiv 0 \subset F^1V \subset F^2V \equiv V. \tag{2.54}$$

On the dual vector space V^*, the Jordan structure also induces a unique filtration. Define F_1V^* to consist of all the elements $\omega \in V^*$ which annihilate the elements $v \in F^1V$,

$$\omega(v) = 0. \tag{2.55}$$

This naturally defines a filtration

$$F_0V^* \equiv 0 \subset F_1V^* \subset F_2V^* \equiv V^*. \tag{2.56}$$

Of course, these filtrations of V and V^* naturally induce the corresponding filtrations of the full tensor algebra over V. Note that the vector space naturally dual to F^1V is *not* a subspace of V^*; instead, it is the coset space V^*/F_1V^*.

This filtration structure on the tensor algebra is to be contrasted with the standard Hodge decomposition of the (complexified) tangent space of a Σ that carries a complex structure, and the subsequent Dolbeault decomposition of the tensor algebra over such vector spaces with a non-degenerate complex structure. In our case, Σ will not be equipped with a complex structure, and the standard language of Riemann surfaces so familiar from relativistic (topological) sigma models and string theory would be unnatural. Instead, we will develop an understanding of the natural structures and symmetries that are induced on various geometrical features of Σ simply by postulating the existence of a Jordan structure.

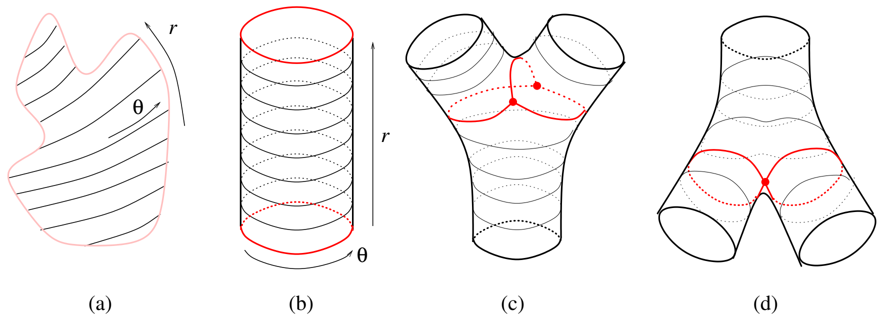

Figure 2.2: Examples of worldsheets with Jordan structures. **(a):** A local open simply-connected neighborhood with a non-singular foliation (see §2.2 and §2.2). **(b):** The open cylinder with the standard non-singular foliation, referred to as "the sleeve" in the body of this paper (see §2.2). We expect that at the boundaries of the cylinder, the sleeve is connected to other sleeves via a singular leaf of the foliation. **(c,d):** Two examples of Jordan structures with a singular leaf, connecting several sleeves to form pieces of higher-genus geometries of Σ. The singular leaf and therefore the Jordan structure in **(c)** is more generic than the one in **(d)**, since the singular leaf in **(d)** comes from that in **(c)** by collapsing one of the three segments of the singular leaf to a point.

Worldsheets with Jordan structures

Our intention is to construct topological sigma models, and later on topological strings, without having to introduce conventional complex structures (or conventional conformal structures) on the worldsheet, replacing them with the Jordan structures instead. In order to prepare for this construction, we need to examine the consequences of the existence of a Jordan structure on Σ and its symmetries, at first locally and then globally.

Jordan structures in a local neighborhood on Σ

Consider an open, simply connected neighborhood $\mathcal{U}$ of a generic point p on the worldsheet, with a non-degenerate Jordan structure defined on $\mathcal{U}$.[4] The filtration on $T_p\Sigma$ extends smoothly over $\mathcal{U}$, and defines a distribution (in the sense of vector-space subspaces) of the tangent bundle to Σ over $\mathcal{U}$. Since this distribution is one-dimensional, it is automatically integrable, and therefore induces a natural unique foliation structure of $\mathcal{U}$. The leaves of this foliation are open intervals. It also makes sense to require that the leaves are compact, given the fact that our construction originated from the tropical limit of a local complex coordi-

[4] "Non-degenerate" here means that ε is chosen smoothly and is nonzero everywhere in $\mathcal{U}$. When we later consider compact Σ of arbitrary genus, we will be naturally led for global topological reasons to allow Jordan structures over compact worldsheets Σ that require point-like "degenerations" of ε, i.e., ε's that exhibit isolated zeros or poles at a finite number of points in Σ. Here we first focus on the non-degenerate case in simply connected open neighborhood $\mathcal{U}$.

nate system, in which the leaves of the foliation were naturally compact and parametrized by the periodic coordinate θ. Ultimately, which types of foliation should be allowed globally is an important dynamical question, which however belongs to the discussion of dynamical gravity. In this paper, we will simply take the fixed foliation of Σ as fixed, and given a priori, without yet making gravity dynamical.

Symmetries of the Jordan structure in a local neighborhood on Σ

Consider again an open, simply connected neighborhood $\mathcal{U}$ of a generic point p on the worldsheet, with a coordinate system r, θ adapted to the preferred foliation induced by the Jordan structure ε. (In this adapted coordinate system, ε takes the canonical form (2.53).)

The group of symmetries that preserve the choice of ε is given by

$$\widetilde{r} = \widetilde{r}(r), \tag{2.57}$$

$$\widetilde{\theta} = \widetilde{\theta}_0(r) + \theta\, \partial_r \widetilde{r}(r), \tag{2.58}$$

with $\widetilde{\theta}_0(r)$ an arbitrary differentiable function of its argument, and the condition $\partial\widetilde{r}/\partial r \neq 0$ separating the group into two disconnected components, labeled by the sign of $\partial\widetilde{r}/\partial r$. Both connected components of the symmetry group preserve the orientation of Σ. We will focus on the Lie algebra generators of the infinitesimal symmetries,

$$\delta r = f(r), \tag{2.59}$$

$$\delta\theta = F(r) + \theta\, \partial_r f(r). \tag{2.60}$$

Thus, the local symmetries of the Jordan structure are generated by the infinite-dimensional Lie algebra whose elements are parametrized by two real, arbitrary projectable differentiable functions $f(r)$ and $F(r)$ on the foliation. The Lie algebra takes the natural form of a semi-direct sum, with a very clear geometric interpretation: While $f(r)$ generates all diffeomorphisms of r, the infinitesimal transformations of θ are r-dependent affine transformations along the leaves of the foliation, with the coefficient of the term linear in θ being uniquely determined by the infinitesimal reparametrization of r and indicating how the infinitesimal diffeomorphisms of r act on the Abelian subalgebra of the r-dependent translations of θ generated by $F(r)$.[5]

Now we are ready to discuss the transformation properties of our proposed localization equations (2.47). That equation is already in a manifestly covariant form under all worldsheet diffeomorphisms generated by any $\xi^\alpha(\sigma^\beta)$, if (as we are assuming) ${\varepsilon_\alpha}^\beta$ transforms as a tensor of rank $(1,1)$, and X and Θ transform as a scalar. As a consequence, the localization equations will take the same special form (2.31) and (2.32) in all coordinate systems in

[5]The reader should be warned that such a Lie algebra and its corresponding Lie group are only formally defined: When the range of r is $\mathbb{R}$, there are many inequivalent definitions that make the group at least a Fréchet space. They depend on the precise conditions imposed on the allowed functions $f(r)$ on $\mathbb{R}$, see for example [96]. In this paper, we will not attempt to identify which one of these mathematically more precisely defined groups should be the best candidate for the symmetries of our path integral.

which the Jordan structure takes the canonical form (2.53), if we postulate that X and Θ transform under (2.59), (2.60) as scalars,

$$\delta X = f(r)\partial_r X + \Big(F(r) + \theta\partial_r f(r)\Big)\partial_\theta X, \tag{2.61}$$

$$\delta\Theta = f(r)\partial_r \Theta + \Big(F(r) + \theta\partial_r f(r)\Big)\partial_\theta \Theta. \tag{2.62}$$

From now on, we will assume such transformation properties for any pair X, Θ representing a tropicalized (complex) target-space dimension.

The infinite-dimensional symmetry algebra of the Jordan structure has some interesting finite-dimensional subalgebras. First of all, the rigid translations along r and θ together with the nonrelativistic boosts

$$\delta\theta = \lambda r, \qquad \delta r = 0, \tag{2.63}$$

form a three-dimensional nilpotent Lie algebra, well-known in the literature as the Heisenberg-Weyl algebra (generated by a canonical p and q pair). Note that the role of the central element $[q, p] = i\hbar$ is played by the generator of translations in θ, along the leaves of the foliation.

In fact, the algebra of all affine symmetries of the Jordan structure (i.e., symmetries acting up to linearly on r and θ) is not three-dimensional, but four-dimensional; it is generated by the generators of the Heisenberg-Weyl algebra, together with the isotropic constant rescalings of r and θ. All four-dimensional real Lie algebras were fully classified in 1963 by Mubarakzyanov [97, 98]; our algebra of all affine symmetries of ${\varepsilon_\alpha}^\beta$ appears on Mubarakzyanov's list as $A_{4.8}$. It is an indecomposable, solvable Lie algebra. In fact, this algebra belongs to a family, parametrized by a real parameter $-1 \leq b \leq 1$, with our case corresponding to $b = 0$. In our physical interpretation, the deformations away from $b = 0$ would correspond to the possibility of making the constant rescaling anisotropic, with a dynamical critical exponent

$$z = 1/(1 + b). \tag{2.64}$$

This means that in such a deformed theory, one would be assigning an anomalous, nonzero scaling dimension to ε. From now on, we will focus on the $b = 0$ case. Note that the central element of the Heisenberg-Weyl algebra is no longer central in $A_{4.8}$.

Symmetries of the Jordan structure on a cylinder

We will be particularly interested in the foliations by compact S^1 leaves, which take locally the form of a $I \times S^1$, with I an open interval. We will refer to such a portion of Σ that respects this $I \times S^1$ foliation structure as a "sleeve".

When we take into account the global topology of the sleeve, it is natural to restrict our class of adapted coordinates such that the θ coordinate (along the compact S^1 leaves of the foliation) is always periodic with periodicity 2π. This additional normalization condition will restrict the infinitesimal symmetries of the Jordan structure just to

$$\delta r = r_0, \tag{2.65}$$
$$\delta\theta = F(r), \tag{2.66}$$

the Lie algebra generated by global translations of r (if the sleeve is infinite and the range of r is $\mathbb{R}$), together with r-dependent rotations of Θ.

One can also ask what happens if the requirement that the map from the sleeve to itself be an isomorphism is relaxed, and one simply requires it to be an endomorphism, not necessarily one-to-one; the endomorphisms of the sleeve that preserve the Jordan structure must respect the periodicity of θ, and are therefore given by

$$\widetilde{r} = nr + r_0, \qquad n \in \mathbb{Z}, \tag{2.67}$$
$$\widetilde{\theta} = \widetilde{\theta}_0(r) + n\theta. \tag{2.68}$$

This is again matching the intuitive expectations from the mathematicians' treatment of tropical manifolds: After applying the forgetful operation of dropping θ, morphisms between tropical manifolds are realized by piece-wise linear maps in r, with the coefficients of the terms strictly linear in r restricted to be integers.

Note that since we have restricted our coordinate systems on the sleeve to respect the periodicity of θ with the period of 2π, this now allows us to associate an invariant length to the sleeve along the r direction, without having to introduce any additional metric. This length is simply defined as $\int dr$, between the two endpoints of the sleeve. Thus, our foliation carries a natural measure along the tropical direction; this construction is closely reminiscent of Thurston's definition of "measured foliations" [99] of two-dimensional surfaces, which plays a central role in Thurston's theory of the geometry of two- and three-manifolds.

Anisotropic conformal symmetries

Simply by asking what are the symmetries of a Jordan structure on Σ, we have found infinite-dimensional symmetry algebras quite reminiscent of nonrelativistic conformal symmetries in two dimensions. In order to clarify in what sense these symmetries should be interpreted in our tropical context as "conformal", we will now relate the symmetries of the Jordan structure to the scaling symmetries of the appropriately defined degenerate metric structures on Σ.

Tropicalization of the metric

Just as we studied what happens with the standard complex structure on Σ under the tropicalization limit, one could play the same game with a nondegenerate worldsheet metric $\widehat{g}_{\alpha\beta}$. (Again, we denote nondegenerate Riemannian metrics by symbols with the hat, reserving symbols such as $g_{\alpha\beta}$ for their degenerate tropicalized limit, which we are about to introduce.) Let us begin with the simplest case, of the flat metric with components $\widehat{g}_{\alpha\beta} = \delta_{\alpha\beta}$ on Σ,

$$\widehat{g}_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = du^2 + dv^2, \tag{2.69}$$

and switch again to the tropical coordinates r and θ,

$$\widehat{g}_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = \frac{1}{\hbar^2} dr^2 + d\theta^2. \tag{2.70}$$

In order to keep the resulting object finite as $\hbar \to 0$, we must renormalize $\widehat{g}_{\alpha\beta}$ by a multiplicative $\hbar^2$ factor, obtaining

$$g_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = dr^2. \tag{2.71}$$

This object is still a symmetric tensor, but it is no longer nondegenerate – instead, it is of rank one on Σ, and it is the natural tropicalized candidate for replacing the relativistic metric in our nonrelativistic environment of tropicalized worldsheets. What are the natural conformal symmetries associated with such a degenerate metric, and are they related to the symmetries of the Jordan structure? The appropriate notion of (anisotropic) conformal symmetry on spaces with foliations was defined in [100]). Given a (degenerate or nondegenerate) metric g its conformal symmetry algebra simply corresponds to those allowed diffeomorphisms that map g to itself up to a (possibly anisotropic) Weyl transformations. In our case, we do not need anisotropic scaling with a dynamical exponent $z \neq 1$, and will simply restrict our attention to isotropic Weyl transformations. The infinitesimal diffeomorphisms ξ^α that map (2.71) to itself up to an infinitesimal Weyl rescaling by $\omega(\sigma^\alpha)$ satisfy

$$g_{\alpha\gamma} \partial_\beta \xi^\gamma + g_{\beta\gamma} \partial_\alpha \xi^\gamma = \omega(\sigma^\gamma) g_{\alpha\beta}. \tag{2.72}$$

In our simplest degenerate case (2.71), these equations reduce to

$$\partial_\theta \xi^r = 0, \qquad 2\partial_r \xi^r = \omega(r, \theta). \tag{2.73}$$

The solutions give the algebra isomorphic to the algebra of *all* foliation-preserving diffeomorphisms of Σ,

$$\xi^r = f(r), \qquad \xi^\theta = F(r, \theta), \qquad \omega = 2\partial_r f. \tag{2.74}$$

a symmetry strictly larger than the symmetry of the Jordan structure defining the foliation.

While it is somewhat intriguing that the algebra of all foliation-preserving diffeomorphisms can be interpreted as the conformal algebra associated with a degenerate metric, and isotropic Weyl transformations, this is not the end of the story. The original metric $\widehat{g}_{\alpha\beta}$ had its inverse metric, which we will denote by $\widehat{h}^{\alpha\beta}$. Writing the inverse metric in the tropical coordinates,

$$\widehat{h}^{\alpha\beta} \frac{\partial}{\partial\sigma^\alpha} \frac{\partial}{\partial\sigma^\beta} = \hbar^2 \left(\frac{\partial}{\partial r}\right)^2 + \left(\frac{\partial}{\partial\theta}\right)^2, \tag{2.75}$$

we see that h in the tropical limit becomes also degenerate,

$$h^{\alpha\beta}\frac{\partial}{\partial\sigma^\alpha}\frac{\partial}{\partial\sigma^\beta} = \left(\frac{\partial}{\partial\theta}\right)^2, \tag{2.76}$$

in such a way that g and h are no longer inverses of each other, but instead satisfy the "mutual invisibility" condition

$$g_{\alpha\beta}h^{\beta\gamma} = 0, \qquad h^{\alpha\beta}g_{\beta\gamma} = 0. \tag{2.77}$$

We can set up an analog of the natural definition of conformal symmetry for $h^{\alpha\beta}$: Which diffeomorphisms of Σ map this degenerate inverse metric to itself up to a $z = 1$ Weyl transformation? This condition in the covariant form is now given by

$$-h^{\alpha\gamma}\partial_\gamma\xi^\beta - h^{\beta\gamma}\partial_\gamma\xi^\alpha = \widetilde{\omega}(\sigma^\gamma)h^{\alpha\beta}, \tag{2.78}$$

for some infinitesimal Weyl rescaling parameter $\widetilde{\omega}$. In components, these conditions reduce to

$$\partial_\theta\xi^r = 0, \qquad -2\partial_\theta\xi^\theta = \widetilde{\omega}(r,\theta). \tag{2.79}$$

They are solved by

$$\xi^r = f(r), \qquad \xi^\theta = -\frac{1}{2}\int\widetilde{\omega}(r,\theta)\,d\theta, \tag{2.80}$$

with $\widetilde{\omega}(r,\theta)$ arbitrary. This symmetry algebra is again adapted to the foliation, and again larger than the symmetry algebra of the Jordan structure.

Weyl transformations and symmetries of the Jordan structure

The symmetries of the Jordan structure will emerge as conformal symmetries of our degenerate metric structure when we allow for the presence of both $g_{\alpha\beta}$ and $h^{\alpha\beta}$, and require that they transform under the Weyl rescalings with the opposite weight. This last condition requires $\widetilde{\omega}(\sigma^\gamma) = -\omega(\sigma^\gamma)$, and one can show that under this requirement, the intersection of the conformal symmetry algebras of g and h treated separately results precisely in the symmetry algebra of the Jordan structure. This gives a clear geometric interpretation of the symmetry algebra of the Jordan structure as a nonrelativistic conformal symmetry associated with isotropic Weyl rescalings of a pair $g_{\alpha\beta}$, $h^{\alpha\beta}$ satisfying the mutual invisibility condition mentioned above.

In retrospect, this result is in fact not geometrically very surprising. Consider a Jordan structure ε on Σ. In the preferred local coordinates, ε takes the form

$$\varepsilon_\alpha{}^\beta d\sigma^\alpha\frac{\partial}{\partial\sigma^\beta} = dr\frac{\partial}{\partial\theta}. \tag{2.81}$$

Here dr is an eigenvector (with eigenvalue zero) of ε acting as an automorphism on $T^*\Sigma$, and $\partial/\partial\theta$ is similarly an eigenvector of ε acting as an automorphism of $T\Sigma$. This eigenvector dr

is determined by ε uniquely, up to a Weyl transformation $dr \to \Omega(r,\theta)dr$, and $d\theta$ is similarly determined up to the Weyl transformation with the opposite weight, $\partial/\partial\theta \to \Omega^{-1}(r,\theta)\partial/\partial\theta$. In turn, dr and $\partial/\partial\theta$ define degenerate symmetric tensors

$$g = dr^2, \qquad h = \left(\frac{\partial}{\partial\theta}\right)^2, \tag{2.82}$$

which are uniquely determined by ε precisely up to the Weyl rescalings $g \to \Omega^2 g$ and $h \to \Omega^{-2}h$, respectively. Thus, we see that the Jordan structure uniquely determines the degenerate pair g and h up to the Weyl transformations, and therefore the conformal symmetries of this pair coincide with the symmetries of the Jordan structure itself.

Symmetries of the localization equations

As we have seen in §2.2, the localization equations are invariant under our anisotropic conformal transformations, if X and Θ transform as conformal scalars, as in (2.62). Besides this conformal symmetry, the localization equations as written in a preferred conformal coordinate system (r,θ)

$$\partial_r X - \partial_\theta\Theta = 0, \qquad \partial_\theta X = 0, \tag{2.83}$$

exhibit an additional important symmetry, independent for each (X,Θ) pair. The first equation is clearly invariant under

$$\delta X(r,\theta) = \partial_\theta\alpha(r,\theta), \tag{2.84}$$

$$\delta\Theta(r,\theta) = \partial_r\alpha(r,\theta), \tag{2.85}$$

for arbitrary $\alpha(r,\theta)$. Requiring that the second equation also be invariant restricts α to be at most linear in θ, and we obtain an intriguing symmetry

$$\delta X(r,\theta) = \alpha_1(r), \tag{2.86}$$

$$\delta\Theta(r,\theta) = \alpha_0(r) + \theta\partial_r\alpha_1(r), \tag{2.87}$$

with $\alpha_0(r)$ and $\alpha_1(r)$ arbitrary projectable functions on the foliation, *i.e.*, functions of only r.

At first glance, the precise status of such a symmetry appears a bit mysterious. It looks like a hybrid between a linear shift symmetry in θ, while it still exhibits an arbitrary dependence on r. If we interpret r as a time variable, this would perhaps suggest an underlying gauge invariance, and associated constraints on the momenta in the canonical quantization.

Admissible singularities in the foliation

When we attempt to define the path integral for the tropological sigma model on a higher-genus worldsheet Σ, inevitably some leaves of the foliation induced by the Jordan structure

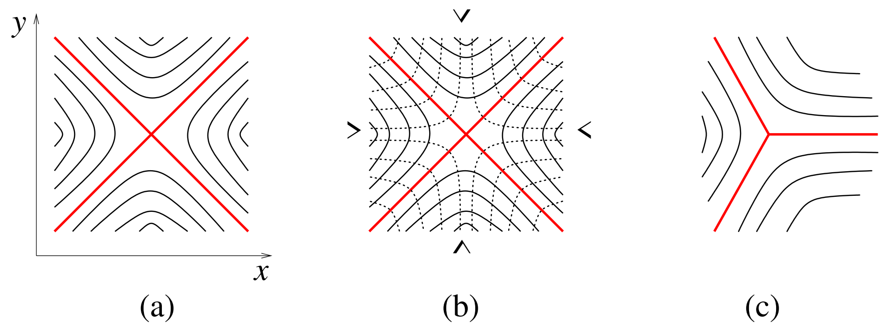

Figure 2.3: Defects in the Jordan structure lead to singular leaves in the associated foliation.

must be singular. This signals the presence of controllable topological defects in the gravity sector of the worldsheet theory. In this paper, we consider the Jordan structure as a non-dynamical given. This still allows us to consider several natural types of defects in the Jordan structure and investigate whether the localization equations continue making sense in the vicinity of such singularities of the foliation. For this, it is important that we have our proposed localization equations written in the covariant form, which does not require the existence of the adapted coordinates (r, θ) around the singular points, where no adapted coordinate systems exist.

Some of the simplest examples of the singular leaves in the foliation, and the associated topological defects in the Jordan structure, are depicted in Figure 2.3.

In physics terms, the singularities in our foliations of the worldsheet should be viewed as topological defects of the dynamical Jordan structure. While the full analysis of the allowed defects and their dynamics would require a more detailed description of the dynamics in the worldsheet gravity sector (which will be developed in the sequel [75]), in this paper we can at least consider the topological classification of such defects, and check to what extent are the localization equations for the sigma model still satisfied near such defects.

First, note that the natural order parameter that classifies singular leaves in the foliation is essentially the same as the order parameter in two-dimensional nematic liquid crystals: It is represented by an unoriented direction in the tangent space to the two-dimensional surface. In our case, it is natural to orient this order parameter along the leaves of the foliation. In the nematic liquid crystal case, the order-parameter manifold is an S^1, implying that the corresponding homotopy groups classifying stable defects are such that the point-like defects on the plane are labeled by one integer, the winding number of the unoriented direction as we circumnavigate around the point-like defect. In our worldsheets with foliations, these types of defects have been also encountered frequently in the mathematical literature on foliated manifolds. Consider a candidate Jordan structure on a compact surface Σ, of genus $g > 1$. The induced foliation will inevitably have some singular leaves. We will choose the Jordan structure such that the number of such singular leaves is finite, and such that the rest of Σ consists of a finite number of sleeves ending with their boundary components on the singular leaves.

Junctions of sleeves

A typical such configuration is locally depicted in Figure 2.3(a), with four sleeves meeting at the singular leaf. We will focus for simplicity on discussing this simple case, but the construction is analogous for any finite number of sleeves meeting. In Figure 2.3(a), nonsingular leaves inside each sleeve are indicated by the curved lines. We wish to investigate suitable junction conditions that such sleeves must satisfy at the singular leaf; this is going to be important for our ability to count the number of instantons contributing to the path integral of the tropological sigma models on higher-genus worldsheets Σ. We will describe the local geometry near the singular point of the singular leaf using an atlas containing five coordinate charts. We will introduce coordinates x, y such that the singular leaf is locally described by $x = \pm y$, and in particular the coordinate system is well-defined at the singular point of the foliation at $x = y = 0$. These are *not* adapted coordinates to the foliation, but are a perfectly legitimate coordinate choice on Σ interpreted as a differentiable manifold, with $x = y = 0$ a smooth point. Next, on the internal portion of each of the four sleeves we introduce natural adapted coordinates. We label the four sleeves by $>, <, \wedge, \vee$ correspondingly (see Figure 2.3(b)). On the first sleeve, we choose adapted coordinates $r_<, \theta_<$, such that $r_<$ is negative inside the sleeve, and approaches zero as we approach the singular leaf. Similarly, $\theta_<$ will represent a coordinate along the leaves of the sleeve, chosen such that $\theta_< = 0$ along the $y = 0$ axis. Analogous adapted coordinate systems are chosen on the remaining three sleeves. Each of these coordinate systems can be naturally continued inside the neighboring two sleeves, leading to the natural transition functions between them. For instance, $r_<$ can be continued to positive values, which for $\theta_<$ positive extends smoothly into the neighboring $\vee$ sleeve (and for $\theta_<$ negative into the $\wedge$ sleeve), with natural transition functions

$$\begin{aligned} r_\vee &= -r_<, & \theta_\vee &= -\theta_<, \\ r_\wedge &= -r_<, & \theta_\wedge &= -\theta_<, \end{aligned} \tag{2.88}$$

and similarly for all the other neighboring pairs of sleeves.

These four adapted charts cover the open domain of Σ around the $x = y = 0$ singular point, with this point removed. In order to complete the description around the singular point, we must specify the transition functions from the four adapted coordinate systems inside the sleeves to the x, y coordinates. On the overlap with the $r_<, \theta_<$ coordinate system, one can for instance take

$$r_< = y^2 - x^2, \qquad \theta_< = 2xy, \tag{2.89}$$

and similarly for the remaining overlaps. (The lines of constant $\theta_<$ are indicated in Figure 2.3(b) as dotted lines.)

This establishes an atlas of smooth charts in the vicinity of the defect in the Jordan structure.

A few comments are in order:

(i) Note that the open portions of the singular leaf of the foliation away from $x = y = 0$ look locally smooth, and indistinguishable from local neighborhoods in regular S^1 leaves of the foliation. The entire singular feature of the singular leaf is associated with the juncture point at $x = y = 0$.

(ii) Having found a suitable coordinate description of the vicinity of the singular point, we can now evaluate the Jordan structure in the coordinates x, y suitable for taking the $x, y \to 0$ limit. We get:

$$\varepsilon_\alpha{}^\beta = \frac{1}{x^2 + y^2} \begin{pmatrix} xy & x^2 \\ -y^2 & -xy \end{pmatrix}. \tag{2.90}$$

We see that this Jordan structure is indeed singular at the origin in the x, y coordinate system: While the values of the components $\varepsilon_\alpha{}^\beta$ obtained as x and y are taken simultaneously to zero are finite, the result depends on the angle with which the singular point is approached.

(iii) Fortunately, for the calculations in the path integral of the tropological sigma model, we do not need the Jordan structure to be nonsingular at the singular point of its foliation, it is sufficient if the localization equations are unambiguously defined, and satisfied, near such singular points. The localization equations in the x, y coordinate system take the following form,

$$\frac{1}{x^2 + y^2} \begin{pmatrix} xy & x^2 \\ -y^2 & -xy \end{pmatrix} \begin{pmatrix} \partial_x \Theta \\ \partial_y \Theta \end{pmatrix} + \begin{pmatrix} \partial_x X \\ \partial_y X \end{pmatrix} = 0, \tag{2.91}$$

$$\frac{1}{x^2 + y^2} \begin{pmatrix} xy & x^2 \\ -y^2 & -xy \end{pmatrix} \begin{pmatrix} \partial_x X \\ \partial_y X \end{pmatrix} = 0, \tag{2.92}$$

and they are naturally solved by maps that have a second-order zero at $x = y = 0$,

$$X = C(y^2 - x^2), \qquad \Theta = 2Cxy, \tag{2.93}$$

for some constant C. This compensates for the singularity of the Jordan structure, and makes the continuation of the solutions of the localization equations unambiguous at the juncture point in the singular leaf. Thus, it appears natural that such mild singularities in the worldsheet Jordan structure should be admissible; they are indeed inevitable for constructing solutions of the localization equations on higher-genus surfaces.

The construction is easily generalized to the case with other values of the defect quantum number around a juncture in the singular leaf. An example with three sleeves meeting at a singular leaf is depicted in Figure 2.3(c). The main novelty in the case when an odd number of sleeves meet at the singular point of the singular leaf is the fact that X and Θ are then antiperiodic as one circumnavigates the singular foliation point. In general, any number of sleeves greater than two can meet at a singular leaf.

Having understood the local geometric structure of worldsheets near singular points of the foliation, we can now construct global solutions of the localization equations, by gluing together the sleeve solutions that we found in (2.36), (2.37) (or their more general, local form (2.34), 2.35)) at the singular leaves where several sleeves meet. This is accomplished with the use of the α symmetries discovered in §2.2. Take for example the global solutions on two neighboring sleeves,

$$X_< = n_< r_<, \qquad \Theta_< = \Theta_{<0}(r) + n_< \theta_<, \tag{2.94}$$

$$X_\vee = n_\vee r_\vee, \qquad \Theta_\vee = \Theta_{\vee 0}(r) + n_\vee \theta_\vee. \tag{2.95}$$

We will consider the nondegenerate case, when $n_<$ and $n_\vee$ are both nonzero integers. To match such solutions on the coordinate patch where they overlap, one can use the $\alpha_0(r)$ symmetry to set $\Theta_{<0} = \Theta_{\vee 0}$ on the overlap, and then use the $\alpha_1(r)$ symmetry to rescale r such that $n_< = n_\vee$ in the local vicinity of the singular point. (If one of the integers $n_<$, $\ldots$, is zero, the corresponding sleeve is entirely mapped to a marked point modulo an α transformation, and can therefore be collapsed to a worldsheet point.)

Iterating this process for all pairs of neighboring sleeves, one constructs a global solution of the localization equation, assuming that a single global topological constraint is satisfied for each singular leaf of the foliation: The map from Σ to M will be continuous (and, indeed, smooth) if at each singular leaf the total winding numbers of all the attached sleeves sum up to zero. Thus, for example, when four leaves meet at one singular leaf (as locally depicted in Figure 2.3(a)), they must globally satisfy

$$n_< + n_\wedge + n_> + n_\vee = 0. \tag{2.96}$$

If the target space has more than one X^I, Θ^I pairs of coordinates, there is one such condition for each pair, at each singular leaf. This precisely reproduces our topological expectations, and matches for example the analogous charge conservation conditions at Type IIB string junctions.

Punctures

Finally, one last type of singularity in the worldsheet Jordan structure should be discussed: a puncture (see Figure 2.4). We use it to simply indicate the presence of a semi-infinite sleeve, whose natural adapted r coordinate goes to ∞ (or $-\infty$). We find it useful to depict it simply as a marked point on Σ (which one can think of as representing a singular leaf of the worldsheet foliation, "at infinity"), with the understanding that the coordinates that would be smooth around such a marked point do not belong to the class of adapted coordinates of the Jordan structure. For example, one can relate the adapted coordinates (r, θ) to coordinates (x, y), well-defined at the puncture at $r \to -\infty$, via

$$x = e^r \cos\theta, \qquad y = e^r \sin\theta. \tag{2.97}$$

Then one finds that the standard Jordan structure on the semi-infinite sleeve is given in the (x, y) coordinates by

$$\varepsilon_\alpha{}^\beta = \frac{1}{x^2 + y^2}\begin{pmatrix} -xy & x^2 \\ -y^2 & xy \end{pmatrix}. \tag{2.98}$$

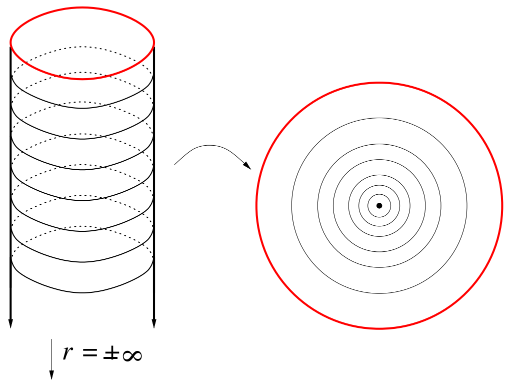

Figure 2.4: It is often convenient to depict the half-infinite sleeve as a punctured disk, even though the local coordinates around the marked point on the disk are not adapted to the Jordan structure.

In these non-adapted coordinates at the puncture, these components of the Jordan structure look superficially similar as those at the singular leaf where four sleeves meet, (2.90). However, despite this superficial similarity, the solutions of the localization equations in the (x, y) coordinates around the puncture do not exhibit zeros, but instead they diverge at the puncture,

$$X(x,y) = \frac{n}{2}\log(x^2+y^2), \tag{2.99}$$

$$\Theta(x,y) = n\arctan\left(\frac{y}{x}\right), \tag{2.100}$$

indicating that the end of the sleeve (when $n \neq 0$) is indeed at infinity.

Does this singular behavior of X and Θ at the puncture mean that punctures should be treated as singularities not belonging to the worldsheet Σ? In fact, the opposite turns out to be true: The punctures can be consistently treated as smooth points in the compact worldsheet Σ, mapped smoothly to the singular foliation leaves of the target space. In order to see that, one needs to switch not only to the (x, y) coordinates near the worldsheet puncture, but also to similar non-adapted coordinates $(\mathcal{X}, \mathcal{Y})$ that cover a neighborhood of the singular leaf at $X = -\infty$ in the target $\mathbb{T}P^1$. We define

$$\mathcal{X} = e^X\cos\Theta, \tag{2.101}$$

$$\mathcal{Y} = e^X\sin\Theta. \tag{2.102}$$

In such non-adapted coordinates defined in the vicinity of the singular leaf at $X = 0$, the standard Jordan structure on the $\mathbb{T}P^1$ takes the form

$$J_i{}^j = \frac{1}{\mathcal{X}^2+\mathcal{Y}^2}\begin{pmatrix} -\mathcal{X}\mathcal{Y} & \mathcal{X}^2 \\ -\mathcal{Y}^2 & \mathcal{X}\mathcal{Y} \end{pmatrix}. \tag{2.103}$$

When expressed in this pair of non-adapted coordinate systems (x, y) on the worldsheet and $(\mathcal{X}, \mathcal{Y})$ on the target space, the localization equations

$$\frac{1}{x^2+y^2}\begin{pmatrix}-xy & x^2\\ -y^2 & xy\end{pmatrix}\begin{pmatrix}\partial_x\mathcal{X} & \partial_x\mathcal{Y}\\ \partial_y\mathcal{X} & \partial_y\mathcal{Y}\end{pmatrix}-\frac{1}{\mathcal{X}^2+\mathcal{Y}^2}\begin{pmatrix}\partial_x\mathcal{X} & \partial_x\mathcal{Y}\\ \partial_y\mathcal{X} & \partial_y\mathcal{Y}\end{pmatrix}\begin{pmatrix}-\mathcal{X}\mathcal{Y} & \mathcal{X}^2\\ -\mathcal{Y}^2 & \mathcal{X}\mathcal{Y}\end{pmatrix}=0 \tag{2.104}$$

look quite complicated, but they and their solutions can be successfully continued through the puncture at $x = y = 0$. In fact, there is an infinite sequence of natural solutions to these nonlinear equations, given by simple monomials in x and y of arbitrarily high degree n,

$$\begin{array}{lll} n=1: & \mathcal{X}=x, & \mathcal{Y}=y,\\ n=2: & \mathcal{X}=x^2-y^2, & \mathcal{Y}=2xy,\\ n=3: & \mathcal{X}=x^3-3xy^2, & \mathcal{Y}=-y^3+3x^2y,\\ \quad\vdots & \quad . & \end{array} \tag{2.105}$$

At higher integer values of n, the structure of these monomial solutions can best be illustrated as follows. Combine the worldsheet and target-space coordinates into complex coordinates,

$$z = x + iy, \qquad \mathcal{Z} = \mathcal{X} + i\mathcal{Y}. \tag{2.106}$$

(This is not meant to imply the existence of any standard complex structure near the worldsheet puncture or near the singular leaf of the $\mathbb{T}P^1$ target space – these complex coordinates are introduced merely for convenience.) In these complex coordinates, the localization equations (2.104) boil down to

$$2\partial_{\bar z}\mathcal{Z}-\frac{z}{\bar z}\partial_z\mathcal{Z}+\frac{\mathcal{Z}}{\bar{\mathcal{Z}}}\partial_{\bar z}\bar{\mathcal{Z}} = 0, \tag{2.107}$$

$$\frac{\bar z}{z}\partial_{\bar z}\mathcal{Z}+\frac{\mathcal{Z}}{\bar{\mathcal{Z}}}\partial_z\bar{\mathcal{Z}} = 0. \tag{2.108}$$

Clearly, these nonlinear equations are not the equations for holomorphicity in standard complex geometry; yet, remarkably, they have an infinite sequence of solutions given simply by holomorphic monomials,

$$\mathcal{Z}_{(n)}(z, \bar z) = z^n. \tag{2.109}$$

These monomial solutions labeled by n represent smooth continuations of the standard sleeve solutions (2.36), (2.37) with winding number n through the puncture. As we see from this analytic form, the localization equations are again smoothly extended through the singular leaf of the foliation at the puncture, and we have a good notion of smoothness at the puncture for the maps to the target space.

Tropical limit as a nonrelativistic limit

In the context of the Litvinov-Maslov dequantization, the idea of the tropical limit emerged in attempts to make sense of the classical asymptotics of quantum systems and their observables. In our two-dimensional topological sigma model context, we have found that the "dequantization" of the tropical geometry does not correspond to the classical limit of our system; instead, the $\hbar \to 0$ limit of the Litvinov-Maslov dequantization corresponds to a *nonrelativistic limit*, in which the worldsheet theory undergoes a $c \to \infty$ contraction, with c the worldsheet speed of light. In this limit, the leaves of the foliation represent the worldsheet spatial dimension, and the direction transverse to the leaves is the worldsheet time. As a result, the symmetries of the tropical geometry naturally acquire a field-theoretical interpretation as nonrelativistic symmetries of the worldsheet theory.

Our quantum topological field theory of the tropological sigma model will of course have its own Planck constant, which will be denoted by e below. This Planck constant has nothing to do with the $\hbar$ of the Litvinov-Maslov dequantization. Based on the standard BRST arguments, the semiclassical approximation $e \to 0$ will be exact in the path integral of our tropological sigma models.

2.3 Tropological sigma models

Having understood the basic geometric features of Jordan structures, and its symmetries, we are now equipped to construct a path integral formulation of our theory using the methods of topological field theories of the cohomological type [72]. In this theory, the worldsheet path integral for the correlation functions of physical observables will localize to the solutions of our localization equations representing tropicalization.

Cohomological BRST construction from the localization equations

We follow the standard logic of cohomological field theories, and use the traditional methods of BRST quantization to construct the path integral [72]. All fields of the BRST quantization fall into multiplets of the BRST charge Q, which satisfies $Q^2 = 0$ and defines physical observables through its cohomology.

Ghosts

Our basic BRST multiplet reflects the underlying bosonic topological symmetry (2.20), transforming Y^i into the corresponding ghost ψ^i:

$$[Q, Y^i] = \psi^i. \tag{2.110}$$

When we use the adapted coordinates X and Θ on the target space, we will use the following simplified notation for the individual components of the ghost field,

$$\psi^X \equiv \psi, \qquad \psi^\Theta \equiv \Psi. \tag{2.111}$$

Thus, in the adapted coordinates the basic BRST multiplet associated with each tropicalized complex dimension of the target space is given by

$$[Q, X] = \psi, \qquad \{Q, \psi\} = 0, \tag{2.112}$$

$$[Q, \Theta] = \Psi, \qquad \{Q, \Psi\} = 0. \tag{2.113}$$

Antighosts and auxiliaries

In order to perform the gauge-fixing of the topological symmetry using the localization equation $E_\alpha{}^i = 0$, we introduce the trivial BRST multiplet

$$\{Q, \chi^\alpha{}_i\} = \mathcal{B}^\alpha{}_i, \tag{2.114}$$

$$[Q, \mathcal{B}^\alpha{}_i] = 0, \tag{2.115}$$

containing the bosonic auxiliary field $\mathcal{B}^\alpha{}_i$ and its antighost superpartner $\chi^\alpha{}_i$ which must be chosen such that the integral

$$\int_\Sigma d^2\sigma\, \mathcal{B}^\alpha{}_i\, E_\alpha{}^i \tag{2.116}$$

is covariantly well-defined. In addition, it is often customary to introduce another BRST exact term in the action, which is quadratic and nondegenerate in the auxiliaries $\mathcal{B}^\alpha{}_i$, so that they can be integrated out by Gaussian integration. This in turn leads to the bosonic part of the action that is nondegenerate and quadratic in $E_\alpha{}^i$.

When we try to repeat this strategy in the tropical case, we encounter several intriguing subtleties, which we now address. Note first that since $E_\alpha{}^i$ is a section of $T^*\Sigma \otimes \Phi^*(TM)$, $\mathcal{B}^\alpha{}_i$ must be a density-valued section of the dual bundle $T\Sigma \otimes \Phi^*(T^*M)$. In addition, since $E_\alpha{}^i$ satisfies (2.50) and therefore has only two independent components, only two out of the four components of $\mathcal{B}^\alpha{}_i$ will couple. How can we formulate this reduction of $\mathcal{B}$ in geometrical terms, and can it be done covariantly?

In the relativistic case, one uses the fact that every tensor $\mathcal{B}^\alpha{}_i$ can be decomposed into its self-dual and anti-self dual part, and simply restricts $\mathcal{B}^\alpha{}_i$ to be (anti)-self-dual. This reduces the number of independent components to two, which are the appropriate ones that couple to the two components of the localization equation. In our case, one can again define the natural tropicalized limit of (anti)-self-duality equations,

$$\varepsilon_\beta{}^\alpha \mathcal{B}^\beta_{\pm i} = \pm J_i{}^k \mathcal{B}^\alpha_{\pm k}. \tag{2.117}$$

However, declaring the auxiliary field to be self-dual or anti-self-dual in this way does not solve our problem: A quick inspection in adapted coordinates shows that either of the two choices throws away one of the components of $\mathcal{B}^\alpha{}_i$ that needs to couple to $E_\alpha{}^i$. This is again related to the observations about vector spaces with Jordan structures that we made above: The structures induced on the vector spaces are often filtrations, not direct-sum decompositions. As a result, tensors of the type $\mathcal{B}^\alpha{}_i$ do not decompose into a sum of $\mathcal{B}^\alpha_+ i$ and $\mathcal{B}^\alpha_-{}_i$ that satisfy (2.117) for the two corresponding sign choices. The reasons can be seen in the adapted coordinates – first of all, both equations in (2.117) require $\mathcal{B}^r{}_\Theta = 0$, so only those tensors that have this component vanishing can be written as a sum of solutions of (2.117). But even then, such a decomposition will not be unique: The $\mathcal{B}^\theta_{+\Theta}$ and $\mathcal{B}^\theta_{-\Theta}$ components are completely unrestricted by (2.117), and setting their sum equal to $\mathcal{B}^\theta{}_\Theta$ leaves an ambiguity. In addition, $\mathcal{B}^r{}_\Theta$ is precisely one of the components that we need to keep around, since $E_r{}^\Theta$ is one of the two nonzero components in the localization equation.

A fully covariant way how to achieve the desired reduction of $\mathcal{B}^\alpha{}_i$ does exist, but it requires the introduction of an additional gauge symmetry. Observe that our action (2.116) exhibits a gauge invariance under

$$\delta\mathcal{B}^\alpha{}_i = f^\alpha_{+i}(\sigma^\beta), \qquad \delta Y^i = 0, \tag{2.118}$$

where $f^\alpha_{+i}(\sigma^\beta)$ satisfies the tropicalized self-duality condition (2.117) with the plus sign choice. Thus, the covariant reduction of the four components in $\mathcal{B}^\alpha{}_i$ cannot be achieved as a reduction to a two-dimensional subspace, but instead, it is covariantly represented as a two-dimensional coset space, defined modulo the gauge transformations (2.118).

The next question to ask is whether – at least in the adapted coordinates – there is a natural gauge-fixing choice for our new gauge symmetry (2.118). One simple choice would seem to suggest itself:

$$\mathcal{B}^\theta{}_X = 0, \qquad \mathcal{B}^r{}_X = -\mathcal{B}^\theta{}_\Theta. \tag{2.119}$$

This is certainly a fine gauge-fixing choice for (2.118), and it appears to mimic the symmetry structure of $E_\alpha{}^i$ that the auxiliaries are coupling to. However, perhaps somewhat surprisingly, this gauge choice turns out to violate the worldsheet conformal invariance!

In order to find another gauge choice that is consistent with our worldsheet conformal invariance, we must abandon the full covariance in the target space. Recall that tropicalized target spaces have no continuous symmetries that would mix various coordinates; in particular, there is no continuous symmetry that would transform the preferred coordinates X and Θ into each other. Until now, we often used more general coordinates Y^i, and implicitly required invariance of our formulas under arbitrary coordinate changes of Y^i. Once we are willing to use the adapted coordinates X and Θ on the target space, it turns out that there is a unique natural gauge-fixing condition, which is still fully covariant under arbitrary coordinate changes on Σ,

$$\mathcal{B}^\alpha{}_X = 0. \tag{2.120}$$

As we will see below, this condition is also uniquely determined by requiring the consistency with worldsheet conformal transformations. From now on, we adopt this gauge-fixing condition, and formulate the theory using the adapted coordinates X, Θ on the target. When we also use the adapted coordinates r, θ on Σ, we will simply refer to the two remaining components of $\mathcal{B}^\alpha{}_i$ using the following simpler notation:

$$\mathcal{B}^r{}_\Theta \equiv B, \qquad \mathcal{B}^\theta{}_\Theta \equiv -\beta. \tag{2.121}$$

One additional subtlety appears when we attempt to add a term to the action quadratic in B and β, to integrate them out. In the worldsheet covariant form, such a term would be given by

$$-\frac{1}{2}\int d^2\sigma\,\gamma_{\alpha\beta}H^{\Theta\Theta}\mathcal{B}^{\alpha}{}_{\Theta}\mathcal{B}^{\beta}{}_{\Theta}. \tag{2.122}$$

Here $H^{\Theta\Theta}$ is the component of the degenerate inverse metric on the target in the tropical limit, which we have indeed found to be non-zero. Finally, Since each $\mathcal{B}^{\alpha}{}_{\Theta}$ transforms under worldsheet diffeomorphisms as a tensor density, $\gamma_{\alpha\beta}$ must itself be a tensor density of weight -1.

In the relativistic case, there would be a natural candidate

$$\widehat{\gamma}_{\alpha\beta}=\frac{1}{\sqrt{\widehat{g}}}\widehat{g}_{\alpha\beta}, \tag{2.123}$$

where $\widehat{g}_{\alpha\beta}$ is a nondegenerate worldsheet metric, and $\widehat{g}$ its determinant. This relativistic $\widehat{\gamma}_{\alpha\beta}$ is sensitive only to the relativistic conformal structure defined by $\widehat{g}_{\alpha\beta}$. In our tropical case, we do not have a nondegenerate worldsheet metric, only its degenerate limit $g_{\alpha\beta}$. Similarly, there is a degenerate limit $\gamma_{\alpha\beta}$ of (2.123), whose only non-zero component is $\gamma_{rr}=1$, much as the only nonzero component of $g_{\alpha\beta}$ in the adapted coordinates is g_{rr}. This $\gamma_{\alpha\beta}$ can be used to covariantly square one component of $\mathcal{B}^{\alpha}{}_{\Theta}$, but not the other. Thus, we can add a BRST invariant and conformally invariant term to our action, given in adapted coordinates simply by

$$S=-\frac{1}{2}\int dt\,d\theta\,B^2. \tag{2.124}$$

B can then be integrated out by the Gaussian integral, to simplify the action; however, β cannot acquire a conformally invariant quadratic term, and therefore the action will stay linearly dependent on β if we wish to maintain our nonrelativistic conformal invariance of the full BRST quantized theory.

The structure of the antighosts $\chi^{\alpha}{}_{i}$ now follows the same pattern as that of the auxiliaries. Only two components of this fermionic tensor will couple to the ghosts ψ^i. In the fully covariant formulation, there is a fermionic superpartner symmetry acting on the antighosts,

$$\delta\chi^{\alpha}{}_{i}=\varphi^{\alpha}_{+i}(\sigma^{\beta}), \tag{2.125}$$

where φ^{α}_{+i} is a fermionic analog of f^{α}_{+i} satisfying the same equation (2.117). It can be uniquely gauge-fixed in a way consistent with worldsheet conformal invariance by setting

$$\chi^{\alpha}{}_{X}=0, \tag{2.126}$$

again using the adapted coordinates X and Θ on the target.

If we in addition choose to use the adapted coordinates r,θ on the worldsheet, the two remaining components of the antighost will be simply denoted by

$$\chi^{r}{}_{\Theta}\equiv\mathfrak{X},\qquad\chi^{\theta}{}_{\Theta}\equiv-\chi. \tag{2.127}$$

(We suggest pronouncing the symbol for the antighost $\mathfrak{X}$ as “capital chi”.)

Having sorted out the structure of our BRST multiplets, we are now ready to construct the path integral using the standard methods of BRST quantization.

Lagrangians

We will define the tropological sigma model in terms of a path integral,

$$\int \mathcal{D}X\,\mathcal{D}\Theta\,\mathcal{D}\psi\,\mathcal{D}\Psi\,\mathcal{D}\chi\,\mathcal{D}\mathfrak{X}\,\mathcal{D}B\,\mathcal{D}\beta\,e^{-S}, \tag{2.128}$$

with the action S being a worldsheet integral of a local Lagrangian which is BRST exact, modulo possibly adding topological terms:

$$\begin{aligned} S &= \frac{1}{e}\int dr\,d\theta\,\left\{Q, \mathfrak{X}(\partial_\theta\Theta - \partial_r X - \tfrac{1}{2}B) + \chi\,\partial_\theta X\right\} \\ &= \frac{1}{e}\int dr\,d\theta\left\{B(\partial_\theta\Theta - \partial_r X) + \beta\partial_\theta X - \frac{1}{2}B^2 - \mathfrak{X}(\partial_\theta\Psi - \partial_r\psi) - \chi\partial_\theta\psi\right\}. \end{aligned} \tag{2.129}$$

As promised, we have introduced the Planck constant of our quantum theory, e. By the standard BRST arguments of cohomological field theories, the semiclassical one-loop approximation in e will be exact, leading to the localization of the path integral to the solutions of the localization equations.

We can integrate out B, to bring the action to a more standard form,

$$S = \frac{1}{e}\int dr\,d\theta\left\{\frac{1}{2}\left(\partial_\theta\Theta - \partial_r X\right)^2 + \beta\partial_\theta X - \mathfrak{X}(\partial_\theta\Psi - \partial_r\psi) - \chi\partial_\theta\psi\right\}. \tag{2.130}$$

This action is invariant under the BRST transformations

$$[Q, X] = \psi, \qquad \{Q, \psi\} = 0, \tag{2.131}$$

$$[Q, \Theta] = \Psi, \qquad \{Q, \Psi\} = 0, \tag{2.132}$$

$$\{Q, \mathfrak{X}\} = \partial_\theta\Theta - \partial_r X, \tag{2.133}$$

$$\{Q, \chi\} = \beta, \qquad [Q, \beta] = 0. \tag{2.134}$$

Both actions are also invariant under the nonrelativistic conformal transformations. Before B has been integrated out, the conformal symmetries act via

$$\delta X = f(r)\partial_r X + \Big(F(r) + \theta\partial_r f(r)\Big)\,\partial_\theta X, \tag{2.135}$$

$$\delta\Theta = f(r)\partial_r\Theta + \Big(F(r) + \theta\partial_r f(r)\Big)\,\partial_\theta\Theta, \tag{2.136}$$

$$\delta B = f(r)\partial_r B + B\partial_r f(r) + \Big(F(r) + \theta\partial_r f(r)\Big)\,\partial_\theta B, \tag{2.137}$$

$$\delta\beta = f(r)\partial_r\beta + \beta\partial_r f(r) + \Big(F(r) + \theta\partial_r f(r)\Big)\,\partial_\theta\beta + B\,\partial_r\Big(F(r) + \theta\partial_r f(r)\Big). \tag{2.138}$$

Note that B and β transform in a way reminiscent of a Jordan pair, with B transforming through terms involving only B, while β transforms via terms involving both β and B. When B is integrated out, the transformation rule for β becomes dependent on X and Θ,

$$\delta\beta = f(r)\partial_r\beta + \beta\partial_r f(r) + \Big(F(r) + \theta\partial_r f(r)\Big)\partial_\theta\beta + (\partial_\theta\Theta - \partial_r X)\partial_r\Big(F(r) + \theta\partial_r f(r)\Big). \tag{2.139}$$

The action (2.130) is then consistently invariant under these reduced conformal transformations not involving B.

For completeness, we list the conformal transformation properties of the remaining two components of $\mathcal{B}^\alpha{}_i$ that we dropped by our gauge choice $\mathcal{B}^\alpha{}_X = 0$ for the gauge symmetry generated by f^α_{+i} in §2.3:

$$\begin{aligned}\delta\mathcal{B}^r{}_X &= f(r)\partial_r\mathcal{B}^r{}_X + \mathcal{B}^r{}_X\partial_r f(r) + \Big(F(r) + \theta\partial_r f(r)\Big)\partial_\theta\mathcal{B}^r{}_X,\\ \delta\mathcal{B}^\theta{}_X &= f(r)\partial_r\mathcal{B}^\theta{}_X + \mathcal{B}^\theta{}_X\partial_r f(r) + \Big(F(r) + \theta\partial_r f(r)\Big)\partial_\theta\mathcal{B}^\theta{}_X - \mathcal{B}^r{}_X\partial_r\Big(F(r) + \theta\partial_r f(r)\Big).\end{aligned}$$

We see that they transform into each other, again as a Jordan pair: $\mathcal{B}^r{}_X$ into itself, and $\mathcal{B}^\theta{}_X$ into itself and $\mathcal{B}^r{}_X$. By inspection of these transformation rules, one can now explicitly confirm several points we made above: (i) It is consistent with worldsheet conformal invariance to fix the f_+ gauge symmetry by setting $\mathcal{B}^\alpha{}_X = 0$; (ii) our naive first choice for the gauge-fixing condition, (2.119), would be inconsistent with conformal invariance, and (iii) while the B^2 term in the action is conformally invariant, no linear combination of β^2 and βB is consistent with the conformal symmetries.

Compared to the standard relativistic case, the action of our tropological sigma model exhibits one additional novelty that needs to be addressed. Unlike its relativistic counterpart, our action has residual gauge symmetries, closely related to the residual symmetries of the localization equations that we found in §2.2. Consider a local simply-connected neighborhood $\mathcal{U}$, with adapted coordinates r, θ. The bosonic sector of (2.130) is invariant under

$$\delta X = \partial_\theta\alpha, \tag{2.140}$$
$$\delta\Theta = \partial_r\alpha, \qquad \delta\beta = 0, \tag{2.141}$$

if the gauge parameter $\alpha(r,\theta)$ is further constrained to satisfy

$$\partial_\theta{}^2\alpha = 0. \tag{2.142}$$

Similarly, the fermionic sector is invariant under a fermionic gauge transformation with Grassmannian gauge parameter $\zeta(r,\theta)$,

$$\delta\psi = \partial_\theta\zeta, \qquad \delta\mathfrak{X} = 0, \tag{2.143}$$
$$\delta\Psi = \partial_r\zeta, \qquad \delta\chi = 0, \tag{2.144}$$

if $\zeta(r,\theta)$ is constrained by

$$\partial_\theta{}^2\zeta = 0. \tag{2.145}$$

The constraint equations on the gauge parameters are solved by r-dependent affine functions of θ,

$$\alpha = \alpha_0(r) + \theta\alpha_1(r), \quad \zeta = \zeta_0(r) + \theta\zeta_1(r), \tag{2.146}$$

yielding the gauge symmetries in the unconstrained form

$$\begin{aligned} \delta X &= \alpha_1(r), \\ \delta\Theta &= \partial_r\alpha_0(r) + \theta\partial_r\alpha_1(r), \\ \delta\psi &= \zeta_1(r), \\ \delta\Psi &= \partial_r\zeta_0(r) + \theta\partial_r\zeta_1(r), \\ \delta\beta &= \delta\mathfrak{X} = \delta\chi = 0. \end{aligned} \tag{2.147}$$

First, a few comments about the nature of these symmetries:

(i) The α and ζ symmetries are a hybrid between a genuine gauge redundancy with an arbitrary dependence on worldsheet coordinates (here exhibited only along r), and a global linear shift symmetry (along θ): The independent parameters α_0, α_1 and ζ_0, ζ_1 are projectable functions on the worldsheet foliation, a phenomenon that has no analog in relativistic theories.

(ii) If we interpret r as (imaginary) worldsheet time – and that is the interpretation here, in the topological context – then α indeed represents a true gauge redundancy, with constraints on the canonical variables in the canonical quantization along r, and a non-uniqueness of the time evolution of a classical solution given fixed initial conditions. If we studied the theory in the cross-channel, using θ as time, the interpretation of the α symmetries would be less clear.

(iii) If α were not constrained to be linear in θ, its action on X and Θ would be as the gauge symmetry in relativistic electromagnetism in two dimensions, with X and Θ playing the role of the components of the electromagnetic gauge field. However, this analogy is incomplete for two reasons: First, our action also contains the β-dependent term (which in the electromagnetic analogy looks a little like a gauge-fixing term), and perhaps more importantly, our Θ is a periodic variable with periodicity 2π.

How should these residual gauge symmetries be treated, and do they require a secondary BRST gauge fixing, leading to a second generation of "ghosts-for-ghosts"? If we decided to treat these symmetries as secondary, in addition to the original topological symmetries (2.20), it would be natural to implement them equivariantly, as for example is the case with the ordinary Yang-Mills gauge symmetries in topological Yang-Mills theories. This would in turn lead to second-generation bosonic ghosts.

A closer look at the topological symmetries (2.20) and our gauge-fixing conditions shows that this is not the case. In the relativistic topological sigma models, the holomorphicity conditions fix the gauge of the topological symmetry (2.20) almost perfectly, leaving only a finite-dimensional space of moduli. On the other hand, in the tropicalized case, our localization equations fix the topological symmetries (2.20) less perfectly, leaving an infinite-dimensional space of classical solutions, with the residual α symmetries. Indeed, recall that on a simply-connected neighborhood $\mathcal{U}$, in the local coordinates, we found that the classical solutions of the localization equations,

$$\begin{aligned} X &= X_0(r), && (2.148)\\ \Theta &= \Theta_0(r) + \theta\partial_r X_0(r), && (2.149) \end{aligned}$$

contain arbitrary functions of r, which are precisely acted on by the residual, unfixed α symmetry.

Similarly, the remaining auxiliary field β exhibits its own α-type symmetry: The action is invariant under r-dependent constant shifts of β,

$$\delta\beta = \alpha_{\mathcal{B}}(r), \tag{2.150}$$

with all other fields invariant. The equations of motion for β are solved by

$$\beta = \beta_0(r). \tag{2.151}$$

While it is not inconceivable to have a path-integral localization to an infinite-dimensional space of solutions, this is not the strategy we wish to pursue here – our intention is to construct a tropical version of topological sigma models which is dependent only on the Jordan structure, but otherwise it is as close as possible to the standard relativistic topological sigma models. Therefore, in a way which will become clear below, we will aim to gauge-fix the residual α gauge symmetry in a simple way, simply by using them to set locally $X_0(r)$, $\Theta_0(r)$ and $\beta_0(r)$ to zero (or some other convenient fixed value). This step will again leave behind only a finite-dimensional moduli space of solutions, and without any need for any secondary ghosts, due to the simplicity of this additional gauge-fixing condition for α.

When we ask the same questions on a sleeve Σ, the α gauge transformation will now have to respect the periodicities of θ and Θ. That reduces the gauge transformation parameter $\alpha_1(r)$ to be a constant $\alpha_{1,0}$. This is in turn consistent with the fact that the classical solutions of the localization equations on the sleeve take the form

$$\begin{aligned} X &= x_0 + nr, \qquad n \in \mathbb{Z}, && (2.152)\\ \Theta &= \Theta_0(r) + n\theta, && (2.153)\\ \beta &= \beta_0(r). && (2.154) \end{aligned}$$

$\Theta_0(r)$ and $\beta_0(r)$ can be set to zero (or other convenient fixed values) using the $\alpha_0(r)$ and $\alpha_{\mathcal{B}}(r)$ gauge symmetries, and x_0 can be set to zero (or another convenient constant value) by the constant residual gauge transformation $\alpha_{1,0}$. The integer winding number n is a gauge invariant under all α gauge transformations.

Observables

Having clarified that our BRST multiplets do not lead to ghost-for-ghosts, it is now clear that the BRST invariant observables of the theory are exactly as in standard, relativistic topological sigma models. With any differential form on M,

$$\omega_{i_1\ldots i_p} dY^{i_1} \wedge \ldots \wedge dY^{i_p} \tag{2.155}$$

we associate a local operator

$$\mathcal{O}_\omega^{(0)} = \omega_{i_1\ldots i_p} \psi^{i_1} \ldots \psi^{i_p}. \tag{2.156}$$

Since no secondary ghosts-for-ghosts were required by the α symmetries, ψ is BRST invariant, therefore $\mathcal{O}_\omega^{(0)}$ is BRST invariant if ω is closed, and BRST trivial if ω is exact.

The Stora-Zumino descent equations then yield a hierarchy of 1-form and 2-form observables on Σ,

$$d\mathcal{O}_\omega^{(0)} = \{Q, \mathcal{O}^{(1)}], \tag{2.157}$$

$$d\mathcal{O}_\omega^{(1)} = \{Q, \mathcal{O}^{(2)}], \tag{2.158}$$

$$d\mathcal{O}_\omega^{(2)} = 0. \tag{2.159}$$

(Here as usual the symbol "$\{Q,\ ]$" denotes a commutator or anticommutator, depending on the statistics of the object in its second entry.)

2.4 Example: The tropical $\mathbb{C}P^1$ model

In the case of relativistic topological sigma models, $\mathbb{C}P^1$ was used as an early test example in [101] to probe the viability and self-consistency of the BRST construction of the path integral. In our case, the tropicalization of this example precisely corresponds to keeping just one pair (X, Θ) in our tropological sigma model, and thus this example enjoys a more privileged status than it did in the relativistic case: it represents the basic building block from which tropicalizations of higher-dimensional manifolds are naturally built.

Observables and correlation functions at tree level

The fundamental observables of the $\mathbb{T}P^1$ theory are given by the BRST cohomology of local operators; just as in the relativistic $\mathbb{C}P^1$ case [101], this space is two-dimensional, spanned by the identity operator and the operator of ghost number two associated with the top-degree cohomology class $[\omega]$ on $\mathbb{C}P^1$, which we will simply refer to as $\mathcal{O}_\omega$. Consider first the relativistic case: We will choose a representative two-form ω in the cohomology class $[\omega]$, and normalize it such that

$$\int_{\mathbb{C}P^1} \omega = 1, \tag{2.160}$$

in which case

$$\int_\Sigma \Phi^*(\omega) = k \in \mathbb{Z} \tag{2.161}$$

is the instanton number of the map Φ, corresponding to the topological winding number of maps $S^1 \to S^1$. The most natural choice of ω is the covariantly constant volume element on $\mathbb{C}P^1$ viewed as a round S^2.

In the tropical case, while the BRST cohomology that we have found is identical to that of the relativistic case, we must be a bit more careful with the choice of a representative of $[\omega]$: We cannot simply choose ω to be $dX \wedge d\Theta$ multiplied by a suitable normalization constant, since the standard integral $\int_{\mathbb{T}P^1} dX \wedge d\Theta$ is infinite. The natural choice is to pick a two-form which is translationally invariant along Θ but not along X, such that the integral converges and can be normalized to 1. The simplest natural choice is to pick a reference point $X_0 \in \mathbb{R}$, and to choose

$$\omega_{\mathbb{T}} = \frac{1}{2\pi}\delta(X - X_0)dX \wedge d\Theta. \tag{2.162}$$

This representative is then correctly normalized, giving

$$\int_{\mathbb{T}P^1} \omega_{\mathbb{T}} = 1, \tag{2.163}$$

where we interpret the integral of a two-form over $\mathbb{T}P^1$ in the classical sense of integrals of classical two-forms over the underlying topological space S^2. The choice of the reference point X_0 is immaterial, as changing it is cohomologically trivial and thus preserves the cohomology class $[\omega]$. Note that the role of the instanton number k in the tropical case,

$$\int_\Sigma \Phi^*(\omega_{\mathbb{T}}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\Theta}{\partial\theta} d\theta = k \in \mathbb{Z}, \tag{2.164}$$

is played by the winding number of the worldsheet periodic coordinate θ around the periodic target space dimension Θ.

It is intriguing that one can take an alternative (but essentially equivalent) perspective on how to choose and interpret the representative of $[\omega]$ in the tropical case: We can insist that $[\omega]$ be represented by the translationally invariant form

$$\omega'_{\mathbb{T}} = \frac{1}{2\pi} dX \wedge d\Theta, \tag{2.165}$$

if we simultaneously change the interpretation of what it means to integrate this form over $\mathbb{T}P^1$ (and of its pull-back via Φ over Σ) – the integral must now be interpreted in the tropical sense [64, 65, 66] along the tropicalized dimension X,

$$\int^{\oplus}_{\mathbb{T}P^1} \omega'_{\mathbb{T}} \equiv \frac{1}{2\pi}\int^{\oplus} dX \int_0^{2\pi} d\Theta = 1. \tag{2.166}$$

Indeed, the tropical version of the integral of a differential form $f(X)dX$ along the one-dimensional tropical variety parametrized by X is simply defined by finding the maximum value of $f(X)$ over the range $\mathcal{R}$ of integration,

$$\int_{\mathcal{R}}^{\oplus} f(X)dX \equiv \max\{f(X), X \in \mathcal{R}\}. \tag{2.167}$$

Since our form has $f(X) = 1$, a constant, our result (2.166) follows from the definition of the tropical integration. Similarly, the instanton number k would result from the tropical integration along the worldsheet dimension r of the pull-back $\Phi^*(\omega'_{\mathbb{T}})$.

Now we wish to calculate the correlation functions of the local observables, first on the worldsheet with genus zero. Let us first recall how the evaluation of the correlation functions of $\mathcal{O}_\omega$'s proceeds in the standard relativistic case with the $\mathbb{C}P^1$ target [101]. Instantons for worldsheets of genus zero are rational complex curves, described by

$$Z(z) = a\frac{(z-b_1)\dots(z-b_k)}{(z-c_1)\dots(z-c_k)}. \tag{2.168}$$

Here $Z \in \mathbb{C}$ is the standard coordinate on $\mathbb{C}P^1$ with the point at infinity removed, and z is the standard worldsheet coordinate on $\Sigma = S^2$ with the point at infinity removed. The integer k is the instanton number, and the moduli space of instantons with the instanton number k is complex $2k+1$-dimensional, parametrized by the complex parameters a, b_i, c_i, with $i = 1, \dots k$. We will introduce a coupling λ, designed such that the contribution from the instantons of instanton number k to any correlation function is weighted by a factor of

$$\lambda^k. \tag{2.169}$$

The generic correlation function will then vanish (for example, due to the existence of unsaturated ghost zero modes), unless we select $2k+1$ points $P_0, P_1, \dots P_{2k}$ on Σ and calculate the correlation function of $2k+1$ observables $\mathcal{O}_\omega$ inserted at these points. Each insertion of $\mathcal{O}_\omega(P_\ell)$ is equivalent to the restriction of the value of $Z(z)$ at $z = P_\ell$ to equal a fixed prescribed value Z_ℓ; the simplest and most convenient choice for these values is to take $Z_\ell = 0$ for $\ell = 1, \dots, k$, $Z_\ell = \infty$ for $\ell = k+1, \dots, 2k$, and finally $Z_0 = 1$. This leaves precisely one rational function of instanton number k, with the prescribed structure of zeros and poles and with the fixed overall normalization. The correlation function at genus $g = 0$ thus is

$$\langle \mathcal{O}_\omega(P_0)\mathcal{O}_\omega(P_1)\dots\mathcal{O}_\omega(P_n)\rangle_{g=0} = \begin{cases} 0, & \text{if } n = 2k-1; \\ \lambda^k, & \text{if } n = 2k. \end{cases} \tag{2.170}$$

(The BRST cohomology of point-like operators in this model of course contains also the operator $\mathcal{O}_1(P)$ that corresponds to the zero-form constant cohomology class [1]; but its effect in the path integral is trivial, in the sense that it does not restrict the instanton at point P in any way, and therefore can be inserted any number of times inside the correlation functions without changing their values as given in (2.170).)

In the tropical case, we also wish to evaluate the family of correlation functions of any number of $\mathcal{O}_\omega$'s, inserted at points P_ℓ of the worldsheet. As we discussed above, the solutions of our localization equations (which one could refer to as "tropical instantons") are also assigned an instanton number k, now defined via (2.164), or equivalently with the use of tropical integration (2.166). For virtually identical reasons as in the relativistic case, for such an instanton of instanton number k to give a nonzero contribution to the correlation function, we need an insertion of $2k+1$ judiciously placed observables $\mathcal{O}_\omega$ so that the instanton is an isolated solution without nontrivial moduli.

While we will see that the result for the correlation functions in the tropical case reassuringly coincides with the relativistic result (2.170), the steps of the evaluation in the tropological sigma model are intriguingly different from the standard relativistic theory. Our instantons will be globally well-defined solutions of our localization equations, mapping Σ (which is topologically a sphere, with a number of marked points where the observables are inserted), to the $\mathbb{T}P^1$ target space. As a starting point, we need to choose a fixed, generic Jordan structure on $\Sigma = S^2$, capable of supporting an instanton of instanton number k. In order for such solutions to exist and to be isolated with no moduli, a certain number of insertions of $\mathcal{O}_\omega$ is again needed. It is easy to see that for a generic Jordan structure, the instanton of instanton number k again requires the insertion of precisely $2k+1$ operators $\mathcal{O}_\omega(P_\ell)$, just as in the relativistic case. We wish to mimic the relativistic evaluation as possible, to stress the parallels and to see the differences in the construction. Thus, following the relativistic example, in order to get a unique instanton with no moduli we will require points $P_1, \ldots P_k$ to be mapped to the tropical zero, $X = -\infty$; similarly, points $P_{k+1}, \ldots P_{2k}$ will be mapped to the tropical infinity, $X = +\infty$; and the remaining point P_0 will be mapped to any finite point in the target, say $X = 0$, $\Theta = 0$.

The generic Jordan structure on Σ for which such an isolated tropological instanton solution of instanton number k exists is depicted in Figure 2.5(a): It is given by a foliated surface with $2k$ punctures $P_1, \ldots P_{2k}$ and one additional marked point P_0 on one of the nonsingular leaves of the foliation. Moreover, the $2k$ punctures (which represent semi-infinite sleeves) are naturally divided into two groups of k, one group considered "incoming" where the natural value of the adapted coordinate $r = -\infty$ and the other group "outgoing", with $r = +\infty$. Generically, such a surface contains junctions of three sleeves at a time. Consecutive singular leaves are connected by finite sleeves, whose lengths represent moduli of the Jordan structure; since gravity is still non-dynamical, these moduli are fixed numbers, chosen to have generic nonzero values.

The unique instanton solution on this surface is constructed as follows. First, the placement of $\mathcal{O}_\omega$ at $P_1, \ldots, P_k$ constrains the values of the instanton solution to be

$$X(r = -\infty) = -\infty, \tag{2.171}$$

with Θ on the individual half-infinite sleeve around each of these k punctures exhibiting the winding number $n = 1$. Similarly, the insertions of $\mathcal{O}_\omega$ at the "outgoing" punctures at

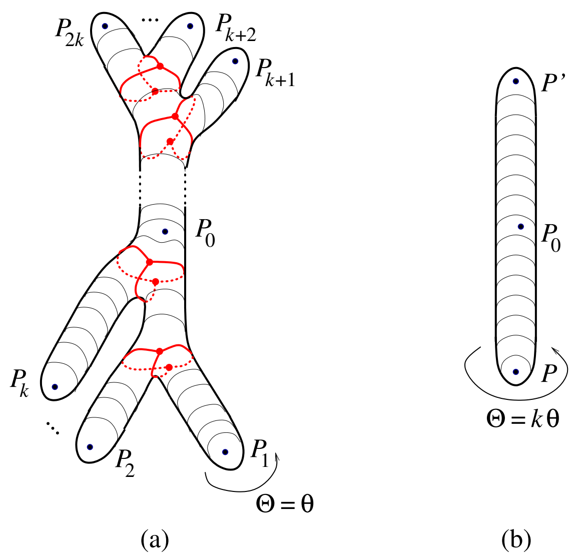

Figure 2.5: The instanton contributing to the $2k+1$-point function. **(a):** A map from the worldsheet with a generic Jordan structure. **(b):** The same instanton on a worldsheet with a non-generic Jordan structure.

$P_{k+1}, \ldots, P_{2k}$ constrain the values

$$X(r = \infty) = +\infty, \tag{2.172}$$

for each of these half-infinite sleeves, again with Θ exhibiting winding number one around each such sleeve. The rest of the instanton is simply constructed by the matching construction at singular leaves where three sleeves meet, as presented in §2.2. The winding numbers are conserved at each such junction, assuring that the map from Σ to the target space is globally well-defined (and in fact smooth). A quick check shows that such a map exhibits the desired instanton number k, defined simply by the worldsheet integral of the pull-back of the two-form ω, either in the conventional or the tropical sense as discussed at the beginning of this subsection.

Is such an instanton solution unique? Not quite yet; as we construct the global surface map by starting at points with $r = -\infty$ on the worldsheet and following the globally defined function r on Σ towards $r = +\infty$, we encounter singular sleeves where two incoming sleeves form one outgoing sleeve, or one incoming sleeve splits into two outgoing ones. Using the α gauge transformations to match the values of X, Θ of the sleeves that meet at a singular leaf, the continuity of $X(r)$ can be maintained throughout the entire construction of the global map, but we are still left with one undetermined zero mode: This instanton still has one modulus X_0, corresponding to the unfixed freedom to shift the target space coordinate by a constant,

$$X(r) \to X(r) + X_0, \tag{2.173}$$

globally everywhere throughout the surface Σ. This remaining modulus will then be eliminated simply by choosing our bulk point P_0 on some generic nonsingular leaf of the foliation, and requiring

$$X(P_0) = 0, \tag{2.174}$$

yielding a unique isolated instanton. Thus, the correlation functions in the tropological $\mathbb{T}P^1$ model at string tree level match exactly the relativistic result, (2.170).

What if we choose a different Jordan structure, with a different number of incoming and outgoing punctures? We claim that the standard BRST arguments (whereby the deformation of the Jordan structure can be viewed at least formally as a BRST-exact operation) suggest the correlation functions will not change. This can be tested and further supported by considering various examples. Consider for example the Jordan structure depicted in Figure 2.5(b), with only one incoming and one outgoing puncture. Again, a unique instanton of instanton number k can be constructed: Now we are forced to put all the $\mathcal{O}_\omega(P_1), \ldots, \mathcal{O}_\omega(P_k)$ at the unique incoming puncture, so we must set

$$P_1 = \ldots = P_k = P. \tag{2.175}$$

The winding number around the unique sleeve must therefore be k. Similarly, all $\mathcal{O}_\omega(P_{k+1}), \ldots,$ $\mathcal{O}_\omega(P_{2k})$ must be placed at the unique outgoing puncture,

$$P_{k+1} = \ldots = P_{2k} = P' \tag{2.176}$$

(otherwise no solution would exist). As on the generic surface from Figure 2.5(a), we must fix the overall zero-mode of X_0 the instanton by setting $X(P_o) = 0$, thus leading to the unique nontrivial instanton solution and hence the same correlation functions that we first obtained with the generic choice of the Jordan structure.

Similarly, if you start with a Jordan structure on S^2 which has more than $2k$ punctures, any solution of the localization equations with instanton number k would have some of these punctures describing sleeves whose winding number must be zero. Such sleeves can then be collapsed to a point, and since they do not restrict the value of the instanton at that point, they correspond to the harmless insertion of the $\mathcal{O}_1$ operator which as we know does not change the value of the correlation functions.

Correlation functions to all loop orders

Having completed the evaluation of correlation functions of BRST invariant point-like operators at genus zero, it is natural to extend the analysis to all string loop orders. In the relativistic case, the structure of topological theories without gravity can be usefully encoded in the axiomatic approach initiated by Atiyah [102] and Segal [103]. In the case of our tropological sigma models, one can similarly anticipate that an appropriate refinement of the Atiyah-Segal axioms will be valid: Our theory turns out to be defined by a more-or-less standard path integral (albeit nonrelativistic), and the theory is not yet coupled to

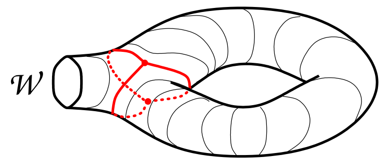

Figure 2.6: The handle operator $\mathcal{W}$ whose insertion raises the genus of Σ by one.

dynamical gravity; therefore, it should exhibit similar axiomatic properties as the relativistic theories before coupling to gravity.

In this section, we will not assume the validity of such axioms, and instead derive the desired all-loop correlation functions for our $\mathbb{T}P^1$ example by direct arguments. The main tool needed to extend our tree-level correlation functions to higher orders is the inclusion of a handle into the foliated worldsheet. Such a handle should be equipped with its own consistent foliation, such that there exists a unique nontrivial instanton at the corresponding instanton number, contributing to the correlation function as in the tree-level case. In analogy with the relativistic case [101], one can then expect that the addition of such a handle would correspond to the insertion of a local "handle operator" $\mathcal{W}$.

It appears that the only such foliated handle represented by a nontrivial $\mathcal{W}$ operator is the torus with one finite sleeve attached, and equipped with a generic foliation structure, as depicted in Figure 2.6. The instanton solution on such a handle can be constructed as follows. The finite sleeve that closes onto itself to form the torus will support a general solution, specified by the winding number n. But to ensure that the loop can be closed in the direction transverse to the foliation requires that $X(r)$ be periodic along that closed loop, forcing $n = 0$. Thus, the entire loop formed by the closed finite sleeve is mapped to a single point in $\mathbb{T}P^1$. Similarly, by the winding-number conservation at the singular sleeve of the handle in Figure 2.6, the winding number of the finite sleeve attached to the torus must vanish, and the sleeve can also be collapsed to a point. Thus, the entire handle will be represented on the rest of the worldsheet by an insertion of a local operator $\mathcal{W}$, without any need to invoke a factorization property of the path integral. As in the relativistic case, this local operator must be given by

$$\mathcal{W} = 2\mathcal{O}_\omega. \tag{2.177}$$

(This follows simply from the matching to the one-loop partition function, interpreted as the torus one-point function of $\mathcal{O}_1$. Since the two-point function $\langle\mathcal{O}_\omega\mathcal{O}_1\rangle_0 = 1$, this means that $\mathcal{W}$ is proportional to $\mathcal{O}_\omega$. Since the same one-loop partition function can also be interpreted as the trace over the two-dimensional space of physical states, the overall normalization must be equal to two, just as in the relativistic case. Finally, the handle operator must be proportional to λ^0, since the winding number through the handle is zero, and therefore adding the handle does not change the value of the instanton number.)

With this handle operator $\mathcal{W}$ at hand, we can add additional handles, as long as the singular leaves of the worldsheet foliation at which these handles are inserted are generic with respect to each other. This again reproduces the relativistic result for higher-genus correlation functions. In particular, the partition function – summed to all orders in the string coupling g_s – gives

$$\langle\langle\mathcal{O}_1\rangle\rangle \equiv \sum_{g=0}^{\infty} g_s^{2g-2}\langle\mathcal{O}_1\rangle_g = \sum_{g=0}^{\infty} g_s^{2g-2}\langle\mathcal{O}_1\mathcal{W}^{g-1}\rangle_0 = \frac{2}{1-4g_s^4\lambda}, \tag{2.178}$$

again matching the well-known relativistic result of the $\mathbb{C}P^1$ model [101].

Should there be nontrivial handle operators associated with more complicated ways how to insert topological handles on foliated surfaces? Consider for example inserting a handle by replacing a finite sleeve in a tree-level diagram with a torus with two finite sleeves attached – a foliated surface with two boundaries that would follow from gluing the two surfaces depicted in Figures 2.2(c) and 2.2(d) along two of their boundary components. A generic Jordan structure on this surface would contain the lengths of the two internal sleeves inside the loops as fixed moduli. In the generic case, the ratio of these lengths is a generic real number, generally not a rational number. But for the solutions along the sleeves to have the correct periodicity along Θ, $X(r)$ along the two sleeves must be fixed to be $n_1 r$ and $n_2 r$, with n_1 and n_2 both integers. The condition of periodicity of X as the loop is getting closed would then require n_1/n_2 to be equal to the generic real ratio of the two lengths of the sleeves, and therefore is not possible to satisfy if the Jordan structure is indeed generic, and if n_1 and n_2 are both nonzero. If, on the other hand, one of the n_1 and n_2 is zero, the corresponding sleeve can be contracted to a point, reducing the construction to the $\mathcal{W}$ operator discussed above. With n_1 and n_2 both nonzero, there is no instanton supported by this handle, and no need to invent a corresponding representation of such a handle by local operators. We are satisfied to see that the correlation functions of the tropological sigma model, to all string loops, appear to have just enough content to reproduce precisely the results for its relativistic cousin.

In this chapter, we focused on the tropicalization of the topological A-model, and developed the theory and its symmetries from first principles of the BRST gauge-fixed path integral formulation. It is natural to ask how the process of tropicalization of the path integral influences various dualities that have been well studied in the relativistic case, such as the relationship with the B-model. We leave the study of possible generalizations of our construction of A-model tropological sigma models open for future study.

Even in the case of the A-model, we contented ourselves in this paper with finding the standard point-like BRST invariant observables of the relativistic A-model (the "quantum cohomology" of the relativistic target) embedded as observables in our tropological sigma model. This leaves an open question of whether our nonrelativistic theory could contain additional, more subtle observables whose existence would solely be possible only as a result of the worldsheet foliation structure. A closely related question, also open for future study, has to do with generalizations to tropological sigma models with worldsheet boundaries, and the classification of topological D-branes in this class of models. In relativistic topological sigma models, topological D-branes and the corresponding worldsheet boundary conditions were first studied in 1990 in [104] [6], in the context of constructing orientifolds of the topological sigma model, not long after the closed-string topological sigma models were first introduced in [63]. The basic class of worldsheet boundary conditions describing topological D-branes was shown to involve the choice of a Lagrangian submanifold of the target space. It should be intriguing to analyze how this construction withstands the process of tropicalization.

Staying within the context of the tropological sigma models constructed in this paper, there are several important open directions to study further. One is the formalization of the amplitudes in the context of an appropriate generalization of the Atiyah-Segal axioms. This question actually has two distinct facets: First, it can be phrased in the context of topological field theories, as studied in Sections 1 to 4 of this paper, and the answer would represent a generalization of Atiyah's axioms [102] for two-dimensional topological field theories. The worldsheet bordisms should correspond to two-dimensional worldsheets with boundaries, carrying the structure of a foliation; the bordisms should respect this foliation structure, in particular the individual boundary components should be identifiable with individual leaves of the foliation. The novelty stems from the fact that, unlike in the relativistic case, we have two different kinds of foliation leaves where states can be studied: The nonsingular leaves of the foliation, and the singular leaves corresponding to the "location along the sleeve at infinity", represented by worldsheet punctures. Perhaps the smoothness property that we discovered around the punctures in §2.2 will be useful in clarifying the question of operator product expansions and factorization when a mixture of nonsingular leaves and punctures is involved in the process.

Alternatively, one can ask this question of axiomatization in the broader context of nonrelativistic conformal theories, as discussed in Section 5. The answer would then provide a generalization of Segal's axioms of relativistic CFTs [103] to nonrelativistic CFTs on worldsheets with foliations. Having such a formalized set of axioms might be useful for extending the list of theories that generalize the construction presented in this paper in ways consistent with the general principles of quantum field theory.

[6] See remarks in Section 3.1 of [105].

Chapter 3

Extensions of the Tropological Sigma Model

In the previous chapter, we have discussed the basic construction behind the Euclidean tropological sigma model. In this chapter, we will discuss their physically relevant analytical continuations into real time for the nonequilibrium string. We also demonstrate that the tropological sigma models admit boundary conditions that permit a tropical analog of branes known as tropical branes and their corresponding canonical quantization which shows that the Hamiltonian associated to these can be described as an infinite tower of particles of increasingly mass. Lastly, we will explicitly develop the case of the tropological sigma models on higher target space dimensions and show that the natural geometries associated to these sigma models are filtered manifolds which admit natural symmetries associated with the noncompact Engel algebra. This leads to the construction of Nil-Equivariant tropological sigma models whose path integrals are expected to be formulated by using compact Nilmanifold quotients of the aforementioned Engel symmetry. Lastly, we present some preliminary unpublished results on how the Maslov dequantization can be extended to some standard topological field theories.

3.1 Analytic Continuations

We begin with the analytic continuations to real worldsheet time. Our preference to interpret the direction along the leaves of the foliation as the worldsheet time originates from broader considerations in physical string theory, concretely from our expectations about the structure of string perturbation theory when the system it describes is not in equilibrium. In quantum field theory, systems out of equilibrium are most naturally formulated with the use of the closed, doubled time contour (known as the Schwinger Keldysh contour), in which the system is first evolved from the far past to the remote future, and then again de-evolved into the remote past. It is an intriguing open question whether the covariant formulation of string perturbation theory can also be extended to the Schwinger Keldysh time, and if so, how.

In the absence of a direct worldsheet construction, this question was addressed in [54, 25, 55] with the use of expected holographic duality to gauge theories in the large-N expansion, for which the formulation away from equilibrium using the Schwinger Keldysh contour is known. The result found in [54, 25, 55] strongly suggests that in nonequilibrium string perturbation theory, the standard expression for observable amplitudes in terms of a sum over contributions from worldsheets Σ_g of genus g, weighted by the appropriate power of the string coupling constant g_s,

$$\mathcal{A} = \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}(\Sigma_g), \tag{3.1}$$

will be refined in an interested and perhaps unexpected way: On the Schwinger Keldysh time contour, each worldsheet Σ_g carries a natural triple decomposition,

$$\Sigma_g = \Sigma^{+}_{g_+} \cup \Sigma^{\wedge}_{g_\wedge} \cup \Sigma^{-}_{g_-}, \tag{3.2}$$

where $\Sigma^{+}_{g_+}$ corresponds to the part of the worldsheet on the forward branch of the time contour, $\Sigma^{-}_{g_-}$ corresponds to the backward part of the time contour, and the "wedge region" $\Sigma^{\wedge}_{g_\wedge}$ is the portion of the worldsheet that connects them, and corresponds to the turn-around late time where the forward and backward branch of the time contour meet. Topologically, the wedge region is connected to the other two parts of this decomposition along a collection of S^1 boundaries, but geometrically, there are hints that these connecting boundaries should be interpreted as nodes in Σ_g. Of course, the total genus g is uniquely determined in terms of the three genera g_+, $g_\wedge$ and g_-, such that the Euler number $\chi(\Sigma_g)$ matches the Euler number obtained from the triple decomposition.

The interesting point is that for the matching to the expectations from the large-N expansion on the dual gauge theory side, not only are Σ^+ and Σ^- expected to be of arbitrary genus, but also the wedge region must be allowed to be of any genus $g_\wedge$. Thus, the perturbative sum (3.2) is refined into a *triple sum* over nontrivial topologies of Σ^+, Σ^-, and also $\Sigma^\wedge$.

Consider now the specific case of critical string or superstring theory. While the worldsheet path-integral description of the theory in regions $\Sigma^\pm$ might be expected to reproduce the standard Polyakov-like path integral, the connecting $\Sigma^\wedge$ region remains quite mysterious from the point of view of the worldsheet dynamics. The robust arguments from the large-N duality to nonequilibrium gauge theories are not refined enough to predict clearly a worldsheet description. However, they do provide one strong hint: While the topology of $\Sigma^{\wedge}_{g_\wedge}$ can be arbitrary (i.e., its genus $g_\wedge = 0, 1, \ldots$), its *geometry* is expected to be highly anisotropic on the worldsheet [25]: The parts of the ribbon diagrams that generate $\Sigma^\wedge$ equip this region with a coarse-grained foliation structure; the leaves of the foliation are segments connecting a point on the boundary with the Σ^+ region to a point on the boundary with the Σ^- region. Moreover, the worldsheet geometry undergoes an anisotropic limit, such that the lengths along the leaves of the foliation go to zero while the lengths in the transverse directions are held fixed. This type of anisotropy expected from the holographic duality of the mysterious wedge region is extremely reminiscent of the geometric structure we have

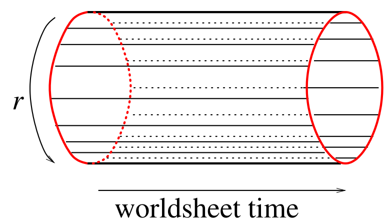

Figure 3.1: The worldsheet foliation in the cross-channel.

seen on the tropicalized worldsheets in this paper, with the coordinate θ along the foliation leaves now interpreted as the worldsheet time.[1] Thus we would like to understand whether the tropological sigma model can be appropriately formulated in real worldsheet time, with propagating degrees of freedom, not just as a bosonic sector of a topological theory. We will consider the real worldsheet time to run *along the leaves of the foliation*, and not in the direction transverse to the foliation (as was the case for the imaginary time in the tropological sigma models).[2] We would like to identify a real-time interpretation of the theory described by (3.5), perhaps by a suitable analytic continuation of the fields and coordinates, such that it satisfies sensible physical requirements, including unitarity, and positivity of energy.

A naive Wick rotation $\theta \to it$ to candidate real time t would not work: S as given in (3.5) is real (for real fields, and real coordinates r, θ). A continuation $\theta \to it$ would make the action complex-valued for real t. In real-time physical theories, one can ensure unitarity by requiring the real-time action to be real. When such a theory is Wick-rotated from real to purely imaginary time, the original condition of unitarity is reformulated as "reflection-positivity": The imaginary-time action can have a non-zero imaginary part, which is constrained to be odd under the time reversal transformation. We will impose the same restrictions on the expected behavior of S in our nonrelativistic case as well.

Now that we have decided that we would like to interpret the direction along the leaves of the foliation as time, it is natural to compactify r on a circle. In addition, we will choose the coordinate r such that it is periodic with periodicity 2π. In this channel, we can study in some more detail the worldsheet conformal symmetries generated by $f(r)$ and $F(r)$, as found above in the tropological context: Unlike the case of noncompact r, there is now a natural countable basis that spans the Lie algebra of conformal symmetries, obtained simply by decomposing the infinitesimal generators $f(r)$ and $F(r)$ into Fourier modes,

$$f(r) = \sum_{\mathbb{Z}} L_m e^{imr}, \qquad F(r) = \sum_{\mathbb{Z}} J_m e^{imr}. \tag{3.3}$$

[1] Tropical geometry has already made its appearance in the study of equilibrium string amplitudes, first in the $\alpha' \to 0$ limit [106] and more recently at general α' [107, 108].

[2] Such theories can then be coupled to a suitable version of nonrelativistic gravity, which would correspond in this channel to the $c \to 0$ nonrelativistic limit, often referred to in the literature as the Carrollian limit.

At the classical level, they satisfy a Lie algebra with a clear geometric interpretation: L_m form the Virasoro algebra of the infinitesimal diffeomorphisms of $r \in \mathbf{S}^1$, while J_m represents an Abelian algebra, transforming under the Virasoro as the Fourier components of a spin-two field. Quantum mechanically, this algebra admits two central charges c and $c_\times$,

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
[J_m, J_n] &= 0, \\
[L_m, J_n] &= (m-n)J_{m+n} + \frac{c_\times}{12}m(m^2-1)\delta_{m+n,0}. \qquad (3.4)
\end{aligned}$$

Note that the Abelian commutation relations between J_m's do not allow a nonzero central extension consistent with the commutation relations involving the L_m generators (see also [109]).

This algebra has been recognized and studied in the literature, in several contexts [110, 111, 112, 113]. It is also recognized as the Bondi-Metzner-Sachs (BMS) algebra of asymptotic symmetries of the Minkowski spacetime at infinity [114, 115, 116, 117, 118], in the special case of the $2+1$ dimensions, and it has been studied extensively in that context [119, 120, 121, 122, 123, 124, 125, 125].

In the relativistic case, the topological sigma models represent much simpler and controllable cousins of the physical theories with propagating degrees of freedom, which could be either the untwisted supersymmetric sigma models, or the sigma models truncated to their bosonic sector. One can naturally ask whether a similar relationship exists between the nonrelativistic tropological sigma models, and theories with propagating local degrees of freedom.

Consider the bosonic sector of our tropological sigma model, with action

$$S = \int dr\, d\theta \left\{ \frac{1}{2}\left(\partial_\theta \Theta - \partial_r X\right)^2 + \beta \partial_\theta X \right\}. \qquad (3.5)$$

The invariance of the action under constant translations in r and θ and the Noether theorem imply the existence of a conserved energy-momentum tensor $T^\alpha{}_\beta$,

$$\begin{aligned}
T^r{}_r &= \frac{1}{2}(\partial_r X)^2 - \frac{1}{2}(\partial_\theta \Theta)^2 - \beta \partial_\theta X, && (3.6) \\
T^r{}_\theta &= (\partial_r X - \partial_\theta \Theta)\partial_\theta X, && (3.7) \\
T^\theta{}_r &= -(\partial_r X - \partial_\theta \Theta)\partial_r \Theta + \beta \partial_r X, && (3.8) \\
T^\theta{}_\theta &= \frac{1}{2}(\partial_\theta \Theta)^2 - \frac{1}{2}(\partial_r X)^2. && (3.9)
\end{aligned}$$

Note some similarities and a few slight conceptual differences compared to the relativistic case: This energy-momentum tensor is conserved, $\partial_\alpha T^\alpha{}_\beta = 0$, but because of the absence of any non-degenerate metric on Σ, it is not natural to raise or lower the indices on $T^\alpha{}_\beta$. The lower index indicates which translation symmetry this conserved current is associated with, while the upper index labels the components of this current. (One could use $g_{\alpha\beta}$ and $h^{\alpha\beta}$ to lower or raise indices, but after this operation, some information contained in $T^{\alpha}{}_{\beta}$ would be inevitably lost.) Note that the trace of the energy-momentum is still well-defined,

$$T^{\alpha}{}_{\alpha} = -\beta\partial_{\theta}X. \tag{3.10}$$

It vanishes on-shell, confirming that our theory exhibits isotropic scaling with dynamical exponent $z = 1$. Note that despite the isotropic scaling, the theory is still sensitive to the worldsheet foliation structure, and the symmetries generated by the energy momentum tensor are indeed in accord with its underlying nonrelativistic conformal symmetry.

Looking for real time in the cross-channel: Conformal symmetries

Since S in (3.5) is already real, we can try to interpret θ itself as *real* time, and see whether its canonical quantization (using methods of Dirac quantization [126]) gives consistent results. Interpreting now θ as time, the canonical momenta P, Π and π conjugate to X, Θ and β are

$$\begin{aligned} P &= \beta, && (3.11)\\ \Pi &= \partial_{\theta}\Theta - \partial_r X, && (3.12)\\ \pi &= 0. && (3.13) \end{aligned}$$

The system can be quantized straightforwardly. The vanishing of π is a constraint, which is very easy to deal with: X and β simply represent an unconstrained canonical pair. The Hamiltonian density and the Hamiltonian are

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \Pi\partial_r X, \qquad H = \int dr\,\mathcal{H}. \tag{3.14}$$

This is of course a Hamiltonian of some free theory. The individual degrees of freedom can be identified by performing a canonical transformation that diagonalizes the Hamiltonian. This canonical transformation is given by

$$\begin{aligned} \widetilde{\Pi} &= \Pi + \partial_r X, \qquad \widetilde{X} = X, && (3.15)\\ \widetilde{P} &= P - \partial_r\Theta, \qquad \widetilde{\Theta} = \Theta. && (3.16) \end{aligned}$$

(Here the first line is suggested by completing the square in the Hamiltonian, and the second is required to make the full transformation canonical, since the new momentum $\widetilde{\Pi}$ does not commute with the old P.) After the canonical transformation, the Hamiltonian density becomes

$$\mathcal{H} = \frac{1}{2}\widetilde{\Pi}^2 - \frac{1}{2}(\partial_r\widetilde{X})^2, \tag{3.17}$$

and we see a problem: This theory with θ interpreted as real time has a real action and is formally unitary, but its energy is not bounded from above or below.

Conformal reductions

Having found that the system represents two decoupled free degrees of freedom, one with a positive-definite energy and the other with a negative-definite energy, it is natural to ask whether we can consistently truncate the system (without violating conformal invariance) to keep only one or the other subsystem.

This should not be done in an *ad hoc* manner, but more systematically – perhaps by invoking a suitable gauge symmetry. We have found that our theory has a subtle gauge invariance represented by the α symmetries, which depend arbitrarily on r but only linearly on θ. In the Lagrangian formalism, it is not quite clear whether or how such hybrid symmetries can be used to reduce the number of degrees of freedom from two to one. We will therefore fall back on the canonical quantization in the Hamiltonian formalism for systems with constraints.

The decoupling of the two subsystems is manifest in the $\widetilde{X}, \widetilde{P}, \widetilde{\Theta}, \widetilde{\Pi}$ coordinates on the phase space. We first attempt to eliminate the $\widetilde{X}, \widetilde{P}$ canonical pair. There are two paths how to do so, leading to the same result: First, we choose to impose

$$\widetilde{X} = 0 \tag{3.18}$$

as a primary constraint. Since the constraint commutes with the Hamiltonian, there is no secondary constraint. Our constraint generates a gauge symmetry, acting via the commutator with the constraint. In particular, under this gauge symmetry, with gauge parametrer $\widetilde{\alpha}_1$, $\widetilde{P}$ transforms simply via

$$\delta P = \widetilde{\alpha}_1. \tag{3.19}$$

Setting $\widetilde{P} = 0$ is then a good gauge choice, yielding the desired reduction to just one degree of freedom, which can be described by the reduced action

$$S_1 = \int dr\, d\theta\, \frac{1}{2}(\partial_\theta \Theta)^2. \tag{3.20}$$

The second path leading to the same result starts instead by imposing the primary constraint

$$\widetilde{P} = 0. \tag{3.21}$$

The commutator with the Hamiltonian reveals that there is now a secondary constraint, $\widetilde{X} = 0$, and no gauge symmetries. The constraints form a second-class pair, which when treated via the Dirac bracket simply eliminates the $\widetilde{X}, \widetilde{P}$ canonical pair, leading to the same reduced theory described by (3.20).

If we instead wish to eliminate the $\widetilde{\Theta}, \widetilde{\Pi}$ canonical pair, we again have two paths: We can choose the primary constraint

$$\widetilde{\Theta} = 0, \tag{3.22}$$

which will imply a secondary constraint $\widetilde{\Pi}$, forming a second-class pair; the Dirac bracket quantization of the system then eliminates the $\widetilde{\Theta}, \widetilde{\Pi}$ pair, leading to one degree of freedom described by the reduced action

$$S_2 = \int dr\, d\theta \left\{ \frac{1}{2}(\partial_r X)^2 + \beta \partial_\theta X \right\}. \tag{3.23}$$

(Now of course the energy of the system would be negative-definite, which we can simply compensate for by reversing the overall sign in front of the original full action and repeating the reduction steps.)

The second path starts with imposing

$$\widetilde{\Pi} = 0 \tag{3.24}$$

as our primary constraint. There is no secondary constraint. The primary constraint is again first-class, generating a gauge symmetry with gauge parameter $\widetilde{\alpha}_2$, acting as

$$\delta\widetilde{\Theta} = \widetilde{\alpha}_2. \tag{3.25}$$

Setting $\Pi = 0$ is a good gauge-fixing condition, reducing the system again to (3.23).

Note that the action of the gauge symmetries $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ in the Hamiltonian quantization is quite reminiscent of the α gauge symmetries that we found in the Lagrangian formalism. How such symmetries should be correctly implemented in the Lagrangian quantization is somewhat obscure, since they are a hybrid between a local and a global symmetry; however, the Hamiltonian quantization with constraints has provided a natural guidance how to interpret and implement such symmetries in the bosonic theory.

Are the two truncations to a system with just one degree of freedom consistent with conformal invariance? A closer look at the conformal transformation rules for all fields reveals a happy fact: The same split into the $\widetilde{X}, \widetilde{P}$ and $\widetilde{\Theta}, \widetilde{\Pi}$ pairs that resulted from our canonical transformation in the Hamiltonian framework would also result if we instead required that the split is conformally invariant in the Lagrangian formalism. Thus, our Hamiltonian quantization has produced two consistent conformal truncations of the original system, to subsystems with Lagrangians (3.20) and (3.23).

Another analytic continuation to real time

Having seen that the original system (3.5) has two consistent conformal truncations, one can naturally wonder whether one can consistently analytically continue the theory so that (i) both $\widetilde{X}$ and $\widetilde{\Theta}$ degrees of freedom are kept, (ii) the theory has a suitably continued conformal symmetry, and (iii) it is unitary, with energy bounded from below. If such a continuation exists, it would also be interesting to see whether it comes from a suitably continued theory in imaginary time, which satisfies the reflection positivity condition on the action.

In the original variables X, Θ, β appearing in (3.5), it is not immediately obvious how to perform such a desired continuation. Wick rotations of r or θ and analytic continuations to some of the fields from real to purely imaginary values do not work. In order to find the correct analytic continuation, let us first compare the theory before and after the canonical transformation to $\widetilde{X}, \widetilde{P}, \widetilde{\Theta}, \widetilde{\Pi}$ variables, and ask how this transformation manifests itself in the Lagrangian formalism. From the known Hamiltonian (3.17), we can reverse-engineer the Lagrangian and express it in the original Lagrangian variables,

$$\begin{aligned} \widetilde{L} &= \widetilde{P}\partial_\theta \widetilde{X} + \widetilde{\Pi}\partial_\theta \widetilde{\Theta} - \mathcal{H} && (3.26) \\ &= \frac{1}{2}(\partial_\theta \Theta)^2 + \frac{1}{2}(\partial_r X)^2 + (\beta - \partial_r \Theta)\partial_\theta X. && (3.27) \end{aligned}$$

This Lagrangian differs from the original Lagrangian in (3.5) simply by adding what is locally a total-derivative term; more precisely, the actions differ globally a topological invariant, given by the integral of the pull-back of $d\Theta \wedge dX$ to Σ:

$$L = \widetilde{L} + \partial_r \Theta \partial_\theta X - \partial_\theta \Theta \partial_r X. \tag{3.28}$$

Perhaps more importantly, in this representation we see a hint how to perform the desired analytic continuation to real time, with positive-definite energy and real action: We need to Wick rotate θ to real time t while keeping X and Θ real, and then we must rotate the shifted field

$$\beta' \equiv \beta - \partial_r \Theta \tag{3.29}$$

from real to purely imaginary values, $\beta' = i\mathscr{B}$, with $\mathscr{B}$ real.

Is such an analytic continuation consistent with the appropriately analytically continued conformal symmetries? After the Wick rotation $\theta = it$, we must make sure that the conformal transformations maintain the reality of r and t. This is achieved by keeping $f(r)$ real, and rotating $F(r) = i\Phi(r)$ to be purely imaginary. Such analytically continued conformal transformations act naturally on X and Θ, while preserving their reality. In addition, on β' they act via

$$\delta\beta' = f(r)\partial_r\beta' + \beta'\partial_r f(r) + \Big(\Phi(r) + t\partial_r f(r)\Big)\partial_t\beta' - i\partial_r X \partial_r \Big(\Phi(r) + t\partial_r f(r)\Big). \tag{3.30}$$

This is indeed consistent with analytically continuing $\beta' = i\mathscr{B}$. The conformal transformations with parameters $f(r)$ and Φ transform $\mathscr{B}$ consistently with the reality of $\mathscr{B}$ while X and Θ are kept real. In contrast, note that the original auxiliary field β transforms after the Wick rotation in a slightly more complicated way than β',

$$\delta\beta = f(r)\partial_r\beta + \beta\partial_r f(r) + \Big(\Phi(r) + t\partial_r f(r)\Big)\partial_t\beta + (\partial_t\Theta - i\partial_r X)\partial_r \Big(\Phi(r) + t\partial_r f(r)\Big). \tag{3.31}$$

The term that contains $\partial_t\Theta$ in this transformation represents an obstruction against analytically continuing β to purely imaginary values while keeping X and Θ real – such a continuation would not be consistent with conformal invariance. This observation makes it clear why the proper analytic continuation of the original theory (3.5) was rather obscure from the perspective of its original variables.

Note also that our analytic continuation will make the topological term in (3.28) purely imaginary; if we desire the real-time action to be fully real including the topological term, we need to analytically continue the coupling constant of this topological term (*i.e.*, its "theta angle") accordingly.

Since we found that the physical interpretation of taking the tropical limit of the relativistic sigma models corresponds to a certain nonrelativistic version of the relativistic topological sigma models, one may naturally wonder whether our Lagrangian path-integral construction simply follows from taking a direct nonrelativistic limit of the standard relativistic topological theory. In the particular example of $\mathbb{T}P^1$ that we focused on in this paper, the answer is yes, although the limit is somewhat subtle – for example, it requires that the standard auxiliary fields that impose the localization to pseudoholomorphic maps must not be integrated out before the nonrelativistic limit is taken (similarly as in the recently studied case in [127]). However, we believe that the direct first-principles construction of the tropological path integral that we followed in this paper has several strong advantages over attempting to take a nonrelativistic limit of the relativistic topological sigma model:

(i) In the case of more general target spaces, to find a prescription for taking a nonrelativistic tropical limit of the topological sigma model is not unique, as it requires a choice of preferred coordinates on the target space. The tropicalization limit then proceeds dimension by dimension, leaving only a discrete group of target-space symmetries (2.39) that we discussed in § 2.1. In the process, the general curved metric on the target space becomes piecewise flat, giving the underlying manifold a piecewise linear polyhedral structure, in the tropical limit. Different choices of preferred coordinate systems on the target space will lead to *a priori* different nonrelativistic theories.

(ii) As we will see in the example presented below, not all tropological sigma models (as defined in this paper) originate from a nonrelativistic limit of a relativistic topological sigma model. (In the example below, the target space will be of an odd real dimension, and thus not obtainable as a limit of a complex target space.) An attempt to classify all possible tropological sigma models thus goes clearly beyond the classification of the relativistic sigma models, and it is useful to develop their path integral intrinsically, based on their nonrelativistic symmetries, without relying on nonrelativistic limits of the relativistic theories.

(iii) With the tools developed in this paper, it is also possible to ask what marginal deformations do our worldsheet theories have, and what kind of nontrivial backgrounds in the tropological target space they correspond to. Again, the classification of possible nontrivial backgrounds in the target space is likely to be wider than the subclass of backgrounds that can be obtained from nonrelativistic limits of backgrounds known in the relativistic case.

In this paper, we focused on the construction of the tropical version of the simplest class of matter systems, the topological sigma models. In the process, we have uncovered a long list of intriguing differences and similarities of the path integral formulation in this class of two-dimensional field theories, in comparison to the standard relativistic topological sigma models, and its close connections with nonrelativistic quantum field theories on surfaces with nondynamical foliations.

On the way towards constructing a full-fledged tropicalized version of the path integral for the topological string, the next step would be to couple such matter systems to the appropriate types of worldsheet topological gravity. Such a coupling is likely to teach us new lessons about the scope of applicability of string theory, perhaps extendable beyond the strict limit of the topological string. Similarly, on the mathematical side, one of our original motivations came from the work of Mikhalkin, which shows an intriguing isomorphism between the results of the standard evaluation of certain Gromov-Witten invariants in the relativistic setting on one hand, and the tropical analog of this enumerative problem on the other. The goal has been to shed new light on this result using the worldsheet path-integral methods. When the result of Mikhalkin is rephrased in the physics language, it is a statement about correlation functions of BRST invariant operators in topological sigma models coupled to topological gravity. Hence, in order to examine this tropicalization of the Gromov-Witten invariants and Mikhalkin's results from the perspective of the path integral, we also need the matter systems studied in this paper to be coupled to a suitable version of topological gravity.

Since the tropicalization of the matter sector on the worldsheet is represented by descending from relativistic surfaces with complex structures to the more singular Jordan structures and their associated worldsheet foliations, it is natural to expect that the coupling of such matter systems to topological gravity should be equivalent to making the Jordan structure (and the worldsheet foliation) dynamical. This will take us to the well-studied realm of nonrelativistic quantum gravities of the Lifshitz type [73, 74], which naturally live on spaces with foliations, albeit without requiring anisotropic worldsheet scaling.

3.2 Tropical Branes

D-branes play a central role in the non-perturbative sector of string theory. Naturally, this raises the question of how branes manifest in a tropical string theory. The exploration of D-brane boundary conditions in both physical and topological string theories was first initiated in [128, 129, 130]. Specifically, we examine the analytically continued tropological sigma model on a worldsheet with boundaries. We consider the strip

$$\Sigma = \mathbb{R} \times [0, \pi], \tag{3.32}$$

where the spatial coordinate r is bounded in the interval $[0, \pi]$, and we regard this as an open tropicalized string theory with boundaries at $r = 0$ and $r = \pi$. We discuss the form of the two types of physically distinct boundary conditions, which we still label as Dirichlet (D) and Neumann (N) by considering variations of the tropicalized action. These boundary conditions are different from the relativistic ones, modifying the mode expansions of the equations of motion. The generators of the anisotropic conformal symmetries also lead to a new type of boundary algebra, discussed in §3.2. Interestingly, these solutions are reminiscent of the Carrollian D-branes found in [131, 132, 133]. Finally, in §3.2, we discuss the open strings solutions with different boundary conditions and the canonical quantization of this theory.

One approach, which we do not pursue in this chapter but mention due to its relevance for Carrollian theories, is to treat θ already as a real time but then the theory's energy is not bounded from below or above [60]. To cure the unboundedness, one has to perform a conformal reduction that reduces the degrees of freedom. These reductions can be achieved consistently by gauging a subgroup of the aforementioned α symmetry. We consider two distinct subgroups that lead to two different theories with actions (see [60] for more details)

$$S_1 = \int_\Sigma drd\theta \left\{ \frac{1}{2}(\partial_\theta \Theta)^2 \right\}, \tag{3.33}$$

$$S_2 = \int_\Sigma drd\theta \left\{ \frac{1}{2}(\partial_r X)^2 + \beta \partial_\theta X \right\}. \tag{3.34}$$

In the literature, these are respectively referred to as the electric and magnetic sectors of Carrollian scalar field theories [134] and their quantization was discussed in [135].

For brane dynamics, we take another approach, which is to add to the Lagrangian action of the tropological sigma model, a topological invariant given by the pull-back of the two form $dX \wedge d\Theta$ to Σ:

$$\begin{aligned} \widetilde{\mathcal{L}} &= \mathcal{L} + \partial_r X \partial_\theta \Theta - \partial_r \Theta \partial_\theta X \\ &= \frac{1}{2}(\partial_\theta \Theta)^2 + \frac{1}{2}(\partial_r X)^2 + (\beta - \partial_r \Theta)\partial_\theta X. \end{aligned} \tag{3.35}$$

We can arrive at this alternative Lagrangian action by directly taking the tropical limit on the bosonic part of the action of the topological A-model,

$$\int_\Sigma d^2 z \left\{ \frac{1}{2} g_{ij} \partial Y^i \bar{\partial} Y^j \right\}, \tag{3.36}$$

instead of through the localization equations. We apply Viro's subtropical deformation to the integral measure, metric and fields. Working in the adapted coordinate system (r, θ) for the worldsheet and (X, Θ) for the target space, we obtain

$$\int drd\theta \left\{ \frac{1}{2}(\partial_r X)^2 + \frac{1}{2}(\partial_\theta \Theta)^2 + \frac{1}{\hbar^2}(\partial_\theta X) + \frac{\hbar^2}{2}(\partial_r \Theta)^2 \right\}, \tag{3.37}$$

up to an overall normalization constant. In the limit $\hbar \to 0$, the third term is divergent. We remove this divergence by introducing an auxiliary field ξ and apply the Hubbard-Stratonovich transformation. This step is inspired by a derivation in [127] of a Galilean limit of the Polyakov action. After the transformation, the action reads

$$\int drd\theta \left\{ \frac{1}{2}(\partial_r X)^2 + \frac{1}{2}(\partial_\theta \Theta)^2 + \xi \partial_\theta X + \frac{\hbar^2}{2}(\xi^2 + (\partial_r \Theta)^2) \right\}. \tag{3.38}$$

We may now take the $\hbar \to 0$ limit without running into any divergences. With the identification $\xi = \beta - \partial_r\Theta$, the following action

$$\int drd\theta \left\{ \frac{1}{2}(\partial_r X)^2 + \frac{1}{2}(\partial_\theta \Theta)^2 + (\beta - \partial_r\Theta)\partial_\theta X \right\}, \tag{3.39}$$

now recovers (3.35).

It is now clear how to analytically continue (3.35) with $\theta \to it$. We keep the fields X and Θ as real valued. In addition to continuing $\theta \to it$, we analytically continue the shifted field $\beta' = \beta - \partial_r\Theta$ as $\beta' = i\mathscr{B}$ wit $\mathscr{B}$ being real valued. After this analytic continuation, the action becomes

$$S = \int_\Sigma dtdr \left\{ \frac{1}{2}\left(\partial_t\Theta\right)^2 - \frac{1}{2}\left(\partial_r X\right)^2 + (\beta - \partial_r\Theta)\partial_t X \right\}. \tag{3.40}$$

This action preserves full conformal invariance. It is also unitary with the energy being bounded from below. We refer to(3.40) as the tropical strings action.

The Euler-Lagrange equations of this action are

$$0 = \partial_r^2 X + \partial_t\partial_r\Theta - \partial_t\beta, \tag{3.41}$$
$$0 = \partial_t^2\Theta - \partial_r\partial_t X, \tag{3.42}$$
$$0 = \partial_t X. \tag{3.43}$$

We use the α symmetry to set $\beta = 0$ (the Lagrange multiplier field β has its own α symmetry: $\delta\beta = \alpha_{\mathcal{B}}(r)$). This gauge fixing condition is a consistent constraint with canonical quantization since the conjugate momentum π to the field β vanishes:

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\beta)} = 0. \tag{3.44}$$

Thus, (3.44) is a primary constraint that, together with the constraint $\beta = 0$, form a pair of second-class type within Dirac's classification [126]. We proceed with quantization by removing this pair of fields (β, π) from the set of non-trivial operators that act on the underlying Hilbert space. After removing that pair from the phase space, the conjugate momenta P and Π to X and Θ are

$$P = -\partial_r\Theta, \tag{3.45}$$
$$\Pi = \partial_t\Theta, \tag{3.46}$$

and the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}\left(\partial_r X\right)^2, \tag{3.47}$$

where

$$H = \int dr\, \mathcal{H}. \tag{3.48}$$

Notice that the momentum conjugate to X is given by $\partial_r \Theta$. This is reminiscent of the situation for open strings charged with respect to a background gauge field [136], in that case the non-zero components of strings along different directions are canonically conjugate to each other. This connection is not that surprising once one recalls that the quantization, in that case, is based on combining components of strings into complexified fields. We comment more about this similarity in [137]. There we discuss the relationship between tropological sigma models and brane quantization, for which the quantization of the symplectic manifold is carried out by the quantization of the strings ending on branes that preserve A-model supersymmetry.

Boundary Conformal Algebra

We now consider a worldsheet with a boundary i.e., the strip given by (3.32). We start with the discussion of the conformal symmetry generators at boundaries and their boundary algebra. Since we want to keep the boundary fixed at $r = 0, \pi$, this requires that the generator δr vanishes there. With r now being compact, we can decompose the infinitesimal generators $f(r)$ and $F(r)$ (in the process of analytic continuation we also redefined $F(r) = iF(r)$ to preserve reality) into Fourier modes

$$f(r) = \sum_{m=1}^{\infty} \widetilde{L}_m \sin(mr), \quad F(r) = \sum_{m\in\mathbf{Z}} J_m e^{imr}, \tag{3.49}$$

where $f(r)$ vanishes at $r = 0, \pi$ to preserve boundaries and the tilde on L_m is used to distinguish these modes from the boundary-less case where

$$f(r) = \sum_{m\in\mathbf{Z}} L_m e^{imr}. \tag{3.50}$$

Without the boundary, the generators L_m and J_m satisfy the $\mathfrak{bms}_{2+1}$ algebra [114, 115, 116, 117, 118] with two central charges c and $c_\times$. This algebra was also shown to arise in the context of Carrollian field theories [110, 111, 112, 113] and flat space holography [138, 139],

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{3.51}$$

$$[J_m, J_n] = 0, \tag{3.52}$$

$$[L_m, J_n] = (m-n)J_{m+n} + \frac{c_\times}{12}m(m^2-1)\delta_{m+n,0}. \tag{3.53}$$

Additionally, this algebra can also be derived as a Wigner-İnönü contraction [140] of two copies of the Virasoro algebra.

We use the L_m generators to construct the boundary version $\widetilde{L}_m$. Comparing (3.49) with (3.50), we find that $\widetilde{L}_m = -\frac{i}{2}(L_m - L_{-m})$. Using this relation, we derive a new boundary algebra with only one central charge surviving

$$[\widetilde{L}_m, \widetilde{L}_n] = \frac{i}{2}\left((m-n)\widetilde{L}_{m+n} - (m+n)\widetilde{L}_{m-n}\right), \tag{3.54}$$

$$[J_m, J_n] = 0, \tag{3.55}$$

$$[\widetilde{L}_m, J_n] = -\frac{i}{2}\left((m-n)J_{m+n} - (m+n)J_{m-n}\right) + \frac{c_J}{12}m(m^2-1)(\delta_{m+n,0} + \delta_{-m+n,0}). \tag{3.56}$$

Boundary Conditions

For a worldsheet with boundaries, the variation of fields for the action (3.40) gives total derivatives with respect to r

$$\int_{\partial\Sigma} dt \left\{-\delta X \partial_r X - \delta\Theta \partial_t X\right\}, \tag{3.57}$$

that we can no longer ignore. Instead, to preserve the equations of motion, we impose boundary conditions.

We discuss two options. One is a novel boundary condition that is a combination of Neumann and Dirichlet boundary conditions, nonetheless, we still refer to them as Neumann boundary conditions in this paper. They are imposed by

$$\partial_r X|_{r=0,\pi} = 0, \quad \partial_t X|_{r=0,\pi} = 0. \tag{3.58}$$

The second is closer to standard Dirichlet boundary conditions, where we fix the variation of fields at the boundaries

$$\delta X|_{r=0,\pi} = 0, \quad \delta\Theta|_{r=0,\pi} = 0. \tag{3.59}$$

Since this has to be satisfied for arbitrary values of t, this implies

$$\partial_t X|_{r=0,\pi} = 0, \quad \partial_t \Theta|_{r=0,\pi} = 0. \tag{3.60}$$

One could also generalize these conditions by introducing background fields and pulling them back. For example, the introduction of a target space 1-form gauge field would modify the N boundary condition by adding terms of the form

$$\int_{\partial\Sigma} dt\{A_X \partial_t X + A_\Theta \partial_t \Theta\}, \tag{3.61}$$

to the variation of the action. We remark that the transition functions that define the gauge field have to be appropriately defined so that the action is invariant under foliation preserving diffeomorphisms and is consistent with the tropical geometry in the target space.

Open Strings Solutions and Canonical Quantization

We construct open strings solutions of (3.41)-(3.43) using the boundary conditions derived above. With two distinct boundary conditions, one can also consider mixed boundary conditions for different fields. Here, we focus on the case where both directions have the same type of boundary conditions. First, consider the case where both boundaries have Dirichlet conditions for both X and Θ: we fix values of X and Θ to be x_0, x_1 and θ_0, θ_1 at boundaries $r = 0, \pi$ respectively. Then, the solutions satisfying (3.59) can be written in the following form

$$X(r,t) = x_0 + \frac{\Delta x}{\pi} r + \sum_{n=1}^{\infty} X_n \sin(2nr), \tag{3.62}$$

$$\Theta(r,t) = \theta_0 + \frac{\Delta \theta}{\pi} r + \sum_{n=1}^{\infty} \Theta_n \sin(2nr) + t \sum_{n=1}^{\infty} 2n X_n (1 - \cos(2nr)), \tag{3.63}$$

where X_n and Θ_n are excitation modes and we defined $\Delta x = x_1 - x_0$, $\Delta\theta = \theta_1 - \theta_0$. Notice that in comparison to the relativistic open strings solutions, X has no dependence on t, which is due to the equation of motion (3.43). Θ depends on t only up to a linear term, whose behavior itself is determined by the X_n oscillation modes up to a constant.

To canonically quantize this theory, we start with the classical Poisson brackets

$$\{X(r,t), P(r',t)\} = \delta(r - r'), \quad \{\Theta(r,t), \Pi(r',t)\} = \delta(r - r'). \tag{3.64}$$

Upon quantization, they get promoted to the following canonical commutation relations

$$[X(r,t), P(r',t)] = i\delta(r - r'), \quad [\Theta(r,t), \Pi(r',t)] = i\delta(r - r'). \tag{3.65}$$

Using the solutions (3.62), (3.63) together with the explicit form of conjugate momenta (3.45), (3.46) we find that the modes X_n, Θ_n satisfy the following commutation relations

$$[X_n, X_m] = 0, \quad [X_n, \Theta_m] = \frac{i}{\pi}\delta_{n,m}. \tag{3.66}$$

From (3.47), one observes that the Hamiltonian only depends on the modes X_n that commute with each other and are already Hermitian. Using (3.48) we can find the Hamiltonian and it takes a simple form in terms of the Fourier modes,

$$H = 2\pi \sum_{n=1}^{\infty} n^2 X_n^2 + \frac{\pi}{2}\left(\frac{\Delta x}{\pi}\right)^2. \tag{3.67}$$

The fact that X_n's commute also implies that there is no need to regularize the ground states oscillation energies as in the relativistic case. In contrast to standard quantum field theory, which yields an infinite number of decoupled oscillatory modes, the theory presented here appears to produce a theory of infinitely many free particles. This outcome aligns with the original physical motivation behind the analytically continued tropological sigma models. These models describe the asymptotic behavior of the moduli space of punctured Riemann surfaces, where it is well known that strings become on-shell and propagate through a tower of increasingly massive physical states [60].

The form of the Dirac brackets (3.66) suggests that the modes Θ_n act as a conjugate momentum of the modes X_n. Therefore, we start by introducing states of the operators X_n and construct the Hilbert space as a tensor product over all modes, for an arbitrary state we can write

$$|X\rangle = |X_1\rangle \otimes |X_2\rangle \otimes |X_3\rangle \dots, \tag{3.68}$$

where each satisfies

$$X_n|X_n\rangle = x_n|X_n\rangle. \tag{3.69}$$

The vacuum is defined as a state for which $x_n = 0,\ \forall n$. Since the Hamiltonian (3.67) is diagonal in the operators X_n, the eigenstates $|X_n\rangle$ are also eigenstates of this Hamiltonian. Now, denoting eigenstates of Θ_n as $|\Theta_n\rangle$, where their construction is analogous to the one above, we can find the overlap to be

$$\langle X_n|\Theta_n\rangle = e^{-i\pi x_n\theta_n}. \tag{3.70}$$

Before, we consider solutions with different boundary conditions, we discuss the case where the string along X is wrapped on a circle S^1 of radius R by identifying points

$$\Delta x \to \Delta x + 2\pi m R, \tag{3.71}$$

with an integer m being a winding number. This periodicity condition shifts the zero point of the Hamiltonian by

$$\left(\frac{\Delta x}{\pi} + 2mR\right)^2. \tag{3.72}$$

Consider another case where the fields (X, Θ) on both boundaries satisfy (3.58). Without any additional constraints, such boundary conditions only fix X. This is a significant difference in contrast to the relativistic case where the boundary conditions restrict all components along which they are enforced. For Θ, only the t dependent part of the solution is impacted by boundary conditions due to the equation of motion (3.41), which relates this part to $\partial_r X$ up to a constant C,

$$X(r,t) = x_0 + \sum_{n=1}^{\infty} X_n \cos(2nr), \tag{3.73}$$

$$\Theta(r,t) = \theta_0 + \sum_{n\neq 0} \Theta_n e^{i2nr} + t\left(C + \sum_{n=1}^{\infty} 2nX_n \sin(2nr)\right). \tag{3.74}$$

As before, the zero mode of X has no conjugate momentum since X is time independent.

The constant C in the solution for Θ can be removed once we insert Wilson line type of terms into the boundary action integral and modify boundary conditions according to (3.61). The addition of boundary terms also changes the conjugate momentum and the quantization. If we keep A_Θ nonzero, the momentum conjugate to Θ (3.46) is now given by

$$\Pi = \partial_t \Theta + A_\pi \delta(r-\pi) - A_0 \delta(r), \tag{3.75}$$

where A_π, A_0 is the value of A_Θ on the boundaries $r = \pi, 0$ respectively. The difference is denoted as $\Delta A = A_\pi - A_0$. Now the Hamiltonian is

$$H = 2\pi \sum_{n=1}^{\infty} n^2 X_n^2 + \frac{\pi}{2} \Delta A^2. \tag{3.76}$$

If we wrap X on S^1 in this case, the Hamiltonian does not change since it does not depend on the momentum conjugate to the zero mode.

One could ask if there exists a tropical analog of T-duality. We would expect that the theory with X field wrapped on S^1 and (DD) boundary conditions is dual to the theory with X wrapped on S^1 with Wilson loops inserted and (NN) boundary conditions. However, the fact that X in the latter case is already periodic implies that the Hamiltonian (3.76) does not have ea term that is inversely proportional to R. We conclude that T-duality in its standard form is absent in this theory. This aligns with the intuition that tropical objects in a tropical geometry base correspond to geometric objects found in the A- and B-models. Therefore, one would not expect to find examples of mirror symmetry that descend to the underlying tropical geometry, as mirror symmetry is fundamentally a relation between the A- and B-models [141]. This suggests that an appropriate tropicalization of the path integrals associated to the B-model would lead to the same tropological sigma models provided by a tropicalization of the A-model. We leave this as an open question for a future work.

In this section, we reviewed the basics of a tropological A-model and discussed its analytic continuation. We examined the resulting theory on the worldsheet with boundaries and introduced a new type of boundary conformal algebra. Additionally, we derived Dirichlet and Neumann modified boundary conditions that arise in this theory and can be imposed at the endpoints of strings, leading to the tropical analog of a D-brane. We analyzed open string solutions with (NN) and (DD) boundary conditions and constructed their mode expansions.

Using canonical quantization, we derived commutator relations for the modes and provided a one-dimensional Hamiltonian, which is simpler than in the relativistic case due to the absence of regularization issues. The Hamiltonian depends only on commuting operators, and there is no energy shift caused by ground state oscillations. This Hamiltonian was interpreted as arising from the infinite tower of increasingly massive string states that one would expect to find when a string goes on-shell. With this Hamiltonian and the quantized operators, one can construct states and compute observables either directly or through an analog of conformal mapping, as in the standard relativistic case. We leave this exploration for a future work.

In comparison to the previous work [60], we demonstrated how tropicalization alters the quantization procedure for open strings. While the system is easier to solve compared to the relativistic case, several intricacies remain to be addressed. The differing boundary conditions for the fields X and Θ leave open the question of mixed boundary conditions. Additionally, we did not investigate the potential new dynamics that might emerge when the target space has a real dimension greater than two. As noted in [60], tropological sigma models allow generalizations to higher dimensions. Surprisingly, due to the nilpotency of higher-order Jordan structures J, these models naturally generalize to odd real dimensions where contact geometry becomes relevant. From J, one can construct actions and impose the boundary conditions discussed here to explore more exotic open string solutions.

Another interesting direction to explore is to discuss tropical D-branes that preserve A-model supersymmetry. In this case, one has to study a theory before the analytic continuation on the worldsheet with boundaries. There are in fact two possible types of branes that one can consider: Lagrangian and coisotropic branes [142] and their construction is dependent on the target space geometry. In [143], these objects are used for the procedure called brane quantization. This method approaches the quantization of a symplectic manifold M by associating an A-model to the complexification of M. We discuss how this procedure changes in the tropical limit in [137].

3.3 Tropological Sigma Models and Higher Dimensional Target Spaces

Throughout this thesis, we have limited our attention to tropological sigma models whose targets are standard tropicalizations of complex manifolds. However, once we have understood the path integral formulation of such theories and their symmetries, it is natural to ask whether the same formalism can be directly extended to a broader class of target-space geometries, going beyond the idea of tropicalization. Here we will indicate some such generalizations, without any attempt at completeness.

The Jordan structures $J_i{}^j$ on tropicalized complex manifolds have a special form: They decompose into two-by-two diagonal blocks, each given in adapted coordinates by (2.44). This limitation is non necessary in our path-integral formulation: We are free to consider target spaces whose Jordan structures contain on the diagonal indecomposable $p \times p$ Jordan blocks with zero eigenvalues,

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \tag{3.77}$$

of arbitrary higher order p, not just the lowest-order $p = 2$ that appeared in tropicalizations of complex manifolds.

Intriguingly, this means that we are not limited to target spaces whose real dimension is even. The simplest example M_3 of such a target space that goes beyond the limits imposed by tropicalizations of complex manifolds is of real dimension three. On M_3, we will use adapted coordinates W, X, Θ, calling them collectively Y^i, with $i = W, X, \Theta$. In these coordinates, the Jordan structure takes the form

$$J_i{}^j = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.78}$$

As a result of this higher-order Jordan structure, M_3 carries a natural structure of a nested, double foliation: One-dimensional leaves of constant X and W are embedded into two-dimensional leaves of constant W.

The localization equations for this system are again given by (2.47), which in the adapted coordinates reduces to

$$\partial_\theta X - \partial_r W = 0, \qquad \partial_\theta \Theta - \partial_r X = 0, \qquad \partial_\theta W = 0, \qquad \partial_\theta X = 0. \tag{3.79}$$

They naturally respect the structure of the nested foliation of the target space. The solutions of the localization equations are locally given by

$$\begin{aligned} W(r,\theta) &= W_0, \\ X(r,\theta) &= X_0(r), \\ \Theta(r,\theta) &= \Theta_0(r) + \theta \partial_r X_0(r). \end{aligned} \tag{3.80}$$

Thus, the solutions look just like those for the (X, Θ) system studied in the bulk of this paper, with each map lying entirely within a two-dimensional leaf of constant W. However, a richer structure emerges when we look at *all* the classical solutions of the bosonic action for the sigma model.

To construct this action, take for simplicity the degenerate metric on M_3 such that $G^{XX} = G^{\Theta\Theta} = 1$ and zero otherwise. The rest of the construction then closely parallels that in §2.3: First, we introduce the auxiliary fields $\mathcal{B}^\alpha{}_i$, and write the action using these auxiliaries as

$$S = \int d^2\sigma \left(\mathcal{B}^\alpha_i E_\alpha{}^i - \frac{1}{2} \gamma_{\alpha\beta} G^{ij} \mathcal{B}^\alpha{}_i \mathcal{B}^\beta{}_j \right). \tag{3.81}$$

Because the number of components of $\mathcal{B}^\alpha{}_i$ is again redundant, this action exhibits a gauge invariance similar to that generated by f^α_{+i} in §2.3. This gauge symmetry can be simply fixed by setting $\mathcal{B}^\alpha_W = 0$, at the cost of abandoning the full covariance in Y^i. Of the remaining four components of $\mathcal{B}^\alpha_i$, two enter the action quadratically, and we can integrate them out. The remaining two enter the action linearly, and we relabel them β_1 and β_2, leading to the simplest form of our conformally invariant action

$$S = \int dr\, d\theta \left\{ \frac{1}{2} (\partial_\theta X - \partial_r W)^2 + \frac{1}{2} (\partial_\theta \Theta - \partial_r X)^2 + \beta_1 \partial_\theta W + \beta_2 \partial_\theta X \right\}. \tag{3.82}$$

The classical equations of motion of this system are locally solved by

$$\begin{aligned}
W(r,\theta) &= W_0(r), \\
X(r,\theta) &= X_0(r), \\
\Theta(r,\theta) &= \Theta_0(r) + \theta\partial_r\Theta_1(r), \\
\beta_1(r,\theta) &= \beta_{1,0}(r) - \theta{\partial_r}^2 W_0(r), \\
\beta_2(r,\theta) &= \beta_{2,0}(r) + \theta\left(\partial_r\Theta_1(r) - {\partial_r}^2 X_0(r)\right).
\end{aligned} \tag{3.83}$$

Note that the general classical solutions are no longer confined to the individual leaf of the codimension-one foliation of fixed W, in contrast to the solutions of the localization equations.

Explicit $\mathbb{CP}^2$ Case

In this subsection, we will study tropological sigma models on the manifold $\mathbb{CP}^2$ and investigate the many inequivalent "tropical limits" that one may consider. We demonstrate that a careful choice of tropical limit results in a target space geometry that is not simply a foliated manifold but instead it is a filtered geometry equipped with a non-holonomic tangent space induced by the filtration. This filtration structure can be seen as an additional global symmetry that acts on the space of fields. For the case of $\mathbb{CP}^2$, we find that this additional symmetry algebra can be identified as a noncompact finite-dimensional nilpotent step 3 Lie algebra known as an Engel algebra. The results are suggestive that tropological sigma models might be able to extract additional invariants beyond the Gromov-Witten invariants for tropicalized Jordan structures.

In §3.3, we provide a quick systematic review of how the tropical limit is implemented in the path integral and show that in low dimensions, the canonical geometric data that tropical field theories are associated with are nilpotent endomorphisms of the tangent bundle known as Jordan structures. We present a classification of Jordan structures in higher dimensions based on the Jordan block decomposition. We argue that the Jordan structure of interest $J_{3,1}$ in this paper can be arrived at through a double Maslov dequantization limit which encodes nested foliations that result in a nontrivial filtration structure. In §3.3, we will give the explicit localization equations and actions for each of Jordan structures. We find that the tropological sigma model associated to $J_{3,1}$ gives rise to the noncompact nilpotent Lie algebra mentioned previously. In §3.4, we show how to regularize the noncompactness of the symmetry group by taking a lattice quotient regularization which yields a nilmanifold global symmetry and construct the equivariant extension of the tropological sigma model. We construct equivariant BRST cohomology and observables. We conclude in §3.4 with a discussion of what new features should be expected in higher dimensions beyond 4 as well as natural follow up questions associated to the Nil-equivariant tropological sigma model.

Higher Order Jordan Structures

As we have reviewed in the previous section, the tropical limit is implemented through a Maslov dequantization, which is done locally. The key geometric data that was created upon by taking a tropical limit was the nilpotent endomorphism of the tangent bundle known as the Jordan structure. Consequently, we begin by classifying all possible degenerations of a complex structure in 2 real dimensions and then extend it to 4-dimensions. Since we are working locally, temporarily, we are able to sidestep the issue of additional global moduli arising.

In two real dimensions, there is a unique complex structure up to a change of basis that you can equip $\mathbb{C}$ with, a matrix representation for the complex structure may be given by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.84}$$

Taking the tropical limit gives a nilpotent endomorphism of nilpotency of a degree two. We will call this a *second order Jordan structure* where the second order refers to the nilpotency degree. Standard arguments in a finite-dimensional linear algebra shows that every nontrivial nilpotent endomorphism of degree 2 is conjugate to a Jordan block

$$J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.85}$$

which is unique, again up to a change of basis. The subscript 2 on J refers to the size of the Jordan block describing the Jordan structure. The most natural way to classify Jordan structures on $\mathbb{C}^n$ for $n \geq 2$ is to use the Jordan block classification of nilpotent matrices.

For the case of $\mathbb{C}^2$, we can give an explicit matrix representation of second order Jordan structures in terms of 4x4 matrices. The only possible nonzero Jordan block decompositions are given by either one block of size two along with two blocks of size 1 or two blocks of size 2. The first possibility has a matrix representation of the form

$$J_{2,1,1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.86}$$

We will use the notation $J_{a,b,c}$ where the indices correspond to the size of the Jordan block and the number of Jordan blocks is given by the number of separated indices. The linear map corresponding to this matrix has rank 1 (in general Jordan block of order k has rank $k-1$) and a 3-dimensional kernel. For the case of two blocks of size 2, we have

$$J_{2,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.87}$$

here the map has rank 2 and a 2-dimensional kernel. These two Jordan structures are not conjugate to each other, and they exhaust all possibilities for a nilpotent endomorphism of degree 2 in 4 dimensions.

However, it turns out that there is another possibility that was mentioned in [60] and we can have Jordan structures of nilpotency degree 3. In other words, we can consider *third order* Jordan structures. In a 4-dimensional vector space, a nilpotent endomorphism of degree 3 is one for which $J^3 = 0$ but $J^2 \neq 0$. In the Jordan classification, the nilpotency index is given by the size of the largest Jordan block. Since the overall dimension is 4, the only possible Jordan block decomposition that achieves this is one Jordan block of size 3 and one block of size 1. Up to conjugation, this form is unique. Thus, there is exactly 1 nilpotent endomorphism of degree 3 on a 4 -dimensional vector space. A matrix representation can be given by

$$J_{3,1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.88}$$

Lastly, we can extend this up to the case of a fourth order Jordan structure. This condition forces the largest Jordan block of J to have size 4. In a 4-dimensional space, the only possibility is to have one single Jordan block of size 4. Thus, up to conjugation, there is exactly one such nilpotent endomorphism. A canonical matrix representation with respect to a suitable basis is:

$$J_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.89}$$

We cannot obtain any higher order Jordan structures for a vector space of dimension 4 but the construction readily generalizes to higher dimensions. In principle, we are able to equip the target space with any of these Jordan structures and study the tropological sigma model on these target spaces but a natural question is: are any of these Jordan structures singled out and which is the natural one to to equip $\mathbb{CP}^2$ with?

It turns out that there is exactly one inequivalent complex structure on a 2-complex dimensional vector space up to a complex linear equivalence. Choosing a matrix representation gives

$$\widehat{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.90}$$

Notice that this higher dimensional complex structure is essentially the tensor product of two lower dimensional complex structures. We will see that we can still construct a unique tropicalization in 4 dimensions. We denote the degenerations by primes. If you take the tropical limit of the first embedded complex structure and leave the second embedded complex structure alone, we obtain

$$J' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3.91}$$

which is not nilpotent and hence is not a Jordan structure. Taking the tropical limit of the 2nd complex structure and leaving the first one alone is equivalent up to a matrix transformation to the first option. One obtains

$$J'' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.92}$$

One can easily find a similarity transform which tells us that J' is equivalent to J'' up to a linear change of coordinates.

Taking a tropical limit such that both embedded complex structures become embedded Jordan structures, we obtain

$$J''' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.93}$$

This linear map is nilpotent and represents a true Jordan structure, hence we see that $J''' = J_{2,2}$ is uniquely picked out from tropicalizing the complex structure on $\mathbb{CP}^2$. We examine all of these Jordan structures in the next section, despite $J_{2,2}$ being picked out from a direct tropical limit. It will turn out that $J_{3,1}$ is the most interesting Jordan structure. The reason for this is due to the fact that one cannot directly arrive at $J_{3,1}$ through a single direct Maslov dequantization. Instead, one must carefully perform two distinct Maslov dequantizations in series in such a way that that two of the entries of the complex structure is killed off and we have $J^3 = 0$. One can achieve this by first performing a Maslov dequantization to zero out one of the entries and obtain a matrix equivalent to J_4, rotating the coordinates and then doing another Maslov dequantization limit kills another entry which results in $J_{3,1}$. Thus, in general, we would expect that each Maslov dequantization induces a distinct foliation structure that when overlapping manifests itself as a nontrivial filtration on the space. Such a Jordan structure has first been described in the 3-dimensional target space case in [60]. We discuss this nontrivial filtration structure in §3.3.

Actions for Tropological Sigma Models on $\mathbb{CP}^2$

From now on, we work with the target space $\widehat{X}$, which has 4 real dimensions and local complex coordinates are denoted as $Z_i, \bar{Z}_i$ with $i = 1, 2$. We will denote the adapted world-sheet coordinates via (r, θ) and the adapted target space coordinates via $Y^i = (X^i, \Theta^i)$. We construct the Maslov dequantization by employing

$$Z^i = e^{\frac{X^i}{\hbar}+i\Theta^i}, \tag{3.94}$$

and taking the limit as $\hbar \to 0$. As discussed in §3.3, we construct a tropological sigma model by formally replacing the Jordan structures in the localization equations and construct a BRST representative for the action. In what follows, we ignore the trivial case of $J_{2,1,1}$ (3.86) as it simply reproduces the original tropological sigma model on tropical projective space $\mathbb{TP}^1$. Instead, we discuss the action associated to $J_{2,2}, J_4, J_{3,1}$ (3.87)-(3.89).

The action of the nilpotent BRST charge on the space of fields is given by

$$\begin{aligned} [Q, X^1] = \psi^1, \quad [Q, X^2] = \psi^2, \\ [Q, \Theta^1] = \varphi^1, \quad [Q, \Theta^2] = \varphi^2. \end{aligned} \tag{3.95}$$

additionally, their action on the foliation-preserving diffeomorphism ghosts $(\psi^1, \psi^2, \varphi^1, \varphi^2)$ is given by

$$\left\{Q, \varphi^1\right\} = \left\{Q, \varphi^2\right\} = \left\{Q, \psi^1\right\} = \left\{Q, \psi^2\right\} = 0. \tag{3.96}$$

The bosonic auxiliary fields $\mathcal{B}_i^\alpha$ and its antighost superpartner χ_i^α are defined as usual and put into a BRST multiplet

$$\begin{aligned} \{Q, \chi^\alpha_I\} = \mathcal{B}^\alpha_I, \\ \{Q, \mathcal{B}^\alpha_I\} = 0. \end{aligned} \tag{3.97}$$

This action is invariant under an additional gauge symmetry that preserves Y^i but modifies $\mathcal{B}_i^\alpha$. We can fix this gauge symmetry in a such way that we preserve conformal invariance of the worldsheet, however, the full conformal invariance of the target space cannot be preserved. We refer reader for further details to [60], and in what follows we pick guage fixing such that

$$\mathcal{B}^\alpha{}_{X^1} = \mathcal{B}^\alpha{}_{X^2} = 0. \tag{3.98}$$

The antighosts χ_i^α, which follow the same pattern as of the auxiliaries, are similarly gauge fixed with the choice

$$\chi^\alpha{}_{X^1} = \chi^\alpha{}_{X^2} = 0. \tag{3.99}$$

The leftover non-zero components of these tensor densities in adapted coordinates are denoted as

$$\begin{aligned} \mathcal{B}^r{}_{\Theta^1} = B_1, \quad \mathcal{B}^r{}_{\Theta^2} = B_2, \quad \mathcal{B}^\theta{}_{\Theta^1} = -\beta_1, \quad \mathcal{B}^\theta{}_{\Theta^2} = -\beta_2, \\ \chi^r{}_{\Theta^1} = \Xi_1, \quad \chi^r{}_{\Theta^2} = \Xi_2, \quad \chi^\theta{}_{\Theta^1} = -\chi_1, \quad \chi^\theta{}_{\Theta^2} = -\chi_2. \end{aligned} \tag{3.100}$$

To integrate out half of auxiliaries, we can add a BRST invariant and conformally invariant term of the form

$$-\frac{1}{2}\int d^2\sigma \gamma_{\alpha\beta}\left(H^{\Theta^1\Theta^1}\mathcal{B}^\alpha_{\Theta^1}\mathcal{B}^\beta_{\Theta^1} + H^{\Theta^2\Theta^2}\mathcal{B}^\alpha_{\Theta^2}\mathcal{B}^\beta_{\Theta^2}\right) = -\frac{1}{2}\int d^2\sigma \left(B_1^2 + B_2^2\right), \tag{3.101}$$

where $\gamma_{\alpha\beta}$ is a tensor density of weight -1 constructed with a worldsheet metric and in the adapted coordinates its leftover component after tropicalization is γ_{rr}. $H^{\Theta^1\Theta^1}$ and $H^{\Theta^2\Theta^2}$ are the non-zero components of the target space metric. However, we cannot add to the action a similar quadratic term for β_1 and β_2 as such term would break worldsheet conformal invariance.

Tropological Sigma Model on $\mathbb{TP}^2$ with $J_{2,2}$

As argued in the previous section, the Jordan structure that is singled from the tropical limit of $\mathbb{CP}^2$ is

$$J_{2,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.102}$$

we view tropical projective space $\mathbb{TP}^2$ as having the same underlying differential structure as $\mathbb{CP}^2$ but instead of being equipped with its standard complex structure, we equip it with $J_{2,2}$. The localization equations are then

$$\begin{aligned}
E^{X^1}_r &= \partial_\theta X^1, & E^{X^1}_\theta &= 0, \\
E^{\Theta^1}_r &= \partial_\theta \Theta^1 - \partial_r X^1, & E^{\Theta^1}_\theta &= -\partial_\theta X^1, \\
E^{X^2}_r &= \partial_\theta X^2, & E^{X^2}_\theta &= 0, \\
E^{\Theta^2}_r &= \partial_\theta \Theta^2 - \partial_r X^2, & E^{\Theta^2}_\theta &= -\partial_\theta X^2,
\end{aligned} \tag{3.103}$$

we find that our corresponding action is

$$\begin{aligned}
S &= \frac{1}{e}\int dr d\theta \left\{ Q, \Xi_j \left(\partial_\theta \Theta^j - \partial_r X^j - \frac{1}{2} B^j \right) + \chi_j \partial_\theta X^j \right\} \\
&= \frac{1}{e}\int dr d\theta \left[B_j \left(\partial_\theta \Theta^j - \partial_r X^j \right) + \beta_j \partial_\theta X^j - \Xi_j \left(\partial_\theta \varphi^j - \partial_r \psi^j \right) - \chi_j \partial_\theta \psi^j - \frac{1}{2} B_j B^j \right]
\end{aligned}$$

Here e is a coupling constant of this quantum theory. This action is quadratic in the auxiliary fields B_j and integrating them out results in the final action

$$\begin{aligned}
S = \frac{1}{e}\int dr d\theta \Big[&\frac{1}{2}\left(\partial_\theta \Theta^1 - \partial_r X^1 \right)^2 + \frac{1}{2}\left(\partial_\theta \Theta^2 - \partial_r X^2 \right)^2 + \beta_j \partial_\theta X^j \\
&- \Xi_j \left(\partial_\theta \varphi^j - \partial_r \psi^j \right) - \chi_j \partial_\theta \psi^j \Big]
\end{aligned}$$

This action is simply a doubling of the standard tropological sigma model on $\mathbb{TP}^1$ that is introduced in [60]. It's clear from the form of the Jordan structure that the underlying geometry is that of a foliated manifold where we have individual foliations running over the (X_1, Θ^1) and (X_1, Θ^2) coordinates. Given this doubling, it's clear from previous arguments that the correlation functions of this theory will match with the standard Gromov-Witten invariants of $\mathbb{CP}^2$.

Tropological Sigma Model on $\mathbb{TP}^2$ with J_4

Now, we consider the fourth order Jordan structure

$$J_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.104}$$

The corresponding localization equations are

$$\begin{aligned} E_r^{X^1} &= \partial_\theta X^1, & E_\theta^{X^1} &= 0, \\ E_r^{\Theta^1} &= \partial_\theta \Theta^1 - \partial_r X^1, & E_\theta^{\Theta^1} &= -\partial_\theta X^1, \\ E_r^{X^2} &= \partial_\theta X^2 - \partial_r \Theta^1, & E_\theta^{X^2} &= -\partial_\theta \Theta^1, \\ E_r^{\Theta^2} &= \partial_\theta \Theta^2 - \partial_r X^2, & E_\theta^{\Theta^2} &= -\partial_\theta X^2, \end{aligned} \tag{3.105}$$

and the action is

$$S = \frac{1}{e}\int drd\theta \left[\frac{1}{2}\left(\partial_\theta \Theta^1 - \partial_r X^1\right)^2 + \frac{1}{2}\left(\partial_\theta \Theta^2 - \partial_r X^2\right)^2 + \beta_j \partial_\theta X^j - \Xi_j\left(\partial_\theta \varphi^j - \partial_r \psi^j\right)\right].$$

The localization equations may be solved locally on a worldsheet patch and yield the solutions

$$\begin{aligned} X^1 &= f_1(r), \quad X^2 = f_3(r), \\ \Theta^1 &= f_2(r), \quad \Theta^2 = f_4(r) + \theta\, \partial_r f_3(r), \end{aligned} \tag{3.106}$$

where the $f_j(r)$ are arbitrary projectable functions. Interestingly, unlike the tropological sigma model for the case where $J_{2,2}$ where we got a doubling, the structure of J_4 seems to totally collapse the (X^1, Θ^1) directions and only leaves room for a tropical winding number for the Θ^2 direction. If we assume that the target space Θ^2 is periodic, then we obtain a winding number w given by

$$\begin{aligned} X^2(r) &= wr \\ \Theta^2(r,\theta) &= f_4(r) + w\theta, \end{aligned} \tag{3.107}$$

It's clear from the local solutions that if the Θ^1 coordinate is also periodic, we won't get a doubling of winding numbers.

Tropological Sigma Model on $\mathbb{TP}^2$ with $J_{3,1}$: Nilpotent Fermionic Symmetries

The localization equations associated to

$$J_{3,1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.108}$$

are

$$\begin{aligned} E^{X^1}_{r} &= \partial_\theta X^1, & E^{X^1}_{\theta} &= 0, \\ E^{\Theta^1}_{r} &= \partial_\theta \Theta^1 \ - \ \partial_r X^1, & E^{\Theta^1}_{\theta} &= -\,\partial_\theta X^1, \\ E^{X^2}_{r} &= \partial_\theta X^2 \ - \ \partial_r \Theta^1, & E^{X^2}_{\theta} &= -\,\partial_\theta \Theta^1, \\ E^{\Theta^2}_{r} &= \partial_\theta \Theta^2, & E^{\Theta^2}_{\theta} &= 0, \end{aligned} \tag{3.109}$$

and the action

$$\begin{aligned} S &= \frac{1}{e}\int_\Sigma drd\theta \left\{Q, \chi^\alpha{}_i E^i{}_\alpha - \frac{1}{2}\Xi_i B^i\right\} \\ &= \frac{1}{e}\int_\Sigma drd\theta \Big[B_1\left(\partial_\theta\Theta^1 - \partial_r X^1\right) + B_2\left(\partial_\theta\Theta^2\right) + \beta_1\left(\partial_\theta X^1\right) \\ &- \frac{1}{2}(B_1^2 + B_2^2) - \Xi_1\left(\partial_\theta\varphi^1 - \partial_r\psi^1\right) - \Xi_2\left(\partial_\theta\varphi^2\right) - \chi_1\left(\partial_\theta\psi^1\right)\Big]. \end{aligned} \tag{3.110}$$

Integrating out the B_1 and B_2 auxiliary fields, one can obtain the following action

$$\begin{aligned} S = \frac{1}{e}\int_\Sigma drd\theta \left[\frac{1}{2}\left(\partial_\theta\Theta^1 - \partial_r X^1\right)^2 + \frac{1}{2}\left(\partial_\theta\Theta^2\right)^2 + \beta_1\left(\partial_\theta X^1\right) - \Xi_1\left(\partial_\theta\varphi^1 - \partial_r\psi^1\right) - \right. \\ \left. \Xi_2\left(\partial_\theta\varphi^2\right) - \chi_1\left(\partial_\theta\psi^1\right)\right]. \end{aligned} \tag{3.111}$$

Unlike the previous tropological sigma models, this one has an additional infinitesimal symmetry which generalizes the α symmetry. The symmetry generators will have a trivial action on some fields but a nontrivial action on others, we will list out only the nontrivial transformations. The first symmetry generator denoted δ_1 gives

$$\begin{aligned} \delta_1 X^1 &= \psi^1, \\ \delta_1 \Theta^1 &= \varphi^1, \\ \delta_1 \Xi_1 &= \partial_\theta\Theta^1 - \partial_r X^1, \\ \delta_1 \chi_1 &= \beta_1. \end{aligned} \tag{3.112}$$

The second symmetry generator is given by

$$\begin{aligned} \delta_2 \varphi^1 &= -\varphi^2, \\ \delta_2 \Xi_2 &= \Xi_1. \end{aligned} \tag{3.113}$$

In addition to δ_1 and δ_2, one is able to generate an independent symmetry generator by taking the appropriate graded commutators $[\delta_1, \delta_2] := \delta_3$, which gives

$$\begin{aligned} \delta_3 \Theta^1 &= \varphi^2, \\ \delta_3 \Xi_2 &= \partial_\theta\Theta^1 - \partial_r X^1, \end{aligned} \tag{3.114}$$

as well as $[\delta_1, \delta_3] := \delta_4$ which gives

$$\begin{aligned} \delta_4 \Xi_1 &= -\partial_\theta\varphi^2, \\ \delta_4 \Xi_2 &= \partial_\theta\varphi^1 - \partial_r\psi^1. \end{aligned} \tag{3.115}$$

We can more explicitly write down the generators Q_i by using functional derivatives acting to the left over the foliated worldsheet

$$Q_i = \int d^2\sigma \delta_i \Phi^A(\sigma) \frac{\overleftarrow{\delta}}{\delta \Phi^A(\sigma)}, \tag{3.116}$$

where Φ^A denotes any field and we included the arrow to underline that the derivative acts from the right. Then, the fermionic symmetry generators may be written down as

$$\begin{aligned}
Q_1 &= \int d^2\sigma \left[\psi^1 \frac{\overleftarrow{\delta}}{\delta X^1} + \varphi^1 \frac{\overleftarrow{\delta}}{\delta \Theta^1} + \left(\partial_\theta \Theta^1 - \partial_r X^1\right) \frac{\overleftarrow{\delta}}{\delta \Xi_1} + \beta_1 \frac{\overleftarrow{\delta}}{\delta \chi_1} \right], \\
Q_2 &= \int d^2\sigma \left[-\varphi^2 \frac{\overleftarrow{\delta}}{\delta \varphi_1} + \Xi_1 \frac{\overleftarrow{\delta}}{\delta \Xi_2} \right], \\
Q_3 &= \int d^2\sigma \left[\varphi^2 \frac{\overleftarrow{\delta}}{\delta \Theta^1} + \left(\partial_\theta \Theta^1 - \partial_r X^1\right) \frac{\overleftarrow{\delta}}{\delta \Xi_2} \right], \\
Q_4 &= \int d^2\sigma \left[-\partial_\theta \varphi^2 \frac{\overleftarrow{\delta}}{\delta \Xi_1} + \left(\partial_\theta \varphi^1 - \partial_r \psi^1\right) \frac{\overleftarrow{\delta}}{\delta \Xi_2} \right].
\end{aligned} \tag{3.117}$$

One can show that these differential operators generate the infinitesimal symmetries and they satisfy

$$[Q_1, Q_2] = Q_3, \quad [Q_1, Q_3] = Q_4, \tag{3.118}$$

with all other commutators vanishing. The symmetry also holds off-shell, if we consider (3.110), where auxiliary fields have not been integrated out yet, we find a simple modification of charges

$$\begin{aligned}
Q_1' &= \int d^2\sigma \left[\psi^1 \frac{\overleftarrow{\delta}}{\delta X^1} + \varphi^1 \frac{\overleftarrow{\delta}}{\delta \Theta^1} + (\partial_\theta \Theta^1 - \partial_r X^1) \frac{\overleftarrow{\delta}}{\delta \Xi_1} + \beta_1 \frac{\overleftarrow{\delta}}{\delta \chi_1} + \left(\partial_\theta \varphi^1 - \partial_r \psi^1\right) \frac{\overleftarrow{\delta}}{\delta B_1} \right], \\
Q_3' &= \int d^2\sigma \left[\varphi^2 \frac{\overleftarrow{\delta}}{\delta \Theta^1} + (\partial_\theta \Theta^1 - \partial_r X^1) \frac{\overleftarrow{\delta}}{\delta \Xi_2} + \partial_\theta \varphi^2 \frac{\overleftarrow{\delta}}{\delta B_1} \right],
\end{aligned}$$

with Q_2 and Q_4 not modified. These charges again satisfy (3.118).

The symmetry algebra (3.118) can be identified to be a free 3-step nilpotent Lie algebra known as an Engel algebra it can also be shown to be isomorphic to $A_{4,1}$ algebra of the list on Mubarakzyanov's classification of low-dimensional real Lie algebras [97, 98]. The Engel algebra admits a local frame composed of two generators X, Y whose Lie brackets commute to give

$$[X, Y] = Z, \quad [Y, Z] = W. \tag{3.119}$$

A natural question would be to ask how does this fermionic global symmetry arise? One can examine that the structure of $J_{3,1}$ provides a nontrivial sequence of filtrations on the tangent bundle. There is a chain of generalized eigenspaces

$$0 \subset \operatorname{im}\left(J_{3,1}^2\right) \subset \operatorname{im}\left(J_{3,1}\right) \subset \ker\left(J_{3,1}\right), \tag{3.120}$$

where $\operatorname{im}\left(J_{3,1}^2\right)$ is one-dimensional, $\operatorname{im}\left(J_{3,1}\right)$ is two-dimensional and $\ker\left(J_{3,1}\right)$ is two-dimensional. Unlike the other Jordan structures, $J_{3,1}$ does not define a distribution that one could then integrate to give a foliation on the manifold instead we have a filtered manifold. We will call a Jordan structure that induces a nontrivial filtration (as opposed to a trivial one), an *exotic* Jordan structure.

Recall that a filtered manifold [144] is defined as a manifold where the tangent bundle comes equipped with a nested sequence of filtration subspaces i.e. a filtration flag. Unlike a foliated manifold where the tangent space is still an abelian group with respect to dilations of the tangent vectors, a filtered manifold has an associated graded Lie algebra known as the symbol algebra which one can think as of nonabelian tangent space. The symbol algebra is constructed as a direct sum of elements in the flag via

$$\operatorname{gr}\left(T_p M\right) = \bigoplus_i F_p^i / F_p^{i-1}, \tag{3.121}$$

where F_p^i denotes the terms of the filtration of the tangent space at a point p.

For the explicit case of $J_{3,1}$, we have the filtration flag constructed by

$$F_p^1 = \ker(J), \quad F_p^2 = \ker\left(J^2\right), \quad F_p^3 = \ker\left(J^3\right) = T_p M. \tag{3.122}$$

Through the rank-nullity theorem, one can check that

$$\dim F_p^1 = 2, \quad \dim F_p^2 = 3, \quad \dim F_p^3 = 4, \tag{3.123}$$

which then defines the Lie-algebras

$$\mathfrak{g}_1 = F_p^1, \quad \mathfrak{g}_2 = F_p^2/F_p^1, \quad \mathfrak{g}_3 = F_p^3/F_p^2. \tag{3.124}$$

This Lie-algebra $\mathfrak{g}$ decomposes into a $2 \oplus 1 \oplus 1$-dimensional graded vector space of the form $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$. The associated graded lie-algebra for the Jordan structure $J_{3,1}$ is then precisely isomorphic to the Engel algebra. In summary, the Jordan structure induces a graded Lie-algebra constructed from the filtration on the tangent bundle which then gives us an additional fermionic symmetry group on the space of fields. Interestingly, the dilations on this space acts in an anisotropic way. If the generating set of vector fields X, Y scale by a factor λ under dilations, then their commutator will scale by λ^2. This shows that not only are the gauge symmetries related to tropological sigma models anisotropic but their global symmetries seem to obtain a similar anisotropic scaling behavior.

3.4 Nil-Equivariant Tropological Sigma Models

It is interesting to observe that the tropicalization generically gives rise to not only nilpotent endomorphisms but also nilpotent noncompact global symmetry groups. In the previous section, we saw that the tropological sigma model equipped with $J_{3,1}$ furnished a representation of this nilpotent symmetry group through fermionic charges. In this section, we will begin by regularizing the noncompact global symmetry and find that one can instead reduce the symmetry group to a compact subgroup by quotienting out through a lattice; the result is a nilmanifold. We then show how to construct an equivariant tropical topological sigma model with respect to this new symmetry and end with a discussion of how the observables and their selection rules are modified.

Compact Nilmanifold Quotients of the Engel Algebra

In order to investigate the consequences of this additional fermionic global symmetry, we would like to turn on a background gauge field associated to the Engel algebra and what modifications the standard Gromov-Witten invariants undergo. This may be implemented by directly constructing an equivariant tropological sigma model which is equivariant with respect to the Engel algebra and so we will explicitly try to construct the Killing vector fields associated to the Engel algebra. There is one caveat however; the symmetry generators will generate a nilpotent and noncompact Lie group G; we will need to see how to quotient or in other words properly gauge fix this in the path integral. In order to do so, we must either find an appropriate compact subgroup of G. We will see that no such compact subgroups exist and we will be forced to quotient out by a lattice Γ to achieve a proper regularization for the symmetry group.

Suppose that one chooses a compact subalgebra K of G. We would like its Lie algebra $\mathfrak{k} \subset \mathfrak{g}$ to also be nilpotent since the Engel algebra itself is nilpotent. As usual, we can equip the Lie algebra of any compact subgroup K with an adjoint-invariant positive definite inner product and require that the subalgebra is skew adjoint with respect to the inner product. An immediate consequence of this is that any matrix representation of the nilpotent Lie algebra will have automatically be zero; this is due to the fact that any skew adjoint linear map is normal and by the spectral theorem we can unitarily diagonalize this map with a purely imaginary spectrum but nilpotency forces all eigenvalues to vanish. Consequently; the Lie subalgebra may only lie in the center of $\mathfrak{g}$ but since the group is simply connected, this means that we will only recover trivial subalgebras. The corresponding subgroups would therefore only be discrete subgroups of G i.e., a lattice $\Gamma \subset G$. It is easier to see this explicitly using a 2x2 matrix and forcing nilpotency. A quick example would be

$$\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}, \tag{3.125}$$

imposing nilpotency forces all eigenvalues to be zero.

As a result, we will instead choose to quotient out by a lattice Γ in G. Doing so will result in a compact nilmanifold G/Γ. In [145] has shown that there any simply connected nilpotents Lie group G will admit a lattice iff its algebra admits a basis with rational structure constants. In particular, we can explicitly show that this is the case for the 4-dimensional Engel algebra via the use of canonical coordinates induced by the exponential map. Let's denote the Engel Lie algebra to be abstractly defined by generators X_1, X_2, X_3, X_4 that satisfy

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \tag{3.126}$$

then its associated Lie Group admits exponential coordinates (x, y, z, w) of the second kind [146] via

$$g(x, y, z, w) = \exp(xX_1)\exp(yX_2)\exp(zX_3)\exp(wX_4), \tag{3.127}$$

one would expect that a canonical lattice would simply be given by

$$\Gamma_0 := \exp(\mathbb{Z}X_1)\exp(\mathbb{Z}X_2)\exp(\mathbb{Z}X_3)\exp(\mathbb{Z}X_4), \tag{3.128}$$

however this lattice is not closed under the group law

$$(x, y, z, w) \cdot (x', y', z', w') = \left(x + x', y + y', z + z' + xy', w + w' + xz' + \frac{1}{2}x^2y'\right). \tag{3.129}$$

One can trace back the problem back to the $\frac{1}{2}$ that appears in the w coordinate due to the Baker-Campbell-Hausdorff formula. In order to fix this, we can either choose to rescale our basis vectors or impose a parity constraint on one of the integer coordinates; we will do the former. We begin by rescaling the 4th basis vector X_4 by the factor 2 and defining the basis

$$E_1 := X_1, \quad E_2 := X_2, \quad E_3 := X_3, \quad E_4 := 2X_4, \tag{3.130}$$

we can then once again write any element of the nilpotent Engel group through the exponential map using coordinates (x, y, z, u) as

$$g(x, y, z, u) = \exp(xE_1)\exp(yE_2)\exp(zE_3)\exp(uE_4), \quad (x, y, z, u) \in \mathbb{R}^4, \tag{3.131}$$

the induced group law is now

$$\begin{aligned}&(x, y, z, u) \cdot (x', y', z', u') \\ &= \left(x + x', y + y', z + z' + xy', u + u' + \frac{1}{2}xz' + \frac{1}{4}x^2y'\right),\end{aligned} \tag{3.132}$$

the corresponding lattice is then

$$\Gamma = \left\{ \left(m, n, p, \frac{1}{4}q\right) \middle| \, m, n, p, q \in \mathbb{Z} \right\} = \exp(\mathbb{Z}E_1)\exp(\mathbb{Z}E_2)\exp(\mathbb{Z}E_3)\exp\left(\frac{1}{4}\mathbb{Z}E_4\right). \tag{3.133}$$

The lattice Γ is a rank-4 free abelian subgroup of G and it induces a corresponding lattice in the Lie-algebra defined up to a rational automorphism. The associated quotient is then a compact nilmanifold known as an Engel nilmanifold $\mathcal{G} = G/\Gamma \cong \mathbb{R}^4/\sim$, explicitly the identification can be seen by multiplying a group element by a lattice element and it produces

$$(x,y,z,u) \sim \left(x+m, y+n, z+p+xn, u+\frac{1}{4}q+\frac{1}{2}xp+\frac{1}{4}x^2n\right), \tag{3.134}$$

for integers n, m, p, q. One can write these down these identifications in terms of linear integer combinations of the following generators

$$\begin{aligned}
(x,y,z,u) &\sim (x+1,y,z,u)\\
(x,y,z,u) &\sim \left(x,y+1,z+x,u+\frac{1}{4}x^2\right)\\
(x,y,z,u) &\sim \left(x,y,z+1,u+\frac{1}{2}x\right)\\
(x,y,z,u) &\sim \left(x,y,z,u+\frac{1}{4}\right).
\end{aligned} \tag{3.135}$$

In order to explicitly construct the Killing vector fields, we want to equip our global symmetry group $\mathcal{G}$ with a left invariant Riemannian metric tensor. For the purposes of a background field, we may pick any left invariant metric tensor to implement this however we note in passing that if we are later interested in making the field dynamical, at first glance, it would seem impossible to gauge the Engel algebra due to the fact that any kinetic term would require an Engel adjoint invariant trace but no such adjoint invariant form exists due to the nilpotency of the Engel algebra.

We begin by computing the left-invariant Maurer-Cartan forms and we obtain

$$\begin{aligned}
\theta^1 &= dx,\\
\theta^2 &= dy,\\
\theta^3 &= dz - xdy,\\
\theta^4 &= du - \frac{1}{2}xdz + \frac{1}{4}x^2dy,
\end{aligned} \tag{3.136}$$

their corresponding structure equations can be seen to be $d\theta^1 = 0, \quad d\theta^2 = 0, \quad d\theta^3 = -\theta^1 \wedge \theta^2, \quad d\theta^4 = -\frac{1}{2}\theta^1 \wedge \theta^3$ and we can construct the left invariant Killing vector fields by solving the duality equation $\theta^a(K_b) = \delta^a_b$ which yields

$$\begin{aligned}
K_1 &= \partial_x,\\
K_2 &= \partial_y + x\partial_z + \frac{1}{4}x^2\partial_u,\\
K_3 &= \partial_z + \frac{1}{2}x\partial_u,\\
K_4 &= \partial_u
\end{aligned} \tag{3.137}$$

and they satisfy (3.126), with X_4 scaled by 1/2. Since this frame is left-invariant, we can guarantee that this automatically descends to the nilmanifold quotient. We will use these Killing vector fields with the understanding that in the next subsection in order to construct an equivariant tropical topological (tropological) sigma model on $\mathbb{TP}^2$ that is equivariant with respect to the nilmanifold $\mathcal{G}$. We will call the corresponding tropological sigma model, a *Nil-Equivariant Tropological Sigma Model*.

Nil-Equivariant Tropological Sigma Model on $\mathbb{TP}^2$

In [63], it was shown that one can couple a relativistic topological sigma model to a background gauge field by taking the BRST operator and making it equivariant with respect to the symmetry group. In order to do this, we modify the transformation of the ghosts fields by shifting it by the Killing vector field. In doing so, we introduce the set of Nil ghosts ϕ^a of Grassmann degree 2.

The field content is summarized in Table 3.1. In order to construct a Nil equivariant

Field Multiplet	Grassmann Parity	Ghost Number	Geometric Origin
$X^i = (X^1, \Theta^1, X^2, \Theta^2)$	even	0	Target Space Coordinates
$\Psi^i = (\psi^1, \varphi^1, \psi^2, \varphi^2)$	odd	+1	Foliation-Preserving Diff Ghosts
$\chi_i^\alpha = (\Xi_1, \chi_1, \Xi_2, \chi_2)$	odd	−1	Antighost Fields
$B_i^\alpha = (B_1, \beta_1, B_2, \beta_2)$	even	0	Auxiliary Fields
ϕ^a	even	+2	Nil Ghosts

Table 3.1: Field content

tropological sigma model, we have to reintroduce the auxiliary field and lift the step 3 Engel symmetry to an off-shell symmetry of (3.110). Our corresponding BRST differential gets promoted to a Nil equivariant BRST differential by explicitly writing

$$\begin{aligned} Q_{\mathcal{G}} X^i &= \Psi^i, \\ Q_{\mathcal{G}} \Psi^i &= \phi^a K_a^{\ i}(X), \\ Q_{\mathcal{G}} \phi^a &= 0. \end{aligned} \tag{3.138}$$

One can check that this equivariant BRST differential squares to a Lie derivative along the Killing vector field $Q^2 = \mathcal{L}_{\phi^a K_a}$ provided that we use the following modification of the antighost multiplet

$$\begin{aligned} Q_{\mathcal{G}} \chi^\alpha{}_i &= \mathcal{B}^\alpha{}_i, \\ Q_{\mathcal{G}} \mathcal{B}^\alpha{}_i &= \mathcal{L}_{\phi^a K_a}(\chi^\alpha{}_i) = \phi^a K_a^{\ j} \partial_j \chi^\alpha{}_i + \partial_i \left(\phi^a K_a^{\ j}\right) \chi^\alpha{}_j. \end{aligned} \tag{3.139}$$

We recall that not all the auxiliary fields and the antighosts are zero in the gauge choice we described in (3.98) and (3.99). In fact the nonzero components were listed in the adapted system of coordinates in (3.100).

The gauge fixing fermion remains as in (3.110)

$$\psi_{\mathrm{gf}} = \chi^{\alpha}{}_{i} E^{i}{}_{\alpha} - \frac{1}{2}\left(\Xi_1 B_1 + \Xi_2 B_2\right) \tag{3.140}$$

and a BRST representative action for the Nil equivariant tropological sigma model is then the original action $S_0 = \int \{Q, \psi_{\mathrm{gf}}\}$, (3.110), plus a coupling S_{Nil} to the background Nil killing vector field of the form

$$S_{\mathrm{Nil}} = \frac{1}{2}\int_{\Sigma}\left(\Xi_1 \mathcal{L}_{\phi^a K_a}\Xi_1 + \Xi_2 \mathcal{L}_{\phi^a K_a}\Xi_2\right), \tag{3.141}$$

which one has to add to make sure that the full action is $Q_{\mathcal{G}}$ invariant. The final localization locus is now the original tropical localization equations intersected with the fixed points of the Nil Killing vector field K_a, the resulting path integrals should produce Nil equivariant Gromov Witten invariants and more generally filtered Gromov Witten invariants.

Nil-Equivariant Observables and Selection Rules

Unlike the standard tropological sigma model, physical observables must now lie in the Nil equivariant cohomology $H^{\bullet}\left(\mathbb{T}P^2, \mathcal{G}\right)$. We recall that in the topological A model, the observables are in correspondence with BRST cohomology classes given by the pullback of deRham cohomology classes of the target. In the case for $\mathbb{CP}^1$, the quantum cohomology ring is generated by the identity operator $\mathcal{O}_{\mathrm{Id}}$ and the symplectic form $\mathcal{O}_{\omega}$. One may explicitly construct these via the Stora-Zumino topological descent equations. On the A-model, the descent equations on $\mathcal{O}_{\omega}$ results in a hierarchy of observables on the worldsheet Σ,

$$\begin{aligned} d\mathcal{O}_{\omega}^{(0)} &= Q\mathcal{O}^{(1)}, \\ d\mathcal{O}_{\omega}^{(1)} &= Q\mathcal{O}^{(2)}, \\ d\mathcal{O}_{\omega}^{(2)} &= 0, \end{aligned} \tag{3.142}$$

where d is the worldsheet exterior derivative.

If one couples the A-model to topological gravity, then additional gravitational descendants $\tau_k\left(\mathcal{O}_a\right) \equiv \eta^k \wedge \mathcal{O}_a$, where k is a positive integer and $a = \{\mathrm{id}, \omega\}$, may appear at higher genus g through taking additional wedge products with diffeomorphism ghosts η. As a result, every correlator in the A-model has the well-known schematic [147] form

$$\left\langle \prod_{i=1}^{n_0} \tau_{m_i}(\mathcal{O}_{\mathrm{id}}) \prod_{j=1}^{n_1} \tau_{k_j}(\mathcal{O}_{\omega}) \right\rangle_{g,d}, \tag{3.143}$$

the above correlator is the Gromov-Witten invariant at genus g and is geometrically an intersection number on the virtual fundamental class of the moduli space of pseudoholomorphic curves of degree d $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, d)$. As one would expect, not all observables will be directly relevant since there is a ghost number anomaly that arises which manifests itself as an index theorem i.e. the Riemann Roch theorem of the Dolbeaut operator. Physically, this is the statement that the ghost and antighost zero modes must match and can be stated as

$$\dim\ker\bar{\partial}_\Phi^\dagger - \dim\ker\bar{\partial}_\Phi = d_X(1-g) + \int_\Sigma \Phi^* c_1\left(T^{1,0}\mathbb{CP}^1\right), \tag{3.144}$$

in the case of $\mathbb{CP}^1$, this explicitly evaluates to $n_\psi - n_\chi = (1-g) + 2d$. As explained in [63], correlators that do not have the correct number of ghost and antighost insertions will automatically vanish effectively giving us a selection rule. In the non-equivariant $\mathbb{CP}^2$ case, the Riemann Roch index theorem states $n_\psi - n_\chi = 2(1-g) + 3d$. It was shown in [60] that the same arguments hold for the tropological sigma model and the correlators match precisely although in this case, the relevant differential operators should be deformed to their tropical counterpart. For the purposes of the Nil-equivariant tropological sigma model, we want to first begin with the family of observables that are generated by $\mathbb{CP}^2$ and then take into the account that we have an additional equivariant symmetry.

Unlike $\mathbb{CP}^1$, the additional complex dimension allows our cohomology ring to admit one more class generated by the symplectic 2-form of $\mathbb{CP}^2$ given by the Fubini-Study metric. If we choose our degree 2 generator such that

$$\int_{\mathrm{CP}^1} \omega_{\mathrm{FS}} = 1, \tag{3.145}$$

where the $\mathbb{CP}^1$ is any projective line within $\mathbb{CP}^2$ then we will have an additional class given by

$$\int_{\mathrm{CP}^2} \omega_{\mathrm{FS}} \wedge \omega_{\mathrm{FS}} = 1. \tag{3.146}$$

If we denote the cohomology class of the symplectic form by $h := [\omega_{\mathrm{FS}}] \in H^2\left(\mathbb{CP}^2; \mathbb{Z}\right)$, then it becomes clear that the cohomology ring can be expressed as

$$H^\bullet\left(\mathbb{CP}^2; \mathbb{Z}\right) = \mathbb{Z}[h]/\left(h^3\right), \tag{3.147}$$

the corresponding A-model observables can then be constructed by formally replacing the differential forms with the appropriate power of ghost fields and yields the collection $\mathcal{O}_{\mathrm{id}}, \mathcal{O}_\omega, \mathcal{O}_{\omega\wedge\omega}$. As previously mentioned, coupling this to topological gravity gives additional gravitational descendants that give us a general genus-g, degree-d correlator of the form

$$\left\langle \prod_a \tau_{m_a}\left(\mathcal{O}_{\mathrm{id}}\right) \prod_b \tau_{n_b}\left(\mathcal{O}_\omega\right) \prod_c \tau_{p_c}\left(\mathcal{O}_{\omega\wedge\omega}\right) \right\rangle_{g,d}, \tag{3.148}$$

the Riemann-Roch theorem then gives the selection rule

$$\sum_a m_a + \sum_b (n_b + 1) + \sum_c (p_c + 2) = 2(1 - g) + 3d. \tag{3.149}$$

If we turn off topological gravity, all the correlators collapse to the usual three point functions. The explicit form of these observables can once again be obtained via the Stora-Zumino topological descent equations.

We will now compute the topological descent observables for $\mathbb{CP}^2$ since we ultimately interested in the explicit form of their equivariant analogs. The identity operator gives rise to a ghost degree zero worldsheet 0-form $\mathcal{O}_1^{(0)} = 1$ which has a trivial topological descent. We will denote the deRham degree by the parenthesized superscript. We move onto the ghost degree two worldsheet 0-form for a closed ω

$$\mathcal{O}_\omega^{(0)} = \frac{1}{2}\iota_\psi \iota_\psi (X^*\omega) = \frac{1}{2}\omega_{ij}\psi^i\psi^j, \tag{3.150}$$

where the components ω_{ij} are given by the pullback of the target symplectic form by X, the first step in the descent prescription then gives the ghost degree two 1-form

$$\mathcal{O}_\omega^{(1)} = \frac{1}{2}\iota_\Psi (X^*\omega) = \omega_{ij}\psi^i\partial_\alpha X^j d\sigma^\alpha, \tag{3.151}$$

$$\mathcal{O}_\omega^{(2)} = \frac{1}{2}X^*\omega = \frac{1}{2}\omega_{ij} dX^i \wedge dX^j. \tag{3.152}$$

Unlike $\mathbb{CP}^1$, we have a new seed observable that arises from the four-form $\Omega = \omega \wedge \omega$, the topological descent equations give

$$\mathcal{O}_{\omega\wedge\omega}^{(0)} = \frac{1}{4!}\left(\iota_\Psi\right)^4 (X^*\Omega) = \frac{1}{4}\omega_{ij}\omega_{kl}\psi^i\psi^j\psi^k\psi^l, \tag{3.153}$$

$$\mathcal{O}_{\omega\wedge\omega}^{(1)} = \omega_{ij}\omega_{kl} dX^i\psi^j\psi^k\psi^l, \tag{3.154}$$

$$\mathcal{O}_{\omega\wedge\omega}^{(2)} = \frac{3}{2}\omega_{ij}\omega_{kl} dX^i \wedge dX^j\psi^k\psi^l. \tag{3.155}$$

In [60], the explicit tropical limit of these observables is able to be extracted through the Maslov dequantization and results in the components of the symplectic form being replaced by a Dirac distribution. As a result, the correlators of these observables in the tropical limit end up having the same deRham structure and ghost structure and the only thing that was needed to be demonstrated in order to completely classify the ring of observables was the cohomology of the target space. Unlike the standard approach to tropical geometry which are based on half-dimensional real algebraic geometries; the Maslov dequantization allows us to probe into tropical geometry by retaining the same dimension for the underlying geometry however it is now equipped with an additional $U(1)$ symmetry that generates the foliations on the geometry. It was shown that this additional $U(1)$ symmetry does not give rise to any additional equivariance and so for the sake of path integral calculations, we identify that the relevant cohomology for the tropical projective space $\mathbb{TP}^2$ is still $H^\bullet\left(\mathbb{CP}^2, \mathbb{Z}\right)$.

In the Nil-equivariant tropological sigma model, our physical observables will still be BRST cohomology classes, however the BRST differential is now promoted to the Nil equivariant BRST differential; the relevant cohomology ring for the tropical observables is now the equivariant cohomology ring $H^\bullet\left(\mathbb{CP}^2, \mathcal{G}\right)$ where $\mathcal{G}$ is our nilmanifold global symmetry. To begin, we need to understand how the Nilmanifold symmetry affects the symplectic form on the target space.

There is a canonical shift in the symplectic form through the moment map. If you have a Lie group G that acts on a target space M by symplectomorphisms with generators K_a, then if the action is Hamiltonian, there exists smooth functions known as moment maps that satisfy $d\mu_a = \iota_{K_a}\omega$, in the Cartan interpretation of equivariant cohomology, one may then shift the symplectic form into the equivariant symplectic form as

$$\omega_{\mathrm{G}} = \omega - \mu_a \phi^a, \tag{3.156}$$

it can be shown that this new symplectic form is equivariantly closed.

For the case of $\mathbb{CP}^2$, one can explicitly construct these moment maps with respect to the Nilmanifold left invariant vector fields that were constructed in (3.137) by solving $d\mu_a = \iota_{K_a}\omega_{\mathrm{FS}}$. As a consequence, our equivariant topological observables are now all shifted by the moment map. The ghost degree 0 observable $\mathcal{O}_{\mathrm{id}}$ still gives zero. However unlike the non-equivariant case, the ghost degree two observable is now modified to

$$\mathcal{O}^{(0)}_{\omega,\mathcal{G}} = \frac{1}{2}\omega_{ij}\psi^i\psi^j - \mu_a\phi^a, \tag{3.157}$$

notice that the entire observable has ghost degree 2. The remaining hierarchy can now be computed via the equivariant topological descent equations as

$$\begin{aligned} dO^{(0)}_{\omega,G} &= Q_G O^{(1)}_{\omega,G}, \\ dO^{(1)}_{\omega,G} &= Q_G O^{(2)}_{\omega,G}, \\ dO^{(0)}_{\Omega,G} &= Q_G O^{(1)}_{\Omega,G}, \\ dO^{(1)}_{\Omega,G} &= Q_G O^{(2)}_{\Omega,G}, \end{aligned} \tag{3.158}$$

where we have the observable $\mathcal{O}_{\Omega,\mathcal{G}}$ is constructed from $\omega_G \wedge \omega_G$ that is now of ghost number 4 and it contains term proportional $\phi^a\phi^b$ since the step 3 nilpotent Lie algebra (3.118) has two non-zero structure constants. One might expect that in addition to the observables given by the equivariant symplectic form, we also have observables that are naturally coming from the additional Nil ghost field ϕ^a by taking a Lie-algebra trace with any integer power n of the ghost field

$$\mathcal{O}^{(0)}_{\phi} = \mathrm{Tr}\phi^{\mathrm{n}}\,. \tag{3.159}$$

Nonetheless, these observables are then expected to generate a ring of observables that are in one to one correspondence with the equivariant cohomology $H^\bullet\left(\mathbb{CP}^2, \mathcal{G}\right) = \mathbb{Z}\left[\phi^a, \widetilde{h}\right] / \left(\widetilde{h}^3\right)$, where $\widetilde{h}$ is now the cohomology class of the equivariant symplectic form. One immediate consequence of the moment map is that our correlators are now polynomials in Grassmann even ϕ^a ghost field due to the fact that our action may be supplemented by the additional term

$$\int_{\Sigma} \left(\Phi^* \omega + \phi^a \mu_a\right). \tag{3.160}$$

We have presented the first systematic steps toward understanding tropological sigma models on higher dimensional targets. By classifying the possible nilpotent degenerations of the complex structures on $\mathbb{CP}^2$, we discovered that a possible candidate for the Jordan structure induces nontrivial filtration which give rise to global symmetries not seen for the tropological sigma model on $\mathbb{CP}^1$ or its direct doubling. We have given additional explicit examples of how the actions of these tropological sigma models can be constructed on these higher dimensional space. The key statement that was made is that the nested Maslov dequantization limit of topological sigma models do not always lead to foliated manifolds but generically yield filtered manifolds with their own algebra. This new filtration structure gives rise to nilpotent symmetries on the space of fields which can then give rise to background gauge fields that allow us to construct equivariant tropological sigma models with a respect to a noncompact nilpotent symmetry group. Although we had originally found Engel symmetry from the overall space of fields, when we restrict to on-shell configurations, it becomes clear that this is a symmetry of the target space as suggested by the filtration arguments and thus should naturally be interpreted as providing an equivariant extension of the tropological sigma model.

Chapter 4

Conclusion and Outlook

The central objective of this thesis has been to take initial steps toward a formulation of string theory genuinely out of equilibrium. The broader claim motivating this work is that the conventional perturbative string framework, while extraordinarily successful in equilibrium and near equilibrium settings, is not yet adequate for describing intrinsically time dependent quantum backgrounds in which the appropriate observables are not transition amplitudes between asymptotic vacua but real time expectation values in evolving density matrices. In ordinary quantum field theory, this conceptual transition is implemented by the Schwinger Keldysh formalism. The guiding premise of this thesis has been that if nonequilibrium string theory exists as a coherent extension of perturbative string theory, then its fundamental worldsheet description must be compatible with an analogous closed time contour structure.

To investigate this proposal in a tractable setting, we then passed to a topological sigma model and studied its tropical limit using localization methods. Within this simplified but highly structured framework, one is able to see explicitly how a cohomological field theory can degenerate onto a tropicalized object whose degrees of freedom are no longer naturally described by given by psuedoholomorphic curves but instead tropical curves.. This led to the construction of what we referred to as the *tropological sigma model* i.e. a tropicalized topological sigma model whose kinematics and dynamics are governed by a randomized extension of tropical geometry. The appearance of tropicalization is especially striking because it suggests that the geometry appropriate to nonequilibrium quantum gravity may be intrinsically singular, stratified, or combinatorial in ways that are largely invisible in static backgrounds. If this is correct, then the problem of nonequilibrium string theory is not merely to generalize known formulas, but to identify a genuinely new mathematical language in which the relevant physical questions can even be posed.

Although the tropological sigma models are not the final object that we are interested in, they are interesting in their own right. The subsequent analysis showed that these tropological sigma models are not empty formal constructions, but possess nontrivial dynamical and geometric structure. In particular, they admit boundary conditions suggesting the existence of tropical branes, and they allow target space geometries that differ substantially from the standard geometric backgrounds familiar from equilibrium sigma models. Rather than arising from conventional smooth target geometries alone, these models naturally accommodate filtered spaces and exotic kinematical structures associated with nilpotent Lie algebras in higher dimensional target spaces. We have investigated how tropological sigma models are affected by these noncompact nilpotent symmetry groups. One might wonder if these nilpotent symmetries give rise to anomalous symmetries and the important question is whether these symmetries may be gauged. At first glance, the nilpotency of the symmetry seems to suggest that no nondegenerate (in fact, nontrivial) adjoint invariant kinetic term can be written down. However, we should not be too quick in concluding that a gauging is impossible due to the fact that the existence of a non-equivariant tropological sigma model itself bases on the fact that we no longer have nondegenerate metric tensors to write kinetic terms i.e., we can nonetheless make sense of a nondegenerate quadratic form through the use of densities which allows us to in principle compute the path integrals through localization. At the very least, we would not expect any gauge anomalies due to the fact that all nilpotent Lie groups are unimodular.

It would be interesting to see the direct evaluation of such a path integral integral through supersymmetric localization to show that the theorem of Mikhalkin that tropicalization preserves Gromov-Witten invariants holds in the equivariant case where the equivariance arises due to some filtration structure on the target space. In this sense, the invariants that are defined in this model seem to go a bit further than the standard Gromov-Witten or their equivariant analogs for standard complex geometries. We conjecture that these new filtered Gromov Witten invariants might be a possible novel invariant for filtered geometries. Unlike relativistic topological sigma models, it has been previously stated in [60] that tropological sigma models can also be directly interpreted in odd-dimensions unlike the standard relativistic case which would provide a possible way to more concretely connect the symplectic geometry of GW invariants with the filtered geometry of contact geometries. We leave this as an open question for future work.

Additionally, the Jordan-block analysis that was done in the previous chapter is easily extended for $\mathbb{CP}^n$ where $n > 2$. Already for $n = 3$, one can easily see that we have much more possible filtered algebras with much more exotic symmetries that will generically be some step nilpotent Lie-algebras which are not usually studied. We have previously mentioned that complex projective space has a unique complex structure but more generic complex manifolds have a moduli space of complex structures that are allowed. In particular, we would expect tropicalized Calabi-Yau three folds to have an interesting moduli space of Jordan structures that generically yield trivial filtrations that don't give any additional symmetry but may give rise to enhanced global symmetries when investigated at a moduli space point that describes an exotic Jordan structure in a similar respect to how enhanced gauge symmetries arise when investigating singularities of Calabi-Yau manifolds[148]. In a similar respect, we saw that allowing for higher dimensions allowed us to consider nested Maslov dequantizations and hence nested foliations which gave rise to anisotropic nontrivial filtration structures. We expect that in higher dimensions, one may encounter much more rich structure, for example in the case of G_2 manifolds and once again we want to emphasize that the question about how to connect these higher dimensional nested tropicalizations to these to higher dimensional analogs of contact geometries is a promising avenue.

Although one in principle can evaluate the correlation functions on these spaces through the standard arguments of cohomological field theory, it would be incredibly interesting to see how what a direct path integral localization calculation would yield in the intermediate steps. In particular, one would expect that the geometric analysis required would be the same that used in analyzing supersymmetric localization on sub-Riemannian geometries as was shown in [149] where we have shown that similar nilpotent structures can appear in the target space in the sub-Riemannian setting.

Furthermore, we have primarily focused on the case of the topological sigma model. The natural question would be to probe what would be the natural coupling to a tropological version of topological gravity. Preliminary work suggests that such a tropological gravity would require a dynamical theory of worldsheet foliations.

In concluding, it should be noted that while the primary focus of this thesis has been the tropicalization relevant to the wedge region, a natural extension of this research entails identifying analogous tropical subsectors within the complete nonequilibrium string. Concurrently with this work, preliminary investigations were conducted to determine the applicability of these frameworks to other topological field theories, particularly 2D BF theory. As a full exposition of these developments falls beyond the scope of this manuscript, the reader is directed to [150] [151] for a comprehensive analysis of tropical limits of other topological field theories.

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